Ma 6b, Problem Set 3

Ethan Wilk

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1 Problem 1

Lemma 1.1. *If $G = (V, E)$ is a graph such that $d(v) \geq n - 1$ for all $v \in V$, then G contains every tree on n vertices.*

Proof. We will prove the lemma by induction on n. For $n = 1, 2$, the result is trivial: Any graph where all vertices have degree at least 0 will have the tree of 1 vertex; similarly, any graph where all vertices have degree at most 1 will have the tree of 2 vertices. We now consider the cases for $n \geq 3$.

Let $G = (V, E)$ be our graph such that $d(v) \geq n-1$ for all $v \in V$, and let T_n be any tree of n vertices with w some endpoint of T_n. Then, from the induction hypothesis, $G - w$ contains a copy of $T_n - w$. The tree $T_n - w$ must have $n-2$ edges and $n-1$ vertices (since all vertices other than the root have indegree 1). But because $d(v) \geq n-1, v \in V$, every vertex in $T_n - w$ must have at least one neighbor not in $T_n - w$, and so indeed G must contain T_n. □

Theorem 1. *For P_t the path with t vertices, we have that*

$$r(K_s, P_t) = (s-1)(t-1) + 1$$

for all natural numbers $s, t \geq 1$.

Proof. We will prove the upper bound of the theorem by showing that $r(K_s, T_t) = (s-1)(t-1)+1$, for T_t a tree on t vertices. Since every (Hamiltonian) path on t vertices is only a particular instance of a tree on t vertices, we have that this necessarily implies $r(K_s, P_t) = (s-1)(t-1)+1$ for any path P_t on t vertices. Let $G = (V, E)$ be our graph.

The lower bound $r(K_s, P_t) \geq (s-1)(t-1)+1$ is trivial to confirm. Take the complete graph $K_{(s-1)(t-1)}$ and split its vertices into $t-1$ clusters $V_1, ..., V_{t-1}$ of $s-1$ vertices each. Denote the ith vertex of V_j by $v_i^{(j)}$. Color the edges from $(v_i^{(j)}, v_i^{(k)})$ blue for all $1 \leq i \leq s-1, 1 \leq j \neq k \leq t-1$. Color all other edges red. Then we will obviously not have a red copy of K_s or a blue copy of P_t, so $r(K_s, P_t) \geq (s-1)(t-1)+1$.

We will now show that $r(K_s, P_t) \leq (s-1)(t-1)+1$. Note that $r(K_s, P_t)$ can be viewed equivalently as the minimum n such that any graph on n vertices has that either $K_s \subset G$ or $P_t \subset G'$ for $G' = (V', E')$ the complement of G (that is, the graph such that $V' = V, e \in E' \iff e \notin E$).

Suppose G is a graph on $(s-1)(t-1)+1$ points. Let $G' = (V', E')$ be the complement of G. Then if G' does not contain a copy of K_s, G' must have chromatic number $\chi(G') \leq s-1$, so $\chi(G) \geq t - 1 + \frac{1}{s-1}$, or $\chi(G) \geq t$. But if any coloring of the vertices of G requires at least t colors so that no two adjacent vertices are of the same color, then there must exist some subgraph H of G with minimum degree $t-1$. By Lemma 1.1, H (and thus G) must contain every tree T_t on t vertices, completing the proof. □

2 Problem 2

Lemma 2.1. *Let $G = (V, E)$ be a graph with $u, v \in V$ two non-adjacent vertices. If there exists some 2-coloring of G such that neither $G - v$ nor $G - u$ contain a monochromatic copy of $K_s, s \in \mathbb{N}$, then G does not either.*

Proof. Suppose there exists a single 2-coloring C of G such that neither $G - v$ nor $G - u$ contain a monochromatic copy of K_s.

