ACM 95b, PS 2

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1 Problem 1

Consider

$$f(x) = 1 - x^2/L^2, 0 < x < L. \tag{1}$$

(a) We first find the sine series coefficients of (1) over $0 < x < L$. To do so, we will set $f(x) = -f(-x)$ for all $-L < x < 0$. From lecture, then, we know

$$a_n = \frac{2}{L}\int_0^L \left[\frac{f(x) - f(-x)}{2}\right] \sin(n\pi x/L) dx = \frac{2}{L}\int_0^L f(x) \sin(n\pi x/L) dx$$

$$= \boxed{\frac{2(\pi^2 n^2 - 2\cos(\pi n) + 2)}{\pi^3 n^3}}.$$

Thus the sine series expansion for f over $0 < x < L$ is given by

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right).$$

(b) We will now find the cosine series coefficients for (1). We first find b_0. From lecture, we know that this is given by

$$b_0 = \frac{1}{L}\int_0^L \left[\frac{f(x) + f(-x)}{2}\right] dx = \frac{1}{L}\int_0^L f(x) dx = \frac{2}{3}.$$

Now we seek $b_n, n > 0$. Using the results from lecture, we compute

$$b_n = \frac{2}{L}\int_0^L \left[\frac{f(x) + f(-x)}{2}\right] \cos(n\pi x/L) dx = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^L \left(\cos\left(\frac{n\pi x}{L}\right) - \frac{x^2}{L^2}\cos\left(\frac{n\pi x}{L}\right)\right) dx$$

$$= \frac{4(\sin(\pi n) - \pi n \cos(\pi n))}{\pi^3 n^3} = b_n.$$

Thus we have that the coefficients of the cosine series expansion of f over $0 < x < L$ are given by

$$\boxed{b_0 = \frac{2}{3}}, \boxed{b_n = \frac{-4\cos(\pi n)}{\pi^2 n^2}},$$

and so the cosine series expansion on $0 < x < L$ is given by

$$\sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right).$$

(c) We know from (a) and (b) that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$$

over $0 < x < L$. Thus, we may write the fully periodic Fourier series for $f(x)$ over $0 < x < L$ as

$$f(x) = \frac{1}{2}(f(x) + f(x)) = \boxed{\frac{1}{2}\left(\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)\right)},$$

maintaining the expressions for a_n and b_n given above.

(d) At $x = 0$, we can see from part (a) that $\boxed{\text{the sine series will vanish}}$ due to the sine terms. This makes sense, since the sine series approximates (and is equal to in limit) f on $0 < x < L$, but $-f(-x)$ from $-L < x < 0$. Since $f(x) \to 1$ as $x \downarrow 0$, but $f(x) \to -1$ as $x \uparrow 0$, there exists a jump discontinuity in f at $x = 0$. As such, we would expect there to be a realization of a Gibbs phenomenon at $x = 0$, resulting in the sine series going to 0 at $x = 0$.

We now compute the value of the cosine series at $x = 0$. From (b), we know it will be given by

$$\begin{aligned}\frac{2}{3} + \sum_{n=1}^{\infty} b_n &= \frac{2}{3} + \frac{4}{\pi^3 n^3}\sum_{n=1}^{\infty} \sin(n\pi) - \frac{4}{\pi^2 n^2}\sum_{n=1}^{\infty} \cos(\pi n) \\ &= \frac{2}{3} + \frac{1}{3} = 1,\end{aligned}$$

so $\boxed{\text{the cosine series is equal to 1}}$ at $x = 0$. This is also expected, since cosine series approximates $f(x)$ on all of $-L < x < L$, and $f(x) \to 1$ as $x \to 0$, so there does not exist a discontinuity at $x = 0$ as the cosine series is in fact continuous.

Finally, $\boxed{\text{the fully periodic Fourier series is equal to 1/2}}$ at $x = 0$. This is also expected, since we found in (c) that the fully periodic Fourier series for f over $0 < x < L$ is simply the average of the sine and cosine series. Thus we would expect that, at $x = 0$, the fully periodic series would evaluate to the average of the sine series at $x = 0$ and the cosine series at $x = 0$, yielding $1/2$.

