Computational Complexity of Smooth Differential Equations*

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Abstract

The computational complexity of the solution h to the ordinary differential equation $h(0) = 0$, $h'(t) = g(t, h(t))$ under various assumptions on the function g has been investigated in hope of understanding the intrinsic hardness of solving the equation numerically. Kawamura has shown that the solution h can be PSPACE-hard even if g is assumed to be Lipschitz continuous and polynomial-time computable. We place further requirements on the smoothness of g and obtain the following results: the solution h can still be PSPACE-hard if g is assumed to be of class C^1; for each $k \geq 2$, the solution h can be hard for the counting hierarchy if g is of class C^k.

1 Introduction

Let $g\colon [0,1] \times \mathbf{R} \to \mathbf{R}$ be continuous and consider the differential equation

$$h(0) = 0 \ , \qquad Dh(t) = g(t, h(t)) \quad t \in [0,1] \ , \tag{1}$$

where Dh denotes the derivative of h. How complex can the solution h be, assuming that g is polynomial-time computable? Here, polynomial-time computability and other notions of complexity are from the field of *Computable Analysis* [9, 16] and measure how hard it is to approximate real functions with specified precision (Sect. 2).

If we put no assumption on g other than being polynomial-time computable, the solution h (which is not unique in general) can be non-computable. Table 1 summarizes known results about the complexity of h under various assumptions (that get stronger as we go down the table). In particular, if g is (globally) Lipschitz continuous, then the (unique) solution h is known to be polynomial-space computable but still can be PSPACE-hard [4]. In this paper, we study the complexity of h when we put stronger assumptions about the smoothness of g.

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Table 1: Complexity of the solution h of (1) assuming g is polynomial-time computable

Assumptions	Upper bounds	Lower bounds
—	—	can be all non-computable [13]
h is the unique solution	computable [2]	can take arbitrarily long time [8, 11]
the Lipschitz condition	polynomial-space [8]	can be PSPACE-hard [4]
g is of class $\mathrm{C}^{(\infty,1)}$	polynomial-space	can be PSPACE-hard (Theorem 1)
g is of class $\mathrm{C}^{(\infty,k)}$ (for each constant k)	polynomial-space	can be CH-hard (Theorem 2)
g is analytic	polynomial-time [12, 10, 3]	—

In numerical analysis, knowledge about smoothness of the input function (such as being differentiable enough times) is often beneficial in applying certain algorithms or simplifying their analysis. However, to our knowledge, this casual understanding that smoothness is good has not been rigorously substantiated in terms of computational complexity theory. This motivates us to ask whether, for our differential equation (1), smoothness really reduces the complexity of the solution.

One extreme is the case where g is analytic: h is then polynomial-time computable (the last row of the table) by an argument based on Taylor series[1] (this does not necessarily mean that computing the values of h from those of g is easy; see the last paragraph of Sect. 4.2). Thus our interest is in the cases between Lipschitz and analytic (the fourth and fifth rows). We say that g is of class $\mathrm{C}^{(i,j)}$ if the partial derivative $D^{(n,m)}g$ (often also denoted $\partial^{n+m}g(t,y)/\partial t^n \partial y^m$) exists and is continuous for all $n \leq i$ and $m \leq j$;[2] it is said to be of class $\mathrm{C}^{(\infty,j)}$ if it is of class $\mathrm{C}^{(i,j)}$ for all $i \in \mathbf{N}$.

Theorem 1. *There exists a polynomial-time computable function $g\colon [0,1]\times[-1,1] \to \mathbf{R}$ of class $\mathrm{C}^{(\infty,1)}$ such that the equation (1) has a* PSPACE*-hard solution $h\colon [0,1] \to \mathbf{R}$.*

Theorem 2. *Let k be a positive integer. There is a polynomial-time computable function $g\colon [0,1] \times [-1,1] \to \mathbf{R}$ of class $\mathrm{C}^{(\infty,k)}$ such that the equation (1) has a* CH*-hard solution $h\colon [0,1] \to \mathbf{R}$, where $\mathsf{CH} \subseteq \mathsf{PSPACE}$ is the Counting Hierarchy (see Sect. 3.2).*

We said $g\colon [0,1] \times [-1,1] \to \mathbf{R}$ instead of $g\colon [0,1] \times \mathbf{R} \to \mathbf{R}$, because the notion of polynomial-time computability of real functions in this paper is defined only when the

[1] As shown by Müller [12] and Ko and Friedman [10], polynomial-time computability of an analytic function on a compact interval is equivalent to that of its Taylor sequence at a point (although the latter is a local property, polynomial-time computability on the whole interval is implied by analytic continuation; see [12, Corollary 4.5] or [3, Theorem 11]). This implies the polynomial-time computability of h, since we can efficiently compute the Taylor sequence of h from that of g.

[2] Another common terminology is to say that g is of class C^k if it is of class $\mathrm{C}^{(i,j)}$ for all i, j with $i + j \leq k$.

domain is a bounded closed region.[3] This makes the equation (1) ill-defined in case h ever takes a value outside $[-1, 1]$. By saying that h is a solution in Theorem 1, we are also claiming that $h(t) \in [-1, 1]$ for all $t \in [0, 1]$. This is no essential restriction, because any pair of functions $g\colon [0, 1] \times \mathbf{R} \to \mathbf{R}$ and $h\colon [0, 1] \to \mathbf{R}$ satisfying the equation could be scaled down in the appropriate way (by, say, an integer bounding the absolute values of h) to make h stay in $[-1, 1]$. In any case, since we are putting stronger assumptions on g than Lipschitz continuity, the solution h, if it exists, is unique.

Whether smoothness of the input function reduces the complexity of the output has been studied for operators other than solving differential equations, and the following negative results are known. The integral of a polynomial-time computable real function can be #P-hard, and this does not change by restricting the input to C^∞ (infinitely differentiable) functions [9, Theorem 5.33]. Similarly, the function obtained by maximization from a polynomial-time computable real function can be NP-hard, and this is still so even if the input function is restricted to C^∞ [9, Theorem 3.7][4]. (Restricting to analytic inputs renders the output polynomial-time computable, again by the argument based on Taylor series.) In contrast, for the differential equation we only have Theorem 2 for each k, and do not have any hardness result when g is assumed to be infinitely differentiable.

