Radiographic 2D/3D image registration with pre-integrated random rays

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1 Introduction

An intensity-based similarity metric s is a function that maps an arbitrary number N of pairs of values $\{\mathbf{x}_i \in \mathbb{R}^2 = \{x_i, y_i\},\ i = 1 \dots N\}$ to a scalar value $S \in \mathbb{R}$, where the images are $X = \{x_i,\ i = 1 \dots N\}$ and $Y = \{y_i,\ i = 1 \dots N\}$.

A metric of the similarity between two images can still be calculated without all the values from both images, each value used just needs to have its corresponding value in the other image.

- Let the fixed X-ray image $X : \mathbb{Z}^2 \to \mathbb{R}$ be approximately interpolated by the function $f : \mathbb{R}^2 \to \mathbb{R},\ f(i,j) \approx x_{ij}$.
- Consider the position of the X-ray source to be fixed relative to the X-ray detector array.
- A ray R through the CT space can be defined as the locus of points

  $$\mathbf{p}(\lambda) = \mathbf{p}_0 + \lambda\hat{\mathbf{d}}$$

  for all $\lambda \in \mathbb{R}$, parametrised by $\{\mathbf{p}_0, \hat{\mathbf{d}}\}$.
- The integral of μ through the CT space can be calculated for such a ray, giving $I \in \mathbb{R}$. This value can be considered an attribute of the ray: $R = \{\mathbf{p}_0, \hat{\mathbf{d}}, I\}$.
- Parametrising the transformation T of the CT in X-ray space, as an anti-clockwise rotation through r followed by a translation by (t_x, t_y):

  $$T = \begin{bmatrix} R & \mathbf{t} \\ 0 & 1 \end{bmatrix}$$

- The corresponding ray in X-ray space is T applied to the ray in CT space:

  $$\mathbf{p}'(\lambda) = \mathbf{p}'_0 + \lambda\hat{\mathbf{d}}' = T\begin{bmatrix} \mathbf{p}_0 \\ 1 \end{bmatrix} + \lambda T\begin{bmatrix} \hat{\mathbf{d}} \\ 0 \end{bmatrix}$$

- The ray will correspond to a valid ray for DRR generation if it passes through the X-ray source position $\mathbf{p}_\mathrm{s}$ in X-ray space:

  $$\left|\left|(\mathbf{p}_\mathrm{s} - \mathbf{p}'_0) \times \hat{\mathbf{d}}'\right|\right| < \epsilon$$

- Note that:

  $$\mathbf{p}'_0 = R\mathbf{p}_0 + \mathbf{t}, \quad \hat{\mathbf{d}}' = R\hat{\mathbf{d}}$$

  $$\Rightarrow (\mathbf{p}_\mathrm{s} - \mathbf{p}'_0) \times \hat{\mathbf{d}}' = \mathbf{p}_\mathrm{s} \times R\hat{\mathbf{d}} - \mathbf{t} \times R\hat{\mathbf{d}} - R\mathbf{p}_0 \times R\hat{\mathbf{d}} = R\Big(\big(R^{-1}(\mathbf{p}_\mathrm{s} - \mathbf{t}) - \mathbf{p}_0\big) \times \hat{\mathbf{d}}\Big)$$

$$\Rightarrow \left|\left|(\mathbf{p}_\mathrm{s} - \mathbf{p}_0') \times \hat{\mathbf{d}}'\right|\right| = \left|\left|\left(T^{-1}\begin{bmatrix}\mathbf{p}_\mathrm{s} \\ 1\end{bmatrix} - \mathbf{p}_0\right) \times \hat{\mathbf{d}}\right|\right| = \left|\left|(\mathbf{p}_\mathrm{s,CT} - \mathbf{p}_0) \times \hat{\mathbf{d}}\right|\right|$$

- If it is valid, the location (u, v) on the DRR it corresponds to will be given by:

$$(u, v) = \left(p_{0x}' - \frac{p_{0z}'}{\hat{d}_z'}\hat{d}_x', p_{0y}' - \frac{p_{0z}'}{\hat{d}_z'}\hat{d}_y'\right)$$

