Gauge. Let V be a vector space. A gauge is a function $p : V \to \mathbb{R}$ that is "half-linear", i.e.

- For $r > 0$, we have $p(rv) = rp(v)$.
- $p(u + v) \leq p(u) + p(v)$.

Main lemma for Hahn-Banach. A linear functional defined on a subspace W of V subordinate to gauge p can be extended to a subordinate linear functional defined on $W \oplus \mathrm{span}(v_0)$.

Proof gist. Show the existence of α such that

$$\tilde{\phi}(w + rv_0) = \phi(w) + r\alpha$$

is subordinate to p. Key trick is a separation of variables.

Hahn-Banach. A linear functional defined on a subspace W of V subordinate to gauge p can be extended to a subordinate linear functional defined on V.

Proof gist. The main lemma extends linear functionals one dimension at a time. Apply Zorn's lemma to it (by considering the family of pairs of vector subspaces and subordinate linear functionals defined on them).

Remark. To show the existence of continous linear functionals on normed vector spaces, let the gauge p be the norm. Then any linear functional subordinate to p is continuous (indeed, Lipschitz).

Quotient norm. If W is a closed subspace of Banach space V, then equip V/W with the norm

$$\|\pi(v)\| = \inf\{\|v - w\| : w \in W\}.$$

Note that if W is not closed, then we only get a semi-norm.

Tweak technique. There are a few miscellaneous statements about quotient norms and spaces, and they often use the following technique:

- Fix $\|\pi(v)\| = 1$. Tweak $v' = v + w$ until you get a specific property. Then you can show that there exists some v' with a specific property such that $\|v'\| = 1$ (or whatever number).

As an exercise, prove the following statement: For any $v \in V$, there is a $\phi \in V'$ such that $\phi(W) = 0$ and $\phi(v) = \|\pi(v)\|$.

Theorem. V/W is a Banach space.

Proof gist. "Pull back" a rapidly Cauchy sequence from V/W to V. Tweak technique is needed.

Theorem. Space of continuous linear operators $\mathcal{B}(V, W)$ is a Banach space.

Proof gist. Let T be the point of convergence of a Cauchy sequence. Show T is bounded and linear (very routine).

For vector space V, the following are equivalant:

- The topology on V is the initial topology from a collection of semi-norms on V.
- The topology on V is translation invariant and has a subbase of convex sets.

When the above scernaios occur, we say the topology on V is **locally convex**.

Alaoglu's theorem. If V is a normed vector space, then the closed unit ball B in V' is compact for the weak-* topology.

Point of confusion. Two different topologies are used in the statement of the theorem. The closed unit ball is defined by the norm-topology, but the actual topology on V' is weak-*.

Proof gist. Use Tynchoff's theorem to construct a compact space and establish a homeomorphism between that compact space and B with the weak-* topology.

More detail. Define $D_v = \{t \in \mathbb{R} : |t| < ||v||\}$. Then consider the map

$$J : B \to \prod_{v \in V}^{\infty} D_v = P$$

given by the (informal) expression

$$J(\phi) \mapsto \prod_{v \in V}^{\infty} \phi(v).$$

By comparing the subbases of B and $J(B)$, we conclude they are homeomorphic. To show $J(B)$ is closed (hence compact), we note each element of P determines a function f. If f is in the closure of $J(B)$, then by approximating f by elements of $J(B)$, we show f is linear and thus an element of $J(B)$.

Hahn-Banach Separation Theorem. Let V be a real normed vector space. Suppose O and C are disjoint convex subsets, with O open. Then there exists a linear functional ϕ that separates O and C, i.e.

$$\phi(O) < t \leq \phi(C)$$

where t is a constant.

