ACM 216, Problem Set 2

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1 Problem 1

Let $X = (X_n)_{n\in\mathbb{N}}$ be a random process with state space $E = \{0, 1, ..., 5\}$ denoting our wealth at time n so $X_0 = 1$. Define the stopping time $\tau = \min\{n : X_n \geq 5 \text{ or } X_n \leq 0\}$. Then we seek $\mathbb{P}[X_\tau = 5|X_0 = 1]$ under strategies 1 and 2.

To compute the survival probability under strategy 1 (the timid strategy), we first recognize that X is a homogeneous Markov chain (HMC). Indeed, from any wealth $0 < X_n < 5$, the transition probability to either $X_n + 1$ or $X_n - 1$ is $\frac{1}{2}$, since the coin is fair. We therefore have constant transition probabilities independent of time and future states dependent solely on the current one, so X is homogeneous.

Since X is homogeneous, we suppress time subscripts so that $\mathbb{P}[X_\tau = 5|X_n = i] = p_i \ \forall\ 0 \leq i \leq 5, 0 \leq n \leq \tau$. Clearly, then, $p_0 = 0, p_5 = 1$. We compute

$$p_1 = \frac{1}{2}(0 + p_2), p_2 = \frac{1}{2}(p_1 + p_3),$$

$$p_3 = \frac{1}{2}(p_4 + p_2), p_4 = \frac{1}{2}(1 + p_3)$$

$$\implies p_1 = \boxed{\frac{1}{5}}$$

is our probability of survival under the timid strategy.

Under the bold strategy, we will bet 1 dollar from wealth 1; 2 from wealth 2; 1 from wealth 4 (since we need only 1 more dollar to arrive at 5); and 2 from wealth 3 (since we would need 2 more dollars to arrive at 5). Each transition will happen as before with probability $\frac{1}{2}$ for non-absorbing states. The transition matrix P of X under this strategy is more complex so for clarity we show it below.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

It follows that X is also an HMC under this strategy, since the transition matrix is independent of time. Then maintaining the same stopping time τ from above and the assignment $\mathbb{P}[X_\tau = 5 | X_n = i] = p_i \; \forall \; 0 \le i \le 5, 0 \le n \le \tau$, we compute

$$p_1 = \frac{1}{2}(0 + p_2), p_2 = \frac{1}{2}(0 + p_4),$$

$$p_4 = \frac{1}{2}(1 + p_3), p_3 = \frac{1}{2}(1 + p_1)$$

$$\implies p_1 = \boxed{\frac{1}{5}}$$

is also our probability of winning under strategy 2.

2 Problem 2

To formalize our system, we create the stochastic process $X = (X_n)_{n \in \mathbb{N}}$ with state space $E = \{0, 1, 2\}$.

We will denote a sunny day as 0, a cloudy day as 1, and a rainy day as 2. Given the problem description, we then have the transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix} \end{array}.$$

Note that since

$$P^2 = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0.25 & 0.375 & 0.375 \\ 0.1875 & 0.4375 & 0.375 \\ 0.1875 & 0.375 & 0.4375 \end{pmatrix} \end{array}.$$

we have that X is irreducible. Since it is possible to return to a cloudy day after a cloudy day, and the chain is irreducible, then state 1 being aperiodic implies the entire chain is aperiodic. Thus since X is irreducible and aperiodic with a finite state space E, it has a unique invariant distribution π.

We solve $\pi P = \pi = (\pi_0, \pi_1, \pi_2), \sum_i \pi_i = 1$ for π :

$$\pi_0 = 0.25\pi_1 + 0.25\pi_2,$$

$$\pi_1 = 0.5\pi_0 + 0.5\pi_1 + 0.25\pi_2,$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\implies \pi = (\pi_0, \pi_1, \pi_2) = (0.2, 0.4, 0.4).$$

Now, it is then simple to see that, since X is ergodic (it is an irreducible, aperiodic HMC with a unique stationary distribution π on a finite state space

E), $\mathcal{L}(X_n) \to \pi$ for $\mathcal{L}(X_n)$ the law of X_n, and we have from the ergodic theorem that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} I\{X_k = i\} = \mathbb{E}_\pi[I\{X_0 = i\}],$$

for $I\{X_k = i\}$ the indicator function for the event $X_k = i, k \geq 0, i \in E$.

Then the answers to (1) and (2) are identical: The long-run proportion of sunny days (equal to the probability that $X_n = 0$ for large n) is $\boxed{0.2}$ and the long-run proportion of cloudy days (equal to the probability that $X_n = 1$ for large n is $\boxed{0.4}$.

3 Problem 3

Let P be the transition matrix of X_n and Q that of Y_n so that, for any states $x_0, x_1 \in E^X$ and $y_0, y_1 \in E^Y$, $\mathbb{P}(X_{n+1} = x_1 | X_n = x_0) = P(x_0, x_1)$ and $\mathbb{P}(Y_{n+1} = y_1 | Y_n = y_0) = Q(y_0, y_1)$. Let $E^Z = E^X \times E^Y$ be the state space of Z.

