RANDOM RESISTOR NETWORKS

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INTRODUCTION

At the most basic level, the study of random resistor networks (RRNs) is a simple extension of the study of site (or bond) percolation problems. In site percolation, sites on a lattice of some geometry are filled or unfilled with probability p. If there is a cluster of filled sites connected by nearest neighbors (i.e. not by only a corner) that connects from the top of the lattice to the bottom of the lattice, we say that the cluster is *percolating*. At some critical threshold p_c, the lattice will always have a percolating cluster. In the study of random resistor networks, we take the additional step of saying that filled sites can conduct electricity, and empty sites cannot. We are then interested in the current that flows through the percolating cluster when a unit voltage is applied across the lattice. This current per unit voltage is called the *conductance* of the lattice. A schematic of a generic 2D RRN is shown in Figure 1, taken from [1].

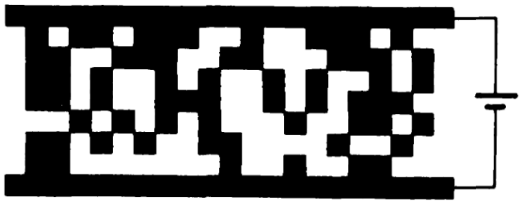

Fig. 1. Diagram of a RRN, taken from [1].

When unit voltage is applied from the top of a 2D lattice to the bottom of the lattice, the conductance is proportional to N, the width of the lattice, and inversely proportional to L, the length of the lattice. The factor of proportionality that relates conductance and geometry is called conductivity, and is denoted by Σ. In 2D, if the lattice is square, these dependencies cancel out, and the conductance is therefore equal to the conductivity Σ. In this work, we focus exclusively on 2D square lattices, and our goal is therefore to study properties of Σ. We are also interested in the mass of the percolating cluster, which is simply the ratio of the number of lattice sites in the percolating cluster to the total number of lattice sites. A naive guess would be that the mass and the conductivity are directly proportional, since larger mass means more conducting material, but as we will see, this is not the case.

DESCRIPTION OF SIMULATION

Our simulation to study RRNs can be broken down into the following main components:

1. Generate a lattice of size L by L with some fraction of sites filled p.
2. Use the Hoshen-Kopelman cluster labeling algorithm to label clusters and identify a percolating cluster, should there be one.
3. Calculate conductivity and mass, both normalized to their values for $p = 1$.
4. Do this many times!

The general result of this procedure is a lattice that looks something like Figure 2.

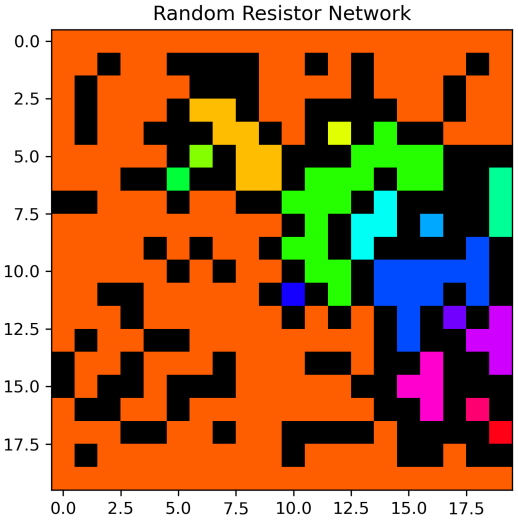

Fig. 2. Sample 20 by 20 RRN with mass=0.5 and conductivity=0.3.

