Research Notes

1 Background

Given a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with finite nodes $|\mathcal{V}| = N$, the discrete online networked control problem with LTI dynamics, TV-costs, and disturbances has the following form:

$$\begin{aligned}
\min_{(x_{0:T})_{\mathcal{V}}, (u_{0:T-1})_{\mathcal{V}}} \ & \sum_{t=1}^{T} \sum_{i \in \mathcal{V}} f_t[i](x_t[i]) + c_t[i](u_{t-1}[i]) \\
\text{s.t. } \ & x_{t+1}[i] = \sum_{j \in \mathcal{N}_{\mathcal{G}}[i]} A[i,j]x_t[j] + B[i,j]u_t[j] + w_t[i], \quad i \in \mathcal{V}, \text{ and } t = 0, ..., T-1, \\
& x_0[i] = \bar{x}[i], \quad i \in \mathcal{V},
\end{aligned} \tag{1}$$

where the neighbourhood of a node i is defined as $\mathcal{N}_{\mathcal{G}}[i] := \{i\} \cup \{j : (i,j) \in \mathcal{E}\}$. $x_t[i]$, $u_t[i]$, and $w_t[i]$ are the state, control, and disturbance/exogenous input of node i at time t; similarly $A[i,j]$ and $B[i,j]$ are system matrices connecting nodes i and j, and are only nonzero when $(i,j) \in \mathcal{E}$. Furthermore, $f_t[i](\cdot)$ and $c_t[i](\cdot)$ are the time-varying costs associated with the state and control of node i and the tuples $(x_{0:T})_{\mathcal{V}}$ and $(u_{0:T-1})_{\mathcal{V}}$ are the states of all nodes from time 0 to T and controls of all nodes for times 0 to $T-1$ respectively.

1.1 Algorithm Description

The original online algorithm we are interested in is based off of Lin's predictive control algorithm in which at every time step t they observe k information tuples I_t to I_{t+k-1} where $I_t := (A, B, w_t, f_{t+1}, c_{t+1})$ and solve the finite-horizon predictive control problem for $t < T - k$:

$$\begin{aligned}
\tilde{\Psi}_t^k(x, \zeta; F) := \operatorname*{arg\,min}_{y_{0:k}, v_{0:k-1}} & \sum_{\tau=0}^{k-1} f_{t+\tau}(y_\tau) + \sum_{\tau=1}^{k} c_{t+\tau}(v_{\tau-1}) + F(y_k) \\
\text{s.t. } & y_{\tau+1} = Ay_\tau + Bv_\tau + \zeta_\tau, \quad \tau = 0, ..., k-1 \\
& y_0 = x,
\end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ is the initial state, $\zeta \in (\mathbb{R}^n)^k$ is a sequence of disturbances, and F is a terminal cost function regularizing the final state. y_τ is the predictive state at time step τ and v_τ is the predictive control action. In the decentralized setting, we consider the truncated optimization problem for each node:

$$\begin{aligned}
& \tilde{\psi}_t^k(x, \zeta, \mathcal{N}_{\mathcal{G}}^\kappa[i]; F) := \\
& \operatorname*{arg\,min}_{(y_{0:k}), (v_{0:k-1})} \sum_{\tau=0}^{k-1} f_{t+\tau}(y_\tau) + \sum_{\tau=1}^{k} c_{t+\tau}(v_{\tau-1}) + F(y_k) \\
& \text{s.t. } \ y_{\tau+1}[j] = \begin{cases} \sum_{m \in \mathcal{N}_{\mathcal{G}}[j]} A[j,m]y_\tau[m] + B[j,m]v_\tau[m] + \zeta_\tau[j], & j \in \mathcal{N}_{\mathcal{G}}^\kappa[i], \\ 0, & \text{else}, \end{cases} \quad \tau = 0, ..., k-1 \\
& \qquad y_0[j] = \begin{cases} x[j], & j \in \mathcal{N}_{\mathcal{G}}^\kappa[i], \\ 0, & \text{else}, \end{cases}
\end{aligned} \tag{3}$$

where the notation $\mathcal{N}_{\mathcal{G}}^\kappa[i]$ is set containing $\{j : d_{\mathcal{G}}(i,j) \le \kappa\}$ (i.e. the kappa-hop neighbourhood of i). We apply the the predictive control $u_t[i] = v_0[i]$ obtained from solving $\tilde{\psi}_t^k(x, \zeta, \mathcal{N}_{\mathcal{G}}^\kappa[i]; F)$ at each time step t for every node i. The main idea of the algorithm in (3) is that each node observes a κ-hop neighbourhood of information (setting all variables outside of the neighbourhood equal to 0) and tries to solve the centralized problem in (2) with its given information.

2 Exponential Decay Property of (2)

We first make some assumptions about our costs:

Assumption 1. *The costs in* (2) *and* (3) *satisfy the following*

1. $f_t(\cdot)$ *is* μ_f*-strongly convex for* $t = 1, ..., T$*, and* L_f*-smooth for* $t = 1, ..., T-1$
2. $c_t(\cdot)$ *is both* μ_c*-strongly convex and* L_c*-smooth for* $t = 1, .., T$*.*
3. $F(\cdot)$ *is a convex,* L_F*-smooth, K-function such that* $\nabla^2 F \succeq \nabla^2 f_t$ *for all* $t = 1, ..., T-1$*.*
4. $f_t(\cdot)$*,* $c_t(\cdot)$*, and* $F(\cdot)$ *are* C^2 *functions for* $t = 1, .., T$
5. $f_t(\cdot)$ *and* $c_t(\cdot)$ *are non-negative and* $f_t(0) = c_t(0) = 0$ *for* $t = 1, .., T$*.*

We also make the following assumption on the system matrices in (2)

Assumption 2. *The system matrices in* (2) *satisfy the following:*

1. $\|A\| \leq L_A$*,* $\|B\| \leq L_B$
2. (A, B) *is* (L, ρ)*-stabilizable for* $L > 1$ *and* $\rho \in (0, 1)$*.*

Where we will often denote $L := \max\{L_f, L_c, L_F, L_A, L_B\} > 1$. and we define stabilizability in the following equivalent fashion:

Definition 1. *We say a system* (A, B) *is* $(L, \rho)-$*stabilizable if there exists a* K *for* $L > 1$ *and* $\rho \in [0, 1)$*, such that* $\|K\| \leq L$ *and* $\|\Phi^t\| \leq L\rho^t$ *for* $t \in \mathbb{Z}_{\geq 0}$ *and where* $\Phi := A - BK$*.*

We note that this definition is equivalent to the standard notion of stabilizability which is: $\exists K$ such that $\mathrm{sr}(\Phi) < 1$.
From the problem set up of (2), we can derive the lagrangian as

$$\mathcal{L}((y_\tau)_\tau, (v_\tau)_\tau, (\lambda_\tau)_\tau) := \sum_{\tau=0}^{k-1} f_{t+\tau}(y_\tau) + \sum_{\tau=1}^{k} c_{t+\tau}(v_{\tau-1}) + F(y_k) + \lambda_0^\top (y_0 - x) + \sum_{\tau=0}^{k-1} \lambda_{\tau+1}^\top (y_{\tau+1} - Ay_\tau - Bv_\tau - \zeta_\tau)$$

So the first order necessary conditions (FONC) are

$$\begin{bmatrix} \nabla_{y_0} f_t(y_0^*) + \lambda_0^* - A^\top \lambda_1^* \\ \nabla_{v_0} c_{t+1}(v_0^*) - B^\top \lambda_1^* \\ \nabla_{y_1} f_{t+1}(y_1^*) + \lambda_1^* - A^\top \lambda_2^* \\ \nabla_{v_1} c_{t+2}(v_1^*) - B^\top \lambda_2^* \\ \vdots \\ \nabla_{y_{k-1}} f_{t+k-1}(y_{k-1}^*) + \lambda_{k-1}^* - A^\top \lambda_k^* \\ \nabla_{v_{k-1}} c_{t+k}(v_{k-1}^*) - B^\top \lambda_k^* \\ \nabla_{y_k} F(y_k^*) + \lambda_k^* \end{bmatrix} = 0$$

We additionally require that the hessian of the lagrangian at the minimum $\nabla_{zz}\mathcal{L}((y_\tau^*)_\tau, (v_\tau^*)_\tau, (\lambda_\tau^*)_\tau)$ be positive semidefinite on the tangent space of the feasible set. More concretely, define $z = (y_\tau, v_\tau)_\tau$ to be a feasible trajectory and let $\tilde{f}(z)$ be the cost in (2), then clearly the second order condition is satisfied given the strong convexity of $\tilde{f}(z)$ from assumption 1. We note that the constraint jacobian of (2) is of the form:

$$J := \begin{bmatrix} I & & & & & & & & \\ -A & -B & I & & & & & & \\ & & -A & -B & I & & & & \\ & & & & \ddots & & & & \\ & & & & -A & -B & I & & \\ & & & & & & -A & -B & I \end{bmatrix},$$

where J is of dimension $n(k+1) \times n(k+1) + mk$. Combining the FONC and the requirement that an optimal trajectory z^* must be feasible (i.e. the KKT conditions), we obtain the following system of equations

$$\begin{bmatrix} \nabla \tilde{f}(z^*) + J^\top \lambda^* \\ Jz^* \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{c} \end{bmatrix},$$

where $\mathbf{c} = \begin{bmatrix} x^\top & \zeta^\top \end{bmatrix}^\top$. We claim that there exists $G(z^*)$ such that $G(z^*)(z^* - z') = \nabla \tilde{f}(z^*) - \nabla \tilde{f}(z')$ and

$$\begin{bmatrix} G(z^*) & J^\top \\ J & 0 \end{bmatrix} \begin{bmatrix} z^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} G(z^*)z' - \nabla \tilde{f}(z') \\ \mathbf{c} \end{bmatrix},$$

for any z'. (Rewriting the proof from Guannan's paper for posterity): The proof immediately follows from the next lemma:

Lemma 1. *for μ-strongly convex and L-smooth function f, there exists symmetric $G(z)$ such that $\mu I \preceq G(z) \preceq LI$ such that $\nabla f(z) - \nabla f(z') = G(z)(z - z')$*