(a) TRUE. $Z_n := (X_n, Y_n)$ is an irreducible, aperiodic HMC. To see that it is homogeneous, observe that

$$\mathbb{P}(Z_{n+1} = (x_1, y_1) | Z_n = (x_0, y_0)) = \mathbb{P}((X_{n+1}, Y_{n+1}) = (x_1, y_1) | (X_n, Y_n) = (y_1, y_0))$$

$$= \frac{\mathbb{P}(X_{n+1} = x_1, Y_{n+1} = y_1, X_n = x_0, Y_n = y_0}{\mathbb{P}(X_n = x_0, Y_n = y_0)}$$

$$= \mathbb{P}(X_{n+1} = x_1 | X_n = x_0)\mathbb{P}(Y_{n+1} = y_1 | Y_n = y_0) = P(x_0, x_1)Q(y_0, y_1),$$

where the penultimate inequality follows from the independence of X_n, Y_n. Then we can define the transition matrix R of Z_n to be $R((x_0, y_0), (x_1, y_1)) = P(x_0, x_1)Q(y_0, y_1)$. It follows that Z_n is a Markov chain. Furthermore, since $P(x_0, x_1), Q(y_0, y_1)$ are independent of time (X_n and Y_n are HMCs) for all $x_0, x_1 \in E^X$ and $y_0, y_1 \in E^Y$, then R must be independent of time as well, and so Z_n is indeed an HMC.

We now show that Z_n is irreducible and aperiodic. Since P, Q are irreducible, we have that for every $x_0, x_1 \in E^X, y_0, y_1 \in E^Y$, there exist n, m such that $P^n(x_0, x_1), Q^m(y_0, y_1) > 0$. Furthermore, since P, Q are aperiodic, there exists some $N > 0$ such that $k > N \implies P^k(x_0, x_0), Q^k(y_0, y_0) > 0$. Then by independence,

$$R^{n+m+k}((x_0, y_0), (x_1, y_1)) = P^{n+m+k}(x_0, x_1)Q^{n+m+k}(y_0, y_1)$$

$$= \left(\sum_{l\in E} P^n(x_0, l)P^{m+k}(l, x_0)\right)\left(\sum_{l\in E} Q^n(y_0, l)Q^{m+k}(l, x_0)\right)$$

$$\geq P^n(x_0, 1)P^{m+k}(x_0, x_1)Q^n(y_0, 1)Q^{m+k}(y_0, y_1) > 0,$$

where the second equality follows from Chapman-Kolmogorov. Then we see that for any states $(x_0, y_0), (x_1, y_1) \in E^Z$, there exists some $c = n + m + k$ such that $R^c((x_0, y_0), (x_1, y_1)) > 0$, and so Z is irreducible.

Recall that x_0, y_0 were arbitrarily chosen. It follows that Z is also aperiodic by taking $x_0 = x_1, y_0 = y_1$ so that $n = m = 0$, and then $R^k((x_0, y_0), (x_0, y_0)) > 0$, as desired.

(b) Since $Z = (Z_n)_{n\in\mathbb{N}}$ is an irreducible, aperiodic HMC on the finite state space $E^Z = E^X \times E^Y$, we have that Z is ergodic. Furthermore, we have that Z admits a unique invariant distribution $\pi_Z \in \mathcal{P}(E)$ so that $\pi_Z R = \pi_Z$.

Then all bounded $f : E^Z \to \mathbb{R}$ (our f is on a finite state space E so f will be bounded in the non-extended real plane) have that $\mu^n f = \mathbb{E}[f(Z_n)] = \mathbb{E}[f(X_n, Y_n)]$, and $\mu^n f \to \pi_Z f$ as $n \to \infty$, for μ^n the law $\mathcal{L}(Z_n)$ of Z_n.

4 Problem 4

We will formalize our system by creating the stochastic process $X = (X_n)_{n\in\mathbb{N}}$ for the number of shoes in the front door of the man's house at time n.

