ACM 95b, PS 1

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1 Problem 1

Consider the IVP
$$y'' + \pi^2 y = 0, 0 \leq x \leq 1, \tag{1}$$
with $y(0) = 1, y'(0) = 0$. This is a second-order, linear, homogeneous ODE with indicial equation
$$r^2 + \pi^2 = 0 \implies r_{1,2} = \pm i\pi.$$

Thus (1) has complex roots, so its general solution is given by
$$y(x) = c_1 \cos(\pi x) + c_2 \sin(\pi x). \tag{2}$$

Using our initial conditions, we see
$$y(0) = 1 = c_1,$$
$$y'(0) = 0 = \pi c_2 \implies c_2 = 0,$$
so our solution to the IVP is then
$$\boxed{y(x) = \cos(\pi x)}$$
over $0 \leq x \leq 1$. (Note that (1) is already in canoncial form, and since π^2 is a constant (and thus is entire), we need not restrict the domain of this solution.)

Now consider the BVP
$$y'' + \pi^2 y = 0, 0 \leq x \leq 1, \tag{3}$$
with $y(0) = 0, y(1) = 1$. We know the general solution to this ODE is given by (2). Applying the boundary conditions, we see
$$y(0) = 0 = c_1,$$
$$y(1) = 1 = -c_1.$$

Contradiction! This system is inconsistent WRT c_1, so indeed the BVP $\boxed{\text{has no solution}}$.

2 Problem 2

Consider the ODE

$$y'' + \lambda y = 0, 0 < x < 1, \tag{4}$$

with $y'(0) = 0, y(1) = 0$.

(a) We will first show that $\lambda = 0$ is not an eigenvalue of the BVP so we may restrict our attention to nonzero solutions. Observe that

$$\lambda = 0 \implies y'' = 0 \implies y(x) = c_1 x + c_2,$$

for c_1, c_2 some constants. Then, applying initial conditions, we get

$$y(1) = 0 = c_1 + c_2,$$

$$y'(0) = 0 = c_1.$$

The only solution to this system is $c_1 = c_2 = 0$, the trivial solution. Thus 0 is not an eigenvalue. We will now find the eigenvalues and functions of the BVP.

Note (4) is a second-order, linear, homogeneous ODE with indicial equation

$$r^2 + \lambda = 0 \implies r = \pm i\sqrt{\lambda},$$

so the general solution to the ODE is then

$$y(x) = c_1 \cos(\sqrt{\lambda}) + c_2 \sin(\sqrt{\lambda}x).$$

Differentiating and applying the first boundary condition, we have

$$y'(x) = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}x),,$$

$$y'(0) = 0 = c_2\sqrt{\lambda}.$$

Since we have established that $\lambda \neq 0$, we will assume $c_2 = 0$. Then, we have from the second initial condition that

$$y(1) = 0 = c_1 \cos(\sqrt{\lambda}).$$

The eigenvalues to this BVP are those values of λ that will set $\cos(\sqrt{\lambda})$ to 0, as we wish to avoid fixing $c_1 = 0$, since $c_1 = c_2 = 0$ would yield the trivial solution. Thus we have that our eigenvalues λ_n are given by

$$\sqrt{\lambda_n} = \frac{\pi}{2} + \pi n \implies \boxed{\lambda_n = \left(\frac{\pi}{2} + n\pi\right)^2}, n \in \mathbb{N},$$

and the associated eigenfunctions to those eigenvalues are

$$\boxed{y_n(x) = \cos\left(\left(\frac{\pi}{2} + n\pi\right)^2 x\right)}.$$

(b) Observe that the eigenvalues of the problem are given by

$$\lambda_n = \left(\frac{\pi}{2} + n\pi\right)^2, n \in \mathbb{N},$$

and the eigenfunctions corresponding to those eigenvalues are

$$y_n(x) = \cos\left(\left(\frac{\pi}{2} + n\pi\right)^2 x\right).$$

Then, since $n \in \mathbb{N} \subset \mathbb{R}$, and $2, \pi \in \mathbb{R}$, and we know that the product, fraction, and sum of real numbers are real, we have that $\lambda_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Similarly, since the cosine of a real function is in fact also real, and $x \in \mathbb{R}$, we have that $y_n(x)$ is also real for all n. Thus in the complex λ-plane, these solutions lie solely on the real axis.

