滑らかな常微分方程式の計算量

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概 要

紊綣荐膊　純ｒ緇　初∽違　幻緇　合綣, **PSPACE** 絎　B＜,
潟篁　, 紊綣荐膊　純ｒ算緇　初∽違　幻緇　合綣, **PSPACE** 絎
B＜　腓冴. 桶腓冴 Lipshitz ＞散羣紹後小合綣　B **PSPACE** 絎
腟　≦宍ｃ.

1　絨

1.1　荐膊　処B

荐膊　処B (Computable Analysis)　膊　醇ｘ茫荐膊茫　羈　B絖
茵　ｃ.

荐膊　処B　, 絎違絎∽違　≪　綵√綣泣　. 障絎 x　x 後違　$\{r_n\}$ ц；
障. y　＞散　$|r_n - x| \leq 2^{-n}$　. r_n　桁　純ｃ, 絎違　ュ　仮綺　,
菴篌弱ゃ菴∽違　≪　ｃ.

絎違∽違　≪　, 絎∽ $f : \mathbf{R} \to \mathbf{R}$　∽違∽違後 蕭∽違　. ｃ　膊　純
≪　腟　ｃ　. ч∽ $f : \mathbf{R} \to \mathbf{R}$ 荐膊　純ｃ　, ュ　x　膕上墾 n 筝
, s_n　$|s_n - f(x)| \leq 2^{-n}$ 羣　阪罘罌 M 綣　. x　泣ゃ冴　;障　, M 贋
x 茯　ｃ　. M 腑荐罘罌違　, 菴篌弱ゃ菴∽違腑荐　.　荐罘罌違
羣狗　, 紊綣 (**P**) 紊綣 (**PSPACE**) 絲上絎∽違　鴻絎臂　ｃ. 喝　臂
ゃ　2 腴　臂　.

1.2　蕁　∫g腥

g 絎∽ $g : [0,1] \times \mathbf{R} \to \mathbf{R}$　札筝　幻緇　合綣.

$$h(0) = 0, \qquad \mathcal{D}^{(1)} h(t) = g(t, h(t)) \quad (t \in [0,1]) \tag{1.1}$$

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表 1.1: ∫ g腥

狗	筝	筝
—	—	荐膊筝　純 [?]
h　g　　茹	荐膊　[?]	篁紙　[?]
Lipshitz ＞散羣	紊綣	紊綣育ｃ　[Kaw10]
$\mathcal{D}^{(0,1)}g$　g	紊綣	紊綣育ｃ [　]
$\mathcal{D}^{(0,i)}g$　g	紊綣	篁　　綣育ｃ [　]
g 茹ｆ	紊綣 [?]	—

絽後小合綣　ｃ, “g 膣　∽違　　, h　祉　　∽違　　”　蕁　ｃ　箴＜.　罕　狗　　幻綳　合綣　B　膊腥吟　[茵 1.1]. g　　狗荐　　翫,　B　膊筝純. g　　茹ｃゃ狗翫, 茹ｃ　膊　純ｃ, 篁紙　. Lipschitz ＞散　幻綳　合綣　B　ｃ篆荐若荀　＞散ｃ. Lipschitz ＞散羣　, 茹ｃ　綣荐膊　純ｃ, 紊綣荐膊絎　. g 茹ｆｃ　, 絽後小合綣　B茹ｆ　　, 紊綣荐膊　純ｃ.

　狗　　≫　ゃ　　, 篁ヤ　ヨ綳.

定理 1.1. 紊綣絎∽ $g(t,y)$, 綳　　純 $\mathcal{D}^{(0,1)}g$　g ｃ, g　幻綳　合綣 (1.1)　B h　**PSPACE** 絎　ｃ　絖　.

定理 1.2. 篁　4.1　, 篁紙　　倶 $k \geq 2$　, 紊綣絎∽ $g(t,y)$, $\mathcal{D}^{(0,k)}g$　g ｃ, g　幻綳　合綣 (1.1)　B h　**PSPACE** 絎　ｃ　絖　.

g ♂綳　　純ｃ　荀羆　　　　. 絎 1.2　算 k　　k 綳　　純　∽違　,

2　羣

2.1　茵

(篋)　倶違　**N**, 贋違　**Z**, 絎違　**R**, 違　**Q**, $0^{\mathbf{N}} = \{0^n \mid n \in \mathbf{N}\}$　；荐. 筝紊育∽ f　i　∽違 $\mathcal{D}^{(i)}f$　；荐. 罕　紊育∽ g　　筝綣違　i , 膃　綣違　j　∽違 $\mathcal{D}^{(i,j)}g$　；荐.

i　　違　綳　羲膊絖 $\mathcal{D}_i$　；荐. 障 j 綳　羲膊絖 $\mathcal{D}_i\mathcal{D}_i\cdots\mathcal{D}_i$　$\mathcal{D}_i^j$　；荐　篋紊育∽ g 膃　綣違　≪ k　g綳　　純ｃ　,

2.2　絎違

絎違　♂　激ゃ,　絖　　桁　筝　純ｃ. ｙ違荐膊　純　≪　ф宴, 羆膕上塑筝, 絎違　篌弱ゃ　仮綺　ц　∽違.