Note that we do this because we can deduce the exact number of pairs of shoes at the back exit if we know the number of pairs in the front. Furthermore, note that, informally,

$$\mathbb{P}(\text{man runs barefoot}) = \mathbb{P}(\text{man leaves from front and front has 0 pairs})$$

$$+\mathbb{P}(\text{man leaves from back and back has 0 pairs}).$$

The man is equally likely to exit from the front or back of the house, and he does so independently of the dispersion of the pairs of shoes, so this expression reduces to

$$\frac{1}{2}\left[\mathbb{P}(\text{0 shoes in front}) + \mathbb{P}(\text{0 shoes in back})\right].$$

Furthermore, since the man has an even-money chance of exiting from the back or the front, the front/back labels of the exits are interchangeable. It follows that $\mathbb{P}(\text{0 shoes in front}) = \mathbb{P}(\text{0 shoes in back})$, and we have that

$$\mathbb{P}(\text{man runs barefoot}) = \mathbb{P}(\text{0 shoes in front (arbitrarily chosen)}).$$

Now, on each day, the man has 4 possible exit/entry combinations: Leave through front, enter through front (call this route 1); leave front, enter back (route 2); leave back, enter back (3); or leave back, enter front (4). Then, we may create the following table to determine the probability that $X_{n+1} = j$ given that $X_n = i$:

$X_n = i$	$X_{n+1} = j$	Routes	Probability
$i = 0$	$j = 0$	1, 2, 3	0.75
$i = 0$	$j = 1$	4	0.25
$0 < i < k$	$i + 1$	4	0.25
$0 < i < k$	i	1, 3	0.5
$0 < i < k$	$i - 1$	2	0.25
$i = k$	k	1, 3, 4	0.75
$i = k$	$k - 1$	2	0.5

This table allows us to create the entire transition matrix for X. We will now see that X has a unique invariant distribution.

First, we note that all states in X communicate, so X is irreducible. Furthermore, it can be seen in the table that at least one state has nonzero probability of returning to itself, so the entire chain is aperiodic. Finally, k is finite, so the state space $E = \{1, 2, ..., k\}$ of X is finite. Then we have that X must have a unique invariant distribution $\pi = \pi P$.

We will now solve for $\pi = (\pi_0, \pi_1, ..., \pi k)$. We have, from the transition probabilities for X given above,

$$\pi_0 = 0.75\pi_0 + 0.25\pi_1,$$

$$\pi_1 = 0.25\pi_0 + 0.5\pi_1 + 0.25\pi_2,$$
$$\pi_2 = 0.25\pi_1 + 0.5\pi_2 + 0.25\pi_3,$$
$$\vdots$$
$$\pi_{k-1} = 0.25\pi_{k-2} + 0.5\pi_{k-1} + 0.25\pi_k,$$
$$\sum_{i=1}^{k} \pi_i = 1.$$

Solving this system of linear equations from the top down, we see

$$\pi_0 = \pi_1,$$

$$\pi_1 = \pi_2,$$
$$\pi_2 = \pi_3,$$
$$\vdots$$
$$\pi_{k-1} = \pi_k.$$

Since $\pi_0 + \pi_1 + ... + \pi_k = 1$, we have that

$$\pi_0 = \boxed{\frac{1}{k+1}}$$

is the long-run proportion of times the man runs barefoot.

5 Problem 5

Let the process $X = (X_n)_{n\in\mathbb{N}}$ denote the urn we are drawing from at time n.

Then the state space E of X is given by $E = \{R, W, B\}$ for the red, white, and blue urns, respectively. Let $f_j(i)$ be the number of balls of color i in urn j. From elementary probability theory, we have that

$$\mathbb{P}[X_{n+1} = i | X_n = j] = \frac{f_j(i)}{f_j(R) + f_j(W) + f_j(B)},$$

and so we may construct the transition matrix P of X to be

$$P = \begin{array}{c c} & \begin{array}{ccc} R & W & B \end{array} \\ \begin{array}{c} R \\ W \\ B \end{array} & \begin{pmatrix} 1/5 & 0 & 4/5 \\ 2/7 & 3/7 & 2/7 \\ 3/9 & 4/9 & 2/9 \end{pmatrix} \end{array}.$$

Thus, since the transition probabilities are not time-dependent, X is a homogeneous Markov chain. Furthermore, since $P^2(0,1) = (4/5)(4/9) = 16/45$, we have that for all $i, j \in E, \exists\ n$ such that $P^n(i,j) > 0$, so X is irreducible. Of course, since $P(0,0) > 0$, we also have that X is aperiodic.

We may thus be sure that X has a unique invariant distribution $\pi = (\pi_R, \pi_W, \pi_B) \in \mathcal{P}(E)$ so that $\pi P = \pi$. We solve for π :

$$\pi_R = \frac{1}{5}\pi_R + \frac{2}{7}\pi_W + \frac{3}{9}\pi_B,$$

$$\pi_W = \frac{3}{7}\pi_W + \frac{2}{7}\pi_B,$$

$$\pi_R + \pi_W + \pi_B = 1$$

$$\implies \pi = \boxed{(0.2809, 0.3146, 0.4045)},$$

so the long-run proportion of red, white, and blue balls is 0.2809, 0.3146, 0.4045, respectively. Since the event that we see a ball of color $i \in E$ at time $n \geq 0$ is equivalent to us visiting urn $i \in E$ at time $n+1$, we have also that since $\mathcal{L}(X_n) \to \pi$, the probability $\mathcal{L}(X_n)$ of selecting a red, white, or blue ball at time n for large n is the same as the boxed solution above.

