ACM 95b, PS 3

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April 2023

1 Problem 1

Consider the BVP

$$y'' + \lambda y = x, 0 \leq x \leq 1, \tag{1}$$

with BCs $y'(0) = y(1) = 0$.

(a) We may use any set of eigenfunctions from Sturm-Liouville ODEs to obtain a set of eigenfunctions for this BVP. To make matters simple, we opt to take those from the homogeneous ODE

$$y'' + \lambda y = 0,$$

with boundary conditions $y'(0) = y(1) = 0$. We know that the general solution to the ODE is given by

$$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x),$$

$$y'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x).$$

Applying our boundary conditions, we have

$$y'(0) = c_2\sqrt{\lambda} \implies c_2 = 0,$$

$$\implies y(1) = 0 = c_1\cos(\sqrt{\lambda}) \implies \lambda_n = \left(\frac{\pi}{2} + \pi n\right)^2, n \in \mathbb{N}$$

are our eigenvalues to the BVP, with associated eigenfunctions $\phi_n(x) = \cos(\sqrt{\lambda_n}x)$.

Thus (1) has eigenfunctions given by ϕ_n with associated eigenvalues $\lambda_n, n \in \mathbb{N}$. We now assume that the solution to the BVP takes the form

$$y(x) = \sum_{n=1}^{\infty} A_n\phi_n,$$

where the A_n's are some constants. Then, differentiating (we can do this due to the homogeneous BCs) and substituting into (1),

$$\sum_{n=1}^{\infty} (\lambda - \lambda_n)A_n\phi_n(x) = x.$$

From orthogonality, we then have that

$$A_n(\lambda - \lambda_n) = f_n,$$

for f_n given by

$$f_n = \frac{\int_0^1 x\phi_n(x)dx}{\int_0^1 \phi_n^2(x)dx} = \frac{2(2\pi\cos(\pi n)(2n+1) - 4)}{(2\pi n + \pi)^2}.$$

Thus our final solution to the BVP is given by

$$y(x) = \sum_{n=1}^{\infty} \frac{f_n}{\lambda - \lambda_n}\phi_n(x),$$

for f_n, λ_n, ϕ_n defined above.

(b) No, there does not exist a solution for which $\lambda = \lambda_n$, as can be seen by the fact that $y(x)$ relies on a division by $\lambda - \lambda_n$ in its summation.

(c) We may again use the eigenfunctions found for the corresponding homogeneous ODE as in part (a), since our boundary conditions have remained the same. Thus we have

$$\lambda_n = \left(\frac{\pi}{2} + n\pi\right)^2,$$

and we assume the solution to the ODE takes the form

$$y(x) = \sum_{n=1}^{\infty} A_n \cos\left(\left(\frac{\pi}{2} + n\pi\right)x\right), A_n = \frac{f_n}{\lambda - \lambda_n} = \frac{f_n}{(3\pi/2)^2 - \lambda_n},$$

for f_n given by

$$f_n = \frac{\int_0^1 (x^2 + Bx)\phi_n(x)dx}{\int_0^1 \phi_n^2(x)dx}.$$

Thus we can see immediately that

$$\lambda_1 = \left(\frac{\pi}{2} + \pi\right)^2 = \left(\frac{3\pi}{2}\right)^2$$

could cause problems for our solution. However, if we can get f_n such that $f_1 = 0$, this problem can be averted. We compute

$$f_1 = \frac{\int_0^1 (x^2 + Bx)\phi_1(x)dx}{\int_0^1 \phi_1^2(x)dx} = \frac{-4(3B\pi(2+3\pi) + 9\pi^2 - 8)}{27\pi^3}$$

$$\implies f_1 = 0 \iff \boxed{B = \frac{8 - 9\pi^2}{6\pi + 9\pi^2}}.$$

(d) Using the results from part (c), we now must simply pass over $n = 1$ in our series solution. This will not affect the solution, as we now know $f_1 = 0$. However, from lecture, we know we can always add back an arbitrary amount of ϕ_1 and still maintain our solution. Thus

$$\boxed{y(x) = c\cos\left(\frac{3\pi x}{2}\right) + \sum_{n>1}^{\infty} \frac{f_n}{(3\pi/2)^2 - \lambda_n} \cos\left(\left(\frac{\pi}{2} + n\pi\right)x\right)},$$

for c some constant.

2 Problem 2

Consider the BVP

$$y'' + \lambda y = x^2, 0 \leq x \leq 1, \tag{2}$$

with $y(0) = 1, y(1) = 0$.

