Ec 172, PS 4

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May 23, 2023

1 Problem 1

Part (a): Let n be even. Find a pure Nash equilibrium or explain why none exist.

Any pure Nash equilibria of this game requires that the players alternate in their choice of action. That is, arbitrarily picking any player in the circle to be player 1, we require that all even-numbered players $i = 2k, 1 \leq k \leq \frac{n}{2}$ select either X or Y. If the even-numbered players select X, all others select Y; if the even-numbered players select Y, all others select X.

Any strategy profile of this form describes a pure Nash equilibrium of the game since the payoff to every player will be 1, and the game offers no utilities greater than 1. Denoting a_i the action of player $i, 1 \leq i \leq n-1$, we have that $a_i \neq a_{i+1}$, and for the edge case, since n is even, but 1 is odd, player n and 1 will also always select different actions, ensuring utilities of 1 for them as well.

Part (b): Let n be odd. Find a pure Nash equilibrium or explain why none exist.

There are no pure Nash equilibria to the game in the case that n is odd. It is clear that the strategy profile from part (a) will not work if n is odd, since then players 1 and n will choose the same action.

Thus if we suppose there is some strategy profile for the game which is also a Nash equilibrium, then it must be the case that the strategy involves at least two adjacent players choosing the same action. But this obviously leads to a contradiction, since the leftmost player in this block could improve her utility from 0 to 1 by switching her action to be different from the player to her right.

Part (c): Find a completely mixed Nash equilibrium for those values of n for which no pure one exists. What is the expected utility to each player?

The completely mixed Nash equilibrium for the case in which n is odd is for every player to select X with probability $\frac{1}{2}$ and Y with probability $\frac{1}{2}$. Then the utility to player $i, 1 \leq i \leq n$, is

$$E(u_i|p_X, q_X) = p_X(1-q_X) + (1-p_X)q_X = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

for p_X the probability that player i selects X and q_X the probability that the player to her right selects X. We can confirm that this is indeed a mixed equilibrium, since

$$\frac{dE(u_i|p_X, q_X)}{dp_X} = 1 - 2q_X \implies q_X = \frac{1}{2}$$

is the probability such that player i's choice of p_X will not affect her expected utility. Thus $p_X = q_X = \frac{1}{2}$ is indeed a completely mixed Nash equilibrium of the game. (It is completely mixed since $p_X = 1 - p_X > 0$.)

Part (d): For those values of n for which no pure Nash equilibria exist, find a correlated equilibrium in which the expected utility to every player is $1 - 1/n$.

Suppose we randomly select one of the n players to be the "loser," and all other players to be winners. We then send a signal to the designated loser that they are to choose X. Then, ignoring the designated loser, there are an even number of players remaining, so the player to the immediate right of the designated loser can choose X and all players thereafter can alternate as in part (a), ensuring utilities of 1 for all but the loser.

In this variant of the game, a correlated equilibrium will be that in which the designated loser obeys the signal all the time and chooses X, and all other players follow the instructions above as well. Then the expected utility of any player i will be

$$E(u_i) = P(\text{being the loser})(0) + P(\text{being a winner})(1) = \frac{n-1}{n} = 1 - \frac{1}{n}.$$

Thus since the probability of being selected the loser is only $\frac{1}{n}$, the expected utility of each player is now $1 - \frac{1}{n}$, which will exceed $\frac{1}{2}$ for all $n > 2$.

2 Problem 2

Consider a game in which two players sit across a table from one another, and a single die is rolled between them. The second player can always see the result of the roll, but the first player is only shown the result with probability $\frac{1}{2}$.

Now suppose that two players play this very exciting game regularly, and the second player has learned that the first player will only raise her eyebrows if she sees the result of the roll. Then suppose the die is rolled, the first player sees the result, and she raises her eyebrows. Then the first player knows that the second players knows the result, because the second player always does; also, the second player sees that the first raised her eyebrows, so the second knows that the first player knows the result; however, the first player, not realizing this trait of hers, will not know that the second player knows she knows.

Formally, in this case, we have that our knowledge space is given by $(N, \Omega, \{T_i\}_i, \{t_i\}_i)$, for $N = \{1, 2\}$, $\Omega = \{0, 1\}$, for 0 the state in which the first player raises her eyebrows and 1 the state in which she does not, $T_1 = \{0, 1\}$, for the cases in which the first player knows the result (1) and in which she does not (0), $T_2 = \{2\}$ the single state in which the second player knows the result, and $t_1 = \omega \in \Omega, t_2 = 2$.