Ma 6b, Problem Set 2

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1 Problem 1

Theorem 1. *If $G = (V, E)$ is a k-connected graph with $k \geq 2$, then every set of k vertices in G lies on a cycle.*

Proof. Denote our arbitrary set of k vertices by $V_k = \{v_1, ..., v_k\} \subset V, k \geq 2$, and take C a cycle in G containing as many of these vertices as possible.

Of course, if C contains all of V_k, then we are done. Otherwise, it must be the case that, without loss of generality (we can always relabel the vertices), C contains $v_1, ..., v_n, n < k$. Then we must show that there exists an algorithm to find a cycle containing $v_1, ..., v_n, v_{n+1}$, so that one may iterate it until all missing vertices are in the cycle.

We provide that algorithm now. Since G is k-connected, we know from Menger's Theorem that every pair of vertices in G must have at least k vertex-disjoint paths between them. Therefore, we may find at least k vertex-disjoint paths from v_{n+1} to endpoints $w_1, ..., w_k$ in C.

Then, choose the set of k paths from v_{n+1} to C that minimizes the number of vertices traversed. Then the endpoints $w_1, ..., w_k$ of these paths will divide C into k segments. But then either one of these segments will be the null set (it will consist of a single edge) or, if all segments contain some vertices, then one of the segments will have none of its vertices in V_k, since $n < k$, but there are in fact k segments.

If one of the segments, say $(w_i, w_j), 0 \leq i \leq j \leq k-1$ consists of a single edge (so, WLOG, $j = i+1$), then we may simply replace the edge (w_i, w_j) with the paths to v_{n+1}, those being $(v_{n+1}, w_i), (v_{n+1}, w_{i+1})$. Then we have a cycle C containing $v_1, ..., v_{n+1}$.

On the other hand, if all segments include vertices in C, then as stated above, it must be the case that one of these segments, say $(w_i, w_j), 0 \leq i \leq j \leq k-1$, does not contain any vertices in V_k. Then simply replace (w_i, w_j) by the paths (w_i, v_{n+1}) and (v_{n+1}, w_{i+1}) and we will again have a cycle containing all of $v_1, ..., v_{n+1}$. We may then iterate this algorithm until all of V_k is in C. □

2 Problem 2

Theorem 2. *Every graph can be embedded in $\mathbb{R}^3$ with all edges straight.*

Proof. We will be able to embed a graph $G = (V, E)$ in $\mathbb{R}^3$ if no four of the mapped vertices are coplanar in $\mathbb{R}^3$; if three are, we know K_3 is planar, so no edges will cross each other except at the vertices. It follows that K_4 is the critical case.

We will therefore show that there exists a function $f : \mathbb{R} \to \mathbb{R}^3$ so that we may map any set of n points from $\mathbb{R}$ to $\mathbb{R}^3$ with no four distinct points in the image of the function lying on the same plane.

Consider the function $f(t) = (t, t^2, t^3)$. Then take any 4 points $w, x, y, z \in \mathbb{R}$ and observe that

$$\begin{pmatrix} f(w) & 1 \\ f(x) & 1 \\ f(y) & 1 \\ f(z) & 1 \end{pmatrix} = \begin{pmatrix} w & w^2 & w^3 & 1 \\ x & x^2 & x^3 & 1 \\ y & y^2 & y^3 & 1 \\ z & z^2 & z^3 & 1 \end{pmatrix},$$

where

$$\det \begin{pmatrix} w & w^2 & w^3 & 1 \\ x & x^2 & x^3 & 1 \\ y & y^2 & y^3 & 1 \\ z & z^2 & z^3 & 1 \end{pmatrix} \neq 0.$$

Therefore, it must be the case that $f(w), f(x), f(y)$, and $f(z)$ are not coplanar; there cannot exist a plane in $\mathbb{R}^3$ containing all of them. Of course, since these points were chosen arbitrarily in $\mathbb{R}$, we have that any four distinct points generated by $f(t)$ will not lie on the same plane in $\mathbb{R}^3$.

