Ec 172, PS 3

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1 Problem 1

Part (a): Show that the quiz game does not have a pure Nash equilibrium.

To have a pure Nash equilibrium, we require that there exists some fixed strategy (i.e., some day) the student may pick to study and some fixed strategy the teacher may pick to host the quiz such that neither may improve their utility.

Suppose by contradiction, then, that there does exist such an $i, 1 \leq i \leq 5$, and $j, 1 \leq j \leq 5$, such that if the student always chooses day i to study, and the teacher always chooses day j to announce the quiz, neither can alter their strategy to improve their utility.

But if $i = j$, the student's utility is 1, so the teacher's is 0. But clearly, then, the teacher should have picked any day $j \neq i$ to receive a utility of 1 (because the student would then have a utility of 0).

Otherwise, if $i \neq j$, then the student's utility is 0, and the teacher's is 1. But the student therefore should have picked day $i = j$ to receive a utility of 1 (because the teacher would then have a utility of 1).

Thus there does not exist any such i, j such that if the student always chooses day i to study and the teacher always chooses day j to host the quiz, neither will be able to improve their utility. It follows that the game has no pure equilibrium.

Part (b): Find a mixed Nash equilibrium for this game.

Denote by $s_i, 1 \leq i \leq 5$, the probability that the student chooses to study on day i. Denote $t_j, 1 \leq j \leq 5$, the probability that the teacher assigns the quiz on day j. Then for u_s, u_t the utilities of the student and the teacher respectively, we have that

$$E(u_s) = s_1 t_1 + ... + s_5 t_5,$$

$$E(u_t) = 1 - E(u_s).$$

We claim that one such mixed Nash equilibrium in this game is for the student to choose $s_i = \frac{1}{5}$, and the teacher to choose $t_i = \frac{1}{5}$ for all $1 \leq i, j \leq 5$.

Under this strategy profile, the expected utility of the student is $\frac{5}{25} = \frac{1}{5}$, so the expected utility of the teacher is $1 - \frac{1}{5} = \frac{4}{5}$. Now suppose the student instead chose a different set of probabilities $s_1, ..., s_5$, where for at least one $i, s_i \neq \frac{1}{5}$. Then

$$E(u_s | t_1, ..., t_5 = 1/5) = \frac{1}{5}(s_1 + ... + s_5).$$

But clearly we must have that $\sum_i s_i = 1$, so the student's expected utility will always be $\frac{1}{5} \cdot 1 = \frac{1}{5}$.

Similarly, since $E(u_t) = 1 - E(u_s)$, if the teacher knows that $s_i = \frac{1}{5}$ for all i, then the expected utility for the teacher will always be

$$E(u_t | s_1, ..., s_5 = 1/5) = 1 - E(u_s) = 1 - \frac{1}{5}(t_1 + ... + t_5) = \frac{4}{5},$$

so the expected utility of the teacher will similarly remain unchanged under any other mixed strategy. Thus the described strategy profile is indeed a mixed Nash equilibrium for the game.

Part (c): Show that if there are infinitely many days then there does not exist a mixed Nash equilibrium. Why does this not violate Nash's Theorem?

Suppose by contradiction that there exists some strategy profile $\sigma = (\sigma_1, \sigma_2)$ that is a mixed Nash equilibrium in this game. In particular, let $\sigma_1(i)$ be the probability that the student chooses to study on day i, and let $\sigma_2(j)$ be the probability the teacher gives the quiz on day j. Note that it cannot be the case that $\sigma_1(i) > 0$ for every $i \geq 1$, since then

$$\sum_{i>1} \sigma_1(i)$$

diverges and is not equal to 1, a contradiction. Thus there must exist some finite set A of days in which $\sigma_1(i^* \in A) = 0$. Thus if the teacher knows this, she must have set $\sigma_2(i^* \in A) > 0, \sum_{i^* \in A} \sigma_2(i^*) = 1$.

But of course, if the student knows this, then she should set $\sigma_1(i^* \in A) > 0$, which creates a new set of days on which the teacher should then choose to assign the test. This set is precisely going to be $\mathbb{N} \backslash A$, or the days on which the student's probability of studying is now zero. Thus both agents would now have deviated from their strategies under σ, implying that their utilities could be improved given the strategy of the opponent. Thus σ is in fact not a mixed Nash equilibrium and the theorem is proved by contradiction.

This, however, does not violate Nash's theorem because this game is not finite. Since the set of potential strategies in this game is countably infinite, if the game were guaranteed to terminate after m days, one can always suggest that both the teacher and the student choose some day to assign the quiz and study that is greater than m. Thus this game is in fact not finite, and so Nash's theorem does not apply.

