Applied Static Analysis

Data Flow Analysis

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Lattice Theory

Many static analyses are based on the mathematical theory of lattices.

The lattice put the facts (often, but not always, sets) computed by an analysis in a well-defined partial order.

Analysis are often **well-defined** functions over lattices and can then be combined and reasoned about.

Example: Sign Analysis

- Let’s assume that we want to compute the sign of an integer value. The analysis should only return the information is definite. I.e.,

- Instead of computing with concrete values, our analysis performs it computations using abstract values:
    - positive (+)
    - negative (-)
    - zero

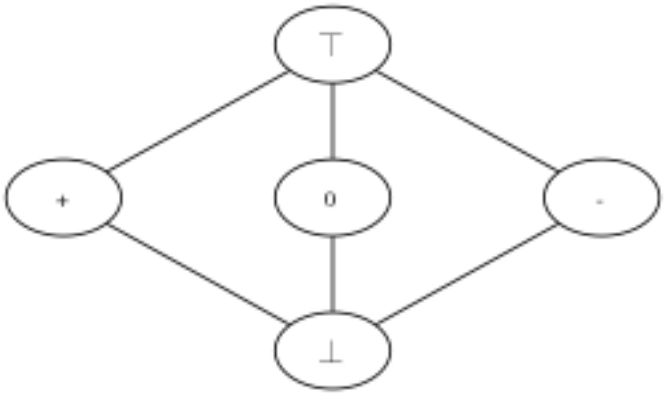
- Additionally, we have to add an abstract value $\top$ that represents the fact that we don’t know the sign of the value.

- Values that are not initialized are represented using $\bot$.

$\top$ is called *top*. (The least precise information. *The sound over-approximation*.)
$\bot$ is called *bottom*. (The most precise information.)

Example: Sign Analysis - the lattice

The lattice for the previous domain is:



The ordering reflects that ⊤ reflects all types of integer values.

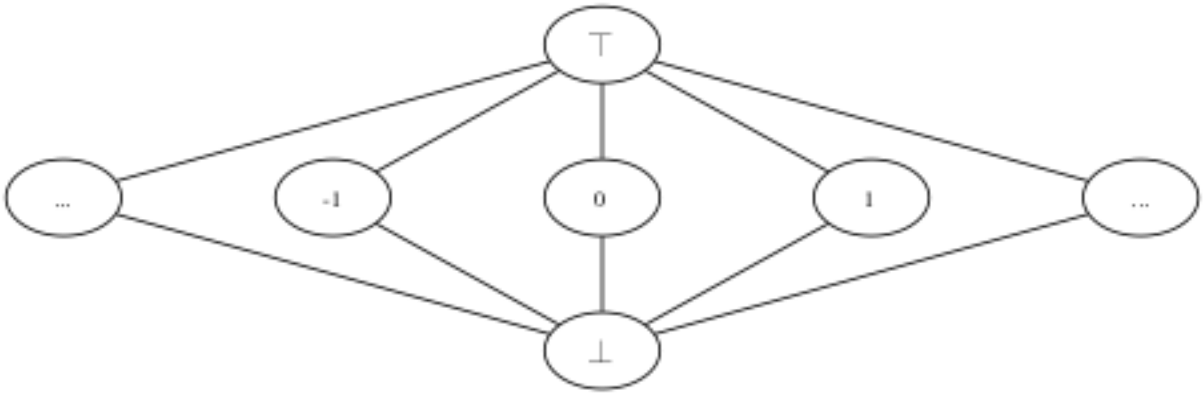
Example: Sign Analysis - example program

```
def select(c : Boolean): Int = {
    val a = 42
    val b = 333
    var x = 0;
    if (c)
        x = a + b;
    else
        x = a - b;
    x
}
```

A possible result of the analysis could be that `a` and `b` are always positive; x is either positive or negative (or zero) ($\top$).

Example: Constant Propagation - the lattice

The lattice would be:



The ordering reflects that ⊤ reflects that the value is any.

Note that this lattice is not finite, but has finite height

Again ⊥ denotes uninitialized (most precise) values and ⊤ denotes that the value is not constant.

