ACM 95b, Final

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1 Problem 1

1.1 Part a

At equilibrium (as $t \to \infty$), we require that $u(x,t)$ is not varying in time, and so $\phi_t = 0$, for $\phi(x)$ the equilibrium solution. Thus our PDE reduces to

$$a^2\phi_{xx} + x^3 = 0, 0 < x < L.$$

Integrating, we see

$$\phi = \frac{1}{a^2}\left(-\frac{x^5}{20} + c_1 x + c_2\right),$$

and using our boundary conditions,

$$\phi(0) = c_2 = 0, \phi(L) = -\frac{L^5}{20} + c_1 L = 0 \implies c_2 = 0, c_1 = \frac{L^4}{20a},$$

so our equilibrium solution is

$$\phi(x) = \frac{1}{a^2}\left(-\frac{x^5}{20} + \frac{L^4}{20}x\right).$$

1.2 Part b

From lecture, we know that for the heat equation with a source, we can find a solution of the form

$$u(x,t) = \phi(x) + \psi(x,t)$$

by using

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2a^2t}{L^2}\right)\sin(n\pi x/L),$$

$$c_n = \frac{2}{L}\int_0^L \sin(n\pi x/L)dx = \frac{2(1-\cos(\pi n)))}{\pi n},$$

and for ϕ the equilibrium solution from part (a).

1.3 Part c

We again require that the equilibrium solution $\phi(x)$ has that $\phi_t = 0$. Then, from (a), we have that

$$\phi(x) = \frac{1}{a^2}\left(-\frac{x^5}{20} + c_1 x + c_2\right),$$
$$\phi'(x) = \frac{1}{a^2}\left(-\frac{x^4}{4} + c_1\right),$$

so applying our new boundary conditions,

$$\phi'(0) = 0 \implies c_1 = 0, \phi'(L) = -\frac{L^4}{4a^2} = \beta.$$

2 Problem 2

2.1 Part a

When $R = G = 0$, we have

$$V_x = -LI_t, I_x = -CV_t,$$

so $V_{xx} = -LI_{tx} = -LI_{xt}$, and substituting in I_x, this becomes

$$V_{xx} = -LI_{xt} = LCV_{tt},$$

which is indeed the 1-D wave equation in $V(x,t)$. We can try to find a solution of the form $V(x+ut)$, for u the speed of the waves. This yields

$$V_{xx} = \frac{1}{u^2}V_{tt} \implies \frac{1}{u^2} = LC,$$

and thus the speed is given by

$$u = \frac{1}{\sqrt{LC}}.$$

2.2 Part b

We can see that, differentiating the second equation wrt t,

$$I_{xt} = I_{tx} = -CV_{tt} - GV_t,$$

and the first wrt x,

$$V_{xx} = -LI_{tx} - RI_x,$$

so substituting this equation into our first, we get that

$$V_{xx} - RCV_t - RGV = LCV_{tt} + LGV_t$$
$$\implies V_{tt} + \left(\frac{R}{L} + \frac{G}{C}\right)V_t + \frac{RG}{CL}V = \frac{1}{CL}V_{xx},$$

so the relationship holds for $\alpha = \frac{R}{L}, \beta = \frac{G}{C}$.

2.3 Part c

Plugging back into the PDE, we get that

$$\sum_{n=1}^{\infty} A_n''(t)\sin(\lambda_n x) + (\alpha+\beta)\sum_{n=1}^{\infty} A_n'(t)\sin(\lambda_n x) + \alpha\beta\sum_{n=1}^{\infty} A_n(t)\sin(\lambda_n x)$$

$$= -c^2\sum_{n=1}^{\infty}\lambda_n^2 A_n(t)\sin(\lambda_n x),$$

so

$$A_n''(t) + (\alpha+\beta)A_n'(t) + \alpha\beta A_n(t) + c^2\lambda_n^2 A_n(t) = 0.$$

We know that $V(0,t) = V(L,t) = 0$, and so

$$0 = \sum_{n=1}^{\infty} A_n(t)\sin(\lambda_n L) \implies \lambda_n = \frac{n\pi}{L}.$$

Now, we can see that the ODE

$$A_n''(t) + (\alpha+\beta)A_n'(t) + \alpha\beta A_n(t) + c^2\lambda_n^2 A_n(t) = 0$$

has characteristic polynomial

$$r^2 + (\alpha+\beta)r + \alpha\beta + c^2\lambda_n^2 = 0$$

$$\implies r = \frac{-(\alpha+\beta) \pm \sqrt{(\alpha+\beta)^2 - 4(\alpha\beta + c^2\lambda_n^2)}}{2}$$

$$= -\frac{\alpha+\beta}{2} \pm \frac{\sqrt{(\alpha-\beta)^2 - 4c^2\lambda_n^2}}{2} = -\frac{\alpha+\beta}{2} \pm i\sqrt{-\frac{(\alpha-\beta)^2}{4} + \frac{c^2\pi^2 n^2}{L^2}}.$$

Then our solution is given by

$$A_n(t) = e^{-dt}(E_n\cos(\omega_n t) + F_n\sin(\omega_n t)),$$

for some coefficients E_n, F_n, and for d, ω_n as defined.

