ACM 95b, PS 4

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May 20, 2023

1 Problem 1

(a) Denote the density of both rods be ρ, the specific heat of the first and second be c_1, c_2, respectively, the lengths of the first and second be L_1, L_2, the heats be Q_1, Q_2, and the proportionality constants be K_1, K_2. Then the diffusion equation for rod 1 is given by

$$\frac{\partial T_1}{\partial t} = \frac{K_1}{c_1 \rho} \frac{\partial^2 T_1}{\partial x^2} + \frac{Q_1}{c_1 \rho}, \tag{1}$$

and the diffusion equation for rod 2 is given by

$$\frac{\partial T_2}{\partial t} = \frac{K_2}{c_2 \rho} \frac{\partial^2 T_2}{\partial x^2} + \frac{Q_2}{c_2 \rho}. \tag{2}$$

We will require conditions at the ends of the rods informing us of their temperatures and fluxes. Since we don't know whether the temperature and flux at the ends of the combined rods are being held constant or changing in a predictable manner, the most general set of boundary conditions to use will be convective. For this first rod, then, we will require the conditions

$$-K_1(0) \frac{\partial T_1}{\partial x}(0, t) = -H_1[T_1(0, t) - T_{1,0}(t)],$$

$$-K_1(L_1) \frac{\partial T_1}{\partial x}(L_1, t) = H_1[T_1(L_1, t) - T_{1,L_1}(t)],$$

for H_1 the heat transfer coefficient of rod 1 and $T_{1,\cdot}(t)$ the temperature of rod 2 at $x = \cdot$. For the second rod, we will require the conditions

$$-K_2(L_1) \frac{\partial T_2}{\partial x}(L_1, t) = -H_2[T_2(L_1, t) - T_{2,L_1}(t)],$$

$$-K_2(L_1 + L_2) \frac{\partial T_2}{\partial x}(L_1 + L_2, t) = H_2[T_2(L_1 + L_2, t) - T_{2,L_1+L_2}(t)].$$

Note that, at $x = L_1$, since we are told that the temperatures of the rods are equal, $T_1(L_1, t) - T_{1,L_1}(t) = T_2(L_1, t) - T_{2,L_1}(t) = 0$. Additionally, since the flux is 0 at the point of contact, $\frac{\partial T_1}{\partial x}(L_1, t) = \frac{\partial T_2}{\partial x}(L_1, t) = 0$.

(b) Since we have two rods, the system we wish to solve is

$$\frac{\partial T_1}{\partial t} = 2\frac{\partial^2 T_2}{\partial x^2}, \frac{\partial T_2}{\partial t} = \frac{1}{2}\frac{\partial^2 T_2}{\partial x^2} + \frac{1}{2},$$

with conditions $T_1(0,t) = 0, T_2(2,t) = 0$. We assume the equilibrium solution will be a function of x only, so we seek $T_1^*(x), T_2^*(x)$ that solve the equations. From our system of PDEs, we get that

$$0 = \frac{\partial^2 T_2}{\partial x^2}, 0 = \frac{1}{2}\frac{\partial^2 T_2}{\partial x^2} + \frac{1}{2}$$
$$\implies T_1(x,t) = ax + b = T_1^*(x),$$
$$T_2(x,t) = -\frac{x^2}{2} + cx + d = T_2^*(x).$$
$$\implies T_1^*(x) = ax, T_2^*(x) = -\frac{x^2}{2} + cx + 2 - 2c,$$

and since $T_1(1,t) = T_2(1,t)$, we have that $a = \frac{3}{2} - c$. Since the rods are in perfect thermal contact, we have also that

$$-2\frac{\partial T_1^*}{\partial x}(1) = -\frac{\partial T_2^*}{\partial x}(1)$$

$$\implies -2a = -3 + 2c = -(-1+c) = 1 - c \implies 3c = 4 \implies c = \frac{4}{3}.$$

Thus our solution to the PDE is

$$\boxed{T_1^*(x) = \frac{1}{6}x, T_2^*(x) = -\frac{x^2}{2} + \frac{4}{3}x - \frac{2}{3}}.$$

