Theorem. The space of real measures $\mathcal{S}$ forms a vector lattice, with the positive elements of the lattice being the finite positive measures.

Proof gist. For $\mu \in \mathcal{S}$, we show that the total variation measure is $|\mu|$ (here, $|\cdot|$ is interpreted in the lattice sense). As we have seen before, this implies that $\mathcal{S}$ is a lattice.

More detail. Let $\|\mu\|$ denote total variation measure. We show $\|\mu\|$ is countably additive by the usual argument (prove the inequality both ways, consider disjoint unions, etc). Then we show $\|\mu\|$ is finite by the following technicalitiy:

- Real (and Banach) measures need to be absolutely convergent by definition, i.e. for disjoint sets E_n,
$$\sum \mu(E_n) = \sum \mu(E_j)$$
where E_j represents a different ordering.
- If $\|\mu\|(E) = \infty$ (i.e. E is unbounded) but $\mu(E) < \infty$, we construct a sequence of disjoint sets whose "partial sums" are not absolutely convergent.

Finally we show $\|\mu\| = |\mu|$ by using the fact
$$\|\mu(E)\| = \sup\{\mu(E_1) - \mu(E_2) : E_1 \sqcup E_2 = E\}.$$

A final detail. The construction of the sequence of disjoint sets mentioned above is as follows:

- split E into two disjoing unbounded sets E_1, F_1 where $|\mu(F_1)| > 1$.
- repeat for $E = E_1$

Our result is $F_1, F_2, F_3 \dots$.

Remark. It is not difficult to see that total variation is also a norm, and this norm plays well with the lattice. In other words, $\mathcal{S}$ is a normed vector lattice.