Theorems 1 and 2 are about the complexity of each solution h. We will also discuss the complexity of the final value $h(1)$ and the complexity of the operator that maps g to h; see Sect. 4.1 and 4.2.

Notation Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ denote the set of natural numbers, integers, rational numbers and real numbers, respectively.

We assume that any polynomial is increasing, since it does not change the meaning of polynomial-time computable or polynomial-space computable.

Let A and B be bounded closed intervals in $\mathbf{R}$. We write $|f| = \sup_{x \in A} f(x)$ for $f\colon A \to \mathbf{R}$. A function $f\colon A \to \mathbf{R}$ is of *class* C^i (or *i-times continuously differentiable*) if all the derivatives $Df, D^2 f, \ldots, D^i f$ exist and are continuous.

For a differentiable function g of two variables, we write $D_1 g$ and $D_2 g$ for the derivatives of g with respect to the first and the second variable, respectively. A function $g\colon A \times B \to \mathbf{R}$ is of *class* $\mathrm{C}^{(i,j)}$ if for each $n \in \{0, \ldots, i\}$ and $m \in \{0, \ldots j\}$, the derivative $D_1^n D_2^m g$ exists and is continuous. A function g is of *class* $\mathrm{C}^{(\infty,j)}$ if it is of class $\mathrm{C}^{(i,j)}$ for all $i \in \mathbf{N}$. When g is of class $\mathrm{C}^{(i,j)}$, we write $D^{(i,j)} g$ for the derivative $D_1^i D_2^j g$.

[3] Although we could extend our definition to functions with unbounded domain [6, Sect. 4.1], the results in Table 1 do not hold as they are, because polynomial-time compubable functions g, such as $g(t, y) = y + 1$, could yield functions h, such as $h(t) = \exp t - 1$, that grow too fast to be polynomial-time (or even polynomial-space) computable. Bournez, Graça and Pouly [1, Theorem 2] report that the statement about the analytic case holds true if we restrict the growth of h appropriately.

[4] The proof of this fact in [9, Theorem 3.7] needs to be fixed by redefining

$$f(x) = \begin{cases} u_s & \text{if not } R(s,t), \\ u_s + 2^{-(p(n)+2n+1)\cdot n} \cdot h_1(2^{p(n)+2n+1}(x - y_{s,t})) & \text{if } R(s,t). \end{cases}$$

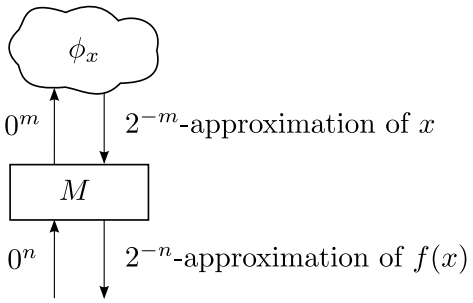

Figure 1: A machine M computing a real function f

2 Computational Complexity of Real Functions

This section reviews the complexity notions in Computable Analysis [9, 16]. We start by fixing an encoding of real numbers by string functions.

Definition 3. A function $\phi\colon \{0\}^* \to \{0,1\}^*$ is a *name* of a real number x if for all $n \in \mathbf{N}$, $\phi(0^n)$ is the binary representation of $\lfloor x \cdot 2^n \rfloor$ or $\lceil x \cdot 2^n \rceil$, where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ mean rounding down and up to the nearest integer.

In effect, a name of a real number x receives 0^n and returns an approximation of x with precision 2^{-n}.

We use *oracle Turing machines* (henceforth just *machines*) to work on these names (Fig. 1). Let M be a machine and ϕ be a function from strings to strings. We write $M^\phi(0^n)$ for the output string when M is given ϕ as oracle and string 0^n as input. Thus we also regard M^ϕ as a function from strings to strings.

Definition 4. Let A be a bounded closed interval of $\mathbf{R}$. A machine M *computes* a real function $f\colon A \to \mathbf{R}$ if for any $x \in A$ and any name ϕ_x of it, M^{ϕ_x} is a name of $f(x)$.

Computation of a function $f\colon A \to \mathbf{R}$ on a two-dimensional bounded closed region $A \subseteq \mathbf{R}^2$ is defined in a similar way using machines with two oracles. A real function is (*polynomial-time*) *computable* if there exists some machine that computes it (in polynomial time). Polynomial-time computability of a real function f means that for any $n \in \mathbf{N}$, an approximation of $f(x)$ with error bound 2^{-n} is computable in time polynomial in n independent of the real number x.

By the time the machine outputs the approximation of $f(x)$ of precision 2^{-n}, it knows x only with some precision 2^{-m}. This implies that all computable real functions are continuous. If the machine runs in polynomial time, this m is bounded polynomially in n. This implies (3) in the following lemma, which characterizes polynomial-time real functions by the usual polynomial-time computability of string functions without using oracle machines.

Lemma 5. *A real function is polynomial-time computable if and only if there exist a polynomial-time computable function* $\phi\colon (\mathbf{Q} \cap [0,1]) \times \{0\}^* \to \mathbf{Q}$ *and polynomial* $p\colon \mathbf{N} \to \mathbf{N}$ *such that for all* $d \in \mathbf{Q} \cap [0,1]$ *and* $n \in \mathbf{N}$,

$$|\phi(d, 0^n) - f(d)| \leq 2^{-n}, \tag{2}$$

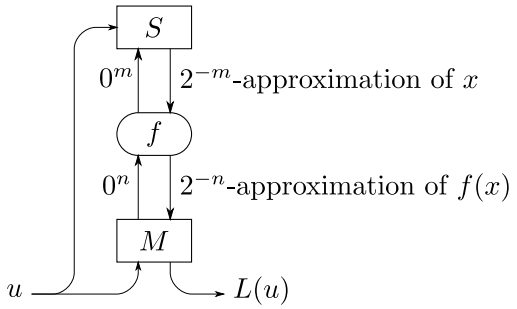

Figure 2: Reduction from a language L to a function $f\colon [0,1] \to \mathbf{R}$

and for all $x, y \in [0,1]$, $n \in \mathbf{N}$,

$$|x - y| \le 2^{-p(n)} \Rightarrow |f(x) - f(y)| \le 2^{-n}, \tag{3}$$

where each rational number is written as a fraction whose numerator and denominator are integers in binary.

To talk about hardness, we define reduction. A language $L \subseteq \{0,1\}^*$ is identified with the function $L\colon \{0,1\}^* \to \{0,1\}$ such that $L(u) = 1$ when $u \in L$.