Then we can take each vertex v_i in any graph $G = (V, E)$ and map $v_i \to f(i), 1 \leq i \leq |V|$, creating straight edges between $f(j), f(k), 1 \leq j < k \leq |V|$ if and only if there exists an edge $e = (v_j, v_k) \in E(G)$. Since we know no four distinct vertices will then map to points in $\mathbb{R}^3$ on the same plane, we can be certain that none of these edges will cross except at the vertices, and thus we will have an embedding in $\mathbb{R}^3$ with straight edges only. □

3 Problem 3

Theorem 3. *If $G = (V, E)$ is a planar graph with finite girth g, then*

$$|E| \leq \frac{g}{g-2}(|V| - 2).$$

Proof. Let $G = (V, E)$ be a planar graph with finite girth g and face set F. Then the length of the shortest cycle in G is g, and for every $f \in F$, the length $l(f)$ of f (the number of sides in f) is bounded below by g. If this were not the case, then $\exists\ f \in F$ such that $l(f) < g$, but then f has fewer than g edges enclosing it, and so we may easily take those edges to form a cycle with fewer edges than g, a contradiction.

Now since $l(f) \geq g$ for all $f \in F$,

$$\sum_{f\in F} l(f) \geq \sum_{f\in F} g = g|F|.$$

From lecture, we know that since each edge will be a side for two faces,

$$\sum_{f\in F} l(f) = 2|E| \geq g|F|.$$

Then, by Euler's formula, $|V| - |E| + |F| = 2$, so

$$\begin{aligned} |F| = 2 - |V| + |E|, 2|E| &\geq g|F| \\ \implies 2|E| &\geq g(2 - |V| + |E|) \\ \implies (|V| - 2)g &\geq |E|(g-2) \\ \implies (|V| - 2)\frac{g}{g-2} &\geq |E|, \end{aligned}$$

as desired. □

Corollary 1. *$K_{3,3}$ is not planar.*

Proof. $K_{3,3}$ has $|V| = 6, |E| = 9$, and $g = 4$, so

$$(|V| - 2)\frac{g}{g-2} = (4)\frac{4}{2} = 8 < 9,$$

so by Theorem 3, $K_{3,3}$ is not planar. □

4 Problem 4

Lemma 1. *Any triangle-free planar graph $G = (V, E)$ has that $|E| \leq 2|V| - 4$.*

Proof. Let F be the set of faces of G, and let f_i be the length of the ith face. Then, since G has no triangles, $f_i \geq 4 \; \forall \; i \in [|F|]$, so

$$\sum_i f_i = 2|E| \geq 4|F|.$$

Then by Euler's formula,

$$\begin{aligned} |V| - |E| + |F| = 2 &\implies |F| = 2 - |V| + |E| \\ &\implies 2|E| \geq 4(2 - |V| + |E|) \\ &\implies 2|V| - 4 \geq |E|. \end{aligned}$$

□

Lemma 2. *Any triangle-free planar graph $G = (V, E)$ has a vertex of degree greater than or equal to 3.*

Proof. Suppose all of $v \in V$ have that $d(v) \geq 4$, for $d(v)$ the degree of vertex v. Then

$$4|V| \leq \sum_{v \in V} d(v) = 2|E|$$

by the handshaking lemma. By Lemma 1, however,

$$2|E| \leq 4|V| - 8,$$

but then $4|V| \leq 2|E| \leq 4|V| - 8$. □

Theorem 4. *Every triangle-free planar graph is 4-colorable.*

Proof. We will prove the claim by induction on the number of vertices. Let $G = (V, E)$ be a triangle-free planar graph with face set F. The base case is trivial: A graph with a single vertex is clearly 4-colorable and is planar with no triangles. Now assume that the claim holds for some $|V| = n$.

By Lemma 2, we can find $v \in V$ so that $d(v) \leq 3$. Then we can choose 4 colors c_0, c_1, c_2, c_3 for each $w \in v \cup N(v)$ so that no two vertices in the set share the same color, where $N(v)$ is the neighbor set of v. Now remove v from the graph. (Though retain the color c_v that was assigned to it.) Then $G - v$ will be a planar graph with no triangles, since otherwise we would have that G had a triangle in it originally, a contradiction. By the induction hypothesis, we may find a 4-coloring of this graph, and so adding v back to the graph, we may assign it the color c_v and be certain it will not share this color with its neighbors.