Proof. We start off by defining the vector-valued trajectory $g(s) = \nabla f(z' + s(z - z'))$. Its derivative is then $g'(s) = \nabla^2 f(z' + s(z - z'))(z - z')$ and so we can write

$$\nabla f(z) - \nabla f(z') := g(1) - g(0) = \int_0^1 g'(s)\mathrm{d}s = \left(\int_0^1 \nabla^2 f(z' + s(z - z'))\mathrm{d}s \right)(z - z') = G(z)(z - z')$$

and by μ-strong convexity and L-smoothness of f, we have that for any $s \in [0, 1]$, $\mu I \preceq G(z) \preceq LI$. $\square$

So now it is sufficient to say that the optimality conditions for (2) follows the KKT-matrix-like form

$$\begin{bmatrix} G(z^*) & J^\top \\ J & 0 \end{bmatrix} \begin{bmatrix} z^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} G(z^*)z' - \nabla \tilde{f}(z') \\ \mathbf{c} \end{bmatrix}. \tag{4}$$

To prove exponential decay of the resulting controller from (2), we define the notion of bandwidth in matrices such that

Definition 2. *Consider a matrix H, a graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, and a partitioning of the entries of H by the nodes in $\mathcal{V}$, that is a sequence of tuples $(I_i, J_j)_{i,j \in \mathcal{V}}$ such that I_i is the set of indices belonging to node i and similarly for J_j. Then we say that H has bandwidth B induced by the graph $\mathcal{G}$ and the partitioning $(I_i, J_j)_{i,j \in \mathcal{V}}$ if B is the smallest nonnegative integer such that the submatrix $H[i, j] = 0$ (Which is the matrix containing indices from the sets I_i and J_j) for $i, j \in \mathcal{V}$ such that $d_{\mathcal{G}}(i, j) > B$.*

and given this definition, from Sungho's paper we have the following theorem:

Theorem 3. *Consider H whose bandwidth B_H is no greater than 1 induced by the graph $\mathcal{G}$ and a partitioning $(I_i, J_j)_{i.j \in \mathcal{V}}$; further, assume that μ_H, $L_H > 0$ satisfy the following inequality about the singular values of H: $\mu_H \leq \sigma(H) \leq L_H$. Then $\|H^{-1}[i, j]\| \leq \alpha \rho^{d_{\mathcal{G}}(i,j)}$ for $i, j \in \mathcal{V}$ where $\rho = \sqrt{\frac{L_H^2 - \mu_H^2}{L_H^2 + \mu_H^2}}$ and $\alpha = \frac{L_H}{\mu_H^2 \rho}$.*

The proof from Sungho's paper is provided here:

Proof. H satisfies the following property $\mu_H^2 I \preceq HH^\top \preceq L_H^2 I$ and so we can construct a matrix whose singular values have magnitude less than 1 in the following way:

$$\frac{2\mu_H^2}{\mu_H^2 + L_H^2} I \preceq \frac{2}{\mu_H^2 + L_H^2} HH^\top \preceq \frac{2L_H^2}{\mu_H^2 + L_H^2} I \iff \frac{\mu_H^2 - L_H^2}{\mu_H^2 + L_H^2} I \preceq I - \frac{2}{\mu_H^2 + L_H^2} HH^\top \preceq \frac{L_H^2 - \mu_H^2}{L_H^2 + \mu_H^2} I.$$

Using this matrix, we can rewrite H^{-1} as

$$\begin{aligned} H^{-1} &= \frac{2}{\mu_H^2 + L_H^2} H^\top \left(\frac{2}{\mu_H^2 + L_H^2} HH^\top \right)^{-1} \\ &= \frac{2}{\mu_H^2 + L_H^2} \sum_{n=0}^{\infty} H^\top \left(I - \frac{2}{\mu_H^2 + L_H^2} HH^\top \right)^n, \end{aligned}$$

for which the term in the sum $H^\top(I - \frac{2}{\mu_H^2+L_H^2}HH^\top)^n$ has bandwidth no more than $(2n+1)B_H = 2n+1$. Extracting the subblocks of H^{-1} we have

$$H^{-1}[i,j] = \frac{2}{\mu_H^2 + L_H^2} \sum_{n=n_0}^{\infty} \left(H^\top \left(I - \frac{2}{\mu_H^2 + L_H^2} HH^\top \right)^n \right) [i,j]$$

where $n_0 = \lceil \frac{d_{\mathcal{G}}(i,j)-1}{2} \rceil$. Observe that for any $n < n_0$, $2n+1 < d_{\mathcal{G}}(i,j)$. Then we take the norm of the sublock and the desired upper bound follows:

$$\begin{aligned} \|H^{-1}[i,j]\| &\le \frac{2}{\mu_H^2 + L_H^2} \sum_{n=n_0}^{\infty} \left\| H^\top \left(I - \frac{2}{\mu_H^2 + L_H^2} HH^\top \right)^n \right\| \\ &\le \frac{2}{\mu_H^2 + L_H^2} \sum_{n=n_0}^{\infty} L_H \left(\frac{L_H^2 - \mu_H^2}{L_H^2 + \mu_H^2} \right)^n \le \frac{L_H}{\mu_H^2} \left(\frac{L_H^2 - \mu_H^2}{L_H^2 + \mu_H^2} \right)^{\lceil \frac{d_{\mathcal{G}}(i,j)-1}{2} \rceil}, \end{aligned}$$

this finishes the proof. □

Applying this line of reasoning to our setting, essentially, we want to show that the pseudo-KKT matrix

$$H := \begin{bmatrix} G(z^*) & J^\top \\ J & 0 \end{bmatrix} \tag{5}$$

follows the hypothesis in theorem 3. We first show that H has bandwidth B_H no greater than 1

Theorem 4. *The pseudo KKT matrix H in* (5) *has bandwidth $B_H = 1$ induced by the graph $\mathcal{G}$ and some partitioning $(I_i, J_j)_{i,j \in \mathcal{V}}$*

Proof. To prove this theorem, we first prove the following lemma:

Lemma 2. *Suppose M is a block matrix such that $M = [M_k]_{k=1:KN}$ (M is composed of KN submatrices M_k) have at most bandwidth 1 induced by the graph $\mathcal{G}$ and a partitioning $(I_i^k, J_j^k)_{i,j \in \mathcal{V}}$, also has bandwidth 1 induced by the same graph $\mathcal{G}$ and a "larger" partitioning $(I_i, J_j)_{i,j \in \mathcal{V}}$*

Proof. To this end, we explicitly construct the larger partitioning $(I_i, J_j)_{i,j \in \mathcal{V}}$. First, we collect all such "sub-partitionings" $(I_i^k, J_j^k)_{i, \in \mathcal{V}}$ for $k = 1, ..., KN$ and map the local indices that belong to said partitioning to their relative indices in the overall block matrix M, that is the set of local indices I_i^k in M_k gets mapped to the relative indices $(I_k)_i$ in M and similarly: $J_j^k \to (J_k)_i$. Then the larger partitioning is the sequence of tuples $(I_i, J_j)_{i,j \in \mathcal{V}}$ such that each set of indices I_i contains all relative indices $((I_k)_{k=1:KN})_i$ for $i \in \mathcal{V}$ and similarly, $J_j := ((J_k)_{k=1:KN})_j$ for $j \in \mathcal{V}$. Then it immediately follows that $M_k[i,j] = 0$ for $d_{\mathcal{G}}(i,j) > 1$ where M_k is partitioned by $(I_i^k, J_j^k)_{i,j \in \mathcal{V}}$ implies that $M[i,j] = 0$ for $d_{\mathcal{G}}(i,j) > 1$ where M is partitioned by $(I_i, J_j)_{i,j \in \mathcal{V}}$. □

Since A and B are networked matrices, J must have bandwidth $B_J = 1$ by the lemma above. It is then sufficient to show that $G(z^*)$ has bandwidth $B_{G(z^*)} = 1$. Since $G(z^*) = \int_0^1 \nabla^2 \tilde{f}(z' + s(z^* - z'))\mathrm{d}s$ and the integral does not affect the sparsity structure of $\nabla^2 \tilde{f}(z' + s(z^* - z'))$ the bandwidth of $G(z^*)$ is equal to the bandwidth of $\nabla^2 \tilde{f}(z' + s(z^* - z'))$. Recall that the hessian of $\tilde{f}$ is the block matrix

$$\nabla^2 \tilde{f}(z^*) = \begin{bmatrix} \nabla^2_{y_0 y_0} f_t(y_0^*) & & & & \\ & \nabla^2_{v_0 v_0} c_{t+1}(v_0^*) & & & \\ & & \ddots & & \\ & & & \nabla^2_{v_{k-1} v_{k-1}} c_{t+k}(v_{k-1}^*) & \\ & & & & \nabla^2_{y_k y_k} F(y_k^*) \end{bmatrix},$$

where in the setting of (1), these submatrices are also diagonal block matrices such that their bandwidths equal 0. Hence H has bandwidth at most $B_H = 1$. □

2.1 Singular values of H

Next, we require upperbounds on the singular values of H. In particular

Theorem 5. *The pseudo-KKT matrix H has the property that $\mu_H I \preceq H \preceq L_H I$ where*

Proof. Suppose that the following uniform regularity assumptions hold:

$$\|H\| \leq L_H, \quad Z^\top G(z^*)Z \succeq \mu I \quad JJ^\top \succeq \beta_J I,$$

we recognize the second and third conditions as the uniform strong second-order sufficiency condition (SSOSC) and uniform linear independence constraint qualification (LICQ) respectively. Z is a nullspace basis matrix of J such that $\mathrm{range}(Z) = \mathrm{null}(J)$. By definition of singular values, the upper bound follows immediately. Since $G(z^*)$ is not necessarily positive definite, we can recover the inverse of H as

$$\begin{aligned}
H^{-1} &= \begin{bmatrix} G(z^*) + \gamma J^\top J & J^\top \\ J & 0 \end{bmatrix}^{-1} \begin{bmatrix} I & \gamma J^\top \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} T^{-1} - T^{-1}J^\top(JT^{-1}J^\top)^{-1}JT^{-1} & T^{-1}J^\top(JT^{-1}J^\top)^{-1} \\ (JT^{-1}J^\top)^{-1}JT^{-1} & -(JT^{-1}J^\top)^{-1} \end{bmatrix} \begin{bmatrix} I & \gamma J^\top \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} T^{-1} - T^{-1}J^\top(JT^{-1}J^\top)^{-1}JT^{-1} & T^{-1}J^\top(JT^{-1}J^\top)^{-1} \\ (JT^{-1}J^\top)^{-1}JT^{-1} & \gamma I - (JT^{-1}J^\top)^{-1} \end{bmatrix}
\end{aligned}$$

where $T := G(z^*) + \gamma J^\top J$ and $\gamma := \frac{2L_H^2/\mu + \mu + L_H}{\beta_J}$. The second equality comes from the schur complement. And so we can obtain a lower bound on the singular values of H as

$$\|H^{-1}\| \leq \|T^{-1}\| + (1 + 2\|JT^{-1}\| + \|JT^{-1}\|^2)\|(JT^{-1}J^\top)^{-1}\| + \gamma,$$