Note that since K_s is a complete graph on s vertices, every vertex in K_s is connected to every other; i.e., $w, x \in V(K_s) \implies e = (w, x) \in E(K_s)$, for $V(K_s)$ the vertex set of K_s and $E(K_s)$ the edge set. Now, if the coloring C on G contains a monochromatic copy of K_s, then either this copy includes u (but not v), v (but not u), or neither u nor v. It cannot contain both because u and v are non-adjacent, and in K_s, every vertex is adjacent to every other.

But then, in each of three cases enumerated above, the monochromatic copy of K_s must be contained in one of $G - v$ or $G - u$, since the copy cannot include both u and v. It follows that if neither $G - v$ nor $G - u$ contain a monochromatic copy of K_s under the coloring C, G cannot contain one either. □

Theorem 2. *A graph with the property that every two-coloring of its edges contains a monochromatic K_s must have at least $\binom{r(s)}{2}$ edges.*

Proof. Suppose $G = (V, E)$ is a graph with fewer than $\binom{r(s)}{2}$ edges. We will show that there exists a 2-coloring of G such that, under this coloring, neither $G - u$ nor $G - v$ contain a monochromatic copy of K_s. The theorem will then follow from the lemma above.

Clearly, we must require that $|V| \geq r(s)$, since $r(s)$ is the minimum $|V|$ such that any coloring on a *complete* graph of $r(s)$ vertices will have a monochromatic copy of K_s. Suppression of any vertex will thus eliminate a critical edge or vertex in the monochromatic copy of K_s for at least one coloring of $K_{r(s)}$, so indeed, $|V| \not< r(s)$. We therefore assume that G has at least $r(s)$ vertices.

We will now proceed by induction on the number of vertices $n = |V|$. Suppose the theorem holds for some $n \geq r(s)$ across all $|E| = m < \binom{r(s)}{2}$. We now consider a graph G with $n + 1$ vertices and $m < \binom{r(s)}{2}$ edges. Let $u, v \in V$ be two non-adjacent vertices in G. (Since G has fewer than $\binom{r(s)}{2}$ edges yet more than $r(s)$ vertices, G is not complete, so such vertices must exist.) We now must show that there exists a 2-coloring of G meeting the requirements given above.

Let $N(u), N(v)$ be the sets of neighbors of u and v, respectively. Add all edges from u to $w \in N(v)$ to $G - v$ and call this graph H_u. Similarly, add all edges from v to $w \in N(u)$ to $G - u$ and call this graph H_v. Note that then H_u and H_v are topologically equivalent, since both will have the same number of vertices with equivalent numbers of edges between respective vertices. (More formally, we can relabel the vertices of one or both graphs such that $e = (a, b) \in E(H_u) \iff e = (a, b) \in E(H_v)$.) Then H_u and H_v are graphs on n vertices, both with fewer than $\binom{r(s)}{2}$ edges, so by the induction hypothesis, it must be the case that there exist 2-colorings of H_u and H_v such that the two graphs do not contain monochromatic copies of K_s.

In fact, since H_u and H_v are, as noted, topologically equivalent, it must be the case that there is a single (equivalent, since vertices may be relabeled as noted above) 2-coloring for both H_u and H_v such that the two graphs do not contain monochromatic copies of K_s. It then becomes easy to apply this coloring, delete the edges that were added to $G-v, G-u$ in forming H_u, H_v, respectively, and conclude that neither $G - u$ nor $G - v$ can contain a monochromatic copy of K_s under this coloring. By Lemma 2.1, this completes the proof. □

3 Problem 3

Lemma 3.1. *If all triples in a set of points $X \subset \mathbb{R}^2$ have the same orientation, then the set is in convex position.*

Proof. Suppose, WLOG, all triples $(x_i, x_j, x_k), i < j < k$ in $X = \{x_1, ..., x_n\}$ are oriented in the counterclockwise direction. Then it is the case that, for any particular triple $(x_i, x_{i+1}, x_{i+2}), 1 \leq i \leq n-2$, the path from x_i to x_{i+1} to x_{i+2} goes in the counterclockwise direction.

