Portfolio Management including Reinsurance

Fall 2017

Outline

Overview: Portfolio Management including Reinsurance

Define S to be (random) obligations that arise from a collection (portfolio) of insurance contracts

- We are particularly interested in probabilities of large outcomes and so formalize the notion of a “heavy-tail” distribution
- How much in assets does an insurer need to retain to meet obligations arising from the random S? A study of “risk measures” helps to address this question
- As with policyholders, insurers also seek mechanisms in order to spread risks. A company that sells insurance to an insurance company is known as a “reinsurer”

Tails of Distributions

- The tail of a distribution (more specifically: the right tail) is the portion of the distribution corresponding to large values of the r.v.
- Understanding large possible loss values is important because they have the greatest impact on the total of losses.
- R.v.'s that tend to assign higher probabilities to larger values are said to be heavier tailed.
- When choosing models, tail weight can help narrow choices or can confirm a choice for a model.

Classification Based on Moments

- One way of classifying distributions:

  are all moments finite, or not?

- The finiteness of all positive moments indicates a (relatively) light right tail.

- The finiteness of only positive moments up to a certain value indicates a heavy right tail.

Classification Based on Moments

- KPW Example 3.9: demonstrate that for the gamma distribution all positive moments are finite but for the Pareto distribution they are not.

- For the gamma distribution

$$\begin{aligned}
\mu_k' &= \int_0^\infty x^k \frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha} dx \\
&= \int_0^\infty (y\theta)^k \frac{(y\theta)^{\alpha-1} e^{-y}}{\Gamma(\alpha)\theta^\alpha} \theta dy \\
&= \frac{\theta^k}{\Gamma(\alpha)} \Gamma(\alpha+k) < \infty \quad \text{for all } k > 0.
\end{aligned}$$

Classification Based on Moments II

- KPW Example 3.9: demonstrate that for the gamma distribution all positive moments exist but for the Pareto distribution they do not.
- For the Pareto distribution

$$
\begin{aligned}
\mu_k^{'} &= \int_0^\infty x^k \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} dx \\
&= \int_\theta^\infty (y-\theta)^k \frac{\alpha\theta^\alpha}{y^{\alpha+1}} dy \\
&= \alpha \cdot \theta^\alpha \int_\theta^\infty \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) y^{j-\alpha-1} (-\theta)^{k-j} dy,
\end{aligned}
$$

for integer values of k.

This integral is finite only if $\int_\theta^\infty y^{j-\alpha-1} dy = \frac{y^{j-\alpha}}{j-\alpha}\Big|_\theta^\infty$ is finite. Finiteness occurs when $j-\alpha < 0$ for $j = 1, \ldots, k$. Or, equivalently, $k < \alpha$.

Classification Based on Moments III

- Pareto is said to have a heavy tail, and gamma has a light tail.

Comparison Based on Limiting Tail Behavior

- Consider two distributions with the same mean.
- If ratio of $S_1(.)$ and $S_2(.)$ diverges to infinity, then distribution 1 has a heavier tail than distribution 2.
- Thus, we examine

$$
\begin{aligned}
\lim_{x\to\infty}\frac{S_1(x)}{S_2(x)} &= \lim_{x\to\infty}\frac{S_1'(x)}{S_2'(x)} \\
&= \lim_{x\to\infty}\frac{-f_1(x)}{-f_2(x)} = \lim_{x\to\infty}\frac{f_1(x)}{f_2(x)}.
\end{aligned}
$$

Comparison Based on Limiting Tail Behavior II

- KPW Example 3.10: demonstrate that Pareto distribution has a heavier tail than the gamma distribution using the limit of the ratio of their density functions.

- We consider

$$
\begin{aligned}
\lim_{x\to\infty} \frac{f_{\text{Pareto}}(x)}{f_{\text{gamma}}(x)} &= \lim_{x\to\infty} \frac{\alpha\theta^{\alpha}(x+\theta)^{-\alpha-1}}{x^{\tau-1}e^{-x/\lambda}\lambda^{-\tau}\Gamma(\tau)^{-1}} \\
&= c \lim_{x\to\infty} \frac{e^{x/\lambda}}{(x+\theta)^{\alpha+1}x^{\tau-1}} \\
&= \infty
\end{aligned}
$$

Exponentials go to infinity faster than polynomials, thus the limit is infinity.