Example: Constant Propagation - example program

```
val z = 3
var x = 1
while(x > 0) {
  if(x == 1) {
    y = 7
  } else {
    y = z + 4
  }
  x = 3
}
```

A possible result of the analysis could be that `y` is always 7 (and `z` 3.)

Partial Orderings

- a partial ordering is a relation $\sqsubseteq: L \times L \to \{true, false\}$, which
  - is reflexiv: $\forall l : l \sqsubseteq l$
  - is transitive: $\forall l_1, l_2, l_3 : l_1 \sqsubseteq l_2 \wedge l_2 \sqsubseteq l_3 \Rightarrow l_1 \sqsubseteq l_3$
  - is anti-symmetric: $\forall l_1, l_2 : l_1 \sqsubseteq l_2 \wedge l_2 \sqsubseteq l_1 \Rightarrow l_1 = l_2$

- a partially ordered set $(L, \sqsubseteq)$ is a set L equipped with a partial ordering $\sqsubseteq$

When $x \sqsubseteq y$ we say x is at least as precise as y or y over-approximates x/y is an over-approximation of y.

Upper Bounds

- for $Y \subseteq L$ and $l \in L$
  - l is an upper bound of Y, if $\forall l' \in Y : l' \sqsubseteq l$
  - l is a **least upper bound** of Y, if $l \sqsubseteq l_0$ whenever l_0 is also an upper bound of Y
- if a least upper bound exists, it is unique ($\sqsubseteq$ is anti-symmetric)
- the least upper bound of Y is denoted $\bigsqcup Y$
  we write: $l1 \sqcup l2$ for $\bigsqcup\{l1, l2\}$

$\sqcup$ is also called the join operator.

Lower Bounds

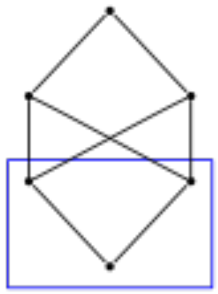
- for $Y \subseteq L$ and $l \in L$
  - l is a lower bound of Y, if $\forall l' \in Y : l \sqsubseteq l'$
  - l is a **greatest lower bound** of Y, if $l_0 \sqsubseteq l$ whenerver l_0 is also a lower bound of Y

- if a greatest lower bound exists, it is unique (⊑ is anti-symmetric)
- the greatest lower bound of Y is denoted $\bigsqcap Y$
  we write: $l1 \sqcap l2$ for $\bigsqcap \{l1, l2\}$

⊓ is also called the meet operator.

Upper/Lower Bounds

A subset Y of a partially ordered set L need not have least upper or greatest lower bounds.



Here, the subset is depicted in blue.

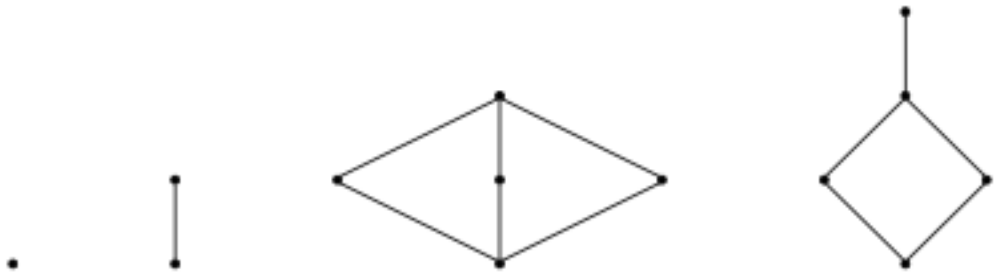
(complete) Lattice

- complete Lattice $L = (L, \sqsubseteq, \sqcap, \bigsqcup, \top, \bot)$
- is a partially ordered set $(L, \sqsubseteq)$ such that each subset Y has a greatest lower bound and a least upper bound.
  - $\bot = \bigsqcup \emptyset = \sqcap L$
  - $\top = \sqcap \emptyset = \bigsqcup L$

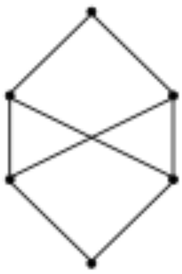
A lattice must have a unique largest element $\top$ and a unique smallest element $\bot$.