3 Problem 3

Consider the second-order ODE

$$P(x)y'' + Q(x)y' + \lambda R(x)y = 0, a \leq x \leq b. \tag{5}$$

(a) Since $P(x)$ does not vanish over $a \leq x \leq b$, we may rewrite this equation in canonical form as

$$y'' + \frac{Q(x)}{P(x)}y' + \lambda\frac{R(x)}{P(x)}y = 0.$$

Now, the desired $\mu(x)$ must have that

$$\mu(x)y'' + \mu(x)\frac{Q(x)}{P(x)}y' + \mu(x)\lambda\frac{R(x)}{P(x)}y = (p(x)y'(x))' + q(x)y + \lambda r(x)y$$
$$= p(x)y'' + p'(x)y + (\lambda r(x) + q(x))y,$$

for some p, q, r. Equating the coefficients on the y'' terms, we get easily that $\boxed{\mu(x) = p(x)}$. Furthermore, equating coefficients on the y' terms, we have that

$$\mu(x)\frac{Q(x)}{P(x)} = p'(x) = \mu'(x).$$

This if a first-order, linear, homogeneous ODE with solution

$$\boxed{\mu(x) = \exp\left(\int \frac{Q}{P}dx\right)},$$

so the required integrating factor in its entirety (including the initial conversion to canonical form) is

$$\boxed{\frac{\mu(x)}{P(x)}}.$$

Finally, equating the coefficients on the y terms, we see that

$$\mu(x)\lambda\frac{R(x)}{P(x)} = \lambda r(x) + q(x).$$

We set $\boxed{q(x) = 0}$ so that

$$\boxed{r(x) = \mu(x)\frac{R(x)}{P(x)}}$$

for $\mu(x)$ as defined above.

(b) Observe that, letting $x = g(\zeta)$,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial x}, \frac{\partial y}{\partial x} = \frac{\partial y}{\partial \zeta}\frac{\partial \zeta}{\partial x}.$$

Then

$$\frac{\partial}{\partial x}\left[p(x)\frac{\partial y}{\partial x}\right] + q(x)y + \lambda r(x)y = 0$$

$$= \frac{\partial}{\partial \zeta}\frac{1}{g'(\zeta)}\left[p(g(\zeta))\frac{\partial y}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right] + q(g(\zeta))y(g(\zeta)) + \lambda r(g(\zeta))y(g(\zeta)) = 0.$$

Multiplying through by $g'(\zeta)$,

$$\frac{\partial}{\partial \zeta}\left[p(g(\zeta))\frac{\partial y}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right] + q(g(\zeta))y(g(\zeta))g'(\zeta) + \lambda r(g(\zeta))y(g(\zeta))g'(\zeta) = 0.$$

We require that

$$g'(\zeta)r(g(\zeta)) = 1 \implies g'(\zeta) = \frac{1}{r(g(\zeta))} \implies \boxed{g(\zeta) = \int \frac{1}{r(g(\zeta))}d\zeta},$$

so

$$\frac{\partial}{\partial \zeta}\left[p(g(\zeta))\frac{\partial y}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right] + q(g(\zeta))y(g(\zeta))g'(\zeta) + \lambda y(g(\zeta)) = 0.$$

Then

$$\boxed{B(\zeta) = \frac{p(g(\zeta))}{g'(\zeta)}, C(\zeta) = g'(\zeta)q(g(\zeta))}.$$

4 Problem 4

Consider the operators

$$L[y] = y'' + p(z)y' + q(z)y, \tag{6}$$

$$M[u] = u'' - (p(z)u)' + q(z)u. \tag{7}$$

(a) We must show that

$$uL[y] - yM[u] = \frac{\partial}{\partial z}(F(u,y)). \tag{8}$$

Observe that the LHS of this expression is

$$\begin{aligned} &uy'' + upy' + quy - yu'' + y(pu)' - yqu \\ &= uy'' + up(z)y' + y(-u'' + p(z)u' + up'(z)). \end{aligned} \tag{9}$$

