Dense subsets of $C_{\mathbb{R}}(X)$. Let X be a compact space, and L a subspace and sublattice of $C(X)$. If L strongly separates points in X, then L is dense in $C(X)$ (for the sup norm).

Remark. Strong seperation implies X is necessarily Hausdorff.

Proof gist. Given $f \in C(X)$, we construct a $g \in L$ such that $\|f - g\| < \varepsilon$. To do this, we use strong separation to generate families of functions in L, then piece these functions together using compactness and the $\wedge$, $\vee$ operators.

Real Stone-Weierstrass. Same statement as before, but instead of L, we consider subalgebra A.

Proof gist. We prove $\bar{A}$ is a subspace and sublattice, then invoke our previous theorem.

Key tricks. There are two key tricks to show that $\bar{A}$ is sublattice. The first is to note

$$f + g = \frac{f + g + |f - g|}{2}, \quad |f| = \sqrt{f^2}.$$

The second is to observe that $\sqrt{t}$ can be uniformly approximated on a compact interval by a polynomial $p(t)$. This uses some power series knowledge.

Extension. We can extend Stone-Weierstrass to the complex case if we force A to be closed under complex conjugation.