Math 140 - Spring 2017 - Final Exam - Version A

You have 110 minutes to complete this final exam. Books, notes and electronic devices are not permitted. Read and follow directions carefully. Show and check all work. Label graphs and include units where appropriate. If a problem is not clear, please ask for clarification.

Multiple Choice 1-10	/30
Free Response 11	/10
Free Response 12	/10
Free Response 13	/10
Free Response 14	/10
Free Response 15	/10
Total	/80

I pledge that I have completed this final exam in compliance with the NYU CAS Honor Code. In particular, I have neither given nor received unauthorized assistance during this exam.

Name Solutions

N Number ____________________

Signature ____________________

Date ____________________

1 Multiple Choice

(30 points) For problems 1-10, circle the best answer choice unless otherwise noted.

1. Define a subspace $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ of $\mathbb{R}^3$ where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then $\dim V =$

$$\vec{v}_3 - \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_4 - \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

V contains standard basis

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

2. Suppose A is a 3×4 matrix. Then the number of solutions to $A\boldsymbol{x} = \boldsymbol{0}$ is

(a) infinite

will always have at least 1 free variable

(b) zero

(c) one

(d) two

(e) need more information to determine the number of solution to $A\boldsymbol{x} = \boldsymbol{0}$

3. Let A be an $n \times n$ matrix with real entries.

Circle all of the statements below that are equivalent to the statement "A is invertible."

(a) 0 is not an eigenvalue of A

(b) $\det A \neq 0$

(c) A is symmetric

(d) rank $A = n$

(e) A is diagonalizable

4. Recall that $\mathbb{M}_{2\times 2}$ is the space of all 2×2 matrices with real entries.

Which of the following subsets of $\mathbb{M}_{2\times 2}$ is NOT a subspace of $\mathbb{M}_{2\times 2}$?

(a) 2×2 upper triangular matrices

(b) 2×2 symmetric matrices

(c) 2×2 invertible matrices I_2 and $-I_2$ both invertible

(d) 2×2 diagonal matrices   but $I_2 + (-I_2) = 0_{2\times 2}$ not invertible

(e) any subset of $\mathbb{M}_{2\times 2}$ is a subspace of $\mathbb{M}_{2\times 2}$

5. Suppose A is an 8×7 matrix in which dim(Nul A) $= 6$. Then rank $A =$

(a) 1   rank A + dim Nul A = n = 7 ⟹ rank A = 1

(b) 2

(c) 6

(d) 7

(e) 8

6. Let $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_r}\}$ and $\{\boldsymbol{v_{r+1}}, \ldots, \boldsymbol{v_n}\}$ be orthonormal bases for subspaces W and $W^\perp$ of $\mathbb{R}^n$, respectively.

Then the union $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_r}, \boldsymbol{v_{r+1}}, \ldots, \boldsymbol{v_n}\}$ is

(a) a linearly independent set but not necessarily orthogonal

(b) an orthonormal set but not necessarily a basis for $\mathbb{R}^n$

(c) a spanning set for $\mathbb{R}^n$ but not necessarily independent

(d) an orthonormal basis for $\mathbb{R}^n$

(e) none of the above

7. Consider the 3×3 matrix $A = \begin{bmatrix} 0 & 1 & k \\ 2 & k & -6 \\ 2 & 7 & 4 \end{bmatrix}$. For what values of k is matrix A invertible?

(a) $k \in \mathbb{R}$

(b) all real k except 2 and 5

(c) $k \geq 0$

(d) $k = 0$

(e) no value of k makes A invertible

$\det A = -1(2\times 4 - 2(-6)) + k(2\times 7 - 2k)$

$= -20 + 14k - 2k^2$

$\det A = 0$ when $-2k^2 + 14k - 20 = 0$

$-2(k-2)(k-5) = 0$

$k = 2$ or 5

8. $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$. What is the corresponding eigenvalue?

(a) 0

(b) 2

(c) 3

(d) 7

(e) need more information to determine eigenvalue associated with eigenvector $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

9. Circle all of the statements below that are true regarding the SVD of an $m \times n$ matrix A.

(a) The singular values of A are given by the square root of the corresponding eigenvalues of $A^T A$

(b) The singular value decomposition of A is unique

(c) If A has singular value decomposition $U\Sigma V^T$ and rank $A = r$, then $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_r\}$ is a basis for Col A

(d) Every A has a singular value decomposition

(e) A^2 has a singular value decomposition A^2 is not defined if $m \neq n$

10. Circle all of the transformations below that are linear. Let $\boldsymbol{v} = (v_1, v_2, v_3)$ be the input vector.

(a) $T(\boldsymbol{v}) = v_1 + v_2 + v_3$

(b) $T(\boldsymbol{v}) = ||\boldsymbol{v}||$

(c) $T(\boldsymbol{v}) = (v_1, v_2)$

(d) $T(\boldsymbol{v}) = v_1 v_2 v_3$

(e) $T(\boldsymbol{v}) = (0, 0, 0)$

2 Free Response

For problems 11-15, show all work and justify each step to receive full credit. Draw a box around your answers.