2.4 Part d

We have that $V(x, t=0) = \delta(x-a)$, so

$$V(x,0) = \sum_{n=1}^{\infty} F_n\sin(\lambda_n x) = \delta(x-a)$$

$$\implies F_n = \frac{2}{L}\int_0^L \delta(x-a)\sin(n\pi x/L)dx = \frac{2}{L}\sin(n\pi a/L).$$

Furthermore, we have that

$$V_t(x,0) = \sum_{n=1}^{\infty}[-dF_n + E_n\omega_n]\sin(\lambda_n x) = 0$$

$$\implies -\frac{2d}{L}\sin(n\pi a/L) + E_n\omega_n = 0 \implies E_n = \frac{2d}{L\omega_n}\sin(n\pi a/L).$$

2.5 Part e

Let $\alpha = \beta$. Then $\omega_n = \frac{n\pi c}{L}, d = \alpha$. We compute

$$\sin(n\pi x/L)\cos(n\pi ct/L) = \frac{1}{2}\left(\sin\left(\frac{n\pi}{L}(x-ct)\right) + \sin\left(\frac{n\pi}{L}(x+ct)\right)\right),$$

and

$$\sin(n\pi x/L)\sin(n\pi ct/L) = \frac{1}{2}\left(\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right).$$

Thus we get that, after setting $\alpha = \beta$,

$$V(x,t) = e^{-\alpha t}\sum_{n=1}^{\infty}\frac{E_n}{2}\sin\left(\frac{n\pi}{L}(x+ct)\right) - \frac{F_n}{2}\cos\left(\frac{n\pi}{L}(x+ct)\right)$$

$$+e^{-\alpha t}\sum_{n=1}^{\infty}\frac{E_n}{2}\sin\left(\frac{n\pi}{L}(x-ct)\right) + \frac{F_n}{2}\cos\left(\frac{n\pi}{L}(x-ct)\right),$$

which is indeed of the form $V(x,t) = e^{-dt}[f(x+ct) + g(x-ct)]$.

3 Problem 3

3.1 Part a

Plugging the series solution into the Laplace equation, we get that

$$\sum_{n=1}^{\infty}\sin(\beta_n\theta)\left[A_n'' + \frac{A_n'(r)}{r} - \frac{A_n(r)\beta_n^2}{r^2}\right] = 0$$

$$\implies \left[A_n'' + \frac{A_n'(r)}{r} - \frac{A_n(r)\beta_n^2}{r^2}\right] = 0.$$

Solving this ODE, we get that

$$A_n(r) = r^{\beta_n}, \text{ or } r^{-\beta_n}.$$

To then find β_n, we note that

$$u_1(r, \theta = \alpha) = 0,$$

so

$$0 = \sum_{n=1}^{\infty}A_n(r)\sin(\beta_n\alpha) \implies \sin(\beta_n\alpha) = 0 \implies \beta_n = \frac{\pi n}{\alpha}.$$

3.2 Part b

Let $x = -\log(r)$. Then

$$u_r = u_x x_r = -\frac{1}{r}u_x, u_{rr} = u_{xx}(x_r)^2 + u_x x_{rr} = \frac{u_{xx}}{r^2} + \frac{u_x}{r^2},$$

so our PDE is then, for $r > 0$, simply

$$u_{xx} + u_{\theta\theta} = 0.$$

Letting $F[\cdot]$ be the sine Fourier transform with respect to x, we can compute

$$F[u_{xx}] = \sqrt{\frac{2}{\pi}}\int_0^\infty u_{xx}\sin(kx)dx$$

$$= \sqrt{\frac{2}{\pi}}\left[-ku\cos(kx)\Big|_0^\infty\right] - k^2F[u] = \sqrt{\frac{2}{\pi}}ku(x=0,\theta) - k^2F[u].$$

Note that $x = 0 \iff r = 1$, so using our boundary condition, our PDE is

$$\sqrt{\frac{2}{\pi}}kg(\theta) - k^2F[u] + F[u]_{\theta\theta} = 0.$$

This is a second-order linear ODE in θ. Using variation of constants, we can find that its solution is given by

$$F[u] = e^{k\theta}\left(c_1 - \frac{\sqrt{2}}{2\sqrt{\pi}}\int_1^\theta e^{-\gamma k}g(\gamma)d\gamma\right) + e^{-k\theta}\left(c_2 + \frac{\sqrt{2}}{2\sqrt{\pi}}\int_1^\theta e^{\gamma k}g(\gamma)d\gamma\right).$$

Finding the solution $u_2(r,\theta)$, then, requires only inverting $F[u]$ with respect to k, and then setting $x = -\log(r)$ in the resulting expression. Thus

$$u_2(r,\theta) = \sqrt{\frac{2}{\pi}}\int_0^\infty F[u]\sin(-k\log(r))dk,$$

for $F[u]$ the Fourier sine transform of u_2 in x given above.