6 Problem 6

Define $X = (X_n)_{n\in\mathbb{N}}$ the length of the longest sequence ending at flip n that aligns with THH, starting from T.

Then $X_n = 0$ corresponds to either the sequence of all heads or the null sequence, $X_n = 1$ corresponds to ...T, $X_n = 2$ to ...TH, and $X_n = 3$ to ...THH. Of course, the state space E of X is then given by $E = \{0, 1, 2, 3\}$.

Define the stopping time $\tau = \min\{t : X_t = 3\}$. Then we seek $\mathbb{E}[\tau|X_0 = 0]$. We are given that the coin is fair, so we can easily create the transition matrix P of X:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \end{pmatrix} \end{array}.$$

It follows that X is a homogeneous Markov chain, since its transition matrix is not time-dependent. We may therefore define

$$m_i = \mathbb{E}[\tau|X_0 = i],$$

for $0 \leq i \leq 3$. We note that, due to the homogeneity of the chain, we also have that, for all $n \geq 0$,

$$\mathbb{E}[\tau|X_{n+1} = i] = \mathbb{E}[\tau|X_n = i] + 1.$$

Then we may compute

$$m_0 = \frac{1}{2}(m_1 + 1) + \frac{1}{2}(m_0 + 1),$$

$$m_1 = \frac{1}{2}(m_2 + 1) + \frac{1}{2}(m_1 + 1),$$

$$m_2 = \frac{1}{2}(1) + \frac{1}{2}(m_1 + 1),$$

$$\implies m_2 = \frac{1}{2}m_1 + 1,$$

$$m_1 = 6 \implies m_0 = 1 + 7 = \boxed{8}$$

is the expected number of flips until we see the sequence THH.

7 Problem 7

Let $X = (X_n)_{n\in\mathbb{N}}$. Given that $X_n = E_j$ for some $n \geq 0, j \in [0, ..., N-1]$, we have that $X_{n+1} \in [E_{j+1}, E_0]$, since the next item we check will either be defective (we return to E_0) or it will not be defective (we move to E_{j+1}).

If we are at $X_n = E_N$, then we will either transition to E_0 with probability rp (the chance of both checking the item and it being defective) or we will return to E_N with probability $1 - rp$ (the chance we either don't check an item at all or we do and it is not defective). Using this information, and the fact that the probability of any item being defective is $p \in (0, 1)$, we can create the following transition matrix for $(X_n)_n$:

$$P = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{N-1} \\ E_N \end{array} \begin{array}{c} \begin{array}{ccccccc} E_0 & E_1 & E_2 & E_3 & \dots & E_{N-1} & E_N \end{array} \\ \begin{pmatrix} p & 1-p & 0 & 0 & \dots & 0 & 0 \\ p & 0 & 1-p & 0 & \dots & 0 & 0 \\ p & 0 & 0 & 1-p & \dots & 0 & 0 \\ p & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ p & 0 & 0 & 0 & \vdots & 0 & 1-p \\ rp & 0 & 0 & 0 & \dots & 0 & 1-rp \end{pmatrix} \end{array}.$$

(a) It follows that $(X_n)_n$ is a homogeneous Markov chain, since the transition probabilities do not change with time. To see that the chain is irreducible, we will show that there is a non-zero probability that, starting from any state, we loop through each state and return to our original state, thus communicating with all other states. Formally, observe that, for $0 \leq i < j < N$, and $0 \leq n < m$,

$$\begin{aligned} &\mathbb{P}[X_m = E_j | X_n = E_i] \\ &\geq \mathbb{P}[X_{n+1} = E_{i+1}, X_{n+2} = E_{i+2}, ..., X_m = E_j] = (1-p)^{m-n}, \end{aligned} \tag{1}$$

where we will replace a $1-p$ for a $1-rp$ if we choose $j = N$. Furthermore, since

$$\begin{aligned} \mathbb{P}[X_m = E_0 | X_n = E_i] &\geq \mathbb{P}[X_{n+1} = E_{i+1}, X_{n+2} = E_{i+2}, ..., X_{m-1} = E_N, X_m = E_0] \\ &= (1-p)^{m-n-1}(rp), \end{aligned} \tag{2}$$

then for any $0 \leq i \leq j \leq N$, and $0 \leq n < m$, we have that $\mathbb{P}[X_m = E_i | X_n = E_j]$ is at least the product of (2) and (1), where (1) will then take $i = 0$.

Thus we have that for all states E_i, E_j, there exists n such that $P^n(i,j) > 0$, and so our chain is irreducible. It follows that because $P(0,0) > 0$, state 0 has period 1, and thus the entire chain is aperiodic, since it is irreducible.

