A Visual Introduction to Curvature

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Abstract

This paper seeks to cover what one would learn in a years long graduate course in Riemannian geometry in a way that is both visually stimulating and enlightening. It is directed primarily to early undergraduate students of math who have familiarity with topics from linear algebra. Much reliance will be given to the reader's understanding of inner product spaces, determinants, linear transformations, as well as a familiarity with the geometric understanding of all these ideas. However, a precocious high schooler could follow along after reading chapters 1,2,3, and 6 of Shelden Axler's Linear Algebra Done Right. As well, practice in computing derivatives and integrals and the geometric meaning therein would be helpful. However, a course in vector calculus is not required.

By no means should this be seen as a full and rigorous exploration into the vast topic of Riemannian and differential geometry; many tomes have be written doing such a task. Instead, it will leverage visual intuition of the space we occupy to make broad and powerful statements about surfaces more generally. Of course, visual intuition is limited by the dimension we occupy, so many generalizations will be stated, without proof, with references to where one could find proofs.

1 Manifolds

Any proper discussion of curvature must first answer the question *what is being curved?* After all, asking how curvy the integers are is meaningless, while asking the same for a sphere is seemingly not. The difference lies in the notion of space and geometry for the latter that the former lacks. A language capturing this intuition is necessary if any progress is to be made in the theory of curvature. Thankfully, mathematicians have developed such a language for discussing all manner of curved surfaces. A complete discussion of the theory and language would require an exploration into topology in a way more rigorous than necessary in developing an intuition for the subject at hand.

We will start the discussion by giving the definition for charts, which intuitively we will think of as a GPS system, or coordinates, allowing us to "walk along" a manifold in a way similar to how a spaceship would navigate through the solar system, or how an ant would walk along a table. It will be clear, by how we define charts, that they are not suitable by themselves to encapsulate even all the surfaces we encounter in our everyday lives, much less those of higher dimensions.

To remedy this, we will "stitch together" different charts to create manifolds, the topic of discussion for the remainder of the paper

Charts

Manifold

Rough and Smooth Manifolds

2 The Measure of a Manifold

The Euclidean Metric

The Riemannian Metric

3 Connections

Vector Fields

Geodesics

4 Curvature