Applied Static Analysis

Data Flow Analysis

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Lattice Theory

Many static analyses are based on the mathematical theory of lattices.

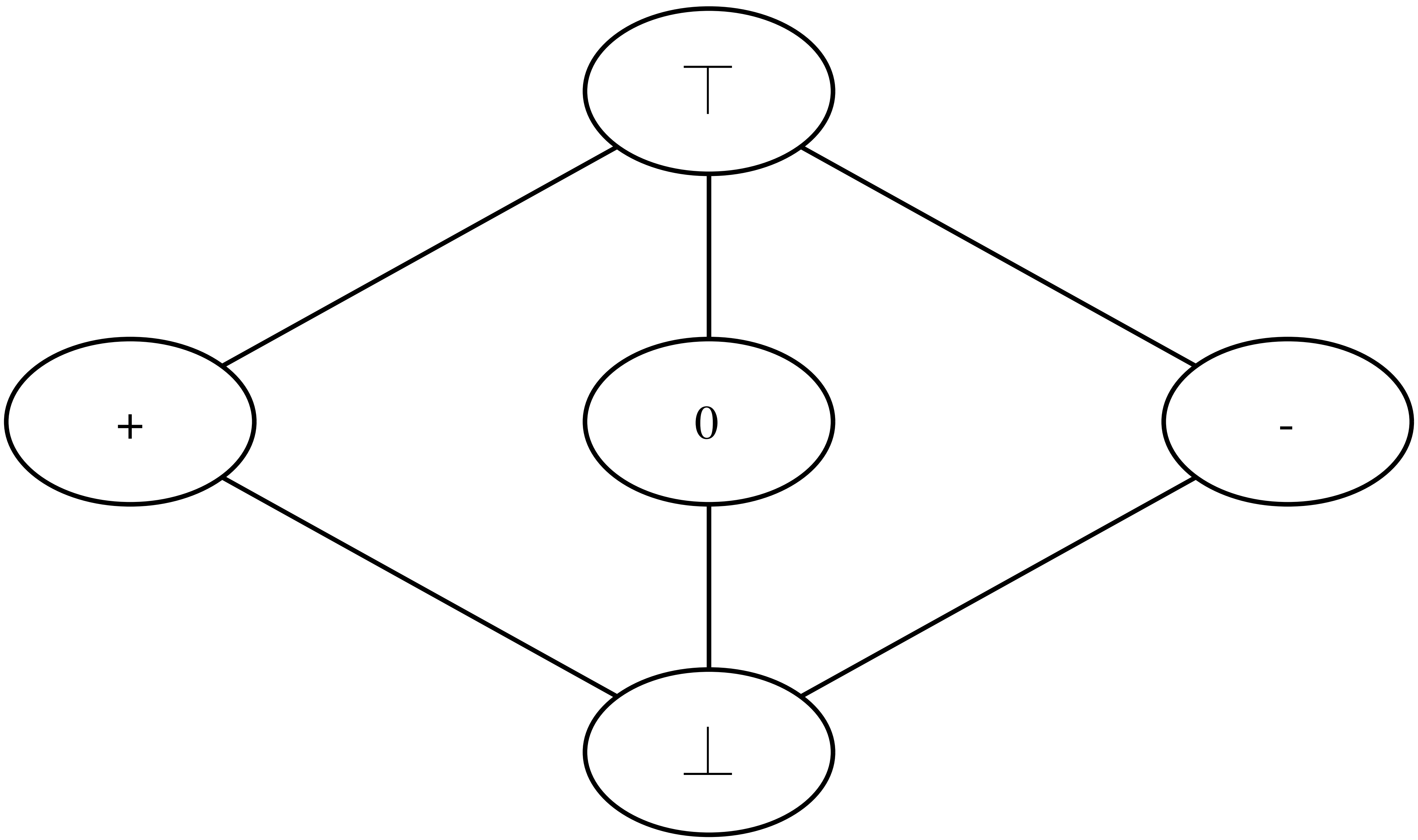
The lattice put the facts (often, but not always, sets) computed by an analysis in a well-defined partial order.

