Functional Analysis

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Contents

1 Lecture 1-21-2020

Definition 1.1. Let V be a vector space. A gauge is a function $p : V \to \mathbb{R}$ that is "half-linear", i.e.

- For $r > 0$, we have $p(rv) = rp(v)$.
- $p(u + v) \leq p(u) + p(v)$.

Theorem 1.1 (Main lemma for Hahn-Banach)**.** *A linear functional defined on a subspace W of V subordinate to gauge p can be extended to a subordinate linear functional defined on $W \oplus span(v_0)$.*

Proof gist. Show the existence of α such that

$$\tilde{\phi}(w + rv_0) = \phi(w) + r\alpha$$

is subordinate to p. Key trick is a separation of variables.

Theorem 1.2 (Hahn-Banach)**.** *A linear functional defined on a subspace W of V subordinate to gauge p can be extended to a subordinate linear functional defined on V.*

Proof gist. The main lemma extends linear functionals one dimension at a time. Apply Zorn's lemma to it (by considering the family of pairs of vector subspaces and subordinate linear functionals defined on them).

Remark. To show the existence of continous linear functionals on normed vector spaces, let the gauge p be the norm. Then any linear functional subordinate to p is continuous.

2 Lecture 1-23-2020

Definition 2.1 (Quotient Norm)**.** If W is a closed subspace of Banach space V, then equip V/W with the norm

$$\|\pi(v)\| = \inf\{\|v - w\| : w \in W\}.$$

Remark. Note that if W is not closed, then we only get a semi-norm.

There are a few miscellaneous statements about quotient norms and spaces, and they often use the following technique:

- Fix $\pi(v)$. Tweak $v \to v + w$ until you get a specific property.

Exercise 1. *Use the tweak technique above to show for any $v \in V$, there is a $\phi \in V'$ such that $\phi(W) = 0$, $\phi(v) = \|\pi(v)\|$, and $\|\phi\| = 1$.*

Theorem 2.1. *V/W is a Banach space.*

Proof gist. "Pull back" a rapidly Cauchy sequence from V/W to V. Tweak technique is needed.

Theorem 2.2. *Space of continuous linear operators $\mathcal{B}(V, W)$ is a Banach space.*

Proof gist. Let T be the point of convergence of a Cauchy sequence. Show T is bounded and linear (very routine).

3 Lecture 1-29-2020

For vector space V, the following are equivalant:

- The topology on V is the initial topology from a collection of semi-norms on V.
- The topology on V is translation invariant and has a subbase of convex sets.

When the above scenarios occur, we say the topology on V is **locally convex**.

Theorem 3.1 (Alaoglu's theorem)**.** *If V is a normed vector space, then the closed unit ball B in V' is compact for the weak-* topology.*

Remark. A possible point of confusion: two different topologies are used in the statement of the theorem. The closed unit ball is defined by the norm-topology, but the actual topology on V' is weak-*.

Proof gist. Use Tynchoff's theorem to construct a compact space and establish a homeomorphism between that compact space and B with the weak-* topology.

More detail. Define $D_v = \{t \in \mathbb{R} : |t| < ||v||\}$. Then consider the map

$$J : B \to \prod_{v \in V}^{\infty} D_v = P$$

given by the (informal) expression

$$J(\phi) \mapsto \prod_{v \in V}^{\infty} \phi(v).$$

By comparing the subbases of B and $J(B)$, we conclude they are homeomorphic. To show $J(B)$ is closed (hence compact), we note each element of P determines a function f. If f is in the closure of $J(B)$, then by approximating f by elements of $J(B)$, we show f is linear and thus an element of $J(B)$.

Theorem 3.2 (Hahn-Banach Extension). *Let V be a real normed vector space. Suppose O and C are disjoint convex subsets, with O open. Then there exists a linear functional ϕ that separates O and C, i.e.*

$$\phi(O) < t \leq \phi(C)$$

where t is a constant.

Proof gist. Use the Hahn-Banach theorem to construct a functional ϕ such that

$$\phi(O - C) < 0.$$

In more detail. $O - C$ is an open convex set not containing the origin, and let $v_0 \in O - C$. Pictorially, we have Figure 1. If we define ϕ on Rv_0 with $\phi(-v_0) = 1$ and make the obvious extension (in this finite dimensional case), then clearly $\phi(O - C) < 0$. The general case is a more technical version of this.

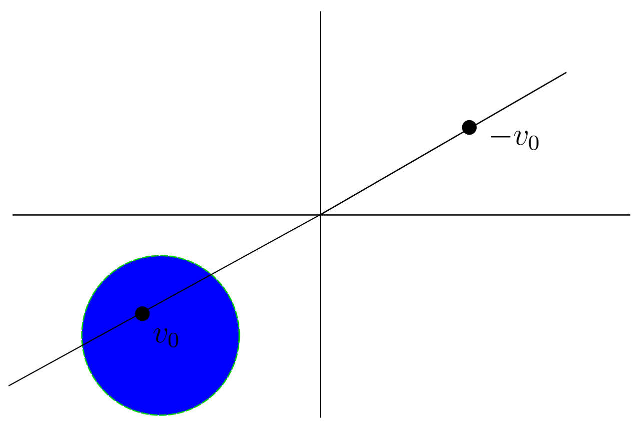

Figure 1: A convex open set

Extra detail. To pick the appropriate gauge to make the extension in the general case, let $U = O - C - v_0$ and define

$$m_u(v) = \inf\{s \in \mathbb{R}^+ : \frac{v}{s} \in U\}.$$

Then proceed as before, but use the gauge m_u to perform the extension.