Proof gist. Use the Hahn-Banach theorem to construct a functional ϕ such that

$$\phi(O - C) < 0.$$

In more detail. $O - C$ is an open convex set not containing the origin, and let $v_0 \in O - C$. Pictorially, we have

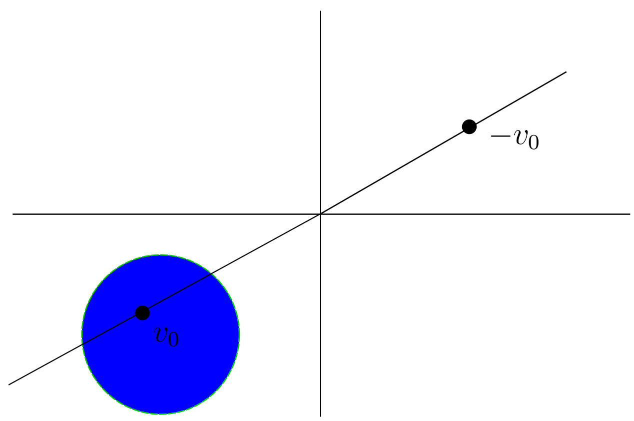

Figure 1: A convex open set

If we define ϕ on Rv_0 with $\phi(-v_0) = 1$ and make the obvious extension (in this finite dimensional case), then clearly $\phi(O - C) < 0$. The general case is a more technical version of this.

Extra detail. To pick the appropriate gauge to make the extension in the general case, let $U = O - C - v_0$ and define

$$m_u(v) = \inf \{s : \frac{v}{s} \in U\}.$$

Then proceed as before, but use the gauge m_u to perform the extension.

Hahn-Banach Extension for spaces over $\mathbb{C}$. Let V be a complex normed vector space, W a subspace, and p a semi-norm. If ϕ is a continuous linear functional on W dominated by p, then there exists a continuous linear extension $\tilde{\phi}$ on V dominated by p.

Proof gist. Use the real Hahn-Banach theorem to construct $\tilde{\phi}$.

More detail. Let $\psi = \Re\phi$ on W, and pretend V is a real vector space (i.e. iv is not a multiple of v). Then use the real version of Hahn-Banach to extend ψ to $\tilde{\psi}$ on V. Now define

$$\tilde{\phi}(v) = \tilde{\psi(v)} - i\tilde{\psi(iv)}$$

and show $\tilde{\phi}$ satisfies the properties we want.

The weak topology is tight. Suppose V is a vector space, and W is a collection of linear functionals on V. Then if ϕ is a continuous linear functional for the W-weak topology, $\phi \in W$.

Key trick. If $\phi_1, \ldots, \phi_n$ are a collection of linear functionals, then

$$\bigcap \ker \phi_i \subset \ker \phi \implies \phi \in \operatorname{span}\{\phi_i\}.$$

To show the LHS, we compare the topology generated by ϕ with the subbase of the weak topology.

Krein-Milman. Let C be a closed convex subset of a locally convex topological vector space V (assumed Hausdorff). Then the convex hull of extreme points of C is C.

Main idea. Consider the poset P of all the faces of C. Then the minimal points of P are the extreme points (need Zorn for existence). Then use this property to prove this theorem.

Key technique. If D is compact convex set, we consider a continuous linear functional ϕ not constant on D (use H-B or H-B separation). Then

$$\{v \in D : \phi(v) \text{ achieves its maximum.}\}$$

is a proper (compact convex) face.

The parallelogram inequality $\|v+w\|^2 + \|v-w\|^2 = 2(\|v\|^2 + \|w\|^2)$ guarantees a closest point to a convex closed set in Hilbert space $\mathcal{H}$. This is how we define orthogonal projection.

The decomposition $\ker\phi + (\ker\phi)^\perp$ is used to construct Reisz Representation.

Theorem. For $\mu(x) < \infty$, $\Big(\mathcal{L}^1(X, \mathcal{S}, \mu)\Big)'$ is isometric and isomorphic to $L^\infty(X, \mathcal{S}, \mu)$.

Main idea. Manuever $\mathcal{L}^1$ into the $\mathcal{L}^2$ setting and apply the Reisz-Representation theorem.