Of course, steps 3 and 4 are non-trivial. Let us first discuss calculating conductivity. There are multiple methods for doing so; one approach by Fogelholm [2] iterates through the lattice, replacing each block of nearest neighbors by a single

site that has a conductivity equivalent to the block. This way, one can reduce the entire lattice down to a single site of equivalent conductivity. Another approach by Kirkpatrick [3] and described in detail in [4] is to combine current conservation and the microscopic form of Ohm's law:

$$\vec{\nabla} \cdot \vec{j} = 0 \tag{1}$$

$$\vec{j} = \sigma \vec{E} = \sigma \vec{\nabla} V \tag{2}$$

Here $\vec{j}$ is the current density, σ is the local conductivity (1 for a filled site, 0 for unfilled), and $V = -\phi$ is the "local" potential. On a lattice with unit spacing, we can write a finite difference expression for the gradient operator as

$$\partial_i j_i(\vec{r}) = j_i(\vec{r} + \vec{e_i}) - j_i(\vec{r})$$

where $\vec{e_i}$ is a unit vector in the i direction. Applying the same operator, we also have

$$\begin{aligned} j_i(\vec{r}) &= \sigma(\vec{r}) \partial_i V(\vec{r}) \\ &= \sigma(\vec{r})(V(\vec{r}) - V(\vec{r} - \vec{e_i})) \end{aligned} \tag{3}$$

which gives

$$\begin{aligned} \partial_i j_i(\vec{r}) = \sigma(r + \vec{e_i})(V(\vec{r} + \vec{e_i}) - V(\vec{r})) \\ - \sigma(\vec{r})(V(\vec{r}) - V(\vec{r} - \vec{e_i})). \end{aligned} \tag{4}$$

Combining (1) and (4) and simplifying yields a prescription for iterating through potentials V through a relaxation method:

$$\begin{aligned} V_{k+1}(x_i, y_j) = \Big[&\sigma(x_i + 1, y_j) V_k(x_i + 1, y_j) \\ &+ \sigma(x_i, y_j) V_k(x_i - 1, y_j) \\ &+ \sigma(x_i, y_j + 1) V_k(x_i, y_j + 1) \\ &+ \sigma(x_i, y_j) V_k(x_i, y_j - 1) \Big] \\ \div \Big[&\sigma(x_i + 1, y_j) + \sigma(x_i, y_j + 1) + 2\sigma(x_i, y_j) \Big] \end{aligned} \tag{5}$$

This formula essentially tells us that the potential at each point is an average of nearby potentials weighted according to the nearby conductivities. If we start with the lattice initialized to have a potential equal to linear gradient from 1 to 0 top to bottom, then we should be able to converge on the correct potential for our lattice using this method. However, using this formula directly will lead to a "critical" slowing down for fraction of sites filled near p_c. The solution to this problem, as shown in Figure 3, is to apply over-relaxation according to

$$\bar{V_{k+1}} = (1 + \alpha) V_{k+1} - \alpha V_k \tag{6}$$

where $\bar{V_{k+1}}$ is the actual updated value, V_{k+1} is determined according to (5), and α is the parameter describing the strength of the relaxation.

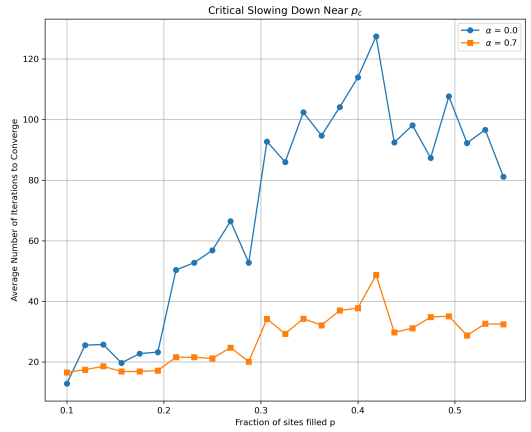

Fig. 3. Number of iterations for potential to converge with L=10 as a fraction of sites filled, using relaxation and over-relaxation.

Once we have the potential, we can apply (3) to calculate the current densities, and then find the total current in the top and bottom row. These should be equal by current conservation, and also equal to the conductivity.