The least upper bound (the greatest lower bound) of a set $Y \subseteq L$ is always an element of L but not necessarily an element of Y.

Valid lattices:



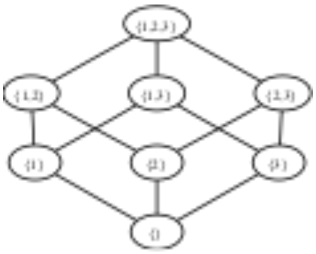
No lattice:



Do ask yourself why the lower diagram is not a lattice?

(complete) Lattice - example

Example $(\mathcal{P}(S), \subseteq)$, $S = \{1, 2, 3\}$



The above diagram is called a *Hasse* diagram.

Every finite set S defines a lattice $(\mathcal{P}S, \subseteq)$ where $\bot = \emptyset$ and $\top = S$, $x \bigsqcup y = x \cup y$, and $x \sqcap y = x \cap y$

A Hasse diagram is a graphical representation of the relation of elements of a partially ordered set.

Height of a lattice

The length of the longest path from $\perp$ to $\top$.

In general, the powerset lattice has height $|S|$.

The height of the previous lattice is 3.

Closure Properties

Construction of complete lattices:

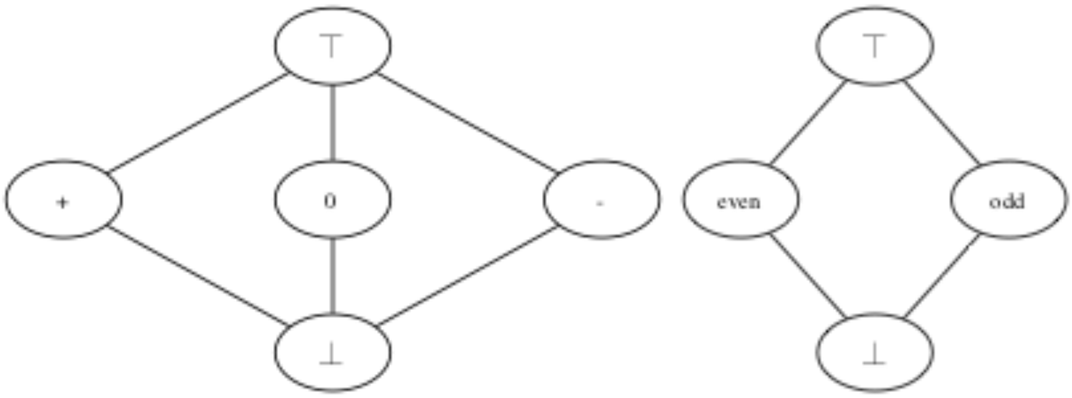
If $L_1, L_2, \ldots, L_n$ are lattices with finite height, then so is the (cartesian) product:

$$L_1 \times L_2 \times \cdots \times L_n =$$
$$(x_1, x_2, \ldots, x_n) | X_i \in L_i$$

$$height(L_1 \times \cdots \times L_n) = height(L_1) + \cdots + height(L_n)$$

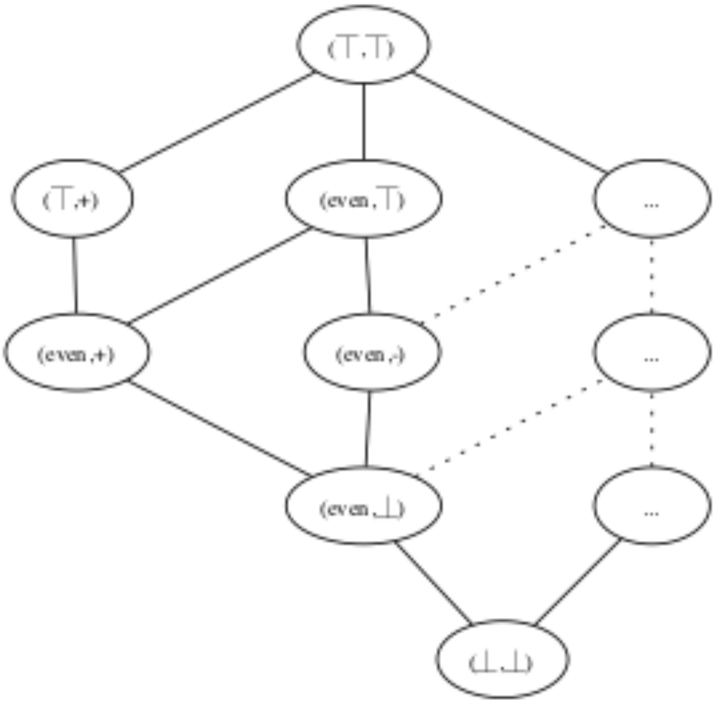
$\sqsubseteq$ is defined pointwise (i.e., $(l_{11}, l_{21}) \sqsubseteq (l_{12}, l_{22})$ iff $l_{11} \sqsubseteq_1 l_{21} \wedge l_{11} \sqsubseteq_2 l_{21}$) and $\bigsqcup$ and $\sqcap$ can be computed pointwise.

