Actuarial Mathematics Analytics

An open text authored by the Actuarial Community

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Contents

Preface

Book Description

Actuarial Mathematics Analytics is an interactive, online, freely available text.

- The online version contains many interactive objects (quizzes, computer demonstrations, interactive graphs, video, and the like) to promote *deeper learning*.
- A subset of the book is available for *offline reading* in pdf and EPUB formats.
- The online text will be available in multiple languages to promote access to a *worldwide audience*.

What will success look like?

The online text will be freely available to a worldwide audience. The online version will contain many interactive objects (quizzes, computer demonstrations, interactive graphs, video, and the like) to promote deeper learning. Moreover, a subset of the book will be available in pdf format for low-cost printing. The online text will be available in multiple languages to promote access to a worldwide audience.

How will the text be used?

This book will be useful in actuarial curricula worldwide. It will cover the loss data learning objectives of the major actuarial organizations. Thus, it will be suitable for classroom use at universities as well as for use by independent learners seeking to pass professional actuarial examinations. Moreover, the text will also be useful for the continuing professional development of actuaries and other professionals in insurance and related financial risk management industries.

Why is this good for the profession?

An online text is a type of open educational resource (OER). One important benefit of an OER is that it equalizes access to knowledge, thus permitting a broader community to learn about the actuarial profession. Moreover, it has the capacity to engage viewers through active learning that deepens the learning process, producing analysts more capable of solid actuarial work. Why is this good for students and teachers and others involved in the learning process?

Cost is often cited as an important factor for students and teachers in textbook selection (see a recent post on the $400 textbook). Students will also appreciate the ability to "carry the book around" on their mobile devices.

Why loss data analytics?

Although the intent is that this type of resource will eventually permeate throughout the actuarial curriculum, one has to start somewhere. Given the dramatic changes in the way that actuaries treat data, loss data seems like a natural place to start. The idea behind the name *loss data analytics* is to integrate classical loss data models from applied probability with modern analytic tools. In particular, we seek to recognize that big data (including social media and usage based insurance) are here and high speed computation s readily available.

Project Goal

The project goal is to have the actuarial community author our textbooks in a collaborative fashion.

To get involved, please visit our Loss Data Analytics Project Site.

Chapter 1

Survival Models and Life Tables

Let (x) denote a life aged x.

1.1 Future Lifetime

- T_x = **time-until-death for (x)**, a continuous random variable (in years).

  T_x is also called the **future lifetime random variable**. T_x may also be written as $T(x)$ or T.

  Special case: T_0 = **age-at-death for (0)**, where (0) denotes a newborn life. Note: T_0 = x + T_x, given survival to age x.

- $F_x(t) = {}_tq_x = Pr(T_x \leq t)$

  This is the **cumulative distribution function of** T_x, "the probability that (x) dies within t years." The q-notation will be used most of the time.

  $F_0(t)$ can also be written more simply as $F(t)$.

- $S_x(t) = {}_tp_x = Pr(T_x > t)$

  This is the **survival function function of** T_x,"the probability that (x) survives for at least t years." The words "at least" are often omitted. The p-notation will be used most of the time.

  $S_0(t)$ can also be written more simply as $S(t)$ or $s(t)$.

- From above: ${}_tq_x + {}_tp_x = 1$.

  "(x) will either survive or die within t years."

- $S_0(x+t) = S_0(x){}_tp_x$

  "The probability that (0) survives $x+t$ years is equivalent to (0) first surviving x years to age x, and then surviving t additional years to age x + t."

- ${}_{u+t}p_x = ({}_up_x)({}_tp_{x+u})$

  "The probability that (x) survives $u+t$ years is equivalent to (x) first surviving u years to age x + u, and then surviving t additional years to age x + u + t."

- Be careful: ${}_{u+t}q_x \neq ({}_uq_x)({}_tq_{x+u})$.

  "The right-hand side implies that it is possible for (x) to die within u years, then somehow come back to life at age x + u in order to die again within t years. This, of course, is not possible and cannot be equal to the left-hand side which is the probability (x) dies within $u+t$ years."

- ${}_{u|t}q_x = {}_{u+t}q_x$ - ${}_{u}q_x = {}_{u}p_x$ - ${}_{u+t}p_x = ({}_{u}p_x)({}_{t}q_{x+u})$

  This is a u-year deferred probability of death, "the probability that (x) dies between ages x + u and x + u + t." Note: ${}_{0|t}q_x = {}_{t}q_x$.

- Note: ${}_{1}q_x$, ${}_{1}p_x$, and ${}_{u|1}q_x$ are written as q_x, p_x, and ${}_{u|}q_x$, respectively.

  q_x may be referred to as a **mortality rate** at age x, and p_x may be referred to as a **survival rate** at age x.

1.2 Force of Mortality

- $\mu_x = \mu(x) =$ **force of mortality at age x**, given survival to age x. This is sometimes called the "hazard rate" or "failure rate."

  $\mu_x = -\frac{\frac{d}{dx}[S_0(x)]}{S_0(x)} = -\frac{d}{dx}[\ln S_0(x)]$

  "This is the instantaneous death rate for a life at age x."

- $\mu_{x+t} = \mu_x(t) =$ **force of mortality at age x + t**, given survival to x + t.

  $\mu_{x+t} = -\frac{\frac{d}{dt}[{}_{t}p_x]}{{}_{t}p_x} = -\frac{d}{dt}[\ln {}_{t}p_x]$

  "This is the instantaneous death rate for a life at age x + t. Here, the variable is time after age x. You could also obtain μ_{x+t} by replacing x in μ_x with x + t."

- ${}_{t}p_x = \exp[-\int_x^{x+t} \mu_s ds] = \exp[-\int_0^t \mu_{x+s} ds]$

- If $c > 0$, then $\mu^*_{x+s} = c\mu_{x+s} \implies {}_{t}p^*_x = ({}_{t}p_x)^c$.

  For constant k, then $\mu^*_{x+s} = \mu_{x+s} + k \implies {}_{t}p^*_x = (e^{-kt})({}_{t}p_x)$.

  The constant k should be such that $\mu^*_{x+s} > 0$.

- $f_x(t) = {}_{t}p_x\mu_{x+t} =$ **probability density function of** T_x.

  This comes from the above formula for μ_{x+t}, recognizing that $f_x(t) = \frac{d}{dt}[{}_{t}q_x] = -\frac{d}{dt}[{}_{t}p_x]$.

- ${}_{t}q_x = \int_0^t {}_{s}p_x\mu_{x+s}ds$

- ${}_{t}p_x = \int_t^\infty {}_{s}p_x\mu_{x+s}ds$

- ${}_{u|t}q_x = \int_u^{u+t} {}_{s}p_x\mu_{x+s}ds$

1.3 Curtate Future Lifetime

- $K_x =$ **curtate future lifetime for (x)**, a discrete random variable.

  $K_x = \lfloor T_x \rfloor =$ integer part of T_x. That is, K_x represents the complete number of future years survived by (x), where any fractional time survived in the year of death is ignored. Note: $K_x = 0, 1, 2\ldots$

  K_x may also be written as $K(x)$ or K.

- ${}_{k|}q_x = Pr(K_x = k) = Pr(k \leq T_x < k+1)$ for $k = 0, 1, 2\ldots$

  This is the **probability mass function of** K_x, "the probability that (x) dies in the (k + 1)st year, between ages x + k and x + k + 1."

- ${}_{k+1}q_x = {}_{0|}q_x + {}_{1|}q_x + \ldots + {}_{k|}q_x$

  "The probability that (x) dies within $k + 1$ years is the sum of the probabilities that (x) dies in the first year, the second year, . . . , the $(k + 1)$st year."

1.4 Other Features of T_x and K_x Distributions

- $\mathring{e}_x = E(T_x) = \int_0^\infty t({}_tp_x)(\mu_{x+t})dt = \int_0^\infty {}_tp_x dt$

  This is the **complete expectation of life for (x)**, the average time-until-death for (x). That is, (x) is expected to die at age x + $\mathring{e}_x$.
- $Var(T_x) = E(T_x^2)$ - $[E(T_x)]^2 = 2\int_0^\infty t\,{}_tp_x dt$ - $[\mathring{e}_x]^2$
- $\mathring{e}_{x:\overline{n}|} = E[min(T_x, n)] = \int_0^n {}_tp_x dt$

  This is the **n-year temporary complete life expectancy for (x)**, the average number of years out of the next n years that (x) survives.

  This expectation helps define the recursion: $\mathring{e}_x = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}$.

  "The average number of future years that (x) survives is the average number of years out of the first n years that (x) survives plus the average number of years (x) survives after the first n years (accounting for the probability that (x) survives the first n years)."

  Similarly: $\mathring{e}_{x:\overline{m+n}|} = \mathring{e}_{x:\overline{m}|} + {}_mp_x\mathring{e}_{x+m:\overline{n}|}$.
- The **100α-th percentile of the distribution of** T_x, π_α, is such that:

  ${}_{\pi_\alpha}q_x = \alpha$ for $0 \le \alpha \le 1$.

  Special case: $\alpha = 0.50$; $\pi_{.50}$ is called the **median future lifetime for (x)**.
- $e_x = E(K_x) = \sum_{k=0}^\infty k({}_{k|}q_x) = \sum_{k=1}^\infty {}_kp_x$

  This is the **curtate expectation of life for (x)**, the average curtate future lifetime for (x).
- $\mathring{e}_x \approx e_x + \frac{1}{2}$
- $Var(K_x) = E(K_x^2)$ - $[E(K_x)]^2 = \sum_{k=1}^\infty (2k-1){}_kp_x$ - $[e_x]^2$
- $e_{x:\overline{n}|} = E[min(K_x, n)] = \sum_{k=1}^n {}_kp_x$

  This is the **n-year temporary curtate life expectancy for (x)**.

  This expectation helps define the recursions: $e_x = e_{x:\overline{n}|} + {}_np_xe_{x+n}$ and

  $e_{x:\overline{m+n}|} = e_{x:\overline{m}|} + {}_mp_xe_{x+m:\overline{n}|}$.

1.5 Special Mortality Laws

1.5.1 de Moivre's Law

T_x has a continuous uniform distribution.

The limiting age is ω such that $0 \le x \le x + t \le \omega$.

- $\mu_x = \frac{1}{\omega - x}$ (Note: $x \neq \omega$)
- $S_0(x) = \frac{\omega - x}{\omega}$
- $F_0(x) = \frac{x}{\omega}$

- ${}_tp_x = \frac{\omega-x-t}{\omega-x}$
- ${}_tq_x = \frac{t}{\omega-x}$
- ${}_{u|t}q_x = \frac{t}{\omega-x}$
- $f_x(t) = {}_tp_x\mu_{x+t} = \frac{1}{\omega-x}$ (Note: $x \neq \omega$)
- $\mathring{e}_x = \frac{\omega-x}{2}$
- $\mathring{e}_{x:\overline{n}|} = n{}_np_x + \frac{n}{2}{}_nq_x$

  "(x) can either survive n years with probability ${}_np_x$, or die within n years with probability ${}_nq_x$. Surviving n years contributes n to the expectation. Dying within n years contributes $\frac{n}{2}$ to the expectation as future lifetime has a uniform distribution - (x), on average, would die halfway through the n-year period."
- $Var(T_x) = \frac{(\omega-x)^2}{12}$
- $e_x = \frac{\omega-x-1}{2}$
- $Var(K_x) = \frac{(\omega-x)^2}{12}$ - $\frac{1}{12}$

1.5.2 Modified/Generalized de Moivre's Law

T_x has a beta distribution.

The limiting age is ω such that $0 \leq x \leq x+t \leq \omega$. Also, there is a parameter $\alpha > 0$.

- $\mu_x = \frac{\alpha}{\omega-x}$ (Note: $x \neq \omega$)
- $S_0(x) = \left(\frac{\omega-x}{\omega}\right)^\alpha$
- $F_0(x) = 1$ - $\left(\frac{\omega-x}{\omega}\right)^\alpha$
- ${}_tp_x = \left(\frac{\omega-x-t}{\omega-x}\right)^\alpha$
- ${}_tq_x = 1$ - $\left(\frac{\omega-x-t}{\omega-x}\right)^\alpha$

  Note: ${}_tq_x \neq \left(\frac{t}{\omega-x}\right)^\alpha$.
- $\mathring{e}_x = \frac{\omega-x}{\alpha+1}$
- $Var(T_x) = \frac{\alpha(\omega-x)^2}{(\alpha+1)^2(\alpha+2)}$

  Note: $\alpha = 1$ results in uniform distribution/de Moivre's Law.

1.5.3 Constant Force of Mortality

T_x has an exponential distribution, $x \geq 0$. There is another parameter that denotes the force of mortality: $\mu > 0$.

- $\mu_x = \mu$
- $S_0(x) = e^{-\mu x}$
- $F_0(x) = 1$ - $e^{-\mu x}$
- ${}_tp_x = e^{-\mu t} = (p_x)^t$

- ${}_tq_x$ = 1 - $e^{-\mu t}$
- $\mathring{e}_x = \frac{1}{\mu}$
- $\mathring{e}_{x:\overline{n}|} = \frac{1-e^{-\mu n}}{\mu}$
- $Var(T_x) = \frac{1}{\mu^2}$
- $e_x = \frac{p_x}{q_x}$
- $Var(K_x) = \frac{p_x}{(q_x)^2}$

  Note1: A constant force of mortality implies that "age does not matter." This can easily be seen from ${}_tp_x = e^{-\mu t}$; x does not appear on the right-hand side.

  Note2: T_x has an exponential distribution implies that K_x has a geometric distribution.

1.5.4 Gompertz's Law

- $\mu_x = Bc^x$ for $x \geq 0$, $B > 0$, $c > 1$
- $S_0(x) = \exp[-\frac{B}{\ln c}(c^x - 1)]$
- $F_0(x)$ = 1 - $\exp[-\frac{B}{\ln c}(c^x - 1)]$
- ${}_tp_x = \exp[-\frac{B}{\ln c}c^x(c^t - 1)]$
- ${}_tq_x$ = 1 - $\exp[-\frac{B}{\ln c}c^x(c^t - 1)]$

Note: $c = 1$ results in a constant force of mortality.

1.5.5 Makeham's Law

- $\mu_x = A + Bc^x$ for $x \geq 0$, $A \geq$ -B, $B > 0$, $c > 1$
- $S_0(x) = \exp[-Ax - \frac{B}{\ln c}(c^x - 1)]$
- $F_0(x)$ = 1 - $\exp[-Ax - \frac{B}{\ln c}(c^x - 1)]$
- ${}_tp_x = \exp[-At - \frac{B}{\ln c}c^x(c^t - 1)]$
- ${}_tq_x$ = 1 - $\exp[-At - \frac{B}{\ln c}c^x(c^t - 1)]$

Note1: $A = 0$ results in Gompertz's Law.

Note2: $c = 1$ results in a constant force of mortality.

1.5.6 Weibull's Law

T_x has a Weibull distribution.

- $\mu_x = kx^n$ for $x \geq 0$, $k > 0$, $n > 0$
- $S_0(x) = \exp[-\frac{k}{n+1}x^{n+1}]$
- $F_0(x)$ = 1 - $\exp[-\frac{k}{n+1}x^{n+1}]$
- ${}_tp_x = \exp[-\frac{k}{n+1}((x+t)^{n+1} - x^{n+1})]$
- ${}_tq_x$ = 1 - $\exp[-\frac{k}{n+1}((x+t)^{n+1} - x^{n+1})]$

1.6 Life Tables

Given a survival model with survival probabilities ${}_tp_x$, one can construct a **life table**, also called a **mortality table**, from some initial age x_0 (usually age 0) to a maximum age ω (a limiting age).

- Let l_{x_0}, the **radix** of the life table, represent the number of lives age x_0.

  l_{x_0} is an arbitrary positive number.

- $l_\omega = 0$.