Analysis are often <u>well-defined</u> functions over lattices and can then be combined and reasoned about.

Example: Sign Analysis

- Let's assume that we want to compute the sign of an integer value. The analysis should only return the information is definite. I.e.,
- Instead of computing with concrete values, our analysis performs it computations using abstract values:
  - positive (+)
  - negative (-)
  - zero
- Additionally, we have to add an abstract value $\top$ that represents the fact that we don't know the sign of the value.
- Values that are not initialized are represented using $\perp$.

Example: Sign Analysis - the lattice



Example: Sign Analysis - example program

```
def select(c : Boolean): Int = {
    val a = 42
    val b = 333
    var x = 0;
    if (c)
        x = a + b;
    else
        x = a - b;
    x
}
```

Partial Orderings

- a partial ordering is a relation $\sqsubseteq: L \times L \to \{true, false\}$, which
  - is reflexiv: $\forall l : l \sqsubseteq l$
  - is transitive: $\forall l_1, l_2, l_3 : l_1 \sqsubseteq l_2 \land l_2 \sqsubseteq l_3 \Rightarrow l_1 \sqsubseteq l_3$
  - is anti-symmetric: $\forall l_1, l_2 : l_1 \sqsubseteq l_2 \land l_2 \sqsubseteq l_1 \Rightarrow l_1 = l_2$
- a partially ordered set $(L, \sqsubseteq)$ is a set L equipped with a partial ordering $\sqsubseteq$

Upper Bounds

- for $Y \subseteq L$ and $l \in L$
  - l is an upper bound of Y, if $\forall l' \in Y : l' \sqsubseteq l$
  - l is a least upper bound of Y, if $l \sqsubseteq l_0$ whenever l_0 is also an upper bound of Y
  - if a least upper bound exists, it is unique ($\sqsubseteq$ is anti-symmetric)
  - the least upper bound of Y is denoted $\bigsqcup Y$ we write: $l1 \sqcup l2$ for $\bigsqcup\{l1, l2\}$