(e) At $x = L$, we can see from (a) that $\boxed{\text{the sine series will vanish}}$ at $x = L$, since

$$\sin\left(\frac{n\pi L}{L}\right) = \sin(n\pi) = 0.$$

Note that $f(x) \to 0$ as $x \uparrow L$ and we have fixed the period of the sine functions in our series to be L, so we would only expect a Gibbs phenomenon if the odd extension of f had that f approached different values as $x \uparrow L$ and $x \downarrow -L$. This is not the case, as both approach 0, and hence we expect the sine series to align with these results.

Similarly, $\boxed{\text{the cosine series will vanish}}$ at $x = L$, since the series will be

$$\frac{2}{3} + \sum_{n=1}^{\infty} \frac{-4\cos^2(\pi n)}{\pi^2 n^2} = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3} - \frac{2}{3} = 0.$$

This result makes sense for the same reason the sine series' result does: The even extension of f has that f will approach 0 both as $x \uparrow L$ and $x \downarrow -L$. Since the period of the cosine functions in our series is L, we expect the series to evaluate to 0 as well.

Finally, $\boxed{\text{the fully periodic series will vanish}}$ at $x = L$. This makes sense because it is merely the average of the sine series and the cosine series at $x = L$.

2 Problem 2

Consider the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1/2, \\ 2 - 2x, & 1/2 < x < 1. \end{cases} \tag{2}$$

The interval on which f is defined is $L = 1$. Using the odd extension of (2), we have from lecture that

$$a_n = \frac{2}{L}\int_0^1 \left[\frac{f(x) - f(-x)}{2}\right] \sin(n\pi x)dx$$

$$= 2\int_0^{1/2} 2x\sin(n\pi x)dx + 2\int_{1/2}^1 (2-2x)\sin(n\pi x)dx$$

$$= 2\left[\frac{2\sin(n\pi/2) - n\pi\cos(n\pi/2)}{\pi^2 n^2} + \frac{2\sin(n\pi/2) - 2\sin(n\pi) + n\pi\cos(n\pi/2)}{\pi^2 n^2}\right]$$

$$= 2\left[\frac{4\sin(n\pi/2)}{\pi^2 n^2}\right] = \frac{8\sin(n\pi/2)}{\pi^2 n^2}.$$

Thus the sine series for (2) over $0 < x < 1$ is given by

$$f(x) = \boxed{\sum_{n=1}^{\infty} \frac{8\sin(n\pi/2)}{\pi^2 n^2}\sin(n\pi x)}.$$

(a) Our Fourier coefficients from above are given by

$$a_n = \frac{8\sin(n\pi/2)}{\pi^2 n^2}.$$

Thus since $0 \leq \sin(n\pi/2) \leq 1$, and $8/\pi^2$ is a constant, the coefficients will decay at a rate of $\frac{1}{n^2}$ with increasing n.

From lecture, we know that if f has $k-1$ continuous derivatives, then the coefficients will decay at least as fast as n^{-k}. Thus since the odd extension of our f is not differentiable (its slope changes instantly at $x = \frac{1}{2}$), we can only state that the coefficients decay at least as fast as $\frac{1}{n}$, which is indeed the case.

3 Problem 3

Consider the ODE

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) - qy + \lambda r y = 0, 0 \leq x \leq 1 \tag{3}$$

with boundary conditions $y(0) = 0, y(1) = 0$.

(a) Expanding (3) and substituting $y = \alpha(x)u(x)$, we get

$$\begin{aligned}
&(p')(\alpha' u + \alpha u') + (p)(\alpha'' u + 2\alpha' u' + \alpha u'') - q\alpha u + \lambda r \alpha u = 0 \\
&= (u'')(\alpha p) + (u')(p'\alpha + 2p\alpha') + (u)(p'\alpha' + p\alpha'' - q\alpha + \lambda r \alpha).
\end{aligned}$$