- $l_{x+t} = (l_x)({}_tp_x)$ for $x_0 \leq x \leq x+t \leq \omega$.

  l_{x+t} represents the **expected number of survivors** to age x + t out of l_x individuals aged x.

- ${}_td_x = l_x$ - $l_{x+t} = (l_x)({}_tq_x)$ for $x_0 \leq x \leq x+t \leq \omega$.

  ${}_td_x$ represents the **expected number of deaths** between ages x and x + t out of l_x lives aged x.

  Note 1: ${}_1d_x$ is written as d_x.

  Note 2: If n = 1, 2. . . , then ${}_nd_x = d_x + d_{x+1} + \ldots + d_{x+n-1}$.

- ${}_td_{x+u} = l_{x+u}$ - $l_{x+u+t} = (l_x)({}_{u|t}q_x)$.

The **Illustrative Life Table** is the life table that is provided to the candidate taking Exam MLC. Some questions from either exam will involve Illustrative Life Table calculations. A web link to this table (and ALL exam tables) is provided for each exam in Appendix A of this study supplement.

1.7 Fractional Age Assumptions

Life Tables are usually defined for integer ages x and integer times t. For a quantity that involves fractional ages and/or fractional times, one has to make an assumption about the survival distribution between integer ages; that is, one has to interpolate the value of the quantity within each year of age. Two common interpolation assumptions follow.

1.7.1 Uniform Distribution of Deaths (UDD)

One linearly interpolates within each year of age. For integer age x and $0 \leq s \leq s+t \leq 1$:

- $l_{x+s} = l_x$ - $sd_x = (1-s)l_x + (s)l_{x+1}$. This is a linear function of s.
- ${}_sq_x = sq_x$
- ${}_sp_x = 1$ - sq_x
- $\mu_{x+s} = \frac{q_x}{1-sq_x}$ (does not hold at $s = 1$)
- $f_x(s) = {}_sp_x\mu_{x+s} = q_x$ (does not hold at $s = 1$)
- ${}_sq_{x+t} = \frac{sq_x}{1-tq_x}$
- $\mathring{e}_x = e_x + \frac{1}{2}$
- $Var(T_x) = Var(K_x) + \frac{1}{12}$
- Note: uniform distribution/de Moivre's Law has the property of UDD across all ages up to the limiting age ω.

  Furthermore, uniform distribution/de Moivre's Law may be expressed as $l_x = k(\omega - x)$ for $0 \leq x \leq \omega$ where $k > 0$.

1.7.2 Constant Force of Mortality

One exponentially interpolates within each year of age. For integer age x and $0 \leq s \leq s+t \leq 1$:

- $l_{x+s} = l_x p_x^s \implies \ln[l_{x+s}] = (1-s)\ln[l_x] + s\ln[l_{x+1}]$. This is an exponential function of s.
- ${}_sp_x = p_x^s$
- ${}_sq_x = 1$ - p_x^s
- $\mu_{x+s} =$ -ln $p_x = \mu_x$ (does not hold at $s = 1$)
- $f_x(s) = {}_sp_x\mu_{x+s} =$ -ln $p_x(p_x^s)$ (does not hold at $s = 1$)
- ${}_sq_{x+t} = 1$ - p_x^s

1.8 Exercises

1.1. Suppose: $F_0(t) = 1$ - $(1 + 0.00026t^2)^{-1}$ for $t \geq 0$.

Calculate the probability that (30) dies between ages 35 and 40.

(A) 0.056 (B) 0.058 (C) 0.060 (D) 0.062 (E) 0.064

1.2. You are given: $s(x) = \frac{10{,}000 - x^2}{10{,}000}$ for $0 \leq x \leq 100$.

Calculate: q_{49}.

(A) 0.009 (B) 0.011 (C) 0.013 (D) 0.015 (E) 0.017

1.3. Consider a population of newborns (lives aged 0). Each newborn has mortality such that:

$S_0(x) = \frac{x^2}{\omega^2}$ - $\frac{2x}{\omega} + 1$ for $0 \leq x \leq \omega$.

It is assumed that ω varies among newborns, and is a random variable with a uniform distribution between ages 85 and 105.

Calculate the probability that a random newborn survives to age 18.

(A) 0.656 (B) 0.657 (C) 0.658 (D) 0.659 (E) 0.660

1.4. Suppose: $S_0(t) = \exp[-\frac{t^2}{2500}]$ for $t \geq 0$.

Calculate the force of mortality at age 45.

(A) 0.036 (B) 0.039 (C) 0.042 (D) 0.045 (E) 0.048

1.5. The probability density function of the future lifetime of a brand new machine is: $f(x) = \frac{4x^3}{27c}$ for $0 \leq x \leq c$.

Calculate: $\mu(1.1)$.

(A) 0.06 (B) 0.07 (C) 0.08 (D) 0.09 (E) 0.10

1.6. You are given:

(i) The probability that (30) will die within 30 years is 0.10.

(ii) The probability that (40) will survive to at least age 45 and that another (45) will die by age 60 is 0.077638.

(iii) The probability that two lives age 30 will both die within 10 years is 0.000096.

(iv) All lives are independent and have the same expected mortality.

Calculate the probability that (45) will survive 15 years.

(A) 0.90 (B) 0.91 (C) 0.92 (D) 0.93 (E) 0.94

1.7. You are given:

(i) $e_{50} = 20$ and $e_{52} = 19.33$

(ii) $q_{51} = 0.035$

Calculate: q_{50}.

(A) 0.028 (B) 0.030 (C) 0.032 (D) 0.034 (E) 0.036

1.8. You are given:

$S_0(x) = \left(\frac{1+0.005(1.1)^x}{1.005}\right)^{-0.2098}$ for $x > 0$.

Calculate the force of mortality at age 30.

(A) 0.0012 (B) 0.0016 (C) 0.0020 (D) 0.0024 (E) 0.0028

1.9. For a population of smokers and non-smokers:

(i) Non-smokers have a force of mortality that is equal to one-half the force of mortality for smokers at each age.

(ii) For non-smokers, mortality follows a uniform distribution with $\omega = 90$.

Calculate the difference between the probability that a 55 year old smoker dies within 10 years and the probability that a 55 year old non-smoker dies within 10 years.

(A) 0.20 (B) 0.22 (C) 0.24 (D) 0.26 (E) 0.28

1.10. You are given:

(i) The standard probability that (40) will die prior to age 41 is 0.01.

(ii) (40) is now subject to an extra risk during the year between ages 40 and 41.

(iii) To account for the extra risk, a revised force of mortality is defined for the year between ages 40 and 41.

(iv) The revised force of mortality is equal to the standard force of mortality plus a term that decreases linearly from 0.05 at age 40 to 0 at age 41.

Calculate the revised probability that (40) will die prior to age 41.

(A) 0.030 (B) 0.032 (C) 0.034 (D) 0.036 (E) 0.038

1.11. An actuary assumes that Jed, aged 40, has the force of mortality:

$\mu_x = \frac{x^2}{c^3-x^3}$ for $0 \le x < c$.

Using μ_x, the actuary calculates the probability that Jed dies within 20 years as 0.06844. However, μ_x is only appropriate for a life with standard mortality. Jed is actually a substandard life with force of mortality:

$\mu_x^* = 3\mu_x = \frac{3x^2}{c^3-x^3}$ for $0 \le x < c$.

Using μ_x^*, calculate the correct value of the probability that Jed dies within 20 years.

(A) 0.16 (B) 0.17 (C) 0.18 (D) 0.19 (E) 0.20

1.12. You are given:

(i) T_x denotes the time-until-death random variable for (x).

(ii) Mortality follows de Moivre's Law with limiting age ω.

(iii) The variance of T_{25} is equal to 352.0833.

Calculate: $\mathring{e}_{40:\overline{10}|}$.

(A) 7.5 (B) 8.0 (C) 8.5 (D) 9.0 (E) 9.5

1.13. You are given: $\mu_x = \frac{1}{\sqrt{80-x}}$ for $0 \leq x < 80$.

Calculate the median future lifetime for (40).

(A) 4.0 (B) 4.3 (C) 4.6 (D) 4.9 (E) 5.2

1.14. You are given:

$$\mu_x = \begin{cases} 0.04 & \text{for } 0 \leq \text{x} < 40 \\ 0.05 & \text{for } 40 \leq \text{x} \end{cases}$$

Calculate: $\mathring{e}_{25}$.

(A) 22 (B) 23 (C) 24 (D) 25 (E) 26

1.15. You are given: ${}_{k|}q_0 = 0.10$ for $k = 0, 1, \ldots, 9$.

Calculate: ${}_5p_2$.

(A) 0.275 (B) 0.325 (C) 0.375 (D) 0.425 (E) 0.475

1.16. For the current model of Zingbot:

(i) $s(x) = \frac{\omega - x}{\omega}$ for $0 \leq x \leq \omega$

(ii) $var[T(5)] = 102.083333$.

For the proposed model of Zingbot, with the same ω as the current model:

(1) $s^*(x) = (\frac{\omega - x}{\omega})^\alpha$ for $0 \leq x \leq \omega$, $\alpha > 0$

(2) $\mu^*_{10} = 0.0166667$.

Calculate the difference between the complete expectation of life for a brand new proposed model of Zingbot and the complete expectation of life for a brand new current model of Zingbot.

(A) 5.9 (B) 6.1 (C) 6.3 (D) 6.5 (E) 6.7

1.17. Mortality for Frodo, age 33, is usually such that:

${}_tp_x = \left(\frac{110-x-t}{110-x}\right)^2$ for $0 \leq t \leq 110 - x$.

However, Frodo has decided to embark on a dangerous quest that will last for the next three years (starting today). During these three years only, Frodo's mortality will be revised so that he will have a constant force of mortality of 0.2 for each year. After the quest, Frodo's mortality will once again follow the above expression for ${}_tp_x$.

Calculate Frodo's revised complete expectation of life.

(A) 15.2 (B) 15.4 (C) 15.6 (D) 15.8 (E) 16.0

1.18. You are given:

$$\mu_x = \begin{cases} 0.020 & \text{for } 20 \leq x < 30 \\ 0.025 & \text{for } 30 \leq x < 42 \\ 0.030 & \text{for } 42 \leq x < 60 \end{cases}$$

Calculate the probability that (26) dies in the 19th year.

(A) 0.015 (B) 0.017 (C) 0.019 (D) 0.021 (E) 0.023

1.19. An actuary has developed a survival model for a widget, denoted by A, such that:

$S_0^A(x) = \frac{(10-x)^2}{100}$ for $0 \le x \le 10$.

The actuary's supervisor notes that the above survival model is incorrect. The correct survival model for a widget, denoted by B, is such that:

$$S_0^B(x) = \begin{cases} \frac{(10-x)^2}{100} & \text{for } 0 \le x < 5 \\ \\ e^{-0.2\ln(4)x} & \text{for x} \ge 5 \end{cases}$$

Calculate: $\mathring{e}_2^B$ - $\mathring{e}_2^A$.

(A) 0.70 (B) 0.72 (C) 0.74 (D) 0.76 (E) 0.78

1.20. For a group of lives aged 40, consisting of 30% smokers and 70% non-smokers, you are given:

(i) For non-smokers, $\mu^N(x) = 0.05$ for $x \ge 40$.

(ii) For smokers, $\mu^S(x) = 0.10$ for $x \ge 40$.

Calculate the 90th percentile of the distribution of the future lifetime of a randomly selected member from this population.

(A) 40 (B) 42 (C) 44 (D) 46 (E) 48

1.21. You are given:

(i) T_x is the time-until-death for (x) random variable.

(ii) The force of mortality is constant.

(iii) $e_x = 15.63$

Calculate the variance of T_x.

(A) 240 (B) 250 (C) 260 (D) 270 (E) 280

1.22. Originally, mortality for Daniel, currently aged 30, is such that:

(i) $e_{30} = 40.78$

(ii) $e_{30:\overline{15}|} = 14.07$

(iii) ${}_{15}p_{30} = 0.8764$ and ${}_{16}p_{30} = 0.8664$

(iv) The limiting age is 100.

Now, it is believed that in the year of age between ages 45 and 46, Daniel will be subject to an additional risk such that the constant 0.15 will be added to the force of mortality $\mu_{45}(t)$ for $0 \le t < 1$.

Calculate the revised value of e_{30} for Daniel, accounting for the additional risk in the year of age between ages 45 and 46.

(A) 36 (B) 37 (C) 38 (D) 39 (E) 40

1.23. For (x):

(i) K is the curtate future lifetime random variable.

(ii)

$${}_{k|}q_x = \begin{cases} 0.20 & \text{for } k = 0,\ 1,\ 2 \\ 0.40 & \text{for } k = 3 \end{cases}$$

Calculate the standard deviation of K.

(A) 1.1 (B) 1.2 (C) 1.3 (D) 1.4 (E) 1.5

1.24. You are given:

(i) $\mu(x) = B(1.05)^x$ for $x \geq 0$, $B > 0$.

(ii) $p_{51} = 0.9877$

Calculate: B.

(A) 0.001 (B) 0.002 (C) 0.003 (D) 0.004 (E) 0.005

1.25. You are given:

(i) The force of mortality for Vivian is $\mu_x^V = \mu$ for $x \geq 0$, $\mu > 0$.

(ii) The force of mortality for Augustine is $\mu_x^A = \frac{1}{90-x}$ for $0 \leq x < 90$.

Calculate μ so that ${}_{10}p_{30}$ is the same for Vivian and Augustine.

(A) 0.016 (B) 0.018 (C) 0.020 (D) 0.022 (E) 0.024

1.26. Consider the following survival function:

$S_0(x) = 0.0125\sqrt{k^2 - x^2}$ for $0 \leq x \leq k$.

Calculate the force of mortality at age 46.

(A) 0.011 (B) 0.013 (C) 0.015 (D) 0.017 (E) 0.019

1.27. For a certain model of widget, the Widget T-1000:

$\mu_x = \frac{2x}{c^2 - x^2}$ for $0 \leq x < c$.

A brand new Widget T-1000 has a complete expectation of life equal to 8 years. Calculate the probability that a one-year old Widget T-1000 survives at least two years but no more than four years.

(A) 0.11 (B) 0.12 (C) 0.13 (D) 0.14 (E) 0.15

1.28. Actuary A and Actuary B are each trying to calculate the 3-year temporary curtate life expectancy for Miguel, aged 60.

Both agree that a standard life has the following force of mortality:

$\mu_x^S = 0.00006 + 0.0000006(1.15)^x$ for $x \geq 0$.

Both also agree that Miguel has a force of mortality that exceeds the force of mortality for a standard life at all ages. However, Actuary A believes that Miguel has force of mortality $1.15\mu_x^S$, while Actuary B believes that Miguel has force of mortality $\mu_x^S + 1.15$.

Calculate the difference between Actuary A's 3-year temporary curtate life expectancy for Miguel and Actuary B's 3-year temporary curtate life expectancy for Miguel.

(A) 2.1 (B) 2.3 (C) 2.5 (D) 2.7 (E) 2.9

1.29. For a population of lives each aged 55 that consists of smokers and non-smokers:

(i) For smokers, $\mu_x = 0.0008x$ for $x \geq 0$.

(ii) The force of mortality for a smoker is twice the force of mortality for a non-smoker at each age.

Calculate the median future lifetime for a non-smoker from this population.

(A) 24.0 (B) 24.5 (C) 25.0 (D) 25.5 (E) 26.0

1.30. Suppose today is January 1, 2014, and Paul has just turned age 35. He has mortality such that:

${}_tp_{35} = (0.95)^t$ for $t \geq 0$.

Calculate the probability that Paul will die in an odd-numbered calendar year.

(A) 0.48 (B) 0.49 (C) 0.50 (D) 0.51 (E) 0.52

1.31. Consider the following life table, where missing entries are denoted by "—":

x	q_x	l_x	d_x
48	—	90,522	—
49	0.007453	89,900.9286	—

Calculate the expected number of deaths between ages 48 and 50.