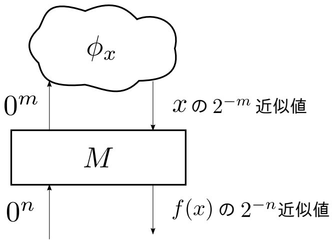

図 2.1: 絎∽違　≪

定義 2.1 (絎違　). ∽ $\phi : 0^{\mathbf{N}} \to \mathbf{Z}$ 絎 $x \in [0,1]$　c　, $\phi(0^n) = \lfloor x \cdot 2^n \rfloor$ 障 $\phi(0^n) = \lceil x \cdot 2^n \rceil$ 羣.

2.3 荐膊　遵∽, 紊綣絎∽

絎∽違　ュ　違, 絎違腦　桁　純c. yュ　違　荐　. ∽違　≪ , 羆臓上墾ュ　∽違　ュc絎違　腑荐 , ∽違　ゃ　篌弱ゃ菴　荐罘罌違　臂　[2.3]. 喝　札筝　臂　.

定義 2.2. 腑荐罘罌 M 絎∽ $f : A \to \mathbf{R}$ 荐膊　, 篁紙　$x \in A$, 篁紙 x　ϕ_x , M^{ϕ_x}　$f(x)$　c.

紊紊育∽違　≪　, 紊違　違腑荐ょ荐罘罌違　c　罕　臂　.

絎∽違荐膊　純c　,　∽違荐膊腑荐罘罌違絖　c. 罕, 絎∽違紊綣荐膊純c　,　∽違荐膊紊綣腑荐罘罌違絖　c.

腑荐罘罌 M　f 荐膊　, 羆臓上墾 n　, x　篌弱ゃ　荀　仮綺 m 絎障, 荐膊　純　∽違　g c. 障 n　m　綽≫　違　菴篌弱ゃ筝 , 荐膊　遵∽違紊綣荐膊　遵∽違　, 腑荐罘罌違　ゃ　劫彰篁　純c.

補題 2.3. $\phi_f : \mathbf{Q} \times 0^{\mathbf{N}} \to \mathbf{Q}$, $m_f : \mathbf{N} \to \mathbf{N}$

$$|\phi_f(d, 0^n) - f(d)| \le 2^{-n} \tag{2.1}$$

$$|x - y| \le 2^{-p_f(m)} \Rightarrow |f(x) - f(y)| \le 2^{-m} \tag{2.2}$$

帥∽違　.

- f 荐膊　純c　, 荐膊　純 ϕ_f, m_f 絖　ゃ c.
- f 紊綣荐膊　純c　, 紊綣荐膊　純 ϕ_f, 紊綣 m_f 絖　ゃ c.

2.4 絎

∽違　腓冴, 育e喝　c絎臂　. 荐茯 L 絎∽ $f : [0,1] \to \mathbf{R}$　綣　純c　, f 荐膊罘罌違　鴻　, ュ u　, 臓上墾 f　, 絎 x_u　荐罔≦c, $f(x_u)$ 篌弱ゃ, u　L　障　紊綣ц膊　純c　c [2.4]. 喝　札筝　臂　.

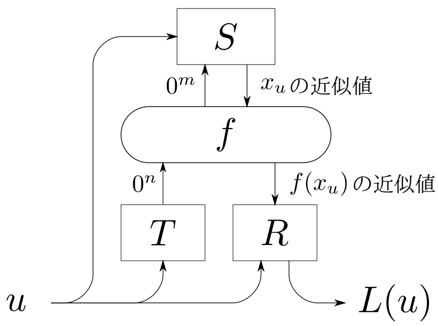

図 2.2: 荐茯 L ∽ f 吾

定義 2.4 (案綣). 荐茯 L 絎∽ $f : [0,1] \to \mathbf{R}$ 綣 純c , 篁紙 絖 u , 篁ヤ羣絎 $x_u \in [0,1]$ 案綣荐膊 純 ∽ R, S, T 絖 .

- $R : N \times N \to \{0,1\}, \quad S : \mathbf{N} \times 0^{\mathbf{N}} \to \mathbf{N}, \quad T : \mathbf{N} \to 0^{\mathbf{N}}$;
- $S(u, \cdot)$ x_u ;
- 篁紙 $f(x_u)$ ϕ

$$L(u) = R(u, \phi(T(u))).$$

荐膊 C , ∽ f C 育c c , 篁紙 C 障荐茯 f 綣 純c c. f C 障 , ゃ障 C 綽腑荐罘罌違 f 荐膊 絖 , f C 絎 c 臂 .

3 綴 初∽違 幻綴 合綣

3.1 ∽ｅゅ蕁

ゅ蕁 ∽c 若吾 c 潟 **PSPACE** 絎 c 紮. 羃恰 札箏 臂 c 若 識荐膊 **PSPACE** 絎 c 腓冴 .

補題 3.1 (荅蕁 4.7. [Kaw10]). 篁紙 茯 $L \in \mathbf{PSPACE}$, 篁紙 絖 u , 篁ヤ羣 絎 $d \geq 2$ 案綣 P, Q, ∽井 $(G_u)_u$, $(H_u)_u$, $(G_u)_u$ 綣荐膊 , $H_u(P(|u|)+1, 2^{Q(|u|)}) = L(u)$ c 絖 .