(a) Let $u(x) = y(x) + \alpha x + \beta(1-x)$, for some constants α, β. We wish to have $u(0) = 1, u(1) = 0 \implies u(0) = 0 = 1 + \beta, u(1) = 0 = \alpha$, so we set $\alpha = 0, \beta = -1$. Thus

$$u(x) = y - (1 - x) = y + x - 1.$$

From this equation, we can see that $u'' = y''$, and so (2) is

$$u'' + \lambda(u - x + 1) = x^2 \implies u'' + \lambda u = \lambda x - \lambda + x^2.$$

Now since the boundary conditions of this ODE are homogeneous, we can substitute the Fourier series for u,

$$u(x) = \sum_{n=1}^{\infty} U_n \sin(n\pi x),$$

directly into it. This will allow us to find a reasonable solution for y as long as $g(x) = \lambda x - \lambda + x^2$ has a Fourier sine series over $0 \leq x \leq 1$, since

$$-\sum_{n=1}^{\infty} U_n \pi^2 n^2 \sin(n\pi x) + \lambda \sum_{n=1}^{\infty} U_n \sin(n\pi x) = \sum_{n=1}^{\infty} U_n \sin(n\pi x)(\lambda - \pi^2 n^2) = g(x).$$

Indeed, $g(x)$ is a polynomial, so it must have a Fourier series over $0 \leq x \leq 1$. We know from lecture that

$$g(x) = \sum_{n=1}^{\infty} G_n \sin(n\pi x), G_n = -\frac{2\lambda}{\pi n} + F_n,$$

for F_n the coefficients of the Fourier sine series of x^2 on $0 \le x \le 1$. We can find these easily from

$$F_n = 2\int_0^1 x^2 \sin(n\pi x)dx = \frac{2((2-\pi^2 n^2)\cos(\pi n) - 2)}{\pi^3 n^3}.$$

We now have the coefficients G_n for the Fourier sine series of g, and so we have that

$$\sum_{n=1}^{\infty} U_n \sin(n\pi x)(\lambda - \pi^2 n^2) = \sum_{n=1}^{\infty} G_n \sin(n\pi x) \implies U_n(\lambda - \pi^2 n^2) = G_n,$$

so

$$u(x) = \sum_{n=1}^{\infty} \frac{G_n}{\lambda - \pi^2 n^2} \sin(n\pi x).$$

It follows finally that

$$y(x) = u(x) - x + 1, \sum_{n=1}^{\infty} U_n \sin(n\pi x)(\lambda - n^2) = g(x)$$

$$\implies \boxed{y(x) = 1 - x + \sum_{n=1}^{\infty} \frac{G_n}{\lambda - \pi^2 n^2} \sin(n\pi x)},$$

for G_n given above.

(b) We will multiply the ODE through by $2\sin(n\pi x)dx$ and integrate both sides to get

$$\int_0^1 2\sin(n\pi x)y''(x)dx + \int_0^1 2\sin(n\pi x)\lambda y(x)dx = \int_0^1 2\sin(n\pi x)x^2 dx$$

$$= 2\int_0^1 \sin(n\pi x)y''(x)dx + \lambda A_n = F_n,$$

for F_n as defined before and A_n the coefficients of the Fourier sine series of $y(x)$ over $0 \le x \le 1$. Integrating by parts twice, we have that

$$2\int_0^1 \sin(n\pi x)y''dx = 2n\pi - n^2\pi^2 A_n$$

$$\implies 2n\pi - n^2\pi^2 A_n + \lambda A_n = F_n \implies \boxed{A_n = \frac{F_n - 2n\pi}{\lambda - n^2\pi^2}},$$

for

$$y(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x).$$

(c) From (a), we have that

$$y(x) = 1 - x + \sum_{n=1}^{\infty} \frac{G_n}{\lambda - \pi^2 n^2} \sin(n\pi x), \tag{3}$$

and from (b),

$$y(x) = \sum_{n=1}^{\infty} \frac{F_n - 2n\pi}{\lambda - n^2\pi^2} \sin(nx). \tag{4}$$