□

5 Problem 5

Lemma 3. *Every planar graph has a vertex of degree at most 5.*

Proof. Let $G = (V, E)$ be our graph. If $|V| \leq 5$, the result is obvious by considering a fully connected graph.

Now suppose by contradiction that there exists a planar graph $G = (V, E)$ with $|V| > 5$ and with all vertices of degree at least 6. Then, for $d(v)$ the degree of vertex $v \in V$,

$$\sum_{v \in V} d(v) = 2|E| \geq 6|V| \implies |E| \geq 3|V|$$

by the handshake lemma. However, then

$$|E| > 3|V| - 6,$$

a contradiction. (From lecture, we know that for a planar graph $G = (V, E)$, $|E| \leq 3|V| - 6$.) □

Theorem 5. *Every planar graph is the union of 3 forests.*

Proof. Let $G = (V, E)$ be a planar graph. We will prove the theorem by induction on the number of vertices in G. If $|V| = 1$, then clearly the theorem holds. Assume it holds for all planar graphs of some number of vertices n. Now consider a planar graph $G = (V, E)$ with $|V| = n + 1$.

By Lemma 3, we may find $v \in V$ such that $d(v) \leq 5$. Then $G - v$ is a graph of n vertices, so we may decompose it into 3 forests, F_0, F_1, F_2. By the pigeonhole principle, $|N(v) \cap F_i| \leq 1$ for some $i \in \{0, 1, 2\}$. But then $F_i \cup v$ is again a forest and so we may replace F_i by it to get three forests whose union is in fact $(G - v) \cup v = G$. □

6 Problem 6

Theorem 6. *A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.*

Proof. We will show equivalently that a graph $G = (V, E)$ is nonplanar if and only if it contains K_5 or $K_{3,3}$ as a minor.

($\Longrightarrow$) Suppose G is nonplanar. Then by Kuratowski's theorem, G contains K_5 or $K_{3,3}$ as a topological minor. All topological minors of G are also minors of G, so K_5 or $K_{3,3}$ is a minor of G.

($\Longleftarrow$) Now suppose G contains K_5 or $K_{3,3}$ as a minor. It was shown in Lecture 10, Lemma 6 that if contraction of some edge in a graph H resulted in a Kuratowski subgraph, then H itself contained a Kuratowski subgraph.

Then if G has K_5 or $K_{3,3}$ as a minor, we may contract and suppress (a special case of edge contraction) to form a sequence of progressively smaller graphs $G_1, ..., G_n$ until G_n contains either K_5 or $K_{3,3}$. Then $G_{n-1}, ..., G_1$ must all have Kuratowski subgraphs and so in fact G contains one itself. Thus G is nonplanar. □

7 Problem 7

Theorem 7. *A graph is outerplanar if and only if it contains neither K_4 nor $K_{2,3}$ as a topological minor.*

Proof. ($\Longrightarrow$) Let $G = (V, E)$ be an outerplanar graph with one or both of K_4 or $K_{2,3}$ as a topological minor. Place a new vertex v in the unbounded face of G and connect all vertices in G to v to create a new graph G'.

Then since G is outerplanar, G' will be planar. (One may always fan out the edges from each $w \in V$ to v so that none of the edges intersect one another; by the Jordan curve theorem, all the edges will be traversing solely through the outside of G, so none must intersect.) Since G contains one or both of K_4 or $K_{2,3}$, then connecting a new vertex v to all of the vertices forming K_4 must create K_5 in G'; similarly, connecting v to all of the vertices forming $K_{2,3}$ will create a $K_{3,3}$ subgraph in G'.

But by Kuratowski's theorem, G' cannot contain K_5 or $K_{3,3}$ as a topological minor since it is planar, a contradiction.

($\Longleftarrow$) Now suppose $G = (V, E)$ contains neither $K_{2,3}$ nor K_4 as a topological minor. Then we may simply add a new vertex v to G as before and connect v to every $w \in V$, creating a new graph G'. But since G did not contain either K_4 or $K_{2,3}$, the addition of this new vertex cannot induce a Kuratowski subgraph in G'. Thus G' is planar.

