ACM 95b, PS 6

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1 Problem 1

Consider the PDE
$$u_{tt} = c^2 u_{xx}, -\infty < x < \infty, \tag{1}$$
with boundary conditions $u(x,0) = 0, u_t(x,0) = p(x)$.

We wish to compute the Fourier transform of both sides of this equation. First note that, for $\mathcal{F}$ the Fourier transform,

$$\mathcal{F}[u_t] = \int_{-\infty}^{\infty} \lim_{h\to 0} \frac{u(x,t+h) - u(x,t)}{h} e^{-ikx} dx$$

$$= \lim_{h\to 0} \frac{1}{h} \left(\int_{-\infty}^{\infty} u(x,t+h) e^{-ikx} dx - \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx \right)$$
$$= \lim_{h\to 0} \frac{1}{h} (\mathcal{F}(u(x,t+h)) - \mathcal{F}(u(x,t))) = \frac{\partial}{\partial t} \mathcal{F}(u(x,t)),$$

where we have assumed that the integrals will converge before applying the Fourier transform in the first place. So

$$\mathcal{F}[u_{tt}] = \frac{\partial^2}{\partial t^2} \mathcal{F}[u(x,t)].$$

We also know from lecture that $\mathcal{F}[u_x] = ik\frac{\partial}{\partial x}\mathcal{F}[u(x,t)] = ik\mathcal{F}[u](k,t)$, so applying the transform to both sides of (1) yields

$$\frac{\partial^2}{\partial t^2} \mathcal{F}[u(x,t)] = -c^2 k^2 \mathcal{F}[u(x,t)].$$

Fixing k, we get that for $v(k,t) = \mathcal{F}[u(x,t)]$,

$$v_{tt} = -c^2 k^2 v,$$

a second-order linear ODE. Trying the solution $v = e^{rt}$, we get that

$$v_{tt} + c^2 k^2 v = r^2 e^{rt} + c^2 k^2 e^{rt} = 0 \implies r_{1,2} = \pm ick.$$

Thus our general solution to the ODE is given by $v(t) = a_1 e^{ickt} + a_2 e^{-ickt}$, where a_1 and a_2 are still allowed to depend on k.

To then recover $u(x,t)$ from v, we must take the inverse Fourier transform of v with respect to k:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} v(t)e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}[a_1(k)e^{ickt} + a_2(k)e^{-ickt}]dk$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} a_1(k)e^{ickt} + \int_{-\infty}^{\infty} a_2(k)e^{-ickt}\right) = a_1(x-ct) + a_2(x+ct).$$

We can already see now that this solution aligns with the D'Alembert solution to (1) shown in lecture. Plugging in our initial conditions, we see that

$$a_1(x) + a_2(x) = 0, -ca_1'(x) + ca_2'(x) = p(x)$$

$$\implies a_2'(x) = \frac{p(x)}{2c}, a_1'(x) = -\frac{p(x)}{2c},$$

so

$$a_1(x) = \int^x \frac{p(x')}{2c}dx' - m, a_2(x) = \int^x \frac{p(x')}{2c}dx' + m,$$

for m some constant. Then our final solution to the PDE is given by

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} p(x')dx'.$$

2 Problem 2

2.1 Part a

Let $U(x,s)$ be the Laplace transform of $u(x,t)$. We know that the Laplace transform of u_{tt} in time (t) will be given by $s^2U(x,s) - su_t(x,0) - u(x,0)$, and the Laplace transform of u_{xx} in time will be given by U_{xx}.

Then, applying the Laplace transform to both sides of the PDE, we get that

$$s^2U(x,s) - su_t(x,0) - u(x,0) = s^2U(x,s) = c^2U_{xx}(x,s).$$

This is a second-order, linear, homogeneous ODE in x. We try the solution $U(x,s) = e^{rx}$, yielding

$$s^2e^{rx} - c^2r^2e^{rx} = 0 \implies r = \pm\frac{s}{c},$$

and so the general equation to this ODE is given by

$$U(x,s) = \alpha(s)e^{-\frac{s}{c}x} + \beta(s)e^{\frac{s}{c}x}.$$

By the assumptions of the Laplace transform, we require that the solution be bounded as $x \to \infty$, so we must have that $\beta(s) = 0$. Then, from the initial conditions,

$$U(0,s) = \alpha(s),$$

where $U(0,s)$ is simply the Laplace transform of the boundary condition

$$u(0,t) = \begin{cases} A\sin(\omega t), & 0 < t \leq \pi/\omega \\ 0, & t > \pi/\omega \end{cases}.$$

Note that this can be written

$$u(0,t) = A\sin(\omega t)H(t) - A\sin(\omega t)H(t-\pi/\omega),$$

for H the Heaviside function, so using Laplace transform tables, we can compute

$$\begin{aligned} U(0,s) &= \mathcal{L}[A\sin(\omega t)(H(t) - H(t-\pi/\omega)] \\ &= A\frac{\omega}{s^2+\omega^2}\left[1 - e^{-\pi s/\omega}\frac{s\sin(\pi)+\omega\cos(\pi)}{s^2+\omega^2}\right] \\ &= \frac{A\omega}{s^2+\omega^2}\left[1+e^{-\pi s/\omega}\right], \end{aligned}$$

so our Laplace transform of the solution is given by

$$U(x,s) = \alpha(s)e^{-sx/c} = \frac{A\omega}{s^2+\omega^2}\left[1+e^{-\pi s/\omega}\right]e^{-sx/c},$$

as desired.