Measures of Risk

- A risk measure is a mapping from the r.v. representing the loss associated with the risks to the real line.
- A risk measure gives a single number that is intended to quantify the risk.
  - For example, the standard deviation is a risk measure.
- Notation: $\rho(X)$.
- We briefly mention:

  - VaR: Value at Risk;
  - TVaR: Tail Value at Risk.

Value at Risk

- Say $F_X(x)$ represents the cdf of outcomes over a fixed period of time, e.g. one year, of a portfolio of risks.
- We consider positive values of X as losses.
- Definition 3.11: let X denote a loss r.v., then the Value-at-Risk of X at the $100p\%$ level, denoted $VaR_p(X)$ or π_p, is the $100p$ percentile (or quantile) of the distribution of X.
- E.g. for continuous distributions we have

$$P(X > \pi_p) \quad = \quad 1 - p.$$

Value at Risk II

- VaR has become the standard risk measure used to evaluate exposure to risk.
- VaR is the amount of capital required to ensure, with a high degree of certainty, that the enterprise does not become technically insolvent.
- Which degree of certainty?
  - 95%?
  - in Solvency II 99.5% (or: ruin probability of 1 in 200).

Value at Risk III

- VaR is not subadditive.

  Subadditivity of a risk measure $\rho(.)$ requires

  $$\rho(X+Y) \leq \rho(X) + \rho(Y).$$

  Intuition behind subadditivity: combining risks is less riskier than holding them separately.

- Example: let X and Y be i.i.d. r.v.'s which are Bern(0.02) distributed.
  - Then, $P(X \leq 0) = 0.98$ and $P(Y \leq 0) = 0.98$. Thus, $F_X^{-1}(0.975) = F_Y^{-1}(0.975) = 0$.
  - For the sum, $X+Y$, we have $P[X+Y=0] = 0.98 \cdot 0.98 = 0.9604$. Thus, $F_{X+Y}^{-1}(0.975) > 0$.
  - VaR is not subadditive, since VaR$(X+Y)$ in this case is larger than VaR(X) + VaR(Y).

Value at Risk IV

- Another drawback of VaR:
  - it is a single quantile risk measure of a predetermined level p;
  - no information about the thickness of the upper tail of the distribution function from VaR_p on;
  - whereas stakeholders are interested in both frequency and severity of default.
- Therefore: study other risk measures, e.g. Tail Value at Risk (TVaR).

Tail Value at Risk

- Definition 3.12: let X denote a loss r.v., then the Tail Value at Risk of X at the $100p\%$ security level, TVaR(p), is the expected loss given that the loss exceeds the $100p$ percentile (or: quantile) of the distribution of X.
- We have (assume continuous distribution)

$$\begin{aligned}\text{TVaR}_p(X) &= E(X|X>\pi_p) \\ &= \frac{\int_{\pi_p}^{\infty} x\cdot f(x)dx}{1-F(\pi_p)}.\end{aligned}$$

- We can rewrite this as [the usual definition of TVaR]

$$\begin{aligned}\text{TVaR}_p(X) &= \frac{\int_{\pi_p}^{\infty} x dF_X(x)}{1-p} \\ &= \frac{\int_{p}^{1} \text{VaR}_u(X)du}{1-p},\end{aligned}$$

using the substitution $F_X(x)$ and thus $F^{-1}(u)$

Tail Value at Risk

- From the definition

$$\mathsf{TVaR}_p(X) = \frac{\int_p^1 \mathsf{VaR}_u(X)du}{1-p},$$

we understand

- TVaR is the arithmetic average of the quantiles of X, from level p on;
- TVaR is ‘averaging high level VaRs’;
- TVaR tells us much more about the tail of the distribution than does VaR alone.

Tail Value at Risk

- Finally, TVaR can also be written as

$$
\begin{aligned}
\mathrm{TVaR}_p(X) &= E(X|X > \pi_p) \\
&= \frac{\int_{\pi_p}^{\infty} x f(x) dx}{1-p} \\
&= \pi_p + \frac{\int_{\pi_p}^{\infty} (x-\pi_p) f(x) dx}{1-p} \\
&= \mathrm{VaR}_p(X) + e(\pi_p),
\end{aligned}
$$

with $e(\pi_p)$ the mean excess loss function evaluated at the $100p$th percentile.

Tail Value at Risk

- We can understand these connections as follows. (Assume continuous r.v.'s)
- The relation

$$\mathrm{CTE}_p(X) \quad = \quad \mathrm{TVaR}_{F_X(\pi_p)}(X),$$

then follows immediately by combining the other two expressions.