Definition 6. A language L *reduces* to a function $f\colon [0,1] \to \mathbf{R}$ if there exists a polynomial-time function S and a polynomial-time oracle Turing machine M (Fig. 2) such that for any string u,

(i) $S(u, \cdot)$ is a name of a real number x_u, and

(ii) $M^\phi(u)$ accepts if and only if $u \in L$ for any name ϕ of $f(x_u)$.

This reduction may look stronger (and hence the reducibility weaker) than the one in Kawamura [4] in that M can make multiple queries adaptively, but this makes no difference, because the lengths of these queries are bounded by a polynomial in $|u|$, and the longest query gets all the information that any shorter query gets.

For a complexity class C, a function f is *C-hard* if all languages in C reduce to f.

3 Proof of the Main Theorems

The proofs of Theorems 1 and 2 proceed as follows. In Sect. 3.1, we define *difference equations*, a discrete version of the differential equations. In Sect. 3.2, we show the PSPACE- and CH-hardness of difference equations with certain restrictions. In Sect. 3.3, we show that these classes of difference equations can be simulated by families of differential equations satisfying certain uniform bounds on higher-order derivatives. In Sect. 3.4, we prove Theorems 1 and 2 by putting these families of functions together to obtain one differential equation with the desired smoothness ($\mathrm{C}^{(\infty,1)}$ and $\mathrm{C}^{(\infty,k)}$).

The idea of simulating a discrete system of limited feedback capability by differential equations was essentially already present in the proof of the Lipschitz version

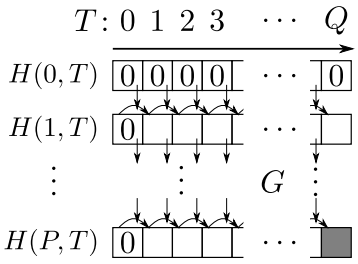

Figure 3: The solution H of the difference equation given by G

[4]. We look more closely at this limited feedback mechanism, and observe that this restriction is one on the *height* of the difference equation. We show that a stronger height restriction allows the difference equation to be simulated by smoother differential equations, leading to the CH-hardness for $\mathrm{C}^{(\infty,k)}$ functions.

3.1 Difference Equations

In this section, we define difference equations, a discrete version of differential equations, and show the PSPACE- and CH-hardness of families of difference equations with different height restrictions.

Let $[n]$ denote $\{0, \ldots, n-1\}$. Let $G\colon [P] \times [Q] \times [R] \to \{-1, 0, 1\}$ and $H\colon [P+1] \times [Q+1] \to [R]$. We say that H is the solution of the *difference equation* given by G if for all $i \in [P]$ and $T \in [Q]$ (Fig. 3),

$$H(i, 0) = H(0, T) = 0 \ , \tag{4}$$

$$H(i+1, T+1) - H(i+1, T) = G(i, T, H(i, T)) \ . \tag{5}$$

We call P, Q and R the *height*, *width* and *cell size* of the difference equation. The equations (4) and (5) are similar to the initial condition $h(0) = 0$ and the equation $Dh(t) = g(t, h(t))$ in (1), respectively. In Sect. 3.3, we will use this similarity to simulate difference equations by differential equations.

We view a family of difference equations as a computing system by regarding the value of the bottom right cell (the gray cell in Fig. 3) as the output. A family $(G_u)_u$ of functions $G_u\colon [P_u] \times [Q_u] \times [R_u] \to \{-1, 0, 1\}$ *recognizes* a language L if for each u, the difference equation given by G_u has a solution H_u and $H_u(P_u, Q_u) = L(u)$. A family $(G_u)_u$ is *uniform* if the height, width and cell size of G_u are polynomial-time computable from u (in particular, they must be bounded by $2^{p(|u|)}$, for some polynomial p) and $G_u(i, T, Y)$ is polynomial-time computable from (u, i, T, Y). A family $(G_u)_u$ has *polynomial height* (and *logarithmic height*, respectively) if the height P_u is bounded by $|u|^{\mathrm{O}(1)}$ (and by $\mathrm{O}(\log |u|)$, respectively). With this terminology, the key lemma in [4, Lemma 4.7] can be written as follows:

Lemma 7. *There exists a* PSPACE*-hard language L that is recognized by some uniform family of functions with polynomial height*[5].

[5]In fact, this language L is in PSPACE, because a uniform family with polynomial height can be

Kawamura obtained the hardness result in the third row of Table 1 by simulating the difference equations of Lemma 7 by Lipschitz continuous differential equations. Likewise, Theorem 1 follows from Lemma 7 by a modified construction that keeps the function in class $\mathrm{C}^{(\infty,1)}$ (Sects. 3.3 and 3.4).

We show further that difference equations restricted to logarithmic height can be simulated by $\mathrm{C}^{(\infty,k)}$ functions for each k (Sects. 3.3 and 3.4). Theorem 2 follows from this simulation and the following lemma.

Lemma 8. *There exists a* CH*-hard language* L *that is recognized by some uniform family of functions with logarithmic height.*

The definition of the counting hierarchy CH, its connection to difference equations and the proof of Lemma 8 will be presented in Sect. 3.2.

3.2 The Counting Hierarchy and Difference Equations of Logarithmic Height

The polynomial hierarchy PH is defined using non-deterministic polynomial-time oracle Turing machines:

$$\Sigma_0^p = \mathsf{P} \ , \qquad \Sigma_{n+1}^p = \mathsf{NP}^{\Sigma_n^p} \ , \qquad \mathsf{PH} = \bigcup_n \Sigma_n^p \ . \tag{6}$$

The counting hierarchy CH is defined similarly using probabilistic polynomial-time oracle Turing machines [15, 14]:

$$\mathsf{C}_0\mathsf{P} = \mathsf{P} \ , \qquad \mathsf{C}_{n+1}\mathsf{P} = \mathsf{PP}^{\mathsf{C}_n\mathsf{P}} \ , \qquad \mathsf{CH} = \bigcup_n \mathsf{C}_n\mathsf{P} \ . \tag{7}$$

It is known that $\mathsf{PH} \subseteq \mathsf{CH} \subseteq \mathsf{PSPACE}$, but we do not know whether $\mathsf{PH} = \mathsf{PSPACE}$.