Theorem 3.3 (Hahn-Banach Extension for Spaces over $\mathbb{C}$). *Let V be a complex normed vector space, W a subspace, and p a semi-norm. If ϕ is a continuous linear functional on W dominated by p, then there exists a continuous linear extension $\tilde{\phi}$ on V dominated by p.*

Proof gist. Use the real Hahn-Banach theorem to construct $\tilde{\phi}$.

More detail. Let $\psi = \Re\phi$ on W, and pretend V is a real vector space (i.e. iv is not a multiple of v). Then use the real version of Hahn-Banach to extend ψ to $\tilde{\psi}$ on V. Now define

$$\tilde{\phi}(v) = \tilde{\psi(v)} - i\tilde{\psi(iv)}$$

and show $\tilde{\phi}$ satisfies the properties we want.

4 Lecture 2-4-2020

Theorem 4.1 (The Weak Topology is Tight). *Suppose V is a vector space, and W is a collection of linear functionals on V. Then if ϕ is a continuous linear functional for the W-weak topology, $\phi \in W$.*

Key trick. If $\phi_1, \ldots, \phi_n$ are a collection of linear functionals, then

$$\bigcap \ker \phi_i \subset \ker \phi \implies \phi \in \operatorname{span}\{\phi_i\}.$$

To show the LHS, we compare the topology generated by ϕ with the subbase of the weak topology.

Theorem 4.2 (Krein-Milman). *Let C be a closed convex subset of a locally convex topological vector space V (assumed Hausdorff). Then the convex hull of extreme points of C is C.*

Main idea. Consider the poset P of all the faces of C. Then the minimal points of P are the extreme points (need Zorn for existence). Then use this property to prove this theorem.

Key technique. If D is compact convex set, we consider a continuous linear functional ϕ not constant on D (use H-B or H-B separation). Then

$$\{v \in D : \phi(v) \text{ achieves its maximum.}\}$$

is a proper (compact convex) face.

5 Lecture 2-6-2020

- The parallelogram inequality $\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2)$ guarantees a closest point to a convex closed set in Hilbert space $\mathcal{H}$. This is how we define orthogonal projection.
- The decomposition $\ker \phi + (\ker \phi)^\perp$ is used to construct Reisz Representation.

6 Lecture 2-11-2020

Theorem 6.1. *For $\mu(x) < \infty$, $\left(\mathcal{L}^1(X, \mathcal{S}, \mu)\right)'$ is isometric and isomorphic to $L^\infty(X, \mathcal{S}, \mu)$.*

Main idea. Manuever $\mathcal{L}^1$ into the $\mathcal{L}^2$ setting and apply the Reisz-Representation theorem.

More detail. Observe for $\mu(X) < \infty$, we have $\left(\mathcal{L}^1(\mu)\right)' \subset \left(\mathcal{L}^2(\mu)\right)'$. Thus by R-R, for every $\phi \in \left(\mathcal{L}^1(\mu)\right)'$ there exists a g such that

$$\phi(f) = \int fg.$$

We then show $g \in \mathcal{L}^\infty(\mu)$. Conversely, we then show every $g \in \mathcal{L}^\infty(\mu)$ defines a continuous linear functional on $\mathcal{L}^1(\mu)$ by the equation above.

Key trick. To g is bounded a.e., we show for a closed set C, we have

$$\frac{1}{\mu(E)} \int_E g \in C, \ \forall \ \mu(E) > 0 \implies g(x) \in C \text{ a.e.}$$

To do this, we show any open ball disjoint with C must have preimage measure 0. Then as the range of g is separable, we take a countable union of all such open balls.

Motivation for first step. For spaces of finite measure, $\mathcal{L}^2 \subset \mathcal{L}^1$. Thus it makes sense for linear functionals on $\mathcal{L}^1$ to also be linear functionals on $\mathcal{L}^1$.

Remark. This theorem can be extended to σ-finite measures (as many statements about finite measures can).

7 Lecture 2-13-2020

Key Equation. Suppose μ, ν are two σ-finite measures. Observe that the map

$$\phi(f) = \int f d\nu$$

is an element of the dual space of $\mathcal{L}^1(\mu + \nu)$. Thus there exists an $h \in \mathcal{L}^\infty(\mu + \nu)$ such that

$$\int f d\nu = \int f h d(\mu + \nu) = \int f h d\mu + \int f h d\nu,$$

i.e.

$$\int f(1 - h) d\nu = \int f h d\mu.$$

Proof gist for Lebesgue Decomp. Observe $\|h\| \leq 1$. Define

$$E = \{x : h(x) = 1\}, \quad F = X \backslash E$$

and use the equation to note that $\nu|_E$ is singular to μ and $\nu|_F$ is absolutely continuous to μ.