Thus we require that

$$p'\alpha + 2p\alpha' = 0 \implies \frac{p'}{p} = -2\frac{\alpha'}{\alpha}$$

$$\implies -2\int \frac{\alpha'}{\alpha}dx = -2\ln(\alpha) + c = \int_0^x \frac{p'}{p}dx = \ln(p)$$

$$\implies \alpha^2 = \frac{c}{p} \implies \alpha = \frac{c}{\sqrt{p}}, c \in R.$$

Now setting $c = 1 \implies \alpha = \frac{1}{\sqrt{p}}$, we have

$$\alpha' = -\frac{p'}{2p^{3/2}}, \alpha'' = -\frac{p''}{2p^{3/2}} + \frac{3(p')^2}{4p^{5/2}},$$

so our expansion can be written

$$(u'')(p\alpha) + (u')(p'\alpha + 2p\alpha') + (u)(p\alpha'' + p'\alpha' - q\alpha + \lambda r\alpha) = 0$$

$$\implies (u'')\sqrt{p} + (u)\left(-\frac{p''}{2p^{1/2}} + \frac{3(p')^2}{4p^{3/2}} - \frac{(p')^2}{2p^{3/2}} - \frac{q}{\sqrt{p}} + \frac{\lambda r}{\sqrt{p}}\right) = 0$$

$$\implies u'' + (u)\left(-\frac{p''}{2p} + \frac{(p')^2}{4p^2} - \frac{q}{p} + \frac{\lambda r}{p}\right) = 0.$$

Thus indeed we have that

$$\boxed{\alpha(x) = \frac{1}{\sqrt{p(x)}}, A(x) = -\frac{p''}{2p} + \frac{(p')^2}{4p^2} - \frac{q}{p}, B(x) = \frac{r}{p}}$$

satisfy the problem constraints.

(b) Setting $u(x) = e^{S(x)}$, we get

$$u'(x) = S'(x)e^{S(x)}, u''(x) = S''(x)e^{S(x)} + (S'(x))^2 e^{S(x)}.$$

Thus we have that

$$\frac{u''}{\lambda} + \frac{A}{\lambda}u + Bu = 0$$
$$= \frac{S''e^S + (S')^2 e^S}{\lambda} + \frac{Ae^S}{\lambda} + Be^S = 0.$$

Dividing through by e^S, this is

$$= \left[\frac{S'' + (S')^2}{\lambda}\right] + \left[\frac{A}{\lambda} + B\right] = 0,$$

as desired.

(c) Assume that S''/λ matches the size of B for fixed x, and as $\lambda \to \infty$, all other terms are dominated by these. Then for large $|\lambda|$,

$$\frac{S''}{\lambda} = -B(x) \implies S'' = -\lambda B(x) \implies S' = -\lambda \int B(x)dx$$

$$\implies (S')^2 = \lambda^2 \left(\int B(x)\right)^2.$$

Now, it should be the case that, as $|\lambda| \to \infty$,

$$\frac{(S')^2}{\lambda} \leq c \cdot \frac{S''}{\lambda}$$

for some constant $c > 0$. However, observe that this would imply

$$\frac{-\lambda}{\lambda} B(x)c \geq \frac{\lambda^2}{\lambda} \left(\int B(x)\right)^2$$

$$\iff -B(x)c \geq \lambda \left(\int B(x)\right)^2.$$

Since $r(x) > 0, p(x) > 0 \implies B(x) > 0$, and we have fixed $c > 0$, it follows that this relationship clearly does not hold as $\lambda \to +\infty$. Thus it must be the case that $(S')^2$ will in fact dominate the S'' as $|\lambda| \to \infty$, and so we may set

$$\frac{(S')^2}{\lambda} = -B(x).$$

(And, of course, it is clear that $A(x) \to 0$ as $|\lambda| \to \infty$, since A has no dependence on λ. Thus $A(x)$ will be dominated by $(S')^2$ as well as $|\lambda| \to \infty$.)