Then because X is an irreducible HMC with finite state space E, we must have that each entry in the invariant distribution will be positive, because otherwise (if all were zero), the probabilities will not sum to one. Since we are operating on a finite state space, we require that any law integrate (sum) to one. Thus the chain must be positive recurrent.

(b) We will now compute the unique stationary distribution $\pi = (\pi_0, ..., \pi_N) \in \mathcal{P}(E), E = \{E_0, ..., E_N\}$ of X. We have that $\pi = \pi P$, so we solve the linear system (omitting one equation due to redundancy):

$$\pi_1 = (1-p)\pi_0,$$

$$\pi_2 = (1-p)\pi_1,$$

$$\vdots$$

$$\pi_{N-1} = (1-p)\pi_{N-2},$$

$$\pi_N = \frac{1-rp}{rp}(1-p)\pi_{N-1},$$

and of course,

$$\sum_{i=0}^{N} \pi_i = 1. \tag{3}$$

Then we have that

$$\begin{aligned}
\pi_1 &= (1-p)\pi_0, \\
\pi_2 &= (1-p)^2\pi_0, \\
\pi_3 &= (1-p)^3\pi_0, \\
&\vdots \\
\pi_{N-1} &= (1-p)^{N-1}\pi_0,
\end{aligned}$$

and

$$\pi_N = \frac{1-rp}{rp}(1-p)^{N-1}\pi_0$$

Inserting these into (3), we get

$$\begin{aligned}
\pi_0\left[\sum_{n=0}^{N-1}(1-p)^n + \frac{1-rp}{rp}(1-p)^{N-1}\right] &= 1, \\
\pi_0[r - r(1-p)^N + (1-rp)(1-p)^{N-1}] &= rp, \\
\pi_0 &= \frac{rp}{r+(1-p)^{N-1}(1-r)},
\end{aligned}$$

and so for $0 \le j \le N-1$,

$$\pi_j = \boxed{\frac{rp(1-p)^j}{r+(1-p)^{N-1}(1-r)}},$$

and

$$\pi_N = \boxed{\frac{(1-p)^{N-1}(1-rp)}{r+(1-p)^{N-1}(1-r)}}.$$

(c) Define $Y = (Y_k)_{k\in\mathbb{N}}$ a Markov chain with state space $E = \{0,1\}$ such that $Y_k = 1$ if the kth item is being checked, and $Y_k = 0$ otherwise. Then, since X is an irreducible HMC with a stationary distribution π on a finite state space E, we may invoke the ergodic theorem to compute

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} I\{Y_k = 1\} = \mathbb{E}_\pi[I\{Y_0 = 1\}] = \mathbb{P}_\pi[Y_0 = 1]$$

$$= \mathbb{P}_\pi[Y_0 = 1|X_0 = E_N]\mathbb{P}_\pi[X_0 = E_N] + \mathbb{P}_\pi[Y_0 = 1|X_0 \neq E_N]\mathbb{P}_\pi[X_0 \neq E_N],$$

so $X_0 = E_N$ corresponds to the process being in phase A and $X_0 \neq E_N$ corresponds to B. Then this expression can be evaluated to be

$$r \times \pi_N + (1 - \pi_N) = \pi_N(r-1) + 1$$

$$= \boxed{\frac{(1-p)^{N-1}(1-rp)}{r + (1-p)^{N-1}(1-r)}(r-1) + 1}$$

$$= \frac{r(p(r-1)(1-p)^N + p - 1)}{(r-1)(1-p)^N + r(p-1)},$$

the long-run proportion of items that will be checked.

(d) We denote the efficiency of the system by γ. Then

$$\gamma = \frac{\text{Long-run \% of detected defectives}}{\text{Long-run \% of defectives in general}}.$$

We are given that the long-run proportion of defectives among all items will be $r \in (0, 1)$, so we must only compute the numerator. We define another process $Z = (Z_k)_{k \in \mathbb{N}}$ with state space $\{0, 1\}$ so $Z_k = 1$ if the kth item is defective and $Z_k = 0$ otherwise. Then the indicator $I\{Y_k = 1, Z_k = 1\}$ is 1 for the event that we detect a defective item.

Once again invoking the ergodic theorem, we get that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} I\{Y_k = 1, Z_k = 1\} = \mathbb{E}_\pi[I\{Y_0 = 1, Z_0 = 1\}]$$

$$= \mathbb{P}_\pi[Y_0 = 1, Z_0 = 1] = \mathbb{P}_\pi[Y_0 = 1]\mathbb{P}_\pi[Z_0 = 1],$$

so

$$\gamma = \frac{r(\pi_N(r-1) + 1)}{r}$$

$$= \boxed{\frac{(1-p)^{N-1}(1-rp)}{r + (1-p)^{N-1}(1-r)}(r-1) + 1}$$

$$= \frac{r(p(r-1)(1-p)^N + p - 1)}{(r-1)(1-p)^N + r(p-1)},$$

is the theoretical efficiency of the system (and the long-run proportion of items that are checked at all!)

