Ec 172, PS 1

Ethan Wilk

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1 Problem 1

Problem: What are the subgame perfect equilibria of the centipede game?

The (only) subgame perfect equilibrium for the centipede game requires that each player defects at every possible opportunity.

It is important to note that there are multiple Nash equilibria for this game, though all such equilibria involve immediate defection by the first player. For instance, suppose the first player opts to defect at round one, three, and five, but to continue at round seven, and the second player opts to always defect at every opportunity. This is a Nash equilibrium, since, given the opponent's strategy, it would be impossible to improve one's own payoff.

However, such a strategy profile is not a subgame perfect equilibrium. To see this, we may consider the subgame with initial node at round seven. From this state, the first player continues, but the second player then stops at round eight. The first player would have earned more money if he had stopped at round seven, so his strategy is not optimal in every subgame.

Note that this reasoning applies to any terminal node in any subgame other than that resulting from immediate defection by the player who makes the first move in that subgame. Thus the only strategy profile which is an equilibrium across all subgames is that in which each player defects whenever possible.

2 Problem 2

Problem (a): Why can the second player not force a victory in tic-tac-toe?

Suppose by contradiction that the second player can force a victory in tic-tac-toe; i.e., the second player has a strategy σ_2 such that, regardless of the first player's choice of strategy σ_1, the second player always wins.

Then suppose the first player picks a square to mark arbitrarily on her first move, and following the second player's action, she follows the second player's strategy as if she were the second player having just observed the first player's choice of action. If that strategy indicates that she should ever have to mark the square she chose arbitrarily at first, she can arbitrarily choose a new square.

(Having more marked squares in tic-tac-toe cannot harm one's prospects of winning the game.)

Since σ_2 is guaranteed to result in a win by assumption, it must then be the case that both the first and second players will win. But this cannot be the case, so we may conclude that such a σ_2 does not exist.

Problem (b): Why can the second player not force a victory in the sweet 15 game?

Consider the following 3×3 magic square:

2	7	6
9	5	1
4	3	8

.

Notice that the square contains all digits 1-9 and exhaustively depicts all triples of the numbers 1-9 that sum to 15 as the rows, columns, and diagonals of the square.

But if this the case, then this game is no different than tic-tac-toe. Players alternate turns, drawing X or O on each square, and the first to get three of her own symbol in a row, column, or diagonal wins the game. Thus by (2a), the proof is complete.

3 Problem 3

Problem (a): Find a subgame perfect equilibrium of the described game.

Denote by $P(i > j)$ the probability that die i yields a number greater than die j when rolled. Thus, from elementary probability theory,

$$P(a > b) = \frac{5}{9}, P(b > a) = \frac{4}{9},$$

$$P(a > c) = \frac{4}{9}, P(c > a) = \frac{5}{9},$$

$$P(b > c) = \frac{5}{9}, P(c > b) = \frac{4}{9}.$$

Note, then, that for every die i, there exists a die $j \in \{a, b, c\}$
i such that $P(j > i) > \frac{5}{9}$. In particular, if the first player chooses a, the second may choose c; if the first chooses b, the second may choose a; and if the first chooses c, the second may choose b.

Regardless of the die the first player picks, the second may select one such that the utility of the second player if $\frac{5}{9}$, and that of the first player becomes $\frac{4}{9}$.

Thus a subgame equilibrium would be for the first player to pick, say, die a, and for the second player to pick die c.

Problem (b): Who has the higher utility? Is there a subgame perfect equilibrium in which the other has higher utility?

As shown in part (a) above, the second player will always be able to receive the higher utility. There does not exist a subgame perfect equilibrium in which the first player has higher utility, because otherwise, the second player could swap her choice of die to receive the higher utility.

Problem (c): N/A.

Warren can't fool Billy.

4 Problem 4

Problem: Find a subgame perfect equilibrium of the extensive-form dollar auction game, as described in section 2.9 of the lecture notes.

The subgame perfect equilibrium of the two-player dollar auction game requires that the first player bids at every opportunity, and the second player holds at every opportunity.

This is an equilibrium in general because neither player will be able to improve his or her utility by changing strategy, given what his or her opponent will do. If the second player chose to bid after the first held, then the first player would continue to bid until either the second player conceded or the first player made it to a bid of \$100.05 (this would happen on round 500). The payoff of the second player would then be -\$99.95, so she could have done better by bidding nothing initially. Clearly, for the first player, knowing that the second will always hold, there is no better option than to bid the minimum amount and pocket the \$0.95 profit. Thus, this is indeed an equilibrium.

Furthermore, this is also a subgame perfect equilibrium. Consider any node in the game tree following n rounds of betting in which it is the first player's turn to move. After those n rounds, the first player will have bet $\$0.05 + \$0.2n$. Knowing the second player will always hold, it is again optimal for the first player to bet, since the profit to the first player will then be $-\$0.05 - \$0.2(n+1) + \$1 = \$0.75 - \$0.2n$, which is greater than $-\$0.05 - \$0.2n$ for all n.

Similarly, after n rounds of betting, if we have chosen a node in which it is the second player's turn to act, then the second player has bid $\$0.15 + \$0.2n$ at this point in time. If she knows that the first player will always continue to raise until hitting \$100.05, it is optimal for her to hold, so her net profit is $-\$0.15 - \$0.2n$, instead of $-\$99.95$. Thus the strategy profile in which the first player always continues to bet and the second player always holds is indeed a subgame perfect equilibrium to the extensive-form dollar auction game.