(A) 1280 (B) 1290 (C) 1300 (D) 1310 (E) 1320

1.32. You are given the following life table, where missing entries are denoted by "—":

x	l_x	q_x	e_x
65	79,354	0.0172	—
66	—	0.0186	—
67	—	—	—
68	74,993	—	14.89
69	—	—	14.22

Calculate the expected number of deaths between ages 67 and 69.

(A) 2970 (B) 3020 (C) 3070 (D) 3120 (E) 3170

1.33. You are given:

(i) $l_x = 1000(\omega - x)$ for $0 \leq x \leq \omega$

(ii) $\mu_{30} = 0.0125$

Calculate: $\mathring{e}_{40:\overline{20}|}$.

(A) 17.1 (B) 17.6 (C) 18.1 (D) 18.6 (E) 19.1

1.34. You are given the following life table, where missing values are indicated by "—":

x	l_x	d_x	p_x
0	1000.0	—	0.875
1	—	125.0	—
2	—	—	—
3	—	—	0.680
4	—	182.5	—
5	200.0	—	—

Calculate: ${}_{2|}q_0$.

(A) 0.16 (B) 0.17 (C) 0.18 (D) 0.19 (E) 0.20

1.35. Woolhouse is currently age 40. Woolhouse's mortality follows 130% of the Illustrative Life Table; that is, q_x for Woolhouse is 130% of q_x in the Illustrative Life Table for x = 40, 41, . . . , 110.

Calculate Woolhouse's 4-year temporary curtate life expectancy.

(A) 3.950 (B) 3.955 (C) 3.960 (D) 3.965 (E) 3.970

1.36. Suppose mortality follows the Illustrative Life Table, and deaths are uniformly distributed within each year of age.

Calculate: ${}_{4.5}q_{40.3}$.

(A) 0.0141 (B) 0.0142 (C) 0.0143 (D) 0.0144 (E) 0.0145

1.37. Suppose mortality follows the Illustrative Life Table, where deaths are assumed to be uniformly distributed between integer ages.

Calculate the median future lifetime for (32).

(A) 44.7 (B) 45.0 (C) 45.3 (D) 45.6 (E) 45.9

1.38. Suppose mortality follows the Illustrative Life Table with the assumption that deaths are uniformly distributed between integer ages.

Calculate: ${}_{0.9}q_{60.6}$.

(A) 0.0130 (B) 0.0131 (C) 0.0132 (D) 0.0133 (E) 0.0134

1.39. You are given the mortality rates:

$q_{30} = 0.020$, $q_{31} = 0.019$, $q_{32} = 0.018$.

Assume deaths are uniformly distributed over each year of age.

Calculate the 1.4-year temporary complete life expectancy for (30).

(A) 1.377 (B) 1.379 (C) 1.381 (D) 1.383 (E) 1.385

1.40. Using the Illustrative Life Table, calculate: ${}_{11|17}q_{42}$.

(A) 0.20 (B) 0.21 (C) 0.23 (D) 0.24 (E) 0.25

1.41. Consider two survival models A and B:

(i) For Model A: $l_x = 1000(\omega_A$ - $x)$ for $0 \leq x \leq \omega_A$

(ii) For Model B: $l_x = 500(\omega_B - x)^\alpha$ for $0 \leq x \leq \omega_B$, $\alpha > 0$

Furthermore:

(1) For Model B, the force of mortality at age 55 is 0.046.

(2) The complete expectation of life for (40) under Model A is 39.615% higher than the complete expectation of life for (40) under Model B.

(3) For Model A, the probability that (45) survives the first 20 years and dies in the subsequent 10 years is 0.20.

For Model B, calculate the probability that (45) dies between ages 65 and 75.

(A) 0.16 (B) 0.17 (C) 0.18 (D) 0.19 (E) 0.20

1.42. Consider a population that consists of 600 lives aged 50 and 520 lives aged 60.

Each life has mortality that follows the Illustrative Life Table, and all lives have independent future lifetime random variables.

Calculate the standard deviation of the total number of survivors to age 80.

(A) 14.7 (B) 15.2 (C) 15.7 (D) 16.2 (E) 16.7

1.43. Suppose:

(i) $q_{70} = 0.04$ and $q_{71} = 0.05$.

(ii) Let UDD denote a uniform distribution of deaths assumption within each year of age, and let CF denote a constant force of mortality within each year of age.

Calculate the probability that (70.6) will die within the next 0.5 years under UDD minus the probability that (70.6) will die within the next 0.5 years under CF.

(A) 0.00008 (B) 0.00010 (C) 0.00012 (D) 0.00014 (E) 0.00016

1.44. You are given:

(i) The force of mortality is constant between integer ages.

(ii) ${}_{0.3}q_{x+0.7} = 0.10$

Calculate: q_x.

(A) 0.24 (B) 0.26 (C) 0.28 (D) 0.30 (E) 0.32

1.45. A life insurer issues Roderick, aged 40, a policy that will pay 10,000 upon survival of a number of years equal to Roderick's median future lifetime. You are given:

(i) $d = 0.04$

(ii) For Roderick: $q_{40+k} = 0.05(1 + k)$ for $k = 0, 1, \ldots, 19$. (Roderick is a very unfortunate individual with respect to his future lifetime distribution.)

(iii) The force of mortality is constant between integer ages.

Calculate the expected present value of the 10,000 payment; that is, calculate the present value of 10,000 times the probability that the 10,000 will be paid to Roderick.

(A) 4100 (B) 4130 (C) 4160 (D) 4190 (E) 4220

1.46. For a mortality table:

(i) $q_{70} = 0.058$

(ii) Deaths are uniformly distributed within each year of age.

(iii) The probability that (71.25) dies within 0.4 years is 0.0252.

Calculate the probability that (70) dies between ages 71.25 and 71.65.

(A) 0.021 (B) 0.023 (C) 0.025 (D) 0.027 (E) 0.029

1.47. Andy, aged 66, has mortality rates that are 3 times higher than mortality rates in the Illustrative Life Table. That is, for $x = 66, 67, \ldots, 110$:

$q_x^* = 3q_x$,

where q_x is a mortality rate in the Illustrative Life Table and q_x^* is Andy's corresponding mortality rate.

Calculate the probability that Andy dies between ages 68 and 70.

(A) 0.11 (B) 0.12 (C) 0.13 (D) 0.14 (E) 0.15

1.8.1 Answers to Exercises

1.1. E 1.26. A

1.2. C 1.27. A

1.3. A 1.28. C

1.4. A 1.29. D

1.5. B 1.30. B

1.6. C 1.31. B

1.7. B 1.32. E

1.8. B 1.33. A

1.9. A 1.34. D

1.10. C 1.35. C

1.11. D 1.36. E

1.12. D 1.37. C

1.13. B 1.38. A

1.14. A 1.39. C

1.15. C 1.40. C

1.16. E 1.41. D

1.17. D 1.42. E

1.18. C 1.43. A

1.19. D 1.44. D

1.20. A 1.45. C

1.21. C 1.46. B

1.22. B 1.47. D

1.23. B

1.24. A

1.25. B

1.9 Past Exam Questions

- Exam MLC, Fall 2015: #1, 2
- Exam MLC, Spring 2015: #1
- Exam MLC, Spring 2014: #1
- Exam MLC, Fall 2013: #24, 25
- Exam 3L, Fall 2013: #1, 2, 3
- Exam MLC, Spring 2013: #20
- Exam 3L, Spring 2013: #1, 2, 3
- Exam MLC, Fall 2012: #3
- Exam 3L, Fall 2012: #1, 2, 3
- Exam MLC, Spring 2012: #2 (MLC Only)
- Exam 3L, Spring 2012: #1, 2, 3
- Exam MLC, Sample Questions: #13, 21, 22, 28, 32, 59, 65, 98, 106, 116, 120, 131, 145, 155, 161, 171, 188, 189, 200, 201, 207, 219, 223, 267 (MLC Only), 276
- Exam 3L, Fall 2011: #1, 2
- Exam 3L, Spring 2011: #1, 2, 3
- Exam 3L, Fall 2010: #1, 2, 3

- Exam 3L, Spring 2010: #1, 2, 3, 4
- Exam 3L, Fall 2009: #1, 2, 3
- Exam 3L, Spring 2009: #1, 3
- Exam 3L, Fall 2008: #12, 13, 14
- Exam 3L, Spring 2008: #13, 14, 15, 16
- Exam MLC, Spring 2007: #1, 21

Chapter 2

Selection

2.1 Key Concepts

For a life table based on an insured population, one must consider for each individual both (i) the age of policy issue and (ii) the time that has elapsed since policy issue. This is because the insurer typically underwrites individuals that purchase a policy. Through the underwriting, the insurer learns additional information about the individual's survival distribution that the insurer would not know for a life randomly drawn from the general population. This additional information must be accounted for in the calculation of various quantities such as survival probabilities for the individual and the value of the individual's policy.

- Consider an individual now aged x + t who purchased a policy at age x. We say that the individual was **selected**, or **select**, at age x (and time $t = 0$).
- The additional information gained from underwriting the above individual, obtained by surveys and/or a medical examination, is assumed to apply for a certain number of years after policy issue called the **select period**.
- Say the select period is d years. For $t < d$, one accounts for the initial selection of the above individual at age x; the individual's current age would be written as

  $$x$$

  + t (the select brackets

  denote the initial age of selection). For $t \geq d$, one no longer accounts for the initial selection of the individual at age x, and the individual's age would be written simply as x + t (with no select brackets

).
- An individual has **select mortality** for ages/times within the select period that differs from the mortality of the general population. An individual has **ultimate mortality** for ages/times beyond the select period where their mortality is assumed to be the same as a life from the general population.
- A life table that accounts for both select and ultimate mortality is called a **select-and-ultimate life table**.
- A life table that ignores selection completely is called an **aggregate life table**.
- The previous formulas for the quantities considered so far, such as survival probabilities, are still valid in the event of selection. One simply has to use information from the select part of the select-and-ultimate life table for ages/times within the select period.

- For example, with a select period of 3 years, ${}_2p_{[x]} = (p_{[x]})(p_{[x]+1})$ and ${}_5p_{[x]} = (p_{[x]})(p_{[x]+1})(p_{[x]+2})(p_{x+3})(p_{x+4})$. The p's with select brackets would come from the select part of the select-and-ultimate life table, and the p's without select brackets would come from the ultimate part.
- An illustrative select-and-ultimate table is the

  Standard Select and Ultimate Survival Model. This table is provided in Appendix D of Dickson et al. For brevity, I will refer to this table as the **Standard Select Survival Model**.

2.2 Exercises

2.1. Mortality follows the select-and-ultimate life table:

Calculate: $10{,}000 {}_{1|}q_{[30]}$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

2.2. Suppose mortality follows the Standard Select Survival Model.

Calculate: ${}_{1|2}q_{[70]+1}$.

(A) 0.025 (B) 0.027 (C) 0.029 (D) 0.031 (E) 0.033

2.3. Consider the following select-and-ultimate life table:

$\mathbf{x}$	$\mathbf{q_{[x]}}$	$\mathbf{q_{[x]+1}}$	$\mathbf{q_{[x]+2}}$	$\mathbf{q_{x+3}}$	$\mathbf{x+3}$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

Assume that deaths follow the uniform distribution of deaths assumption between integer ages.

Calculate: ${}_{1.6}q_{[61]+0.75}$.

(A) 0.1855 (B) 0.1856 (C) 0.1857 (D) 0.1858 (E) 0.1859

2.4. A select-and-ultimate life table with a select period of 2 years is based on probabilities that satisfy the following relationship:

$q_{[x-i]+i} = \frac{3}{5-i} \times q_x$ for $i = 0, 1$.

You are given that $l_{68} = 10{,}000$, $q_{66} = 0.026$, and $q_{67} = 0.028$.

Calculate: $l_{[65]+1}$.

(A) 10,414 (B) 10,451 (C) 10,479 (D) 10,493 (E) 11,069

2.5. Suppose mortality follows the Standard Select Survival Model.

Calculate: $e_{[60]:\overline{5}|}$.

(A) 4.928 (B) 4.932 (C) 4.936 (D) 4.940 (E) 4.944

2.6. Consider a select-and-ultimate life table with a 2-year select period.

You are given:

(i) $l_{[35]} = 1500$

(ii) $l_{36} = 1472.31$

(iii) $q_{[35]} = 0.0240$

(iv) $q_{[35]+1} = 0.0255$

Calculate: $l_{35}({}_{1|}q_{35})$.

(A) 42 (B) 44 (C) 46 (D) 48 (E) 50

2.7. Quinn is currently age 60. He was selected by the PlzDntDie Life Insurance Company one year ago. Quinn has mortality that follows a select-and-ultimate life table with a 2-year select period:

(i) The ultimate part of the model is such that:

$\mu_x = 0.0002(1.1)^x$ for $x \geq 0$.

(ii) The select part of the model is such that:

$\mu_{[x]+s} = (0.8)^{2-s}\mu_{x+s}$ for $x \geq 0$, $0 \leq s \leq 2$.

Calculate the probability that Quinn survives to age 61.

(A) 0.938 (B) 0.944 (C) 0.950 (D) 0.956 (E) 0.962

2.8. You are given:

(i) Mortality follows the Standard Select Survival Model.

(ii) Deaths are uniformly distributed over each year of age.

Calculate: $\mathring{e}_{[75]+1:\overline{1.3}|}$.

(A) 1.277 (B) 1.280 (C) 1.283 (D) 1.287 (E) 1.290

2.9. For a select and ultimate life table with a 1-year select period:

(i) $\mu_{[55]+t} = 0.5\mu_{55+t}$ for $0 \leq t \leq 1$

(ii) $e_{55} = 18.02$

(iii) $e_{[55]} = 18.33$

Calculate: e_{56}.

(A) 17.60 (B) 17.65 (C) 17.70 (D) 17.75 (E) 17.80

2.10. Consider a population of lives each age 55 and selected at that age, where 70% are non-smokers and 30% are smokers.

The force of mortality is:

$\mu_{[55]+t} = 0.009t(0.4 + 0.1S)$ for $t \geq 0$,

where $S = 0$ for a non-smoker and $S = 1$ for a smoker.

Calculate the probability that a randomly chosen life from the above population will die before age 75.

(A) 0.51 (B) 0.52 (C) 0.53 (D) 0.54 (E) 0.55

2.11. For a select-and-ultimate life table with a 2-year select period:

(i) Ultimate mortality follows the Illustrative Life Table.

(ii) $q_{[x]} = 0.5q_x$, where q_x is from the Illustrative Life Table.

(iii) $q_{[x]+1} = 0.25q_{x+1}$, where q_{x+1} is from the Illustrative Life Table.

Calculate the probability that a select life aged 70 will die after age 75.

(A) 0.79 (B) 0.81 (C) 0.83 (D) 0.85 (E) 0.87

2.12. Consider the setup in Exercise 2.11. Calculate the 4-year temporary curtate expectation of life for a select life aged 70.

(A) 3.79 (B) 3.81 (C) 3.83 (D) 3.85 (E) 3.87

2.2.1 Answers to Exercises

2.1. C

2.2. B

2.3. C

2.4. D

2.5. E

2.6. C

2.7. B

2.8. C

2.9. B

2.10. D

2.11. D

2.12. A

2.3 Past Exam Questions

- Exam MLC, Fall 2014: #20
- Exam MLC, Spring 2014: #2
- Exam MLC, Fall 2013: #3
- Exam MLC, Spring 2013: #19
- Exam MLC, Fall 2012: #2
- Exam MLC, Spring 2012: #1, 13
- Exam MLC, Sample Questions: #66, 73, 136, 168
- Exam MLC, Spring 2007: #18

Chapter 3

Annuities I

3.1 Key Concepts

A **life annuity** policy provides payments to an **annuitant** each period while that person survives.

