Ec 172, PS 2

By Ethan Wilk

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1 Problem 1

Problem (a): Find the equilibria in the voter turnout game when N^a and N^b are the same size.

If $|N^a| = |N^b|$, the only equilibrium of this game requires that all members of N^b vote for b, and all members of N^a vote for a.

First, we note that it is never optimal for a voter to vote for the candidate from the opposing party. Say the voter is from N^a (we can always relabel the a and b scripts, so the logic holds both ways). Then if a wins, the voter receives utility $1 - c, c \in (0, 1/2)$; if a loses, the voter receives utility $-c$; and if a ties, the voter receives utility $1/2 - c$.

If, however, the voter had abstained, then those same utility payoffs would have been $1, 0$, and $1/2$, respectively. If the voter had voted for a, then the payoffs would have remained the same as when the voter voted for b, but having more votes for candidate a can never harm the voter's prospective utility. Thus voters in N^a (or N^b) will only vote for candidate a (or b) or abstain.

We now show that the strategy profile proposed is indeed an equilibrium. Observe that, under the profile, all voters will receive a utility payoff of $1/2-c > 0, c \in (0, 1/2)$. If any voter in, say, N^a knows that all other voters in N^a will vote for a and all voters in N^b will vote for b, then by abstaining, the voter receives a utility payoff of $0 < 1/2-c$. The same logic clearly applied to members of N^b. Thus this is indeed an equilibrium of the game.

We now show that this is the only equilibrium of the game. Suppose k_a voters from N^a choose to abstain, and k_b from N^b do the same. Then if $k_a > k_b$, candidate b will win. But then it would have been favorable for any voter in N^a who did not abstain to have abstained. If instead $k_a = k_b$, then it would have been favorable for any voter in N^a who abstained to have instead voted for a, as such a voter's utility would have increased from $1/2$ to $1-c > 1/2$. Thus this is not an equilibrium of the game.

Problem (b): Find the equilibria in the voter turnout game when $|N^a|$ is greater than $|N^b|$.

If $|N^a| > |N^b|$, then the voter turnout game has no equilibria. If, say, m_a voters from N^a decided to vote and m_b from N^b decided to do the same, then clearly if $m_a < m_b$, those who voted from N^a would have improved their payoffs if they had abstained. On the other hand, if $m_a > m_b$, then those from N^b who voted would have improved their payoffs if they had abstained. Finally, if $m_a = m_b$, then any voter in N^a who abstained would have improved her payoff if she chose to vote. Thus there is no such m_a, m_b such that no individual would have had an alternative strategy that would not have improved her payoff.

2 Problem 2

Problem (a): An equilibrium is said to be symmetric if all players choose the same strategy. Find a symmetric pure Nash equilibrium of the Cournot competition game.

The symmetric equilibrium of the Cournot competition game is for all firms to produce

$$q^* = \frac{A - C}{2Bn}.$$

We can see this, since the payoff of any given firm will be

$$u_i(q_1, ..., q_n) = q_i P\left(\sum_i q_i\right) - c(q_i)$$

$$= q^* P(nq^*) - c(q^*) = q^*(A - Bnq^*) - Cq^*.$$

Optimizing, we get

$$\frac{\partial u_i}{\partial q^*} = A - 2Bnq^* - C = 0$$

$$\implies q^* = \frac{A - C}{2Bn}.$$

We can confirm that this corresponds to the point of a local maximum for u_i by computing

$$\frac{\partial^2 u_i}{\partial q^{*2}} = -2Bn < 0,$$

since $B > 0$. Thus, if any firm i knows that all other firms will be producing q^* units of the good, then the firm will optimize its own utility by also producing q^*, so this is indeed a symmetric equilibrium.

Problem (b): Imagine that an organized crime boss is brought in to enforce a cartel policy that maximizes the total utility of the companies. By how much does their total utility increase?

We will denote $Q = \sum_i q_i$ the total quantity produced by the companies in the cartel, and $U = \sum_i u_i$ the total utility of the companies. Then

$$U = P(Q)Q - CQ = Q(A - BQ) - CQ$$

$$\implies \frac{\partial U}{\partial Q} = A - 2BQ - C \implies Q = \frac{A - C}{2B}$$

is the value of Q that maximizes the total utility of the companies in the cartel. (This can again be confirmed by noting that the second derivative of U WRT Q is negative.) Note that this is achievable by simply having every firm produce q^* as indicated in part (a). Thus the total utility of the companies would not change, as this result is only a general case of the result from (a).

3 Problem 3

Problem (a): Show that this game (modified matching pennies) has a pure Nash equilibrium.