8 Problem 8

(a) Consider the stochastic process $X = (X_n)_{n \in \mathbb{N}}$ with state space $E = \{0, 1, 2\}$, where X_n is our wealth at time n modulo 3.

Then, since we are always playing game B, we have that if $X_n = 0$ yields a $\frac{1}{10} - \epsilon$ chance of winning, moving us to $X_{n+1} = 1$. We then have a $\frac{9}{10} + \epsilon$ chance of losing, moving to $X_{n+1} = 2$. From $X_n = 1$, we play with the other coin, so we have a $\frac{3}{4} - \epsilon$ chance of winning, moving to $X_{n+1} = 2$, and a $\frac{3}{4} + \epsilon$ chance of losing, moving to $X_{n+1} = 0$. These same latter transition probabilities hold for $X_n = 2$ since we play with the same coin, but we move to 0 if we win and 1 if we lose. Therefore our transition matrix is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 0.095 & 0.905 \\ 0.255 & 0 & 0.745 \\ 0.745 & 0.255 & 0 \end{pmatrix} \end{array}.$$

Then since

$$P^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.6985 & 0.2308 & 0.0708 \\ 0.5550 & 0.2142 & 0.2308 \\ 0.0650 & 0.0708 & 0.8642 \end{pmatrix} \end{array},$$

this HMC is irreducible, since $P^2(i,j) > 0$ for all states $i, j \in E$. Then consider that it is possible to start from state 0, move to 1, and then back to 0, a total of 2 steps ($0 \to 1 \to 0$). However, we could also have followed the path $0 \to 1 \to 2 \to 0$, a total of 3 steps. Thus $GCD(2,3) \implies$ state 0 has period 1, and since the chain is irreducible, the chain is aperiodic.

Thus there exists a unique invariant distribution $\pi = (\pi_0, \pi_1, \pi_2)$ so that $\pi = \pi P$. Solving for π, we see

$$\pi_0 = 0.255\pi_1 + 0.745\pi_2,$$

$$\pi_1 = 0.105\pi_0 + 0.255\pi_2,$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\implies \pi = (0.3836, 0.1543, 0.4621).$$

Then, since X is an irreducible and aperiodic HMC with finite state space E, the ergodic theorem tells us that, letting $n \to \infty$,

$$\mathbb{P}[X_n = i] = \mathbb{P}_\pi[X_n = i],$$

regardless of the initial distribution. Then we have that, over the long-run,

$$\mathbb{P}[\text{winning}] = \mathbb{P}[X_{n+1} = 1 | X_n = 0]\mathbb{P}_\pi[X_n = 0]$$

$$+\mathbb{P}[X_{n+1} = 2 | X_n = 1]\mathbb{P}_\pi[X_n = 1]$$

$$+\mathbb{P}[X_{n+1} = 0 | X_n = 2]\mathbb{P}_\pi[X_n = 2]$$

$$= (0.095)(0.3836) + (0.745)(0.1543) + (0.745)(0.4621) = \boxed{0.4957},$$

as desired.

(b) Denote B the transition matrix of our wealth modulo 3 when playing game B as previously given. We write the transition matrix A of X when playing on the game A only:

$$A = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 0.495 & 0.505 \\ 0.505 & 0 & 0.495 \\ 0.495 & 0.505 & 0 \end{pmatrix} \end{array}.$$

Since we play the games in a sequence $A, A, B, B, A, A, B, B, ...$, we must consider the matrices B^2A^2 to understand the transitions in the state of X between each block of games played in the sequence B, B, A, A (more on this in a second). This is given below

$$B^2A^2 = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0.4238 & 0.3108 & 0.2654 \\ 0.3888 & 0.3052 & 0.3060 \\ 0.2702 & 0.2637 & 0.4660 \end{pmatrix} \end{array}.$$

We will also require the transition matrix BA^2B to understand the transitions in the state of X between each block played in the sequence B, A, A, B. (Seen another way, these are the transition probabilities of X for the second B in the first block, where we start from the first A.) This is

$$BA^2B = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0.4238 & 0.1457 & 0.4305 \\ 0.3872 & 0.1411 & 0.4718 \\ 0.2654 & 0.1045 & 0.6301 \end{pmatrix} \end{array}.$$

Note that only the probabilities of winning or losing in game B will alter as a function of X_n, but the probabilities of win or loss in game A will be the same no matter our starting wealth, since the same coin is used in every game. Thus we only are interested in the distribution of X_n before starting game B, hence our choice of transition matrices given above.

Our attack plan, then, is to find (and confirm existence/uniqueness of) the invariant distributions for X under both B^2A^2 and BA^2B. We will then compute the long-term proportion of games won by considering the block B, B, A, A played repeatedly. We will directly compute the probability of winning either 4, 3, 2, 1, or 0 games in this block using the invariant distribution, and then compute the expected number of wins in a game in that block as our final answer. Let's start!