Annuities can be described as either (i) **continuous**: the payments are made continuously each year while the annuitant survives or (ii) **discrete**: the payments are made at the beginning or the end of each period while the annuitant survives. If the payments are made at the beginning of each period, the policy is an **annuity-due**; if the payments are made at the end of each period, the policy is an **annuity-immediate**.

Types of Life Annuities:

- **Whole Life Annuity:** Provides payments each period while the annuitant survives.
- **Temporary Life Annuity:** Provides payments each period while the annuitant survives for at most n years after policy issue. This is also called a **term annuity**.
- **Deferred Life Annuity:** The annuitant must survive a u-year deferral period after policy issue in order for any payments to be made. A **deferred whole life annuity** provides payments each period while the annuitant survives if the annuitant first survives the u-year deferral period; that is, payments are made after u years while the annuitant survives. A **deferred temporary life annuity** provides payments each period while the annuitant survives for at most n years after first surviving the u-year deferral period; that is, payments are made between u years and $u + n$ years after policy issue while the annuitant survives.
- **Certain and Life Annuity:** Is guaranteed to provide payments for the first n years after policy issue, regardless of whether the annuitant survives or dies within the n-year period. If the annuitant survives the first n years, the annuity continues to provide payments each period for as long as the annuitant survives. This is also called a **guaranteed annuity**.

  For each of these annuities, the following will be considered:

  - The **present value of the annuity**, Y, are the payments discounted for interest between policy issue and each payment date. This is a random variable, as the number of payments is a function of the future lifetime of the annuitant.
  - The **expected present value of the annuity**, $E(Y)$, are the payments discounted for both interest and survival between policy issue and each potential payment date. $E(Y)$ is with respect to the distribution of the annuitant's future lifetime.

    $E(Y)$ will be written differently for each type of life annuity considered. In addition, $E(Y)$ can also be called the **actuarial present value of the annuity**, the **single premium**, the **net single premium**, or the **single benefit premium**.

– Secondary characteristics of the distribution of Y that will be of interest include the variance of Y and percentiles of the distribution of Y.

3.2 Life Annuity Formulas

This section provides key formulas for different life annuities. Note:

- Life annuities have continuous and discrete versions. In the continuous case, the payments are made continuously each year up until the moment of death of the annuitant (T_x years after policy issue). The discrete case can be : (i) annual or (ii) m-thly (Exam MLC only). In (i), the payments are made at either the beginning of each year (for a total of $K_x + 1$ payments), or payments are made at the end of each year (for a total of K_x payments).

  It is possible that the number of payments made could be subject to a finite term and/or a deferral period. It is also possible that there may be a guaranteed number of payments.

- The general formula for the present value of a continuous life annuity on (x) with payment rate π_t at time t (> 0) is:

  $Y = \int_0^{T_x} \pi_t v^t dt = \int_0^{T_x} \pi_t e^{-\delta t} dt.$

  The general formula for the expected present value of a continuous life annuity on (x) with payment rate π_t at time t (> 0) is:

  $E(Y) = \int_0^{\infty} \pi_t v^t {}_tp_x dt = \int_0^{\infty} \pi_t e^{-\delta t} {}_tp_x dt = \int_0^{\infty} \pi_t ({}_tE_x) dt.$ Furthermore:

  "Say the payment rate is π_t at time t. Then the present value of this benefit at time t is $\pi_t v^t dt$, and the expected present value of this benefit is $\pi_t v^t {}_tp_x dt$ ((x) has to survive to time t in order for $\pi_t dt$ to be made at that time). Integrating over all possible payment times provides the overall expected present value. This is the **current payment approach.**"

- The general formula for the present value of an annual life annuity-due that pays π_k at time k ($k = 0$, 1, 2, ...) is:

  $Y = \sum_{k=0}^{K_x} \pi_k v^k.$

  The general formula for the expected present value of an annual life annuity-due that pays π_k at time k ($k = 0$, 1, 2, ...) is:

  $E(Y) = \sum_{k=0}^{\infty} \pi_k v^k {}_kp_x = \sum_{k=0}^{\infty} \pi_k ({}_kE_x).$ Furthermore:

  "Say the payment is π_k at time k. Then the present value of this benefit is $\pi_k v^k$, and the expected present value of this benefit is $\pi_k v^k {}_kp_x$. Summing over all possible payment times provides the overall expected present value. This is the current payment approach."

  Note: The general formulas for the present value and expected present value of an annual life annuity-immediate on (x) are similar, expect there would be no payment at $k = 0$.

- The International Actuarial Notation for $E(Y)$ often contains an a, which indicates that the expected present value is for a life annuity. For example, $\ddot{a}_x$ denotes the expected present value of a whole life annuity-due of 1 per year on (x).

- ${}^2a$ is a evaluated at double the force of interest.

- ${}^2a$ is NOT equal to $E(Y^2)$. This implies that $Var(Y)$ is NOT equal to ${}^2a$ - a^2 (which will usually be negative).

- For each annuity, a (total) payment rate of 1 per year is assumed. If the payment rate is R, notation and formulas are adjusted. For example, the expected present value of a continuous whole life annuity of R per year on (x) is $E(Y) = R\bar{a}_x = \int_0^{\infty} R({}_tE_x)$.

- Recursion formulas take an expected present value and decomposes it into the sum of two expected present values: the expected present value of the annuity during the first period plus the expected present value of the annuity if the annuitant survives the first period.

  For example: $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$. "The right-hand side breaks up the expected present value of a whole life annuity-due of 1 per year on (x) into the very first payment of 1 at issue plus the expected present value of all of the remaining payments of 1 at issue."

- For many annuities, $Var(Y)$ is too difficult to calculate. Therefore, $Var(Y)$ is omitted for several of the following annuities.

- The **actuarial accumulated value** is the expected present value of a payment or payments divided by a discount factor. For example:

  $\ddot{s}_{x:\overline{n}|} = \frac{1}{{}_nE_x}\ddot{a}_{x:\overline{n}|}$

  = the actuarial accumulated value at time n of an n-year temporary life annuity-due of 1 per year on (x).

3.3 Level Annuities

3.3.1 Whole Life Annuity of 1 per Year on (x)

Continuous Whole Life Annuity:

- $Y = \bar{a}_{\overline{T_x}|} = \frac{1-v^{T_x}}{\delta}$

  So: $Y = \frac{1-Z}{\delta}$, where Z is the present value random variable for a continuous whole life insurance of 1 on (x).

- $E(Y) = \bar{a}_x = \int_0^\infty {}_tE_x dt = \frac{1-\bar{A}_x}{\delta}$
  - With a constant force of mortality: $\bar{a}_x = \frac{1}{\mu+\delta}$

- It is also true that: $\bar{a}_x = \int_0^\infty (\bar{a}_{\overline{t}|})\,{}_tp_x\mu_{x+t}dt$

  "Say the moment of death of (x) occurs at time t. Then the present value of the payments is $\bar{a}_{\overline{t}|}$, and the expected present value of the payments is $(\bar{a}_{\overline{t}|})\,{}_tp_x\mu_{x+t}dt$ ((x) has to survive t years and then immediately die for the present value to be $\bar{a}_{\overline{t}|}$). Integrating over all times of death provides the overall expected present value."

- $Var(Y) = \frac{{}^2\bar{A}_x - [\bar{A}_x]^2}{\delta^2}$

- $F_Y(y) = Pr[Y \leq y] = Pr[\bar{a}_{\overline{T_x}|} \leq y] = {}_cq_x$ where $c =$ -$\frac{\ln(1-\delta y)}{\delta}$
  - For de Moivre's Law (Uniform Distribution):

    $F_Y(y) =$ -$\frac{\ln(1-\delta y)}{\delta(\omega-x)}$ for $0 \leq y \leq \bar{a}_{\overline{\omega-x}|}$. If the annual payment is R, replace y with $\frac{y}{R}$.
  - Constant force of mortality: $F_Y(y) = 1$ - $(1-\delta y)^{\frac{\mu}{\delta}}$ for $0 \leq y \leq \frac{1}{\delta}$. If the annual payment is R, replace y with $\frac{y}{R}$.

  Note: The 100α-th percentile of the distribution of Y, y_α, solves:

  $F_Y(y_\alpha) = \alpha$ for $0 \leq \alpha \leq 1$.

Annual Whole Life Annuity:

Whole Life Annuity-Due:

- $Y_d = \ddot{a}_{\overline{K_x+1|}} = \frac{1-v^{K_x+1}}{d}$

  So: $Y_d = \frac{1-Z}{d}$, where Z is the present value random variable for an annual whole life insurance of 1 on (x).

- $E(Y_d) = \ddot{a}_x = \sum_{k=0}^{\infty} {}_kE_x = \frac{1-A_x}{d}$
  - With a constant force of mortality: $\ddot{a}_x = \frac{1+i}{q+i}$
- It is also true that: $\ddot{a}_x = \sum_{k=0}^{\infty} (\ddot{a}_{\overline{k+1|}})_{k|}q_x$
- $Var(Y_d) = \frac{{}^2A_x - [A_x]^2}{d^2}$
- Recursion: $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$

Whole Life Annuity-Immediate:

- $Y_i = a_{\overline{K_x|}} = \frac{1-v^{K_x}}{i}$

  So: $Y_i = \frac{1-(1+i)Z}{i}$, where Z is the present value random variable for an annual whole life insurance of 1 on (x).

- $E(Y_i) = a_x = \sum_{k=1}^{\infty} {}_kE_x = \ddot{a}_x$ - 1
- $E(Y_i) = a_x = \frac{1-(1+i)A_x}{i}$
- $Var(Y_i) = \frac{{}^2A_x - [A_x]^2}{d^2}$ (same as an annual whole life annuity-due)
- Recursion: $a_x = vp_x(1 + a_{x+1})$

3.3.2 Temporary Life Annuity of 1 per Year on (x)

Continuous Temporary Life Annuity:

- $$Y = \begin{cases} \bar{a}_{\overline{T_x|}} & \text{for } T_x \leq \text{n} \\ \bar{a}_{\overline{n|}} & \text{for } T_x > \text{n} \end{cases}$$

  $= \frac{1-Z}{\delta}$,

  where Z is the present value random variable for a continuous n-year endowment insurance of 1 on (x).

- $E(Y) = \bar{a}_{x:\overline{n|}} = \int_0^n {}_tE_x dt = \frac{1-\bar{A}_{x:\overline{n|}}}{\delta}$
  - With a constant force of mortality: $\bar{a}_{x:\overline{n|}} = \frac{1}{\mu+\delta}[1 - \exp[-(\mu+\delta)n]]$
- $Var(Y) = \frac{{}^2\bar{A}_{x:\overline{n|}} - [\bar{A}_{x:\overline{n|}}]^2}{\delta^2}$

Annual Temporary Life Annuity:

Temporary Life Annuity-Due:

- $$Y_d = \begin{cases} \ddot{a}_{\overline{K_x+1|}} & \text{for } K_x = 0,\ 1,\ ...,\ \text{n - 1} \\ \ddot{a}_{\overline{n|}} & \text{for } K_x = \text{n, n + 1, ...} \end{cases}$$

  $= \frac{1-Z}{d}$,

  where Z is the present value random variable for an annual n-year endowment insurance of 1 on (x).

- $E(Y_d) = \ddot{a}_{x:\overline{n|}} = \sum_{k=0}^{n-1} {}_kE_x = \frac{1-A_{x:\overline{n|}}}{d}$

- $Var(Y_d) = \frac{{}^2A_{x:\overline{n}|} - [A_{x:\overline{n}|}]^2}{d^2}$
- Recursion: $\ddot{a}_{x:\overline{n}|} = 1 + vp_x\ddot{a}_{x+1:\overline{n-1}|}$

Temporary Life Annuity-Immediate:

- $$Y_i = \begin{cases} a_{\overline{K_x}|} & \text{for } K_x = 0, 1, ..., n - 1 \\ a_{\overline{n}|} & \text{for } K_x = n, n + 1, ... \end{cases}$$
- $E(Y_i) = a_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kE_x = \ddot{a}_{x:\overline{n}|}$ - 1 + ${}_nE_x$
- Note: $\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$
- $Var(Y_i) = \frac{{}^2A_{x:\overline{n+1}|} - [A_{x:\overline{n+1}|}]^2}{d^2}$
- Recursion: $a_{x:\overline{n}|} = vp_x(1 + a_{x+1:\overline{n-1}|})$

3.3.3 Deferred Whole Life Annuity of 1 per Year on (x)

Continuous Deferred Whole Life Annuity:

- $$Y = \begin{cases} 0 & \text{for } T_x \leq u \\ v^u\bar{a}_{\overline{T_x-u}|} & \text{for } T_x > u \end{cases}$$
- $E(Y) = {}_{u|}\bar{a}_x = \int_u^\infty {}_tE_x = \bar{a}_x$ - $\bar{a}_{x:\overline{u}|} = {}_uE_x\bar{a}_{x+u}$

Annual Deferred Whole Life Annuity:

Deferred Whole Life Annuity-Due:

- $$Y_d = \begin{cases} 0 & \text{for } K_x = 0, 1, ..., u - 1 \\ v^u\ddot{a}_{\overline{K_x+1-u}|} & \text{for } K_x = u, u + 1, ... \end{cases}$$
- $E(Y_d) = {}_{u|}\ddot{a}_x = \sum_{k=u}^{\infty} {}_kE_x = \ddot{a}_x$ - $\ddot{a}_{x:\overline{u}|} = {}_uE_x\ddot{a}_{x+u}$
- Recursion: ${}_{u|}\ddot{a}_x = 0 + vp_x({}_{u-1|}\ddot{a}_{x+1})$

Deferred Whole Life Annuity-Immediate:

- $$Y_i = \begin{cases} 0 & \text{for } K_x = 0, 1, ..., u - 1 \\ v^u a_{\overline{K_x-u}|} & \text{for } K_x = u, u + 1, ... \end{cases}$$
- $E(Y_i) = {}_{u|}a_x = \sum_{k=u+1}^{\infty} {}_kE_x = a_x$ - $a_{x:\overline{u}|} = {}_uE_xa_{x+u}$
- Note: ${}_{u|}\ddot{a}_x = {}_{u|}a_x + {}_uE_x$
- Recursion: ${}_{u|}a_x = 0 + vp_x({}_{u-1|}a_{x+1})$

3.3.4 Deferred Temporary Life Annuity of 1 per Year on (x)

Continuous Deferred Temporary Life Annuity:

- $$Y = \begin{cases} 0 & \text{for } T_x \leq u \\ v^u\bar{a}_{\overline{T_x-u}|} & \text{for } u < T_x \leq u + n \\ v^u\bar{a}_{\overline{n}|} & \text{for } T_x > u + n \end{cases}$$

- $E(Y) = {}_{u|n}\bar{a}_x = {}_{u|}\bar{a}_{x:\overline{n}|} = \int_u^{u+n} {}_tE_x dt = \bar{a}_{x:\overline{u+n}|}$ - $\bar{a}_{x:\overline{u}|} = {}_uE_x\bar{a}_{x+u:\overline{n}|}$

Annual Deferred Temporary Life Annuity:

Deferred Temporary Life Annuity-Due:

- $$Y_d = \begin{cases} 0 & \text{for } K_x = 0, 1, ..., u - 1 \\ v^u\ddot{a}_{\overline{K_x+1-u}|} & \text{for } K_x = u, u + 1, ..., u + n - 1 \\ v^u\ddot{a}_{\overline{n}|} & \text{for } K_x = u + n, u + n + 1, ... \end{cases}$$

- $E(Y_d) = {}_{u|n}\ddot{a}_x = {}_{u|}\ddot{a}_{x:\overline{n}|} = \sum_{k=u}^{u+n-1} {}_kE_x = \ddot{a}_{x:\overline{u+n}|}$ - $\ddot{a}_{x:\overline{u}|} = {}_uE_x\ddot{a}_{x+u:\overline{n}|}$
- Recursion: ${}_{u|n}\ddot{a}_x = 0 + vp_x({}_{u-1|n}\ddot{a}_{x+1})$