To get a proper estimate of the lower bound above, we must estimate the bounds on the singular values of $T = G(z^*) + \gamma J^\top J$. Given that $H \preceq L_H$, then it is clear that $T \preceq L_H(1 + \gamma L_H)$. $\underline{\lambda}(T)$ can be obtained via the optimization problem $\min_{\|v\|=1} v^\top(G(z^*) + \gamma J^\top J)v$ where, we can express v as

$$v = Yv_y + Zv_z$$

where Y is the matrix whose columns form an orthonormal basis for the row space of J and and Z is the matrix whose columns form an orthonormal basis for the null space of J. We must then have that $\|v_y\|^2 + \|v_z\|^2 = 1$. We then have that the minimization problem can be lower bounded as

$$\begin{aligned}
& v^\top(G(z^*) + \gamma J^\top J)v \\
&= v_y^\top Y^\top(G(z^*) + \gamma J^\top J)Yv_y + v_z^\top Z^\top G(z^*)Zv_z + 2v_y^\top Y^\top G(z^*)Zv_z \\
&\geq -L_H\|v_y\|^2 + \gamma\underline{\lambda}(JYY^\top J^\top)\|v_y\|^2 + \mu\|v_z\|^2 - 2L_H\|v_y\|\|v_z\| \\
&\geq (\gamma\beta_J - L_H - \mu)\|v_y\|^2 - 2L_H\|v_y\| + \mu \\
&\geq \mu - \frac{L_H^2}{\mu\beta_J - L_H - \mu} = \mu/2.
\end{aligned}$$

Where the first equality comes from $JZ = 0$. The first inequality comes from orthogonality of Y and Z, the lower and upper bounds on the singular values of $G(z^*)$ and J, and the property that $\underline{\lambda}(M^\top M) = \underline{\lambda}(MM^\top)$ for any square matrix M. The second inequality comes from orthogonality of Y, the SSOSC conditions, and the fact that $1 \geq \|v_z\|^2 = 1 - \|v_y\|^2 \geq 0$. Finally the last line comes from noting that $\gamma\beta_J - L_H - \mu = 2L_H^2/\mu > 0$ from where $L_H > 0$ by the LICQ and SSOSC, and taking the derivative of the quadratic form in the previous line and setting it equal to zero. Thus, since $\mu/2 \preceq T \preceq L_H(1+\mu L_H)$, we can estimate the lower bound on the singular values of H as

$$\|H^{-1}\| \leq \frac{2}{\mu} + \left(1 + \frac{4L_H}{\mu} + \frac{4L_H^2}{\mu^2}\right)\frac{L_H(1 + \gamma L_H)}{\beta_J} + \gamma = \mu_H.$$

Where the upper bound on $\|(JT^{-1}J^\top)^{-1}\|$ follows as

$$\begin{aligned}\|(JT^{-1}J^\top)^{-1}\| &= \|(JT^{-1/2}T^{-1/2}J^\top)^{-1}\| \\ &= \|(T^{-1/2}(J^\top J)^{1/2}(J^\top J)^{1/2}T^{-1/2})^{-1}\| \\ &= \|T^{1/2}(J^\top J)^{-1}T^{1/2}\| \leq \|T\|\|(J^\top J)^{-1}\| \leq L_H(1+\mu L_H)/\beta_J.\end{aligned}$$

It then remains to show that the uniform regularity assumptions hold.

Lemma 3. *For the H matrix in (5), the following regularity conditions hold under assumptions 1 and 2:*

$$\|H\| \leq L_H, \quad Z^\top G(z^*)Z \succeq \mu I, \quad JJ^\top \succeq \beta_J I$$

Proof. First denote $G_{f_{t+\tau}}(y_\tau^*) := \int_0^1 \nabla^2_{y_\tau y_\tau} f_{t+\tau}(y'_\tau + s(y^*_\tau - y'_\tau))\mathrm{d}s$, $G_{c_{t+\tau+1}}(v^*_\tau) := \int_0^1 \nabla^2_{v_\tau v_\tau} c_{t+\tau+1}(v'_\tau + s(v^*_\tau - v'_\tau))\mathrm{d}s$, and $G_F(y^*_k) := \int_0^1 \nabla^2_{y_k y_k} F(y'_k + s(y^*_k - y'_k))\mathrm{d}s$. We will denote these unambiguously as $G_{f_{t+\tau}}$, $G_{c_{t+\tau+1}}$, and G_F unambiguously. Then the $G(z^*)$ matrix is

$$G(z^*) = \begin{bmatrix} G_{f_t} & & & & & \\ & G_{c_{t+1}} & & & & \\ & & \ddots & & & \\ & & & G_{f_{t+k-1}} & & \\ & & & & G_{c_{t+k}} & \\ & & & & & G_F \end{bmatrix}. \tag{6}$$

Proof of $\|H\| \leq L_H$: Let I_τ be the index set corresponding to $(y_\tau, v_\tau, \lambda_\tau)$, then the matrix H can be split up as

$$H[I_\tau, I_{\tau'}] = \begin{cases} \begin{bmatrix} G_{f_{t+\tau}} & & I \\ & G_{c_{t+\tau+1}} & \\ I & & \end{bmatrix}, & \tau = \tau' \neq k, \\ \begin{bmatrix} & & \mathbf{0} \\ & & \mathbf{0} \\ -A & -B & \end{bmatrix}, & \tau - \tau' = 1,\ \tau \neq k \\ \mathbf{0}, & \tau - \tau' > 1, \\ \begin{bmatrix} G_F & I \\ I & \end{bmatrix}, & \tau = \tau' = k, \\ \begin{bmatrix} & & \mathbf{0} \\ -A & -B & \end{bmatrix}, & \tau - \tau' = 1,\ \tau = k, \end{cases}$$

where each sublock denotes rows corresponding to the indices of $(y_\tau, v_\tau, \lambda_\tau)$ and the columns correspond to the indices of $(y_{\tau'}, v_{\tau'}, \lambda_{\tau'})$. The case for $t < t'$ is similar due to the symmetry of H and is not shown here. Then, from the following Lemma

Lemma 4. *Given a matrix M and a partitioning $(I_i, J_j)_{i,j \in \mathcal{V}}$, the following holds:*

$$\|M\| \leq \left(\max_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} \|M[i,j]\|\right)^{1/2} \left(\max_{j \in \mathcal{V}} \sum_{i \in \mathcal{V}} \|M[i,j]\|\right)^{1/2}.$$

Whose proof is in Sungho's previous paper (*Exponential Decay of Sensitivity in Graph-Structured Nonlinear Programs*). Using the above result, we see that the upper bound on the norm of H is $L_H = 2L+1$.

Proof of $JJ^\top \succeq \beta_J I$: Consider the column operation:

$$\begin{bmatrix} I & & & & \\ -A & -B & I & & \\ & & \ddots & & \\ & & -A & -B & I \end{bmatrix} \begin{bmatrix} I & & & & \\ -K & I & & & \\ & & \ddots & & \\ & & -K & I & \\ & & & & I \end{bmatrix} = \begin{bmatrix} I & & & & \\ -\Phi & -B & I & & \\ & & \ddots & & \\ & & -\Phi & -B & I \end{bmatrix},$$

where K is an (L,ρ)-stabilizing gain of (A,B). Denote the matrix on the RHS as E and the inverse of the column operation as J_2, then $JJ^\top = EJ_2J_2^\top E^\top$. Then denote a submatrix of the RHS as J_1 whose form is:

$$J_1 := \begin{bmatrix} I & & & \\ -\Phi & I & & \\ & & \ddots & \\ & & -\Phi & I \end{bmatrix}, \quad J_1^{-1} = \begin{bmatrix} I & & & \\ \Phi & I & & \\ \vdots & & \ddots & \\ \Phi^k & \cdots & \Phi & I \end{bmatrix}, \quad J_2 = \begin{bmatrix} I & & & & \\ -K & I & & & \\ & & \ddots & & \\ & & -K & I & \\ & & & & I \end{bmatrix}^{-1},$$

since J_1 is a submatrix of E, we have that $JJ^\top \succeq J_1J_2J_2^\top J_1^\top \succeq \underline{\lambda}(J_2J_2^\top)J_1J_1^\top \succeq \|J_1^{-1}\|^{-2}\|J_2^{-1}\|^{-2}I$. From Lemma 4, we see that $\|J_1^{-1}\| \leq \frac{L}{1-\rho}$ and similarly, $\|J_2^{-1}\| \leq L+1$ since K is an (L,ρ) stabilizing gain. Then we achieve the lower bound $FF^\top \succeq \frac{(1-\rho)^2}{L^2(1+L)^2}I = \beta_J I$

Proof of $Z^\top G(z^*)Z \succeq \mu I$: We know that any $G_{f_{t+\tau}} = \int_0^1 \nabla^2_{y_\tau y_\tau} f_{t+\tau}(y'_\tau + s(y_{\tau^*} - y'_\tau))\mathrm{d}s \succeq \mu_f I$, $\nabla^2 F \succeq \nabla^2 f_t$, and $G_{c_{t+\tau+1}} = \int_0^1 \nabla^2_{v_\tau v_\tau} c_{t+\tau+1}(v'_\tau + s(v^*_\tau - v'_\tau))\mathrm{d}s \succeq \mu_c I$. Set $\mu := \min\{\mu_c, \mu_f\}$ then it immediately follows that $G(z^*) \succeq \mu$ and so, $Z^\top G(z^*)Z \succeq \mu$. □

This finishes the proof of Theorem 5. □

2.2 Analysis of the node problem (3)

We see that the problem in (3) can be written more succinctly as

$$\begin{aligned}\tilde{\psi}_t^k(x,\zeta,\mathcal{N}_{\mathcal{G}}^{\kappa}[i];F) := \underset{(y_{0:k}),(v_{0:k})}{\arg\min} & \sum_{\tau=0}^{k-1} f_{t+\tau}(y_\tau) + \sum_{\tau=1}^{k} c_{t+\tau}(v_{\tau-1}) + F(y_k) \\ \text{s.t. } & y_{\tau+1} = A^{i,\kappa} y_\tau + B^{i,\kappa} v_\tau + \zeta_\tau^{i,\kappa}, \quad \tau = 0,...,k-1 \\ & y_0 = x^{i,\kappa},\end{aligned} \tag{7}$$