Starting at $i = 1$ and ending at $i = n - 2$, then, we may traverse across all such triples in X to create a path oriented entirely in the counterclockwise direction. It is easy to see that this path will in fact be the convex hull for X. (To handle the final case, observe that (x_1, x_{n-1}, x_n) must be oriented in the counterclockwise direction, so we may simply complete the hull by drawing a final counterclockwise path from x_n to x_1.) □

Theorem 3 (Erdös-Szekeres)**.** *For any $n \in \mathbb{N}$, there exists N such that any set of N points in general position in the plane contains a subset of size n in convex position, that is, a convex $n-$gon.*

Proof. We will show that, for any $n \in \mathbb{N}$, there exists N so that any set of N points in general position in the plane has a subset X of size n in which all triples in X have the same orientation. For $n = 1, 2$, it is trivial to confirm the theorem (the desired convex subsets are a singleton and a set of points in the plane). We thus assume $n \geq 3$.

From Ramsey's theorem, we know that for any natural numbers $m, l \geq k \geq 2$, there exists some N^* such that every red/blue-coloring of the edge set of the $k-$uniform complete hypergraph $K_{N^*}^{(k)}$ contains either a red copy of $K_m^{(k)}$ or a blue copy of $K_l^{(k)}$. Let $m, l = n, k = 3$, so any red/blue-coloring of the edge set of $K_{N^*}^{(3)}$ contains a monochromatic copy of $K_n^{(3)}$.

Create the set X by taking any N^* points in the plane, and color triples of X blue if they are oriented in the counterclockwise direction and red otherwise. Then, we will either have a blue copy or a red copy of $K_n^{(3)}$, that is, a subset of n points in which all triples of points are of the same orientation. By Lemma 3.1, this subset is convex, and so the proof is complete. □

4 Problem 4

Theorem 4. *For any n, there exists N such that any collection of N points in the plane will contain either a convex $n-$gon or n points on a line.*

Proof. From the Erdös-Szekeres theorem, we know that for any natural number n, there exists N^* such that any set of N^* points in general position in the plane will contain a convex $n-$gon. Thus if we take $N \geq N^*$ points in general position in the plane, we are guaranteed to find a convex $n-$gon. We must now consider the case when the N points are not in general position.

From Ramsey's theorem for complete, $k-$uniform hypergraphs, we know that for any natural numbers $m, l \geq k \geq 2$, there exists $R(m, l, k) = R$ such that every red/blue coloring of the edge set of $K_R^{(k)}$ contains either a red copy of $K_m^{(k)}$ or a blue copy of $K_l^{(k)}$.

Take $m = N^*, l = n$, and $k = 3$, and set $N \geq \max\{R(N^*, n, 3), N^*\}$. Then we have that any 2-coloring of $K_N^{(3)}$ will contain either a monochromatic copy of $K_{N^*}^{(3)}$ or $K_n^{(3)}$. In particular, we may color the edges (i.e., the subsets of the vertex set $[N]$, each of size 3) red if the 3 points forming the edge lie on the same line in the plane and blue otherwise.

Then, if we get a red copy of $K_n^{(3)}$, it must be the case that all n points in that copy are collinear, since all triples forming it were colored red, meaning any set of 3 points in it must lie on the same line. On the other hand, if we get a blue copy of $K_{N^*}^{(3)}$, then it must be the case that we have a convex n-gon, since no three of the points in this copy will be collinear (none of the triples are red), and so we may easily apply the Erdös-Szekeres theorem as noted above. Thus taking any $N \geq \max\{R(N^*, n, 3), N^*\}$ points in the plane, we are guaranteed to have either n points on a line or a convex $n-$gon. □

5 Problem 5

Theorem 5. *Any graph with $2m+x, x \in \{0, 1\}$ vertices and at least $m(m+x)+1$ edges contains at least m triangles.*

Proof. Let $G = (V, E)$ be any graph with m vertices and at least $m(m+x)+1$ edges. We will prove the theorem by induction on the number of vertices in G.