Deferred Temporary Life Annuity-Immediate:

- $$Y_i = \begin{cases} 0 & \text{for } K_x = 0, 1, ..., u - 1 \\ v^u a_{\overline{K_x-u}|} & \text{for } K_x = u, u + 1, ..., u + n - 1 \\ v^u a_{\overline{n}|} & \text{for } K_x = u + n, u + n + 1, ... \end{cases}$$

- $E(Y_i) = {}_{u|n}a_x = {}_{u|}a_{x:\overline{n}|} = \sum_{k=u+1}^{u+n} {}_kE_x = a_{x:\overline{u+n}|}$ - $a_{x:\overline{u}|} = {}_uE_x a_{x+u:\overline{n}|}$
- Recursion: ${}_{u|n}a_x = 0 + vp_x({}_{u-1|n}a_{x+1})$

3.3.5 Certain and Life Annuity of 1 per Year on (x)

Continuous Certain and Life Annuity:

- $$Y = \begin{cases} \bar{a}_{\overline{n}|} & \text{for } T_x \le \text{n} \\ \bar{a}_{\overline{T_x}|} & \text{for } T_x > \text{n} \end{cases}$$

- $E(Y) = \bar{a}_{\overline{\overline{x:\overline{n}|}}} = \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x$

Annual Certain and Life Annuity:

Certain and Life Annuity-Due:

- $$Y_d = \begin{cases} \ddot{a}_{\overline{n}|} & \text{for } K_x = 0, 1, ..., \text{n - 1} \\ \ddot{a}_{\overline{K_x+1}|} & \text{for } K_x = \text{n, n + 1, ...} \end{cases}$$

- $E(Y_d) = \ddot{a}_{\overline{\overline{x:\overline{n}|}}} = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x$
- Recursion: $\ddot{a}_{\overline{\overline{x:\overline{n}|}}} = 1 + vq_x\ddot{a}_{\overline{n-1}|} + vp_x\ddot{a}_{\overline{\overline{x+1:\overline{n-1}|}}}$

Certain and Life Annuity-Immediate:

- $$Y_i = \begin{cases} a_{\overline{n}|} & \text{for } K_x = 0, 1, ..., \text{n - 1} \\ a_{\overline{K_x}|} & \text{for } K_x = \text{n, n + 1, ...} \end{cases}$$

- $E(Y_i) = a_{\overline{\overline{x:\overline{n}|}}} = a_{\overline{n}|} + {}_{n|}a_x$
- Recursion: $a_{\overline{\overline{x:\overline{n}|}}} = v + vq_x a_{\overline{n-1}|} + vp_x a_{\overline{\overline{x+1:\overline{n-1}|}}}$

3.4 Exercises

5.1. Assume: $\mu_x(t) = 0.02$ for $t > 0$ and $\delta = 0.05$.

Calculate: $\bar{a}_{x:\overline{15}|}$.

(A) 8.70 (B) 8.90 (C) 9.10 (D) 9.30 (E) 9.50

5.2. Assume mortality follows: $l_x = 100(110 \text{ - } x)$ for $0 \leq x \leq 110$, and $d = 0.05$.

Calculate: ${}_{15|}\ddot{a}_{45}$.

(A) 4.40 (B) 4.60 (C) 4.80 (D) 5.00 (E) 5.20

5.3. You are given:

(i) $\mu = \delta = c$, where c is a positive constant.

(ii) ${}^2\bar{a}_x = \frac{25}{3}$

Calculate: $Var(\bar{a}_{\overline{T(x)}|})$.

(A) 50 (B) 52 (C) 54 (D) 56 (E) 58

5.4. You are given:

(i) ${}_{k|}q_{35} = 0.005(k + 1)$ for $k = 0, 1, 2, 3$.

(ii) $i = 0.05$

Calculate the actuarial present value of a 4-year temporary life annuity-due of 100 per year on (35).

(A) 350 (B) 360 (C) 370 (D) 380 (E) 390

5.5. You are given:

(i) $\delta = 0.05$

(ii)

$$\mu_x = \begin{cases} 0.05 & \text{for } 0 \leq \text{x} < 50 \\ 0.08 & \text{for x} \geq 50 \end{cases}$$

Calculate the actuarial present value of a continuous whole life annuity of 1 per year on (30).

(A) 9.10 (B) 9.30 (C) 9.50 (D) 9.70 (E) 9.90

5.6. You are given:

$$\delta_t = \begin{cases} 0.05 & \text{for } 0 \leq \text{t} < 10 \\ 0.07 & \text{for t} \geq 10 \end{cases}$$

$$\mu_x(t) = \begin{cases} 0.02 & \text{for } 0 \leq \text{t} < 10 \\ 0.03 & \text{for t} \geq 10 \end{cases}$$

Calculate the expected present value of a continuous 10-year certain and life annuity on (x) of 1 per year.

(A) 11.60 (B) 11.90 (C) 12.20 (D) 12.50 (E) 12.80

5.7. You are given:

(i) $\mu_x(t) = 0.02$ for $t > 0$

(ii) $\delta = 0.06$

Calculate: $\Pr[\bar{a}_{\overline{T_x}|} > \bar{a}_x]$.

(A) 0.42 (B) 0.55 (C) 0.63 (D) 0.84 (E) 0.91

5.8. You are given:

(i) The force of mortality is constant.

(ii) $\bar{A}_x = 0.428571$

(iii) $\bar{A}^{1}_{x:\overline{10}|} = 0.215749$

(iv) Y is the present value random variable for a continuous 10-year temporary life annuity of 500 per year on (x).

Calculate: $E(Y)$.

(A) 3550 (B) 3600 (C) 3650 (D) 3700 (E) 3750

5.9. You are given:

(i) Mortality for a standard life aged 40, denoted as S, is such that:

$q_x^S = 0.032$ for $x = 40, 41, 42, \ldots$

(ii) Mortality for a certain life aged 40, denoted as C, is such that:

$q_{40}^C = 0.048$ and $q_x^C = 0.032$ for $x = 41, 42, 43, \ldots$

(iii) $d = 0.05$

Calculate: $\ddot{a}_{40}^S$ - $\ddot{a}_{40}^C$.

(A) 0.16 (B) 0.17 (C) 0.18 (D) 0.19 (E) 0.20

5.10. You are given:

(i) $\mu_x(t)$ is the force of mortality associated with the Illustrative Life Table.

(ii) $i = 0.05$

Calculate the single benefit premium for a 3-year temporary life annuity-immediate of 1000 per year on (30) payable annually, assuming that the force of mortality used is equal to $\mu_{30}(t) + 0.20$ for $0 \leq t \leq 3$.

(A) 1860 (B) 1900 (C) 1940 (D) 1980 (E) 2020

5.11. A fund is created such that:

(i) There are 60 lives each age 30.

(ii) Each life receives payments of 100 per year for life, payable annually, beginning immediately.

(iii) Mortality follows the Illustrative Life Table.

(iv) The lifetimes are independent.

(v) $i = 0.06$

(vi) The amount of the fund is determined, using the normal approximation, such that the probability that the fund is sufficient to make all payments is 99%.

Calculate the initial amount of the fund.

(A) 98,000 (B) 98,500 (C) 99,000 (D) 99,500 (E) 100,000

5.12. For a group of individuals all age x:

(i) 35% are smokers and 65% are non-smokers.

(ii) The constant force of mortality for smokers is 0.08.

(iii) The constant force of mortality for non-smokers is 0.04.

(iv) $\delta = 0.06$

Calculate $Var[\bar{a}_{\overline{T(x)}|}]$ for an individual chosen at random from this group.

(A) 25 (B) 26 (C) 27 (D) 28 (E) 29

5.13. Suppose Z is the present value random variable for a 2-year pure endowment of 1 on (x). You are given:

(i) $v = 0.95$ and $p_x = 0.98$

(ii) $A_x = 0.45$ and $\ddot{a}_{x+2} = 10.68$

Calculate: $Var(Z)$.

(A) 0.045 (B) 0.050 (C) 0.055 (D) 0.060 (E) 0.065

5.14. Cody, age 25, and Ted, age 30, have each won the actuarial lottery:

(i) Cody has decided to collect his winnings via a 20-year temporary life annuity-due, which pays 400,000 each year.

(ii) Ted has decided to collect his winnings via a 20-year certain and life annuity-due, which pays K each year.

(iii) Mortality for both Cody and Ted follows the Illustrative Life Table, and $i = 0.06$.

The expected present values of Cody's annuity and Ted's annuity are both equal. Calculate: K.

(A) 281,000 (B) 286,000 (C) 291,000 (D) 295,000 (E) 299,000

5.15. Consider a special whole life annuity on (x) which pays R at the beginning of the first year, $2R$ at the beginning of the second year, and $3R$ at the beginning of each year thereafter. You are also given:

(i) The actuarial present value of this annuity is 3333.

(ii) $i = 0.05$

(iii) $p_x = 0.98$ and $p_{x+1} = 0.97$

(iv) $\ddot{a}_{x+2} = 31.105$

Calculate: R.

(A) 30 (B) 35 (C) 40 (D) 45 (E) 50

5.16. Let Y denote the present value random variable for a whole life annuity on (x) of R per year payable continuously each year:

(i) $\delta = 0.050$

(ii) $\mu_x(t) = 0.035$ for $t \geq 0$

(iii) The expected value of Y is 1.64% of the variance of Y.

Calculate: R.

(A) 12 (B) 14 (C) 16 (D) 18 (E) 20

5.17. You are given the following portfolio of mutually independent lives:

(i) 50 lives age 65 purchase a whole life annuity-immediate with an annual payment of 30,000.

(ii) 20 lives age 75 purchase a whole life annuity-immediate with an annual payment of 20,000.

Mortality follows the Illustrative Life Table, and $i = 0.06$.

Let S be the present value for the total payments on the portfolio.

Calculate the 95th percentile of the distribution of S, in millions, using the normal approximation.

(A) 16.9 (B) 17.2 (C) 17.5 (D) 17.8 (E) 18.1

5.18. You are given:

(i) $\mu_x = \frac{1}{90-x}$ for $0 \leq x < 90$

(ii) $i = 0$

Calculate: $\ddot{a}_{30}$.

(A) 29.0 (B) 29.5 (C) 30.0 (D) 30.5 (E) 31.0

5.19. Paul, aged 35, has just taken out a home mortgage loan where he will pay 12,000 at the end of each year for 25 years.

Paul was also required to purchase a life insurance policy that will pay any remaining payments should he die within the 25-year period.

Paul has mortality that follows the Illustrative Life Table. The effective annual interest rate is 6%.

Calculate the expected present value of the life insurance policy.

(A) 5150 (B) 5250 (C) 5350 (D) 5450 (E) 5550

5.20. You are given:

(i) Mortality follows:

(ii) $d = 0.03$

Calculate the probability that the present value of a 5-year temporary life annuity-due of 500 per year on (30) exceeds its actuarial present value.

(A) 0.45 (B) 0.50 (C) 0.55 (D) 0.60 (E) 0.65

5.21. You are given:

(i) $\delta = 0.04$

(ii) $\mu_x = 0.0003(1.05)^x$ for $x \geq 0$

Calculate the expected present value of a 2-year deferred 2-year temporary life annuity-immediate of 100 per year on (34).

(A) 167 (B) 169 (C) 171 (D) 173 (E) 175

5.22. For a 10-year deferred 10-year continuous temporary life annuity of 1000 per year on (x):

(i)

$$\delta_t = \begin{cases} 0.06 & \text{for t} \leq 6 \\ 0.07 & \text{for t} > 6 \end{cases}$$

(ii)

$$\mu_x(t) = \begin{cases} 0.025 & \text{for t} \leq 6 \\ 0.035 & \text{for t} > 6 \end{cases}$$

Calculate the single benefit premium for this annuity.

(A) 2320 (B) 2360 (C) 2400 (D) 2440 (E) 2480

5.23. Consider a policy on (40) that provides the following benefits:

(i) A whole life annuity-due of 500 per year payable annually.

(ii) A death benefit of 5000 payable at the end of the year of death.

Furthermore:

(iii) $i = 0.06$

(iv) Mortality follows the Illustrative Life Table.

Calculate the standard deviation of the present value random variable for this policy.

(A) 515 (B) 530 (C) 545 (D) 560 (E) 575

5.24. You are given:

(i)

(ii) $i = 0.05$

Calculate the actuarial accumulated value at the end of the fifth year of a 5-year temporary life annuity-immediate of 100 per year payable annually on (70).

(A) 310 (B) 425 (C) 540 (D) 665 (E) 785

5.25. Consider a whole life annuity-due of 1 per year payable annually on (x):

(i) $v = 0.965$

(ii) ${}_{10}p_x = 0.920$

(iii) $\ddot{a}_{x+11} = 11.36$

Suppose q_{x+10} is increased to $q_{x+10} + 0.100$.

Calculate the change in the expected present value of the annuity.

(A) - 0.71 (B) - 0.41 (C) - 0.26 (D) 0.26 (E) 0.58

5.26. Consider the following special annuity on (x) payable annually:

(i) The payment at time k, π_k, is such that:

$$\pi_k = \begin{cases} 1000vq_{x+k} & \text{for } k = 0, 1, ..., 19 \\ 0 & \text{for } k = 20, 21, ... \end{cases}$$

(ii) $i = 0.045$

(iii) ${}_{20}p_x = 0.945$

(iv) $a_x = 18.23$

(v) $a_{x+20} = 13.94$

Calculate the expected present value of this annuity.

(A) 30 (B) 32 (C) 34 (D) 36 (E) 38

5.27. Consider a policy on (x) that provides the following benefits:

(i) 1000 per year payable continuously each year while (x) is alive.

(ii) S payable at the moment of death of (x).

Furthermore:

(iii) $\delta = 0.05$

(iv) $\mu_x(t) = 0.03$ for $t > 0$.

(v) W denotes the present value random variable for this policy.

Determine the value of S that minimizes the variance of W.

(A) 5000 (B) 10,000 (C) 15,000 (D) 20,000 (E) 25,000

3.4.1 Answers to Exercises

5.1. D 5.26. B

5.2. B 5.27. D

5.3. B

5.4. C

5.5. D

5.6. E

5.7. C

5.8. B

5.9. D

5.10. A

5.11. C

5.12. A

5.13. A

5.14. E

5.15. C

5.16. E

5.17. B

5.18. D

5.19. C

5.20. B

5.21. D

5.22. D

5.23. E

5.24. E

5.25. A

3.5 Past Exam Questions

- Exam MLC, Spring 2015: #7
- Exam MLC, Fall 2014: #5
- Exam MLC, Spring 2014: #6
- Exam MLC, Fall 2013: #1, 5
- Exam 3L, Fall 2013: #12, 13
- Exam MLC, Spring 2013: #21
- Exam 3L, Spring 2013: #12

- Exam 3L, Fall 2012: #12
- Exam MLC, Spring 2012: #15
- Exam MLC, Sample Questions: #11, 25, 35, 45, 55, 63, 67, 79, 86, 88, 113, 114, 126, 130, 140, 146, 166, 186, 192, 196, 209, 210, 229, 285
- Exam 3L, Fall 2011: #12
- Exam 3L, Spring 2011: #12
- Exam 3L, Spring 2010: #15
- Exam 3L, Fall 2009: #12
- Exam 3L, Spring 2009: #12, 13
- Exam 3L, Fall 2008: #20, 21
- Exam MLC, Spring 2007: #2, 17, 24, 29

Chapter 4

Annuities II

4.1 m-thly Annuities-Due

Here, we consider a **discrete** life annuity where each payment is provided at the beginning of the m-th of a year, conditional on survival. The value of m is typically equal to 2 (half-year), 4 (quarter of a year), or 12 (month).

