ACM 95b, PS 5

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1 Problem 1

Consider the 2D heat equation

$$u_t = k\nabla^2 u, 0 < \theta < \pi, 0 < r < r_0$$

with boundary conditions $u(r_0, \theta, t) = u(r, 0, t) = u_\theta(r, \pi, t) = 0$ and initial condition $u(r, \theta) = 1$.

We know that we can solve the 2D heat equation in polar coordinates by applying a separation of variables technique: We assume the solution $u(r, \theta, t)$ will take the form $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$.

From lecture, we know that

$$R(r) = J_m(j_{mn} r),$$

for J_m the mth Bessel function of the first kind and j_{mn} those values such that $J_m(j_{mn}) = 0, n \in \mathbb{N}$.[1] We also have that

$$T(t) = e^{-k\lambda_{mn}^2 t}.$$

Finally, we know $\Theta(\theta)$ must satisfy the ODE

$$\Theta_{\theta\theta} + \mu^2\Theta = 0, \Theta'(\pi) = 0,$$

for μ a constant. The solution to this ODE must also be periodic with period 2π, since we should expect that $u(r, \theta, t) = u(r, \theta + 2\pi, t)$, even on a semicircle. Thus the solution to the ODE is given by

$$\Theta(\theta) = \sin(\theta(2m + 1)/2).$$

Putting these results together, we have that $u(r, \theta, t)$ must be of the form

$$u(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} J_m(\lambda_{mn} r)\sin(\theta(2m + 1)/2)e^{-k\lambda_{mn}^2 t}$$

[1] We may use the result from lecture since we have only changed our domain from a circle to semicircle; this does not affect the solution to the ODEs on r or t.

for B_{mn} some undetermined coefficients. Finally, then, since we also need that $u(r, \theta, t = 0) = 1$,

$$1 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn} J_m(\lambda_{mn} r) \sin(\theta(2m+1)/2)$$

$$\implies J_m(\lambda_{mn'} r) \sin(\theta(2m'+1)/2)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn} J_m(\lambda_{mn} r) \sin(\theta(2m+1)/2) J_m(\lambda_{mn'} r) \sin(\theta(2m'+1)/2),$$

for $m', n' \in \mathbb{N}$. From orthogonality of the eigenfunctions, we then have that

$$\int_0^{r_0} \int_0^{\pi} J_m(\lambda_{mn'} r) \sin(\theta(2m'+1)/2) r d\theta dr$$

$$= A_{m'n'} \int_0^{r_0} \int_0^{\pi} (J_{m'}(\lambda_{m'n'} r) \sin(\theta(2m'+1)/2)^2 r d\theta dr$$

$$\implies A_{mn} = \frac{\int_0^{r_0} \int_0^{\pi} J_m(\lambda_{mn} r) \sin(\theta(2m+1)/2) r d\theta dr}{\int_0^{r_0} \int_0^{\pi} (J_m(\lambda_{mn} r) \sin(\theta(2m+1)/2))^2 r d\theta dr}.$$

Following notation from class, we can write this solution as

$$A_{mn} = \frac{1}{W_{mn}} \int_0^{r_0} \int_0^{\pi} J_m(\lambda_{mn} r) \sin(\theta(2m+1)/2) r d\theta dr,$$

for

$$W_{mn} = \frac{\pi}{2} N_{mn}, m \neq 0,$$

where

$$N_{mn} = \int_0^{r_0} (J_m(\lambda_{mn} r))^2 r dr.$$

2 Problem 2

Consider the 3D heat equation

$$u_t = k \nabla^2 u$$

with boundary conditions $u(a, \theta, \phi, t) = 0$.

For $u(r, \theta, \phi, 0) = f(r, \theta, \phi)$, we know from lecture that the solution to this PDE is given by

$$u(r, \theta, \phi, t) = \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lmn} j_l(\lambda_{ln} r) Y_{lm}(\theta, \phi) e^{-k \lambda_{ln}^2 t},$$

for A_{lmn} some coefficients (given below), j_l the spherical Bessel functions which are regular at the origin,

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}, l = 0, 1, 2, ..., -l \leq m \leq l,$$

where the P_l^m's are generalized Legendre polynomials and $\lambda_{ln} = \frac{\xi_{ln}}{r_0}$ for $\xi_{ln}, n \in \mathbb{N}$ the zeroes of $j_l(\cdot)$.