More detail. Observe for $\mu(X) < \infty$, we have $\big(\mathcal{L}^1(\mu)\big)' \subset \big(\mathcal{L}^2(\mu)\big)'$. Thus by R-R, for every $\phi \in \big(\mathcal{L}^1(\mu)\big)'$ there exists a g such that

$$\phi(f) = \int fg.$$

We then show $g \in \mathcal{L}^\infty(\mu)$. Conversely, we then show every $g \in \mathcal{L}^\infty(\mu)$ defines a continuous linear functional on $\mathcal{L}^1(\mu)$ by the equation above.

Key trick. To g is bounded a.e., we show for a closed set C, we have

$$\frac{1}{\mu(E)} \int_E g \in C, \ \forall \ \mu(E) > 0 \implies g(x) \in C \text{ a.e.}$$

To do this, we show any open ball disjoint with C must have preimage measure 0. Then as the range of g is separable, we take a countable union of all such open balls.

Motivation for first step. For spaces of finite measure, $\mathcal{L}^2 \subset \mathcal{L}^1$. Thus it makes sense for linear functionals on $\mathcal{L}^1$ to also be linear functionals on $\mathcal{L}^1$.

Extension remarks. This theorem can be extended to σ-finite measures (as many statements about finite measures can).

Key Equation. Suppose μ, ν are two σ-finite measures. Observe that the map

$$\phi(f) = \int f d\nu$$

is an element of the dual space of $\mathcal{L}^1(\mu + \nu)$. Thus there exists an $h \in \mathcal{L}^\infty(\mu + \nu)$ such that

$$\int f d\nu = \int f h d(\mu + \nu) = \int f h d\mu + \int f h d\nu.$$

Lebesgue Decomposition. Observe $\|h\| \leq 1$. Define

$$E = \{x : h(x) = 1\}, \quad F = X \backslash E$$

and use the equation to note that $\nu|_E$ is singular to μ and $\nu|_F$ is absolutely continuous to μ.

Radon-Nikodyn. Noting

$$\int f(1-h) d\nu = \int f h d\mu,$$

with a clever choice of f we conclude

$$\nu(G) = \int \chi_G d\nu = \int \frac{h \chi_G}{1-h} d\mu$$

when the set $\{x : h(x) = 1\}$ has measure 0, i.e. ν is absolutely continuous with respect to μ (c.f. above).

Theorem. For $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and finite measure space X, there is an isometric bijection between the positive elements of $\left(\mathcal{L}^p(\mu)\right)'$ and $\mathcal{L}^q(\mu)$.

Proof gist. For positive $\phi \in \left(\mathcal{L}^p(\mu)\right)'$, observe

$$\nu(E) = \int \phi(\chi_E) d\mu$$

is a measure absolutely continuous to μ. Then by Radon-Nikodyn, there exists a positive g such that

$$\nu(E) = \int_E g d\mu.$$

We show that $g \in \mathcal{L}^q(\mu)$. The other direction comes from Holder.

Remark. In the usual fashion, this can be extended to the σ-finite case.

Lattices! Here's a way to remember the two lattice operations: pretend that the shapes represent the set of bounds.

- The $\wedge$ shape looks like one point is greater than all the points, so this represents the greatest lower bound.
- Similarly, $\vee$ represents the least upper bound.

In order of increasing structure, we have: lattice ordered (abelian) group $\to$ lattice ordered vector space $\to$ lattice ordered normed vector space. A partial order is compatible with a normed vector space V if

- For $v, w \in V$, $v, w \geq 0 \implies v + w \geq 0$.
- For $v \in V$, $r \in \mathbb{R}$, $v \geq 0, r \geq 0 \implies rv \geq 0$.
- For $v \in V$, $\|v\| = \||v|\|$. (c.f. below for definition of $|\cdot|$).
- For $v, w \in V$, $0 \leq v \leq w \implies \|v\| \leq \|w\|$.

The definitions of all the structures mentioned can be interpolated from this.

Here are some key properties and definitions of lattices (with the appropriate structure):

- We can impose an order structure in a group by defining a collection of elements to be positive. (This collection has to satisfy some additional properties, evidently not all collections work).
- We define $v^+ = v \wedge 0$, $v^- = (-v) \wedge 0$, $v^+ + v^- = |v|$.
- For function spaces, we usually use the order: $f \geq 0$ means $f(x) \geq 0$ for all x. For spaces of functionals, we usually use the order $f \geq 0$ means $f(x) \geq 0$ if $x \geq 0$ (for the function space order).