Let X_1 be the process of our wealth modulo 3 when playing the sequence B, B, A, A, so X_1 has transition matrix B^2A^2. Then since all entries in B^2A^2 are positive, X_1 is irreducible, and since $\mathbb{P}[X_{n+1} = 0 | X_n = 0] > 0$, the chain is aperiodic. The same facts hold for the sequence of games considered one time unit later: B, A, A, B. Denote the process of our wealth modulo 3 under this sequence as X_2. Then both X_1 and X_2 are irreducible, aperiodic HMCs with finite state space $E = \{0, 1, 2\}$.

We now seek the invariant distribution $\alpha = \alpha A^2 B^2 = (\alpha_0, \alpha_1, \alpha_2)$ for X_1. We have

$$\alpha_0 = 0.4238\alpha_0 + 0.3888\alpha_1 + 0.2702\alpha_2,$$
$$\alpha_1 = 0.3108\alpha_0 + 0.3052\alpha_1 + 0.2637\alpha_2,$$
$$\alpha_0 + \alpha_1 + \alpha_2 = 1$$
$$\implies \alpha = (0.3603, 0.2928, 0.3469).$$

Similarly, denoting $\beta = \beta B A^2 B = (\beta_0, \beta_1, \beta_2)$ for X_2. We have

$$\beta_0 = 0.4238\beta_0 + 0.3872\beta_1 + 0.2654\beta_2,$$
$$\beta_1 = 0.1457\beta_0 + 0.1411\beta_1 + 0.1045\beta_2,$$
$$\beta_0 + \beta_1 + \beta_2 = 1$$
$$\implies \beta = (0.3310, 0.1227, 0.5442).$$

Then we may compute that, as $n \to \infty$,

$$\mathbb{P}[\text{win first B}] = \mathbb{P}[\text{win first B}|X_n = 0]\mathbb{P}_\alpha[X_n = 0]$$
$$+\mathbb{P}[\text{win first B}|X_n = 1]\mathbb{P}_\alpha[X_n = 1]$$
$$+\mathbb{P}[\text{win first B}|X_n = 2]\mathbb{P}_\alpha[X_n = 2] = 0.5108.$$

Similarly,

$$\mathbb{P}[\text{win second B}] = \mathbb{P}[\text{win second B}|X_n = 0]\mathbb{P}_\beta[X_n = 0]$$
$$+\mathbb{P}[\text{win second B}|X_n = 1]\mathbb{P}_\beta[X_n = 1]$$
$$+\mathbb{P}[\text{win second B}|X_n = 2]\mathbb{P}_\beta[X_n = 2] = 0.5285.$$

Of course, since the probability of winning game A will not change as a function of our wealth modulo 3, we may simply write that

$$\mathbb{P}[\text{win A}|X_n = i \in E] = 0.495.$$

Now it is simply a matter of computing all $2^4 = 16$ combinations of wins/losses possible in the sequence B, B, A, A and summing the wins, weighting them by their probabilities of occurrence, and summing all terms to arrive at the expected number of wins in the block:

$$\mathbb{E}[\text{wins in sequence } B, B, A, A] = 4\mathbb{P}[\text{winning all } B, B, A, A]$$
$$+3(\mathbb{P}[\text{winning exactly three of } B, B, A, A])$$
$$+2(\mathbb{P}[\text{winning exactly two of } B, B, A, A])$$
$$+1(\mathbb{P}[\text{winning only one of } B, B, A, A])$$

$$= 4(0.5108)(0.5285)(0.4950^2) + 3[(0.5108)(0.5285)(0.4950)(0.505) + ...$$
$$+(0.4892)(0.5285)(0.495)(0.495)] + 2[(0.5108)(0.5285)(0.505^2) + ...$$
$$+(0.4892)(0.4715)(0.495)(0.495)]$$
$$+1[(0.4892)(0.4715)(0.505)(0.495) + ...(0.5108)(0.4715)(0.505)(0.505)]$$
$$= \boxed{0.5073},$$

as expected.

9 Problem 9

Let $X = (X_n)_{n\in\mathbb{N}}$ be a stochastic process tracking the position of the particle at time n.

We assign X the state space $E = \{0, 1, 2, 3\}$, so $X_n = i, 0 \leq i \leq 3$ signals that the particle at a vertex i steps away from v (in the shortest-path sense). Note that it will not cause problems to group vertices in this way because the cube is symmetric, so vertices in each distance grouping will be equivalent under rotations and translations from the perspective of vertex v.

We now provide the transition matrix P of X :

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/4 & 3/4 & 0 & 0 \\ 1/4 & 1/4 & 2/4 & 0 \\ 0 & 2/4 & 1/4 & 1/4 \\ 0 & 0 & 3/4 & 1/4 \end{pmatrix} \end{array}.$$

This matrix can be obtained easily by considering any single vertex in each vertex grouping and filling the transition probabilities of that vertex into the respective entry, since all vertices are equivalent from our perspective as explained above. Clearly, then, X is homogeneous, since its transition probabilities are not time-dependent.

