A possible point of confusion: μ_ϕ denotes both the measure and the content. Use context.

Here, we prove that the measurable sets of μ_ϕ contains the Borel σ-algebra, and thus can be restricted to the Borel σ-algebra. This requires some of the following concepts:

If μ is a measure or outer measure, then μ is:

- inner regular if the measures of sets can be approximated from below by the measures of compact sets,
- outer regular if the measures of sets can be approximated from above by the measures of open sets.

The content μ_ϕ is inner regular for open sets! However, we require a slightly modified definition of inner regularity as μ_ϕ is defined only on the open sets. What we mean is

$$\mu_\phi(U) = \sup\{\mu_\phi(V) : V \text{ open}, \ \bar{V} \subset U\}, \quad U \text{ open}.$$

Theorem. Open sets are measurable in μ_ϕ.

Proof gist. For open set U, we want to show for any $A \subset X$, we have

$$\mu_\phi^*(A - U) + \mu_\phi^*(A \cap U) = \mu_\phi^*(A).$$

Use inner regularity for open sets to prove for A open, then extend using outer regularity.

More detail. To prove for A open, we need to use LCH Urysohn's friend and finite additivity. The key is that $A - U$ is not necessarily open, so we need to do a few tricks before applying finite additivity.

Inner regularity for open sets. Because μ_ϕ is inner regular for open sets, this extends to μ_ϕ^*. It is also easy to see that μ_ϕ^* is outer regular.