Proof gist for Radon-Nikodyn. When the set $\{x : h(x) = 1\}$ has measure 0 (i.e. ν is absolutely continuous w/r/t μ), we choose $f = \chi_G$ and note

$$\nu(G) = \int \chi_G d\nu = \int \frac{h\chi_G}{1-h} d\mu.$$

Theorem 7.1. *For $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and finite measure space X, there is an isometric bijection between the positive elements of $\left(\mathcal{L}^p(\mu)\right)'$ and $\mathcal{L}^q(\mu)$.*

Proof gist. For positive $\phi \in \left(\mathcal{L}^p(\mu)\right)'$, observe $\nu(E) = \phi(\chi_E)$ is a measure absolutely continuous to μ. Then by Radon-Nikodyn, there exists a positive g such that

$$\nu(E) = \int_E g d\mu.$$

We show that $g \in \mathcal{L}^q(\mu)$. The other direction comes from Holder and the following observation proved via simple functions:

$$\phi(f) = \int f d\nu = \int f g d\mu$$

Remark. In the usual fashion, this can be extended to the σ-finite case.

8 Lecture 2-18-2020

Lattices! Here's a way to remember the two lattice operations: pretend that the shapes represent the set of <u>bounds</u>.

- The $\wedge$ shape looks like one point is greater than all the points, so this represents the greatest lower bound.
- Similarly, $\vee$ represents the least upper bound.

In order of increasing structure, we have: lattice ordered (abelian) group $\rightarrow$ lattice ordered vector space $\rightarrow$ lattice ordered normed vector space. A partial order is compatible with a normed vector space V if

- For $v, w \in V$, $v, w \geq 0 \implies v + w \geq 0$.
- For $v \in V$, $r \in \mathbb{R}$, $v \geq 0, r \geq 0 \implies rv \geq 0$.

- For $v \in V$, $\|v\| = \||v|\|$. (c.f. below for definition of $|\cdot|$).
- For $v, w \in V$, $0 \le v \le w \implies \|v\| \le \|w\|$.

The definitions of all the structures mentioned can be interpolated from this.

Here are some key properties and definitions of lattices (with the appropriate structure):

- We can impose an order structure in a group by defining a collection of elements to be positive. (This collection has to satisfy some additional properties, evidently not all collections work).
- We define $v^+ = v \wedge 0$, $v^- = (-v) \wedge 0$, $v^+ + v^- = |v|$.
- For function spaces, we usually use the order: $f \ge 0$ means $f(x) \ge 0$ for all x. For spaces of functionals, we usually use the order $f \ge 0$ means $f(x) \ge 0$ if $x \ge 0$ (for the function space order).

9 Lecture 2-20-2020

Theorem 9.1 (Dense Subsets of $C_{\mathbb{R}}(X)$)**.** *Let X be a compact space, and L a subspace and sublattice of $C(X)$. If L strongly separates points in X, then L is dense in $C(X)$ (for the sup norm).*

Remark. Strong seperation implies X is necessarily Hausdorff.

Proof gist. Given $f \in C(X)$, we construct a $g \in L$ such that $\|f - g\| < \varepsilon$. To do this, we use strong separation to generate families of functions in L, then piece these functions together using compactness and the $\wedge$, $\vee$ operators.

Theorem 9.2 (Real Stone-Weierstrass)**.** *Same statement as before, but instead of L, we consider subalgebra A.*

Proof gist. We prove $\bar{A}$ is a subspace and sublattice, then invoke our previous theorem.

Key tricks.. There are two key tricks to show that $\bar{A}$ is sublattice. The first is to note

$$f + g = \frac{f + g + |f - g|}{2}, \quad |f| = \sqrt{f^2}.$$

The second is to observe that $\sqrt{t}$ can be uniformly approximated on a compact interval by a polynomial $p(t)$. This uses some power series knowledge.

Remark. We can extend Stone-Weierstrass to the complex case if we force A to be closed under complex conjugation.

10 Lecture 2-25-2020

Theorem 10.1. *The dual of a normed vector lattice V is a normed vector lattice (with the order usually associated with functionals).*

Proof gist. Given $\phi \in V'$, we independently construct $\phi^+ \in V'$ and show $\phi^+ = \phi \vee 0$. The translation properties of ordered vector spaces then shows V' is lattice ordered. We then check the rest of the properties of a normed vector lattice (some inequalities and bashing required).

Construction of ϕ^+. We first define ϕ^+ on V^+ by

$$\phi^+(v) = \sup\{\phi(x) : 0 \leq x \leq v\}$$

and prove it is linear. Then we linearly extend ϕ^+ to V and show it is continuous.

Some techniques used.

- Prove something for positive v and extend via $v = v^+ - v^-$.
- Prove some inequality for variables x and y satisfying some condition. The inequality holds if we take the supremum over x and y with this condition.
- Use the fact that V is a normed vector lattice!

Remark. We already showed that if p, q are Holder conjugates with $1 < p, q < \infty$, then there is an isometric bijection between $(\mathcal{L}^p)'$ and $\mathcal{L}^q$. The theorem above extends this statement to all of $(\mathcal{L}^p)'$ and $\mathcal{L}^q$.

11 Lecture 2-27-2020

Theorem 11.1. *The space of real measures $\mathcal{S}$ forms a vector lattice, with the positive elements of the lattice being the finite positive measures.*

Proof gist. For $\mu \in \mathcal{S}$, we show that the total variation measure is $|\mu|$ (here, $|\cdot|$ is interpreted in the lattice sense). As we have seen before, this implies that $\mathcal{S}$ is a lattice.

More detail. Let $\|\mu\|$ denote total variation measure. We show $\|\mu\|$ is countably additive by the usual argument (prove the inequality both ways, consider disjoint unions, etc). Then we show $\|\mu\|$ is finite by the following technicalitiy:

- Real (and Banach) measures need to be absolutely convergent by definition, i.e. for disjoint sets E_n,
$$\sum \mu(E_n) = \sum \mu(E_j)$$

where E_j represents a different ordering.