For $m = 1$ the theorem clearly holds, since any graph with $1, 2$ vertices will not have any triangles. Assume the theorem holds for some $m - 1 > 0$ so that we must show that the theorem holds for m.

We claim that there must exist a vertex in G of degree at most m. If this is not the case, then we have that

$$(m+1)|V| = (m+1)(2m+x) \leq \sum_{v \in V} d(v) = 2|E| = 2m(m+x)+2$$

$$\implies 2m^2 + xm + 2m + x \leq 2m^2 + 2mx + 2$$
$$\implies 2m \leq x(m-1) + 2, m \leq 1,$$

a contradiction. Label v a vertex in G with $d(v) \leq m$. Then consider $G' = G - v$.

If G was a graph with $2m+1$ vertices ($x = 1$), then G' will have $2m$ vertices. Furthermore, G' will have $m(m+1)+1-d(v) \geq m(m+1)+1-m = m^2+1$ edges. We may thus delete edges from G' until it has $m(m-1)+1$ edges. We may then apply the induction hypothesis on this graph to assert that it will at least m triangles. Adding the vertex and other edges back to form G will not alter this property, so we are done.

Now consider the case that G was a graph with $2m$ vertices ($x = 0$). Then either $d(v) < m$ or $d(v) = m$. If $d(v) < m$, then G' has $m^2+1-d(v) > m^2+1-m$ edges. Then we can delete edges in G' until we have a graph with $m(m-1)+1$ edges but still $m-1$ vertices. We may also choose at least one deleted edge to be from a triangle T in G'. Then when we apply the induction hypothesis to get at least $m-1$ triangles in the subgraph H of $m-1$ vertices and $m(m-1)+1$ edges, we will also get the extra triangle T whose edge was deleted in the creation of H, and so G must have at least m triangles.

Finally, we consider the same case as in the above paragraph but with $d(v) = m$. Then G' has $2m-1$ vertices and $m(m-1)+1$ edges. Let W be the set of neighbors of v in G' and let Z be the remaining $n-1$ vertices in G', so $Z = G' \backslash W$.

If there exist $w, z \in W$ that are adjacent then clearly $\{w, z, v\}$ will form a triangle in G that is not in G', so we may conclude that G has at least m triangles. Otherwise, all vertices in W must have at most $n-1$ neighbors in G'. If there exists a $y \in W$ with less than or equal to $n-2$ neighbors in G', then y has less than or equal to $n-1$ vertices in G, so we can easily use y instead of v to reapply the reasoning for the $d(v) < m$ case.

We thus assume that every vertex $y \in W$ has $m-1$ neighbors in G'. Then W, Z and the edges between them will form a copy of $K_{m,m-1}$, yielding $m(m-1)$ edges of G'. Then there must remain some edge $e = (x, z)$ in Z. But then we can form a triangle between $\{x, y, z\}$ for every $y \in W$, thus yielding n triangles in G and completing our proof. □

Corollary 5.1. *Any graph with n vertices and at least $\lfloor \frac{n^2}{4} \rfloor + 1$ edges contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.*

Proof. Apply Theorem 5 with $n = 2m+x$, noting that $m(m+x) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1 = \lfloor \frac{n^2}{4} \rfloor + 1$. □

Remark. *The above inequality is sharp for $n \geq 3$.*

Suppose we have a graph on $n \geq 3$ vertices with $\lfloor \frac{n^2}{4} \rfloor$ edges. In line with the maximal case for Mantel's theorem, first partition the vertices into two disjoint sets V_1, V_2 of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$, respectively. Form the complete bipartite graph between these two sets, so we have $\lfloor \frac{n^2}{4} \rfloor$ edges and no triangles. Now add a single edge between two vertices in V_2. Since those vertices will both connect to every vertex in V_1, we will now have $|V_1| = \lfloor \frac{n}{2} \rfloor$ triangles in this graph.