- Where the DRR is located on the x, y-plane, with $u = 0$, $v = 0$ corresponding to its centre.
- Given a transformation T, any ray $R = \{\mathbf{p}_0, \hat{\mathbf{d}}, I\}$ can provide an intensity pair for the similarity function as follows:

$$\{x_i', y_i'\} = \left\{f\left(p_{0x}' - \frac{p_{0y}'}{d_y'}d_x'\right), I\right\} \text{ if } \left|\left|(\mathbf{p}_\mathrm{s} - \mathbf{p}_0') \times \hat{\mathbf{d}}'\right|\right| < \epsilon,\ \mathbf{p}_0' = T\begin{bmatrix}\mathbf{p}_0 \\ 1\end{bmatrix},\ \hat{\mathbf{d}}' = T\begin{bmatrix}\hat{\mathbf{d}} \\ 0\end{bmatrix}$$

- This can be made continuous by considering the number of intensity pairs as a continuous value $N = \sum_i n_i$, to which each can contribute $n \in [0, 1]$:

$$n = \exp\left(-\alpha \left|\left|(\mathbf{p}_\mathrm{s} - \mathbf{p}_0') \times \hat{\mathbf{d}}'\right|\right|^2\right)$$

  for some scaling value α.
- So, a number $M \sim 10^6$ of rays can be calculated, $\mathbf{p}_0$ and $\hat{\mathbf{d}}$ for each generated randomly, and I calculated for each through the CT data. Each ray will need about 28 bytes, so a total memory use of the order of 100 MB.
- For an iteration of the optimisation at a transformation T, the source position is calculated in CT space, and used to calculated the weight of each ray in parallel:

$$n_i = \exp\left(-\alpha \left|\left|(\mathbf{p}_\mathrm{s,CT} - \mathbf{p}_{0i}) \times \hat{\mathbf{d}}_i\right|\right|^2\right)$$

- For $\alpha \sim 3000$, rays further than 3 cm from the source position will have small n_i, so for a patient - detector distance of ~ 1 m, and assuming a uniform distribution of ray intersections through the 1 m radius spherical surface centred on the patient, about $\frac{\pi \cdot 0.03^2}{4\pi \cdot 1^2} \approx \frac{1}{4000}$ of the rays $\sim \frac{M}{10^3} \sim 10^3$ will be significant.
- The value pair contributions $\{x_i', y_i', n_i\}$ are then calculated for these rays only:

$$\{x_i', y_i', n_i\} = \left\{f\left(p_{0x}' - \frac{p_{0z}'}{\hat{d}_z'}\hat{d}_x', p_{0y}' - \frac{p_{0z}'}{\hat{d}_z'}\hat{d}_y'\right), I, \exp\left(-\alpha \left|\left|(\mathbf{p}_\mathrm{s,CT} - \mathbf{p}_0) \times \hat{\mathbf{d}}\right|\right|^2\right)\right\}$$

$$\mathbf{p}_0' = T\begin{bmatrix}\mathbf{p}_0 \\ 1\end{bmatrix},\ \hat{\mathbf{d}}' = T\begin{bmatrix}\hat{\mathbf{d}} \\ 0\end{bmatrix}$$

- And the final similarity S is then calculated as

$$S = S(\{x_0, y_0, n_0\}, \{x_1, y_1, n_1\}, \dots),\ x_i = n_i x_i',\ y_i = n_i y_i'$$

- Consider the transformation to be parametrised by some values $\boldsymbol{\theta}$.

