Reisz-Markov. If ϕ is a positive radon measure on X, then

$$\phi(f) = \int f d\mu_\phi,$$

where μ_ϕ is inner regular for open sets, outer regular, the σ-algebra of μ_ϕ contains the Borel sets.

The proof of the properties of μ_ϕ are easy. The proof of the integral representation can be divided into two parts.

Part I: ϕ and $\int d\mu_\phi$ play well with each other. Let $f \in C_c(X)$. Then

$$\chi_A \le f \le \chi_B \implies \int \chi_A d\mu_\phi = \mu_\phi(A) \le \phi(f) \le \mu_\phi(B) = \int \chi_B d\mu.$$

Proof gist. We prove each inequality separately. We always first prove for open sets, and then generalize.

More detail. We need to play around with the following tricks.

- Vector lattice tricks (see lecture 3-3-2020).
- Positive radon measures are continuous for the inductive limit topology! So if $\{f_n\} \to f$ converges in the inductive limit topology, then $\phi(f_n) \to f$.

Part II. For $f \in C_c(X)$ and $\varepsilon > 0$,

$$|\phi(f) - \int f d\mu_\phi| < \varepsilon.$$

From an intuitive perspective, it is not hard to see that Part I is an important tool to craft out Part II. The harder part is the formalization and fleshing out the details.

Proof gist. Break apart f into the sum of many small functions, apply Part I to the small functions, and the use the triangle inequality to put everthing back.

More detail. The explicit construction is to break up the range of f. Define $f_n = f \wedge n\varepsilon$, let $g_n = f_{n+1} - f_n$, and observe

$$\varepsilon\{x : f(x) > (n+1)\varepsilon\} \le g_n \le \varepsilon\{x : f(x) > n\varepsilon\}.$$