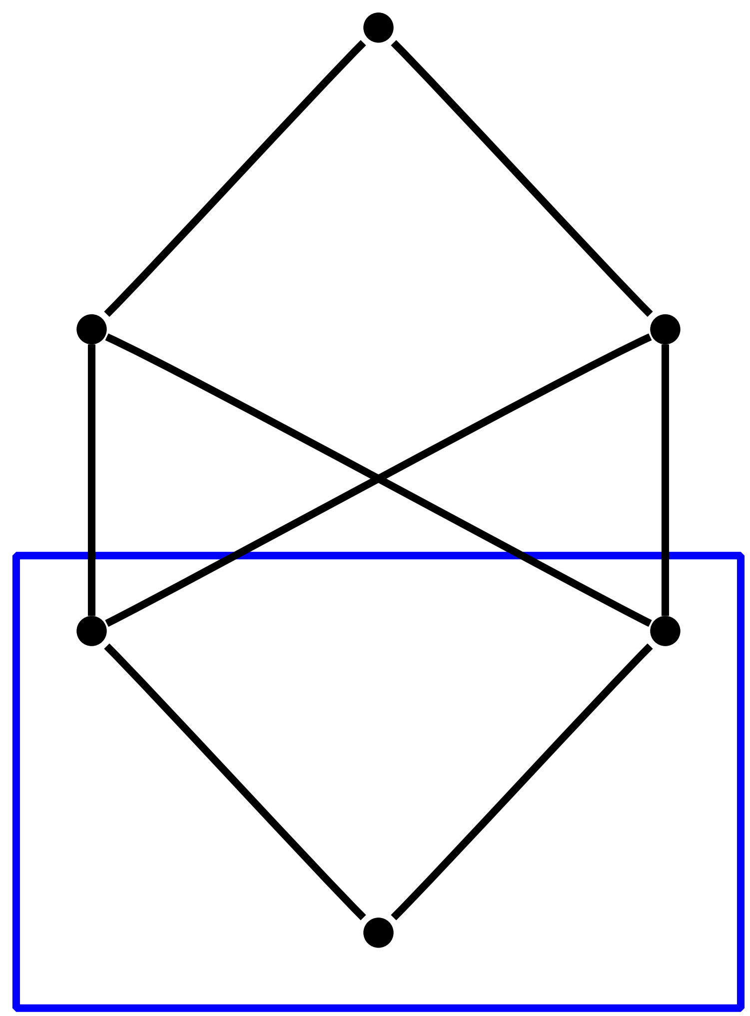
Lower Bounds

- for $Y \subseteq L$ and $l \in L$
  - l is a lower bound of Y, if $\forall l' \in Y : l \sqsubseteq l'$
  - l is a greatest lower bound of Y, if $l_0 \sqsubseteq l$ whenerver l_0 is also a lower bound of Y
  - if a greatest lower bound exists, it is unique ($\sqsubseteq$ is anti-symmetric)
  - the greatest lower bound of Y is denoted $\bigsqcap Y$

    we write: $l1 \sqcap l2$ for $\bigsqcap\{l1, l2\}$

Upper/Lower Bounds

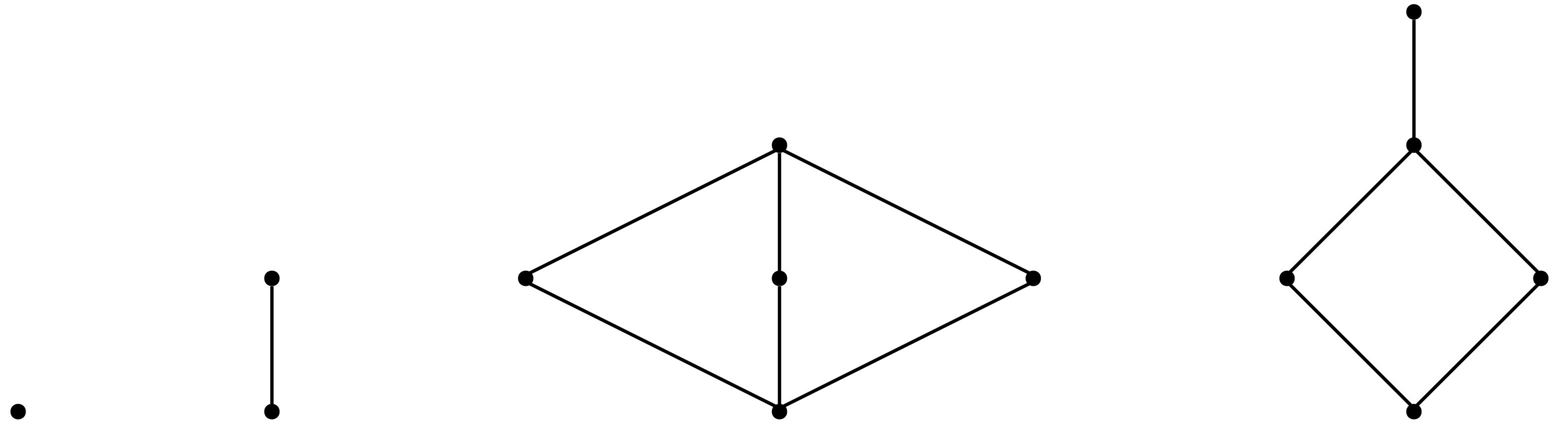
A subset Y of a partially ordered set L need not have least upper or greatest lower bounds.