We now compute the Fourier sine series for $1-x$ over $0 \leq x \leq 1$. We know from (a) that the Fourier sine series of $x-1$ over $0 \leq x \leq 1$ has coefficients

$$h_n = -\frac{2}{\pi n},$$

so the coefficients of the sine series for $1-x$ will simply be $-h_n$. Furthermore, in (a), we defined

$$G_n = -\frac{2\lambda}{\pi n} + F_n,$$

so combining these results, (3) is

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} \frac{2}{\pi n} \sin(n\pi x) + \sum_{n=1}^{\infty} \left(\frac{-2\lambda}{\pi n(\lambda - \pi^2 n^2)} + \frac{F_n}{\lambda - \pi^2 n^2} \right) \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} \left(\frac{2}{\pi n} + \frac{-2\lambda}{\pi n(\lambda - \pi^2 n^2)} + \frac{F_n}{\lambda - \pi^2 n^2} \right) \sin(n\pi x). \end{aligned}$$

Observe that

$$\begin{aligned} \frac{2}{\pi n} + \frac{-2\lambda}{\pi n(\lambda - \pi^2 n^2)} + \frac{F_n}{\lambda - \pi^2 n^2} &= \frac{2(\lambda - \pi^2 n^2) - 2\lambda + F_n \pi n}{\pi n(\lambda - \pi^2 n^2)} \\ &= \frac{-2\pi^2 n^2 + F_n \pi n}{\pi n(\lambda - \pi^2 n^2)} = \frac{-2\pi n + F_n}{\lambda - \pi^2 n^2}, \end{aligned}$$

so the coefficients on sine series (3) from part (a) indeed match those on series (4) from part (b).

3 Problem 3

Consider the BVP

$$y'' + \frac{4}{x} y' + \frac{2}{x^2} y = x, 1 \leq x \leq 2,$$

with BCs $y(1) = y(2) = 0$.

(a) We first find two linearly independent solutions to the correpsonding homogeneous ODE

$$G'' + \frac{4}{x} G' + \frac{2}{x^2} G = 0.$$

Assume G takes the form $G = x^\alpha$. Then

$$\alpha(\alpha-1)x^{\alpha-2} + 4\alpha x^{\alpha-2} + 2x^{\alpha-2} = 0$$

$$\implies x^{\alpha-2}(\alpha^2 + 3\alpha + 2) = 0 \implies \alpha_1, \alpha_2 = -1, -2.$$

Thus the general solution to the homogeneous ODE is given by

$$y(x) = \frac{c_1}{x} + \frac{c_2}{x^2}.$$

We need one solution to satisfy $y_1(1) = 0$, and the other to satisfy $y_2(2) = 0$. This can be accomplished by setting

$$y_1(x) = \frac{1}{x} - \frac{1}{x^2}, y_2(x) = \frac{1}{x} - \frac{2}{x^2}.$$

Note also that we cannot solve for c_1, c_2 in the general solution $y(x)$ such that $y(1) = y(2) = 0$. Thus there exists a solution for the Green's function in the original problem. We now compute the Wronskian of y_1, y_2, given by

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

$$= \left(\frac{1}{x} - \frac{1}{x^2}\right)\left(-\frac{1}{x^2} + \frac{4}{x^3}\right) - \left(-\frac{1}{x^2} + \frac{2}{x^3}\right)\left(\frac{1}{x} - \frac{2}{x^2}\right) = \frac{1}{x^4}.$$

Then we know from lecture that the corresponding Green's function to the BVP will be given by

$$\boxed{G(x, x') = \begin{cases} A(x')y_1(x)y_2(x') & x < x' \\ A(x')y_2(x)y_1(x') & x > x' \end{cases}},$$

for $A(x) = \frac{1}{W(x)}$, and y_1, y_2, W given above.

(b) The formal solution to the inhomogeneous problem for general $f(x)$ is given by

$$y(x) = \int_1^x G(x, x')f(x')dx' + \int_x^2 G(x, x')f(x')dx',$$

for $G(x, x')$ given above.

(c) The Green's function for this problem is given in part (a), and the solution to the problem is given in part (b), for $f(x) = x$. Then we simply compute

$$y(x) = \underbrace{\int_1^x A(x')y_2(x)y_1(x')f(x')dx'}_{=y_1(x)\int_1^x (x')^5 y_1(x')dx'} + \underbrace{\int_x^2 A(x')y_1(x)y_2(x')f(x')dx'}_{=y_2(x)\int_x^2 (x')^5 y_2(x')dx'}$$

$$= \boxed{\frac{x^5 - 31x + 30}{20x^2}}.$$

Indeed we can check that, using this $y(x)$,

$$y'' + \frac{4}{x}y' + \frac{2}{x^2}y = \frac{10x^5 - 62x + 120 + 62x - 120}{10x^4} = x,$$

and $y(1) = y(2) = 0$.