(i) $G_u : [P(|u|)] \times [2^{Q(|u|)}] \times [d] \to \{-1, 0, 1\}$;

(ii) $H_u : [P(|u|)+1] \times [2^{Q(|u|)}+1] \to [d]$;

(iii) 篁紙 $i \in [P(|u|)], T \in [2^{Q(|u|)}]$

- $H_u(i, 0) = H_u(0, T) = 0$
- $H_u(i+1, T+1) = H_u(i+1, T) + G_u(i, T, H_u(i, T))$.

c $d = 4$ c c, 綴 茫 輝箏 .

3.2 ∽eゅ蕁罔≦c∽井

篁紙 茯 $L \in \mathbf{PSPACE}$, 絖 u , 筝荐 膊罔≦c $L(u)$ 荐膊緇 純 ∽ g_u 罕.

補題 3.2. 篁紙 茯 $L \in \mathbf{PSPACE}$, 紊綣 λ , 紊綣 ρ, ∽井 $(g_u)_u, (h_u)_u$, $(g_u)_u$ 綣荐膊 純c, 篋我絖 u 札筝羣 絖 .

(i) $g_u : [0,1] \times [-1,1] \to \mathbf{R}, \quad h_u : [0,1] \to [-1,1]$;

(ii) 篁紙 $y \in [-1,1]$ $g_u(0,y) = g_u(1,y) = 0$;

(iii) h_u g_u 幻緇 合綣 B;

(iv) $\mathcal{D}^{(0,1)}g$ g;

(v) $|\mathcal{D}^{(0,1)}g| \le 2^{-\lambda(|u|)-|u|}$;

(vi) $h_u(1) = 2^{-\rho(|u|)}L(u)$.

蕁 ⊢ , c ず 羯 綣絎∽ $f : [0,1] \to \mathbf{R}$ 絨ャ.

補題 3.3 (荅蕁 3.6. [Ko91])**.** 篁ヤ羣紊綣♂緇 遵∽ $f : [0,1] \to \mathbf{R}$ 絖 .

(i) $f(0) = 0, \quad f(1) = 1$;

(ii) 篁紙 $n \ge 1$ $f^{(n)}(0) = f^{(n)}(1) = 0$;

(iii) f $[0,1]$ y 茯水;

(iv) 篁紙 $n \ge 1$ $f^{(n)}$ 綣絎∽.

荅蕁 3.2 ⊢. $d, P, Q, (G_u)_u, (H_u)_u$ 荅蕁 3.1 罕 臂 . 鴻 $P(u)$ 蚊 , $G_u(i,T,Y) \ne 0$ 羣 i T 1ゃ c. i $j_u l(T)$; 障. 篁紙 i $G_u(i,T,Y) = 0$ $j_u(T)$ 篁紙 ゃ . 札筝 皿絎 c.

$$H_u(i, 2^{Q(|u|)}) = \begin{cases} L(u) & (i = P(|u|)) \\ 0 & (i < P(|u|)) \end{cases} \tag{3.1}$$

$$G_u(i, 2 \cdot 2^{Q(|u|)} - 1 - T, Y) = \begin{cases} 0 & (i = P(|u|) - 1) \\ -G_u(i,T,Y) & (i < P(|u|) - 1) \end{cases} \tag{3.2}$$

$$H_u(i, 2 \cdot 2^{Q(|u|)} - T) = \begin{cases} H_u(P(|u|), 2^{Q(|u|)}) & (i = P(|u|)) \\ H_u(i,T) & (i < P(|u|)) \end{cases} \tag{3.3}$$

苓蕁 3.3 f , 絎 c 篁紙 $x \in [0,1]$ $|\mathcal{D}^{()}1f(x)| \le 2^c$ 羣絨 倶違 . 絎 $d' = \lceil \log(4d+1) \rceil$, $B = 2^{\lambda(|u|)+Q(|u|)+|u|+c+d'}$, $(t,y) \in [0,1] \times [-1,1]$, 倶 N, $\theta \in [0,1]$, 贋 Y, $\eta \in [-1/4, 3/4]$ $t = (T+\theta)2^{-Q(|u|)}$, $y = (Y+\eta)B^{-j_u(T)}$ 羣 絎.

,

$$g_u^*(t,Y) = \frac{2^{Q(|u|)}\pi \sin(\theta\pi)}{2B^{j_u(T)+1}} G_u\left(j_u(T), T, Y \bmod 2^{d'}\right) \tag{3.4}$$

g_u, h_u 篁ヤ 臂 .

$$g_u(t,y) = \begin{cases} g_u^*(t,Y) & (\eta \le \frac{1}{4}) \\ (1 - f(2\eta - 1/2))g_u^*(t,Y) + f(2\eta - 1/2)g_u^*(t, Y+1) & (\eta > \frac{1}{4}) \end{cases} \tag{3.5}$$

$$h_u(t) = \sum_{i=0}^{P(|u|)} \frac{H_u(i,T)}{B^i} + \frac{1-\cos(\theta\pi)}{2} \cdot \frac{G_u(j_u(T), T, H_u(j_u(T), T))}{B^{j_u(T)+1}} \tag{3.6}$$

筝荐 臂 g_u, h_u 苓蕁 3.2 ф ц蟹羣 腓冴. (i) , (ii) . $(g_u)_u$ 紊綣荐 腨 純c 蕁 c ず.