- If $\|\mu\|(E) = \infty$ (i.e. E is unbounded) but $\mu(E) < \infty$, we construct a sequence of disjoint sets whose "partial sums" are not absolutely convergent.

Finally we show $\|\mu\| = |\mu|$ by using the fact

$$\|\mu(E)\| = \sup\{\mu(E_1) - \mu(E_2) : E_1 \sqcup E_2 = E\}.$$

A final detail. The construction of the sequence of disjoint sets mentioned above is as follows:

- split E into two disjoing unbounded sets E_1, F_1 where $|\mu(F_1)| > 1$.
- repeat for $E = E_1$

Our result is $F_1, F_2, F_3 \dots$.

Remark. It is not difficult to see that total variation is also a norm, and this norm plays well with the lattice. In other words, $\mathcal{S}$ is a normed vector lattice.

12 Lecture 3-3-2020

Some observations. Working in a lattice space allows us to use a few tricks:

- We can decompose $x = x^+ - x^-$, so we need only study positive elements. We have already seen this many times.
- Given x, we can create elements like $x \vee 1$, $x \wedge 1$, etc. More generally, we can "combine" multiple elements. This is a powerful property for construction (c.f. Stone Weierstrass).

The following toy example will show the power of the second trick. Then we will use the second trick to prove partitions of unity.

Theorem 12.1 (A Toy Example)**.** *Suppose μ, ν are real measures such that $\mu \wedge \nu = 0$. Then μ and ν are mutually singular.*

Proof gist. If not, construct a strictly positive measure smaller than both.

More detail. Afer applying Lebesgue Decomposition and then R-N, we get

$$\int_E h d\mu = \nu_{ac}(E).$$

If $h \neq 0$ a.e., consider the strictly positive measure induced by $h \wedge 1$.

Theorem 12.2 (Partitions of Unity for LCH Spaces)**.** *If X is LCH, then for any compact $C \subset X$ and open cover θ_i of C, there exists positive functions $f_i \in C_C(X)$ such that $\sum f_i = 1$ on C and supp$(f_i) \subset \theta_i$.*

Remark. Support is defined as the closure of the carrier.

Proof gist. Use LCH Urysohn's lemma (and friend) to construct a function

$$g = \sum g_i \geq 1.$$

Then consider

$$\frac{g}{g \vee 1}.$$

Construction of first equation. We first show there exists closed sets $B_i \subset \theta_i$ such that $C \subset \cup B_i$. (Need LCH Urysohn's friend). Then use LCH Urysohn's lemma to define $g_i(x) = 1$ for $x \in B_i$ and $\operatorname{supp}(g_i) \subset \theta_i$.

- **LCH Urysohn's friend (used to prove LCH Urysohn).** X LCH, subset C compact, U open, $C \subset U$. Then there exists open V, $\bar{V}$ compact, such that $C \subset V \subset \bar{V} \subset U$.
- **LCH Urysohn's lemma.** If subset C is compact and $C \subset \theta$ is open, then there exists a continuous $f : X \to [0, 1]$ with $f(x) = 1$ for $x \in C$ and $f(x) = 0$ for $x \notin \theta$.

Sorry, I come up with weird names.

13 Lecture 3-5-2020

In the next few lectures, we wish to study the dual space of $C_c(X)$, where X is LCH. We call a positive linear functional on $C_c(X)$ a positive Radon measure (PRM). It is our end goal to characterize all PRMs as integrals (Riesz-Markov), so we begin by constructing a measure for every PRM.

Motivation. In general, for a positive linear functional ϕ, we have

$$\phi(f) = \int f d\mu_\phi, \quad \mu_\phi(E) = \phi(\chi_E).$$

We have seen this type of construction when studying $(\mathcal{L}^p)'$. However, as PRMs are only defined on continuous functions, we have to play some technical tricks.

Definition 13.1. If ϕ is a PRM, define μ_ϕ on open sets by

$$\mu_\phi(U) = \sup\{\phi(f) : 0 \leq f \leq \chi_U,\ \operatorname{supp}\{f\} \subset U\}.$$

It is not too hard to show μ_ϕ is a content, i.e. a "finitely additive measure." (Use partitions of unity to show countable subadditivity.) We then do the standard extension: content $\mu_\phi \to$ outer measure $\mu_\phi^* \to$ measure μ_ϕ.

Remark. A possible point of confusion: μ_ϕ denotes both the measure and the content. Use context.

14 Lecture 3-10-2020

Here, we prove that the measurable sets of μ_ϕ contains the Borel σ-algebra, and thus can be restricted to the Borel σ-algebra. This requires some of the following concepts:

Definition 14.1. If μ is a measure or outer measure, then μ is:

- inner regular if the measures of sets can be approximated from below by the measures of compact sets,
- outer regular if the measures of sets can be approximated from above by the measures of open sets.

The content μ_ϕ is inner regular for open sets! However, we require a slightly modified definition of inner regularity as μ_ϕ is defined only on the open sets. What we mean is

$$\mu_\phi(U) = \sup\{\mu_\phi(V) : V \text{ open}, \ \bar{V} \subset U\}, \quad U \text{ open}.$$

Theorem 14.1. *Open sets are measurable in μ_ϕ.*

Proof gist. For open set U, we want to show for any $A \subset X$, we have

$$\mu_\phi^*(A - U) + \mu_\phi^*(A \cap U) = \mu_\phi^*(A).$$

Use inner regularity for open sets to prove for A open, then extend using outer regularity.