6 Problem 6

Lemma 6.1. *Let G be a non-bipartite, triangle-free graph of n vertices, and let C be the shortest cycle of odd length in G. Then every vertex in $G \backslash C$ will have at most 2 neighbors in C.*

Proof. We first note that from PS 1, Problem 1, we know that there will be at least one cycle of odd length in any non-bipartite graph G. Let C_k to be the smallest of these cycles so C_k is of length k.

We will prove the claim by contradiction: Suppose there exists $v \in G \backslash C_k = V$ such that v has 3 neighbors in C_k (we will easily confirm the case for v having greater than 3 neighbors after this initial proof). Then the neighbors of v in C_k split C_k into 3 paths P_1, P_2, P_3. Note that all 3 paths must be of length greater than 1 (to maintain that G is triangle-free) and that the paths together must contain all k edges in C_k.

Since k must be odd, we know that the length l of one of the paths (say, WLOG, P_1) must be odd and less than $k-2$. But then we may join P_1 with the two edges to v, yielding an odd cycle of length $l+2 < k$, a contradiction. □

Theorem 6. *The number of edges in a triangle-free non-bipartite graph with n vertices is at most*

$$\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil + 1.$$

Proof. Let G be any graph that is triangle-free and not bipartite. From PS 1, Problem 1, we know that G must have a cycle of odd length. Let C_k be the shortest cycle of odd length in G, and suppose it is of length k.

We will prove the theorem by induction on the number of vertices $n \geq 5$ in G. (For $n = 1, 2, 3, 4$, we require that either the graph have a triangle or be bipartite.) For $n = 5$, the inequality is exact: A graph with 5 vertices that is non-bipartite and triangle-free must have at most a cycle of length 5 and so it contains $\left\lfloor \frac{5-1}{2} \right\rfloor \left\lceil \frac{5-1}{2} \right\rceil + 1 = 5$ edges. We will now consider $n \geq 6$.

Let V be the set of vertices in G but not in C_k. By Lemma 6.1, we know that every $v \in V$ has at most 2 neighbors in C_k. Thus we know that the number of edges between V and C_k is at most $2|V|$.

Furthermore, from Mantel's Theorem, we know the number of edges between vertices in V is at most $\lfloor \frac{|V|^2}{4} \rfloor$, since any subgraph of G must also be triangle-free. Finally, we know that C_k must have exactly k edges, since no two vertices in C_k can share an edge unless it is in the cycle path itself (since otherwise there would be a shorter cycle of odd length on one of the two parts of C_k separated by the extra edge between the two vertices in C_k). We then have that the number of edges in G will be at most

$$\left\lfloor \frac{|V|^2}{4} \right\rfloor + 2|V| + k = \left\lfloor \frac{(n-k)^2}{4} \right\rfloor + 2(n-k) + k$$

$$\leq \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 2(n-5) + 5 = \left\lfloor \frac{n^2 - 2n + 4 + 1}{4} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} + 1 \right\rfloor$$

$$= \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1 = \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil + 1,$$

where the first inequality follows from the fact that we have considered $n \geq 6 \implies k \geq 5$.

□

7 Problem 7

Theorem 7. *Let S be a set of diameter 1 in the plane. The number $n = |S|$ of points in S whose distance is greater than $\frac{1}{\sqrt{2}}$ is at most $\lfloor \frac{n^2}{3} \rfloor$.*

Proof. Let $S = \{v_1, v_2, ..., v_n\} \subset \mathbb{R}^2$ be a set of points in a circle of diameter 1 in $\mathbb{R}^2$. Construct the graph $G = (V, E)$ by taking $V = S$ and adding an edge $e = (v_i, v_j)$ if the distance between v_i and v_j is greater than $\frac{1}{\sqrt{2}}$.