2 Problem 2

Part (a): Using only the Intermediate Value Theorem (i.e., without using Nash's Theorem or Brouwer's Theorem), prove that this game has a mixed (or pure)

Nash equilibrium.

We will prove a stronger theorem to confirm the one given. In particular, we will show that any finite game in which every player has exactly two strategies to choose from will have at least one mixed Nash equilibrium, regardless of the utility functions governing each agent's decisions.

Let $p = (p_1, ..., p_n)$ be a mixed strategy profile in this game. Define $r_i(\sigma_i) = \max\{u_i(\sigma_i, p_i) - u_i(p_i, p_{-i}, 0\}$ to be player i's expected utility from switching from strategy p_{-i} to σ_i. Thus we seek to show that there exists some $\sigma = (\sigma_1, ..., \sigma_n)$ such that $r_i(\sigma_i) = 0 \; \forall \; i \in N$. We will create the auxiliary function

$$h_i(p_i(x_i)) = \frac{p_i(x_i) + r_i(x_i)}{\sum_x (p_i(x) + r_i(x))} \tag{1}$$

to accomplish this.

Note that the mixed strategy of any player i may be described simply by $\sigma_i(a)$, that is, the probability with which player i chooses strategy a, since this implies $\sigma_i(b) = 1 - \sigma_i(a)$. Thus we have that

$$h_i(\sigma_i(a)) = \frac{\sigma_i(a) + r_i(a)}{\sigma_i(a) + r_i(a) + \sigma_i(b) + r_i(b)} = \frac{\sigma_i(a) + r_i(a)}{1 + r_i(a) + r_i(b)}.$$

We can see that $h_i : [0, 1] \to [0, 1]$, since the function takes one-dimensional probabilities as inputs and outputs real numbers in the range $[0, 1]$, as can easily be seen in the function's representation in (1). Furthermore, h_i is continuous over $[0, 1]$, since h_i is linear in $\sigma_i(a)$ (to see this, denote $1 + r_i(a) + r_i(b)$ some constant c and note that then $h_i(x) = \frac{x}{c} + d$, for d some other constant). Lastly, we see $[0, 1]$ is clearly a compact set, as it is both closed and bounded.

Observe that

$$h_i(0) = \frac{r_i(a)}{1 + r_i(a) + r_i(b)} > 0, h_i(1) = \frac{r_i(a)}{1 + r_i(a) + r_i(b)} < 1,$$

so if we consider the function $g_i(x) = h_i(x) - x$, then $g_i(0) > 0, g(1) < 0$. Since h_i is continuous, g_i is continuous, and thus from the IVT there exists some α_i^* such that $g_i(\alpha_i^*) = 0 \implies h_i(\alpha_i^*) = \alpha_i^*$. Thus every h_i has a fixed point in $[0, 1]$.

Now set $\alpha_i^* = \sigma_i(a)$, that is, the probability with which i chooses strategy a, so we have that

$$h_i(\sigma_i(a)) = \sigma_i(a) = \frac{\sigma_i(a) + r_i(a)}{1 + r_i(a) + r_i(b)}. \tag{2}$$

Note that there are two possibilities for player i's decision in the strategy profile p: Either player i chose strategy a, or player i chose b. We first assume player i chose a. Then it follows that the expected utility $r_i(a)$ that player i receives from switching from strategy a to a is 0, and so (2) is

$$h_i(\sigma_i(a)) = \sigma_i(a) = \frac{\sigma_i(a)}{1 + r_i(b)} \implies \sigma_i(a)(1 + r_i(b)) = \sigma_i(a) \implies r_i(b) + 1 = 1,$$

so $r_i(b) = 0$, as desired. On the other hand, suppose that under p, player i chose strategy b. Then $r_i(b) = 0$, so (2) becomes

$$h_i(\sigma_i(a)) = \sigma_i(a) = \frac{\sigma_i(a) + r_i(a)}{1 + r_i(a)} \implies \sigma_i(a)(r_i(a)) = r_i(a) \implies r_i(a) = 0,$$

since we know that $\sigma_i(a) \neq 1$, as $\sigma_i(a) = 1$ is not a fixed point for h_i.

Thus, regardless of player i's decision under the strategy profile p, we have that we can find $\sigma_i(a)$ (the probability with which player i will select strategy a) such that $r_i(a) = r_i(b) = 0$ (the expected utility to i from changing her strategy is 0). Thus there exists at least one mixed Nash equilibrium in this game.