Two basic domains



Creating the cross-product

Creating the cross product of the sign and even-odd lattices.



Properties of Functions

A function $f : L_1 \to L_2$ between partially ordered sets is **monotone** if:

$$\forall l, l' \in L_1 : l \sqsubseteq_1 l' \Rightarrow f(l) \sqsubseteq_2 f(l')$$

The composition of monotone functions is monotone. However, being monotone does not imply being extensive ($\forall l \in L : l \sqsubseteq f(l)$). A function that maps all values to $\perp$ is clearly monotone, but not extensive.

The function f is **distributiv** if:

$$\forall l_1, l_2 \in L_1 : f(l_1 \sqcup l_2) = f(l_1) \sqcup f(l_2)$$

Chains

A subset $Y \subseteq L$ of a partially ordered set $L = (L, \sqsubseteq)$ is a chain if

$$\forall l_1, l_2 \in Y : (l_1 \sqsubseteq l_2) \vee (l_2 \sqsubseteq l_1)$$

That is, the values l_1 and l_2 are comparable; a chain is a possibly empty subset of L that is totally ordered.

The chain is finite if Y is a finite subset of L.

A sequence $(l_n)_n = (l_n)_{n \in N}$ of elements in L is an ascending chain if

$$n \leq m \Rightarrow l_n \sqsubseteq l_m$$

A descending chain is defined accordingly.

A sequence $(l_n)_n$ eventually stabilizes iff $\exists n_0 \in N : \forall n \in N : n \geq n_0 \Rightarrow l_n = l_{n_0}$

Interval analysis - example

Let's compute the range of values that an integer variable can assume at runtime.

```
var x = 0
while (true) {
  x = x + 1
  println(x)
}
```

Ask yourself how the lattice would look like?

Ascending/Descending Chain Condition

- A partially ordered set L satisfies the Ascending Chain Condition if and only if all ascending chains eventually stabilize.
- A partially ordered set L satisfies the Descending Chain Condition if and only if all descending chains eventually stabilize.

(A lattice must not be finite to satisfy the ascending chain condition).

Fixed Point

- $l \in L$ is a fixed point for f if $f(l) = l$
- A least fixed point $l_1 \in L$ for f is a fixed point for f where $l_1 \sqsubseteq l_2$ for every fixed point $l_2 \in L$ for f.

Equation system

$x_1 = F_1(x_1, \ldots, x_n)$

$\vdots$

$x_n = F_2(x_1, \ldots, x_n)$

where x_i are variables and $F_i : L^n \to L$ is a collection of functions. If all functions are monotone then the system has a unique least solution which is obtained as the least-fixed point of the function $F : L^n \to L^n$ defined by:

$F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$

In a lattice L with finite height, every monotone function f has a unique least fixed point given by: $fix(f) = \bigcup_{i \geq 0} f^i(\bot)$.

Data-flow analysis: Available Expressions

Determine for each program point, which expressions must have already been computed and not later modified on all paths to the program point.

The following discussion of data-flow analyses uses the more common equational approach.

Available Expressions - Example

```
         def m(initialA: Int, b: Int): Int = {
/*pc 0*/  var a = initialA

/*pc 1*/  var x = a + b;

/*pc 2*/  val y = a * b;

/*pc 3*/  while (y > a + b) {

/*pc 4*/    a = a + 1

/*pc 5*/    x = a + b

          }
/*pc 6*/  a + x
         }
```

The expression `a + b` is available every time execution reaches the test (`pc 4`).

