Final Exam Problem Bank

exAClior$_{\text{laptop}}$

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1 Total [36/50]

2 General [7/7]

2.1 Sakurai [4/4]

2.2 Likharev [3/3]

2.2.1 4.4: Power of Sigma Matrices

Problem 4.4. Calculate the following expressions,

(i) $(\mathbf{c} \cdot \boldsymbol{\sigma})^n$, and then

(ii) $(b\mathrm{I} + \mathbf{c} \cdot \boldsymbol{\sigma})^n$, for the scalar product $\mathbf{c} \cdot \boldsymbol{\sigma}$ of the Pauli matrix vector $\boldsymbol{\sigma} \equiv \mathbf{n}_x \boldsymbol{\sigma}_x + \mathbf{n}_y \sigma_y + \mathbf{n}_z \boldsymbol{\sigma}_z$ by an arbitrary c -number vector $\mathbf{c}$, where $n \geqslant 0$ is an integer, and b is an arbitrary scalar c number. (Change this to bI + i $|\mathrm{c}|$ σ, to use exponential form)

Hint: For task (ii), you may like to use the binomial theorem2, and then transform the result in a way enabling you to use the same theorem backwards.

2.2.2 4.19: Eigenvalue and Measurement

Problem 4.19. In a certain basis, the Hamiltonian of a two-level system is described by the matrix

$$\mathrm{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \quad \text{with } E_1 \neq E_2,$$

while the operator of some observable A of this system, by the matrix

$$\mathrm{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

For the system's state with the energy definitely equal to E_1, find the possible results of measurements of the observable A and the probabilities of the corresponding measurement outcomes.

2.2.3 4.23: Anticommutation and Eigenvalue

A certain state γ is an eigenstate of each of two operators $\hat{A}$ and $\hat{B}$. What can be said about the corresponding eigenvalues a and b, if the operators anticommute?

3 Atomic Physics [12/12]

3.1 Sakurai [5/5]

3.1.1 2.10: Time Evolution

Let $|a'\rangle$ and $|a''\rangle$ be eigenstates of a Hermitian operator A with eigenvalues a' and a'', respectively ($a' \neq a''$). The Hamiltonian operator is given by

$$H = \left|a'\right\rangle \delta \left\langle a''\right| + \left|a''\right\rangle \delta \left\langle a'\right|$$

where δ is just a real number.

1. Clearly, $|a'\rangle$ and $|a''\rangle$ are not eigenstates of the Hamiltonian. Write down the eigenstates of the Hamiltonian. What are their energy eigenvalues?

2. Suppose the system is known to be in state $|a'\rangle$ at $t = 0$. Write down the state vector in the Schrödinger picture for $t > 0$.

3. What is the probability for finding the system in $|a''\rangle$ for $t > 0$ if the system is known to be in state $|a'\rangle$ at $t = 0$?

4. Can you think of a physical situation corresponding to this problem?

3.1.2 2.23: Operator Algebra

Make the definitions

$$J_\pm \equiv h a_\pm^\dagger a_\mp, \quad J_z \equiv \frac{\bar{h}}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right), \quad N \equiv a_+^\dagger a_+ + a_-^\dagger a_-$$

where \$a $_\pm$\$and $a_\pm^\dagger$ are the annihilation and creation operators of two independent simple harmonic oscillators satisfying the usual simple harmonic oscillator commutation relations. Also make the definition

$$\mathbf{J}^2 \equiv J_z^2 + \frac{1}{2}\left(J_+ J_- + J_- J_+\right).$$

Prove

$$\left[J_z, J_\pm\right] = \pm\bar{h} J_\pm, \quad \left[\mathbf{J}^2, J_z\right] = 0, \quad \mathbf{J}^2 = \left(\frac{\bar{h}^2}{2}\right) N \left[\left(\frac{N}{2}\right) + 1\right]$$

*

3.1.3 3.23: Angular Momentum Operator

The wave function of a particle subjected to a spherically symmetrical potential $V(r)$ is given by

$$\psi(\mathbf{x}) = (x + y + 3z) f(r).$$

1. Is ψ an eigenfunction of $\mathbf{L}^2$? If so, what is the 1-value? If not, what are the possible values of l we may obtain when $\mathbf{L}^2$ is measured?

2. What are the probabilities for the particle to be found in various m_l states?

3. Suppose it is known somehow that $\psi(\mathbf{x})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.

3.1.4 5.1: Simple Perturbation Theory

A simple harmonic oscillator (in one dimension) is subjected to a perturbation

$$H_1 = bx$$

where b is a real constant.