Tail Value at Risk

- TVaR is a coherent risk measure, see e.g. Foundations of Risk Measurement course.
- Thus, $\text{TVaR}(X+Y) \leq \text{TVaR}(X) + \text{TVaR}(Y)$.
- When using this risk measure, we never encounter a situation where combining risks is viewed as being riskier than keeping them separate.

Tail Value at Risk

- KPW Example 3.18 *(Tail comparisons)* Consider three loss distributions for an insurance company. Losses for the next year are estimated to be on average 100 million with standard deviation 223.607 million. You are interested in finding high quantiles of the distribution of losses. Using the normal, Pareto, and Weibull distributions, obtain the VaR at the 90%, 99%, and 99.99% security levels.

- Solution

  Normal distribution has a lighter tail than the others, and thus smaller quantiles.

  Pareto and Weibull with $\tau < 1$ have heavy tails, and thus relatively larger extreme quantiles.

Tail Value at Risk

- Example 3.18 *(Tail comparisons)* Consider three loss distributions for an insurance company. Losses for the next year are estimated to be on average 100 million with standard deviation 223.607 million. You are interested in finding high quantiles of the distribution of losses. Using the normal, Pareto, and Weibull distributions, obtain the VaR at the 99%, 99.9%, and 99.99% security levels.

```
> qnorm(c(0.9,0.99,0.999),mu,sigma)
[1] 386.5639 620.1877 790.9976
> qpareto(c(0.9,0.99,0.999),alpha,s)
[1]  226.7830  796.4362 2227.3411
> qweibull(c(0.9,0.99,0.999),tau,theta)
[1]  265.0949 1060.3796 2385.8541
```

Tail Value at Risk

- We learn from Example 3.18 that results vary widely depending on the choice of distribution.
- Thus, the selection of an appropriate loss model is highly important.
- To obtain numerical values of VaR or TVaR:
  - estimate from the data directly;
  - or use distributional formulas, and plug in parameter estimates.

Tail Value at Risk

- When estimating VaR directly from the data:
  - use R to get quantile from the empirical distribution;
  - R has 9 ways to estimate a VaR at level p from a sample of size n, differing in the way the interpolation between order statistics close to np .
- When estimating TVaR directly from the data:
  - take average of all observations that exceed the threshold (i.e. π_p);
- Caution: we need a large number of observations (and a large number of observations $> \pi_p$) in order to get reliable estimates.
- When not may observations in excess of the threshold are available:
  - construct a loss model;
  - calculate values of VaR and TVaR directly from the fitted

Tail Value at Risk

- For example

$$\begin{aligned}
\text{TVaR}_p(X) &= E(X|X > \pi_p) \\
&= \pi_p + \frac{\int_{\pi_p}^{\infty}(x-\pi_p)f(x)dx}{1-p} \\
&= \pi_p + \frac{\int_{-\infty}^{\infty}(x-\pi_p)f(x)dx - \int_{-\infty}^{\pi_p}(x-\pi_p)f(x)dx}{1-p} \\
&= \pi_p + \frac{E(X) - \int_{-\infty}^{\pi_p} xf(x)dx - \pi_p(1-F(\pi_p))}{1-p} \\
&= \pi_p + \frac{E(X) - E[\min(X,\pi_p)]}{1-p} = \pi_p + \frac{E(X) - E(X \wedge \pi_p)}{1-p},
\end{aligned}$$

see Appendix A for those expressions.

Reinsurance

- A major difference between reinsurance and primary insurance is that a reinsurance program is generally tailored more closely to the buyer
- There are two major types of reinsurance
  - Proportional
  - Excess of Loss
- A proportional treaty is an agreement between a reinsurer and a ceding company (the reinsured) in which the reinsurer assumes a given percent of losses and premium.

Proportional Reinsurance

- A proportional treaty is an agreement between a reinsurer and a ceding company (the reinsured) in which the reinsurer assumes a given percent of losses and premium.
- The simplest example of a proportional treaty is called *Quota Share*.
  - In a quota share treaty, the reinsurer receives a flat percent, say 50%, of the premium for the book of business reinsured.
  - In exchange, the reinsurer pays 50% of losses, including allocated loss adjustment expenses
  - The reinsurer also pays the ceding company a ceding commission which is designed to reflect the differences in underwriting expenses incurred.
- The amounts paid by the direct insurer and the reinsurer are defined as follows:

$$Y_{insurer} = cX \qquad Y_{reinsurer} = (1-c)X$$

Note that $Y_{insurer} + Y_{reinsurer} = X$.