Available Expressions - gen/kill functions

- An **expression is killed** in a block if any of the variables used in the (arithmetic) expression are modified in the block. The function $kill : Block \rightarrow \mathcal{P}(ArithExp)$ produces the set of killed arithmetic expressions.
- A **generated expression** is a non-trivial (arithmetic) expression that is evaluated in the block and where none of the variables used in the expression are later modified in the block. The function $gen : Block \rightarrow \mathcal{P}(ArithExp)$ produces the set of generated expressions.

Typically, a block is a single statement in a program's three-address code.

The underlying lattice is the powerset of all expression of the program.

If you know that a function is pure or a field is (effectively) final it is directly possible to also take these expressions into consideration.

Available Expressions - data flow equations

Let S be our program and $flow$ be a flow in the program between two statements (pc_i, pc_j).

$$AE_{entry}(pc_i) = \begin{cases} \emptyset & \text{if } i = 0 \\ \bigcap AE_{exit}(pc_h) | (pc_h, pc_i) \in flow(S) & otherwise \end{cases}$$

$$AE_{exit}(pc_i) = (AE_{entry}(pc_i) \backslash kill(block(pc_i)) \cup gen(block(pc_i)))$$

Available Expressions - Example continued I

```
/*pc 1*/  var x = a + b;
/*pc 2*/  val y = a * b;
/*pc 3*/  while (y > a + b) {
/*pc 4*/    a = a + 1
/*pc 5*/    x = a + b
          }
```

The kill/gen functions:

pc	kill	gen
1	$\emptyset$	$\{a+b\}$
2	$\emptyset$	$\{a*b\}$
3	$\emptyset$	$\{a+b\}$
4	$\{a+b, a*b, a+1\}$	$\emptyset$
5	$\emptyset$	$\{a+b\}$

We get the following equations:

$$AE_{entry}(pc_1) = \emptyset$$
$$AE_{entry}(pc_2) = AE_{exit}(pc_1)$$
$$AE_{entry}(pc_3) = AE_{exit}(pc_2) \cap AE_{exit}(pc_5)$$
$$AE_{entry}(pc_4) = AE_{exit}(pc_3)$$
$$AE_{entry}(pc_5) = AE_{exit}(pc_4)$$
$$AE_{exit}(pc_1) = AE_{entry}(pc_1) \cup \{a+b\}$$
$$AE_{exit}(pc_2) = AE_{entry}(pc_2) \cup \{a*b\}$$
$$AE_{exit}(pc_3) = AE_{entry}(pc_3) \cup \{a+b\}$$
$$AE_{exit}(pc_4) = AE_{entry}(pc_4) \backslash \{a+b, a*b, a+1\}$$
$$AE_{exit}(pc_5) = AE_{entry}(pc_5) \cup \{a+b\}$$

Data-flow analysis: Naive algorithm

> to solve the combined function: $F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$:

This is just a conceptual algorithm which is not to be implemented:

$$x = (\perp, \ldots, \perp); do\{\ t = x; x = F(x); \}while(x \neq t);$$

Data-flow analysis: Chaotic Iteration

> We exploit the fact that our lattice has the structure L^n to compute the solution $(x_1, \dots, x_n)$:

$x_1 = \bot; \dots; x_n = \bot; \mathit{while}(\exists i : x_i \neq F_i(x_1, \dots, x_n))\{\ x_i = F_i(x_1, \dots, x_n); \}$

Available Expressions - Example continued II

```
/*pc 1*/  var x = a + b;
/*pc 2*/  val y = a * b;
/*pc 3*/  while (y > a + b) {
/*pc 4*/    a = a + 1
/*pc 5*/    x = a + b
          }
```

Solution:

pc	kill	gen
1	$\emptyset$	$\{a+b\}$
2	$\{a+b\}$	$\{a+b, a*b\}$
3	$\{a+b\}$	$\{a+b\}$
4	$\{a+b\}$	$\emptyset$
5	$\emptyset$	$\{a+b\}$

References