Then we may project G' onto a sphere and back onto the plane to find an arrangement of the graph such that v is in the unbounded face. Then v connects to every vertex of G' and G' is planar, so the suppression of v to return to G will maintain a planar graph with the property that all its vertices will be on the boundary of the unbounded face; that is, G will be an outerplanar graph. $\square$

8 Problem 8

Theorem 8. *Suppose that the edges of K_n are 2-colored. Then there exist two monochromatic paths P_1 and P_2 such that $V(P_1) \cup V(P_2) = V(K_n)$.*

Proof. We will prove the theorem by induction on the number of vertices n in K_n. For $n = 1$, we clearly have that the union of the single vertex with itself will be the entire vertex set, and there are no edges for coloring, so we are done. Now assume that the theorem holds for some n.

Let G' be a graph of K_n with some (arbitrarily chosen) 2-coloring of its edges and two monochromatic paths P_1' and P_2' for which $V(P_1') \cup V(P_2') = V(G')$. Then add a new vertex $v \notin V(G')$ to G' and connect it to every other vertex in G' to create a new graph G.

Since every vertex in G' lies on some monochromatic path, we may take some $w \in V(G')$ on, say, without loss of generality, the blue path (the other being red). Then if we wish to color the edge (v, w) blue, we may simply extend the blue path to hit v after w, return to w, and then continue as it had before.

Now suppose we wish to color (v, w) red. Then it must be the case that either there are at least 2 vertices on the red path in G' or there are none. If there are none, we simply note that v will now be on the red path, and any red edge going out from it will now include the endpoints as being on the red path. We will be able to maintain the blue path since G' is fully connected, so we may simply jump the now-red vertices to create the new blue path and have both paths collectively cover all vertices.

Now suppose there are at least 2 red vertices. Then if we color the edges from v such that a blue edge goes to any node on the blue path or a red edge goes to any node on the red path, then we are done since we may simply extend the respective monochromatic path as explained above.

In the final case, if the colors of the edges from v are all mismatched with the colors of the vertices (i.e., the colors of the paths the vertices are on) at the endpoints, observe that it must be the case that there exists a node u on the, say, without loss of generality, red path with (v, u) a blue edge such that u has a blue edge extending from it. This follows since G' is fully connected, so if there exist vertices on both the red and blue paths, then there must be a node at which the two paths cross. It is then easy to extend that (again WLOG) blue path to include both u and v, and either reassign the paths of the vertices on the red path to be on the blue (if they are now isolated) or leave them as is if they are connected to the rest of the red path. □

9 Problem 9

Theorem 9. *For every $r \in \mathbb{N}, \exists\ n$ such that every connected graph with at least n vertices contains either $K_r, K_{1,r}$, or P_r as an induced subgraph.*

Proof. We will provide a proof by contradiction. First assume that every r has an associated n satisfying the conditions of the theorem. Now suppose that for some r, n, $G = (V, E)$ is a connected graph with at least n vertices such that G does not contain $K_r, K_{1,r}$, or P_r as an induced subgraph.

Then observe that the maximum degree d^* of a vertex in G must be less than the Ramsey number $r^* = R(r, 2)$. If this were not the case, we may take any vertex $v \in V(G)$ with $d(v) \geq r^*$ and consider the subgraph G' induced by $N(v)$, the neighbor set of v. Since $|V(G')| \geq r^*$, then we must have that either every pair of vertices in G' share an edge, or that none of the vertices share an edge. In either case, we will obtain a subgraph of either K_r (from G') or $K_{1,r}$ (from $G' \cup v$), a contradiction.

Furthermore, since P_r cannot be an induced subgraph of G, we must have that the shortest path between any two vertices in G is bounded above by r. Denote the number of vertices in the shortest path between any two vertices in G by δ so $\delta < r$.

