Partial Differential Equations

Edward Zeng

May 8, 2020

Contents

1 Divergence Theorem

A vector field is a smooth map from $\mathbb{R}^n \to \mathbb{R}^n$.

Definition 1.1. Let F be the vector field given by $F = (f_1, \dots, f_n)$. The divergence of F is given by

$$\nabla \cdot F = \sum_{i=1}^{n} \frac{\partial f_i}{\partial e_i}.$$

Physical interpretation. $\nabla \cdot F(x)$ measures the "outwardness" of the vector field arrows at x. In other words, the amount of vector field arrows x spews out minus the amount of arrows x absorbs.

Theorem 1.1 (Divergence Theorem)**.** *Let $B \subset \mathbb{R}^n$, and let n be the normal vector on ∂B. Then*

$$\int_B \nabla \cdot F = \int_{\partial B} F \cdot n.$$

Physical interpretation. Integrating divergence over B gives the net difference of vector field arrows spewed out and absorbed in B. This is, intuitively, the amount of vector field arrows crossing the boundary, i.e. flux.

Remark. We do not go into the conditions we impose on B (compact smooth orientable MwB, etc.). Most "nice" objects work.

2 Balance Laws

Continuous mechanics is the study of continuous substances. We present a general method from which PDEs in this field often arise.

Suppose we have a mysterious substance S in $\mathbb{R}^n$.

- Let $u(x, t)$ describe the density of S.
- Let $Q(x)$ be a vector field that describes how S flows. This usually comes from a physical law.
- Let $f(x, t)$ describe the rate of creation/desruction of S. This becomes the nonhomogeneous part of the PDE.

We pick a region $\Omega \subset \mathbb{R}^n$. Then

$$\frac{d}{dt} \int_\Omega u dV = \begin{pmatrix} \text{rate of in/out} \\ \text{flow on } \partial\Omega \end{pmatrix} + \begin{pmatrix} \text{rate of creation} \\ \text{or destruction in} \\ \Omega \end{pmatrix} = -\int_{\partial\Omega} Q \cdot n dA + \int_\Omega f dV.$$

Applying the Divergence theorem and Leibiniz's rule, we get

$$\int_\Omega u_t dV = -\int_\Omega \nabla \cdot Q dV + \int_\Omega f dV.$$

As Ω is an arbitrary region, we conclude that $u_t = -\nabla \cdot Q + f$.

3 Heat Equation

3.1 Statement

The heat equation is

$$u_t = k u_{xx}, \quad k > 0.$$

It is derived via balance laws with Fourier's law of heating, i.e. heat flows in the direction of steepest descent.

> *Physical Interpretation.* The heat equation can be interpreted as: "the change in heat is proportional to the concavity of the heat profile." In other words, the change in temperature at point p is proportional to how much hotter/colder the surrounding temperature is.

3.2 Heat Equation on $\mathbb{R}$

Because the heat equation is easiest to study without boundary conditions, we first study the heat equation on $\mathbb{R}$.

3.2.1 Fundamental Solution

An important solution to the heat equation on $\mathbb{R}$ for $t > 0$ is given by the Gaussian:

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{\frac{-x^2}{4kt}}.$$

Φ is called the kernel kernel/fundamental solution and is often derived via an Ansatz.

> *Remark.* Φ is undefined for $t = 0$. It is only a solution for $t > 0$ but it approaches the Dirac delta as $t \to 0$.

3.2.2 Initial Conditions

Now suppose we are given the initial conditions $u(x,0) = g(x)$. First observe that $\Phi(x-y,t)g(y)$ is a solution to the heat equation. By linearity, this suggests that

$$\Phi * g = \int_{\mathbb{R}} \Phi(x-y,t) g(y) dy$$

is a solution to the heat equation.

Note $\Phi * g$ is not defined at $t = 0$. But because Φ limits to the Dirac delta as $t \to 0$, thus $\Phi * g \to g$ as $t \to 0$. It is in this sense that we have solved the IVP.

3.2.3 Nonhomogeneous Equation

The nonhomogeneous heat equation is

$$u_t = ku_{xx} + f(x,t).$$

Physical interpretation. Heat is being generated at a **rate** of $f(x,t)$ (in addition to regular heat flow).

This is a standard application of Duhamel's principle. The key idea behind Duhamel is that heat generation at a rate r over the time period $[s, s+\varepsilon]$ can be approximated by adding

$$v(x,t,s) = \begin{cases} 0 & t < s \\ \tilde{u}(x, t-s) & t > s, \end{cases}$$

where $\tilde{u}(x,t)$ is the solution to the homogeneous wave equation with the initial condition $\tilde{u}(x,0) = \varepsilon r$.

Thus, by linearity, the solution to nonhomogeneous heat equation with the initial condition $u(x,0) = 0$ can be well approximated by adding together the solutions of many homogeneous heat equations. We take the limit of this approximation, turn a Riemann sum into an integral, and get

$$u(x,t) = \int_0^t \int_{\mathbb{R}} \Phi(x-y, t-s) f(y,s) dy ds.$$

3.3 Maximum Principle

The maximum principle can be used to show uniqueness of the heat equation solution and continual dependence on data. Just consider the difference of two solutions.

Theorem 3.1 (Maximum Principle)**.** *The maximum temperature is always achieved on the boundary or initial conditions.*

Remark. This makes physical sense. Heat spreads out.

Key trick. Consider the pertubation $u + \varepsilon x^2$ to remove a possible degenerate maximum.

3.4 Heat Equation on an Interval

We want to solve the IVP

$$u_t = ku_{xx},$$
$$u(x,0) = f(x),$$

for $t \geq 0$, $x \in [0, 2\pi]$ and either homogeneous Neumann or Dirichlet boundary conditions.

Physical interpretation. We are studying heat propagation over a finite insulated rod. Homogeneous Neumann BC $\implies$ the ends of the rod are insulated; homogeneous Dirichlet BC $\implies$

- the ends are attached to isothermal heat sinks.

3.4.1 Separation of Variables

We solve the heat equation on an interval with a powerful Ansatz technique called separation of variables, i.e. we assume

$$u(x,t) = A(x)B(t).$$

Then the heat equation becomes $A''(x)B(t) = A(x)B'(t)$ (assume $k = 1$). Rearranging, we get

$$\frac{A(x)}{A''(x)} = \frac{B(t)}{B'(t)} = c, \quad c \text{ constant}.$$

Therefore one solution is

$$A(x) = e^{\sqrt{c}x}, B(t) = e^{ct}$$

where $\sqrt{c}$ is imaginary when $c < 0$. Because homogeneous Neumann and Dirichlet BCs imply heat cannot increase to ∞, thus we may assume $c = -\lambda^2 < 0$, and a solution to the heat equation becomes

$$u(x,t) = e^{-\lambda^2 t + i\lambda x}.$$

We take linear combinations of this solution (varying λ) to satisfy the initial and BCs.

> *Remark.* We can satisfy the initial conditions because of Fourier series. Satisfying BCs requires more care: use a sine or cosine Fourier series.