h_u g_u 幻緇 合綣 B c 腓冴. 障 h_u ゃ B. (3.6) ょ $i \le j_u(T)$ 荐 $B^{j_u(T)}$. $i > j_u(T)$ 荐,

$$\begin{aligned} \sum_{i > j_u(T)} \frac{H_u(i,T)}{B^i} &\le \sum_{i > j_u(T)} \frac{d-1}{B^i} = \sum_{i > j_u(T)} \frac{d-1}{B^{i-j_u(T)}} B^{-j_u(T)} \\ &\le \sum_{i > j_u(T)} \frac{(d-1)}{(4d+1)^{i-j_u(T)}} B^{-j_u(T)} \\ &= \frac{d-1}{4d} B^{-j_u(T)} \end{aligned}$$

篋ょ 偽絲上ゃ

$$\left| \frac{1-\cos(\theta\pi)}{2} \cdot \frac{G_u(j_u(T), T, H_u(j_u(T), T))}{B^{j_u(T)+1}} \right| \le \frac{1}{B^{j_u(T)+1}} \le \frac{B^{-j_u(T)}}{4d+1} \tag{3.7}$$

c $h_u(t) = (Y+\eta)B^{-j_u(T)}$ 羣 $\eta \in [-1/4, 1/4]$ 綂 . ,

$$Y = \sum_{i=0}^{j_u(T)} H_u(i,T) \cdot B^{j_u(T)-i}. \tag{3.8}$$

B $2^{d'}$ 違 , $Y \bmod 2^{d'} = H_u(j_u)$. (3.5)$Y$ η 篁ｅヤ,

$$\begin{aligned} g_u(t, h_u(t)) &= \frac{2^{Q(|u|)}\pi \sin(\theta\pi)}{2B^{j_u(T)+1}} G_u(j_u(T), T, H_u(j_u(T), T)) \\ &= \mathcal{D}^{(1)} h_u(t). \end{aligned}$$

c h_u g_u 幻緇 合綣 B.
$g_u y$ $\ll$ 小 純 c,

$$\mathcal{D}^{(0,1)}g(t,y) = \begin{cases} 0 & (\eta \le \frac{1}{4}) \\ 2B^{j_u(T)}\mathcal{D}^{(1)}f(2\eta - 1/2)\cdot(g_u^*(t,Y+1) - g_u^*(t,Y)) & (\eta > \frac{1}{4}) \end{cases} \tag{3.9}$$

c $\mathcal{D}^{(0,1)}g$ g.
$|g_u^*(t,Y)| \le 2^{Q(|u|)}\pi/(2B^{j_u(T)+1}) \le 2^{Q(|u|)+1}/B^{j_u(T)+1}$,

$$\begin{aligned} \left|\mathcal{D}^{(0,1)}g\right| &\le 2B^{j_u(T)}\cdot|\mathcal{D}^{(1)}f(2\eta-1/2)|\cdot 2\cdot\frac{2^{Q(|u|)+1}}{B^{j_u(T)+1}} \\ &\le \frac{2B^{j_u(T)}\cdot 2^c\cdot 2^{Q(|u|)+2}}{B^{j_u(T)+1}} \\ &= \frac{2^{Q(|u|)+c+3}}{B} \quad \le \quad 2^{-\lambda(|u|)-|u|} \end{aligned} \tag{3.10}$$

(vii)

$$\begin{aligned} h_u(1) &= \frac{H_u(P(|u|), 2^{Q(|u|)})}{B^{P(|u|)}} \\ &= \frac{L(u)}{2^{P(|u|)(\lambda(|u|)+Q(|u|)+|u|+c+d')}} \end{aligned} \tag{3.11}$$

, $\rho(k) = P(k)(\lambda(k) + Q(k) + |u| + c + d')$ 腴. $\square$

3.3 絎 1.1 ⊢

証明. L **PSPACE** 絎 茯, $\lambda(k) = 2k+2$. **PSPACE** 絎 茯 L
蕁 3.2 , ρ, $(g_u)_u$, $(h_u)_u$ 緇. $(g_u)_u$ 綣絎∽井 , $|g_u(t,y)| \le 2^{\gamma(|u|)-|u|}$
羣 綣 γ 絖 . u

$$\varLambda_u = 2^{\lambda(|u|)}, \quad \varGamma_u = 2^{\gamma(|u|)} \tag{3.12}$$

$$c_u = 1 - \frac{1}{2^{|u|}} + \frac{2\bar{u}+1}{\varLambda_u}, \quad l_u^{\mp} = c_u \mp \frac{1}{\varLambda_u} \tag{3.13}$$

. $\bar{u} \in \{0, \ldots, 2^{|u|}-1\}$ u 篋我違 B.
∽ g, h $t \in [0,1]$, $y \in \mathbf{R}$, 筝 臂 .

$$g\left(l_u^{\mp} \pm \frac{t}{\varLambda_u}, \frac{y}{\varLambda_u \varGamma_u}\right) = \begin{cases} \pm\frac{1}{\varGamma_u}\left(g_u(t,1) + \mathcal{D}^{(0,1)}g_u(t,1)(y-1)\right) & (1 < y) \\ \pm\frac{g_u(t,y)}{\varGamma_u} & (-1 \le y \le 1) \\ \pm\frac{1}{\varGamma_u}\left(g_u(t,-1) + \mathcal{D}^{(0,1)}g_u(t,-1)(y+1)\right) & (y < -1) \end{cases} \tag{3.14}$$

$$h\left(l_u^{\mp} \pm \frac{t}{\varLambda_u}\right) = \frac{h_u(t)}{\varLambda_u \varGamma_u}. \tag{3.15}$$

篁紙 $y \in \mathbf{R}$ $g(1,y) = h(1) = 0$ 臂 .

gh 絎 1.1 ф∽違　ц蟹羣　腓冴.