Dense subsets of $C_{\mathbb{R}}(X)$. Let X be a compact space, and L a subspace and sublattice of $C(X)$. If L strongly separates points in X, then L is dense in $C(X)$ (for the sup norm).

Remark. Strong seperation implies X is necessarily Hausdorff.

Proof gist. Given $f \in C(X)$, we construct a $g \in L$ such that $\|f - g\| < \varepsilon$. To do this, we use strong separation to generate families of functions in L, then piece these functions together using compactness and the $\wedge$, $\vee$ operators.

Real stone-Weierstrass. Same statement as before, but instead of L, we consider subalgebra A.

Proof gist. We prove $\bar{A}$ is a subspace and sublattice, then invoke our previous theorem.

Key tricks. There are two key tricks to show that $\bar{A}$ is sublattice. The first is to note

$$f + g = \frac{f + g + |f - g|}{2}, \quad |f| = \sqrt{f^2}.$$

The second is to observe that $\sqrt{t}$ can be uniformly approximated on a compact interval by a polynomial $p(t)$. This uses some power series knowledge.

Extension. We can extend Stone-Weierstrass to the complex case if we force A to be closed under complex conjugation.

Theorem. The dual of a normed vector lattice V is a normed vector lattice (with the order usually associated with functionals).

Proof gist. Given $\phi \in V'$, we independently construct $\phi^+ \in V'$ and show $\phi^+ = \phi \vee 0$. The translation properties of ordered vector spaces then shows V' is lattice ordered. We then check the rest of the properties of a normed vector lattice (some inequalities and bashing required).

Construction of ϕ^+ We first define ϕ^+ on V^+ by

$$\phi^+(v) = \sup\{\phi(x) : 0 \leq x \leq v\}$$

and prove it is linear. Then we linearly extend ϕ^+ to V and show it is continuous.

Some techniques used.

- Prove something for positive v and extend via $v = v^+ - v^-$.
- Prove some inequality for variables x and y satisfying some condition. The inequality holds if we take the supremum over x and y with this condition.
- Use the fact that V is a normed vector lattice!

Remark. We already showed that if p, q are Holder conjugates with $1 < p, q < \infty$, then there is an isometric bijection between $(\mathcal{L}^p)'$ and $\mathcal{L}^q$. The theorem above extends this statement to all of $(\mathcal{L}^p)'$ and $\mathcal{L}^q$.

Theorem. The space of real measures $\mathcal{S}$ forms a vector lattice, with the positive elements of the lattice being the finite positive measures.

Proof gist. For $\mu \in \mathcal{S}$, we show that the total variation measure is $|\mu|$ (here, $|\cdot|$ is interpreted in the lattice sense). As we have seen before, this implies that $\mathcal{S}$ is a lattice.

More detail. Let $\|\mu\|$ denote total variation measure. We show $\|\mu\|$ is countably additive by the usual argument (prove the inequality both ways, consider disjoint unions, etc). Then we show $\|\mu\|$ is finite by the following technicalitiy:

- Real (and Banach) measures need to be absolutely convergent by definition, i.e. for disjoint sets E_n,
$$\sum \mu(E_n) = \sum \mu(E_j)$$
where E_j represents a different ordering.
- If $\|\mu\|(E) = \infty$ (i.e. E is unbounded) but $\mu(E) < \infty$, we construct a sequence of disjoint sets whose "partial sums" are not absolutely convergent.

Finally we show $\|\mu\| = |\mu|$ by using the fact

$$\|\mu(E)\| = \sup\{\mu(E_1) - \mu(E_2) : E_1 \sqcup E_2 = E\}.$$

A final detail. The construction of the sequence of disjoint sets mentioned above is as follows:

- split E into two disjoing unbounded sets E_1, F_1 where $|\mu(F_1)| > 1$.
- repeat for $E = E_1$

Our result is $F_1, F_2, F_3 \dots$.