As step 4 implies, we are interested in running this simulation many times in order to extract information about the behavior of the conductivity. In particular, we want to study the scaling of conductivity as a function of L at p_c. We must first determine p_c as follows:

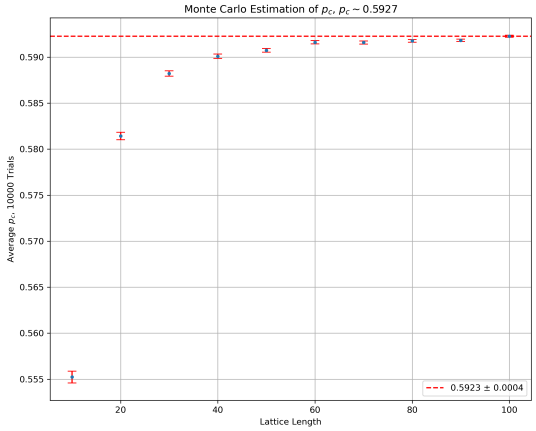

Fig. 4. Estimation of critical fraction at various lattice sizes. Error bars are statistical error.

1. Pick a lattice size L and some desired number of trials, over which we will loop.

2. For each loop, use a bisection method to estimate the probability p_c at which a percolating cluster is always formed. Initialize variables high, low, p and p_{old}.

- Generate a percolation lattice of size L with percolation probability p.
- Label the clusters in the lattice using the Hoshen-Kopelman algorithm and check if there is a spanning cluster.
- If a spanning cluster is found, set the high value of the bisection to p, otherwise set the low value to p.
- Update p_{old} to the current value of p.
- Recalculate p as the midpoint of the updated low and high values.
- Continue until the change in p is less than a specified tolerance.

3. Store the output of each loop in order to accumulate statistics at each lattice size L.

4. Do this for multiple lattice sizes.

The outcome of such a simulation is shown in Figure 4; the result is very similar to the literature value for p_c. In Figure 5, we also see the expected finite size scaling of the standard deviation Δ of the simulations. We are now equipped to study conductivity as desired.

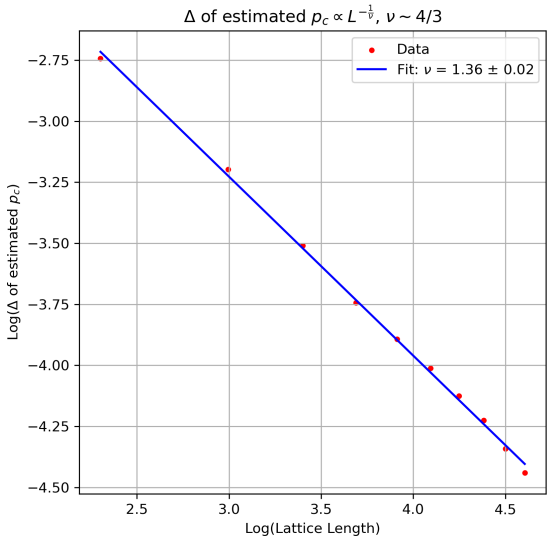

Fig. 5. Dependence of the standard deviation of the estimated critical fraction, Δ, on lattice size.

RESULTS

As mentioned in the previous section, our first and primary goal is to study Σ at p_c, in order to find the critical exponent μ/ν in the known finite size relation

$$\Sigma \propto L^{-\frac{\mu}{\nu}}. \tag{7}$$

This is important because we have already determined ν in the previous section, which means we by combining results we are actually able to determine μ without needing to actually simulate extremely large lattice sizes. Due to the added complexity of having to calculate potential and the conductivity, very large system sizes are not accessible as they were before. However, this should not matter, since we want to determine the dependency on system size anyway. If we plot the results on a log-log plot, the slope will tell us the critical exponent. Results of this for $L \in [3, 11]$ are shown in Figure 6. We see that the value determined by our simulations is in agreement with the established literature value for 2D square lattices.

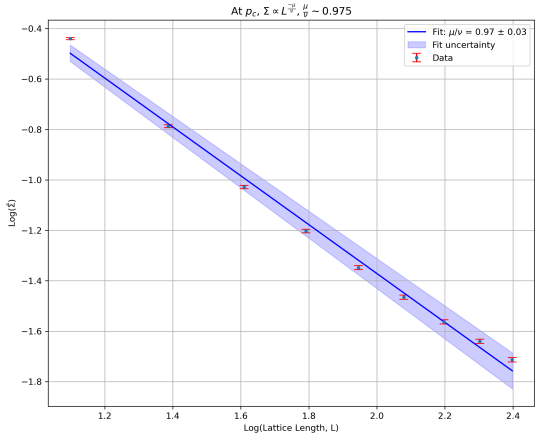

Fig. 6. Estimation of the L dependence of Σ at p_c log-log scale.