1. Calculate the energy shift of the ground state to lowest nonvanishing order.

2. Solve this problem exactly and compare with your result obtained in (a).

3.1.5 5.7: Simple Harmonic Oscillator and Perturbation Theory (a&b only)

Consider an isotropic harmonic oscillator in two dimensions. The Hamiltonian is

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2\right).$$

1. What are the energies of the three lowest-lying states? Is there any degeneracy?

2. We now apply a perturbation

$$V = \delta m\omega^2 xy$$

where δ is a dimensionless real number much smaller than unity. Find the zerothorder energy eigenket and the corresponding energy to first order [that is, the unperturbed energy obtained in (a) plus the first-order energy shift] for each of the three lowest-lying states. *

3.2 Likharev [4/4]

3.2.1 2.1: Momentum Operator

Problem 2.1. The initial wave packet of a free 1D particle is described by Eq. (2.20) of the lecture notes:

$$\Psi(x,0) = \int a_k e^{ikx} dk$$

(i) Obtain a compact expression for the expectation value $\langle p \rangle$ of the particle's momentum. Does $\langle p \rangle$ depend on time?

(ii) Calculate $\langle p \rangle$ for the case when the function $|a_k|^2$ is symmetric with respect to some value k_0.

3.2.2 5.2: Ladder Operator and Heisenberg Picture

Problem 5.2. A spin- 1/2 is placed into an external magnetic field, with a timeindependent orientation, its magnitude $B(t)$ being an arbitrary function of time. Find explicit expressions for the Heisenberg operators and the expectation values of all three Cartesian components of the spin, as functions of time, in a coordinate system of your choice.

3.2.3 5.9: Fock State and Ladder Operator

Problem 5.9. For a 1D harmonic oscillator with mass m and frequency ω_0, calculate:

(i) all matrix elements $\langle n \left| \hat{x}^3 \right| n' \rangle$,

and (ii) the diagonal matrix elements $\langle n \left| \hat{x}^4 \right| n \rangle$, where n and n' are arbitrary Fock states.

Note,

$$\begin{aligned} \langle n' | \hat{x} | n \rangle &= \frac{x_0}{\sqrt{2}} \left[n^{1/2} \delta_{n',n-1} + (n+1)^{1/2} \delta_{n',n+1} \right] \\ &\equiv \left(\frac{\hbar}{2m\omega_0} \right)^{1/2} \left[n^{1/2} \delta_{n',n-1} + (n+1)^{1/2} \delta_{n',n+1} \right] \end{aligned}$$

and $\langle n'|\hat{x}^2|n\rangle = \frac{x_0^2}{2}\left\{[n(n-1)]^{1/2}\delta_{n',n-2}\right.$
$\left. +[(n+1)(n+2)]^{1/2}\delta_{n',n+2} + (2n+1)\delta_{n',n}\right\}.$

3.2.4 5.23: Ladder Operator and Angular Momentum

In the basis of the common eigenstates of the operators $\hat{L}_z$ and $\hat{L}^2$, described by kets $|l,m\rangle$:

(i) calculate the matrix elements $\left\langle l,m_1\left|\hat{L}_x\right|l,m_2\right\rangle$ and $\left\langle l,m_1\left|\hat{L}_x^2\right|l,m_2\right\rangle$;

(ii) spell out your results for the diagonal matrix elements (with $m_1 = m_2$) and their y-axis counterparts;

and (iii) calculate the diagonal matrix elements $\left\langle l,m\left|\hat{L}_x\hat{L}_y\right|l,m\right\rangle$ and $\left\langle l,m\left|\hat{L}_y\hat{L}_x\right|l,m\right\rangle$.

3.3 The Quantum Mechanics Solver [3/3]

- which operators commute with $\vec{\sigma_b}$

3.3.1 6.1: Hyperfine Splitting

Give the degeneracy of the ground state if one neglects the magnetic interaction between the nucleus and the external electron. We note

$$|m_{\mathrm{e}};m_{\mathrm{n}}\rangle = |\text{ electron : } s_{\mathrm{e}} = 1/2, m_{\mathrm{e}}\rangle \otimes |\text{ nucleus : } s_{\mathrm{n}}, m_{\mathrm{n}}\rangle$$

a basis of the total spin states (external electron + nucleus).

3.3.2 6.2: Energy Level of Hyperfine Splitting

We now take into account the interaction between the electron magnetic moment μ_{e} and the nuclear magnetic moment μ_{n}. As in the hydrogen atom, one can write the corresponding Hamiltonian (restricted to the spin subspace) as:

$$\hat{H} = \frac{A}{\hbar^2}\hat{\boldsymbol{S}}_{\mathrm{e}}\cdot\hat{\boldsymbol{S}}_{\mathrm{n}},$$

where A is a characteristic energy, and where $\hat{\boldsymbol{S}}_{\mathrm{e}}$ and $\hat{\boldsymbol{S}}_{\mathrm{n}}$ are the spin operators of the electron and the nucleus, respectively. We want to find the eigenvalues of this Hamiltonian.