We will now compute the integral of (8) with respect to z using integration by parts. We get

$$\begin{aligned} F(u,y) &= \int (uy'' + up(z)y' + y(-u'' + p(z)u' + up'(z)))dz \\ &= uy' - \int (y'u'dz) + up(z)y - \int [yu'p + p'uy]dz - yu' \\ &+ \int u'y'dz + ypu - \int u[yp' + py']dz + ypu - \int p(z)[y'u + u'y]dz. \end{aligned}$$

This last expression simplifies to

$$\begin{aligned} &uy' - yu' + 3ypu - 2\int (yu'p + yup' + y'up)dz \\ &= uy' - yu' + 3ypu - 2ypu = \boxed{uy' - yu' + ypu} = F(u,y). \end{aligned}$$

(b) Suppose we have $u \neq 0$ such that

$$M[u] = u'' - (up)' + uq = 0.$$

Then

$$u'' + uq = u'p + up' \implies u' + \underbrace{\int u(z)q(z)dz}_{\text{Denote this } f(z)} = u(z)p(z).$$

Since $uL[y] = uy'' + upy' + uqy$,

$$uL[y] = uy'' + [u' + f(z)]y' + uqy. \tag{10}$$

Recall that we seek solutions to $uL[y] = 0$. Observe that (10) can be written

$$(uy')' + (fy)',$$

so we seek solutions to

$$(uy')' + (fy)' = 0.$$

Integrating, this is simply

$$uy' + fy = k, k \in \mathbb{R},$$

a first-order, linear, homogeneous ODE. By the existence & uniqueness theorem for first-order, linear ODEs, we know that this ODE will have a unique solution wherever $u(z) \neq 0$. Furthermore, from Abel's Theorem, we know that the Wronskian $W(y_1, y_2)$ of any two solutions y_1, y_2 to $uL[y] = 0$ must satisfy

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = Ce^{-\int u(z)p(z)dz},$$

yielding a second, linearly independent solution y_2 given the y_1 found above.

(c) Let

$$L[y] = y'' + [\lambda - q(z)]y, \tag{11}$$

so

$$L[u] = u'' + [\lambda - q(z)]u.$$

Then, we have that

$$uL[y] - yL[u] = uy'' + u[\lambda - q]y - yu'' - y[\lambda - q]u$$

$$= uy'' - yu'' = \frac{\partial}{\partial z}(F(u, y)),$$

so

$$F(u, y) = \int uy''dz - \int yu''dz$$

$$= uy' - \int y'u'dz - yu' + \int u'y'dz = \boxed{uy' - yu'},$$

and thus by the definition of adjoint operators given, we have that $L[y]$ is indeed self-adjoint.

(d) Let

$$L[y] = y''' + p_1(z)y'' + p_2(z)y' + p_3(z)y,$$
$$M[u] = -u''' + (p_1(z)u)' - (p_2(z)u)' + p_3(z)u.$$

We will show that these two operators are adjoint in the same we approached parts (a) and (c).

We wish to find $F(u, y)$ such that

$$uL[y] - yM[u] = \frac{\partial}{\partial z}F(u, y).$$

Observe that

$$uL[y] - yM[u] = uy''' + yu''' + up_1y'' - yp_1u'' - yup_1'' + up_2y' + yp_2u' + yup_2' - 2yu'p_1'.$$

Then we must compute

$$F(u, y) = \int (uL[y] - yM[u])dz$$

$$= uy'' - u'y' + \int y'u''dz + yu'' - \int u''y'dz + up_1y'$$

$$-\int (y'up_1'dz - y'p_1u')dz - yp_1u' + \int (u'yp_1' + u'p_1y')dz - yup_1'$$

$$+\int (p_1'yu' + p_1'uy')dz - \int 2yu'p_1'dz$$

$$+3up_2y - 2\int (y'up_2 + yu'p_2 + yup_2')dz.$$

Expanding and simplifying, we get that this is

$$\boxed{uy'' + yu'' - u'y' + up_1y' - yp_1u' - yup_1' + yup_2 = F(u,y)},$$

so indeed, $L[y]$ and $M[u]$ are adjoint operators.