Each level of the counting hierarchy has a complete problem defined as follows. For every formula $\phi(X)$ with the list X of l free propositional variables, we write

$$\mathsf{C}^m X \phi(X) \longleftrightarrow \sum_{X \in \{0,1\}^l} \phi(X) \geq m \ , \tag{8}$$

where $\phi(X)$ is identified with the function $\phi\colon \{0,1\}^l \to \{0,1\}$ such that $\phi(X) = 1$ when $\phi(X)$ is true. For lists $X_1, \ldots, X_n$ of variables and a formula $\phi(X_1, \ldots, X_n)$ with all free variables listed, we define

$$\langle \phi(X_1, \ldots, X_n), m_1, \ldots, m_n \rangle \in \mathsf{C}_n B_{\mathrm{be}} \longleftrightarrow \mathsf{C}^{m_n} X_n \cdots \mathsf{C}^{m_1} X_1 \phi(X_1, \ldots, X_n) \ . \tag{9}$$

Lemma 9 ([15, Theorem 7])**.** *For every* $n \geq 1$*, the problem* $\mathsf{C}_n B_{\mathrm{be}}$ *is* $\mathsf{C}_n\mathsf{P}$*-complete.*

simulated in polynomial space.

We define the problem $\mathsf{C}_{\log}B_{\mathrm{be}}$ by

$$\langle 0^{2^n}, u\rangle \in \mathsf{C}_{\log}B_{\mathrm{be}} \longleftrightarrow u \in \mathsf{C}_n B_{\mathrm{be}} \ . \tag{10}$$

This problem $\mathsf{C}_{\log}B_{\mathrm{be}}$ is CH-hard, because any problem A in some level $\mathsf{C}_n\mathsf{P}$ of the counting hierarchy reduces to $\mathsf{C}_n B_{\mathrm{be}}$ via some polynomial-time function F_n by Lemma 9, and hence to $\mathsf{C}_{\log}B_{\mathrm{be}}$ by

$$u \in A \longleftrightarrow F_n(u) \in \mathsf{C}_n B_{\mathrm{be}} \longleftrightarrow \langle 0^{2^n}, F_n(u)\rangle \in \mathsf{C}_{\log}B_{\mathrm{be}} \ . \tag{11}$$

We use $\mathsf{C}_{\log}B_{\mathrm{be}}$ as the language L in Lemma 8:

Proof of Lemma 8. We construct a logarithmic-height uniform function family $(G_u)_u$ recognizing $\mathsf{C}_{\log}B_{\mathrm{be}}$. Let $u = \langle 0^{2^n}, \langle \phi(X_1, \ldots, X_n), m_1, \ldots, m_n\rangle\rangle$, where n, $m_1, \ldots, m_n$ are nonnegative integers and ϕ is a formula. (If u is not of this form, then $u \notin \mathsf{C}_{\log}B_{\mathrm{be}}$.)

We write $l_i = |X_i|$ and $s_i = i + \sum_{j=1}^{i} l_j$. For each $i \in \{0, \ldots, n\}$ and $Y_{i+1} \in \{0,1\}^{l_{i+1}}, \ldots, Y_n \in \{0,1\}^{l_n}$, we write $\phi_i(Y_{i+1}, \ldots, Y_n)$ for the truth value of the subformula $\mathsf{C}^{m_i}X_i \cdots \mathsf{C}^{m_1}X_1\phi(X_1, \ldots, X_i, Y_{i+1}, \ldots, Y_n)$, so that $\phi_0 = \phi$ and $\phi_n() = \mathsf{C}_{\log}B_{\mathrm{be}}(u)$. We regard the quantifier C^m as a function from $\mathbf{N}$ to $\{0,1\}$:

$$C^m(x) = \begin{cases} 1 & \text{if } x \geq m \ , \\ 0 & \text{if } x < m \ . \end{cases} \tag{12}$$

Thus,

$$\phi_{i+1}(Y_{i+2}, \ldots, Y_n) = C^{m_{i+1}}\left(\sum_{X_{i+1}\in\{0,1\}^{l_i}} \phi_i(X_{i+1}, Y_{i+2}, \ldots, Y_n)\right) \ . \tag{13}$$

For $T \in \mathbf{N}$, we write T_i for the ith digit of T written in binary (T_0 being the least significant digit), and $T_{[i,j]}$ for the string $T_{j-1}T_{j-2}\cdots T_{i+1}T_i$.

For each $(i, T, Y) \in [n+1] \times [2^{s_n}+1] \times [2^{|u|}]$, we define $G_u(i, T, Y)$ as follows. The first row is given by

$$G_u(0, T, Y) = (-1)^{T_{s_1}}\phi(T_{[1,s_1]}, T_{[s_1+1,s_2]}, \ldots, T_{[s_{n-1}+1,s_n]}) \ , \tag{14}$$

and for $i \neq 0$, we define

$$G_u(i, T, Y) = \begin{cases} (-1)^{T_{s_{i+1}}}C^{m_i}(Y) & \text{if } T_{[1,s_i+1]} = 10\cdots 0 \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{15}$$

This family $(G_u)_u$ is clearly uniform, and its height $n+1$ is logarithmic in $|u|$.

Let H_u be the solution (as defined in (4) and (5)) of the difference equation given by G_u. We prove by induction on i that $H_u(i, T) \in [2^{l_i}]$ for all T, and that

$$G_u(i, V, H_u(i, V)) = (-1)^{V_{s_{i+1}}}\phi_i(V_{[s_i+1,s_{i+1}]}, \ldots, V_{[s_{n-1}+1,s_n]}) \tag{16}$$

if $V_{[1,s_i+1]} = 10\cdots 0$ (otherwise it is immediate from the definition that $G_u(i, V, H_u(n, V)) = 0$).

