Hahn-Banach Separation Theorem. Let V be a real normed vector space. Suppose O and C are disjoint convex subsets, with O open. Then there exists a linear functional ϕ that separates O and C, i.e.

$$\phi(O) < t \leq \phi(C)$$

where t is a constant.

Proof gist. Use the Hahn-Banach theorem to construct a functional ϕ such that

$$\phi(O - C) < 0.$$

In more detail. $O - C$ is an open convex set not containing the origin, and let $v_0 \in O - C$. Pictorially, we have

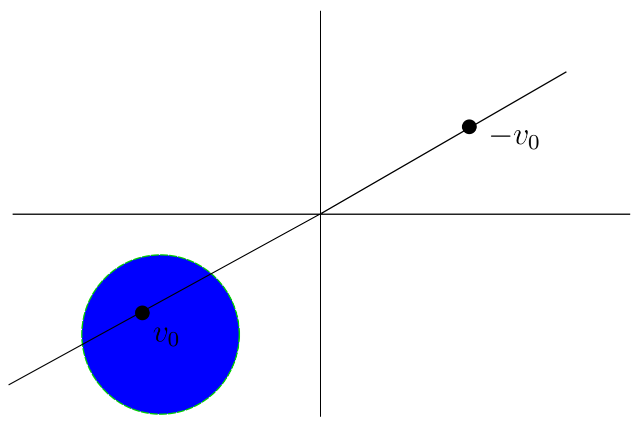

Figure 1: A convex open set

If we define ϕ on Rv_0 with $\phi(-v_0) = 1$ and make the obvious extension (in this finite dimensional case), then clearly $\phi(O - C) < 0$. The general case is a more technical version of this.

Extra detail. To pick the appropriate gauge to make the extension in the general case, let $U = O - C - v_0$ and define

$$m_u(v) = \inf\{s \in \mathbb{R}^+ : \frac{v}{s} \in U\}.$$

Then proceed as before, but use the gauge m_u to perform the extension.

Hahn-Banach Extension for spaces over $\mathbb{C}$. Let V be a complex normed vector space, W a subspace, and p a semi-norm. If ϕ is a continuous linear functional on W dominated by p, then there exists a continuous linear extension $\tilde{\phi}$ on V dominated by p.

Proof gist. Use the real Hahn-Banach theorem to construct $\tilde{\phi}$.

More detail. Let $\psi = \Re\phi$ on W, and pretend V is a real vector space (i.e. iv is not a multiple of v). Then use the real version of Hahn-Banach to extend ψ to $\tilde{\psi}$ on V. Now define

$$\tilde{\phi}(v) = \tilde{\psi(v)} - i\tilde{\psi(iv)}$$

and show $\tilde{\phi}$ satisfies the properties we want.