11. (10 points) Each of the following equations determines a plane in $\mathbb{R}^3$.

$$x_1 + 4x_2 - 5x_3 = 0$$
$$2x_1 - x_2 + 8x_3 = 9$$

(a) Find the complete solution $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$ of the linear system above.

$$\left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{array}\right] \xrightarrow[R_2 - 2R_1 \to R_2]{} \left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{array}\right]$$

$$-9x_2 + 18x_3 = 9 \implies x_2 = -1 + 2x_3$$

$$x_1 + 4x_2 - 5x_3 = 0 \implies x_1 = -4x_2 + 5x_3$$
$$= 4 - 3x_3$$

$$\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 \\ -1 + 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

(b) Based on your answer to part (a), do the two planes intersect? If so, describe their intersection.

Yes, they intersect in a line parallel to $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ and passing through the point $(4, -1, 0)$.

12. (10 points) Let $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$

(a) Find an orthogonal basis for Col A.

$$\vec{q}_1' = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{q}_2' = \vec{v}_2 - \frac{\vec{q}_1' \cdot \vec{v}_2}{\vec{q}_1' \cdot \vec{q}_1'} \vec{q}_1' = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$\therefore$ an orthogonal basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \right\}$

(b) Suppose $\boldsymbol{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$. Compute the least squares solution $\hat{\boldsymbol{x}}$ to $A\boldsymbol{x} = \boldsymbol{b}$.

$$A^T A \hat{\vec{x}} = A^T \vec{b}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \vec{x} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 11 & 14 \end{array}\right] \xrightarrow[R_2 - R_1 \to R_2]{} \left[\begin{array}{cc|c} 3 & 3 & 6 \\ 0 & 8 & 8 \end{array}\right]$$

$$8x_2 = 8 \implies x_2 = 1$$

$$3x_1 + 3x_2 = 6 \implies x_1 = 2 - x_2 = 2 - 1 = 1$$

$$\therefore \hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

13. (10 points) Compute A^{2017} where $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$. (*Hint: Diagonalize A.*)

Since A is triangular, its eigenvalues are the diagonal entries 1 and -1.

- $\lambda_1 = 1$:

$$\left[\begin{array}{cc|c} 1-1 & 0 & 0 \\ 2 & -1-1 & 0 \end{array}\right]$$

$$2x_1 - 2x_2 = 0 \implies x_1 = x_2$$

$$\vec{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- $\lambda_2 = -1$:

$$\left[\begin{array}{cc|c} 1-(-1) & 0 & 0 \\ 2 & -1-(-1) & 0 \end{array}\right] \xrightarrow[R_2 - R_1 \to R_2]{} \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$2x_1 = 0 \implies x_1 = 0$$

$$\vec{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$A^{2017} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}^{2017} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^{2017} & \\ & (-1)^{2017} \end{bmatrix} \begin{bmatrix} 1 & \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

14. (10 points) Find the matrix Σ (but not U or V) from the singular value decomposition of the matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 1 & 0 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$0 = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$$

$$\Longrightarrow (3-\lambda)^2 = 1 \quad \Longrightarrow 3-\lambda = \pm 1 \Longrightarrow \lambda_1 = 4, \ \lambda_2 = 2$$

$$\Longrightarrow \sigma_1 = \sqrt{\lambda_1} = \sqrt{4} = 2$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$$

$$\therefore \ \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

15. (10 points) Recall the following:

- $\mathbb{P}_k$ is the space of all polynomials of degree k or lower
- Power Rule for Integration
  - Indefinite integrals: $\int x^k\,dx = \frac{1}{k+1}x^{k+1} + C = F(x)$ for $k \neq -1$
  - Definite integrals: $\int_0^t x^k\,dx = F(t) - F(0) = \frac{1}{k+1}t^{k+1}$ for $k \neq -1$

Define a transformation $T\colon \mathbb{P}_2 \longrightarrow \mathbb{P}_3$ by the integration formula

$$T(p(x)) = \int_0^x p(t)\,dx$$

(a) Compute $T(3 + 2x + x^2)$.

$$T(3+2x+x^2) = 3x + x^2 + \tfrac{1}{3}x^3$$

(b) Show T is a linear transformation.

$$T[p(x)+q(x)] = \int_0^x (p(t)+q(t))\,dx$$
$$= \int_0^x p(t)\,dx + \int_0^x q(t)\,dx$$
$$= T[p(x)] + T[q(x)]$$

$$T[cp(x)] = \int_0^x cp(t)\,dx = c\int_0^x p(t)\,dx = cT[p(x)]$$

(c) Find a matrix A to represent T with respect to standard bases $\{1, x, x^2\} \subset \mathbb{P}_2$ and $\{1, x, x^2, x^3\} \subset \mathbb{P}_3$.

$$T(1) = x \rightsquigarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x) = \tfrac{1}{2}x^2 \rightsquigarrow \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$$

$$T(x^2) = \tfrac{1}{3}x^3 \rightsquigarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$