For $i = 0$, the claims follow from (14). For $i > 0$, assume (16). We have

$$H_u(i+1,T) = \sum_{V=0}^{T-1} G_u(i,V,H_u(i,V)) \ . \tag{17}$$

Since the assumption (16) implies that flipping the bit $V_{s_{i+1}}$ of any V reverses the sign of $G_u(i,V,H_u(i,V))$, most of the summands in (17) cancel out. The terms that can survive satisfy that $V_{[1,s_i+1]} = 10\cdots0$ and that V is between the numbers whose binary representations are $T_{s_n}\ldots T_{s_{i+1}+1}00\ldots0$ and $T_{s_n}\ldots T_{s_{i+1}+1}01\ldots1$. Since these terms are 0 or 1, $H_u(i+1,T) \in [2^{l_i}]$. Then if $T_{[1,s_{i+1}+1]} = 10\cdots0$,

$$H_u(i+1,T) = \sum_{X\in\{0,1\}^{l_i}} \phi_i(X, T_{[s_{i+1}+1,s_{i+2}]},\ldots,T_{[s_{n-1}+1,s_n]}) \ . \tag{18}$$

By this equation, (13) and (15),

$$\begin{aligned} G_u(i+1,T,H_u(i+1,T)) &= (-1)^{T_{s_{i+2}}} C^{m_{i+1}}(H_u(i+1,T)) \\ &= (-1)^{T_{s_{i+2}}} \phi_{i+1}(T_{[s_{i+1}+1,s_{i+2}]},\ldots,T_{[s_{n-1}+1,s_n]}) \ , \end{aligned} \tag{19}$$

completing the induction steps.

By substituting n for i and 2^{s_n} for T in (16), we get $G_u(n,2^{s_n},H_u(n,2^{s_n})) = \phi_n() = \mathsf{C}_{\log}B_{\mathrm{be}}(u)$. Hence $H_u(n+1,2^{s_n}+1) = \mathsf{C}_{\log}B_{\mathrm{be}}(u)$. □

3.3 Families of Real Functions Simulating Difference Equations

We show that certain families of smooth differential equations can simulate PSPACE- or CH-hard difference equations stated in previous section.

Before stating Lemmas 10 and 11, we extend the definition of polynomial-time computability of real function to families of real functions. A machine M *computes* a family $(f_u)_u$ of functions $f_u\colon A \to \mathbf{R}$ indexed by strings u if for any $x \in A$ and any name ϕ_x of x, the function taking v to $M^{\phi_x}(u,v)$ is a name of $f_u(x)$. We say a family of real functions $(f_u)_u$ is polynomial-time if there is a polynomial-time machine computing $(f_u)_u$.

Lemma 10. *There exist a* CH*-hard language L and a polynomial μ, such that for any $k \geq 1$ and polynomial γ, there are a polynomial ρ and families $(g_u)_u$, $(h_u)_u$ of real functions such that $(g_u)_u$ is polynomial-time computable and for any string u:*

(i) $g_u\colon [0,1]\times[-1,1] \to \mathbf{R}$, $h_u\colon [0,1] \to [-1,1]$;

(ii) $h_u(0) = 0$ *and* $Dh_u(t) = g_u(t,h_u(t))$ *for all* $t \in [0,1]$;

(iii) g_u *is of class* $\mathrm{C}^{(\infty,k)}$;

(iv) $D^{(i,0)}g_u(0,y) = D^{(i,0)}g_u(1,y) = 0$ *for all* $i \in \mathbf{N}$ *and* $y \in [-1,1]$;

(v) $\left|D^{(i,j)}g_u(t,y)\right| \leq 2^{\mu(i,|u|)-\gamma(|u|)}$ *for all* $i \in \mathbf{N}$ *and* $j \in \{0,\ldots,k\}$*;*

(vi) $h_u(1) = 2^{-\rho(|u|)}L(u)$.

Lemma 11. *There exist a* PSPACE*-hard language* L *and a polynomial* μ*, such that for any polynomial* γ*, there are a polynomial* ρ *and families* $(g_u)_u$*,* $(h_u)_u$ *of real functions such that* $(g_u)_u$ *is polynomial-time computable and for any string* u *satisfying (i)–(vi) of Lemma 10 with* $k = 1$*.*

In Lemmas 10 and 11, we have the new conditions (iii)–(v) about the smoothness and the derivatives of g_u that were not present in [4, Lemma 4.1]. To satisfy these conditions, we construct g_u using the smooth function f in following lemma.

Lemma 12 ([9, Lemma 3.6])**.** *There exist a polynomial-time function* $f\colon [0,1] \to \mathbf{R}$ *of class* C^∞ *and a polynomial* s *such that*

(i) $f(0) = 0$ *and* $f(1) = 1$*;*

(ii) $D^n f(0) = D^n f(1) = 0$ *for all* $n \geq 1$*;*

(iii) f *is strictly increasing;*

(iv) $D^n f$ *is polynomial-time computable for all* $n \geq 1$*;*

(v) $|D^n f| \leq 2^{s(n)}$ *for all* $n \geq 1$*.*

Although the existence of the polynomial s satisfying the condition (v) is not stated in [9, Lemma 3.6], it can be shown easily.

We will prove Lemma 10 using Lemma 8 as follows. Let a function family $(G_u)_u$ be as in Lemma 8, and let $(H_u)_u$ be the family of the solutions of the difference equations given by $(G_u)_u$. We construct h_u and g_u from H_u and G_u such that $h_u(T/2^{q(|u|)}) = \sum_{i=0}^{p(|u|)} H_u(i,T)/B^{d_u(i)}$ for each $T = 0, \ldots, 2^{q(|u|)}$ and $Dh_u(t) = g_u(t, h_u(t))$. The polynomial-time computability of $(g_u)_u$ follows from that of $(G_u)_u$. We omit the analogous and easier proof of Lemma 11.

Proof of Lemma 10. Let L and $(G_u)_u$ be as in Lemma 8, and let a function family $(H_u)_u$ be the solution of the difference equation given by $(G_u)_u$. Let f and s be as in Lemma 12.

By a similar argument to the beginning of the proof of [4, Lemma 4.1], we may assume that there exist polynomial-time functions p, j_u and polynomials q, r satisfying the following properties:

$$G_u\colon [p(|u|)] \times [2^{q(|u|)}] \times [2^{r(|u|)}] \to \{-1,0,1\} \ , \tag{20}$$

$$H_u(i, 2^{q(|u|)}) = \begin{cases} L(u) & \text{if } i = p(|u|) \ , \\ 0 & \text{if } i < p(|u|) \ , \end{cases} \tag{21}$$

$$G_u(i,T,Y) \neq 0 \to i = j_u(T) \ . \tag{22}$$

Since G_u has logarithmic height, there exists a polynomial σ such that $(k+1)^{p(x)} \leq \sigma(x)$.