The pure Nash equilibrium of this game is for both players to select strategy A. Indeed, if player one knows player two will play A, she can also play A to receive a payoff of 2. All other payoffs (1 or 2) assuming player one plays A are less than or equal to this. Similarly, if player two knows player one will play A, she should also receives a payoff of 2 from playing A, which is higher than any other payoff under the assumption that player one plays A. Thus both players opting to play strategy A is indeed a pure Nash equilibrium of the game.

Problem (b): What are the weakly dominated strategies?

For player one, strategy A is weakly dominated by H, since player one's payoff from choosing H will always be greater than or equal to her payoff from choosing A.

Problem (c): Iteratively remove the weakly dominated strategies. What is the resulting game? What are its pure Nash equilibria?

We remove the strategy for player one listed in part (b), resulting in the payoff matrix shown below.

	H	T	A
H	(1, 0)	(0, 1)	(2, 0)
T	(0, 1)	(1, 0)	(1, 0)

Now, it is apparent that player two selecting A is also weakly dominated by player two selecting strategy H. Elimination of this strategy results in the final payoff matrix shown below.

	H	T
H	(1, 0)	(0, 1)
T	(0, 1)	(1, 0)

Note that this is exactly the same as the payoff matrix from the original matching pennies game, which we know has no pure Nash equilibrium.

4 Problem 4

Problem (a): For every possible value of b, c and f, find all the mixed Nash equilibria.

Suppose the taxpayer chooses to cheat with probability p_c, and the IRS chooses to audit with probability p_a. Then the expected payoff to the taxpayer will be

$$E(P_{taxpayer}) = p_c p_a(-f) + p_c(1-p_a)(b)$$
$$= -fp_cp_a + bp_c - bp_cp_a$$
$$\implies \frac{\partial E(P_{taxpayer})}{\partial p_c} = -f + b - bp_a.$$

We will define $p_a^* = \frac{b}{b+f}$ the probability of an audit at which any change in the taxpayer's choice of p_c will not affect her expected payoff. If it is the case that $p_a > p_a^*$, then the taxpayer will choose $p_c = 0$; if $p_a < p_a^*$, then the taxpayer will choose $p_c = 1$ to maximize her expected payoff.

Similarly, the expected payoff to the auditor will be

$$E(P_{IRS}) = p_cp_a(f-c) + (1-p_c)p_a(-c)$$

$$= fp_cp_a - cp_cp_a - cp_a + cp_cp_a = fp_cp_a - cp_a$$
$$\implies \frac{\partial E(P_{IRS})}{\partial p_a} = fp_c - c.$$

We will then define $p_c^* = \frac{c}{f}$ the probability of the taxpayer cheating at which the auditor's choice of p_a will not affect the IRS's expected payoff. If $p_c > p_c^*$, then the IRS will choose $p_a = 1$; otherwise, if $p_c < p_c^*$, then the IRS will choose $p_a = 0$ to maximize her expected payoff.

Thus it follows that the mixed Nash equilibrium for this game requires that

$$p_a = p_a^* = \frac{b}{b+f}, p_c = p_c^* = \frac{c}{f}.$$

Problem (b): In what direction does the equilibrium probability of an audit change as a function of b, c and f? How about the probability of cheating?

At equilibrium,

$$p_a = \frac{b}{b+f}.$$

Thus c does not affect the equilibrium probability of an audit. However, as f increases, this probability decreases, and as b increases, it increases.

Similarly, at equilibrium,

$$p_c = \frac{c}{f}.$$

Thus b does not affect the equilibrium probability of cheating. However, as c increases, the probability increases, and as f increases, it decreases.

Problem (c): In what direction do the players' equilibrium expected utilities change as a function of b, c, and f?

From (a), we have that

$$E(P_{taxpayer}) = -fp_cp_a + bp_c - bp_cp_a,$$

$$E(P_{IRS}) = fp_cp_a - cp_a$$

are the expected utilities of the taxpayer and the IRS, respectively. At equilibrium, we have established $p_a = p_a^*, p_b = p_b^*$, so these are

$$E(P_{taxpayer}) = -f\frac{c}{f}\frac{b}{b+f} + b\frac{c}{f} - b\frac{c}{f}\frac{b}{b+f}$$

$$= \frac{c}{f}\left(-\frac{b}{b+f}(f+b) + b\right) = 0.$$

$$E(P_{IRS}) = f\frac{c}{f}\frac{b}{b+f} - c\frac{b}{b+f} = c\left(\frac{b}{b+f} - \frac{b}{b+f}\right) = 0.$$

Thus at equilibrium, neither the taxpayer's nor the IRS's utility is affected by b, c, or f.