Tensors, Self Adjoint Linear Operators

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Today, I decided to take a look at Halmos's book *Finite-Dimensional Vector Spaces*. I was pleasantly surprised by the presentation of the material, and I would like to share a few observations and notes.

Tensors

Recall the construction of the tensor product of vector spaces U and V goes something like this:

$$U \times V \to \mathbb{F}\langle U \times V \rangle \to U \otimes V$$

where the last step involves quotienting out a subspace. (Here, $\mathbb{F}\langle . \rangle$ denotes the free space.) I've always wanted see alternate contruction, which is why I was excited to see Halmos's simple (and obvious!) solution.

Definiton. $U \otimes V$ is the space of bilinear forms $f : U \oplus V \to \mathbb{F}$.

Similarly, we can define the k-fold exterior algebra of U as the space of alternating multilinear forms on $U \oplus U \oplus \cdots \oplus U$. Personally, I prefer this simpler construction over the one I was taught.

Self-Adjoint Linear Operators

Recall now that a linear map $A : U \to V$ induces the **dual map** $A' : V' \to U'$. Halmos calls this the **adjoint**. The key result here is that the matrix of A with respect with some choice of bases for U and V is the transpose of the matrix A' with respect to the corresponding dual bases.

Now, if U is a (finite-dimensional) inner product space, there is a natural bijection between U and U'. More specifically:

Theorem. Suppose $y' \in U'$, where U' is the dual space of a (finite-dimensional) inner product space U. There there exists a unique vector $y \in U$ such that

$$y'(x) = (x, y), \quad \forall x \in U.$$

(Here, $(.,.)$ denotes the inner product.)

In fact, the bijection between U and U' is a conjugate isomorphism, i.e. it happens that

$$(ay')(x) = (x, \bar{a}y).$$

It also happens that this bijection induces an inner product on U', so we will often denote the dual space U' as U^* instead to emphasize this. We observe now that for any linear operator

$A : U \to U$, our usual definition of the adjoint $A' : U^* \to U^*$ induces a linear map $A^* : U \to U$ via the identification between U and U^*. (Yes, the notation is slightly off. A' is an operator on U^*. A^* is an operator on U.)

Unlike before, the matrix of A^* is the *conjugate* transpose of the matrix of A (with respect to corresponding bases). This is where the idea of conjugate transpose comes from! (To be sure, the matrix of A' is the transpose of A, but the matrix of A^* is the conjugate transpose of A, even though A' naturally induces A^*.)

Halmos makes an important heuristic observation here:

the space of linear operators on U acts like the field of complex numbers.

For instance, $A \mapsto A^*$ is analogous to conjugation. From this, we deduce that the set of self-adjoint operators (i.e. **Hermitian matrices**) is analogous to the set of real numbers! (Similarly, the unitary matrices are analogous to the unit circle, the skew-symmetric matrices are analogous to imaginary numbers, etc.)

From this, I am starting to understand the reason why Hermitian matrices were so emphasized in the study of Toeplitz forms. Halmos's heuristic analogy isn't perfect (division breaks down a little), but does hold some fundamental truths. Hint: eigenvalues of Hermitian matrices are always real.