We construct the families of real functions $(g_u)_u$ and $(h_u)_u$ simulating G_u and H_u in the sense that $h_u(T/2^{q(|u|)}) = \sum_{i=0}^{p(|u|)} H_u(i,T)/B^{d_u(i)}$, where the constant B and the function $d_u\colon [p(|u|)+1] \to \mathbf{N}$ are defined by

$$B = 2^{\gamma(|u|)+r(|u|)+s(k)+k+3} \ , \qquad d_u(i) = \begin{cases} \sigma(|u|) & \text{if } i = p(|u|) \ , \\ (k+1)^i & \text{if } i < p(|u|) \ . \end{cases} \tag{23}$$

For each $(t,y) \in [0,1] \times [-1,1]$, there exist unique $N \in \mathbf{N}$, $\theta \in [0,1)$, $Y \in \mathbf{Z}$ and $\eta \in [-1/4, 3/4)$ such that $t = (T+\theta)2^{-q(|u|)}$ and $y = (Y+\eta)B^{-d_u(j_u(T))}$. Using f and a polynomial s of Lemma 12, we define $\delta_{u,Y}\colon [0,1] \to \mathbf{R}$, $g_u\colon [0,1] \times [-1,1] \to \mathbf{R}$ and $h_u\colon [0,1] \to [-1,1]$ by

$$\delta_{u,Y}(t) = \frac{2^{q(|u|)} D f(\theta)}{B^{d_u(j_u(T)+1)}} G_u\big(j_u(T), T, Y \bmod 2^{r(|u|)}\big) \ , \tag{24}$$

$$g_u(t,y) = \begin{cases} \delta_{u,Y}(t) & \text{if } \eta \le \frac{1}{4} \ , \\ (1 - f(\frac{4\eta-1}{2}))\delta_{u,Y}(t) + f(\frac{4\eta-1}{2})\delta_{u,Y+1}(t) & \text{if } \eta > \frac{1}{4} \ , \end{cases} \tag{25}$$

$$h_u(t) = \sum_{i=0}^{p(|u|)} \frac{H_u(i,T)}{B^{d_u(i)}} + \frac{f(\theta)}{B^{d_u(j_u(T)+1)}} G_u\big(j_u(T), T, H_u(j_u(T),T)\big) \ . \tag{26}$$

We will verify that $(g_u)_u$ and $(h_u)_u$ defined above satisfy all the conditions stated in Lemma 10. Polynomial-time computability of $(g_u)_u$ can be verified using Lemma 5. The condition (i) is immediate from (25) and (26). The condition (ii) is verified by a similar argument to [4, Lemma 4.1].

It is easy to verify that g_u is of class $\mathrm{C}^{(\infty,\infty)}$ since we construct δ_u from G_u and g_u from δ_u by connecting them with f of class $\mathrm{C}^{(\infty,\infty)}$. This means that the condition (iii) holds. The derivative $D^i\delta$ is given by for each $i \in \mathbf{N}$,

$$D^i\delta_{u,Y}(t) = \frac{2^{(i+1)q(|u|)} D^{i+1} f(\theta)}{B^{d_u(j_u(T)+1)}} G_u\big(j_u(T),\ T,\ Y \bmod 2^{r(|u|)}\big) \ . \tag{27}$$

The derivative $D^{(i,j)}g$ is given by for each $i \in \mathbf{N}$ and $j = 0$,

$$D^{(i,0)}g_u(t,y) = \begin{cases} D^i\delta_{u,Y}(t) & \text{if } \eta \le \frac{1}{4} \ , \\ \left(1 - f\left(\frac{4\eta-1}{2}\right)\right) D^i\delta_{u,Y}(t) + f\left(\frac{4\eta-1}{2}\right) D^i\delta_{u,Y+1}(t) & \text{if } \frac{1}{4} < \eta \ , \end{cases} \tag{28}$$

and for each $i \in \mathbf{N}$ and $j \in \{1,\dots,k\}$,

$$D^{(i,j)}g_u(t,y) = \begin{cases} 0 & \text{if } -\frac{1}{4} < \eta < \frac{1}{4} \ , \\ (2B^{d_u(j_u(T))})^j D^j f(\frac{4\eta-1}{2})(D^i\delta_{u,Y+1}(t) - D^i\delta_{u,Y}(t)) & \text{if } \frac{1}{4} < \eta < \frac{3}{4} \ . \end{cases} \tag{29}$$

Substituting $t = 0, 1$ ($\theta = 0$) into (28), we get $D^{(i,0)}g_u(0,y) = D^{(i,0)}g_u(1,y) = 0$, so the condition (iv) holds.

We show that the condition (v) holds with $\mu(x,y) = (x+1)q(y)+s(x+1)$. Note that μ is a polynomial and independent of k and γ. Since $|D^i\delta_{u,Y}(t)| \le 2^{(i+1)q(|u|)+s(i+1)} B^{-d_u(j_u(|u|)+1)}$ by (24), for all $i \in \mathbf{N}$ and $j \in \{0,\dots,k\}$, we have

$$|D^{(i,j)}g_u| \le 2^k B^{k \cdot j_u(T)} 2^{s(k)} \cdot 2 \cdot \frac{2^{(i+1)q(|u|)+s(i+1)}}{B^{d_u(j_u(|u|)+1)}} \le \frac{2^{\mu(i,|u|)+s(k)+k+1}}{B} \le 2^{\mu(i,|u|)-\gamma(|u|)}$$

by (28), (29) and our choice of B.

We have (vi) with $\rho(x) = \sigma(x) \cdot (\gamma(x) + r(x) + s(k) + k + 3)$, because

$$h_u(1) = \frac{H_u(p(|u|), 2^{q(|u|)})}{B^{d_u(p(|u|))}} = \frac{L(u)}{2^{\sigma(|u|)\cdot(\gamma(|u|)+r(|u|)+s(k)+k+3)}} = 2^{-\rho(|u|)}L(u) \ . \tag{30}$$

□

To prove Lemma 11, let L and $(G_u)_u$ be as Lemma 7, and let $(H_u)_u$ be the solution of the difference equation given by $(G_u)_u$. Define $(g_u)_u$ and $(h_u)_u$ as (25) and (26) with $d_u(i) = i$. It is shown in the same way as above that they meet all the conditions stated in Lemma 11.