障 g 紊綣荐膊　純 c　腓冴. 荅蕁[1]　　ず. T, Y　 ゃ $g(T,Y)$ 羆　, $T = l_u^{\mp} \pm t/\Lambda_u$, $Y = y/\Lambda_u \Gamma_u$ 羣 $u, \pm(\mp), t, y$, 紊綣ц膊　純 c .

罨＜ g　y　 ≪　小　純 c , 絨∽違 y, t　 ≪ g　　c　　. 咲 x　篋綣違 ≪　小,

$$\mathcal{D}^{(0,1)} g\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}, \frac{y}{\Lambda_u \Gamma_u}\right) = \begin{cases} \pm \Lambda_u \mathcal{D}^{(0,1)} g_u(t,1) & (1 < y) \\ \pm \Lambda_u \mathcal{D}^{(0,1)} g_u(t,y) & (-1 < y < 1) \\ \pm \Lambda_u \mathcal{D}^{(0,1)} g_u(t,-1) & (y < -1). \end{cases} \tag{3.16}$$

c $\mathcal{D}^{(0,1)} g(t,1) = \pm \Lambda_u \mathcal{D}^{(0,1)} g_u(t,1)$, $\mathcal{D}^{(0,1)} g(t,-1) = \pm \Lambda_u \mathcal{D}^{(0,1)} g_u(t,-1)$ c , y　 ≪　小　. $\mathcal{D}^{(0,1)} g_u$　 g c , y　 ≪　g　　.

t 葱御劫吾　g c　 ゃ. 篁紙 $[0,1)$　 違　 u $t \in [0,1]$ 絖　 $l_u^{\mp} \pm t/\Lambda_u$ 就 ц ; . $t \in (0,1)$　　 $\mathcal{D}^{(0,1)} g_u(t,y)$　 g c , t 葱御劫檎 g . $t = 0, 1$　　, $y \in [-1,1]$　　$g_u(0,y) = g_u(1,y) = 0$ $\mathcal{D}^{(0,1)} g_u(0,y) = \mathcal{D}^{(0,1)} g_u(1,y) = 0$ c $t = 0, 1$　　 g . $g(1,y) = 0$ $\mathcal{D}^{(0,1)} g(1,y) = 0$. 障 $\left|\mathcal{D}^{(0,1)} g_u\right| \le 2^{\lambda(|u|)-|u|}$,

$$\lim_{t \to 1-0} \left|\mathcal{D}^{(0,1)} g\right| = \lim_{|u| \to \infty} \left|\Lambda_u \mathcal{D}^{(0,1)} g_u\right| \le \lim_{|u| \to \infty} \left|2^{-|u|}\right| = 0. \tag{3.17}$$

c $\mathcal{D}^{(0,1)} g$　 g .

hg　幻緇　合綣　B c　腓冴. $h(0) = 0$,　$\mathcal{D}^{(1)} h(1) = 0 = g(1, h(1))$.

$$\begin{aligned} & h'(l_u^{\mp} \pm t/\Lambda_u) \\ & = \pm \frac{h_u'(t)}{\Lambda_u \Gamma_u} \\ & = \pm \frac{g_u\left(t, h_u(t)\right)}{\Gamma_u} \\ & = g\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}, \frac{h_u(t)}{\Lambda_u \Gamma_u}\right) \\ & = g\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}, h\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}\right)\right). \end{aligned} \tag{3.18}$$

L h　　純 c　腓冴.

$$h(c_u) = \frac{h_u(1)}{\Lambda_u \Gamma_u} = \frac{L(u)}{2^{\lambda(|u|)+\gamma(|u|)+\rho(|u|)}} \tag{3.19}$$

ゃ障 R, S, T 篁ヤ　　臂　　,　.

$$R(u,v) = v \tag{3.20}$$

$$S(u, 0^n) = \lfloor 2^n c_u \rfloor \text{ 茵　絖,} \tag{3.21}$$

$$T(u) = 0^{\lambda(|u|)+\gamma(|u|)+\rho(|u|)} \tag{3.22}$$

L **PSPACE** 絎　c , h **PSPACE** 絎. □

[1] 羣 y

4 篁紙緇　　初∽違　幻緇　合綣

篁紙緇　　純　∽違　幻緇　合綣　B, 篁　　**PSPACE** 絎　c　　荐惹.

4.1 c 若　　識荐膊

c 若　　識荐膊絎臂　. c 若　　識荐膊　絎 d, ∽ $P : \mathbf{N} \to \mathbf{N}$, 紊綣 $Q : \mathbf{N} \to \mathbf{N}$, ∽井 $(G_u)_u, (H_u)_u$, 5ょ $M = \langle d, P, Q, (G_u)_u, (H_u)_u \rangle$ c .

- $(G_u)_u$　綣荐膊　;
- $P(x) = O(\log x)$ ゅ綣荐膊　;
- $G_u : [P(|u|)] \times \left[2^{Q(|u|)}\right] \times [d] \to \{-1, 0, 1\}$;
- $H_u : [P(|u|) + 1] \times \left[2^{Q(|u|)}\right] \to [d]$;
- 篁紙 $i \in [P(|u|)]$, $T \in \left[2^{Q(|u|)}\right]$
  - $H_u(i, 0) = H_u(0, T) = 0$
  - $H_u(i + 1, T + 1) = H_u(i + 1, T) + G_u(i, T, Hu(i, T))$.

c 若　　識荐膊 M 荐茯 L 茯荔　　算　絖 u $H_u(P(|u|), 2^{Q(|u|)}) = L(u)$ 羣.