We then also have the initial condition

$$f(r, \theta, \phi) = \sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lmn} j_l(\lambda_{ln} r) Y_{lm}(\theta, \phi),$$

which we can then rearrange and solve by leveraging orthogonality to find that the coefficients A_{lmn} are given by

$$A_{lmn} = \frac{1}{N_{nl}}\int_0^{r_0} r^2 dr \int_0^{\pi} \sin(\theta) d\theta \int_0^{2\pi} d\phi f(r, \theta, \phi) j_l(\lambda_{ln} r) Y_{lm}^*(\theta, \phi).$$

The solution to (a) can then be found by simply setting $f(r, \theta, \phi) = r$ in the equations above. In this case, we want Y_{lm} to have no dependence on θ and ϕ, so using a lookup table of the zeroes of Y_{lm}, we see that we require $l = m = 0$.

The solution to (b) can be found by setting $f(r, \theta, \phi) = F(r)\sin(\theta)\cos(\phi)$. Then from a lookup table, we see that the only values of l, m that will allow us to have $\sin(\theta)\cos(\phi)$ in $Y_{lm}(\theta, \phi)$ will be $l = m = 1$.

3 Problem 3

We know from lecture that the minimum eigenvalue will be that which minimizes

$$\lambda = \frac{\int_0^1 \int_0^{2\pi} \left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2\right] r dr d\theta}{\int_0^1 r \int_0^{2\pi} \phi^2 dr d\theta},$$

for $\phi(r, \theta)$ our trial function. We will use the trial function $\phi(r, \theta) = r\cos(\theta)$, which is clearly regular at the origin (in Cartesian, this is simply $\phi(x, y) = x$). Then,

$$\phi_r = \cos(\theta), \phi_\theta = -r\sin(\theta),$$

so

$$\int_0^1 \int_0^{2\pi} \phi_r^2 r d\theta dr + \int_0^1 \int_0^{2\pi} \frac{1}{r}\phi_\theta^2 d\theta dr = 1.57 + 3.14,$$

and

$$\int_0^1 \int_0^{2\pi} r\phi^2 d\theta dx = 0.785.$$

Then our approximation for the minimum eigenvalue is

$$\frac{1.57 + 3.14}{0.785} = 6.$$

From lecture, we know that the correct answer is given by

$$\lambda_{11} = \pi^2 + \pi^2 = j_{01}^2 = 5.783.$$

4 Problem 4

Consider the Sturm-Liouville ODE

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) - q(x)\phi(x) + \lambda r(x)\phi(x) = 0, a < x < b$$

with boundary conditions $\phi(a) = \phi(b) = 0$.

We can multiply this ODE through by $\phi(x)dx$ to get

$$\phi(x)\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)dx - q(x)\phi(x)^2dx + \lambda r(x)\phi(x)^2dx = 0$$

$$\implies \lambda = \frac{-\int_a^b \phi \frac{d}{dx}\left(p(x)\phi'(x)\right)dx + \int_a^b q(x)\phi(x)^2dx}{\int_a^b r(x)\phi(x)^2dx}.$$

Integrating by parts, we can see that

$$-\int_a^b \phi\frac{d}{dx}\left(p(x)\phi'(x)\right)dx = -\phi(x)p(x)\phi'(x)\Big|_a^b + \int_a^b p(x)(\phi'(x))^2dx$$

$$= \int_a^b p(x)(\phi'(x))^2dx,$$

so indeed

$$\lambda = \frac{\int_a^b p(x)(\phi'(x))^2dx + \int_a^b q(x)\phi(x)^2dx}{\int_a^b r(x)\phi(x)^2dx}$$

is a Rayleigh quotient of the original ODE with the given boundary conditions.