Surplus Share Proportional Treaty

- Another proportional treaty is known as *Surplus Share*; these are common in property business.
- A surplus share treaty allows the reinsured to limit its exposure on any one risk to a given amount (the *retained line*).
- The reinsurer assumes a part of the risk in proportion to the amount that the insured value exceeds the retained line, up to a given limit (expressed as a multiple of the retained line, or "number" of lines).
- For example, let the retained Line be \$100,000 and let the given limit be 4 lines (\$400,000). Then, if X is the loss, the reinsurer's portion is $\min(400000, (X - 100000)_+)$.

Excess of Loss Reinsurance

- Under this arrangement, the direct insurer sets a retention level $M(> 0)$ and pays in full any claim for which $X \leq M$.
- The direct insurer retains an amount M of the risk.
- For claims for which $X > M$, the direct insurer pays M and the reinsurer pays the remaining amount $X - M$.
- The amounts paid by the direct insurer and the reinsurer are defined as follows:

$$Y_{insurer} = \begin{cases} X & X \leq M \\ M & X > M \end{cases} = \min(X, M) = X \wedge M$$

$$Y_{reinsurer} = \begin{cases} 0 & X \leq M \\ X - M & X > M \end{cases} = \max(0, X - M)$$

Note that $Y_{insurer} + Y_{reinsurer} = X$.

Relations with Personal Insurance

- We have already seen the needed tools to handle reinsurance in the context of personal insurance
    - For a proportional reinsurance, the transformation $Y_{insurer} = cX$ is the same as a "coinsurance" adjustment in personal insurance
    - For excess of loss reinsurance, the transformation $Y_{reinsurer} = \max(0, X - M)$ is the same as an insurer's payment with a deductible M and $Y_{insurer} = \min(X, M) = X \wedge M$ is equivalent to what a policyholder pays with deductible M.
- Reinsurance applications suggest introducing *layers of coverage*, a (small) mathematical extension.

Layers of Coverage

- Instead of simply an insurer and reinsurer or an insurer and a policyholder, think about the situation with all three parties, a policyholder, insurer, and reinsurer, who agree on how to share a risk.
- In general, we consider k parties. If $k = 4$, it could be an insurer and three different reinsurers.
- Consider a simple example:
  - Suppose that there are $k = 3$ parties. The first party is responsible for the first 100 of claims, the second responsible for claims from 100 to 3000, and the third responsible for claims above 3000.
  - If there are four claims in the amounts 50, 600, 1800 and 4000, they would be allocated to the parties as follows:

Layer	Claim 1	Claim 2	Claim 3	Claim 4	Total
(0, 100]	50	100	100	100	350
(100, 3000]	0	500	1700	2900	5100
(3000, ∞)	0	0	0	1000	1000
Total	50	600	1800	4000	6450

Layers of Coverage II

- Mathematically, partition the positive real line into k intervals using the cut-points $0 = c_0 < c_1 < \cdots < c_{k-1} < c_k = \infty$.
  - The jth interval is $(c_{j-1}, c_j]$.
- Let Y_j be the amount of risk shared by the jth party
- To illustrate, if a loss x is such that $c_{j-1} < x \le c_j$, then

$$\left(\begin{array}{c} Y_1 \\ Y_2 \\ \vdots \\ Y_j \\ Y_{j+1} \\ \vdots \\ Y_k \end{array} \right) = \left(\begin{array}{c} c_1 - c_0 \\ c_2 - c_1 \\ \vdots \\ x - c_{j-1} \\ 0 \\ \vdots \\ 0 \end{array} \right)$$

- More succinctly, we can write

$$Y_j = \min(X, c_j) - \min(X, c_{j-1})$$

Layers of Coverage III

- With the expression $Y_j = \min(X, c_j) - \min(X, c_{j-1})$, we see that the jth party is responsible for claims in the interval $(c_{j-1}, c_j]$
- Note that $X = Y_1 + Y_2 + \cdots + Y_k$
- The parties need not be different.
  - For example, suppose that a policyholder is responsible for the first 500 of claims and all claims in excess of 100,000. The insurer takes claims between 100 and 100,000.
  - Then, we would use $c_1 = 100$, $c_2 = 100000$.
  - The policyholder is responsible for $Y_1 = \min(X, 100)$ and $Y_3 = X - \min(X, 100000) = \max(0, X - 100000)$.
- See the Wisconsin Property Fund site for more info on layers of reinsurance, `https://sites.google.com/a/wisc.edu/local-government-property-insurance-fund/home/reinsurance`