1 Haar Measure

Definition 1.1. Haar Measure A Haar measure is a left-multiplication invariant Borel measure defined on a locally compact topological group G.

We describe how to compute Haar measures for G naturally homeomorphic to open subsets $U \subset \mathbb{R}^n$. The key is to guess that the Haar measure is a Radon measure absolutely continuous to the Lebesgue measure. In other words,

$$\phi(f) = \int_G f(x)w(x)dx.$$

where dx is the inherited Lebesgue measure on G. Invariance under left multiplication means

$$\phi(f) = \int_G f(x)w(x)dx = \int_G f(yx)w(x)dx = \phi(f(y \cdot \_)).$$

Let T be the change of variables $x \to yx$. When G is identified with U, we can compute $S(x) = \det(dT_x)$. Then our equation becomes

$$\int_G f(x)w(x)dx = \int_G f(x)w(y^{-1}x)|S(x)|dx.$$

We can apply the change of variables formula because we are working in the Lebesgue measure. Then it remains to guess a w such that

$$w(x) = w(y^{-1}x)|S(x)|.$$

Remark. Haar proved Haar measures are unique up to multiplication by a constant.

Remark. One can also derive right-invariant Haar measures by composing the left-invariant Haar measure with the group inverse.