Theorem. For $\mu(x) < \infty$, $\Big(\mathcal{L}^1(X, \mathcal{S}, \mu)\Big)'$ is isometric and isomorphic to $L^\infty(X, \mathcal{S}, \mu)$.

Main idea. Manuever $\mathcal{L}^1$ into the $\mathcal{L}^2$ setting and apply the Reisz-Representation theorem.

More detail. Observe for $\mu(X) < \infty$, we have $\big(\mathcal{L}^1(\mu)\big)' \subset \big(\mathcal{L}^2(\mu)\big)'$. Thus by R-R, for every $\phi \in \big(\mathcal{L}^1(\mu)\big)'$ there exists a g such that

$$\phi(f) = \int fg.$$

We then show $g \in \mathcal{L}^\infty(\mu)$. Conversely, we then show every $g \in \mathcal{L}^\infty(\mu)$ defines a continuous linear functional on $\mathcal{L}^1(\mu)$ by the equation above.

Key trick. To g is bounded a.e., we show for a closed set C, we have

$$\frac{1}{\mu(E)} \int_E g \in C,\ \forall\ \mu(E) > 0 \implies g(x) \in C \text{ a.e.}$$

To do this, we show any open ball disjoint with C must have preimage measure 0. Then as the range of g is separable, we take a countable union of all such open balls.

Motivation for first step. For spaces of finite measure, $\mathcal{L}^2 \subset \mathcal{L}^1$. Thus it makes sense for linear functionals on $\mathcal{L}^1$ to also be linear functionals on $\mathcal{L}^1$.

Extension remarks. This theorem can be extended to σ-finite measures (as many statements about finite measures can).