We introduce the operators $\hat{S}_{\mathrm{e},\pm} = \hat{S}_{\mathrm{e},x} \pm i\hat{S}_{\mathrm{e},y}$ and $\hat{S}_{\mathrm{n},\pm} = \hat{S}_{\mathrm{n},x} \pm i\hat{S}_{\mathrm{n},y}$.

(a) Show that

$$\hat{H} = \frac{A}{2\hbar^2}\left(\hat{S}_{\mathrm{e},+}\hat{S}_{\mathrm{n},-} + \hat{S}_{\mathrm{e},-}\hat{S}_{\mathrm{n},+} + 2\hat{S}_{\mathrm{e},z}\hat{S}_{\mathrm{n},z}\right)$$

(b) Show that the two states

$$|m_{\mathrm{e}} = 1/2; m_{\mathrm{n}} = s_{\mathrm{n}}\rangle \quad \text{and} \quad |m_{\mathrm{e}} = -1/2; m_{\mathrm{n}} = -s_{\mathrm{n}}\rangle$$

are eigenstates of $\hat{H}$, and give the corresponding eigenvalues.

(c) What is the action of $\hat{H}$ on the state $|m_{\mathrm{e}} = 1/2; m_{\mathrm{n}}\rangle$ with $m_{\mathrm{n}} \neq s_{\mathrm{n}}$? What is the action of $\hat{H}$ on the state $|m_{\mathrm{e}} = -1/2; m_{\mathrm{n}}\rangle$ with $m_{\mathrm{n}} \neq -s_{\mathrm{n}}$? (d) Deduce from these results that the eigenvalues of $\hat{H}$ can be calculated by diagonalizing 2×2 matrices of the type:

$$\frac{A}{2}\begin{pmatrix} m_{\mathrm{n}} & \sqrt{s_{\mathrm{n}}\left(s_{\mathrm{n}}+1\right) - m_{\mathrm{n}}\left(m_{\mathrm{n}}+1\right)} \\ \sqrt{s_{\mathrm{n}}\left(s_{\mathrm{n}}+1\right) - m_{\mathrm{n}}\left(m_{\mathrm{n}}+1\right)} & -\left(m_{\mathrm{n}}+1\right) \end{pmatrix}.$$

3.3.3 7.3.1: Zeeman Effect

The system is placed in a constant uniform magnetic field $\boldsymbol{B}$ directed along the z axis. The additional Zeeman Hamiltonian has the form

$$\hat{H}_{\mathrm{Z}} = \omega_1\hat{S}_{1z} + \omega_2\hat{S}_{2z}$$

where $\omega_1 = -\gamma_1 B$ and $\omega_2 = -\gamma_2 B$. 7.3.1 Matrix representation of the Zeeman Hamiltonian

(a) Taking into account the result of question 7.2.2 and setting $\omega = -\gamma B$, write the action of $\hat{H}_Z$ on the basis states $\{|\sigma_1, \sigma_2\rangle\}$.

(b) Write in terms of A and $\hbar\omega$ the matrix representation of

$$\hat{H} = \hat{H}_{\mathrm{SS}} + \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{Z}}$$

in the basis $\{|S, m\rangle\}$ of the total spin of the two particles.

(c) Give the numerical value of $\hbar\omega$ in eV for a field $B = 1$ T. Is it easy experimentally to be in a strong field regime, i.e. $\hbar\omega \gg A$?

4 Quantum Information [10/10]

4.1 Bacon Final [4/4]

4.1.1 P1

Problem 1: One Qubit! (30 pts) In this problem you have been given a single qubit which has the wave function given by the ket $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$.

(a) (4 pts) What is the bra $\langle\psi|$ corresponding to this ket?

(b) (4 pts) Is this wave function normalized? This is, does $\langle\psi \mid \psi\rangle = 1$?

(c) (6pts) If you measure this qubit in the computational basis, $|0\rangle, |1\rangle$, what are the probabilities of these two outcomes?

(d) (6 pts) Suppose we apply the unitary

$$U = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{bmatrix}$$

to the qubit in the wave function $|\psi\rangle$. What is the new wave function $U|\psi\rangle$?

(e) (6pts) Recall that the Hadamard matrix is

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

If we first apply U from part (d) followed by the Hadamard matrix, this is the same as applying the unitary HU. What is the two by two matrix HU ? (f) (4 pts) Suppose we start with one qubit which has the wave function $|\psi\rangle$. Next we apply U from part (d). Then we apply the Hadamard H. What is the final qubit wave function? That is, what is $HU|\psi\rangle$?

