An Introduction to Two Inference Methods in Propositional Logic: Model Checking and Resolution

Propositional logic is a system of knowledge representation. A Knowledge Base (KB) stores a series of declarative sentences (propositions) describing what is known about a world.

At the core of propositional logic is the atomic proposition; a declarative statement that can only be true or false. Let's say we live in a world with only two Boolean variables. In our world, it is either raining or not raining, and my windshield wipers are either on or not on. We'll use the following to represent the atomic propositions:

R: "It is raining"
W: "My windshield wipers are on"

The negative of each statement is represented with the "not" symbol (¬):

¬ R: "It is NOT raining"
¬ W: "My windshield wipers are NOT on"

It follows that there are only four possible states of our world:

Possible states:

It is raining (R) and	My windshield wipers are on (W)
It is raining (R) and	My windshield wipers are not on (¬ W)
It is not raining (¬ R) and	My windshield wipers are on (W)
It is not raining (¬ R) and	My windshield wipers are not on (¬ W)

Within propositional logic, we can connect atomic propositions to form more complex representations of knowledge (complex propositions). Atomic propositions are combined using the logical connectives: "and" (∧), "or" (∨) "implies" (⇒) and "if and only if" (⇔).

These connectors have the following semantic meaning:

R ∧ W: "It is raining AND my windshield wipers are on"
R ∨ W: "It is raining OR my windshield wipers are on"
R ⇒ W: "If it is raining, then my windshield wipers are on"

R ⇔ W: “If it is raining, then my windshield wipers are on, AND if my windshield wipers are on, then it is raining.”

The goal of propositional logic is not simply to represent and store knowledge, but also to *infer* knowledge. In other words, given what we know explicitly, what else do we know implicitly?

To solve the problem of inferred knowledge, we first add what we know about our world to the KB. Remember, this is our set of propositions which are immutable logical statements. In our world, we know that whenever it is raining, our windshield wipers are on. We add the following information to our KB:

1. R ⇒ W

We also know it is raining, and add that to our KB:

2. R

In this case, based on what is in our KB, we know explicitly that (1) “If it is raining, then my windshield wipers are on” and (2) “It is raining”. Can we infer that our windshield wipers are on?

Let’s return to the fact that there are only four possible states of our world:

Possible states:

It is raining (R) and	My windshield wipers are on (W)
It is raining (R) and	My windshield wipers are not on (¬ W)
It is not raining (¬ R) and	My windshield wipers are on (W)
It is not raining (¬ R) and	My windshield wipers are not on (¬ W)

Based on the first proposition in our KB, we know that “if it is raining, my windshield wipers are on”. We observe that the second possible state (it is raining and my windshield wipers are not on) is inconsistent with that knowledge. We eliminate it from the possible states of our world.

Possible states:

It is raining (R) and	My windshield wipers are on (W)
~~It is raining (R) and~~	~~My windshield wipers are not on (¬ W)~~

It is not raining (¬ R) and	My windshield wipers are on (W)
It is not raining (¬ R) and	My windshield wipers are not on (¬ W)

Based on the second proposition in our KB, we know that it is raining. Consequently, we eliminate the states in which it is not raining.

Possible states:

It is raining (R) and	My windshield wipers are on (W)
~~It is raining (R) and~~	~~My windshield wipers are not on (¬ W)~~
~~It is not raining (¬ R) and~~	~~My windshield wipers are on (W)~~
~~It is not raining (¬ R) and~~	~~My windshield wipers are not on (¬ W)~~

Now, in every possible remaining state of our world, are my windshield wipers on? Yes. Based on the knowledge in our KB, we can *infer* that my windshield wipers are on without explicit knowledge.

Now let's examine another example world. Once again, we declare in our world, whenever it is raining, my windshield wipers are on. We add the following information to our KB:

1. $R \Rightarrow W$

This time, we know that my windshield wipers are on. Let's add that to the KB.

2. W

Can we infer that it is raining? Again, we will start with our four possible states of the world.

Possible states:

It is raining (R) and	My windshield wipers are on (W)
It is raining (R) and	My windshield wipers are not on (¬ W)
It is not raining (¬ R) and	My windshield wipers are on (W)
It is not raining (¬ R) and	My windshield wipers are not on (¬ W)

As in the last example, we can eliminate the second state since it is inconsistent with our knowledge of the world.