Where we define

$$A^{i,\kappa}[j,m] := \begin{cases} A[j,m], & j \in \mathcal{N}_{\mathcal{G}}^{\kappa}[i], \\ 0, & \text{else,}\end{cases}$$

and similarly for $B^{i,\kappa}, \zeta^{i,\kappa}$, and $x^{i,\kappa}$. We will henceforth leave out the i in the notation above and will implicitly assume some node $i \in \mathcal{V}$. Denote the solution to the problem above as z^κ and the corresponding dual variables as λ^κ, then the kkt conditions are

$$\begin{bmatrix} \nabla \tilde{f}(z^\kappa) + (J^\kappa)^\top \lambda^\kappa \\ J^\kappa z^\kappa \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \begin{bmatrix} x^\kappa \\ \zeta^\kappa \end{bmatrix} \end{bmatrix}, \tag{8}$$

where we define

$$J^\kappa := \begin{bmatrix} I & & & & \\ -A^\kappa & -B^\kappa & I & & \\ & & \ddots & & \\ & & -A^\kappa & -B^\kappa & I \end{bmatrix}.$$

We aim to describe the difference between the solutions to (4) in which the relationship between the two is described by the following theorem

Theorem 6. *Denote the solution to the kkt conditions of the centralized problem in* (2) q^c *and* q^κ *for that of the decentralized problem in* (7) *then under assumptions 1, 2, and 7, we have that the norm difference between the two problems at node* i *is*

$$\|q^c[i] - q^\kappa[i]\| \le \Gamma(\|x\| + \|\zeta\|)\delta^\kappa,$$

where $\delta := \frac{\rho+1}{2}$ *and* $\Gamma := \frac{2\alpha^2\delta L}{(1-\delta)^2}\left(\sup_{d\in\mathbb{Z}_+}(\rho/\delta)^d p(d)\right)^2 p(1)$ *where* ρ *and* α *are as in theorem 3 and* $p(\cdot)$ *is the subexponential function described in assumption 7.*

Proof. We first derive a nicer form for the difference between the two trajectories as

$$\begin{bmatrix} \nabla\tilde{f}(z^c) - \nabla\tilde{f}(z^\kappa) + (J^c)^\top\lambda^c - (J^\kappa)^\top\lambda^\kappa \\ J^c z^c - J^\kappa z^\kappa \end{bmatrix} = \begin{bmatrix} G(z^c - z^\kappa) + (J^c)^\top\lambda^c - (J^\kappa)^\top\lambda^\kappa \\ J^c z^c - J^\kappa z^\kappa \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \begin{bmatrix} x^\perp \\ \zeta^\perp \end{bmatrix} \end{bmatrix},$$

note that the first equality is due to Lemma 1. Further note that we have relabeled the solution to (4) as $\begin{bmatrix} z^\top & \lambda^\top \end{bmatrix}^\top = \begin{bmatrix} (z^c)^\top & (\lambda^c)^\top \end{bmatrix}^\top$ and similarly $J^c = J$. We define $x^\perp := x - x^\kappa$ and $\zeta^\perp := \zeta - \zeta^\kappa$. Note that this system of equations can be written more succinctly as:

$$H^c \begin{bmatrix} z^c \\ \lambda^c \end{bmatrix} - H^\kappa \begin{bmatrix} z^\kappa \\ \lambda^\kappa \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \begin{bmatrix} x^\perp \\ \zeta^\perp \end{bmatrix} \end{bmatrix},$$

where

$$H^c := \begin{bmatrix} G & (J^c)^\top \\ J^c & \mathbf{0} \end{bmatrix}, \quad H^\kappa := \begin{bmatrix} G & (J^\kappa)^\top \\ J^\kappa & \mathbf{0} \end{bmatrix}.$$

Define $q^c := \begin{bmatrix} z^c \\ \lambda^c \end{bmatrix}$ and similarly for q^κ, then

$$H^c q^c - H^\kappa q^\kappa = H^c(q^c - q^\kappa) - (H^\kappa - H^c)q^\kappa = \begin{bmatrix} \mathbf{0} \\ \begin{bmatrix} x^\perp \\ \zeta^\perp \end{bmatrix} \end{bmatrix}$$

$$q^c - q^\kappa = (H^c)^{-1} \begin{bmatrix} \mathbf{0} \\ \begin{bmatrix} x^\perp \\ \zeta^\perp \end{bmatrix} \end{bmatrix} - (H^c)^{-1}(H^c - H^\kappa) \begin{bmatrix} z^\kappa \\ \lambda^\kappa \end{bmatrix},$$

observe that the matrix $H^c - H^\kappa$ is simply

$$H^\perp := H^c - H^\kappa = \begin{bmatrix} \mathbf{0} & (J^\perp)^\top \\ J^\perp & \mathbf{0} \end{bmatrix},$$

then, recall the constraints in (3)

$$y_{\tau+1}[j] = \begin{cases} \sum_{m \in \mathcal{N}_\mathcal{G}[j]} A[j,m]y_\tau[m] + B[j,m]v_\tau[m] + \zeta_\tau[m], & j \in \mathcal{N}_\mathcal{G}^\kappa[i], \\ 0, & \text{else}, \end{cases}, \quad \tau = 0, ..., k-1$$

and

$$y_0[j] = \begin{cases} x_0[j], & j \in \mathcal{N}_\mathcal{G}^\kappa[i] \\ 0, & \text{else}, \end{cases}$$

which imply that any feasible trajectory to (3) must have all $y_\tau[j] = 0$ for all $\tau = 0, ..., k$ and $j \notin \mathcal{N}_\mathcal{G}^\kappa[i]$, and WLOG $v_\tau[j]$ must also be 0 for all $\tau = 0, ..., k-1$ and $j \notin \mathcal{N}_\mathcal{G}^\kappa[i]$.
Recall that the FONC for (7) are

$$\begin{bmatrix} \nabla f_t(y_0^\kappa) + \lambda_0^\kappa - (A^\kappa)^\top \lambda_1^\kappa \\ \nabla c_{t+1}(v_0^\kappa) - (B^\kappa)^\top \lambda_1^\kappa \\ \nabla f_{t+1}(y_1^\kappa) + \lambda_1^\kappa - (A^\kappa)^\top \lambda_2^\kappa \\ \vdots \\ \nabla c_{t+k}(v_{k-1}^\kappa) - (B^\kappa)^\top \lambda_k^\kappa \\ \nabla F(y_k^\kappa) + \lambda_k^\kappa \end{bmatrix} = \mathbf{0}$$

for the optimal trajectory $(z^\kappa, \lambda^\kappa) = ((y_\tau^\kappa, v_\tau^\kappa), (\lambda_\tau)^\kappa)$. For $j \notin \mathcal{N}_\mathcal{G}^\kappa[i]$, we see that the FONC for these nodes are

$$\begin{bmatrix} \nabla f_t(y_0^\kappa)[j] + \lambda_0^\kappa[j] \\ \nabla f_{t+1}(y_1^\kappa)[j] + \lambda_1^\kappa[j] \\ \vdots \\ \nabla f_{t+k-1}(y_{k-1}^\kappa)[j] + \lambda_{k-1}^\kappa[j] \\ \nabla F(y_k^\kappa)[j] + \lambda_k^\kappa[j] \end{bmatrix} = \mathbf{0}.$$

If each $f_{t+\tau}(y_\tau^\kappa) := \sum_{m \in \mathcal{V}} f_{t+\tau}[m](y_\tau^\kappa[m]) = \sum_{m \in \mathcal{V}} f_{t\tau}(y_\tau^\kappa)[m]$ where each nodal cost $f_{t+\tau}[m]$ satisfies the same assumptions in Assumption 1, then each $\nabla f_{t+\tau}(y_\tau^\kappa)[j]$ vanishes for any $j \notin \mathcal{N}_\mathcal{G}^\kappa[i]$, and therefore the optimal $\lambda_\tau^\kappa[j]$ must be $\mathbf{0}$ as well. Under this decentralized form of the cost in (7), we have that

$$q^c - q^\kappa = (H^c)^{-1} \begin{bmatrix} \mathbf{0} \\ \begin{bmatrix} x^\perp \\ \zeta^\perp \end{bmatrix} \end{bmatrix} - (H^c)^{-1} H^\perp q^\kappa.$$

Since $J^\perp$ has the form

$$J^\perp = \begin{bmatrix} \mathbf{0} & & & & \\ -A^\perp & -B^\perp & \mathbf{0} & & \\ & & \ddots & & \\ & & -A^\perp & -B^\perp & \mathbf{0} \end{bmatrix},$$

observe that its norm is upper bounded by $2L$ by Lemma 4. Each matrix has the following structure

$$A^{\perp}[j,m] = \begin{cases} A[j,m], & d_{\mathcal{G}}(i,j) > \kappa, \\ \mathbf{0}, & \text{else}, \end{cases}$$

where the term $H^{\perp}q^{\kappa}$ has non-zero terms like $H^{\perp}[j,m]q^{\kappa}[m]$ where $d_{\mathcal{G}}(i,m) \le \kappa$ and $d_{\mathcal{G}}(i,j) > \kappa$. For (j,m) such that $d_{\mathcal{G}}(j,m) \le 1$, $H^{\perp}[j,m]$ is nonzero and 0 otherwise. Thus, all such m must be on the boundary of the the kappa-hop neighbourhood which we will denote $\partial\mathcal{N}_{\mathcal{G}}^{\kappa}[i] := \{j \colon d_{\mathcal{G}}(i,j) = \kappa\}$. To get an upper bound on the norm difference of the solutions q^c and q^{κ} at node i, we make the following assumption:

Assumption 7. *There exists a subexponential function $p(\cdot)$ such that*

$$|\{j \in \mathcal{V} \colon d_{\mathcal{G}}(i,j) = d\}| \le p(d), \quad \forall i \in \mathcal{V},$$

and so, the bound follows:

$$\begin{aligned}
&\|q^c[i] - q^{\kappa}[i]\| \\
&= \left\| \sum_{j\in\mathcal{V}} (H^c)^{-1}[i,j] \left(\mathbf{c}^{\perp}[j] - H^{\perp}q^{\kappa}[j]\right) \right\| \\
&\le \sum_{j\in\mathcal{V}\setminus\mathcal{N}_{\mathcal{G}}^{\kappa}[i]} \|(H^c)^{-1}[i,j]\| \left(\|\mathbf{c}^{\perp}[j]\| + \sum_{m\in\mathcal{N}_{\mathcal{G}}^{\kappa}[i]\cap\mathcal{N}_{\mathcal{G}}[j]} \|H^{\perp}[j,m]q^{\kappa}[m]\| \right) \\
&\le \alpha \sum_{j\in\mathcal{V}\setminus\mathcal{N}_{\mathcal{G}}^{\kappa}[i]} \rho^{d_{\mathcal{G}}(i,j)} \left(\|\mathbf{c}^{\perp}[j]\| + \sum_{m\in\partial\mathcal{N}_{\mathcal{G}}^{\kappa}[i]\cap\mathcal{N}_{\mathcal{G}}[j]} \sum_{k\in\mathcal{N}_{\mathcal{G}}^{\kappa}[i]} \|H^{\perp}[j,m]\|\|\mathbf{H}^{-1}[m,k]\|\|\mathbf{c}^{\kappa}[k]\| \right) \\
&\le \alpha \sum_{d=\kappa+1}^{\infty} (\rho/\delta)^d p(d)\delta^d \left(\|\mathbf{c}^{\perp}\| + 2\alpha L \sum_{m\in\partial\mathcal{N}_{\mathcal{G}}^{\kappa}[i]\cap\mathcal{N}_{\mathcal{G}}[j]} \sum_{k\in\mathcal{N}_{\mathcal{G}}^{\kappa}[i]} \rho^{d_{\mathcal{G}}(m,k)} \|\mathbf{c}^{\kappa}[k]\| \right) \\
&\le \frac{\alpha\delta}{1-\delta} (\sup_{d\in\mathbb{Z}_+} (\rho/\delta)^d p(d))\delta^{\kappa} \left(\|\mathbf{c}^{\perp}\| + \frac{2\alpha L}{1-\delta} \left(\sup_{d\in\mathbb{Z}_+} (\rho/\delta)^d p(d) \right) p(1)\|\mathbf{c}^{\kappa}\| \right) \\
&\le \Gamma(\|\mathbf{c}^{\perp}\| + \|\mathbf{c}^{\kappa}\|)\delta^{\kappa}.
\end{aligned}$$

where α and ρ are defined as in Theorem 3, $\delta := \frac{\rho+1}{2}$, $\mathbf{H}$ is the KKT matrix corresponding to the decentralized problem (7) (as defined in Lemma 1 with $z' = 0$), and $\Gamma := \frac{2\alpha^2\delta L}{(1-\delta)^2} \left(\sup_{d\in\mathbb{Z}_+} (\rho/\delta)^d p(d) \right)^2 p(1)$. In the third inequality we use the fact that $\|H^{\perp}\| \le 2L$, Assumption 7, and Theorem 3. And in the fourth inequality, we use the fact that $p(d)$ is subexponential, assumption 7, and the following inequality:

$$\sum_{k\in\mathcal{N}_{\mathcal{G}}^{\kappa}[i]} \rho^{d_{\mathcal{G}}(m,k)} \le \sum_{k\in\mathcal{V}} \rho^{d_{\mathcal{G}}(m,k)} \le \sum_{d=0}^{\infty} (\rho/\delta)^d p(d)\delta^d \le \left(\sup_{d\in\mathbb{Z}_+} (\rho/\delta)^d p(d) \right) \frac{1}{1-\delta}.$$

□

3 Stability and Regret of (3)

We have the following upper bound on the performance of the two controllers (2) and (7) such that

$$\|u^c(x)[i] - u^{\kappa}[i](x)\| = \|\tilde{\Psi}_t^k(x,\zeta;F)_{v_0[i]} - \tilde{\psi}_t^k(x,\zeta,\mathcal{N}_{\mathcal{G}}^{\kappa}[i];F)_{v_0[i]}\| \le \Gamma\left(\|x\| + \|\zeta\|\right)\delta^{\kappa}$$

where u^c denotes the centralized controller obtained from (2) and u^κ is the decentralized controller obtained from (3). More explicitly, we have that

$$u^c(x) := \tilde{\Psi}_t^k(x,\zeta;F)_{v_0}, \quad u^\kappa[i](x) := \tilde{\psi}_t^k(x,\zeta,\mathcal{N}_\mathcal{G}^\kappa[i];F)_{v_0[i]},$$

where $u^\kappa(x) := [u^\kappa[i](x)]_{i\in\mathcal{V}}$. The closed loop dynamics are then

$$\begin{aligned} x_{t+1}^c &= Ax_t^c + Bu^c(x_t^c) + w_t, \\ x_{t+1}^\kappa &= Ax_t^\kappa + Bu^\kappa(x_t^\kappa) + w_t, \end{aligned}$$

where we note that $x_{t+1}^c = \tilde{\Psi}_t^k(x_t,\zeta_t;F)_{y_1}$ in which $\zeta_t = w_{t:t+k-1}$.

3.1 Stability

The centralized predictive control algorithm (2) generates the trajectory directly from the optimization problem, that is, $x_{t+1}^c = \tilde{\Psi}_t^k(x_t^c;F)_{y_1}$. For the decentralized control algorithm in (3), x_{t+1} is generated from the dynamics above: $x_{t+1} = Ax_t + Bu^\kappa(x_t) + w_t$, and the control input produced by the algorithm is $u^\kappa[i](x_t) := \tilde{\psi}_t^k(x_t,\mathcal{N}_\mathcal{G}[i];F)_{v_0[i]}$. In order to derive a stability result for the trajectory generated by the algorithm in (3), it would be helpful to derive a bound on $\|x_{t+1}\|$ in terms of $\tilde{\Psi}_t^k(x_t;F)_{y_1}$ and an error term which is exponentially decaying. For $k-1 \le t \le T-k$, the norm of x_{t+1} has upper bound

$$\begin{aligned} \|x_{t+1}\| &= \left\|\tilde{\Psi}_t^k(x_t;F)_{y_1} + B(u^c(x_t) - u^\kappa(x_t))\right\| \le \|\tilde{\Psi}_t^k(x_t;F)_{y_1}\| + L\sum_{i\in\mathcal{V}}\|\tilde{\Psi}_t^k(x_t;F)_{v_0[i]} - \tilde{\psi}_t^k(x_t,\mathcal{N}_\mathcal{G}^\kappa[i];F)_{v_0[i]}\| \\ &\le \|\tilde{\Psi}_t^k(x_t;F)_{y_1}\| + LN\Gamma\left(\|x_t\| + \|\zeta^t\|\right)\delta^\kappa \\ &\le \sum_{m=0}^{k-2}\|\tilde{\Psi}_{t-m}^k(x_{t-m};F)_{y_{m+1}} - \tilde{\Psi}_{t-m-1}^k(x_{t-m-1};F)_{y_{m+2}}\| + \|\tilde{\Psi}_{t-k+1}^k(x_{t-k+1};F)_{y_k}\| + C_t\delta^\kappa, \end{aligned}$$

where $C_t := LN\Gamma(\|x_t\|+\|\zeta^t\|)$ and $\zeta^t := w_{t:t+k-1}$. Observe that $\tilde{\Psi}_{t-m-1}^k(x_{t-m-1};F)_{y_{m+2}} = \tilde{\Psi}_{t-m}^{k-1}(x_{t-m}^c;F)_{y_{m+1}}$ where

$$x_{t-m}^c = \tilde{\Psi}_{t-m-1}^k(x_{t-m-1};F)_{y_1},$$

Thus, we need to obtain a bound on

$$\|\tilde{\Psi}_{t-m}^k(x_{t-m};F)_{y_{m+1}} - \tilde{\Psi}_{t-m}^{k-1}(x_{t-m}^c;F)_{y_{m+1}}\|$$

which follows from the following lemma

Lemma 5. *Define the centralized optimization problem with terminal state:*

$$\begin{aligned} \Psi_t^k(x,\zeta,z) := \underset{y_{0:k},v_{0:k-1}}{\arg\min} &\sum_{\tau=0}^{k-1} f_{t+\tau}(y_\tau) + \sum_{\tau=1}^{k} c_{t+\tau}(v_{\tau-1}) \\ \text{s.t. } & y_{\tau+1} = Ay_\tau + Bv_\tau + \zeta_\tau, \quad \tau = 0,...,k-1 \\ & y_0 = x,\ y_k = z \end{aligned} \tag{9}$$

Then we have that

$$\begin{aligned} &\|\tilde{\Psi}_{t-m}^k(x_{t-m};F)_{y_{m+1}} - \tilde{\Psi}_{t-m}^{k-1}(x_{t-m}^c;F)_{y_{m+1}}\| \\ &\le C\left(N\lambda^{m+1}L\Gamma(\|x_{t-m-1}\| + \|\zeta^{t-m-1}\|)\delta^\kappa + \lambda^{k-m-1}\left(C\lambda^{k-1}(\|x_{t-m}\| + C\lambda\|x_{t-m-1}\| + \frac{2C}{1-\lambda}D) + \frac{4C}{1-\lambda}D\right)\right) \end{aligned}$$

Proof. Let

$$z := \tilde{\Psi}_{t-m}^k(x_{t-m};F)_{y_{k-1}}, \quad z' := \tilde{\Psi}_{t-m}^{k-1}(x_{t-m}^c;F)_{y_{k-1}},$$

then, leveraging the principle of optimality, we have

$$
\begin{aligned}
&\|\tilde{\Psi}^{k}_{t-m}(x_{t-m};F)_{y_{m+1}} - \tilde{\Psi}^{k-1}_{t-m}(x^c_{t-m};F)_{y_{m+1}}\| = \|\Psi^{k-1}_{t-m}(x_{t-m},z)_{y_{m+1}} - \Psi^{k-1}_{t-m}(x^c_{t-m},z')_{y_{m+1}}\| \\
&\le C\left(\lambda^{m+1}\|x_{t-m}-x^c_{t-m}\| + \lambda^{k-m-1}\|z-z'\|\right) \\
&\le C\left(\lambda^{m+1}\|x_{t-m}-x^c_{t-m}\| + \lambda^{k-m-1}\left(C\lambda^{k-1}(\|x_{t-m}\|+\|x^c_{t-m}\|) + \frac{4C}{1-\lambda}D\right)\right),
\end{aligned}
$$

