Key Equation. Suppose μ, ν are two σ-finite measures. Observe that the map

$$\phi(f) = \int f d\nu$$

is an element of the dual space of $\mathcal{L}^1(\mu + \nu)$. Thus there exists an $h \in \mathcal{L}^\infty(\mu + \nu)$ such that

$$\int f d\nu = \int f h d(\mu + \nu) = \int f h d\mu + \int f h d\nu,$$

i.e.

$$\int f(1 - h) d\nu = \int f h d\mu.$$

Lebesgue Decomposition. Observe $\|h\| \leq 1$. Define

$$E = \{x : h(x) = 1\}, \quad F = X \backslash E$$

and use the equation to note that $\nu|_E$ is singular to μ and $\nu|_F$ is absolutely continuous to μ.

Radon-Nikodyn. When the set $\{x : h(x) = 1\}$ has measure 0 (i.e. ν is absolutely continuous w/r/t μ), we choose $f = \chi_G$ and note

$$\nu(G) = \int \chi_G d\nu = \int \frac{h\chi_G}{1 - h} d\mu.$$

Theorem. For $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and finite measure space X, there is an isometric bijection between the positive elements of $\left(\mathcal{L}^p(\mu)\right)'$ and $\mathcal{L}^q(\mu)$.

Proof gist. For positive $\phi \in \left(\mathcal{L}^p(\mu)\right)'$, observe

$$\nu(E) = \int \phi(\chi_E) d\mu$$

is a measure absolutely continuous to μ. Then by Radon-Nikodyn, there exists a positive g such that

$$\nu(E) = \int_E g d\mu.$$

We show that $g \in \mathcal{L}^q(\mu)$. The other direction comes from Holder and the following observation proved via simple functions:

$$\phi(f) = \int f d\nu = \int f g d\mu$$

Remark. In the usual fashion, this can be extended to the σ-finite case.