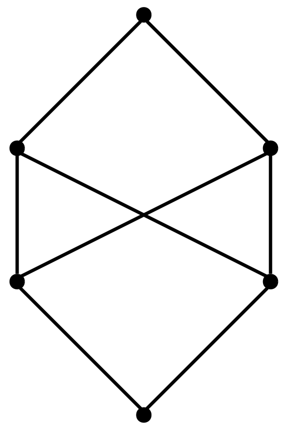
(complete) Lattice

- complete Lattice $L = (L, \sqsubseteq, \bigsqcap, \bigsqcup, \top, \bot)$
- is a partially ordered set $(L, \sqsubseteq)$ such that each subset Y has a greatest lower bound and a least upper bound.
  - $\bot = \bigsqcup \emptyset = \bigsqcap L$
  - $\top = \bigsqcap \emptyset = \bigsqcup L$

Valid lattices:

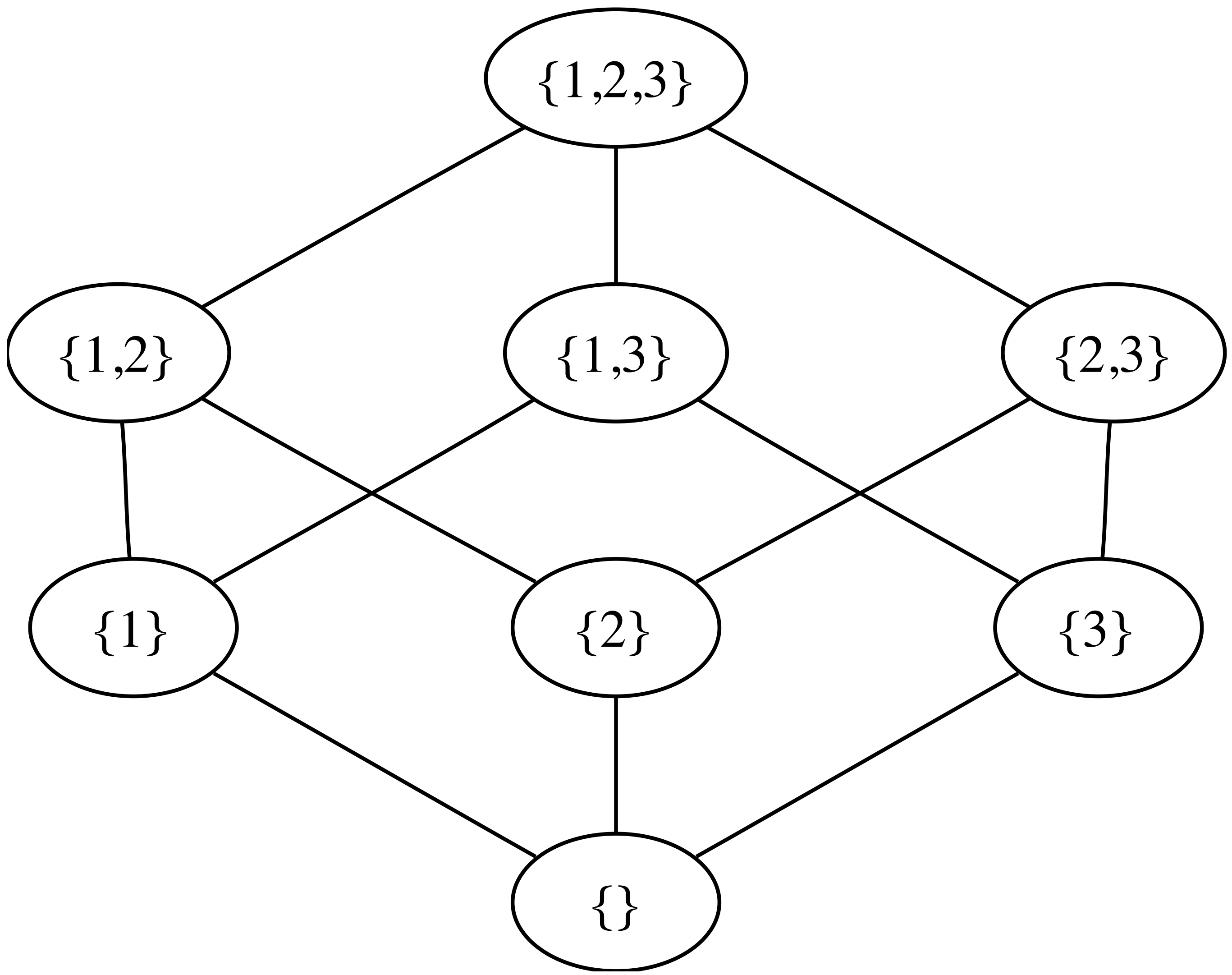


No lattice:



(complete) Lattice - example

Example $(\mathcal{P}(S), \subseteq)$, $S = \{1, 2, 3\}$



Height of a lattice

The length of the longest path from $\perp$ to $\top$.

In general, the powerset lattice has height $|S|$.

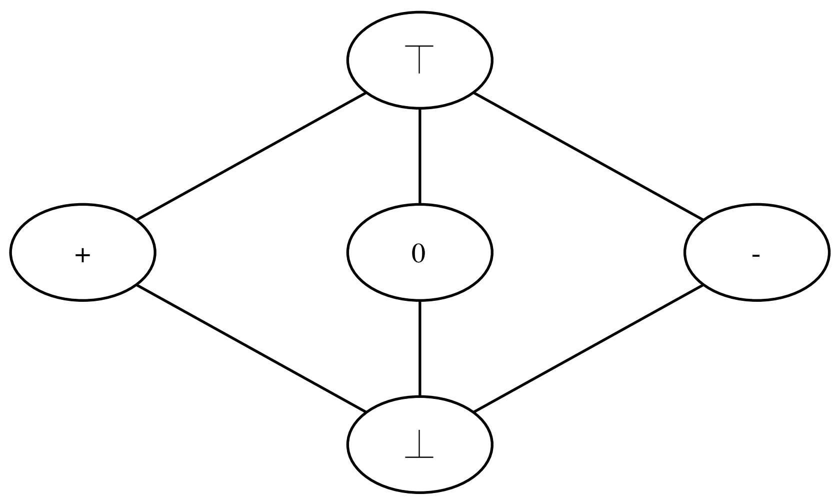
Closure Properties

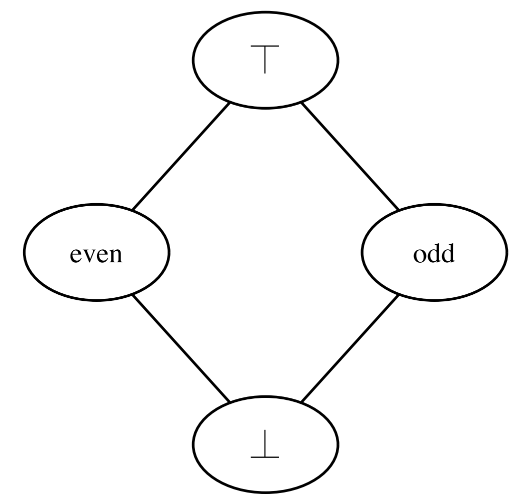
If $L_1, L_2, \ldots, L_n$ are lattices with finite height, then so is the (cartesian) product:

$$L_1 \times L_2 \times \cdots \times L_n = \{(x_1, x_2, \ldots, x_n) | X_i \in L_i\}$$