Possible states:

It is raining (R) and	My windshield wipers are on (W)
~~It is raining (R) and~~	~~My windshield wipers are not on (¬ W)~~
It is not raining (¬ R) and	My windshield wipers are on (W)
It is not raining (¬ R) and	My windshield wipers are not on (¬ W)

We also know that my windshield wipers are on. We can eliminate the fourth state, where my windshield wipers are not on.

Possible states:

It is raining (R) and	My windshield wipers are on (W)
~~It is raining (R) and~~	~~My windshield wipers are not on (¬ W)~~
It is not raining (¬ R) and	My windshield wipers are on (W)
~~It is not raining (¬ R) and~~	~~My windshield wipers are not on (¬ W)~~

Based on the explicit knowledge in our KB, can we state implicitly, or infer that it is raining? No. We can only infer it is raining if it is raining in all possible remaining states. In this example, we cannot infer it is raining because there is still one possible state of our world, given the KB, in which it is not raining (state 3).

We have just stepped through a very basic algorithm for inference in propositional logic problems called *model checking*. The algorithm is as follows (using our first example):

1. Get a list of symbols representing the variables ('R' and 'W')
2. Enumerate all possible combinations of the variables ({R, W}, {R, ¬W}, {¬R, W}, {¬R, ¬W})
3. Eliminate every possible combination that is inconsistent with the Knowledge Base ({R,W}, {~~R, ¬W~~}, {~~¬R, W~~}, {~~¬R, ¬W~~})
4. Check if what you are trying to infer ('W') is true in every remaining state. (True)
5. If it is True, we can infer this knowledge given the Knowledge Base.

Model checking works, but what happens when our world becomes more complex? If our world grows from 2 to 8 Boolean variables, we've suddenly gone from a world with 4 possible states to a world with 2^8 or 256 possible states. Our problem grows exponentially and quickly becomes intractable.

A more efficient approach to inference in propositional logic is proof by resolution. Resolution is based on the principle of proof by contradiction. The basic concept is to start with the KB of known propositions. In proof by contradiction, we add the negation of what we are trying to infer to the KB and show that the result is unsatisfiable, or illogical. Since the negation cannot be true, it follows that the inference we are trying to prove must be true.

This is tricky so let's walk through an example. We'll simplify our world further to one Boolean variable: it is either raining (R) or it is not raining (¬R). Again, we will start by adding what we know to our KB. Let's say we know it is raining.

1. R

Now let's try to prove it's raining. Yes, we've stated explicitly that it is raining. Try not to focus on that. Imagine that fact is buried within a series of complex logic statements in our KB. In proof by contradiction, we are trying to show that the opposite of what we are trying to prove is incompatible with what we already know. To do this, we start with the KB, add the negation of the proposition we are trying to prove (¬R), and show that the result is illogical, or unsatisfiable. Here, our KB has only one proposition (R). Adding the negation (¬R) results in the following logical statement:

$R \wedge \neg R$

In English: "it is raining AND it is not raining". In propositional logic, propositions can only ever be true or false. A statement cannot be true and false simultaneously. To assert "it is raining AND it is not raining" is illogical, or unsatisfiable. Since we've shown the negation of our proposition cannot be consistent with what is known, we can infer the proposition based on the KB. Proof by contradiction: it must be true because it cannot be false. It must be raining, because asserting it is not raining is inconsistent with what is known.

This process is much more efficient than enumerating all possible worlds and checking the proposition is true in every possible state. We simply systematically search the logical statements in our KB for an unsatisfiable set. At a high level, we have introduced another basic algorithm for inference in propositional logic problems called *resolution*. The algorithm is as follows:

1. Start with the KB (R)

2. Add the negation of the proposition to be inferred to the KB ($\neg$R)
3. Systematically look for illogical combinations of the propositions (R $\wedge$ $\neg$R)
4. If one is found, you can infer that R is True
5. If one is not found, you cannot infer that R is True

For a more detailed technical overview of these algorithms, please refer to Chapter 7 of the textbook *Artificial Intelligence: A Modern Approach* by Stuart Russell and Peter Norvig.