From Turán's theorem, we know that if $|E| > \lfloor \frac{n^2}{3} \rfloor$, then G must contain a copy of K_4. To complete the proof, we will now show that it is not possible for G to contain a copy of K_4, so $|E| \leq \lfloor \frac{n^2}{3} \rfloor$.

Suppose by contradiction that G does contain a copy of K_4. Then it must be the case that we have a subset $H = \{v, w, x, y\} \subset S$ in which all $\binom{4}{2} = 6$ pairs of points in H are a distance greater than $\frac{1}{\sqrt{2}}$ apart. However, observe that if we place any 4 points in $\mathbb{R}^2$ at the corners of a square of side length $\frac{1}{\sqrt{2}}$, then the diagonals of that square have length exactly equal to 1. Any enlargement of the side lengths of this square would result in an enlargement of at least one of these diagonal lengths to be greater than 1.[1] But since S is assumed to have diameter 1, this cannot occur in S, and thus such a set H cannot exist. □

Remark. Theorem 7 is sharp for $n \geq 2$.

[1] Note that this observation follows from the law of cosines, i.e. $c = \sqrt{a^2 + b^2 - 2ab\cos(\theta)}$, for c the diagonal of the triangle ABC with side lengths a, b, c, and θ the angle between A and B. If a or b is increased in this triangle, clearly c will increase, since θ will remain constant.

Observe that any $n \geq 2$ can be expressed as either $3k, 3k+1$, or $3k-1$, for some $k \in \{1, 2, 3, ...\}$.

Suppose $n = 3k, k \in \mathbb{N}$. Partition the n points so that groups of k points are placed arbitrarily close to the vertices of a triangle with side lengths $\frac{\sqrt{3}}{2}$. Then since $\frac{\sqrt{3}}{2} > \frac{1}{\sqrt{2}}$, all the points in any group will be greater than $\frac{1}{\sqrt{2}}$ from the points in any other, so we have k^3 pairs of points that are greater than $\frac{1}{\sqrt{2}}$ apart. Note that

$$\left\lfloor \frac{n^2}{3} \right\rfloor = 3k^2,$$

so we are sharp.

Similarly, if $n = 3k+1$, place $k+1$ points at one vertex and k at the two others. Then we have $k(k+1) + k(k+1) + k^2 = 3k^2 + 2k$ pairs of points greater than $\frac{1}{\sqrt{2}}$ apart, and

$$\left\lfloor \frac{n^2}{3} \right\rfloor = \left\lfloor \frac{9k^2 + 6k + 1}{3} \right\rfloor = 3k^2 + 2k.$$

Finally, if $n = 3k-1$, place $k-1$ points at one vertex and k at the two others. Then we have $k(k-1) + k(k-1) + k^2 = 3k^2 - 2k$ pairs of points greater than $\frac{1}{\sqrt{2}}$ apart, and

$$\left\lfloor \frac{n^2}{3} \right\rfloor = \left\lfloor \frac{9k^2 - 6k + 1}{3} \right\rfloor = 3k^2 - 2k.$$

8 Problem 8

Lemma 8.1. *If $G = (V, E)$ is a $d-$regular graph with adjacency matrix A, then $d \in spec(A)$.*

Proof. Assume $A \in \mathcal{M}^{n \times n}$. Since A is $d-$regular, we have that the sum of every row in A will be d. Thus the sum of every row in $A - dI$ will be 0. But then $0 \in \text{spec}(A - dI)$ with eigenvector $(k, ..., k)^\top, k \in \mathbb{R}$, so $A - dI$ is singular and thus $\det(A - dI) = 0$ as required. $\square$

Lemma 8.2. *If $G = (V, E)$ is a bipartite graph with adjacency matrix A, then $\lambda \in spec(A) \implies -\lambda \in spec(A)$.*