Where C and λ are constants from Yiheng's results and $D = \sup_{t\in\mathbb{Z}_+}\|w_t\|$, we apply the Lipschitz result (theorem 3.3), and recall that

$$
\begin{aligned}
\|x_{t-m}-x^c_{t-m}\| &= \|B(u^\kappa_{t-m-1}(x_{t-m-1}) - u^c_{t-m-1}(x_{t-m-1}))\| \\
&\le LN\Gamma(\|x_{t-m-1}\| + \|\zeta^{t-m-1}\|)\delta^\kappa,
\end{aligned}
\tag{10}
$$

note that this result holds for any $x_{t-m}-x^c_{t-m}$ such that x_{t-m} and x^c_{t-m} are generated by $\tilde{\psi}^{k'}_t(x_{t-m-1})_{v_0}$ and $\tilde{\Psi}^{k'}_t(x_{t-m-1})_{v_0}$ (i.e. for arbitrary time horizon k'). Then, we can bound $\|x^c_{t-m}\|$ via the lipschitz result

$$
\begin{aligned}
\|x^c_{t-m}\| &= \|\tilde{\Psi}^{k}_{t-m-1}(x_{t-m-1};F)_{y_1}\| \\
&\le C\lambda\|x_{t-m-1}\| + \frac{2C}{1-\lambda}D
\end{aligned}
$$

Finally, we get that

$$
\begin{aligned}
&\|\tilde{\Psi}^{k}_{t-m}(x_{t-m};F)_{y_{m+1}} - \tilde{\Psi}^{k-1}_{t-m}(x^c_{t-m};F)_{y_{m+1}}\| \\
&\le C\left(N\lambda^{m+1}L\Gamma(\|x_{t-m-1}\|+\|\zeta^{t-m-1}\|)\delta^\kappa + \lambda^{k-m-1}\left(C\lambda^{k-1}(\|x_{t-m}\| + C\lambda\|x_{t-m-1}\| + \frac{2C}{1-\lambda}D) + \frac{4C}{1-\lambda}D\right)\right)
\end{aligned}
$$

$\square$

Returning to the upper bound on the norm of x_{t+1} for $k-1 \le t \le T-k$, we have that

$$
\begin{aligned}
\|x_{t+1}\| \le \sum_{m=0}^{k-2} & CN\lambda^{m+1}L\Gamma(\|x_{t-m-1}\|+\|\zeta^{t-m-1}\|)\delta^\kappa \\
&+ C\lambda^{k-m-1}\left(C\lambda^{k-1}(\|x_{t-m}\|+C\lambda\|x_{t-m-1}\|) + (2+C\lambda^{k-1})\frac{2C}{1-\lambda}D\right) \\
&+ C\lambda^{k}\|x_{t-k+1}\| + \frac{2C}{1-\lambda}D + LN\Gamma(\|x_t\|+\|\zeta^t\|)\delta^\kappa
\end{aligned}
$$

$$
\begin{aligned}
\|x_{t+1}\| \le C\sum_{m=0}^{k-2}&\left((L_N\lambda^{m+1}\delta^\kappa + C^2\lambda^{2k-m-1})\|x_{t-m-1}\| + C\lambda^{2k-m-2}\|x_{t-m}\|\right) + L_N\|x_t\|\delta^\kappa \\
&+ C\lambda^k\|x_{t-k+1}\| + \left(1 + \frac{C(2+C)+(1+C)L_N\delta^\kappa}{1-\lambda}\right)\frac{2C}{1-\lambda}D,
\end{aligned}
$$

where we now take $D := \sup_{t\in\mathbb{Z}_+}\|\zeta^t\|$ and $L_N := LN\Gamma$.
Similarly, for $t \le k-1$, we have

$$
\begin{aligned}
\|x_{t+1}\| &\le \|\tilde{\Psi}^k_t(x_t;F)_{y_1}\| + L_N(\|x_t\|+D)\delta^\kappa \\
&\le \sum_{m=0}^{t-2}\|\tilde{\Psi}^{k}_{t-m}(x_{t-m};F)_{y_{m+1}} - \tilde{\Psi}^{k}_{t-m-1}(x_{t-m-1};F)_{y_{m+2}}\| + \|\tilde{\Psi}^k_0(x_0;F)_{y_{t+1}}\| + L_N(\|x_t\|+D)\delta^\kappa \\
&= \sum_{m=0}^{t-2}\|\tilde{\Psi}^{k}_{t-m}(x_{t-m};F)_{y_{m+1}} - \tilde{\Psi}^{k-1}_{t-m}(x^c_{t-m};F)_{y_{m+1}}\| + \|\tilde{\Psi}^k_0(x_0;F)_{y_{t+1}}\| + L_N(\|x_t\|+D)\delta^\kappa \\
&\le \sum_{m=0}^{t-2} CL_N\lambda^{m+1}\delta^\kappa(\|x_{t-m-1}\|+D) + C\lambda^{k-m-1}\left(C\lambda^{k-1}\left(\|x_{t-m}\|+C\lambda\|x_{t-m-1}\|\right) + (2+C\lambda^{k-1})\frac{2C}{1-\lambda}D\right) \\
&+ C\|x_0\| + \frac{2C}{1-\lambda}D + L_N(\|x_t\|+D)\delta^\kappa
\end{aligned}
$$

Writing out the terms more explicitly, we have

$$\|x_{t+1}\| \le C \sum_{m=0}^{t-2} \left((L_N \lambda^{m+1} \delta^\kappa + C^2 \lambda^{2k-m-1}) \|x_{t-m-1}\| + C\lambda^{2k-m-2} \|x_{t-m}\| \right) + L_N \|x_t\| \delta^\kappa$$
$$+ C\|x_0\| + \left(1 + \frac{C(2+C) + (1+C)L_N\delta^\kappa}{1-\lambda}\right) \frac{2C}{1-\lambda} D$$

Take k and κ to be such that, for small $\xi > 0$

$$C\left(\lambda^k + L_N\delta^\kappa + \sum_{m=0}^{k-2} L_N \lambda^{m+1}\delta^\kappa + C\lambda^{2k-m-2} + C^2\lambda^{2k-m-1}\right) \le 1-\xi,$$

and

$$C\left(L_N\delta^\kappa + \sum_{m=0}^{t-2} L_N \lambda^{m+1}\delta^\kappa + C\lambda^{2k-m-2} + C^2\lambda^{2k-m-1}\right) \le 1-\xi,$$

Then, for $t \le T-k$ we have by induction:

$$\|x_{t+1}\| \le \frac{C}{\xi}(1-\xi)^{\max(0,t-k+1)}\|x_0\| + \left(1 + \frac{C(2+C)+(1+C)(1-\xi)}{1-\lambda}\right)\frac{2C}{1-\lambda}\frac{D}{\xi}$$

The case for $t \le k-1$ is easily verifiable, what remains then is to prove the induction from $k \le t \le T-k$. WLOG Let $t \ge 2k-1$, then

$$\|x_{t+1}\| \le \sum_{m=0}^{k-2} \left(CL_N\lambda^{m+1}\delta^\kappa + C^3\lambda^{2k-m-1}\right)\left(\frac{C}{\xi}(1-\xi)^{t-m-k-1}\|x_0\| + B\frac{1-\xi}{\xi}\right)$$
$$+ C^2\lambda^{2k-m-2}\left(\frac{C}{\xi}(1-\xi)^{t-m-k}\|x_0\| + B\frac{1-\xi}{\xi}\right) + L_N\delta^\kappa\left(\frac{C}{\xi}(1-\xi)^{t-k}\|x_0\| + B\frac{1-\xi}{\xi}\right)$$
$$+ C\lambda^k\left(\frac{C}{\xi}(1-\xi)^{t-2k+1}\|x_0\| + B\frac{1-\xi}{\xi}\right) + B(1-\xi)$$
$$\le \frac{C}{\xi}(1-\xi)^{t-k+1}\Bigg(\sum_{m=0}^{k-2}\left(CL_N\lambda^{m+1}\delta^\kappa + C^3\lambda^{2k-m-1}\right)(1-\xi)^{-m-2} + C^2\lambda^{2k-m-2}(1-\xi)^{-m-1}$$
$$+ L_N\delta^\kappa(1-\xi)^{-1} + C\lambda^k(1-\xi)^{-k}\Bigg)\|x_0\| + B\frac{1-\xi}{\xi}$$
$$\le \frac{C}{\xi}(1-\xi)^{t-k+1}\Bigg(\sum_{m=0}^{k-1} C\lambda^m(1-\xi)^{-m}\frac{L_N\delta^\kappa}{1-\xi} + (C(C\lambda+1-\xi))\sum_{m=0}^{k-2} C\lambda^{2k-m-2}(1-\xi)^{-m-2}$$
$$+ C\lambda^k(1-\xi)^{-k}\Bigg)\|x_0\| + B(1-\xi)/\xi$$
$$\le \frac{C}{\xi}(1-\xi)^{t-k+1}\|x_0\| + B\frac{1-\xi}{\xi},$$

where $B := \left(1 + \frac{C(2+c)+(1+C)}{1-\lambda}\right)\frac{2C}{1-\lambda}$ and we choose ξ, κ, and k such that

$$\frac{L_N C}{1-\xi}\left(\sum_{m=0}^{k-1}\left(\frac{\lambda}{1-\xi}\right)^m\right)\delta^\kappa + \frac{C^2(C\lambda+1-\xi)}{(\lambda(1-\xi))^2}\lambda^{2k}\left(\sum_{m=0}^{k-2}(\lambda(1-\xi))^{-m}\right) + C\left(\frac{\lambda}{1-\xi}\right)^k \le 1$$

letting $r := \frac{\lambda}{1-\xi}$, we take the following upper bound

$$\frac{L_N C}{1-\xi}\frac{1-r^k}{1-r}\delta^\kappa + \frac{C^2(C\lambda+1-\xi)}{(\lambda(1-\xi))^2}\frac{(\lambda(1-\xi))^{-k}-1}{(\lambda(1-\xi))^{-1}-1}\lambda^{2k} + Cr^k \le 1,$$

to this end we can take $\xi \leq 1-\lambda$ and

$$k \geq \frac{\log \frac{\lambda(1-\xi)(1-\lambda(1-\xi))}{3C^2(C\lambda+1-\xi)}}{\log \frac{\lambda}{1-\xi}}, \quad \kappa \geq \frac{\log \frac{1-\xi-\lambda}{3LN\Gamma C}}{\log \delta}$$

and to satisfy the condition on the coefficients, we can take

$$k \geq \frac{\log\left(\frac{1-\xi}{2\left(\frac{C(1-\lambda)+C^2+C^3}{1-\lambda}\right)}\right)}{\log \lambda} \quad \text{and} \quad \kappa \geq \frac{\log\left(\frac{(1-\xi)(1-\lambda)}{2CL\Gamma N}\right)}{\log \delta},$$