3.4 Proof of the Main Theorems

Using the function families $(g_u)_u$ and $(h_u)_u$ obtained from Lemmas 10 or 11, we construct the functions g and h in Theorems 1 and 2 as follows. Divide $[0, 1)$ into infinitely many subintervals $[l_u^-, l_u^+]$, with midpoints c_u. We construct h by putting a scaled copy of h_u onto $[l_u^-, c_u]$ and putting a horizontally reversed scaled copy of h_u onto $[c_u, l_u^+]$ so that $h(l_u^-) = 0$, $h(c_u) = 2^{-\rho'(|u|)}L(u)$ and $h(l_u^+) = 0$ where ρ' is a polynomial. In the same way, g is constructed from $(g_u)_u$ so that g and h satisfy (1). We give the details of the proof of Theorem 2 from Lemma 10, and omit the analogous proof of Theorem 1 from Lemma 11.

Proof of Theorem 2. Let L and μ be as Lemma 10. Define $\lambda(x) = 2x + 2$, $\gamma(x) = \mu(x, x)+x\lambda(x)$ and for each u let $\Lambda_u = 2^{\lambda(|u|)}$, $c_u = 1-2^{-|u|}+2\bar{u} + 1/\Lambda_u$, $l_u^{\mp} = c_u \mp 1/\Lambda_u$, where $\bar{u} \in \{0, \ldots, 2^{|u|} - 1\}$ is the number represented by u in binary notation. Let ρ, $(g_u)_u$, $(h_u)_u$ be as in Lemma 10 corresponding to the above γ.

We define

$$g\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}, \frac{y}{\Lambda_u}\right) = \begin{cases} \pm\sum_{l=0}^{k} \frac{D^{(0,l)}g_u(t,1)}{l!}(y-1)^l & \text{if } 1 < y \ , \\ \pm g_u(t, y) & \text{if } -1 \le y \le 1 \ , \\ \pm\sum_{l=0}^{k} \frac{D^{(0,l)}g_u(t,-1)}{l!}(y+1)^l & \text{if } 1 < y \ , \end{cases} \tag{31}$$

$$h\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}\right) = \frac{h_u(t)}{\Lambda_u} \tag{32}$$

for each string u and $t \in [0, 1)$, $y \in [-1, 1]$. Let $g(1, y) = 0$ and $h(1) = 0$ for any $y \in [-1, 1]$.

It can be shown similarly to the Lipschitz version [4, Theorem 3.2] that g and h satisfy (1) and g is polynomial-time computable. Here we only prove that g is of class $\mathrm{C}^{(\infty,k)}$. We claim that for each $i \in \mathbf{N}$ and $j \in \{0, \ldots, k\}$, the derivative $D_1^i D_2^j g$ is given by

$$D_1^i D_2^j g\left(l_u^{\mp} \pm \frac{t}{\Lambda_u}, \frac{y}{\Lambda_u}\right) = \begin{cases} \pm\Lambda_u^{i+j}\sum_{l=j}^{k} \frac{D^{(i,l)}g_u(t,1)}{(l-j)!}(y-1)^l & \text{if } y < -1 \ , \\ \pm\Lambda_u^{i+j} D^{(i,j)}g_u(t, y) & \text{if } -1 \le y \le 1 \ , \\ \pm\Lambda_u^{i+j}\sum_{l=j}^{k} \frac{D^{(i,l)}g_u(t,-1)}{(l-j)!}(y+1)^l & \text{if } 1 < y \end{cases} \tag{33}$$

for each $l_u^{\mp} \pm t/\Lambda_u \in [0,1)$ and $y/\Lambda_u \in [-1,1]$, and by $D_1^i D_2^j g(1,y) = 0$. This is verified by induction on $i+j$. The equation (33) follows from calculation (note that this means verifying that (33) follows from the definition of g when $i = j = 0$; from the induction hypothesis about $D_2^{j-1} g$ when $i = 0$ and $j > 0$; and from the induction hypothesis about $D_1^{i-1} D_2^j g$ when $i > 0$). That $D_1^i D_2^j g(1,y) = 0$ is immediate from the induction hypothesis if $i = 0$. If $i > 0$, the derivative $D_1^i D_2^j g(1,y)$ is by definition the limit

$$\lim_{s \to 1-0} \frac{D_1^{i-1} D_2^j g(1,y) - D_1^{i-1} D_2^j g(s,y)}{1-s} \ . \tag{34}$$

This can be shown to exist and equal 0, by observing that the first term in the numerator is 0 and the second term is bounded, when $s \in [l_u^-, l_u^+]$, by

$$\begin{aligned} |D_1^{i-1} D_2^j g(s,y)| &\le \Lambda_u^{i-1+j} \sum_{l=j}^{k} |D^{(i-1,l)} g_u| \cdot (\Lambda_u + 1)^l \\ &\le \Lambda_u^{i-1+j} \cdot k \cdot 2^{\mu(i-1,|u|)-\gamma(|u|)} \cdot (2\Lambda_u)^k \\ &\le 2^{(i-1+j+k)\lambda(|u|)+2k+\mu(i-1,|u|)-\gamma(|u|)} \le 2^{-2|u|} \le 2^{-|u|+1}(1-s) \ , \end{aligned} \tag{35}$$

where the second inequality is from Lemma 10 (v) and the fourth inequality holds for sufficiently large $|u|$ by our choice of γ. The continuity of $D_1^i D_2^j g$ on $[0,1) \times [-1,1]$ follows from (33) and Lemma 10 (iv). The continuity on $\{1\} \times [-1,1]$ is verified by estimating $D_1^i D_2^j g$ similarly to (35). □

4 Other Versions of the Problem

4.1 Complexity of the Final Value

In the previous section, we considered the complexity of the solution h of the ODE as a real function. Here we discuss the complexity of the final value $h(1)$ and relate it to tally languages (subsets of $\{0\}^*$), as did Ko [8] and Kawamura [3, Theorem 5.1] for the Lipschitz continuous case.

We say that a language L *reduces to* a real number x if there is a polynomial-time oracle Turing machine M such that M^ϕ accepts L for any name ϕ of x. Note that this is the same as the reduction in Definition 6 to a constant function with value x.