$$height(L_1 \times \cdots \times L_n) = height(L_1) + \cdots + height(L_n)$$

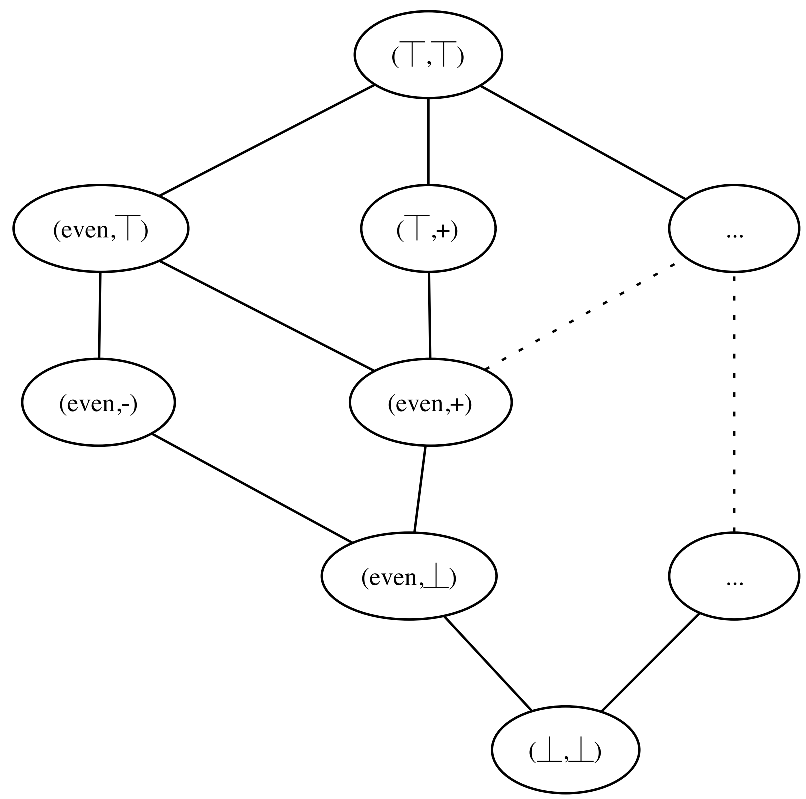
Two basic domains





Creating the cross-product

Creating the cross product of the sign and even-odd lattices.



Properties of Functions

A function $f : L_1 \to L_2$ between partially ordered sets is monotone if:

$$\forall l, l' \in L_1 : l \sqsubseteq_1 l' \Rightarrow f(l) \sqsubseteq_2 f(l')$$

The function f is distributiv if:

$$\forall l_1, l_2 \in L_1 : f(l_1 \sqcup l_2) = f(l_1) \sqcup f(l_2)$$

Chains

A subset $Y \subseteq L$ of a partially ordered set $L = (L, \sqsubseteq)$ is a chain if
$\forall l_1, l_2 \in Y : (l_1 \sqsubseteq l_2) \vee (l_2 \sqsubseteq l_1)$

The chain is finite if Y is a finite subset of L.

A sequence $(l_n)_{n \in N}$ of elements in L is an ascending chain if
$n \leq m \Rightarrow l_n \sqsubseteq l_m$

A sequence $(l_n)_n$ eventually stabilizes iff
$\exists n_0 \in N : \forall n \in N : n \geq n_0 \Rightarrow l_n = l_{n_0}$

Ascending/Descending Chain Condition

- A partially ordered set L satisfies the Ascending Chain Condition if and only if all ascending chains eventually stabilize.
- A partially ordered set L satisfies the Descending Chain Condition if and only if all descending chains eventually stabilize.

Fixed Point

- $l \in L$ is a fixed point for f if $f(l) = l$
- A least fixed point $l_1 \in L$ for f is a fixed point for f where $l_1 \sqsubseteq l_2$ for every fixed point $l_2 \in L$ for f.

Equation system

$$x_1 = F_1(x_1, \ldots, x_n)$$

$$\vdots$$

$$x_n = F_2(x_1, \ldots, x_n)$$

where x_i are variables and $F_i : L^n \to L$ is a collection of functions. If all functions are monotone then the system has a unique least solution which is obtained as the least-fixed point of the function $F : L^n \to L^n$ defined by:

$$F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$$

In a lattice L with finite height, every monotone function f has a unique least fixed point given by:
$fix(f) = \bigcup_{i \geq 0} f^i(\bot)$.

Data-flow analysis: Available Expressions

Determine for each program point, which expressions must have already been computed and not later modified on all paths to the program point.

