Proof. Set $j = 1$. WLOG assume $\lambda_i(A) = 0$. Then the formula to be proved is

$$|v_{i,1}|^2 \prod_{k=1;k\neq i}^{n} \lambda_k(A) = \det(M_1). \tag{1}$$

First we note any Hermitian matrix A can be diagonalized by a unitary transformation

$$T^*AT = \begin{pmatrix} \lambda_1(A) & & \\ & \lambda_2(A) & \\ & & \ddots \end{pmatrix}$$

where the column vectors of T form a basis of unit eigenvectors,

$$T = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n).$$

Now observe the following lemma, which follows directly from how determinants are computed.

Lemma 1 *Let*

$$A_t = A + \begin{pmatrix} t & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{11}+t & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $\det(A_t) = \det(A) + t\det(M_1)$.

Applying T-transformation to A_t, we obtain

$$T^*A_tT = \begin{pmatrix} \lambda_1(A) & & \\ & \lambda_2(A) & \\ & & \ddots \end{pmatrix} + \begin{pmatrix} t\overline{v_{1,1}}v_{1,1} & t\overline{v_{1,1}}v_{2,j} & \dots \\ t\overline{v_{2,1}}v_{1,1} & t\overline{v_{2,1}}v_{2,j} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

When we take the determinant on both sides (using Lemma 1 on the LHS) and collect the t-linear terms, we obtain (1).