Some observations. Working in a lattice space allows us to use a few tricks:

- We can decompose $x = x^+ - x^-$, so we need only study positive elements. We have already seen this many times.
- Given x, we can create elements like $x \vee 1$, $x \wedge 1$, etc. More generally, we can "combine" multiple elements. This is a powerful property for construction (c.f. Stone Weierstrass).

The following toy example will show the power of the second trick. Then we will use the second trick to prove partitions of unity.

Toy example (a generalization of Hahn Decomp). Suppose μ, ν are real measures such that $\mu \wedge \nu = 0$. Then μ and ν are mutually singular.

Proof gist. If not, construct a strictly positive measure smaller than both.

More detail. Afer applying Lebesgue Decomposition and then R-N, we get

$$\int_E h d\mu = \nu_{ac}(E).$$

If $h \neq 0$ a.e., consider the strictly positive measure induced by $h \wedge 1$.

Partitions of Unity for LCH Spaces. If X is LCH, then for any compact $C \subset X$ and open cover θ_i of C, there exists positive functions $f_i \in C_C(X)$ such that $\sum f_i = 1$ on C and $\text{supp}(f_i) \subset \theta_i$.

Remark. Support is defined as the closure of the carrier.

Proof gist. Use LCH Urysohn's lemma (and friend) to construct a function

$$g = \sum g_i \geq 1.$$

Then consider

$$\frac{g}{g \vee 1}.$$

Construction for first equation. We first show there exists closed sets $B_i \subset \theta_i$ such that $C \subset \cup B_i$. (Need LCH Urysohn's friend). Then use LCH Urysohn's lemma to define $g_i(x) = 1$ for $x \in B_i$ and $\text{supp}(g_i) \subset \theta_i$.

- *LCH Urysohn's friend* (used to prove LCH Urysohn). X LCH, subset C compact, $C \subset U$ open. Then there exists open V, $\bar{V}$ compact, such that $C \subset V \subset \bar{V} \subset U$.
- *LCH Urysohn's lemma.* If subset C is compact and $C \subset \theta$ is open, then there exists a continuous $f : X \to [0, 1]$ with $f(x) = 1$ for $x \in C$ and $f(x) = 0$ for $x \notin \theta$.

Sorry, I come up with weird names.