(a) We define $\tau = \min\{n : n > 0, X_n = 0\}$, so we seek $\mathbb{E}[\tau|X_0 = 0]$. Since X is homogeneous, for any $k \in E$,

$$\mathbb{E}[\tau|X_n = k] = \mathbb{E}[\tau|X_{n+s} = k] + s,$$

so we suppress the time subscript to define $m_k = \mathbb{E}[\tau|X_0 = k]$. Then

$$m_0 = 0.75(m_1 + 1) + 0.25(0 + 1),$$

$$m_1 = 0.25(0 + 1) + 0.5(m_2 + 1) + 0.25(m_1 + 1),$$
$$m_2 = 0.25(m_2 + 1) + 0.5(m_1 + 1) + 0.25(m_3 + 1),$$
$$m_3 = 0.25(m_3 + 1) + 0.75(m_2 + 1)$$
$$\implies m_3 = \frac{40}{3}, m_2 = 12, m_1 = \frac{28}{3} \implies m_0 = \boxed{8}$$

is the expected number of steps for the particle to return to v.

(b) We now set $\tau = \min\{n : X_n = 3\}$, since w is the unique vertex that is 3 steps away from v. Then following the notation of (a), we define $m_k = \mathbb{E}[\tau | X_0 = k]$, and we seek m_0. We have that

$$m_0 = 0.75(m_1 + 1) + 0.25m_0,$$

$$m_1 = 0.25(0+1) + 0.5(m_2+1) + 0.25(m_1+1),$$
$$m_2 = 0.25(m_2+1) + 0.5(m_1+1) + 0.25(0+1)$$
$$\implies m_2 = \frac{28}{3}, m_1 = 12, m_0 = \boxed{\frac{40}{3}}$$

is the expected time to first hit w from v. (This could also have been intuitively confirmed by the fact that in part (a), we found the expected time to return to v from w as $\frac{40}{3}$. The symmetry of the cube allows for these results to align.)

(c) We create a new Markov chain $Z = (Z_n)_{n \in \mathbb{N}}$ with state space $E = \{v, w\}$, since these two states are all that are relevant to our problem.

Then we consider the chance that X starts at v and hits w before v. Define $\tau = \min\{n : X_n = v \text{ or } X_n = w\}$. Since X is homogeneous, denoting $p_k = \mathbb{P}[X_\tau = v | X_0 = k]$, we compute

$$p_0 = 0.25(0) + 0.75p_1,$$

$$p_1 = 0.25p_1 + 0.5p_2 + 0.25(0),$$
$$p_2 = 0.25(1) + 0.25p_2 + 0.5p_1,$$
$$\implies p_0 = \frac{3}{10}$$

as the probability p of hitting w before returning to v. By symmetry, this is the probability of hitting v before returning to w when starting from w. Then our transition matrix Q for Z is given by

$$Q = \begin{array}{c} \\ v \\ w \end{array} \begin{array}{c} \begin{array}{cc} v & w \end{array} \\ \begin{pmatrix} 7/10 & 3/10 \\ 3/10 & 7/10 \end{pmatrix} \end{array}.$$

Using this transition matrix, we can easily compute the probability that we return to v after n visits to w as $p^2(1-p)^{n-1}$. Then, starting from v,

$$\mathbb{E}[\text{visits to } w \text{ before returning to } v] = \sum_{n=1}^{\infty} (n+1)(1-p)^n p^2$$
$$= \boxed{1}.$$

10 Problem 10

We have that, for $E = \{0, 1, 2, ...\}$ an infinite state space and $P \in \mathcal{P}(E)$,

$$\sum_{i \in E} p_{i,j} = \sum_{j \in E} p_{i,j} = 1.$$

Then we first seek $\pi = (\pi_0, \pi_1, ...)$ such that $\pi P = \pi$. Set $\pi = \mathbf{1}$. Then $\forall\ j \in E$,

$$\sum_{j \in E} \pi_j p_{i,j} = \sum_{j \in E} p_{i,j} = 1,$$

so indeed π is invariant under P, i.e. $\pi P = \pi$. Since P is irreducible, to find the invariant distribution π^* for P, we must simply normalize π such that $\pi^* = \frac{\pi}{K}$, for $K = \sum_{i \in E} \pi_i$. But

$$\pi = \mathbf{1} \implies K = \sum_{i \in E} \pi_i = \sum_{i=0}^{\infty} 1 = \infty,$$

so it follows that $\pi^* = \mathbf{0}$. Thus since $\pi_i^* = 0 \not> 0$ for every $i \in E$, P cannot be positive recurrent