Let A be an $n \times n$ Hermitian matrix, with eigenvalues $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$. Let v_i be a unit eigenvector corresponding to the eigenvalue $\lambda_i(A)$, and let $v_{i,j}$ be the j-th component of v_i. Then

$$|v_{i,j}|^2 \prod_{k=1;k\neq i}^{n} (\lambda_k(A) - \lambda_i(A)) = \prod_{k=1}^{n-1} (\lambda_k(M_j) - \lambda_i(A))$$

where M_j is the $(n-1) \times (n-1)$ Hermitian matrix formed by deleting the j-th row and column from A.

Proof. We start by making a few simplifying assumptions. Set $j = 1$ and fix i. Note that if $\lambda_i(A) \neq 0$, we can instead consider the Hermitian matrix $A - \lambda_i(A)I$, so we may suppose $\lambda_i(A) = 0$. Therefore, the identity becomes

$$|v_{i,1}|^2 \prod_{k=1;k\neq i}^{n} \lambda_k(A) = \det(M_1).$$

Suppose $A = (a_{mn})$ and choose $v_1, \ldots v_n$ to be a basis of unit eigenvectors corresponding to the eigenvalues $\lambda_1(A), \ldots \lambda_n(A)$. Then let

$$A_t = \begin{pmatrix} a_{11} + t & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad V = \begin{pmatrix} v_{1,1} & v_{2,1} & \cdots \\ v_{1,2} & v_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Observe that $\det(A_t) = t \det(M_1)$. Thus it suffices to show

$$\frac{\det(A_t)}{t|v_{i,1}|^2 \prod_{k=1;k\neq i}^{n} \lambda_k(A)} \to 1 \tag{1}$$

as $t \to 0$. (Note we assume A has all nonzero eigenvalues except $\lambda_i(A)$, or else the identity is trivial.) To do so, note that V^*AV is the diagonal matrix of eigenvalues (here V^* is the conjugate tranpose). Hence

$$V^* A_t V = \begin{pmatrix} t|v_{1,1}|^2 + \lambda_1(A) & t\overline{v_{1,1}} v_{2,j} & \cdots \\ t\overline{v_{2,1}} v_{1,1} & t|v_{2,j}|^2 + \lambda_2(A) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Keeping in mind that $\lambda_i(A) = 0$ and V is unitary, we get

$$\det(A_t) = t|v_{1,i}|^2 \prod_{k=1;k\neq i}^{n} \lambda_k(A) + t^2 \Big(\cdots\cdots \Big)$$

which proves (1).