Then we take the max of both conditions on k and κ to attain ISS for $t \leq T-k$.
For the case that $T-k \leq t \leq T-1$, we have that

$$\begin{aligned}\|x_{t+1}\| &= \left\|\tilde{\Psi}_t^{T-t}(x_t;0)_{y_1} + B\left(\left(\tilde{\psi}_t^{T-t}(x_t;0)_{v_0[i]}\right)_{i\in\mathcal{V}} - \tilde{\Psi}_t^{T-t}(x_t;0)_{v_0}\right)\right\| \\ &\leq \sum_{m=0}^{t+k-T-1}\left\|\tilde{\Psi}_{t-m}^{T-t+m}(x_{t-m})_{y_{m+1}} - \tilde{\Psi}_{t-m-1}^{T-t+m+1}(x_{t-m-1})_{y_{m+2}}\right\| + \|\tilde{\Psi}_{T-k}^k(x_{T-k})_{y_{t+k-T+1}}\| + L_N(\|x_t\|+D)\delta^\kappa\end{aligned}$$

Using the same observation as before, but in this case

$$\tilde{\Psi}_{t-m-1}^{T-t+m+1}(x_{t-m-1})_{y_{m+2}} = \tilde{\Psi}_{t-m}^{T-t+m}(x_{t-m}^{c'})_{y_{m+1}}$$

Where $x_{t-m}^{c'} := \tilde{\Psi}_{t-m-1}^{T-t+m+1}(x_{t-m-1})_{y_1}$, we can directly apply the lipschitz result:

$$\begin{aligned}\|x_{t+1}\| &\leq \sum_{m=0}^{t+k-T-1} C\lambda^{m+1}\|x_{t-m} - x_{t-m}^{c'}\| + C\lambda^{t+k-T+1}\|x_{T-k}\| + \frac{2C}{1-\lambda}D + L_N(\|x_t\|+D)\delta^\kappa \\ &\leq \sum_{m=0}^{t+k-T} C\lambda^m LN\Gamma(\|x_{t-m}\|+D)\delta^\kappa + C\lambda^{t+k-T+1}\|x_{T-k}\| + \frac{2C}{1-\lambda}D\end{aligned}$$

Taking the coefficients

$$\sum_{m=0}^{t+k-T} C\lambda^m LN\Gamma\delta^\kappa \leq 1-\xi$$

We have that

$$\begin{aligned}\|x_{t+1}\| &\leq \frac{C}{\xi}\lambda^{t+k-T+1}\|x_{T-k}\| + \frac{4C}{(1-\lambda)\xi}D \\ &\leq \frac{C}{\xi^2}\lambda^{t+k-T+1}\left(C(1-\xi)^{T-2k}\|x_0\| + B(1-\xi)\right) + \frac{4C}{(1-\lambda)\xi}D\end{aligned}$$

3.2 Regret

We note that, in order to bound the regret, we must take differences between $f_t(x_t) - f_t(x_t^*) + c_{t+1}(u_t) - c_{t+1}(u_t^*)$ where (x_t, u_t) are generated by the controls from (3) and (x_t^*, u_t^*) are the offline optimal control $\tilde{\Psi}_0^T(x_0;0)$. Here, we attempt to obtain the difference of costs via $u^\kappa(x_t) := (\tilde{\psi}_t^{k'}(x_t)_{v_0[i]})_{i\in\mathcal{V}}$ and $\Psi_t^1(x_t, x_{t+1})_{v_0}$ where $k' := \min(k, T-t)$. We first bound $c_{t+1}(\Psi_t^1(x_t,x_{t+1})_{v_0})$ and $c_{t+1}((\tilde{\psi}_t^{k'}(x_t)_{v_0[i]})_{i\in\mathcal{V}})$ by applying the following result for $\eta > 0$:

$$c_{t+1}(x) - (1+\eta)c_{t+1}(x') \leq \frac{L}{2}\left(1+\frac{1}{\eta}\right)\|x-x'\|^2,$$

note that this upper bound holds for f_t as well, thus,

$$c_{t+1}((\tilde{\psi}_t^{k'}(x_t)_{v_0[i]})_{i\in\mathcal{V}}) - (1+\eta)c_{t+1}(\Psi_t^1(x_t,x_{t+1})_{v_0}) \leq \frac{L}{2}\left(1+\frac{1}{\eta}\right)\|(\tilde{\psi}_t^{k'}(x_t)_{v_0[i]})_{i\in\mathcal{V}} - \Psi_t^1(x_t,x_{t+1})_{v_0}\|^2$$

We can bound the norm of the difference as

$$\begin{aligned}\|u^\kappa(x_t) - \Psi_t^1(x_t, x_{t+1})_{v_0}\| &\leq \|\Psi_t^1(x_t, x_{t+1})_{v_0} - \tilde{\Psi}_t^{k'}(x_t)_{v_0}\| + \|\tilde{\Psi}_t^{k'}(x_t)_{v_0} - u^\kappa(x_t)\| \\ &\leq \|\Psi_t^1(x_t, x_{t+1})_{v_0} - \tilde{\Psi}_t^{k'}(x_t)_{v_0}\| + N\Gamma\left(\|x_t\| + D\right)\delta^\kappa\end{aligned}$$

then, using the principle of optimality, we have that

$$\tilde{\Psi}_t^k(x_t; F)_{v_0} = \Psi_t^1(x_t, x_{t+1}^c)_{v_0},$$

where $x_{t+1}^c = \tilde{\Psi}_t^{k'}(x_t)_{y_1}$, so then all we need to bound is the norm of the difference between the controls $\|\Psi_t^1(x_t, x_{t+1})_{v_0} - \Psi_t^1(x_t, x_{t+1}^c)_{v_0}\|$. We prove a general Lemma for the difference in controls produced by the optimization problems with terminal states in (9).

Lemma 6. *Given the one step terminal constraint optimization problem defined in* (9)*, we have that*

$$\|\Psi_t^1(x_t, x_{t+1})_{v_0} - \Psi_t^1(x_t', x_{t+1}')_{v_0}\| \leq \Gamma\left(\|x_t - x_t'\| + \|x_{t+1} - x_{t+1}'\|\right)$$

Proof. We recall that $\Psi_t^1(x_t, x_{t+1})$ has lagrangian

$$\mathcal{L}(z, \lambda) = f_t(y_0) + f_{t+1}(y_1) + c_{t+1}(v_0) + \lambda_1^\top(y_1 - Ay_0 - Bv_0 - \zeta_0) + \lambda_0^\top(y_0 - x_t) + \lambda_2^\top(y_1 - x_{t+1})$$

whose kkt conditions are then

$$\nabla\tilde{f}(z) + J^\top\lambda = \begin{bmatrix}\nabla f_t(y_0) + \lambda_0 - A^\top\lambda_1 \\ \nabla c_{t+1}(v_0) - B^\top\lambda_1 \\ \nabla f_{t+1}(y_1) + \lambda_2 + \lambda_1\end{bmatrix} = 0$$

and

$$Jz = \begin{bmatrix} I & & \\ -A & -B & I \\ & & I\end{bmatrix}\begin{bmatrix}y_0 \\ v_0 \\ y_1\end{bmatrix} = \begin{bmatrix}x_t \\ \zeta_t \\ x_{t+1}\end{bmatrix}$$

We can get similar KKT conditions for $z' = (y_0', v_0', y_1')$ and with corresponding x_t' and x_{t+1}'. Taking the difference of the two and applying Lemma 1, we obtain

$$H_1\begin{bmatrix}z - z' \\ \lambda - \lambda'\end{bmatrix} = \begin{bmatrix}G_c & J^\top \\ J & 0\end{bmatrix}\begin{bmatrix}z - z' \\ \lambda - \lambda'\end{bmatrix} = \begin{bmatrix} 0 \\ \begin{bmatrix} x_t - x_t' \\ 0 \\ x_{t+1} - x_{t+1}'\end{bmatrix}\end{bmatrix}$$

Since G_c is a submatrix of G in Theorem 5, the first 2 conditions hold trivially, So what remains to be shown is that $JJ^\top \succeq \beta_J I$: Consider the following column operation:

$$\begin{bmatrix} I & & \\ -A & -B & I \\ & & I\end{bmatrix}\begin{bmatrix} I & & \\ K & I & \\ & & I\end{bmatrix} = \begin{bmatrix} I & & \\ -\Phi & -B & I \\ & & I\end{bmatrix}$$

$$\|\Psi_t^1(x_t, x_{t+1})_{v_0} - \Psi_t^1(x_t', x_{t+1}')_{v_0}\| \leq \|z - z'\| \leq \Gamma(\|x_t - x_t'\| + \|x_{t+1} - x_{t+1}'\|)$$