We will now show $\delta \geq r$ to arrive at a contradiction. Suppose $r \geq 2$ so $d^* \geq 2$. Note that the size of the set of vertices $\{w : d(v, w) = d\}$, for $d(v, w)$ the distance between v and w, must be less than or equal to $d^*(d^* - 1)^{d-1}$. (It will only be equal to $d^*(d^* - 1)^{d-1}$ in the event that each vertex u in the path between each v and w has maximal degree and none of the vertices in the path have edges connecting to other vertices in the graph; the latter part is guaranteed by the fact that we consider only shortest paths.) Then

$$(d^*)^r < |V(G)| \leq 1 + \sum_{i=1}^{\delta-1} d^*(d^* - 1)^{i-1}$$

$$\leq \sum_{i=0}^{\delta-1} (d^*)^i \leq (d^*)^\delta \implies r \leq \delta,$$

as desired.

□

10 Problem 10

Lemma 4. *There exists a q-coloring of a complete graph of 2^q vertices which does not contain a monochromatic triangle.*

Proof. Let $n = 2^q$ so we consider K_n. Label each of the vertices of K_n by a unique, q-digit bitstring (e.g., $\{0, 0, ..., 0\}, \{1, 0, ..., 0\}$, etc.). Color the edge between two vertices v and w to be $1 \leq i \leq q$ if the bitstrings of v and w first differ on index i, starting from 1.

Now suppose there exists a monochromatic triangle in this graph of color j between vertices $u = (u_1, ..., u_q), v = (v_1, ..., v_q), w = (w_1, ..., w_q)$. Then it must be the case that $u_j = v_j = w_j$, and $u_i \neq v_i \neq w_i, i < j$. But this is obviously not possible, since $u_i, v_i, w_i \in \{0, 1\}$, so it cannot be the case that all three fail to align at all indices prior to j, and only align at j. □

Lemma 5. $r_q(3) \leq 2 + q(r_{q-1}(3) - 1)$.

Proof. Consider the q-colored graph K_n and choose any vertex $v \in V(K_n)$.

By the pigeonhole principle, we must have some color $1 \leq i \leq q$ that will occur at least $m = \lceil \frac{n-1}{q} \rceil$ times along the edges including v. If the color i then appears on any edge between the ends of these i-colored edges, we will have a triangle of color i. Otherwise, we will have a $(q-1)$-colored subgraph of K_m, and so if $r_{q-1}(3) \leq m$, we will have that this subgraph contains a monochromatic triangle of some other color. Thus if

$$r_{q-1}(3) < \frac{n-1}{q} + 1 \implies q(r_{q-1}(3) - 1) + 2 \leq n,$$

then we may find a monochromatic triangle in any q-coloring of K_n. We therefore take the minimum n such that this holds and use it as our upper bound on $r_q(3)$:

$$r_q(3) \leq q(r_{q-1}(3) - 1) + 2.$$

□

Lemma 6. $r_q(3) \leq 1 + \sum_{i=0}^{q} \frac{q!}{i!}$.

Proof. From Lemma 5, we know $r_q(3) \leq 2 + q(r_{q-1}(3) - 1)$. Observe that $r_1(3) = 3, r_2(3) \leq 6$, and assume the theorem holds for some $q - 1$. Then we have that

$$r_q(3) \leq 2 + q(r_{q-1}(3) - 1) \leq 2 + q\left(1 + \sum_{i=0}^{q-1} \frac{(q-1)!}{i!} - 1\right)$$

$$= 2 + \sum_{i=0}^{q-1} \frac{q!}{i!} = 1 + \frac{q!}{q!} + \sum_{i=0}^{q-1} \frac{q!}{i!} = 1 + \sum_{i=0}^{q} \frac{q!}{i!}.$$

□

Theorem 10. *The q-color Ramsey number $r_q(3)$ satisfies $2^q \leq r_q(3) \leq \lfloor eq! \rfloor + 1$ for all $q \geq 2$.*

Proof. The lower bound follows directly from Lemma 4.

We will now confirm the upper bound. From Lemma 6, we have that, from the Taylor series expansion of e,

$$r_q(3) \leq 1 + \sum_{i=0}^{q} \frac{q!}{i!} = 1 + q!\left(e - \sum_{i=q+1}^{\infty} \frac{1}{i!}\right) < 1 + q!e,$$

so $r_q(3) \leq 1 + \lfloor q!e \rfloor$, since $r_q(3) \in \mathbb{N}$.

$\square$