3 Exercice 3

3.1

f est une densité de probabilité donc

$$\begin{aligned} &\int_{\mathbb{R}} f = 1 \\ \iff &\int_0^1 k x^a \mathrm{d}x = 1 \\ \iff &k \left[\frac{x^{a+1}}{a+1} \right]_0^1 = 1 \\ \iff &k(\frac{1}{a+1} - 0) = 1 \\ \iff &k = a+1 \end{aligned}$$

3.2

$$\begin{aligned} E(X^n) &= \int_{\mathbb{R}} x^n k x^a \mathrm{d}x \\ &= (a+1) \left[\frac{x^{n+a+1}}{n+a+1} \right]_0^1 \\ &= \frac{a+1}{n+a+1} \end{aligned}$$

$$E(X) = \frac{a+1}{a+2}$$

$$\begin{aligned} \operatorname{Var} X &= E(X^2) - E(X)^2 \\ &= \frac{a+1}{a+3} - \left(\frac{a+1}{a+2} \right)^2 \\ &= \end{aligned}$$

3.3

- Si $x < 0$, $F_X(x) = 0$
- Si $x > 1$, $F_X(x) = 1$

- Sinon

$$\begin{aligned}
F_X(x) &= P(X < x) \\
&= \int_0^x f \\
&= k \int_0^x y^a \mathrm{d}y \\
&= k \left[\frac{y^{a+1}}{a+1} \right]_0^x \\
&= \frac{(a+1)x^{a+1}}{a+1} \\
&= x^{a+1}
\end{aligned}$$

La fonction $-\ln$ est bijective de $]0,1]$ dans $]0,1]$. Y admet donc une densité de probabilité f_Y

$$Y = -\ln X \iff X = e^{-Y}$$

Donc

$$f_Y(x) = f_X(e^{-x}) \left| \frac{\mathrm{d}(e^{-y})}{\mathrm{d}y} \right| = (a+1)e^{-(a+1)y} \mathbf{1}_{\mathbb{R}^+}(y)$$

3.4

$$\begin{aligned}
E(Y^n) &= \int_{\mathbb{R}^+} y^n f(y) \mathrm{d}y \\
&= \int_{\mathbb{R}^+} y^n (a+1) e^{-(a+1)y} \mathrm{d}y \\
&= \int_{\mathbb{R}^+} \left(\frac{u}{a+1} \right)^n \cancel{(a+1)} e^{-u} \frac{du}{\cancel{a+1}} \qquad \text{avec } u = (a+1)y,\ dy = \frac{du}{a+1} \text{ car } (a+1) \,\mathrm{id}\ C^1 \text{ bijective} \\
&= \frac{1}{(a+1)^n} \int_{\mathbb{R}^+} u^n e^{-u} \mathrm{d}u \\
&= \frac{1}{(a+1)^n} \Gamma(n+1) \\
&= \frac{n!}{(a+1)^n}
\end{aligned}$$

$$\begin{aligned}
E(Y) &= \frac{1}{a+1} \\
\operatorname{Var} Y &= E(Y^2) - E(Y)^2 \\
&= \frac{2}{(a+1)^2} - \frac{1}{(a+1)^2} \\
&= \frac{1}{(a+1)^2}
\end{aligned}$$

3.5

$$Z = \left| X - \frac{1}{2} \right| \iff X = \begin{cases} \frac{1}{2} - X & \text{si } X \leq \frac{1}{2} \\ X - \frac{1}{2} & \text{sinon} \end{cases}$$

$x \mapsto |x - \frac{1}{2}|$ est bijective de $[0, \frac{1}{2}]$ et de $[\frac{1}{2}, 1]$ dans de $[0, \frac{1}{2}]$ et $[\frac{1}{2}, 1]$

1re bijection

$$z = \frac{1}{2} - x \implies$$

...

$$f_Z(z) = (a+1)\left[(\frac{1}{2} - z)^a + (\frac{1}{2} + z)^a\right] \mathbf{1}_{[0, \frac{1}{2}]}$$

5 Files d'attente

$X \coprod Y$, $X \sim \mathcal{E}(\lambda)$, $Y \sim \mathcal{E}(\mu)$

$$\begin{aligned} F_T(t) &= P(T < t) \\ &= P(\inf(X, Y) < t) \\ &= 1 - P(\inf(X, Y) \geq t) \\ &= 1 - P(X \geq t)P(Y \geq t) \qquad \text{car } X \coprod Y \\ &= 1 - F_X(t)F_Y(t) \end{aligned}$$

Or

$$\begin{aligned} F_T(t) &= \int_0^1 \lambda e^{-\lambda x} \mathrm{d}x \\ &\vdots \\ &= 1 - e^{-(\lambda+\mu)t} \end{aligned}$$

Donc $\forall (X, Y) \sim (\mathcal{E}(\lambda), \mathcal{E}(\mu)),\ X \coprod Y \implies \inf(X, Y) \sim \mathcal{E}(\lambda + \mu)$

5.1

Notons T_A (resp. T_B, T_C) le temps en service de A (resp. B, C)

$$T_C = T = \inf(T_A, T_B)$$

5.2

Temps moyen dans le système de C:

$$\begin{aligned} Tempsmoyend'attente + tempsmoyenenservice &= E(T) + E(T_C) \\ &= \frac{1}{\lambda_A + \lambda_B} + \frac{1}{\lambda_C} \end{aligned}$$