Inductive Limit Topology

Notation

Let X be LCH. Let $C_c(X)$ be the subset of $C(X)$ with compact support. For $S \subset X$, let $C_\infty(S)$ be the subset of $C(S)$ that "vanishes at infinity," i.e. ε-bounded outside a compact set.

Definition of ILT

The ILT on $C_c(X)$ is the strongest topology such that the natural inclusion

$$C_\infty(U) \to C_c(X)$$

is continuous for all open U with compact closure. (Here, we equip $C_\infty(U)$ with the $\|\cdot\|_\infty$ norm.)

Remark. *In general, there is no natural inclusion from $C(S) \to C_c(X)$ for any $S \subset X$. Nor is there a natural inclusion $C_\infty(U) \to C_c(X)$ if U is not open or does not have compact closure.*

Motivation

Why are we concerned with the ILT? Note that $C_c(X)$ with $\|\cdot\|_\infty$ norm makes all the maps $C_\infty(U) \to C_c(X)$ continuous, so the $\|\cdot\|_\infty$ topology is weaker than the ILT.

This makes the ILT topology easier to work with because the ILT allows for more continuous linear functionals than $\|\cdot\|_\infty$. (Think about it, a stonger topology means more continuous maps can be defined on it.) For instance, every positive Radon measure is continuous for the ILT, but this is not the case for $\|\cdot\|_\infty$.

Exercise 1. *For $X = \mathbb{R}$, what is an example of a open set in the ILT that is not an open set for $\|\cdot\|_\infty$? Hint: consider the positive Radon measure given by*

$$\phi(f) = \int f d\lambda$$

where λ is the usual measure on $\mathbb{R}$. Note ϕ is discontinuous for $\|\cdot\|_\infty$, and consider $\phi^{-1}(0,1)$.

Criterion for Continuity

Suppose T is a linear functional defined on $C_c(X)$. Then T is continuous for the ILT if for all U open with compact closure, the map

$$T|_{C_\infty(U)} : C_\infty(U) \to \mathbb{R}$$

is continuous. ($C_\infty(U)$ is equipped with the $\|\cdot\|_\infty$ norm.) Think about why.