2.2 Part b

We have that

$$U(x,s) = \frac{A\omega}{s^2+\omega^2}(e^{-\frac{s}{c}x} + e^{-s(\pi/\omega + x/c)}),$$

so we compute the inverse Laplace transforms $\mathcal{L}^{-1}$ of both terms, yielding

$$\begin{aligned} \mathcal{L}^{-1}\left[U(x,s)\right] &= H\left(t-\frac{x}{c}\right)\sin\left(\omega\left(t-\frac{x}{c}\right)\right) \\ +H\left(t-\frac{\pi}{\omega}-\frac{x}{c}\right)\sin\left(\omega\left(t-\frac{\pi}{\omega}-\frac{x}{c}\right)\right) &= u(x,t). \end{aligned}$$

2.3 Part c

We wish to show that the solution above is of the form $a_1(x-ct)+a_2(x+ct)$. We can see that

$$-\frac{x-ct}{c} = t = \frac{x}{c},$$

so we can write

$$a_1(\gamma) = H(-\gamma/c)\sin(\omega(-\gamma/c)) + H(-\gamma/c-\pi/\omega)\sin(\omega(-\gamma/c-\pi/\omega)),$$

and $a_2 = 0$ to see that for $\gamma = x - ct$,

$$a_1(\gamma = x-ct)u(x,t),$$

so this is indeed a solution of the D'Alembert form.

3 Problem 3

We are given the boundary condition

$$\phi(r=1,\theta)=\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\theta)}{n^6}.$$

We can see that

$$\int_{\theta=0}^{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\theta)}{n^6}d\theta=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^6}\int_{0}^{2\pi}\cos(n\theta)d\theta=0.$$

Thus when we integrate $\phi(r=1,\theta)$ over θ from 0 to 2π, we see that this integral is 0. Since our Laplace PDE yields a function of the potential/heat/etc. over a disc in electrostatic/thermal/etc. equilibrium, it must be the case that the total potential over the border of any concentric circle centered at $r=0$ in our domain (the unit disc) will be equal to 0.

This corresponds to the fact that the energy/potential/heat/etc. flowing across any concentric circle must be equal to that flowing out when the system is in equilibrium as assumed by the Laplace equation, and we know the energy/potential/heat flowing out to the boundary of the disc totals to 0. This implies the singular point $r=0$ must have total potential 0, that is, $\phi(r=0,\theta)=0$.

4 Problem 4

4.1 Part a

As in problem 1, we compute the Fourier transforms with respect to x of both sides of the PDE, yielding

$$-k^2\mathcal{F}[\phi]+\frac{\partial^2}{\partial y^2}\mathcal{F}[\phi]=0$$

$$\implies \Phi_{yy}(k,y)-k^2\Phi(k,y)=0 \implies \Phi=\alpha(k)e^{-ky}+\beta(k)e^{ky},$$

for $\mathcal{F}[\cdot]$ the Fourier transform with respect to x and k the transformed x variable in the Fourier domain. To find α,β, we use the initial and boundary conditions:

$$\Phi(k,0)=\alpha(k)+\beta(k) \implies \beta(k)=-\alpha(k)$$

$$\implies \Phi(k,H)=\mathcal{F}[g(x)]=\alpha(k)[e^{-kH}-e^{kH}]$$

$$\implies \alpha(k)=\frac{\mathcal{F}[g(x)]}{e^{-kH}-e^{kH}},$$

and so our solution in its entirety is given by

$$\Phi(k,y)=\frac{\mathcal{F}[g]}{e^{-kH}-e^{kH}}[e^{-ky}-e^{ky}].$$

4.2 Part b

We know from the convolution theorem that for arbitrary functions $g, h, g * h = \mathcal{F}^{-1}[\mathcal{F}[f]\mathcal{F}[g]]$, for $g * h$ the convolution of g and h. Then, we have that

$$\Phi(k, y) = \mathcal{F}[g]\left(\frac{e^{-ky} - e^{ky}}{e^{-kH} - e^{kH}}\right)$$

$$\implies \phi(x, y) = \mathcal{F}^{-1}[\Phi(k, y)] = \mathcal{F}^{-1}\left[\mathcal{F}[g]\left(\frac{e^{-ky} - e^{ky}}{e^{-kH} - e^{kH}}\right)\right]$$

$$\implies \phi(x, y) = g(x) * \mathcal{F}^{-1}\left[\frac{e^{-ky} - e^{ky}}{e^{-kH} - e^{kH}}\right],$$

where $\mathcal{F}^{-1}[\cdot]$ is the integral representation of the inverse Fourier transform,

$$\mathcal{F}^{-1}\left[\frac{e^{-ky} - e^{ky}}{e^{-kH} - e^{kH}}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ky} - e^{ky}}{e^{-kH} - e^{kH}} e^{ikx} dk.$$

Then, letting

$$h(x, y) = \mathcal{F}^{-1}\left[\frac{e^{-ky} - e^{ky}}{e^{-kH} - e^{kH}}\right],$$

we can write

$$\phi(x, y) = \int_{-\infty}^{\infty} g(x')h(x - x', y)dx'.$$