仮定 4.1. 篁紙　茯 $L \in \mathbf{PSPACE}$　　L 茯荔 c 若　　識荐膊 絖　.

ゃ障 c 若　　識荐膊 **PSPACE** 絎　c　篁　.

4.2 ∽ e ゅ蕁罔≦ c ∽井

荐惹　1 緇　　純　　. 篁紙　茯 $L \in \mathbf{PSPACE}$, 絖 u　, 筝荐　c 若　識荐膊罔≦ c $L(u)$ 荐膊 篁紙緇　　純　∽ g_u 罕.

補題 4.2. 篁　4.1　, 篁紙　　倶 $k \geq 2$, 篁紙　茯 $L \in \mathbf{PSPACE}$, 篁紙綣 λ　, ∽ $\rho : \mathbf{N} \to \mathbf{N}$　∽井 g_u, h_u , $\rho, (g_u)_u$　綣荐膊　純 c , 篋我絖 u　札筝羣　絖　.

(i) $g_u : [0, 1] \times [-1, 1] \to \mathbf{R}$,　$h_u : [0, 1] \to [-1, 1]$;

(ii) 篁紙 $y \in [-1, 1]$　　$g_u(0, y) = g_u(1, y) = 0$;

(iii) h_u g_u　幻緇　合綣　B;

(iv) $\mathcal{D}^{(0,k)} g_u$　　g;

(v) 篁紙 $i \in \{0, \ldots, k\}$　　$\left|\mathcal{D}^{(0,i)} g_u(t, y)\right| \leq \Lambda_u^{-i} 2^{-|u|}$;

(vi) $h_u(1) = 2^{-\rho(|u|)}L(u)$.

$\Lambda_u = 2^{\lambda(|u|)}$.

証明. 篁 4.1 L 茯荔 $M = \langle d, P, Q, (G_u)_u, (H_u)_u \rangle$ 緇. 札箏 皿絎.

$$H_u(i, 2^{Q(|u|)}) = \begin{cases} L(u) & (i = P(|u|)) \\ 0 & (i < P(|u|)). \end{cases} \tag{4.1}$$

荅蕁 3.3 f , 絎 c 篁紙 $i \in \{0, \ldots, k\}$, 篁紙 $x \in [0,1]$ $|\mathcal{D}^{(i)}f(x)| \leq 2^c$ 羣絨 倶違 . 絎 $d' = \lceil \log(4d+1) \rceil$, $B = 2^{Q(|u|)+k\lambda(|u|)+(k-1)+|u|+c+d'+k}$, $(t, y) \in [0,1] \times [-1,1]$, 倶 N, $\theta \in [0,1]$, 贋 Y, $\eta \in [-1/4, 3/4]$ $t = (T+\theta)2^{-Q(|u|)}$, $y = (Y+\eta)B^{-(k+1)^{j_u(T)}}$ 羣 .

,

$$g_u^*(t, Y) = \frac{2^{Q(|u|)}\pi\sin(\theta\pi)}{2B^{(k+1)^{(j_u(T)+1)}}} G_u\left(j_u(T), T, Y \bmod 2^{d'}\right) \tag{4.2}$$

g_u, h_u 篁ヤ 臂 .

$$g_u(t, y) = \begin{cases} g_u^*(t, Y) & (\eta \leq \frac{1}{4}) \\ (1 - f\left(2\eta - \frac{1}{2}\right) g_u^*(t, Y) + f\left(2\eta - \frac{1}{2}\right) g_u^*(t, Y+1) & (\eta > \frac{1}{4}) \end{cases} \tag{4.3}$$

$$h_u(t) = \sum_{i=0}^{P(|u|)} \frac{H_u(i, T)}{B^{(k+1)^i}} + \frac{1 - \cos(\theta\pi)}{2} \cdot \frac{G_u(j_u(T), T, H_u(j_u(T), T))}{B^{(k+1)^{(j_u(T)+1)}}} \tag{4.4}$$

箏荐 臂 g_u, h_u 荅蕁 4.2 ф ц 蟹羣 腓冴. (i) , (ii) . $(g_u)_u$ 紊綣荐 膊 純 c 蕁 c ず.

h_u g_u 幻緇 合綣 B c 腓冴. 障 h_u ゃ B . (4.4) ょ $i \leq j_u(T)$ 荐 $B^{(k+1)^{j_u(T)}}$. $i > j_u(T)$ 荐,

$$\begin{aligned} \sum_{i > j_u(T)}^{P(|u|)} \frac{H_u(i, T)}{B^{(k+1)^i}} &\leq \sum_{i > j_u(T)}^{\infty} \frac{d-1}{B^{(k+1)^i}} \\ &\leq \sum_{i > j_u(T)}^{\infty} \frac{d-1}{B^i} = \sum_{i > j_u(T)} \frac{d-1}{B^{i - j_u(T)}} B^{-j_u(T)} \\ &\leq \sum_{i > j_u(T)} \frac{(d-1)}{(4d+1)^{i - j_u(T)}} B^{-j_u(T)} \\ &= \frac{d-1}{4d} B^{-j_u(T)} \end{aligned}$$