Now we are ready for our main theorem.

Theorem 14.2 (Reisz-Markov)**.** *If ϕ is a positive radon measure on X, then*

$$\phi(f) = \int f d\mu_\phi,$$

where μ_ϕ is inner regular for open sets, outer regular, the σ-algebra of μ_ϕ contains the Borel sets.

Proof details for integral representation. Show ϕ and $\int d\mu_\phi$ well approximate each other for simple scenarios, i.e.

$$\chi_A \leq f \leq \chi_B \implies \int \chi_A d\mu_\phi \leq \phi(f) \leq \int \chi_B d\mu.$$

Generalize to arbitrary $f \in C_c(X)$ by breaking f apart into many small functions, applying the above equation to each of the functions, and then putting everything back with the triangle inequality.

The proof of the properties of μ_ϕ are easy.

15 Lecture 3-12-2020

Regularity properties of measures are important because they allow us to approximate arbitrary measurable sets with "nicer" sets. Thus we can often extend properties on nicer sets to a more general class.

Here are some implications about regularity:

- μ Borel measure, outer regular, inner regular for open sets $\implies$ inner regular for σ-finite Borel sets. (Borel measure = defined on Borel sets and finite on compact sets.)
- μ Borel measure, open sets σ-compact $\implies$ μ is regular (i.e. both inner and outer regular).

Intuition. The proofs of these implications are a bit technical (but not hard). But intuitively, the statements all say something about approximation by "nice" sets, so there is no surprise the statements are related.

In light of R-M, this means that positive Radon measures correspond to "well-behaved" Borel measures.

From R-M, we can deduce R-M-like statements (sometimes also called R-M) for the duals of $C_\infty(X)$, and when X is compact, $C(X)$.

16 Lecture 3-19-2020

Motivation for integral operators come from matrix multiplication. (Side note: integral operators are to matrices as convolution (considered as a type of integral operator) are to Toeplitz matrices. The relation between convolution and FT is the reason that Toeplitz matrices are diagonalized by the FFT.)

Definition 16.1 (Integral Operators)**.** For $K \in \mathcal{L}^p(X \times Y)$, define $T_K : \mathcal{L}^q(Y) \to \mathcal{L}^p(X)$ by

$$T_K(f)(x) = \int K(x,y) f(y) dy.$$

The above definition makes sense because of Holder (q is chosen to be the Holder conjugate of p). Also, as Fubini/Tonelli are important when working with integral operators, we usually assume we are working in a σ-finite space.

17 Lecture 3-31-2020

Definition 17.1. A compact operator between two normed vector spaces is a linear map that maps bounded sets to totally bounded sets.

Remark. A equivalent definition is a operator that maps the unit ball to a totally bounded set.

Theorem 17.1. *Integral operators are compact.*

Proof gist. Prove for a simple case ($K = \chi_{E\times F}$) and then extend to for general K via approximation.

Base Case. Note $K = \chi_{E\times F} \implies T_K$ has rank 1.

Key trick for generalization. Suppose $\mathcal{R}$ is a ring that generates the σ-ring $\mathcal{S}$. By approximating elements in $\mathcal{S}$ by elements of $\mathcal{R}$, we show that every element in $\mathcal{L}^p(X, \mathcal{S}, \mu)$ can be approximated by ISFs in $(X, \mathcal{R}, \mu)$. (This is a general statement; X has no relation to the X in the definition of compact operators.)

18 Lecture 4-2-2020

Theorem 18.1. *If $T \in \mathcal{B}(\mathcal{H})$ is a self-adjoint compact operator, then $\|T\|$ and/or $-\|T\|$ are eigenvalues.*

Proof gist. Construct a sequence $\{\zeta_n\}$ such that $\|\zeta_n\| = 1$ and $T(\zeta_n) - \|T\|\zeta_n \to 0$ or $T(\zeta_n) + \|T\|\zeta_n \to 0$ (This is called an "approximate eigenvector"). Use compactness to select the limit ζ of a subsequence of ζ_n.

More detail. For self-adjoint operators, note $\sup\{|\langle T(\zeta), \zeta\rangle| : \|\zeta\| \le 1\} = \|T\|$. This is how we construct the sequence.

Theorem 18.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a self-adjoint compact operator. Then*

$$\mathcal{H} = \overline{\bigoplus \mathcal{H}_\lambda}, \quad \lambda \text{ eigenvalue}$$

where $\mathcal{H}_\lambda$ is the λ-eigenspace.

Proof gist. Let
$$\mathcal{K} = \overline{\bigoplus \mathcal{H}_\lambda}, \quad \lambda \text{ eigenvalue}.$$
Note T is invariant over $\mathcal{K}^\perp$, so $T_{\mathcal{K}^\perp} = T_*$ makes sense. As T_* has eigenvalues $\|T_*\|$ or $-\|T_*\|$, we conclude that $\mathcal{K}^\perp = \{0\}$.

Remark. This is one example of studying operator by studying the invariant subspaces. This is a powerful technique.

From the theroem above, we can construct a orthonormal basis of eigenvectors (i.e. the spectral thereom).

19 Lecture 4-7-2020

We first develop a important theorem regarding the decomposition of compact operators.

Definition 19.1. A linear map $T : V \to W$ is a partial isometry if $V = U \oplus U^\perp$, T is an isometry on U, and $T(U^\perp) = 0$.