Note:

- An m-thly life annuity-due is such that each payment is made at the beginning of each m-th of a year (for a total of $m(K_x^{(m)} + \frac{1}{m})$ payments, in years).
- The general exact formula for the expected present value of an m-thly life annuity-due on (x) that pays $\pi_{\frac{k}{m}}$ at time $\frac{k}{m}$ years ($k = 0, 1, 2, \dots$) is:

  $E(Y_d) = \sum_{k=0}^{\infty} \pi_{\frac{k}{m}} v^{\frac{k}{m}} {}_{\frac{k}{m}}p_x = \sum_{k=0}^{\infty} \pi_{\frac{k}{m}} ({}_{\frac{k}{m}}E_x)$. Furthermore:

  "Say the payment is $\pi_{\frac{k}{m}}$ at time $\frac{k}{m}$. Then the present value of this benefit is $\pi_{\frac{k}{m}} v^{\frac{k}{m}}$, and the expected present value of this benefit is $\pi_{\frac{k}{m}} v^{\frac{k}{m}} {}_{\frac{k}{m}}p_x$ ((x) has to survive to time $\frac{k}{m}$ in order for $\pi_{\frac{k}{m}}$ to be made at that time). Summing over all possible payment times provides the overall expected present value."
- The International Actuarial Notation for $E(Y_d)$ often contains an $\ddot{a}^{(m)}$, which indicates that the expected present value is for an m-thly life annuity-due. For example, $\ddot{a}_x^{(m)}$ denotes the expected present value of a whole life annuity-due of 1 per year on (x), payable in equal installments of $\frac{1}{m}$ at the beginning of each m-th of the year.
- ${}^2\ddot{a}^{(m)}$ is $\ddot{a}^{(m)}$ evaluated at double the force of interest.
- Often, we do not use the exact formulas to calculate expected present values for an m-thly life annuity-due. Rather, we approximate these expected present values from the corresponding annual life annuity-due expected present values using one of two assumptions:
  - **UDD**: deaths are uniformly distributed within each year of age. In UDD formulas that approximate m-thly life annuities-due, we use the following functions:

    $\alpha(m) = \frac{id}{i^{(m)}d^{(m)}}$ and $\beta(m) = \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}$

    Let m approach infinity. In UDD formulas that approximate continuous life annuities, we use the following functions:

    $\alpha(\infty) = \frac{id}{\delta^2}$ and $\beta(\infty) = \frac{i-\delta}{\delta^2}$.

You are provided a table of $\alpha(m)$ and $\beta(m)$ for various values of m at $i = 0.06$ during Exam MLC. Please refer to the web link to Exam MLC tables provided in Appendix A of this study supplement.

  - **Woolhouse's Formula**: based on series expansions. For example, Woolhouse's Formula with three terms for an m-thly whole life annuity-due of 1 per year on (x) is:

    $\ddot{a}_x^{(m)} = \ddot{a}_x$ - $\frac{m-1}{2m}$ - $\frac{m^2-1}{12m^2}(\delta + \mu_x)$, and

    Woolhouse's Formula with two terms for an m-thly whole life annuity-due of 1 per year on (x) is:

    $\ddot{a}_x^{(m)} = \ddot{a}_x$ - $\frac{m-1}{2m}$ (this approximates the UDD formula).

    Furthermore, μ_x can be approximated as -$\frac{1}{2}(\ln p_{x-1} + \ln p_x)$.

- Note: The formulas discussed below reduce to the corresponding annual life annuity-due formulas in **Annuities I** when $m = 1$.
- Note: While unlikely, it is possible that m-thly life annuities-immediate could be tested; each payment is made at the end of each m-th of a year (for a total of $m(K_x^{(m)})$ payments, in years).

  You can use the following relations:

  - $a_x^{(m)} = \ddot{a}_x^{(m)}$ - $\frac{1}{m}$
  - $a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)}$ - $\frac{1}{m}(1$ - ${}_nE_x)$
  - ${}_{n|}a_x^{(m)} = a_x^{(m)}$ - $a_{x:\overline{n}|}^{(m)}$
  - $a_{\overline{x:\overline{n}|}}^{(m)} = a_{\overline{n}|}^{(m)} + {}_{n|}a_x^{(m)}$

4.1.1 m-thly Whole Life Annuity-Due of 1 per Year on (x)

- $Y_d = \ddot{a}_{\overline{K_x^{(m)}+\frac{1}{m}}|}^{(m)} = \frac{1-v^{K_x^{(m)}+\frac{1}{m}}}{d^{(m)}}$

  So: $Y_d = \frac{1-Z}{d^{(m)}}$, where Z is the present value random variable for an m-thly whole life insurance of 1 on (x).

- $E(Y_d) = \ddot{a}_x^{(m)} = \sum_{k=0}^{\infty} \frac{1}{m}\, {}_{\frac{k}{m}}E_x = \frac{1-A_x^{(m)}}{d^{(m)}}$
- $Var(Y_d) = \frac{{}^2A_x^{(m)}-[A_x^{(m)}]^2}{[d^{(m)}]^2}$
- Recursion: $\ddot{a}_x^{(m)} = \frac{1}{m} + v^{\frac{1}{m}}\, {}_{\frac{1}{m}}p_x \ddot{a}_{x+\frac{1}{m}}^{(m)}$
- UDD: $\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x$ - $\beta(m)$; $\bar{a}_x = \alpha(\infty)\ddot{a}_x$ - $\beta(\infty)$
- Woolhouse's Formula with 3 terms: $\ddot{a}_x^{(m)} = \ddot{a}_x$ - $\frac{m-1}{2m}$ - $\frac{m^2-1}{12m^2}(\delta + \mu_x)$;

  $\bar{a}_x = \ddot{a}_x$ - $\frac{1}{2}$ - $\frac{1}{12}(\delta + \mu_x)$

4.1.2 m-thly Temporary Life Annuity-Due of 1 per Year on (x)

- $$Y_d = \begin{cases} \ddot{a}_{\overline{K_x^{(m)}+\frac{1}{m}}|}^{(m)} & \text{for } K_x^{(m)} = 0,\ \frac{1}{m},\ ...,\ \text{n - } \frac{1}{m} \\ \ddot{a}_{\overline{n}|}^{(m)} & \text{for } K_x^{(m)} = \text{n, n + } \frac{1}{m},\ ... \end{cases}$$

$= \frac{1-Z}{d^{(m)}}$

where Z is the present value random variable for an m-thly n-year endowment insurance of 1 on (x).

- $E(Y_d) = \ddot{a}^{(m)}_{x:\overline{n}|} = \sum_{k=0}^{mn-1} \frac{1}{m}\,{}_{\frac{k}{m}}E_x = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$
- $Var(Y_d) = \frac{{}^2A^{(m)}_{x:\overline{n}|}-[A^{(m)}_{x:\overline{n}|}]^2}{[d^{(m)}]^2}$
- Recursion: $\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m} + v^{\frac{1}{m}}\,{}_{\frac{1}{m}}p_x \ddot{a}^{(m)}_{x+\frac{1}{m}:\overline{n-\frac{1}{m}}|}$
- UDD: $\ddot{a}^{(m)}_{x:\overline{n}|} = \alpha(m)\ddot{a}_{x:\overline{n}|}$ - $\beta(m)(1-{}_nE_x)$;

  $\bar{a}_{x:\overline{n}|} = \alpha(\infty)\ddot{a}_{x:\overline{n}|}$ - $\beta(\infty)(1-{}_nE_x)$
- Woolhouse's Formula with 3 terms:

  $\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|}$ - $\frac{m-1}{2m}(1-{}_nE_x)$ - $\frac{m^2-1}{12m^2}(\delta+\mu_x-{}_nE_x(\delta+\mu_{x+n}))$;

  $\bar{a}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|}$ - $\frac{1}{2}(1-{}_nE_x)$ - $\frac{1}{12}(\delta+\mu_x-{}_nE_x(\delta+\mu_{x+n}))$

  I would not memorize these UDD and Woolhouse's formulas. Just know the m-thly whole life results and use:

  $\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_x$ - ${}_nE_x\ddot{a}^{(m)}_{x+n}$.

4.1.3 m-thly Deferred Whole Life Annuity-Due of 1 per Year on (x)

- $$Y_d = \begin{cases} 0 & \text{for } K_x^{(m)} = 0, \frac{1}{m}, ..., \text{u - } \frac{1}{m} \\ v^u\ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}-u}|} & \text{for } K_x^{(m)} = \text{u, u + } \frac{1}{m}, ... \end{cases}$$

- $E(Y_d) = {}_{u|}\ddot{a}^{(m)}_x = \sum_{k=mu}^{\infty} \frac{1}{m}\,{}_{\frac{k}{m}}E_x = \ddot{a}^{(m)}_x$ - $\ddot{a}^{(m)}_{x:\overline{u}|} = {}_uE_x\ddot{a}^{(m)}_{x+u}$
- Recursion: ${}_{u|}\ddot{a}^{(m)}_x = 0 + v^{\frac{1}{m}}\,{}_{\frac{1}{m}}p_x({}_{u-\frac{1}{m}|}\ddot{a}^{(m)}_{x+\frac{1}{m}})$
- UDD: ${}_{u|}\ddot{a}^{(m)}_x = \alpha(m)_{u|}\ddot{a}_x$ - $\beta(m)_uE_x$; ${}_{u|}\bar{a}_x = \alpha(\infty)_{u|}\ddot{a}_x$ - $\beta(\infty)_uE_x$
- Woolhouse's Formula with 3 terms: ${}_{u|}\ddot{a}^{(m)}_x = {}_{u|}\ddot{a}_x$ - $\frac{m-1}{2m}{}_uE_x$ - $\frac{m^2-1}{12m^2}{}_uE_x(\delta+\mu_{x+u})$;

  ${}_{u|}\bar{a}_x = {}_{u|}\ddot{a}_x$ - $\frac{1}{2}{}_uE_x$ - $\frac{1}{12}{}_uE_x(\delta+\mu_{x+u})$

  I would not memorize these UDD and Woolhouse's formulas. Just know the m-thly whole life results and use: ${}_{n|}\ddot{a}^{(m)}_x = {}_nE_x\ddot{a}^{(m)}_{x+n}$.

4.1.4 m-thly Deferred Temporary Life Annuity-Due of 1 per Year on (x)

- $$Y_d = \begin{cases} 0 & \text{for } K_x^{(m)} = 0, \frac{1}{m}, ..., \text{u - } \frac{1}{m} \\ v^u\ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}-u}|} & \text{for } K_x^{(m)} = \text{u, u + } \frac{1}{m}, ..., \text{u + n - } \frac{1}{m} \\ v^u\ddot{a}^{(m)}_{\overline{n}|} & \text{for } K_x^{(m)} = \text{u + n, u + n + } \frac{1}{m}, ... \end{cases}$$

- $E(Y_d) = {}_{u|n}\ddot{a}^{(m)}_x = \sum_{k=mu}^{m(u+n)-1} \frac{1}{m}\,{}_{\frac{k}{m}}E_x = \ddot{a}^{(m)}_{x:\overline{u+n}|}$ - $\ddot{a}^{(m)}_{x:\overline{u}|} = {}_uE_x\ddot{a}^{(m)}_{x+u:\overline{n}|}$
- Recursion: ${}_{u|n}\ddot{a}^{(m)}_x = 0 + v^{\frac{1}{m}}\,{}_{\frac{1}{m}}p_x({}_{u-\frac{1}{m}|n}\ddot{a}^{(m)}_{x+\frac{1}{m}})$

4.1.5 m-thly Certain and Life Annuity-Due of 1 per Year on (x)

- $$Y_d = \begin{cases} \ddot{a}^{(m)}_{\overline{n}|} & \text{for } K_x^{(m)} = 0,\ \frac{1}{m},\ ...,\ \text{n - } \frac{1}{m} \\ \ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|} & \text{for } K_x^{(m)} = \text{n, n + } \frac{1}{m},\ ... \end{cases}$$

- $E(Y_d) = \ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{\overline{n}|} + {}_{n|}\ddot{a}^{(m)}_x$
- Recursion: $\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m} + v^{\frac{1}{m}}\,{}_{\frac{1}{m}}q_x \ddot{a}^{(m)}_{\overline{n-\frac{1}{m}}|}$

  $- \ v^{\frac{1}{m}}\,{}_{\frac{1}{m}}p_x \ddot{a}^{(m)}_{x+\frac{1}{m}:\overline{n-\frac{1}{m}}|}$

4.2 Varying Annuities

- **Annually Increasing Annuity:** A life annuity where the annual payment is increased arithmetically for each year the annuitant survives. For example, an annually increasing temporary life annuity-due pays R at the beginning of the first year, $2R$ at the beginning of the second year given survival of the annuitant,$\ldots$, nR at the beginning of year n given survival of the annuitant.
- **Annually Decreasing Annuity:** A life annuity where the annual payment is decreased arithmetically for each year the annuitant survives. For example, an annually decreasing temporary life annuity-due pays nR at the beginning of the first year, $(n-1)R$ at the beginning of the second year given survival of the annuitant,$\ldots$, R at the beginning of year n given survival of the annuitant.
- **Geometrically Increasing Annuity:** A life annuity where the annual payment is increased at a compound rate of j per year for each year the annuitant survives. For example, a geometrically increasing whole life annuity-due pays R at the beginning of the first year, $R(1+j)$ at the beginning of the second year given survival of the annuitant, $R(1+j)^2$ at the beginning of the third year given survival of the annuitant, etc.

4.2.1 Annually Increasing Whole Life Annuity on (x)

Continuous Annually Increasing Whole Life Annuity:

- Provides 1 continuously during the first year, 2 continuously during the second year, 3 continuously during the third year, etc.
- $E(Y) = (I\bar{a})_x = \sum_{k=0}^{\infty}(k+1)_{k|}\bar{a}_{x:\overline{1}|}$

Continuously Increasing Whole Life Annuity:

- The payment rate at time t is t.
- $E(Y) = (\bar{I}\bar{a})_x = \int_0^\infty t\,{}_tE_x dt$
  - With a constant force of mortality: $(\bar{I}\bar{a})_x = \frac{1}{(\mu+\delta)^2}$.

Annually Increasing Whole Life Annuity-Due:

- Provides 1 at the beginning of the first year, 2 at the beginning of the second year, 3 at the beginning of the third year, etc.
- $E(Y_d) = (I\ddot{a})_x = \sum_{k=0}^{\infty}(k+1)_kE_x$
  - With a constant force of mortality: $(I\ddot{a})_x = \frac{(1+i)^2}{(q+i)^2}$

- Recursion: $(I\ddot{a})_x = \ddot{a}_x + vp_x(I\ddot{a})_{x+1}$

Annually Increasing Whole Life Annuity-Immediate:

- Provides 1 at the end of the first year, 2 at the end of the second year, 3 at the end of the third year, etc.
- $E(Y_i) = (Ia)_x = \sum_{k=1}^{\infty} k\,{}_kE_x$
- Recursion: $(Ia)_x = a_x + vp_x(Ia)_{x+1}$

4.2.2 Annually Increasing Temporary Life Annuity on (x)

Continuous Annually Increasing Temporary Life Annuity:

- Provides 1 continuously during the first year, 2 continuously during the second year, . . . , n continuously during year n.
- $E(Y) = (I\bar{a})_{x:\overline{n}|} = \sum_{k=0}^{n-1}(k+1)\,{}_{k|}\bar{a}_{x:\overline{1}|}$

Continuously Increasing Temporary Life Annuity:

- The payment rate at time t is t during the first n years.
- $E(Y) = (\bar{I}\bar{a})_{x:\overline{n}|} = \int_0^n t\,{}_tE_x dt$

Annually Increasing Temporary Life Annuity-Due:

- Provides 1 at the beginning of the first year, 2 at the beginning of the second year, . . . , n at the beginning of year n.
- $E(Y_d) = (I\ddot{a})_{x:\overline{n}|} = \sum_{k=0}^{n-1}(k+1)\,{}_kE_x$
- Recursion: $(I\ddot{a})_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} + vp_x(I\ddot{a})_{x+1:\overline{n-1}|}$

Annually Increasing Temporary Life Annuity-Immediate:

- Provides 1 at the end of the first year, 2 at the end of the second year, . . . , n at the end of year n.
- $E(Y_i) = (Ia)_{x:\overline{n}|} = \sum_{k=1}^{n} k\,{}_kE_x$
- Recursion: $(Ia)_{x:\overline{n}|} = a_{x:\overline{n}|} + vp_x(Ia)_{x+1:\overline{n-1}|}$

4.2.3 Annually Decreasing Temporary Life Annuity on (x)

Continuous Annually Decreasing Temporary Life Annuity:

- Provides n continuously during the first year, n - 1 continuously during the second year, . . . , 1 continuously during year n.
- $E(Y) = (D\bar{a})_{x:\overline{n}|} = \sum_{k=0}^{n-1}(n-k)\,{}_{k|}\bar{a}_{x:\overline{1}|}$

  Also: $(I\bar{a})_{x:\overline{n}|} + (D\bar{a})_{x:\overline{n}|} = (n+1)\bar{a}_{x:\overline{n}|}$.