Available Expressions - Example

```
         def m(initialA: Int, b: Int): Int = {
/*pc 0*/  var a = initialA

/*pc 1*/  var x = a + b;

/*pc 2*/  val y = a * b;

/*pc 3*/  while (y > a + b) {

/*pc 4*/    a = a + 1

/*pc 5*/    x = a + b

          }
/*pc 6*/  a + x
         }
```

Available Expressions - gen/kill functions

- An <u>expression is killed</u> in a block if any of the variables used in the (arithmetic) expression are modified in the block. The function $\boldsymbol{kill : Block \rightarrow \mathcal{P}(ArithExp)}$ produces the set of killed arithmetic expressions.
- A <u>generated expression</u> is a non-trivial (arithmetic) expression that is evaluated in the block and where none of the variables used in the expression are later modified in the block. The function $\boldsymbol{gen : Block \rightarrow \mathcal{P}(ArithExp)}$ produces the set of generated expressions.

Available Expressions - data flow equations

Let $\boldsymbol{S}$ be our program and $\boldsymbol{flow}$ be a flow in the program between two statements $(\boldsymbol{pc_i}, \boldsymbol{pc_j})$.

$$AE_{entry}(pc_i) = \begin{cases} \emptyset & \text{if } i = 0 \\ \bigcap \{AE_{exit}(pc_h) | (pc_h, pc_i) \in flow(S)\} & otherwise \end{cases}$$

$$AE_{exit}(pc_i) = (AE_{entry}(pc_i) \backslash kill(block(pc_i)) \cup gen(block(pc_i)))$$

Available Expressions - Example continued I

The kill/gen functions:

pc	kill	gen
1	$\emptyset$	$\{a+b\}$
2	$\emptyset$	$\{a*b\}$
3	$\emptyset$	$\{a+b\}$
4	$\{a+b, a*b, a+1\}$	$\emptyset$
5	$\emptyset$	$\{a+b\}$

We get the following equations:

$$AE_{entry}(pc_1) = \emptyset$$

$$AE_{entry}(pc_2) = AE_{exit}(pc_1)$$

$$AE_{entry}(pc_3) = AE_{exit}(pc_2) \cap AE_{exit}(pc_5)$$

$$AE_{entry}(pc_4) = AE_{exit}(pc_3)$$

$$AE_{entry}(pc_5) = AE_{exit}(pc_4)$$

$$AE_{exit}(pc_1) = AE_{entry}(pc_1) \cup \{a+b\}$$

$$AE_{exit}(pc_2) = AE_{entry}(pc_2) \cup \{a*b\}$$

$$AE_{exit}(pc_3) = AE_{entry}(pc_3) \cup \{a+b\}$$

$$AE_{exit}(pc_4) = AE_{entry}(pc_4) \backslash \{a+b, a*b, a+1\}$$

$$AE_{exit}(pc_5) = AE_{entry}(pc_5) \cup \{a+b\}$$

Data-flow analysis: Naive algorithm

to solve the combined function:

$F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$:

$$x = (\perp, \ldots, \perp);$$
$$do\{$$
$$t = x;$$
$$x = F(x);$$
$$\}while(x \neq t);$$

Data-flow analysis: Chaotic Iteration

We exploit the fact that our lattice has the structure $\boldsymbol{L}^n$ to compute the solution $(x_1, \ldots, x_n)$:

$$
\begin{array}{l}
x_1 = \bot; \ldots, ; x_n = \bot; \\
while(\exists i : x_i \neq F_i(x_1, \ldots, x_n))\{ \\
x_i = F_i(x_1, \ldots, x_n); \\
\}
\end{array}
$$

Available Expressions - Example continued II

Solution:

pc	kill	gen
1	$\emptyset$	$\{a + b\}$
2	$\{a + b\}$	$\{a + b, a * b\}$
3	$\{a + b\}$	$\{a + b\}$
4	$\{a + b\}$	$\emptyset$
5	$\emptyset$	$\{a + b\}$