Proof. Since G is bipartite, observe that we may express its adjacency matrix in block form as

$$A = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix},$$

for B some adjacency matrix. Let $\lambda \in \text{spec}(A)$ with eigenvector $v = (x, y)^\top$. Then

$$\begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix},$$

and so

$$By = \lambda x, B^\top x = \lambda y.$$

We then have that

$$\begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -By \\ B^\top x \end{pmatrix} = -\lambda \begin{pmatrix} x \\ -y \end{pmatrix},$$

completing the proof. □

Theorem 8. *If G is a d-regular graph, then $-d$ is an eigenvalue of G if and only if G is bipartite.*

Proof. Let $G = (V, E)$ be a d-regular graph with adjacency matrix A. By Lemma 8.1, d is an eigenvalue of G.

($\Longrightarrow$) Suppose $-d$ is an eigenvalue of A with associated eigenvector $(v_1, ..., v_n)$. Then for all $i \in V$,

$$-dv_i = \sum_{j \in N(i)} v_j,$$

for $N(i)$ the neighbor set of $i \in V$. Denote the maximum magnitude of the components of the eigenvector by $M = \max(|v_1|, ..., |v_n|)$, and let $P = \{i : v_i = M\}, Q = \{i : v_i = -M\}$. Then we have that $\forall\ p \in P$,

$$-dv_p = -dM = \sum_{l \in N(p)} v_l.$$

But, since $v_l \geq -M$ for all $1 \leq l \leq n$ (and thus $\sum_{l \in N(p)} v_l \geq -dM$), we may only have that

$$\sum_{l \in N(p)} v_l = -dM$$

if $v_l = -M$ for all $l \in N(p)$. In other words, we require that every neighbor l of p is an element of Q. Similarly, for all $q \in Q$, we require that all neighbors $w \in N(q)$ are also elements of P. Thus we have created two vertex sets P, Q which form a bipartite connected component: Any edge in $P \cup Q$ must be between one vertex in P and one in Q. It follows that, assuming G is indeed connected, G must be bipartite.

($\Longleftarrow$) Suppose G is bipartite. Then by Lemma 8.2, since d is an eigenvalue of A, $-d$ is as well. □

9 Problem 9

Theorem 9. *The number of triangles in a graph $G = (V, E)$ with adjacency matrix A is given by*

$$|T| = \frac{1}{6} \sum_i \lambda_i^3,$$

for $\{\lambda_i\}_i$ the eigenvalues of A and T the set of triangles in G.

Proof. The proof follows from elementary linear algebra. Assume $A \in \mathcal{M}^{n\times n}$. We know that $(A^3)_{ij}, 1 \leq i,j \leq n$ will be nonzero if and only if there exists a path of length 3 (formed from 3 edges) from vertex i to j.

Then $(A^3)_{ii} \neq 0 \iff$ there exists a cycle of length 3 starting (and ending at) vertex i, so $\mathrm{tr}(A^3) = \sum_{i=1}^n (A^3)_{ii}$ is the total number of cycles of length 3 in G. Since $\lambda \in \mathrm{spec}(A) \iff \lambda^3 \in \mathrm{spec}(A^3)$, we have that

$$\mathrm{tr}(A^3) = \sum_i \lambda_i^3,$$

for $\{\lambda_i\}_i$ the eigenvalues of A.

Note, however, that we do not solely seek the number of cycles of length 3 in G: This overcounts the number of triangles. Specifically, since the graph is undirected, the trace of A^3 will double-count triangles starting at i by including both the path from, say, i to j to k as well as the path from i to k to j, which form the same triangle. Furthermore, the trace will triple-count by including each of the cycles starting at i, j, and k in the triangle (i,j,k), so in total, the trace overcounts by a factor of $3 \cdot 2 = 6$. We therefore have that, for T the set of unique triangles in G,

$$|T| = \frac{1}{6}\mathrm{tr}(A^3) = \frac{1}{6}\sum_i \lambda_i^3.$$

□

10 Problem 10

Lemma 10.1 (C_4 property)**.** *A friendship graph has no cycle of length 4. Moreover, the shortest path length between any 2 vertices in a friendship graph is at most 2.*

Proof. Let $G = (V,E)$ be a friendship graph (i.e., a graph in which every pair of vertices has exactly one common neighbor). Suppose by contradiction that G has a cycle $C = (v,w,x,y,v), v,w,x,y \in V$ of length 4. Then v and x have 2 common neighbors, a contradiction.