Definition 19.2. For compact operator $T \in \mathcal{B}(\mathcal{H})$, define $|T| = \sqrt{T^*T}$.

Remark. The square root is well defined because T^*T is a positive compact operator, so it has basis of eigenvectors of positive eigenvalues. Take the square root of the eigenvalues, but keep the corresponding eigenspaces.

Theorem 19.1 (Polar Decomposition)**.** *For compact operator $T \in B(\mathcal{H})$, we have*
$$T = V|T|,$$
where $V \in \mathcal{B}(\mathcal{H})$ is a partial isometry.

Proof gist. Define $V(|T|(x)) = T(x)$, extend to closure, and map the orthogonal complement to 0.

Remark. Polar decomposition is really SVD in disguise. If you diagonalize $|T|$ and make some trivial tweaks, you get compact SVD.

Theorem 19.2. *Compact operators are well approximated by finite rank operators.*

Proof gist. Apply polar decomposition and then show $|T|$ can be well approximated by finite rank operators.

Approximation of $|T|$. Recall for compact operators that nonzero eigenspaces are finite and there are only finitely many eigenvalues $> \varepsilon$ for a given $\varepsilon > 0$.

20 Lecture 4-9-2020

We take a brief hiatus from Hilbert spaces to talk about certain topological notions of "volume".

Definition 20.1. A subset S of topological space X is nowhere dense if $\bar{S}$ contains no nonempty open sets. S is meager if it is a countable union of nowhere dense sets.

Definition 20.2. A topological space X is a Baire space if it is not a countable union of nowhere dense sets.

Remark. Intuitively, a Baire space has a certain "volume." Nowhere dense subsets and meager subsets do not.

Theorem 20.1 (Baire Category Theorem)**.** *LCH spaces and complete metric spaces are Baire spaces.*

Key Trick. Let C_i be a collection of nowhere dense sets. Construct a sequence of open sets B_i such that $\bar{B}_1 \supset \bar{B}_2 \ldots$ and $C_i \cap \bar{B}_i = \emptyset$. Then show $\cap \bar{B}_i \neq \emptyset$. (Use F.I.P. or Cauchy sequences.)

Theorem 20.2 (Open Mapping Theorem)**.** *Let $T : \mathcal{B}(V, W)$ for Banach spaces V, W. If T is surjective, then T is an open map.*

Proof gist. Use Baire Category Theorem to show $T(Ball(0,1))$ is not meager. Then use contintuity and Cauchy sequences (completeness and pullback!) to show $T(Ball(0,1))$ contains a open set.

One consequence of the open mapping theorem is the closed graph theorem, which states that a linear map between Banach spaces V and W with a closed graph (closed w/r/t one of the usual equivalent norms in $V \oplus W$ like $\max\{\|\cdot\|_v, \|\cdot\|_w\}$) is bounded. From this, one can deduce that all self-adjoint operators are bounded (Hellinger's Theorem).

21 Lecture 4-14-2020

We now return to Hilbert spaces operators. The following is an interesting representation trick for positive operators.

Theorem 21.1 (Positive Operator Representation)**.** *If $T \in \mathcal{B}(\mathcal{H})$ is a positive operator, then there exists positive $S \in \mathcal{B}(\mathcal{H})$ such that $T = S^2$. We often denote $S = \sqrt{T}$.*

Remark. (1) We define T to be positive if $\langle Tv, v\rangle \geq 0$ for all $v \in \mathcal{H}$. (2) Our previous use of $\sqrt{}$ is a specific instance of this definition. In particular, we can now define $|T|$ for any $T \in \mathcal{B}(\mathcal{H})$, not just compact T. (3) We will prove this theorem later.

Now we are ready to introduce the concept of trace.

Definition 21.1 (Trace)**.** Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then define the trace

$$\tau(T) = \sum \langle Te_i, e_i\rangle$$

where $\{e_i\}$ is any orthonormal basis.

Remark. The difficulty of defining trace in general are technicalities surrounding the convergence of the sum. Therefore we first define trace for the positive operators and then extend to a larger class of operators.

Some technical details. To show $\tau(T)$ is irrespective of (orthonormal) basis comes from the positive operator representation trick followed by Parseval's identity.

We now extend the idea of trace.

Definition 21.2. Let

- $\mathfrak{m}_\tau \subset \mathcal{B}(\mathcal{H})$ be the $\mathbb{C}$-span of the positive operators with finite trace.
- $\mathfrak{n}_\tau \subset \mathcal{B}(\mathcal{H})$ be the set of elements S s.t. $S^*S \in \mathfrak{m}_\tau$.

Trace can be naturally defined on $\mathfrak{m}_\tau$ (note it may be complex but not infinite). We would like to study $\mathfrak{m}_\tau$ and $\mathfrak{n}_\tau$. To do so, we go to a more general setting.

Definition 21.3. Let A be a C*-algebra. A weight is a function $\omega : A^+ \to \mathbb{R}^+ \cup \infty$ such that $\omega(a+b) = \omega(a) + \omega(b)$ and $\omega(ra) = r\omega(a)$ when $r \geq 0$.

Remark. In this generalization, we replace $\mathcal{B}(\mathcal{H})$ with A and τ with ω. $\mathfrak{m}_\omega$ and $\mathfrak{n}_\omega$ are defined analogously.