□

Then, we can compare $u^\kappa(x_t)$ to the optimal offline control input u_t^* as

$$\begin{aligned}&c_{t+1}(u^\kappa(x_t)) - (1+\eta')c_{t+1}(\widehat{u}_t) + (1+\eta')\left(c_{t+1}(\widehat{u}_t) - (1+\eta')c_{t+1}(u_t^*)\right) \\ &\leq \frac{L}{2}\left(1 + \frac{1}{\eta'}\right)\left(\|u^\kappa(x_t) - \widehat{u}_t\|^2 + (1+\eta')\|\widehat{u}_t - u_t^*\|^2\right) \\ &\leq L\left(1 + \frac{1}{\eta'}\right)\left(N^2\Gamma^2(\|x_t\| + D)^2\delta^{2\kappa} + \Gamma^2\|x_{t+1} - x_{t+1}^c\|^2 + (1+\eta')\Gamma^2\left(\|x_t - x_t^*\|^2 + \|x_{t+1} - x_{t+1}^*\|^2\right)\right) \\ &\leq 2L^3N^2\Gamma^4\left(1 + \frac{1}{\eta'}\right)\left((\|x_t\| + D)^2\delta^{2\kappa} + (1+\eta')(\|x_t - x_t^*\|^2 + \|x_{t+1} - x_{t+1}^*\|^2)\right)\end{aligned}$$

where we take $1+\eta = (1+\eta')^2$ and the second inequality comes from applying Lemma 6 and the third inequality from the bound on $\|x_t - x_t^c\|$. Then, we can write the Regret between our algorithm and the optimal trajectory as

$$
\begin{aligned}
&\text{cost(ALG)} - (1+\eta)\text{cost(OPT)} \\
&= \sum_{t=0}^{T-1} (f_{t+1}(x_{t+1}) - (1+\eta) f_{t+1}(x_{t+1}^*)) + (c_{t+1}(u_t) - (1+\eta) c_{t+1}(u_t^*)) \\
&\le 2L^3 N^2 \Gamma^4 \left(1 + \frac{1}{\eta'}\right) \sum_{t=0}^{T-1} \left(\|x_{t+1} - x_{t+1}^*\|^2 + (\|x_t\| + D)^2 \delta^{2\kappa} + (1+\eta')\left(\|x_t - x_t^*\|^2 + \|x_{t+1} - x_{t+1}^*\|^2\right)\right) \\
&\le 6L^3 N^2 \Gamma^4 \left(1 + \frac{1}{\eta'}\right)(1+\eta') \sum_{t=0}^{T} ((\|x_t\| + D)^2 \delta^{2\kappa} + \|x_t - x_t^*\|^2)
\end{aligned}
$$

We know that $\|x_t\| = O\left(\frac{\|x_0\|+D}{\xi}\right)$ for $t \le T-k$, so, to attain an upper bound on $\|x_t - x_t^*\|$, let's consider the trajectory $\hat{x}_t$ generated by $\tilde{\Psi}_0^T(x_0; F)$ (i.e., the trajectory generated by the predictive control with horizon T) such that for $t \le T-k-1$

$$
\begin{aligned}
&\|x_{t+1} - \hat{x}_{t+1}\| = \|x_{t+1} - \tilde{\Psi}_0^T(x_0; F)_{y_{t+1}}\| \\
&\le \|x_{t+1} - \tilde{\Psi}_t^{T-t}(x_t)_{y_1}\| + \sum_{m=0}^{t-1} \|\tilde{\Psi}_{t-m}^{T-t+m}(x_{t-m})_{y_{m+1}} - \tilde{\Psi}_{t-m-1}^{T-t+m+1}(x_{t-m-1})_{y_{m+2}}\| \\
&\le \|x_{t+1} - \tilde{\Psi}_t^{k}(x_t)_{y_1}\| + \|\tilde{\Psi}_t^{k}(x_t)_{y_1} - \tilde{\Psi}_t^{T-t}(x_t)_{y_1}\| + \sum_{m=0}^{t-1} C\lambda^{m+1} \|x_{t-m} - \tilde{\Psi}_{t-m-1}^{T-t+m+1}(x_{t-m-1})_{y_1}\| \\
&\le LN\Gamma(\|x_t\| + D)\delta^\kappa + \frac{2C^2}{\lambda(1-\lambda^2)} \lambda^{2k} \|x_t\| + \frac{4C^2}{\lambda(1-\lambda)^2} \lambda^k D \\
&+ \sum_{m=0}^{t-1} C\lambda^{m+1} \left(\|x_{t-m} - \tilde{\Psi}_{t-m-1}^{k}(x_{t-m-1})_{y_1}\| + \|\tilde{\Psi}_{t-m-1}^{k}(x_{t-m-1})_{y_1} - \tilde{\Psi}_{t-m-1}^{T-t+m+1}(x_{t-m-1})_{y_1}\|\right) \\
&\le LN\Gamma(\|x_t\| + D)\delta^\kappa + \frac{2C^2}{\lambda(1-\lambda^2)} \lambda^{2k} \|x_t\| + \frac{4C^2}{\lambda(1-\lambda)^2} \lambda^k D \\
&+ \sum_{m=0}^{t-1} C\lambda^{m+1} \left(LN\Gamma(\|x_{t-m-1}\| + D)\delta^\kappa + \frac{2C^2}{\lambda(1-\lambda^2)} \lambda^{2k} \|x_{t-m-1}\| + \frac{4C^2}{\lambda(1-\lambda)^2} \lambda^k D\right),
\end{aligned}
$$

where the 3rd inequality comes from Yiheng's result for $t \le T-k$:

$$
\|\tilde{\Psi}_t^{k}(x_t)_{y_1} - \tilde{\Psi}_t^{T-t}(x_t)_{y_1}\| \le \frac{2C^2}{\lambda(1-\lambda^2)} \lambda^{2k} \|x_t\| + \frac{4C^2}{\lambda(1-\lambda)^2} \lambda^k D.
$$

Since we can take the coefficients to be

$$
\sum_{m=0}^{t-1} C\lambda^{m+1} \left(LN\Gamma + \frac{2C^2}{\lambda(1-\lambda^2)} + \frac{4C^2}{\lambda(1-\lambda)^2}\right) \le \frac{2C^2}{\lambda(1-\lambda^2)(1-\lambda)^2} (LN\Gamma + 3) = O(1),
$$

we have that

$$
\|x_{t+1} - \hat{x}_{t+1}\| = O\left(\left(D + \frac{\|x_0\| + D}{\xi}\right)\delta^\kappa + \left(D + \frac{\lambda^k(\|x_0\| + D)}{\xi}\right)\lambda^k\right).
$$

For $t \le T-k-1$:

$$
\|\hat{x}_{t+1} - x_{t+1}^*\| \le C\lambda^k \left(2C\lambda^T \|x_0\| + \frac{4C}{1-\lambda} D\right)
$$

And so, we have that

$$\|x_{t+1} - x_{t+1}^*\| = O\left(\left(D + \frac{\|x_0\| + D}{\xi}\right)\delta^\kappa + \left(D + \frac{\lambda^k(\|x_0\| + D)}{\xi}\right)\lambda^k\right).$$

We can also obtain a bound for $T - k \leq t \leq T - 1$ in a more direct way

$$\begin{aligned}
&\|x_{t+1} - x_{t+1}^*\| = \|x_{t+1} - \tilde{\Psi}_0^T(x_0; 0)_{y_{t+1}}\| \\
&\leq \|x_{t+1} - \tilde{\Psi}_t^{T-t}(x_t; 0)_{y_1}\| + \sum_{m=0}^{t-1} \|\tilde{\Psi}_{t-m}^{T-t+m}(x_{t-m}; 0)_{y_{m+1}} - \tilde{\Psi}_{t-m-1}^{T-t+m+1}(x_{t-m-1}; 0)_{y_{m+2}}\| \\
&\leq LN\Gamma(\|x_t\| + D)\delta^\kappa + \sum_{m=0}^{t-1} C\lambda^{m+1}\|x_{t-m} - \tilde{\Psi}_{t-m-1}^{T-t+m+1}(x_{t-m-1})_{y_1}\| \\
&\leq LN\Gamma(\|x_t\| + D)\delta^\kappa + \\
&\sum_{m=0}^{t-1} C\lambda^{m+1}\left(\|x_{t-m} - \tilde{\Psi}_{t-m-1}^{k'}(x_{t-m-1})_{y_1}\| + \|\tilde{\Psi}_{t-m-1}^{k'}(x_{t-m-1})_{y_1} - \tilde{\Psi}_{t-m-1}^{T-t+m+1}(x_{t-m-1})_{y_1}\|\right) \\
&\leq \sum_{m=0}^{t} C\lambda^m LN\Gamma(\|x_{t-m}\| + D)\delta^\kappa + \sum_{m=t-1+k-T}^{t-1} C\lambda^{m+1}\left(\frac{2C^2}{\lambda(1-\lambda^2)}\lambda^{2k}\|x_{t-m-1}\| + \frac{4C^2}{\lambda(1-\lambda)^2}\lambda^k D\right) \\
&= \sum_{m=0}^{t} C\lambda^m LN\Gamma(\|x_{t-m}\| + D)\delta^\kappa + \sum_{m=0}^{T-k} C\lambda^{t+k-T+m}\left(\frac{2C^2}{\lambda(1-\lambda^2)}\lambda^{2k}\|x_{T-k-m}\| + \frac{4C^2}{\lambda(1-\lambda)^2}\lambda^k D\right)
\end{aligned}$$

where $k' = \min(k, T - t + m + 1)$, and thus for $t \geq T - k$, we take the coefficients to be $O(1)$ and get

$$\|x_{t+1} - x_{t+1}^*\| = O\left(\left(D + \frac{\|x_0\| + D}{\xi^2}\right)\delta^\kappa + \left(D + \frac{\lambda^k(\|x_0\| + D)}{\xi}\right)\lambda^k\right),$$

note that we use the big O bound of $\|x_t\|$ for $t \geq T - k + 1$. So the overall regret, can be attained as

$$\begin{aligned}
&\mathbf{cost}(\text{ALG}) - (1 + \eta')^2\mathbf{cost}(\text{OPT}) \\
&\leq 6L^3N^2\Gamma^4\left(1 + \frac{1}{\eta'}\right)\sum_{t=0}^{T}\left((\|x_t\| + D)^2\delta^{2\kappa} + (1 + \eta')\|x_t - x_t^*\|^2\right) \\
&\leq \left(2 + \eta' + \frac{1}{\eta'}\right)O\left(\left(\left(D + \frac{\|x_0\| + D}{\xi^2}\right)^2\delta^{2\kappa} + \left(D + \frac{\lambda^k(\|x_0\| + D)}{\xi}\right)^2\lambda^{2k}\right)T\right)
\end{aligned}$$

Yiheng derived that the optimal cost of $\tilde{\Psi}_0^T(x_0; 0)$ is

$$\mathbf{cost}(\text{OPT}) = O(D^2T + \|x_0\|^2)$$

Taking $\eta' = \Theta(\max(\lambda^k, \delta^\kappa))$, we get

$$\begin{aligned}
&\mathbf{cost}(\text{ALG}) - \mathbf{cost}(\text{OPT}) \\
&= O\left(\left(\left(D + \frac{\|x_0\| + D}{\xi^2}\right)^2\delta^\kappa + \left(D + \frac{\lambda^k(\|x_0\| + D}{\xi}\right)^2\lambda^k\right)T + \max(\lambda^k, \delta^\kappa)\|x_0\|^2\right).
\end{aligned}$$

Correspondingly, taking $k = \kappa = \Theta(\log T)$ gives us $o(1)$ regret.