2 Problem 2

Let $\xi = x^\alpha t$. We wish to find a function $f(\xi)$ such that $T = f(\xi)$ is a solution to the heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}.$$

We have that

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial t} = x^\alpha,$$
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} = \alpha x^{\alpha-1}t$$
$$\implies \frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial^2 f}{\partial \xi^2}\right)(\alpha x^{\alpha-1}t)^2 + \frac{\partial f}{\partial \xi}\alpha(\alpha-1)x^{\alpha-2}t.$$

Plugging these results into the original equation, we get that we require

$$\frac{\partial f}{\partial \xi}x^{\alpha} = k(\alpha x^{\alpha-1}t)^2\frac{\partial^2 f}{\partial \xi^2} + k\alpha(\alpha-1)x^{\alpha-2}t\frac{\partial f}{\partial \xi}$$

$$= k(\alpha\xi x^{-1})^2\frac{\partial^2 f}{\partial \xi^2} + k\alpha(\alpha-1)\xi x^{-2}\frac{\partial f}{\partial \xi}$$

$$\implies \frac{\partial f}{\partial \xi}x^{\alpha+2} = k\xi^2\alpha^2\frac{\partial^2 f}{\partial \xi^2} + k\alpha(\alpha-1)\xi\frac{\partial f}{\partial \xi}.$$

To make the x term vanish, we may simply set $\boxed{\alpha = -2}$. Then $f(\xi)$ will be the solution to the ODE

$$f'(\xi) = 4k\xi^2 f''(\xi) + 6k\xi f'(\xi) \iff 4k\xi^2 f''(\xi) + f'(\xi)(6k\xi - 1) = 0$$

$$\iff f''(\xi) + f'(\xi)\left(\frac{6k\xi - 1}{4k\xi^2}\right) = 0.$$

We will set $\omega = f'(\xi)$. Then we solve

$$\omega' + \omega\left(\frac{6k\xi - 1}{4k}\right) = 0$$

$$\implies \omega = f'(\xi) = A\exp\left(\int \frac{1-6k\xi}{4k\xi^2}d\xi\right)$$

$$\implies \boxed{f(\xi) = \int A\exp\left(\int \frac{1-6k\xi}{4k\xi^2}d\xi\right)d\xi}.$$

3 Problem 3

(a) We wish to find functions $\alpha_n(t), \beta_n(x)$ such that

$$\phi(x,t) = \sum_{n=0}^{\infty} \alpha_n(t)\beta_n(x).$$

We will first find solutions to

$$\beta_t = a^2\beta_{xx}, \beta(0) = \beta_x(L) = 0.$$

We know from lecture that the general solution to this equation will be formed from $\beta(x) = \sin(\lambda x)$ and $\beta(x) = \cos(\lambda x)$. Using the conditions, we see

$$\beta(x) = \sin(\lambda x) \implies \beta(0) = 0,$$

$$\beta'(x) = \lambda\cos(\lambda x) \implies \beta'(L) = 0 \iff \lambda = \frac{\pi + 2\pi n}{2L}, n \in \mathbb{N}$$

satisfies the BCs, while $\cos(\lambda x)$ cannot satisfy the first. Thus an appropriate expansion is to set

$$\beta_n(x) = \sin\left(x\left(\frac{\pi + 2\pi n}{2L}\right)\right)$$

for the equation given at the top of this part.

(b) Using our result for $\beta_n(x)$ from above, we may plug in our series representation of $\phi(x,t)$ to the PDE

$$\phi_t = a^2\phi_{xx} + w(x,t)$$

to see that

$$\sum_{n=0}^{\infty} \alpha_n'(t)\sin(\lambda_n x) = a^2\sum_{n=0}^{\infty} \lambda_n^2\alpha_n(t)(-\sin(\lambda_n x)) + \sum_{n=0}^{\infty} w_n \sin(\lambda_n x)$$

$$\implies \sum_{n=0}^{\infty} [\alpha_n'(t) + a^2\lambda_n^2\alpha_n(t) - w_n]\sin(\lambda x) = 0$$