22 Lecture 4-16-2020

We construct $L^2(A, w)$ in the following manner:

- Observe $\langle a, b\rangle := \omega(b^*a)$ is an pre-inner product on vector space $\mathfrak{n}_\omega$. (Use the polarization identity for *-algebras to show $b^*a \in \mathfrak{m}_\omega$.)

- Apply a quotient $\mathfrak{n}_\omega \to \mathfrak{n}_\omega/\mathcal{N}_\omega$ to turn the pre-inner product into an inner product.
- Complete $\mathfrak{n}_\omega/\mathcal{N}_\omega$ to get $L^2(A, w)$, a Hilbert space.

Theorem 22.1 (C* Representation)**.** *There is a *-embedding from A into $\mathcal{B}(L^2(A, w))$.*

Proof gist. The embedding is given by $a \to L_a$, where L_a is left multiplication by a.

Key trick. To show L_a is bounded, note $b^*cb \leq \|c\|b^*b$. As ω is positive, this eventually implies

$$\langle L_a b, L_a b\rangle \leq \|a^*a\|\langle b, b\rangle.$$

Note that Theorem 22.1 hints at the representation of arbitrary C*-algebras as a C*-algebra of bounded operators on a Hilbert space (big Gelfand-Naimark theorem).

We now turn back to trace.

Theorem 22.2. *We have:*

- $\mathfrak{m}_\tau = \{T \in \mathcal{B}(\mathcal{H}) : \tau(|T|) < \infty\} = \mathcal{B}^1(\mathcal{H}) \subset \mathcal{B}_c(\mathcal{H})$
- $\mathfrak{n}_\tau = \{T \in \mathcal{B}(\mathcal{H}) : \tau(|T|^2) < \infty\} = \mathcal{B}^2(\mathcal{H}) \subset \mathcal{B}_c(\mathcal{H})$

Key Trick. To prove $\mathcal{B}^1(\mathcal{H})$ and $\mathcal{B}^2(\mathcal{H})$ consist of compact operators, observe

$$\|T\| \leq \tau(T), \quad T \geq 0.$$

Use this link between norm and trace to build finite rank approximations.

Key Trick. Once $\mathcal{B}^1(\mathcal{H}) \subset \mathcal{B}_c(\mathcal{H})$ is established, apply polar decomposition ($T = V|T|$), reverse polar decomposition ($V^*T = |T|$), and the fact that $\mathfrak{m}_\tau$ is an ideal to prove the first statement. The second statement is obvious.

Definition 22.1. The operators in $\mathcal{B}^2(\mathcal{H})$ are called Hilbert-Schmidt operators.

23 Lecture 4-21-2020

We describe what we call the finite projection trick, which can be roughly described as follows:

- To study T, we study PT, where P is a finite projection. (Or TP, PTP, etc.)
- Using finite dimensionality, we prove a statement for PT that is independent of P.
- We limit $P \to I$ by projecting to bigger and bigger spaces to prove that statement for T.

Remark. The motif of this trick is used ubiquitously in math: proving for a simpler scenario, and then building up.

Theorem 23.1. *$\mathcal{B}^2(\mathcal{H})$ is a Hilbert algebra with inner product $\langle S, T\rangle = \tau(T^*S)$.*

Proof gist. Show $\mathcal{B}^2(\mathcal{H}) = \mathfrak{n}_\tau = \mathfrak{n}_\tau/\mathcal{N}_\tau = L^2(\mathcal{B}(\mathcal{H}), \tau)$ (c.f. Theorem 22.1). Completeness (the last equality) is the only difficulty: if $\|\cdot\|_2$ denotes the norm on $\mathcal{B}^2(\mathcal{H})$, observe

$$\|T\| \leq \|T\|_2$$

so a $\|T\|_2$ - Cauchy sequence converges to T_* in $\|\cdot\|$. Use the finite projection trick to show it converges to T_* in $\|\cdot\|_2$.

More detail. We use the finite projection trick because $S_i \to S$ in $\|\cdot\| \implies \tau(PS_i) \to \tau(PS)$ if P is a finite projection. It is not nessarily true that $\tau(S_i) \to \tau(S)$.

Notice that $\|T\|_2 \to \tau(|T|^2)$, which is the obvious choice of a norm if you think about the definition of $\mathcal{B}^2(\mathcal{H})$. This motivates the next theorem:

Theorem 23.2. *$\mathcal{B}^1(\mathcal{H})$ is a Banach space with the norm $\|T\|_1 = \tau(|T|)$.*

Proof gist. Play with inequalities to show $\|\cdot\|_1$ is a norm. Completeness follows in the same manner as before: observe $\|T\| \leq \|T\|_1$ and apply the finite projection trick.

More detail. To show $\|\cdot\|_1$ satisfies Δ-inequality, apply reverse polar decomposition and note

$$\tau(MT) \leq \|M\|\|T\|_1$$

if $M \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{B}^1(\mathcal{H})$.

24 Lecture 4-23-2020

We begin by introducing a notation for rank 1 operators.

Definition 24.1. For $\xi, \eta \in \mathcal{H}$, let $D_{\xi\eta}(\zeta) = \langle \zeta, \eta\rangle \xi$

By Riesz-Representation, all rank 1 operators can be represented as so.

Theorem 24.1. *$\mathcal{B}(\mathcal{H}) = (\mathcal{B}^1(\mathcal{H}))'$, where $M \mapsto \tau(M\cdot)$.*

Surjectivity. To show all continuous functionals are of the form $\tau(M\cdot)$, suppose $\phi \in (\mathcal{B}^1(\mathcal{H}))'$.