篋ょ 偽絲上ゃ

$$\left| \frac{1 - \cos(\theta\pi)}{2} \cdot \frac{G_u(j_u(T), T, H_u(j_u(T), T))}{B^{(k+1)^{(j_u(T)+1)}}} \right| \leq \frac{1}{B^{j_u(T)+1}} \leq \frac{B^{-j_u(T)}}{4d+1} \tag{4.5}$$

c $h_u(t) = (Y+\eta)B^{-j_u(T)}$ 羣 $\eta \in [-1/4, 1/4]$ 絖 . ,

$$Y = \sum_{i=0}^{j_u(T)} H_u(i,T) \cdot B^{j_u(T)-i}. \tag{4.6}$$

B $2^{d'}$ 違 , $Y \bmod 2^{d'} = H_u(j_u)$. (4.3)$Y$ η 篁 e ャ,

$$\begin{aligned} g_u(t, h_u(t)) &= \frac{2^{Q(|u|)}\pi \sin(\theta\pi)}{2B^{j_u(T)+1}} G_u(j_u(T), T, H_u(j_u(T), T)) \\ &= \mathcal{D}^{(1)} h_u(t). \end{aligned} \tag{4.7}$$

c h_u g_u 幻緇 合綣 B.

$$\mathcal{D}^{(0,i)} g(t,y) = \tag{4.8}$$

$$\begin{cases} 0 & (\eta \le \frac{1}{4}) \\ 2^i B^{i\cdot(k+1)^{j_u(T)}} \cdot \mathcal{D}^{(i)} f\left(2\eta - \frac{1}{2}\right) \cdot (g_u^*(t, Y+1) - g_u^*(t,Y)) & (\eta > \frac{1}{4}) \end{cases}$$

c $\mathcal{D}^{(0,i)} g$ g

(v) 腓冴. $|g_u^*| \le \left| \frac{2^{Q(|u|)}\pi \sin(\theta\pi)}{2B^{(k+1)^{(j_u(T)+1)}}} \right| \le \frac{2^{Q(|u|)+1}}{B^{(k+1)^{(j_u(T)+1)}}}$, $i = 0$

$$|\mathcal{D}^{(0,0)} g| = |g| \le \frac{2^{Q(|u|)+1}}{B^{(k+1)^{(j_u(T)+1)}}} \le \frac{2^{Q(|u|)+1}}{B^{(k+1)}} \le 2^{-|u|} \tag{4.9}$$

$i \in \{1, \ldots, k\}$,

$$\begin{aligned} |\mathcal{D}^{(0,i)} g| &\le 2^i \cdot B^{i\cdot(k+1)^{j_u(T)}} \cdot 2^c \cdot (g_u^*(t, Y+1) - g_u^*(t,Y)) \\ &\le 2^{c+k} \cdot B^{k(k+1)^{j_u(T)}} \cdot 2 \cdot \frac{2^{Q(|u|)+1}}{B^{(k+1)^{(j_u(T)+1)}}} \\ &\le \frac{2^{Q(|u|)+c+k+2}}{B} \le 2^{-i\lambda(|u|)-|u|} = \Lambda_u^{-i} 2^{-|u|} \end{aligned} \tag{4.10}$$

(vii)

$$\begin{aligned} h_u(1) &= \frac{H_u(P(|u|), 2^{Q(|u|)})}{B^{(k+1)^{P(|u|)}}} \\ &= \frac{L(u)}{2^{(k+1)^{P(|u|)}(Q(|u|)+k\lambda(|u|)+|u|+c+d'+k)}} \end{aligned} \tag{4.11}$$

, $\rho(x) = (k+1)^{P(x)}(Q(x) + k\lambda(x) + x + c + d' + k)$. $P(|u|) = O(\log|u|)$ P 綣荐膊 純 , ρ 綣荐膊 . □

4.3 絎 1.2 ⊢

証明. L **PSPACE** 絎 茯, $\lambda(k) = 2k+2$. **PSPACE** 絎 茯 L 蕁 4.2 , ρ, $(g_u)_u$, $(h_u)_u$ 緇.

$$c_u = 1 - \frac{1}{2^{|u|}} + \frac{2\bar{u}+1}{\Lambda_u}, \qquad l_u^{\mp} = c_u \mp \frac{1}{\Lambda_u} \tag{4.12}$$

. $\bar{u} \in \{0,\ldots,2^{|u|}-1\}$ u 篋我違 B.

∽ g,h $t \in [0,1]$, $y \in \mathbf{R}$,筝 臂 .

$$g\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}, \frac{y}{\Lambda_u}\right) = \begin{cases} \pm g_u(t,1) \pm \mathcal{D}^{(0,1)} g_u(t,1)(y-1) & (1<y) \\ \pm g_u(t,y) & (-1 \le y \le 1) \\ \pm g_u(t,-1) \pm \mathcal{D}^{(0,1)} g_u(t,-1)(y+1) & (y<-1) \end{cases} \tag{4.13}$$

$$h\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}\right) = \frac{h_u(t)}{\Lambda_u}. \tag{4.14}$$

篁紙 $y \in \mathbf{R}$ $g(1,y) = h(1) = 0$ 臂 .

gh 絎 1.1 ф∽違 ц蟹羣 腓冴.

