Recently, Terrence Tao posted a few proofs of the following theorem on his blog:

> Let A be an $n \times n$ Hermitian matrix, with eigenvalues $\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)$. Let v_i be a unit eigenvector corresponding to the eigenvalue $\lambda_i(A)$, and let $v_{i,j}$ be the j-th component of v_i. Then
>
> $$|v_{i,j}|^2 \prod_{k=1;k\neq i}^{n} (\lambda_k(A) - \lambda_i(A)) = \prod_{k=1}^{n-1} (\lambda_k(M_j) - \lambda_i(A))$$
>
> where M_j is the $(n-1) \times (n-1)$ Hermitian matrix formed by deleting the j-th row and column from A.

Here, I will provide an alternate proof.

Proof. We start by making a few simplifying assumptions. Set $j = 1$ and fix i. Note that if $\lambda_i(A) \neq 0$, we can instead consider the Hermitian matrix $A - \lambda_i(A)I$, so we may suppose $\lambda_i(A) = 0$. Therefore, the identity becomes

$$|v_{i,1}|^2 \prod_{k=1;k\neq i}^{n} \lambda_k(A) = \det(M_1).$$

Recall that every Hermitian operator has a basis of eigenvectors. Therefore, if 0 is an eigenvalue of multiplicity > 1, then the kernel of A has dimension > 1. As a result, any principal minor has a kernel with dimension > 0, and the above identity trivially holds. Hence we may assume A has all nonzero eigenvalues except $\lambda_i(A)$.

Suppose $A = (a_{mn})$. Consider now the matrix

$$A_t = \begin{pmatrix} a_{11} + t & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Observe that $\det(A_t) = t \det(M_1)$. Thus, to prove the theorem, it suffices to show

$$\frac{\det(A_t)}{t|v_{i,1}|^2 \prod_{k=1;k\neq i}^{n} \lambda_k(A)} \to 1$$

as $t \to 0$. We do just that. (Remark: the denominator is nonzero. Why?).

From linear algebra, we know that every Hermitian matrix can be diagonalized by a change of basis represented by a unitary matrix. In particular, if $v_1, v_2, \dots, v_n$ form a basis of unit eigenvectors, then

$$\begin{pmatrix} \overline{v_{1,1}} & \overline{v_{1,2}} & \cdots \\ \overline{v_{2,1}} & \overline{v_{2,2}} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} A \begin{pmatrix} v_{1,1} & v_{2,1} & \cdots \\ v_{1,2} & v_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \lambda_1(A) & & \\ & \lambda_2(A) & \\ & & \ddots \end{pmatrix}.$$

Now note that

$$\begin{pmatrix} \overline{v_{1,1}} & \overline{v_{1,2}} & \cdots \\ \overline{v_{2,1}} & \overline{v_{2,2}} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} A_t \begin{pmatrix} v_{1,1} & v_{2,1} & \cdots \\ v_{1,2} & v_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \lambda_1(A) & & \\ & \lambda_2(A) & \\ & & \ddots \end{pmatrix} + \begin{pmatrix} t\overline{v_{1,1}}v_{1,1} & t\overline{v_{1,1}}v_{2,j} & \cdots \\ t\overline{v_{2,1}}v_{1,1} & t\overline{v_{2,1}}v_{2,j} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

We may rewrite the resulting matrix as

$$B_t = \begin{pmatrix} t|v_{1,1}|^2 + \lambda_1(A) & t\overline{v_{1,1}}v_{2,j} & \cdots \\ t\overline{v_{2,1}}v_{1,1} & t|v_{2,j}|^2 + \lambda_2(A) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Keeping in mind that $\lambda_i(A) = 0$, we get

$$\det(B_t) = t|v_{1,i}|^2 \prod_{k=1;k\neq i}^{n} \lambda_k(A) + t^2\Big(\cdots\cdots\Big).$$

As $\det(A_t) = \det(B_t)$, thus we have shown what we set out to prove.