Remark. It is not difficult to see that total variation is also a norm, and this norm plays well with the lattice. In other words, $\mathcal{S}$ is a normed vector lattice.

Some observations. Working in a lattice space allows us to use a few tricks:

- We can decompose $x = x^+ - x^-$, so we need only study positive elements. We have already seen this many times.
- Given x, we can create elements like $x \vee 1$, $x \wedge 1$, etc. More generally, we can "combine" multiple elements. This is a powerful property for construction (c.f. Stone Weierstrass).

The following toy example will show the power of the second trick. Then we will use the second trick to prove partitions of unity.

Toy example (a generalization of Hahn-Banach). Suppose μ, ν are real measures such that $\mu \wedge \nu = 0$. Then μ and ν are mutually singular.

Proof gist. If not, construct a strictly positive measure smaller than both.

More detail. Afer applying Lebesgue Decomposition and then R-N, we get

$$\int_E h d\mu = \nu_{ac}(E).$$

If $h \neq 0$ a.e., consider the strictly positive measure induced by $h \wedge 1$.

Partitions of Unity for LCH Spaces. If X is LCH, then for any compact $C \subset X$ and open cover θ_i of C, there exists positive functions $f_i \in C_C(X)$ such that $\sum f_i = 1$ on C and $\text{supp}(f_i) \subset \theta_i$.

Remark. Support is defined as the closure of the carrier.

Proof gist. Use LCH Urysohn's lemma (and friend) to construct a function

$$g = \sum g_i \geq 1.$$

Then consider

$$\frac{g}{g \vee 1}.$$

Construction for first equation. We first show there exists closed sets $B_i \subset \theta_i$ such that $C \subset \cup B_i$. (Need LCH Urysohn's friend). Then use LCH Urysohn's lemma to define $g_i(x) = 1$ for $x \in B_i$ and $\text{supp}(g_i) \subset \theta_i$.

- *LCH Urysohn's friend* (used to prove LCH Urysohn). X LCH, subset C compact, $C \subset U$ open. Then there exists open V, $\bar{V}$ compact, such that $C \subset V \subset \bar{V} \subset U$.
- *LCH Urysohn's lemma.* If subset C is compact and $C \subset \theta$ is open, then there exists a continuous $f : X \to [0, 1]$ with $f(x) = 1$ for $x \in C$ and $f(x) = 0$ for $x \notin \theta$.

Sorry, I come up with weird names.

In the next few lectures, we wish to study the dual space of $C_c(X)$, where X is LCH. We call a positive linear functional on $C_c(X)$ a positive Radon measure (PRM). It is our end goal to characterize all PRMs as integrals (Riesz-Markov), so we begin by constructing a measure for every PRM. An overview of the construction is as follows: PRM $\to$ content $\to$ outer measure $\to$ measure.

If ϕ is a PRM, define μ_ϕ on open sets by

$$\mu_\phi(U) = \sup\{\phi(f) : 0 \leq f \leq \chi_U,\ \text{supp}\{f\} \subset U\}.$$

Then μ_ϕ is a **content**, i.e. it satisfies

- If U is open with $\bar{U}$ compact, then $\mu_\phi(U)$ is finite. This is because if U is open with $\bar{U}$ compact, then $C_\infty(U) \subset C_c(X)$.
- μ_ϕ is monotone. Easy.
- μ_ϕ is countably subadditive. Use partition of unity to break up a large function to one defined on many small domains.
- μ_ϕ is finitely additive. Not hard.

A content can be thought about "finitely additive measure." We extend content μ_ϕ to an outer measure by defining

$$\mu_\phi^*(A) = \inf\{\mu_\phi(U) : A \subset U\}.$$

Countable subadditivity follows from definition and countable subadditivity of ν. We then use Caratheodory's theorem to filter the outer measure into a measure, which we also denote as μ_ϕ (note μ_ϕ denotes both the content and the measure).

A possible point of confusion: μ_ϕ denotes both the measure and the content. Use context.