障 g 紊綣荐膊 純c 腓冴. 茖蕁[2] ず. T, Y ゃ $g(T,Y)$ 羆 , $T = l_u^{\mp} \pm t/\Lambda_u$, $Y = y/\Lambda_u$ 羣 $u, \pm(\mp), t, y$, 紊綣ц膊 純c.

罨< g y ≪ 小 純c, 絨∽違 y, t ≪ g c . 咲x 篋綣違 ≪ 小 $i \in 1,\ldots,k$

$$\mathcal{D}^{(0,i)} g\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}, \frac{y}{\Lambda_u}\right) = \begin{cases} \pm \Lambda_u^i \mathcal{D}^{(0,i)} g_u(t,1) & (1<y) \\ \pm \Lambda_u^i \mathcal{D}^{(0,i)} g_u(t,y) & (-1<y<1) \\ \pm \Lambda_u^i \mathcal{D}^{(0,i)} g_u(t,-1) & (y<-1). \end{cases} \tag{4.15}$$

c $\mathcal{D}^{(0,i)} g(t,1) = \pm \Lambda_u^i \mathcal{D}^{(0,i)} g_u(t,1)$, $\mathcal{D}^{(0,i)} g(t,-1) = \pm \Lambda_u^i \mathcal{D}^{(0,i)} g_u(t,-1)$ c, y ≪ i 綴 純 i ∽違 g.

t 荵御劫吾 i ∽違 g c ゃ. 篁紙 $[0,1)$ 違 u $t \in [0,1]$ 絖 $l_u^{\mp} \pm t/\Lambda_u$ 就ц;. $t \in (0,1)$ $\mathcal{D}^{(0,i)} g_u(t,y)$ g c, t 荵御劫檎g. $t = 0,1$, $y \in [-1,1]$ $g_u(0,y) = g_u(1,y) = 0$ $\mathcal{D}^{(0,i)} g_u(0,y) = \mathcal{D}^{(0,i)} g_u(1,y) = 0$ c $t = 0,1$ g. $g(1,y) = 0$ $\mathcal{D}^{(0,i)} g(1,y) = 0$. 障 $\left|\mathcal{D}^{(0,i)} g_u\right| \le \Lambda_u^{-i} 2^{-|u|}$,

$$\lim_{t \to 1-0} \left|\mathcal{D}^{(0,i)} g\right| = \lim_{|u| \to \infty} \left|\Lambda_u^i \mathcal{D}^{(0,i)} g_u\right| \le \lim_{|u| \to \infty} \left|2^{-|u|}\right| = 0. \tag{4.16}$$

c $\mathcal{D}^{(0,i)} g$ g.

hg 幻綴 合綣 B c 腓冴. $h(0) = 0$, $\mathcal{D}^{(1)} h(1) = 0 = g(1, h(1))$.

$$\begin{aligned} & h'(l_u^{\mp} \pm t/\Lambda_u) \\ & = \pm \frac{h_u'(t)}{\Lambda_u} \\ & = \pm g_u\left(t, h_u(t)\right) \\ & = g\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}, \frac{h_u(t)}{\Lambda_u}\right) \\ & = g\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}, h\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}\right)\right). \end{aligned} \tag{4.17}$$

[2] 羣 y

L h 純c 腓冴.

$$h(c_u) = \frac{h_u(1)}{\Lambda_u} = \frac{L(u)}{2^{\lambda(|u|)+\rho(|u|)}} \tag{4.18}$$

ゃ障 R, S, T 篁ヤ 臂 , .

$$R(u, v) = v \tag{4.19}$$

$$S(u, 0^n) = \lfloor 2^n c_u \rfloor \text{ 茵 絖}, \tag{4.20}$$

$$T(u) = 0^{\lambda(|u|)+\rho(|u|)} \tag{4.21}$$

L **PSPACE** 絎 c , h **PSPACE** 絎.

□

5 絲

5.1 荔域

筝綣 純 ∽違 幻綣 合綣 B **PSPACE** 絎 恭 c ず, 篋綣 巡札筝 狗荐 翫 ≪ 皿絎 4.1 . 筝綣 純 綣 純 蟹 c , 篋綣 巡札筝 ≪ **PSPACE** 絎 恭 荐惹c c . 荐惹 醇c 蚊 ゃ 皿絎 4.1 腓冴 c , 荀 皿絎 絎 1.2 c c .

蕁 皿絎 4.1 E綵. c若 訓 育篏帥 , G_u ュ $T \in [2^{Q(|u|)}]$ ゃ, 絲丈育篏帥 c .

5.2 茯臥

篁紙綣 純 ∽違 幻綣 合綣 B **PSPACE** 絎 荐惹 膃 蕁c . 篁 4.1 ⊢ 純 , 絎 1.2 , “篁 4.1 ,” 荐 , 荐惹. , c若 識 荐膊篁紙綣 純 ∽違 幻綣 合綣 罔≦c c 荐膊 紊ч c 篆荐若 . ゃ障障 障c ャ **PSPACE** 絎 膊, 篁紙綣 純 ∽違 幻綣 合綣ф一 c c 醇c 罧c .

篋吟 B紊綣絎∽違 , 茹ｆ c ＞散 c絖 , 篋 g ♂g綣 純c ゃc , 茹c 腸 障.

障 g 筝綣 t ≪ c g c 粋羆, 綣 純 B , 眼 狗 c .

6　茗萛

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