- The derivatives of each value w.r.t. to the transformation parameters are:

$$\frac{\partial x_i}{\partial \theta_j} = x_i' \frac{\partial n_i}{\partial \theta_j} + n_i \frac{\partial x_i'}{\partial \theta_j}$$

$$\frac{\partial x_i'}{\partial \theta_j} = \frac{\partial f}{\partial u}(u,v)\frac{\partial u}{\partial \theta_j} + \frac{\partial f}{\partial v}(u,v)\frac{\partial v}{\partial \theta_j}$$

$$u = p_{0x}' - \frac{p_{0z}'}{\hat{d}_z'}\hat{d}_x', \quad v = p_{0y}' - \frac{p_{0z}'}{\hat{d}_z'}\hat{d}_y'$$

$$\frac{\partial u}{\partial \theta_j} = \frac{\partial p_{0x}'}{\partial \theta_j} - \beta \hat{d}_x' - \frac{p_{0z}'}{\hat{d}_z'}\frac{\partial \hat{d}_x'}{\partial \theta_j}, \quad \frac{\partial v}{\partial \theta_j} = \frac{\partial p_{0y}'}{\partial \theta_j} - \beta \hat{d}_y' - \frac{p_{0z}'}{\hat{d}_z'}\frac{\partial \hat{d}_y'}{\partial \theta_j}$$

$$\beta = \frac{\partial}{\partial \theta_j}\left(\frac{p_{0z}'}{\hat{d}_z'}\right) = \frac{\hat{d}_z' \frac{\partial p_{0z}'}{\partial \theta_j} - p_{0z}' \frac{\partial \hat{d}_z'}{\partial \theta_j}}{\hat{d}_z'^2}$$

$$\frac{\partial n_i}{\partial \theta_j} = -2\alpha n_i \Big((\mathbf{p}_{\mathrm{s,CT}} - \mathbf{p}_0) \times \hat{\mathbf{d}} \Big) \cdot \left(\frac{\partial \mathbf{p}_{\mathrm{s,CT}}}{\partial \theta_j} \times \hat{\mathbf{d}} \right)$$

$$= -2\alpha n_i \left((\mathbf{p}_{\mathrm{s,CT}} - \mathbf{p}_0) \cdot \frac{\partial \mathbf{p}_{\mathrm{s,CT}}}{\partial \theta_j} - \Big((\mathbf{p}_{\mathrm{s,CT}} - \mathbf{p}_0) \cdot \hat{\mathbf{d}} \Big) \left(\frac{\partial \mathbf{p}_{\mathrm{s,CT}}}{\partial \theta_j} \cdot \hat{\mathbf{d}} \right) \right)$$

- The derivative of the similarity metric w.r.t. a transformation parameter θ_j is

$$\frac{\partial S}{\partial \theta_j} = \sum_i \left(\frac{\partial S}{\partial x_i}\left(x_i' \frac{\partial n_i}{\partial \theta_j} + n_i \frac{\partial x_i'}{\partial \theta_j} \right) + \frac{\partial S}{\partial y_i}\left(y_i' \frac{\partial n_i}{\partial \theta_j} + n_i \frac{\partial y_i'}{\partial \theta_j} \right) + \frac{\partial S}{\partial n_i}\frac{\partial n_i}{\partial \theta_j} \right)$$

- A common similarity metric is the zero-normalised cross-correlation:

$$\mathrm{ZNCC}(X,Y) = \frac{1}{N\sigma_X\sigma_Y}\sum_i (x_i - \mu_X)(y_i - \mu_Y) = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{N\sum x_i^2 - \left(\sum x_i\right)^2}\sqrt{N\sum y_i^2 - \left(\sum y_i\right)^2}}$$

  Which we can modify into a 'weighted' zero-normalised cross-correlation:

$$\mathrm{WZNCC}(\{x_i', y_i', n_i\}\dots) = \frac{1}{\sigma_X\sigma_Y\sum_i n_i}\sum_i n_i(x_i' - \mu_X)(y_i' - \mu_Y)$$

$$= \frac{\sum n_i \sum n_i x_i' y_i' - \sum n_i x_i' \sum n_i y_i'}{\sqrt{\sum n_i \sum n_i x_i^2 - (\sum n_i x_i)^2}\sqrt{\sum n_i \sum n_i y_i^2 - (\sum n_i y_i)^2}}$$

- Blur fixed image according to alpha?