(d) Suppose $S(0) = 0$ (we will check the validity of this assumption shortly). We then have that

$$S'(x) = \pm i\sqrt{|\lambda|}\sqrt{\frac{r(x)}{p(x)}}$$

$$\implies \int_0^x dS(x') = S(x) - S(0) = S(x) = \pm i\sqrt{|\lambda|}\int_0^x \sqrt{\frac{r(x')}{p(x')}}dx'.$$

Now see that indeed $S(0) = 0$, since the integral of any function from 0 to 0 is 0, so this result is sound. Then recall from (a) that $y(x) = \alpha(x)u(x)$, and

$$\alpha(x) = \frac{1}{\sqrt{p(x)}}.$$

Then, in (b), we used the substitution

$$u(x) = e^{S(x)},$$

where we have just found that

$$S(x) = \pm i\sqrt{|\lambda|}\int_0^x \sqrt{\frac{r(x')}{p(x')}}dx'.$$

These represent two distinct roots to the homogeneous ODE in u from (a), so we can formulate a general solution to the ODE as any linear combination of these values of u, yielding

$$u(x) = c_1 \exp\left(i\sqrt{|\lambda|}\int_0^x \sqrt{\frac{r(x')}{p(x')}}dx'\right) + c_2 \exp\left(-i\sqrt{|\lambda|}\int_0^x \sqrt{\frac{r(x')}{p(x')}}dx'\right)$$

$$\implies y(x) = \alpha(x)u(x) =$$

$$= \frac{c_1}{\sqrt{p(x)}} \exp\left(i\sqrt{|\lambda|}\int_0^x \sqrt{\frac{r(x')}{p(x')}}dx'\right) + \frac{c_2}{\sqrt{p(x)}} \exp\left(-i\sqrt{|\lambda|}\int_0^x \sqrt{\frac{r(x')}{p(x')}}dx'\right),$$

as desired.

(e) Applying the boundary conditions $y(0) = 0, y(1) = 0$, we get

$$y(0) = 0 = \frac{c_1}{\sqrt{p(0)}}\exp(0) + \frac{c_2}{\sqrt{p(0)}}\exp(0) \implies c_1 = -c_2,$$

$$y(1) = 0 = \frac{c_1}{\sqrt{p(1)}} \exp\left(i\sqrt{|\lambda|}\int_0^1 \sqrt{\frac{r(x')}{p(x')}}dx'\right)$$

$$+\frac{c_2}{\sqrt{p(1)}} \exp\left(-i\sqrt{|\lambda|}\int_0^1 \sqrt{\frac{r(x')}{p(x')}}dx'\right).$$

From complex analysis, we know the RHS of this equation is

$$0 = -\frac{2c_2}{\sqrt{p(1)}}\sin\left(\sqrt{|\lambda|}\int_0^1\sqrt{\frac{r(x')}{p(x')}}dx'\right)$$

$$\implies \sqrt{|\lambda|}\int_0^1\sqrt{\frac{r(x')}{p(x')}}dx' = \pi n, n = 1, 2, 3, ...$$

$$\implies \boxed{\lambda_n = \frac{\pi^2 n^2}{\left(\int_0^1\sqrt{\frac{r(x')}{p(x')}}dx'\right)^2}}.$$

We may now plug these values of λ_n back into our solution for y from (d) to get

$$y(x) = -\frac{2c_2}{\sqrt{p(x)}}\sin\left(\sqrt{|\lambda_n|}\int_0^x\sqrt{\frac{r(x')}{p(x')}}dx'\right)$$

$$= \boxed{\frac{C}{\sqrt{p(x)}}\sin\left(\pi n\frac{\int_0^x\sqrt{\frac{r(x')}{p(x')}}dx'}{\int_0^1\sqrt{\frac{r(x')}{p(x')}}dx'}\right)}.$$

(f) Using the result from (e), we have that the WKB approximation for large λ of the solution to the equation from class is given by

$$y(x) = \frac{C}{\sqrt{x}}\sin\left(n\pi\int_0^x 1dx\right).$$

(In this example, $p(x) = x, r(x) = x, q(x) = 0$.) We have that

$$\boxed{y(x) = \frac{C}{\sqrt{x}}\sin(n\pi x)}.$$

Compared to the plots shown in lecture for the eigenfunctions of this ODE, the approximation (for large n, which imply large λ) is also sinusoidal (this is by construction, of course). Furthermore, the approximation decays in amplitude as $x \to \infty$, since $y(x) \to 0$.