Furthermore, the length of the shortest path between any two vertices $v, w \in V$ must be 2, since if it were any longer, v and w would not have any common neighbors. □

Lemma 10.2. *If a friendship graph $G = (V,E)$ on $|V| = n$ vertices has that, for all $v \in V$, $d(v) < n-1$, for $d(v)$ the degree of vertex v, then G is regular. Moreover, we must have that $n = k^2 - k + 1$.*

Proof. Take $v, w \in V$ two non-adjacent vertices with $d(v) = k$, for $d(v)$ the degree of v. Suppose $N(v) = \{x_1, ..., x_k\}$ is the neighbor set of v. There is exactly one $1 \leq i \leq k$ such that $x_i \in N(w)$. Furthermore, x_i and v must have exactly one common neighbor $x_j \in N(v), 1 \leq j \leq k$.

The vertex w will also have a single common neighbor y_l with every $x_l \in N(v)$. Note that each element in this set $\{y_l\}_l$ of common neighbors must be distinct, since otherwise we would create a cycle $y_p \to x_p \to v \to x_m \to y_p$ of length 4, for $y_m = y_p, p \neq m$. This would violate the C_4 property of the friendship graph (Lemma 10.1). Thus, we may assert that $d(w) \geq k = d(v)$. However, since we may apply the exact same argument after swapping v and w, we have that $d(v) = k \geq d(w) \implies d(v) = k = d(w)$.

Then, by construction, since x_i is the only shared neighbor between v and w, any other vertex in G will be adjacent to (at most) one of v or w, and so since the degrees of all vertices are equal, every vertex must have degree $d(v) = k$. This proves that G is k-regular. To count its vertices, we note that v has k neighbors, each of degree k, so we start with k^2 vertices. We have k-counted v across the neighbors of v, so we subtract the $k-1$ overcounts to arrive at $k^2 - k + 1$ total vertices in G. □

Theorem 10 (Friendship theorem)**.** *If every pair of people in a group have exactly one friend in common, then there is someone in the group who is friends with everyone.*

Proof. We wish to prove that any friendship graph (that is, a graph $G = (V, E), |V| = n$ such that every pair $v, w \in V$ has exactly one common neighbor) has a vertex with degree $n-1$. Suppose by contradiction that $G = (V, E)$ is a friendship graph with all vertices of degree less than $n-1$.

By Lemma 10.2, G is k-regular and $|V| = n = k^2 - k + 1$. Note that we may assume $k > 2$, since if $k = 1, 2$, the resultant graphs will be a single vertex and a triangle, respectively. Let A be the adjacency matrix of G. Then the sum of each row in A will be k, and for every pair of rows, there will be exactly one column in which the rows share a 1 (since every pair of vertices will have exactly one common neighbor). It is easy to see that

$$A^2 = \begin{pmatrix} k & 1 & \dots & 1 \\ 1 & k & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & k \end{pmatrix},$$

so the eigenvalues of A^2 are $k^2, k-1$, and those of A are $k, \pm\sqrt{k-1}$. The multiplicity of the eigenvalue k is 1, so we must have that $\sqrt{k-1}$ and $-\sqrt{k-1}$ have multiplicities $r, s, r+s = n-1, r \neq s$, respectively. Then

$$\operatorname{tr}(A) = 0 = k + r\sqrt{k-1} - s\sqrt{k-1}$$

$$\implies (r-s)^2(k-1) = k^2.$$

But then we must have that $k - 1 | k^2 \iff k = 2$, a contradiction. Thus G must have a vertex of degree $n-1$.

□