Continuously Decreasing Temporary Life Annuity:

- The payment rate at time t is n - t during the first n years.
- $E(Y) = (\bar{D}\bar{a})_{x:\overline{n}|} = \int_0^n (n-t)\,{}_tE_x dt$

  Also: $(\bar{I}\bar{a})_{x:\overline{n}|} + (\bar{D}\bar{a})_{x:\overline{n}|} = n\bar{a}_{x:\overline{n}|}$.

Annually Decreasing Temporary Life Annuity-Due:

- Provides n at the beginning of the first year, n - 1 at the beginning of the second year, ..., 1 at the beginning of year n.
- $E(Y_d) = (D\ddot{a})_{x:\overline{n}|} = \sum_{k=0}^{n-1}(n-k)_k E_x$

  Also: $(I\ddot{a})_{x:\overline{n}|} + (D\ddot{a})_{x:\overline{n}|} = (n+1)\ddot{a}_{x:\overline{n}|}$.
- Recursion: $(D\ddot{a})_{x:\overline{n}|} = n + vp_x(D\ddot{a})_{x+1:\overline{n-1}|}$

Annually Decreasing Temporary Life Annuity-Immediate:

- Provides n at the end of the first year, n - 1 at the end of the second year, ..., 1 at the end of year n.
- $E(Y_i) = (Da)_{x:\overline{n}|} = \sum_{k=1}^{n}(n-k+1)_k E_x$

  Also: $(Ia)_{x:\overline{n}|} + (Da)_{x:\overline{n}|} = (n+1)a_{x:\overline{n}|}$.
- Recursion: $(Da)_{x:\overline{n}|} = nvp_x + vp_x(Da)_{x+1:\overline{n-1}|}$

4.2.4 Geometrically Increasing Life Annuity on (x)

- Consider an annual whole life annuity-due where the payment at time 0 is 1, the payment at time 1 is $(1+j)$, the payment at time 2 is $(1+j)^2$, etc. There is an effective annual interest rate of i.

  Define the modified interest rate: $i_\pi = \frac{1+i}{1+j}$ - 1. Then:

  The expected present value of the above annual whole life annuity-due is $\ddot{a}_x$ at i_π.
- Consider an annual n-year temporary life annuity-due where the payment at time 0 is 1, the payment at time 1 is $(1+j)$, the payment at time 2 is $(1+j)^2$, ..., the payment at time n - 1 is $(1+j)^{n-1}$. There is an effective annual interest rate of i.

  Define the modified interest rate: $i_\pi = \frac{1+i}{1+j}$ - 1. Then:

  The expected present value of the above annual n-year temporary life annuity-due is $\ddot{a}_{x:\overline{n}|}$ at i_π.

4.3 Exercises

6.1. You are given:

(i) Mortality follows the Illustrative Life Table.

(ii) Deaths are uniformly distributed over each year of age.

(iii) $i = 0.06$

Calculate: $\ddot{a}^{(12)}_{35:\overline{20}|}$.

(A) 11.0 (B) 11.2 (C) 11.5 (D) 11.8 (E) 12.0

6.2. You are given:

(i) Mortality follows a select-and-ultimate table, 3-year select period.

(ii) $\ddot{a}_{[40]+1} = 19.2297$

(iii) $i = 0.045$

(iv) $p_{[40]} = 0.9987$

(v) $\mu_{[40]+1} = 0.001321$

Using Woolhouse's formula with three terms, calculate: ${}_{1|}\ddot{a}^{(4)}_{[40]}$.

(A) 18.0 (B) 18.1 (C) 18.2 (D) 18.3 (E) 18.4

6.3. Consider a special 20-year temporary life annuity-due on (30) with annual payments:

(i) The payment for the beginning of year $(k + 1)$ is: $\pi_k = (1.04)^k$ for $k = 0, 1, 2, \ldots, 19$.

(ii) $i = 0.06$

(iii) $l_x = 100$ - x for $0 \leq x \leq 100$

Calculate the single benefit premium.

(A) 14.6 (B) 14.7 (C) 14.8 (D) 14.9 (E) 15.0

6.4. You are given:

(i) $l_{62} = 8{,}982{,}404$, $l_{63} = 8{,}915{,}575$, and $l_{64} = 8{,}842{,}735$

(ii) $v = 0.9569$

(iii) $(I\ddot{a})_{62} = 158.94$

(iv) $(I\ddot{a})_{64} = 145.55$

Calculate: $\ddot{a}_{63}$.

(A) 13.2 (B) 13.4 (C) 13.6 (D) 13.8 (E) 14.0

6.5. A fund is established to provide annuity benefits to 500 independent lives all age 35.

You are given:

(i) On January 1, 2012, each life is issued a single premium whole life annuity. The total payment for each year is 12,000, which is payable in equal monthly installments in advance.

(ii) Each life has mortality that follows the Illustrative Life Table.

(iii) $i = 0.06$

(iv) Deaths are uniformly distributed within each year of age.

Calculate the amount needed in the fund on January 1, 2012, in millions, so that the probability, as determined by the normal approximation, is 0.99 that the fund will be sufficient to provide these benefits.

(A) 90.0 (B) 90.5 (C) 91.0 (D) 91.5 (E) 92.0

6.6. Consider a special life annuity issued to Jenn, aged 37:

(i) There is a deferral period of 10 years. If Jenn dies during the deferral period, 80% of the net single premium is refunded without interest at the end of the year of death.

(ii) During the 15-year period starting at the end of the deferral period, 1000 is payable at the beginning of each month while Jenn is alive. If Jenn is still alive 25 years after issue, 3000 is payable at the beginning of each month for life.

(iii) Mortality follows the Illustrative Life Table.

(iv) Deaths are uniformly distributed over each year of age.

(v) $i = 0.06$

Calculate the net single premium.

(A) 135,500 (B) 136,100 (C) 136,700 (D) 137,300 (E) 137,900

6.7. Consider a 15-year certain and life annuity-due of 24,000 per year on (65) payable monthly (actual payments are 2000 per month):

(i) Mortality follows the Illustrative Life Table.

(ii) $i = 0.06$

(iii) Deaths are uniformly distributed over each year of age.

Calculate the expected present value of this annuity.

(A) 267,900 (B) 268,400 (C) 268,900 (D) 269,400 (E) 269,900

6.8. You are given:

(i) Y_1 is the present value random variable for a 10-year temporary life annuity-due of 1 per year on a select life aged 40 payable quarterly.

(ii) Y_2 is the present value random variable for a 10-year certain and life annuity-due of 1 per year on a select life aged 40 payable quarterly.

(iii) $i = 0.05$

(iv) Mortality follows the Standard Select Survival Model.

(v) Woolhouse's formula with three terms is used to approximate quarterly expected present values.

Calculate the variance of the sum of Y_1 and Y_2.

(A) 3.8 (B) 3.9 (C) 4.0 (D) 4.1 (E) 4.2

6.9. Consider a special increasing 3-year temporary life annuity-due payable annually on (x):

(i) The payment for the first year is 1000, the payment for the second year is 3000, and the payment for the third year is 7000.

(ii) ${}_kp_x = (0.97)^k$ for $k = 0, 1, 2$.

(iii) $i = 0.04$

(iv) Y is the present value random variable for this annuity.

Calculate the standard deviation of Y.

(A) 1904 (B) 1920 (C) 1936 (D) 1952 (E) 1968

6.10. You are given a life annuity-due on (55) payable monthly. 100 is payable each month during the first 10 years; 300 is payable each month after the first 10 years.

Mortality follows the Illustrative Life Table, and $i = 0.06$. Woolhouse's formula with two terms is used to approximate monthly expected present values. Calculate the expected present value of this annuity.

(A) 23,710 (B) 24,210 (C) 24,710 (D) 25,210 (E) 25,710

6.11. For a special continuous 10-year deferred life annuity on (55):

(i) Mortality follows the Illustrative Life Table, and $i = 0.06$.

(ii) Woolhouse's formula with two terms is used to determine annuity expected present values.

(iii) 36,000 is payable continuously each year between ages 65 and 75.

(iv) 22,000 is payable continuously each year after age 75.

Calculate the expected present value of this annuity.

(A) 145,000 (B) 146,000 (C) 147,000 (D) 148,000 (E) 149,000

6.12. You are given:

(i) The following select-and-ultimate life table:

(ii) $v = 0.97$

Calculate the variance of the present value of a 3-year temporary life annuity-due of 1000 per year on [35] payable annually.

(A) 4600 (B) 4700 (C) 4800 (D) 4900 (E) 5000

4.3.1 Answers to Exercises

6.1. C

6.2. A

6.3. B

6.4. E

6.5. C

6.6. D

6.7. C

6.8. B

6.9. A

6.10. D

6.11. B

6.12. C

4.4 Past Exam Questions

- Exam MLC, Fall 2012: #19
- Exam MLC, Spring 2012: #30
- Exam MLC, Sample Questions: #7, 284

Chapter 5

Premium Calculation I

5.1 Key Concepts

The policyholder often pays for a life insurance or life annuity with multiple payments to the insurer over time called **premiums**.

5.1.1 Terminology

- **Fully Continuous Insurance:** a continuous insurance that is funded with a continuous annuity of premiums.
- **Fully Discrete Insurance:** an annual insurance that is funded by an annual annuity-due of premiums.
- **Semi-Continuous Insurance:** a continuous insurance that is funded by an annual annuity-due of premiums.

Net Loss-at-Issue:

- The first step to determining the premiums that the policyholder should pay to fund the benefits of a particular policy is to determine the appropriate **net loss-at-issue random variable**, ignoring policy expenses:

  $L_0 = {}_0L = L =$

  Present value of future benefits at issue - Present value of future premiums at issue

  $= PVFB@0$ - $PVFP@0$

- The net loss-at-issue may be written with a superscript, as L_0^n. "Net" will often be omitted if there is no expense information provided.
- Loss is random because $PVFB@0$ and $PVFP@0$ each depend on the future lifetime of the policyholder.
- There will be a **loss** on a policy if the amount the insurer pays out in benefits is higher than the amount the insurer collects in premiums; $L_0 > 0$ if $PVFB@0 > PVFP@0$. There will be a **profit** on a policy if the amount the insurer pays out in benefits is smaller than the amount the insurer collects in premiums; $L_0 < 0$ if $PVFB@0 < PVFP@0$.

5.1.2 Premium Principles

- A **premium principle** is a rule that manipulates the loss-at-issue random variable in some way to generate premiums.
- Premiums calculated using net loss-at-issue random variable are called **net premiums**.
- A common premium principle is the **Equivalence Principle:**
  - Premiums are determined such that: $E(L_0) = 0$.
  - Under this principle, the insurer charges premiums so that, on average, there will be neither a loss or a profit on the policy. Clearly, this is not the case in practice...
  - Using the formula for L_0, the equivalence principle can also be stated as: $E(PVFB@0) = E(PVFP@0)$. This should be your starting point for complicated problems involving the equivalence principle.
  - On Exam LC, and Exam MLC prior to 2014, net premiums determined via the equivalence principle are called **benefit premiums**. Starting in 2014, the term "net premium" will be equivalent to "benefit premium" on Exam MLC unless otherwise indicated; that is, a net premium is a premium calculated using the equivalence principle without expenses. In this supplement, both "benefit premium" and "net premium" will be used interchangeably.

5.2 Equivalence Principle

Benefit Premiums

5.2.1 Fully Continuous Insurance of 1 on (x)

- For each fully continuous insurance,

  $L_0^n = PVFB@0$ - (Benefit Premium)$\frac{PVFP@0}{\text{Benefit Premium}}$, using the appropriate entries for a value of T_x.
- In the "Benefit Premium" column, the left hand side of the equals sign gives the actuarial notation for the benefit premium. Those taking Exam MLC do not have to know this notation, and can denote the benefit premium in each row as P.
- Furthermore, if the face amount is S, both sides of the equation in the "Benefit Premium" column should be multiplied by S.
- For each fully continuous insurance, the benefit premium was determined by $E(L_0^n) = E(PVFB@0)$ - (Benefit Premium)$\frac{E(PVFP@0)}{\text{Benefit Premium}} = 0$.
- $\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x} = \frac{\delta \bar{A}_x}{1-\bar{A}_x} = \frac{1}{\bar{a}_x}$ - δ.
- $\bar{P}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} = \frac{\delta \bar{A}_{x:\overline{n}|}}{1-\bar{A}_{x:\overline{n}|}} = \frac{1}{\bar{a}_{x:\overline{n}|}}$ - δ.
- With a constant force of mortality: $\bar{P}(\bar{A}_x) = \bar{P}(\bar{A}^1_{x:\overline{n}|}) = \mu$.
- For a couple of the insurances in the table, there are analytic formulas for the variance of the net loss-at-issue.
  - For a fully continuous whole life insurance of 1 on (x):

    $Var(L_0^n) = (1 + \frac{\bar{P}(\bar{A}_x)}{\delta})^2({}^2\bar{A}_x - [\bar{A}_x]^2)$.
    * With a constant force of mortality: $Var(L_0^n) = \frac{\mu}{\mu+2\delta}$.

- For a fully continuous n-year endowment insurance of 1 on (x):

  $Var(L_0^n) = (1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}|})}{\delta})^2({}^2\bar{A}_{x:\overline{n}|} - [\bar{A}_{x:\overline{n}|}]^2)$.
- If the benefit is S, multiply each of the above $Var(L_0^n)$ formulas by S^2.
- These formulas for $Var(L_0^n)$ are true for any type of premium, not just a benefit premium, **except** for the constant force of mortality formula.
- For any other type of fully continuous insurance, use: $Var(L_0^n) = E[(L_0^n)^2]$ - $(E[L_0^n])^2$. If the equivalence principle is used to determine premiums, then: $Var(L_0^n) = E[(L_0^n)^2]$.

- The equivalence principle can also determine benefit premiums for continuous annuities. For example: $\bar{P}({}_{n|}\bar{a}_x) = \frac{{}_{n|}\bar{a}_x}{\bar{a}_{x:\overline{n}|}}$.