Here, we prove that the measurable sets of μ_ϕ contains the Borel σ-algebra, and thus can be restricted to the Borel σ-algebra. This requires some of the following concepts:

If μ is a measure or outer measure, then μ is:

- inner regular if the measures of sets can be approximated from below by the measures of compact sets,
- outer regular if the measures of sets can be approximated from above by the measures of open sets.

The content μ_ϕ is inner regular for open sets! However, we require a slightly modified definition of inner regularity as μ_ϕ is defined only on the open sets. What we mean is

$$\mu_\phi(U) = \sup\{\mu_\phi(V) : V \text{ open}, \ \bar{V} \subset U\}, \quad U \text{ open}.$$

Theorem. Open sets are measurable in μ_ϕ.

Proof gist. For open set U, we want to show for any $A \subset X$, we have

$$\mu_\phi^*(A - U) + \mu_\phi^*(A \cap U) = \mu_\phi^*(A).$$

Use inner regularity for open sets to prove for A open, then extend using outer regularity.

More detail. To prove for A open, we need to use LCH Urysohn's friend and finite additivity. The key is that $A - U$ is not necessarily open, so we need to do a few tricks before applying finite additivity.

Inner regularity for open sets. Because μ_ϕ is inner regular for open sets, this extends to μ_ϕ^*. It is also easy to see that μ_ϕ^* is outer regular.

Reisz-Markov. If ϕ is a positive radon measure on X, then

$$\phi(f) = \int f d\mu_\phi,$$

where μ_ϕ is inner regular for open sets, outer regular, the σ-algebra of μ_ϕ contains the Borel sets.

The proof of the properties of μ_ϕ are easy. The proof of the integral representation can be divided into two parts.

Part I: ϕ and $\int d\mu_\phi$ play well with each other. Let $f \in C_c(X)$. Then

$$\chi_A \leq f \leq \chi_B \implies \int \chi_A d\mu_\phi = \mu_\phi(A) \leq \phi(f) \leq \mu_\phi(B) = \int \chi_B d\mu.$$

Proof gist. We prove each inequality separately. We always first prove for open sets, and then generalize.

More detail. We need to play around with the following tricks.

- Vector lattice tricks (see lecture 3-3-2020).
- Positive radon measures are continuous for the inductive limit topology! So if $\{f_n\} \to f$ converges in the inductive limit topology, then $\phi(f_n) \to f$.

Part II. For $f \in C_c(X)$ and $\varepsilon > 0$,

$$|\phi(f) - \int f d\mu_\phi| < \varepsilon.$$

From an intuitive perspective, it is not hard to see that Part I is an important tool to craft out Part II. The harder part is the formalization and fleshing out the details.

Proof gist. Break apart f into the sum of many small functions, apply Part I to the small functions, and the use the triangle inequality to put everthing back.

More detail. The explicit construction is to break up the range of f. Define $f_n = f \wedge n\varepsilon$, let $g_n = f_{n+1} - f_n$, and observe

$$\varepsilon\{x : f(x) > (n+1)\varepsilon\} \leq g_n \leq \varepsilon\{x : f(x) > n\varepsilon\}.$$

Inductive Limit Topology

Notation

Let X be LCH. Let $C_c(X)$ be the subset of $C(X)$ with compact support. For $S \subset X$, let $C_\infty(S)$ be the subset of $C(S)$ that "vanishes at infinity," i.e. ε-bounded outside a compact set.

Definition of ILT

The ILT on $C_c(X)$ is the strongest topology such that the natural inclusion

$$C_\infty(U) \to C_c(X)$$

is continuous for all open U with compact closure. (Here, we equip $C_\infty(U)$ with the $\|\cdot\|_\infty$ norm.)

Remark. *In general, there is no natural inclusion from $C(S) \to C_c(X)$ for any $S \subset X$. Nor is there a natural inclusion $C_\infty(U) \to C_c(X)$ if U is not open or does not have compact closure.*

Motivation

Why are we concerned with the ILT? Note that $C_c(X)$ with $\|\cdot\|_\infty$ norm makes all the maps $C_\infty(U) \to C_c(X)$ continuous, so the $\|\cdot\|_\infty$ topology is weaker than the ILT.

