In the next few lectures, we wish to study the dual space of $C_c(X)$, where X is LCH. We call a positive linear functional on $C_c(X)$ a positive Radon measure (PRM). It is our end goal to characterize all PRMs as integrals (Riesz-Markov), so we begin by constructing a measure for every PRM. An overview of the construction is as follows: PRM $\to$ content $\to$ outer measure $\to$ measure.

If ϕ is a PRM, define μ_ϕ on open sets by

$$\mu_\phi(U) = \sup\{\phi(f) : 0 \leq f \leq \chi_U,\ \mathrm{supp}\{f\} \subset U\}.$$

Then μ_ϕ is a **content**, i.e. it satisfies

- If U is open with $\bar{U}$ compact, then $\mu_\phi(U)$ is finite. This is because if U is open with $\bar{U}$ compact, then $C_\infty(U) \subset C_c(X)$.
- μ_ϕ is monotone. Easy.
- μ_ϕ is countably subadditive. Use partition of unity to break up a large function to one defined on many small domains.
- μ_ϕ is finitely additive. Not hard.

A content can be thought about "finitely additive measure." We extend content μ_ϕ to an outer measure by defining

$$\mu_\phi^*(A) = \inf\{\mu_\phi(U) : A \subset U\}.$$

Countable subadditivity follows from definition and countable subadditivity of ν. We then use Caratheodory's theorem to filter the outer measure into a measure, which we also denote as μ_ϕ (note μ_ϕ denotes both the content and the measure).