5.2.2 Fully Discrete Insurance of 1 on (x)

- For each fully discrete insurance,

  $L_0^n = PVFB@0$ - (Benefit Premium)$\frac{PVFP@0}{\text{Benefit Premium}}$, using the appropriate entries for a value of K_x.
- Recall, K_x can only take on non-negative integer values. So, $K_x < n \implies K_x = 0, 1, \ldots, n$ - 1.
- In the "Benefit Premium" column, the left hand side of each equation gives the actuarial notation for the benefit premium. Those taking Exam MLC only have to know the notation for the whole life, n-year term, n-year pure endowment, and n-year endowment rows; and can denote other benefit premiums as P.
- Furthermore, if the face amount is S, both sides of the equation in the "Benefit Premium" column should be multiplied by S.
- For each fully discrete insurance, the benefit premium was determined by $E(L_0^n) = E(PVFB@0)$ - (Benefit Premium)$\frac{E(PVFP@0)}{\text{Benefit Premium}} = 0$.
- P_x (the benefit premium for a fully discrete whole life insurance of 1 on (x)) should not be confused with p_x (the probability that (x) survives to age x + 1).
- $P_x = \frac{A_x}{\ddot{a}_x} = \frac{dA_x}{1-A_x} = \frac{1}{\ddot{a}_x}$ - d.
- $P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{dA_{x:\overline{n}|}}{1-A_{x:\overline{n}|}} = \frac{1}{\ddot{a}_{x:\overline{n}|}}$ - d.
- With a constant force of mortality: $P_x = P^1_{x:\overline{n}|} = vq$.
- Argue that the following **3-Premium** equations are valid:
  - ${}_nP_x$ - $P^1_{x:\overline{n}|} = A_{x+n}\ P_{x:\overline{n}|}{}^{1}$
  - $P_{x:\overline{n}|}$ - ${}_nP_x = [1$ - $A_{x+n}]\ P_{x:\overline{n}|}{}^{1}$
  - $P^1_{x:\overline{n}|} + P_{x:\overline{n}|}{}^{1} = P_{x:\overline{n}|}$
- For a couple of the insurances in the table, there are analytic formulas for the variance of the net loss-at-issue.
  - For a fully discrete whole life insurance of 1 on (x):

    $Var(L_0^n) = (1 + \frac{P_x}{d})^2({}^2A_x - [A_x]^2)$.
    * With a constant force of mortality: $Var(L_0^n) = \frac{pq}{q+i^2+2i}$.
  - For a fully discrete n-year endowment insurance of 1 on (x):

    $Var(L_0^n) = (1 + \frac{P_{x:\overline{n}|}}{d})^2({}^2A_{x:\overline{n}|} - [A_{x:\overline{n}|}]^2)$.

  - If the benefit is S, multiply each of the above $Var(L_0^n)$ formulas by S^2.
  - These formulas for $Var(L_0^n)$ are true for any type of premium, not just a benefit premium, **except** for the constant force of mortality formula.
  - For any other type of fully discrete insurance, use: $Var(L_0^n) = E[(L_0^n)^2]$ - $(E[L_0^n])^2$. If the equivalence principle is used to determine premiums, then: $Var(L_0^n) = E[(L_0^n)^2]$.

- The equivalence principle can also determine benefit premiums for discrete annuities. For example:

  $P({}_{n|}\ddot{a}_x) = \frac{{}_{n|}\ddot{a}_x}{\ddot{a}_{x:\overline{n}|}}$.

5.2.3 Semi-Continuous Insurance of 1 on (x)

- You can obtain this table by taking the table for **Fully Continuous Insurance of 1 on (x)** and replacing the continuous premium annuity with an annual annuity-due.
- For example, a semi-continuous n-year term insurance of 1 on (x) is:
- Those taking Exam MLC do NOT have to know the actuarial notation for semi-continuous benefit premiums; P is sufficient.
- Exam MLC Only: With a uniform distribution of deaths (UDD) in each year of age:
  - $P(\bar{A}_x) = \frac{i}{\delta} P_x$
  - $P(\bar{A}^1_{x:\overline{n}|}) = \frac{i}{\delta} P^1_{x:\overline{n}|}$
  - $P(\bar{A}_{x:\overline{n}|}) = \frac{i}{\delta} P^1_{x:\overline{n}|} + P_{x:\overline{n}|}{}^1$

5.3 Exercises

7.1. On January 1, 2010, Pat purchases a 5-year deferred whole life insurance of 100,000 payable at the end of the year of death. Premiums of 4000 are payable at the beginning of each year for the first 5 years, and i = 0.05.

Calculate the loss-at-issue if Pat dies on September 30, 2016.

(A) 48,370 (B) 52,884 (C) 53,756 (D) 57,209 (E) 62,187

7.2. Stefano, age 60, purchases a whole life insurance of 1,000,000:

(i) The death benefit is payable at the moment of death.

(ii) Premiums of 50,000 are payable at the beginning of each year for as long as Stefano is alive.

(iii) $i = 0.05$

(iv) L is the loss-at-issue random variable.

Calculate the value of L if Stefano dies at age 61.5.

(A) 764,059 (B) 809,410 (C) 819,138 (D) 831,810 (E) 879,429

7.3. Consider a fully continuous whole life insurance of 1000 on (x).

Assume $\delta = 0.08$ and $\mu_x(t) = 0.04$ for $t \geq 0$.

Calculate the level annual benefit premium.

(A) 30 (B) 35 (C) 40 (D) 45 (E) 50

7.4. Paul, age 31, purchases a fully discrete 20-year endowment insurance of 1000. Assume mortality follows the Illustrative Life Table, and $i = 0.06$.

Calculate: $1000P_{31:\overline{20}|}$.

(A) 21 (B) 23 (C) 25 (D) 27 (E) 29

7.5. Consider the following special fully discrete whole life insurance on (50):

(i) The death benefit is 80,000 before age 65; the death benefit is 150,000 thereafter.

(ii) The level annual net premium is $2P$ before age 65; the level annual net premium is P thereafter.

(iii) Mortality follows the Illustrative Life Table, and $i = 0.06$.

Calculate the annual net premium payable before age 65.

(A) 1300 (B) 1800 (C) 2300 (D) 2700 (E) 3200

7.6. Consider a fully discrete 5-payment 10-year endowment insurance of 1000 on (70):

(i) Mortality follows the Illustrative Life Table.

(ii) $i = 0.06$

Calculate the level annual net premium.

(A) 140 (B) 145 (C) 150 (D) 155 (E) 160

7.7. For a fully continuous whole life insurance of 5000 on (x):

(i) The force of mortality is a constant.

(ii) $\delta = 0.05$

(iii) L is the loss-at-issue random variable based on level annual benefit premiums.

(iv) The standard deviation of L is 2236.07

Calculate the level annual benefit premium.

(A) 100 (B) 125 (C) 150 (D) 175 (E) 200

7.8. A fully continuous whole life insurance of 10,000 on (x) is issued with premiums determined by the equivalence principle.

You are also given:

(i) $\mu_x(t) = 0.02$ for $t \geq 0$

(ii) $\delta = 0.05$

Calculate the probability that the loss-at-issue is positive.

(A) 0.30 (B) 0.33 (C) 0.36 (D) 0.39 (E) 0.42

7.9. An insurer has just issued each of 100 independent lives aged 35 a fully discrete 20-year endowment insurance of 1000 with level annual benefit premiums. Each life has mortality that follows the Illustrative Life Table. The effective annual interest rate is 0.06.

Using the normal approximation, determine the fund amount at issue, h, that is necessary so that the insurer is 99% sure that the sum of the 100 loss-at-issue random variables associated with the endowment insurances will not exceed h.

(A) 2100 (B) 2200 (C) 2300 (D) 2400 (E) 2500

7.10. For a special fully discrete whole life insurance on (35):

(i) The death benefit is equal to 2000 plus the return of all benefit premiums paid in the past without interest.

(ii) $\ddot{a}_{35} = 19.93$

(iii) $(IA)_{35} = 5.58$

(iv) $i = 0.045$

Calculate the level annual benefit premium for this insurance.

(A) 20 (B) 22 (C) 24 (D) 26 (E) 28

7.11. You are given:

(i) The level annual benefit premium for a fully discrete 20-year term insurance of 5000 on (x) is 75.

(ii) The level annual benefit premium for a fully discrete 20-year endowment insurance of 5000 on (x) is 200.

(iii) The level annual benefit premium for a fully discrete 20-payment whole life insurance of 5000 on (x) is 150.

Calculate the actuarial present value of a fully discrete whole life insurance of 5000 on (x + 20).

(A) 2000 (B) 2500 (C) 3000 (D) 3500 (E) 4000

7.12. For a life insurance on (x):

(i) 1000 is payable at the end of the year of death if death occurs in the first ten years; 2000 is payable at the end of the year of death if death occurs in the next ten years; otherwise, the death benefit is 0.

(ii) Level annual benefit premiums are payable at the beginning of each year for the first 20 years.

(iii) $d = 0.10$, and $q_x = 0.03$ for all integer ages x.

Calculate the level annual benefit premium.

(A) 31 (B) 33 (C) 35 (D) 37 (E) 39

7.13. For a special fully discrete 3-year term insurance on (x):

(i) The death benefit: $b_{k+1} = 500(k + 1)$ for $k = 0, 1, 2$

(ii) $q_{x+k} = 0.02(k + 1)$ for $k = 0, 1, 2$

(iii) $i = 0.03$

Use the equivalence principle to calculate the level annual premium for this insurance.

(A) 41 (B) 42 (C) 43 (D) 44 (E) 45

7.14. You are given:

(i) $l_x = 100(110 - x)^2$ for $0 \leq x \leq 110$

(ii) $i = 0$

Calculate the level annual benefit premium for a fully discrete 5-payment 15-year term insurance of 1 on (30): ${}_5P^1_{30:\overline{15}|}$.

(A) 0.06 (B) 0.07 (C) 0.08 (D) 0.09 (E) 0.10

7.15. A fully discrete 5-year endowment insurance of 1000 was just issued to Math Mage, aged 30. In determining the level annual benefit premium, it was assumed that $i = 0.06$ and that Math Mage had mortality that follows the Illustrative Life Table.

Shortly after issuing the 5-year endowment insurance, it was discovered that Math Mage had been cursed by Hattendorf. In calculating the level annual benefit premium, it should have been assumed that $i = 0.06$ and that Math Mage had mortality such that the actual force of mortality was $\mu_{30}(t) + 0.10$ for $0 < t < 5$, where $\mu_{30}(t)$ is the force of mortality associated with the Illustrative Life Table.

Calculate the difference between the benefit premium that Math Mage should be paying calculated using the correct mortality (based on $\mu_{30}(t) + 0.10$) and the benefit premium actually payable by Math Mage calculated using the incorrect mortality (Illustrative Life Table).

(A) 40 (B) 50 (C) 60 (D) 70 (E) 80

7.16. Consider a 5-year deferred whole life annuity-due on (60) with an annual payment of 10,000. You are given:

(i) Benefit premiums are payable at the beginning of each year during the first five years. The benefit premium payable in each of years one and two is half of the benefit premium payable in each of years three, four, and five.

(ii) $d = 0.04306$

(iii) $q_{60} = 0.006155$ and $q_{61} = 0.006765$

(iv) ${}_5E_{60} = 0.77282$

(v) $\ddot{a}_{65} = 13.4662$ and $\ddot{a}_{62:\overline{3}|} = 2.8513$

Calculate the benefit premium payable in each of years one and two.

(A) 14,300 (B) 14,600 (C) 14,900 (D) 15,200 (E) 15,500

7.17. Consider a fully continuous whole life insurance of 100,000 on (30):

(i) If the level annual premium is π_1, the standard deviation of the loss-at-issue random variable is 55,621.49.

(ii) If the level annual premium is π_2, the standard deviation of the loss-at-issue random variable is 49,441.32.

(iii) π_1 is 1.5 times π_2.

(iv) $\delta = 0.04$

(v) Z is the present value random variable for the continuous whole life insurance of 100,000 on (30).

Calculate the standard deviation of Z.

(A) 37,100 (B) 38,200 (C) 39,300 (D) 40,400 (E) 41,500

7.18. Bruce and Lucius, both aged x, have each just purchased a fully discrete 3-year term insurance of 1000:

(i) Bruce pays a benefit premium of 175.72 each year. If Bruce dies in the second year after policy issue, the loss-at-issue is 559.27.

(ii) Lucius pays non-level annual benefit premiums. The first benefit premium is 100, the second benefit premium is 175, and the third benefit premium is P.

(iii) Each life has mortality such that: ${}_{k|}q_x = (0.3)^{k+1}$ for $k = 0, 1, 2$.

(iv) The effective annual interest rate is i.

Calculate: P.

(A) 285 (B) 315 (C) 345 (D) 375 (E) 405

7.19. Consider a special 20-year deferred whole life annuity-due of 5000 per year on (45) payable annually:

(i) Level annual benefit premiums are payable at the beginning of the year during the first 20 years after policy issue.

(ii) There is a death benefit during the premium-paying period, payable at the end of the year of death, that is equal to the return of all benefit premiums previously paid with interest at 6%.

(iii) $i = 0.06$

(iv) Mortality follows the Illustrative Life Table.

Calculate the benefit premium.

(A) 1180 (B) 1210 (C) 1240 (D) 1270 (E) 1300

7.20. You are given:

(i) A fully discrete 2-year deferred, 3-year term insurance of 1000 is issued to a life aged x.

(ii) Level annual premiums are only payable during the first two years.

(iii) The level annual premium is determined such that the average loss-at-issue is zero.

(iv) $v = 0.90$

(v) ${}_{k|}q_x = 0.05(1 + k)$ for $k = 0, 1, 2, 3, 4$; ${}_{5|}q_x = 0.25$

Calculate the median loss-at-issue.

(A) 193 (B) 226 (C) 258 (D) 295 (E) 331

7.21. Paul is attempting to determine the level annual benefit premium for a fully discrete 20-year endowment insurance of 10,000 on (55). Paul assumes the following:

(i) $i = 0.06$

(ii) Paul is not sure of the future lifetime distribution of (55). He believes there is an 80% probability that the mortality of (55) follows the Illustrative Life Table, and that there is a 20% probability that the mortality of (55) is such that ${}_{k|}q_{55} = 0.05$ for $k = 0, 1, 2, \ldots, 19$.

Determine the level annual benefit premium.

(A) 384 (B) 414 (C) 444 (D) 474 (E) 504

7.22. Suppose today is January 1, 2014, and Paul has just turned age 35. He has mortality such that:

${}_tp_{35} = (0.95)^t$ for $t \geq 0$.

Paul has just been issued a special fully discrete whole life insurance:

(i) $i = 0.07$

(ii) The death benefit if Paul dies in an odd-numbered calendar year is 5000.

(iii) The death benefit if Paul dies in an even-numbered calendar year is 10,000.

(iv) Benefit premiums are payable annually, where the benefit premium for an odd-numbered calendar year is double the benefit premium for an even-numbered calendar year.

Calculate the benefit premium payable during an even-numbered calendar year.

(A) 220 (B) 240 (C) 260 (D) 280 (E) 300

5.3.1 Answers to Exercises

7.1. B

7.2. D

7.3. C

7.4. D

7.5. D

7.6. C

7.7. B

7.8. D

7.9. E

7.10. A

7.11. C

7.12. B

7.13. D

7.14. B

7.15. A

7.16. B

7.17. A

7.18. B

7.19. D

7.20. C

7.21. B

7.22. B

5.4 Past Exam Questions

- Exam MLC, Fall 2015: #10
- Exam MLC, Spring 2015: #8, 11
- Exam MLC, Fall 2014: #7
- Exam MLC, Spring 2014: #8
- Exam MLC, Fall 2013: #14, 15, 16
- Exam MLC, Spring 2013: #1, 3, 18
- Exam 3L, Spring 2013: #13, 14
- Exam MLC, Fall 2012: #20, 22
- Exam 3L, Spring 2012: #12
- Exam MLC, Spring 2012: #3, 6
- Exam MLC, Sample Questions: #6, 14, 29, 40, 47, 51, 76, 84, 92, 96, 97, 99, 111, 119, 127, 142, 154, 157, 172, 174, 184, 204, 221, 228, 309
- Exam 3L, Spring 2010: #16
- Exam 3L, Fall 2008: #22
- Exam 3L, Spring 2008: #23
- Exam MLC, Spring 2007: #4

Bibliography