This makes the ILT topology easier to work with because the ILT allows for more continuous linear functionals than $\|\cdot\|_\infty$. (Think about it, a stonger topology means more continuous maps can be defined on it.) For instance, every positive Radon measure is continuous for the ILT, but this is not the case for $\|\cdot\|_\infty$.

Exercise 1. *For $X = \mathbb{R}$, what is an example of a open set in the ILT that is not an open set for $\|\cdot\|_\infty$? Hint: consider the positive Radon measure given by*

$$\phi(f) = \int f d\lambda$$

where λ is the usual measure on $\mathbb{R}$. Note ϕ is discontinuous for $\|\cdot\|_\infty$, and consider $\phi^{-1}(0,1)$.

Criterion for Continuity

Suppose T is a linear functional defined on $C_c(X)$. Then T is continuous for the ILT if for all U open with compact closure, the map

$$T|_{C_\infty(U)} : C_\infty(U) \to \mathbb{R}$$

is continuous. ($C_\infty(U)$ is equipped with the $\|\cdot\|_\infty$ norm.) Think about why.

Reisz-Markov

Here, we wish to provide a intuitive presentation of Reisz-Markov and a motivation of its proof.

Suppose ϕ is a positive linear functional on $C_c(\mathbb{R})$, i.e. a positive Radon measure. It is not far-fetched to hope for a representation like

$$\phi(f) = \int f d\mu_\phi.$$

After all, integration plays a large part in the dual spaces of L^q. So we ask ourselves: if such a representation existed, what would μ_ϕ look like?

Let f be the following function:

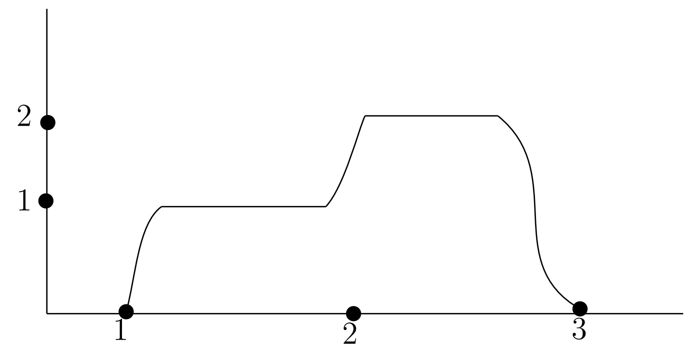

Figure 1: A Simple Example

By linearity, we can decompose $\phi(f) \approx 1 \cdot \phi(g_{(1,2)}) + 2 \cdot \phi(g_{(2,3)})$ where g_E is the continuous analog to the charateristic function χ_E, i.e.

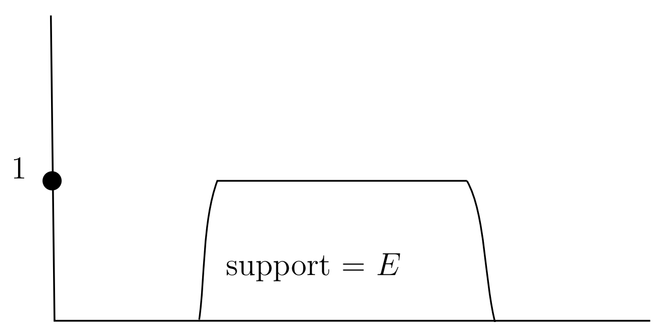

Figure 2: Decomposition

This suggests that we should set $\mu_\phi(E) = \phi(g_E)$. If you think about it (intuitively), this indeed turns ϕ into an integral. Hence, we start our construction of μ_ϕ by defining

$$\mu_\phi(U) = \sup\{\phi(f) : 0 \leq f \leq \chi_U,\ f \in C_c(X),\ \mathrm{supp}\{f\} \subset U\}$$

for open sets U. (Note as ϕ is positive, this supremum is really just approximating χ_U by continuous functions.)