Ec 172, PS 6

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1 Problem 1

(a) Let M, T be the bids of Michael and Thierno, respectively, so $M, T \sim U[0,1]$. Then Nishka will receive payment equal to the second highest bid in the auction. That is,

$$P_N = \min\{M, T\},$$

for P_N Nishka's expected revenue. From elementary probability theory,[1] the expected minimum of n random variables sampled independently from $U[0,1]$ is $\frac{1}{n+1}$, so for $n = 2$,

$$E(P_N) = E(\min\{M, T\}) = \frac{1}{3}.$$

(b) Nishka's expected revenue will now be

$$E(P_N) = E(P_N | T \leq b_r \leq M)P(T \leq b_r \leq M)$$

$$+E(P_N | M \leq b_r \leq T)P(M \leq b_r \leq T) + E(P_N | b_r \leq T \leq M)P(b_r \leq T \leq M)$$

$$+E(P_N | b_r \leq M \leq T)P(b_r \leq M \leq T).$$

By symmetry, $P(T \leq b_r \leq M) = P(M \leq b_r \leq T) = b_r(1 - b_r)$, and $P(b_r \leq T \leq M) = P(b_r \leq M \leq T) = (1 - b_r)^2/2$. Thus

$$E(P_N) = b_r(1 - b_r)(E(P_N | T \leq b_r \leq M) + E(P_N | M \leq b_r \leq T))$$

$$+(1 - b_r)^2(E(P_N | b_r \leq T \leq M) + E(P_N | b_r \leq M \leq T))/2.$$

If $T \leq b_r \leq M$, then Michael will pay Thierno's bid, which has expectation $E(T | T \leq b_r) = \frac{b_r}{2}$. Thus

$$E(P_N | T \leq b_r \leq M) = \frac{b_r}{2}.$$

[1] Or ACM 116.

From the same reasoning, $E(P_N|M \leq b_r \leq T) = \frac{b_r}{2}$. Furthermore, if $b_r \leq M \leq T$, then Thierno will pay Michael's bid, which has expectation

$$E(M|b_r \leq M \leq T) = b_r + \frac{1-b_r}{3} = \frac{1}{3} + \frac{2b_r}{3},$$

so $E(P_N|b_r \leq T \leq M) = E(P_N|b_r \leq M \leq T) = \frac{1+2b_r}{3}$. Thus, combining these results,

$$E(P_N) = b_r(1-b_r)\,(b_r) + (1-b_r)^2\left(\frac{1+2b_r}{3}\right)$$

$$= b_r^2 - b_r^3 + \frac{(1-2b_r+b_r^2)(1+2b_r)}{3} = \frac{1-4b_r^3}{3}.$$

(c) Based on the result from (b), we have that

$$\frac{\partial E(P_N)}{\partial b_r} = -4b_r^2.$$

This is equal to 0 when $b_r = 0$, so since

$$\frac{\partial^2 E(P_N)}{\partial b_r^2} = -8b_r,$$

then we have that $b_r = 0$ is a local max of Nishka's expected revenue, and thus the maximal expected revenue she can obtain is $\frac{1}{3}$.

2 Problem 2

(a) Let Moya's valuations for the loaf and butter, respectively, to be $M_l, M_s \sim U[0,1]$. Then the expected revenue P_L of Lilly is

$$E(P_L) = (b_l + b_s)P(M_l \geq b_l, M_s \geq b_s) + b_l P(M_l \geq b_l, M_s < b_s)$$

$$+ b_s P(M_s \geq b_s, M_l < b_l)$$

$$= (b_l + b_s)(1-b_l)(1-b_s) + b_l(1-b_l)(b_s) + b_s(1-b_s)(b_l)$$

$$= b_l - b_l^2 + b_s - b_s^2 = b_l(1-b_l) + b_s(1-b_s).$$

(b) The maximum of any equation of the form $p(1-p), 0 \leq p \leq 1$ is attained at $p = \frac{1}{2}$, so the maximum expected revenue Lilly can achieve is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

(c) We assume that these conditions imply Moya will only purchase the bundle if the sum of her valuation of the bread and butter are less than b_b. In this case,

$$E(P_L) = b_b P(b_b \leq M_l + M_s).$$

The distribution of the sum of two uniformly distributed, independent random variables is not a standard one. It is often easiest to represent the random variables on a 2D grid and shade the area in which favorable outcomes of the experiment occur to compute the desired probability on the sum of the variables. In our case, setting M_s to be the x-axis and M_l the y, we shade the area given by the inequality $M_l \geq -M_s + b_b$, for b_b some constant.

We consider only the area that falls in the square $S = [0,1] \times [0,1]$, since these are the only attainable values of $M_l \times M_s$. Then this area is given by

$$1 - \frac{b_b^2}{2}, 0 \leq b_b \leq 2,$$

and so since the area of S is 1, our desired probability is the same as the area. Thus

$$E(P_L) = b_b - \frac{b_b^3}{2}.$$

(d) Using the result from (c), we find that Lilly achieves a maximal expected payoff of $\frac{2}{3}\sqrt{\frac{2}{3}}$ at $b_b = \sqrt{\frac{2}{3}}$.