Gauge. Let V be a vector space. A gauge is a function $p : V \to \mathbb{R}$ that is "half-linear", i.e.

- For $r > 0$, we have $p(rv) = rp(v)$.
- $p(u + v) \leq p(u) + p(v)$.

Main lemma for Hahn-Banach. A linear functional defined on a subspace W of V subordinate to gauge p can be extended to a subordinate linear functional defined on $W \oplus \mathrm{span}(v_0)$.

Proof gist. Show the existence of α such that

$$\tilde{\phi}(w + rv_0) = \phi(w) + r\alpha$$

is subordinate to p. Key trick is a separation of variables.

Hahn-Banach. A linear functional defined on a subspace W of V subordinate to gauge p can be extended to a subordinate linear functional defined on V.

Proof gist. The main lemma extends linear functionals one dimension at a time. Apply Zorn's lemma to it (by considering the family of pairs of vector subspaces and subordinate linear functionals defined on them).

Remark. To show the existence of continous linear functionals on normed vector spaces, let the gauge p be the norm. Then any linear functional subordinate to p is continuous (indeed, Lipschitz).