We also want to determine the relationship between conductivity and mass as a function of p. Our simulation result is shown in Figure 7, and a similar experimental result obtained by randomly punching holes in conducting paper is shown in Figure 8. From both results, it is clear that the naive assumption that mass and conductivity should be directly proportional is very wrong. Why is this the case? From careful inspection of the sample lattice in Figure 2, or any sample generated from our program, one can easily convince oneself that the bulk of the material in the percolating cluster is not actually carrying current. Much of the mass is contained within so called "dead ends" sticking out of the percolating cluster; this effect is even more pronounced at larger lattice sizes. Thus most of the mass is not contributing to the conductivity, which explains the large discrepancy between the two.

CONCLUSIONS

Overall, we have successfully implemented a basic model of RRNs, and reproduced well-known results from the literature. This model could serve as the basis for using RRNs to study

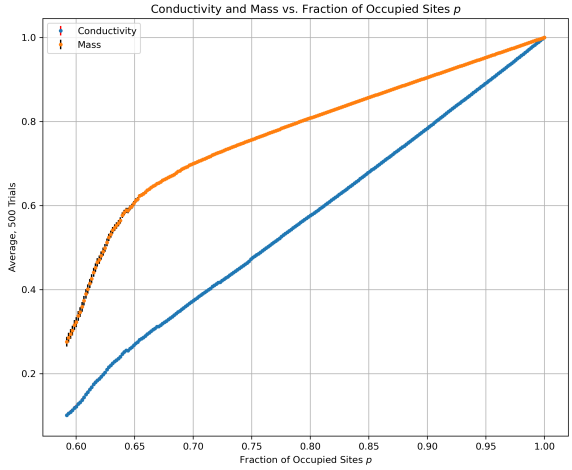

Fig. 7. Dependence of conductivity and mass on fraction of occupied sites, as determined by simulation of L=40.

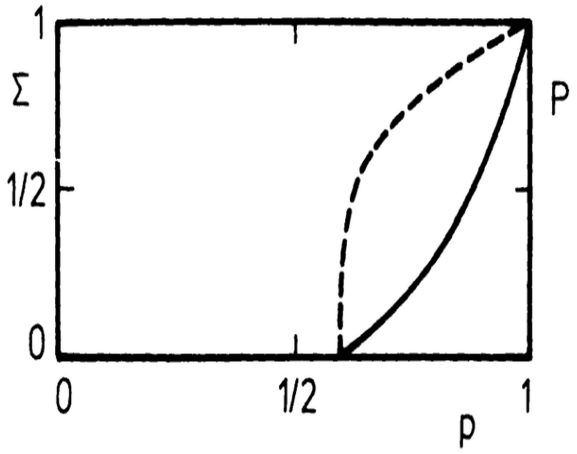

Fig. 8. Σ (solid) and mass (dashed) vs. p, as determined by experiment, taken from [5].

the electrical conductivity of a material, provided that it is some mix of insulating and conducting material that can be modeled as randomly distributed. One could for example use these simulations to determine at what threshold it transitions from being insulating to conducting. Another potential use for these simple simulations is to determine failure points in an electrical network. If one has a lattice made up of individual resistors that have some finite current tolerance, it is important to understand how often a resistor is singly connected, meaning it receives a large amount of current. In this case, this would be directly related to the likelihood that the network will fail. Such a study could easily be built on the work provided here! Other possibilities include studying a mix of superconductors and resistors, or other more exotic situations, as well as of course investigating the same parameters explored here, but for different geometries or numbers of dimensions. Due to their simplicity, RRNs provide an elegant and friendly introduction to simulating electrical conductivity.

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