We constuct M such that ϕ and $\tau(M\cdot)$ agree on all rank 1 operators, and thus $\phi = \tau(M\cdot)$. To do this, we apply the Riesz-Representation of bounded sesquilinear forms to find an M such that

$$\langle M\xi, \nu\rangle = \phi(D_{\xi\eta}).$$

Theorem 24.2. $\mathcal{B}^1(\mathcal{H}) = (\mathcal{B}_c(\mathcal{H}))'$*, where* $T \mapsto \tau(\cdot T)$*.*

Surjectivity. To show all continuous functionals are of the form $\tau(\cdot T)$, suppose $\psi \in (\mathcal{B}_c(\mathcal{H}))'$. Maneuver into the setting $\psi \in (\mathcal{B}^2(\mathcal{H}))'$ and as $\mathcal{B}^2(\mathcal{H})$ is a Hilbert space, apply Riesz-Representation to get

$$\psi(S) = \tau(TS) = \tau(ST)$$

on $\mathcal{B}^2(\mathcal{H})$. (Play around to show the commutativity.) Then apply finite projection trick and polar decomposition to show $T \in \mathcal{B}^1(\mathcal{H})$. As $\mathcal{B}^2(\mathcal{H})$ is dense in $\mathcal{B}^1(\mathcal{H})$, this result extends to $\mathcal{B}^1(\mathcal{H})$.

Remark. If we see trace as an integral, then we obtain some familiar expressions. $\tau(M\cdot)$ becomes

$$\int M \cdot d\mu$$

and $\tau(\cdot T)$ becomes

$$\int \cdot T d\mu.$$

Moreover, $\tau(T^*S)$ becomes

$$\int T^* S d\mu.$$

This suggests that the motivation for some expressions involving trace (or weights in general) come from seeing trace (and weights) as some sort of integral.

25 Lecture 4-28-2020

In the last two lectures of this semester, we develop the theory needed to prove Theorem 21.1, i.e. the ability to take square roots of positive operators. Throughout, $\mathcal{P}$ will denote the algebra of polynomials over $\mathbb{C}$, and σ will denote the spectrum.

In this lecture, we develop some results related to the spectrum.

Theorem 25.1 (Spectral Mapping Theorem)**.** *Suppose* $T \in \mathcal{B}(\mathcal{H})$ *and* $p \in \mathcal{P}$*. Then*

$$\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}.$$

Key Trick. Some simple factorizations.

The following is an important technique for operator algebras. It is obvious - the only technicality is the convergence of the LHS.

Theorem 25.2. *Suppose $T \in \mathcal{B}(\mathcal{H})$ and $\|T\| < 1$. Then*

$$(1 - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

A consequence of this theorem is that $\|\sigma(T)\|_\infty \leq \|T\|$ for $T \in \mathcal{B}(\mathcal{H})$. Our final result relates the spectrum to approximate eigenvalues because approximate eigenvalues are easier to work with.

Theorem 25.3. *Suppose $T \in \mathcal{B}(\mathcal{H})$. Then*

(a) If λ is an approximate eigenvalue, then $\lambda \in \sigma(T)$.

(b) If $\lambda \in \sigma(T)$, then it is an approximate eigenvalue of T or T^*.

Proof overview. (a) is simple. For (b), use the Open Mapping Theorem and the relations between the kernels and ranges of T and T^* to link the approximate eigenvalue 0 to invertibility.

Remark. Some relation between approximate eigenvalues and the spectrum is almost guaranteed - both "roughly describe" eigenvalues.

26 Lecture 4-30-2020

We start with a crucial observation.

Lemma 26.1. *Suppose $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint and $p \in \mathcal{P}_{\mathbb{R}}$. Then $\|p\|_\infty = \|p(T)\|$, where p on the LHS is seen as an element of $C(\sigma(T))$.*

Proof. Note $\|p\|_\infty = \|\sigma(p(T))\|_\infty$ by Spectral mapping theorem, and $\|\sigma(p(T))\|_\infty = \|p(T)\|$ because $p(T)$ is self-adjoint + results from last time. □

From this lemma, we have established a partial mapping from $C(\sigma(T))$ into $\mathcal{B}(\mathcal{H})$ that is so far an isometry. Because the spectrum of T is real, applying Stone-Weierstrass, we can extend the mapping to an isometry from $C_{\mathbb{R}}(\sigma(T))$ into $\mathcal{B}(\mathcal{H})$. Considering real and imaginary parts separately, we can extend further to an isometry from $C(\sigma(T))$ into $\mathcal{B}(\mathcal{H})$. Thus we have a rough proof of the following (check a few details to make rigorous):

Theorem 26.2 (The Continuous Functional Calculus)**.** *Suppose $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint. Then $p \to p(T)$ extends to an isometric homomorphism from $C(\sigma(T))$ into $\mathcal{B}(\mathcal{H})$ whose range is the C^* algebra generated by T and I.*

Remark. The isometry above is still denoted $p \to p(T)$.

This theorem is incredibly powerful because we can construct linear operators with certain properties just by picking an element of $C(\sigma(T))$. For instance, to prove Theorem 21.1, we simply choose $\sqrt{x} \in C(\sigma(T))$. This is very cool! (From another persective, we define $\sqrt{T}$ by approximating with polynomials of T).