$$\implies \alpha_n'(t) + a^2\lambda_n^2\alpha_n(t) - w_n(t) = 0,$$

where we have expanded $w(x,t)$ into its corresponding Fourier series using the $\beta_n(x)$'s as the basis of the expansion. The coefficients of this expansion are therefore given by

$$w_n = \frac{2}{L}\int_0^L w(x,t)\sin(\lambda_n x)dx.$$

This ODE is first-order and linear; its solution is given explicitly by

$$\alpha_n(t) = c_n \exp(-a^2\lambda_n^2 t) + \exp(-a^2\lambda_n^2 t)\int_0^t \exp(a^2\lambda_n^2\tau)w_n(\tau)d\tau,$$

where from lecture, we know

$$c_n = \frac{2}{L}\int_0^L \phi(x,0)\sin(\lambda x)dx = \frac{2}{L}\int_0^L \cos(\pi x/L)\sin(\lambda x)dx = \frac{4(2n+1)}{\pi(4n(n+1)-3)}.$$

(c) For $w(x,t) = 1$, we have that

$$w_n = \frac{2}{L}\int_0^L \sin(\lambda_n x)dx = \frac{4}{2\pi n + \pi}.$$

Then, from (b),

$$\alpha_n(t) = c_n \exp(-a^2\lambda_n^2 t) + \exp(-a^2\lambda_n^2 t)\int_0^t \exp(a^2\lambda_n^2\tau)\frac{4}{2\pi n + \pi}d\tau,$$

where we see that the integral on the RHS reduces to

$$\frac{16L^2(1 - e^{-a^2\lambda_n^2 t})}{a^2(2\pi n + \pi)^3}$$

Since all other terms in $\alpha_n(t)$ decay at least as fast as $\frac{1}{n^3}$, and this term decays at least as fast as $\frac{1}{n^3}$, then we have that our solution is uniformly convergent.

4 Problem 4

We will solve the problem using a series of functions $\beta_n(x)$ to represent the solution as a series

$$\phi(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\beta_n(x).$$

We will choose $\beta_n(x) = \sin(n\pi x/L)$ though the boundary conditions are not homogeneous. We will write our solution as

$$\phi(x,t) = \psi(x,t) + 2I_0(x) + I_L(x),$$

where $I_0(0) = 1, I_0(L) = 0$, and $I_L(0) = 0, I_L(L) = 1$. (Here we have set $L = l$.) We choose

$$I_0(X) = 1 - x/L, I_L(x) = x/L.$$

We then have

$$\phi(x,t) = \psi(x,t) + [2 - \frac{x}{L}].$$

Substituting into the PDE, we have

$$\psi_t = a^2\psi_{xx}$$

with boundary conditions $\psi(0,t) = \psi(L,t) = 0$, and initial condition $\psi(x,0) = -2 + \frac{x}{L}$. The solution to this problem can be represented by a Fourier series expansion over the $\beta_n(x)$ terms as

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n \exp(-\lambda_n^2 a^2 t)\beta_n(x), \lambda_n = \frac{n\pi}{L},$$

with

$$\beta_n(x) = \sin(n\pi x/L), c_n = \frac{2}{L}\int_0^L (-2 + \frac{x}{L})\beta_n(x))dx = \frac{-4 + \cos(\pi n)}{\pi n}.$$

Thus we can see that as $t \to \infty$, $\psi(x,t) \to 0$, and so

$$\phi(x,t) = \psi(x,t) + 2(1 - x/L) + x/L = \psi(x,t) + 2 - x/L \to 2 - x/L.$$

To check that this is the equilibrium solution to the PDE, we can assume that the equilibrium solution ϕ_{eq} is only a function of x to get that

$$0 = a^2\phi_{xx} \implies \phi_{xx} = 0 \implies \phi(x) = cx + d,$$

for c, d some constants, where

$$\phi(0) = 2 \implies d = 2, \phi(L) = cL + 2 = 1 \implies c = -\frac{1}{L},$$

so indeed

$$\phi_{eq}(x) = 2 - \frac{x}{L},$$

and thus $\phi(x,t) \to \phi_{eq}(x)$ as $t \to \infty$.