Ec 172, PS 5 Rework

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1 Problem 1

Part (a): What is the common knowledge algebra?

Each player knows all possible outcomes Ω from the coin tosses and envelope assignments, so $\Omega \in \Sigma_C$, and thus $\emptyset \in \Sigma_C$. Furthermore, since both Rachel and Max know that $b \in \{0,1\}$, then if Rachel knows $k+b=y$ from her signal, then she will know that either $k=y$ or $k=y-1$, while if Max knows $k-b+1=x$ from his signal, then he will know that either $k=x$ or $k=x-1$. In particular, if $b=0$, then $k+1=y+1=x$, and if $b=1$, then $k+1=x+1=y$.

Thus, we can see that the set of states in which Rachel knows that k is either c or $c-1$ for $c>1$ some constant is equal to the set of states of the world in which either $k \in \{c, c-1\}$. Thus the event $k \in \{c, c-1\}$ is self-evident to Rachel. Similarly, the set of states in which Max knows that k is either c or $c-1$ is equal to the set of states in which $k \in \{0,1\}$, so this event is also self-evident to Max, meaning it should be included in the common knowledge algebra Σ_C. In particular, all subsets of states of the world following the form $\{i, i+1\} \times \{0,1\} \subset \Omega, i \geq 1$ should be included in Ω.

Finally, to ensure Σ_C meets the requirements of an algebra, we must include all complements and unions of the sets $\{i, i+1\} \times \{0,1\}, i \geq 1$. This is not unjustified, since if Rachel and Max both know that $k \in \{i, i+1\}$ when $k \in \{i, i+1\}$, then they will also know that $k \notin \{i, i+1\}^C$ when $k \in \{i, i+1\}$; additionally, if Rachel and Max know that $k \in \{i, i+1\}$ when $k \in \{i, i+1\}$ and $k \in \{j, j+1\}$ when $k \in \{j, j+1\}$ and $k \in \{m, m+1\}$ when $k \in \{m, m+1\}$, etc., then they will know $k \in \{i, i+1, j, j+1, m, m+1, \text{ etc.}\}$ when $k \in \{i, i+1, j, j+1, m, m+1, \text{ etc.}\}$.

Part (b): For each possible value of Rachel's signal, calculate her conditional expected gain from trading. Do the same for Max.

We assume Rachel peeked at her envelope, so she knows 10^{k+b}. We must therefore find

$$E(t_M | t_R) - t_R,$$

her expected gain from trading after seeing her note. We recognize that, given 10^{k+b}, Rachel will know that either $b=0$ or $b=1$, so if $k+b=y$, then either

$k = y$ or $k = y-1$. Thus $P(k = y|k+b = y) + P(k = y-1|k+b = y) = 1$. We know from the priors that

$$P(k = y) = \left(\frac{1}{2}\right)^{y+1}, P(k = y-1) = \left(\frac{1}{2}\right)^{y}$$

$$\implies P(k = y|k+b = y) = \frac{(1/2)^{y+1}}{(1/2)^{y+1} + (1/2)^{y}} = \frac{1}{3}$$

$$\implies P(k = y-1|k+b = y) = \frac{2}{3}.$$

Thus we have that the conditional probability that $b = 0$ given 10^{k+b} is $\frac{1}{3}$, while the probability $b = 1$ is then $\frac{2}{3}$. Thus

$$E(10^{k-b+1}|10^{k+b}) = 10^{y+1}\frac{1}{3} + 10^{y-1}\frac{2}{3} = 34 \cdot 10^{y-1},$$

so

$$E(t_M|t_R) - t_R = 34 \cdot 10^{y-1} - 10^{y} = 10^{y-1} \cdot 24 = 10^{k+b-1} \cdot 24$$

is Rachel's expected gain from trading with Max. Similarly, for Max, the expected gain from a trade will be

$$E(t_R|t_M) - t_M,$$

where, letting $x = k - b + 1$, we have that either $k = x$ or $k = x - 1$. We have that $P(k = x|x = k-b+1) + P(k = x-1|x = k-b+1) = 1$, so

$$P(k = x|x = k-b+1) = \frac{(1/2)^{x+1}}{(1/2)^{x+1} + (1/2)^{x}} = \frac{1}{3}$$

$$\implies P(k = x-1|x = k-b+1) = \frac{2}{3},$$

and thus Max's expected gain will be

$$E(t_R|t_M) - 10^{x} = 10^{x+1}\frac{1}{3} + 10^{x-1}\frac{2}{3} - 10^{x}$$

$$= 34 \cdot 10^{x-1} - 10^{x} = 24 \cdot 10^{x-1} = 24 \cdot 10^{k-b}.$$

Part (c): What is Rachel's expected gain from trading before she sees her signal (i.e., before she looks at her note)?

Rachel's signal is given by $t_R(k, b) = 10^{k+b}$, and Max's is given by $t_M(k, b) = 10^{k+1-b}$. If Rachel trades, her utility will become t_M. Before Rachel sees her note, though, she has no information about k, b, or $k+b$. We must therefore consider all possible outcomes $(k, b) \in \Omega$ to compute

$$E(t_M - t_R) = \sum_{k=1}^{\infty}\sum_{b=0}^{1} \mu(k, b)(t_M(k, b) - t_R(k, b))$$

$$= \sum_{k=1}^{\infty} \sum_{b=0}^{1} \left(\frac{1}{2}\right)^{k+1} (10^{k-b+1} - 10^{k+b})$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k+1} (10^{k+1} - 10^{k} + 10^{k} - 10^{k+1}) = 0,$$

her expected gain from trading before she sees her note.

Part (d): Will Rachel want to trade before looking at her note?
No, for the reasoning given in (c).