Theorem 13. *For any tally language $T \in \mathsf{PSPACE}$, there are a polynomial-time computable function $g\colon [0,1] \times [-1,1] \to \mathbf{R}$ of class $\mathrm{C}^{(\infty,1)}$ and a function $h\colon [0,1] \to \mathbf{R}$ satisfying (1) such that L reduces to $h(1)$.*

Theorem 14. *Let k be a positive integer. For any tally language $T \in \mathsf{CH}$, there are a polynomial-time computable function $g\colon [0,1] \times [-1,1] \to \mathbf{R}$ of class $\mathrm{C}^{(\infty,k)}$ and a function $h\colon [0,1] \to \mathbf{R}$ satisfying (1) such that L reduces to $h(1)$.*

We can prove Theorem 14 from Lemma 10 in the same way as the proof of [3, Theorem 5.1]. We skip the proof of Theorem 13 since it is similar.

Proof. Let T be any tally language in CH and k be any positive integer, and let L and μ be as Lemma 10. Define $\lambda(x) = x + 1$, $\gamma(x) = \mu(x, x) + x\lambda(x)$ and let ρ, $(g_u)_u$, $(h_u)_u$ be as in Lemma 10 corresponding to the γ. Since L is CH-complete, there are a polynomial-time function F such that $T(0^i) = L(F(0^i))$ for all i.

Let $l_n = 1 - 2^n$ and $\bar{\rho}(n) = \sum_{i=0}^{n-1} \rho(|F(0^i)|)$. Define g and h as follows: when the first variable is in $[0, 1)$, let

$$g\left(l_n + \frac{t}{2^{n+1}}, \frac{2m + (-1)^m y}{2^{2n+\gamma(n)+\bar{\rho}(n)}}\right) = \frac{g_{F(0^n)}(t, y)}{2^{n-1+\gamma(n)+\bar{\rho}(n)}} \ , \tag{36}$$

$$h\left(l_n + \frac{t}{2^{n+1}}\right) = \frac{h_{F(0^n)}(t)}{2^{2n+\gamma(n)+\bar{\rho}(n)}} + \sum_{i=0}^{n-1} \frac{T(0^i)}{2^{2i+\gamma(i)+\bar{\rho}(i+1)}} \tag{37}$$

for each $n \in \mathbf{N}$, $t \in [0, 1]$, $y \in [-1, 1]$ and $m \in \mathbf{Z}$; when the first variable is 1, let

$$g(1, y) = 0 \ , \tag{38}$$

$$h(1) = \sum_{n=0}^{\infty} \frac{T(0^n)}{2^{2n+\gamma(n)+\bar{\rho}(n+1)}} \ . \tag{39}$$

It can be proved similarly to the proof of Theorem 2 that g is polynomial-time computable and of class $\mathrm{C}^{(\infty,k)}$ and that g and h satisfy (1). The equation (39) implies that T reduces to $h(1)$. □

4.2 Complexity of Operators

Both Theorems 1 and 2 state the complexity of the solution h under the assumption that g is polynomial-time computable. But how hard is it to "solve" differential equations, i.e., how complex is the operator that takes g to h? To make this question precise, we need to define the complexity of operators taking real functions to real functions.

Recall that, to discuss complexity of real functions, we used string functions as names of elements in $\mathbf{R}$. Such an encoding is called a *representation* of $\mathbf{R}$. Likewise, we now want to encode real functions by string functions to discuss complexity of real operators. In other words, we need to define representations of the class $\mathrm{C}_{[0,1]}$ of continuous functions $h\colon [0, 1] \to \mathbf{R}$ and class $\mathrm{CL}_{[0,1]\times[-1,1]}$ of Lipschitz continuous functions $g\colon [0, 1] \times [-1, 1] \to \mathbf{R}$. The notions of computability and complexity depend on these representations. Following [6], we use $\delta_\square$ and $\delta_{\square\mathrm{L}}$ as the representations of $\mathrm{C}_{[0,1]}$ and $\mathrm{CL}_{[0,1]\times[-1,1]}$, respectively. It is known that $\delta_\square$ is the canonical representation of $\mathrm{C}_{[0,1]}$ in a certain sense [5], and $\delta_{\square\mathrm{L}}$ is the representation defined by adding to $\delta_\square$ the information on the Lipschitz constant.

Since these representations use string functions whose values have variable lengths, we use *second order polynomials* to bound the amount of resources (time and space) of machines [6], and this leads to the definitions of second-order complexity classes (e.g. **FPSPACE**, polynomial-space computable), reductions (e.g. $\leq_{\mathrm{W}}$, polynomial-time Weihrauch reduction), and hardness. Combining them with the representations of real functions mentioned above, we can restate our theorems in the constructive form as follows.

Let *ODE* be the operator mapping a real function $g \in \mathrm{CL}_{[0,1]\times[-1,1]}$ to the solution $h \in \mathrm{C}_{[0,1]}$ of (1). The operator *ODE* is a partial function from $\mathrm{CL}_{[0,1]\times[-1,1]}$ to $\mathrm{C}_{[0,1]}$ (it is partial because the trajectory may fall out of the interval $[-1,1]$, see the paragraph following Theorem 2). In [6, Theorem 4.9], the $(\delta_{\square\mathrm{L}}, \delta_{\square})$-**FPSPACE**-$\leq_{\mathrm{W}}$-completeness of *ODE* was proved by modifying the proof of the results in the third row of Table 1. Theorem 1 can be rewritten in a similar way. That is, let ODE_k be the operator *ODE* whose input is restricted to class $\mathrm{C}^{(\infty,k)}$. Then we have:

Theorem 15. *The operator* ODE_1 *is* $(\delta_{\square\mathrm{L}}, \delta_{\square})$-**FPSPACE**-$\leq_{\mathrm{W}}$-*complete.*

To show this theorem, we need to verify that the information used to construct functions in the proof of Theorem 1 can be obtained easily from the inputs. We omit the proof since it does not require any new technique. Theorem 15 automatically implies Theorem 1 because of [6, Lemmas 3.7 and 3.8].

In contrast, the polynomial-time computability in the analytic case (the last row of Table 1) is *not* a consequence of a statement based on $\delta_{\square}$. It is based on the calculation of the Taylor coefficients, and hence we only know how to convert the Taylor sequence of g at a point to that of h. In other words, the operator *ODE* restricted to the analytic functions is not $(\delta_{\square\mathrm{L}}, \delta_{\square})$-**FP**-computable, but $(\delta_{\mathrm{Taylor}}, \delta_{\mathrm{Taylor}})$-**FP**-computable, where δ_{Taylor} is the representation that encodes an analytic function using its Taylor coefficients at a point in a suitable way. More discussion on representations of analytic functions and the complexity of operators on them can be found in [7].

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