4.1.2 P2

Problem 2: Two Qubits! (40 pts) In this problem we have been given two qubits with the wave function $|\phi\rangle = \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{2}|10\rangle$. (a) (3 pts) What is the bra $\langle\phi|$? (b) (4 pts) If we measure $|\phi\rangle$ in the computational basis for two qubits, what are the probabilities of the four outcomes, $|00\rangle, |01\rangle, |10\rangle$, and $|11\rangle$? (c) (5pts) Recall that the single qubit not operator is $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the single qubit identity operator is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Write out the two qubit unitary matrix $I \otimes X$ in the computational basis. (d) (5 pts) Write out the two qubit unitary matrix $X \otimes X$ in the computational basis. (e) (5 pts) Suppose we apply the unitary matrix $X \otimes X$ to $|\phi\rangle$. What is the resulting two qubit state $(X \otimes X)|\phi\rangle$? (f) (5 pts) Suppose that we feed $|\phi\rangle$ into the following circuit What is the resulting two qubit wave function? (g) (3 pts) Return now to $|\phi\rangle$. Suppose we are given two qubits with this wave function and we measure the first of these two qubits in the computational basis, $|0\rangle, |1\rangle$. What are the probabilities of these two outcomes? (h) (3 pts) Recall that the Bell basis are given by the four two qubit kets

$$
\begin{aligned}
|\Psi_+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \\
|\Phi_+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).
\end{aligned}
$$

Suppose that we measure $|\phi\rangle$ in the Bell basis. What are the probabilities of the four outcomes $|\Phi_+\rangle, |\Phi_-\rangle, |\Psi_+\rangle$, and $|\Psi_-\rangle$? (i) (7 pts) Suppose in addition to $|\phi\rangle$, we have a third qubit whose wave function is $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, i.e. we start with $|\phi\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$. Suppose you now measure the first and third qubits of this system in the Bell basis. After this measurement, the wave function of the second qubit will be separable from the wave function of the first and the third qubit. Suppose that you got the outcome corresponding to $|\Phi_+\rangle$: what would the wave function of the second qubit then be?

4.1.3 P3

Problem 3: Outer Products (10 pts) (a) (3 pts) Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Write out the two by two matrix $|+\rangle\langle+|$ in the computational basis, $|0\rangle, |1\rangle$. (b) (4 pts) Let $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$. Suppose that we are given two qubits whose wave function is $\frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle$. If we measure the first of these qubits in the $|+i\rangle, |-i\rangle$ basis, what are the probabilities of these two outcomes? What is the wave function of these two qubits after the measurement for each of these two possible outcomes? (c) (3 pts) Define the four dimensional matrix $V = |00\rangle \left\langle 00 \middle| + e^{\frac{2\pi i}{3}} \middle| 01 \right\rangle \langle 01| + \frac{1}{\sqrt{2}}(|10\rangle\langle 10| + i|11\rangle\langle 10| + i|10\rangle\langle 11| + |11\rangle\langle 11|)$. Is this matrix unitary? That is does $VV^\dagger = I$?

4.1.4 P4

Problem 4: n Qubits! (10 pts) In this problem we will deal with n qubits. (a) (4 pts) Suppose that we have n qubits which have the wave function $|0\rangle = |0, 0, \ldots, 0\rangle$. If we now apply the n qubit Pauli X to these n qubits, $X^{\otimes n}$ (where X is defined Problem 2), what is the resulting n qubit state? Your answer should be a single computational basis state. (b) (4 pts) Recall the definition of the Hadamard from Problem 1. Suppose we apply the n qubit Hadamard, $H^{\otimes n}$ to $(X^{\otimes n})\,|0\rangle$. (H is defined in Problem 1.) What is the resulting n qubit wave function? Express it as a sum over computational basis kets, i.e. in the form $\sum_{x=0}^{2^n-1} a_x|x\rangle$, where a_x is some function of x which you must find a formula for. (c) (2 pts) $H^{\otimes n} X^{\otimes n} H^{\otimes n}$ can be expressed as $A^{\otimes n}$. What is A ?

4.2 MIT P3 [3/3]

4.2.1 P1

1. Measurements and uncertainty.

(a) Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M, with corresponding eigenvalue m. What is the average observed value of M, and the standard deviation? (b) Suppose we have qubit in the state $|0\rangle$, and we measure the observable X (i.e. σ_x). What is the average value of X ? What is the standard deviation of X ?

4.2.2 P4

1. Schmidt decompositions. Consider a composite system consisting of two qubits. Find the Schmidt decompositions of the states

$$
\begin{aligned}
|\phi_1\rangle &= \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}} \\
|\phi_2\rangle &= \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \\
|\phi_3\rangle &= \frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2} \\
|\phi_4\rangle &= \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}}.
\end{aligned}
$$

4.2.3 P5

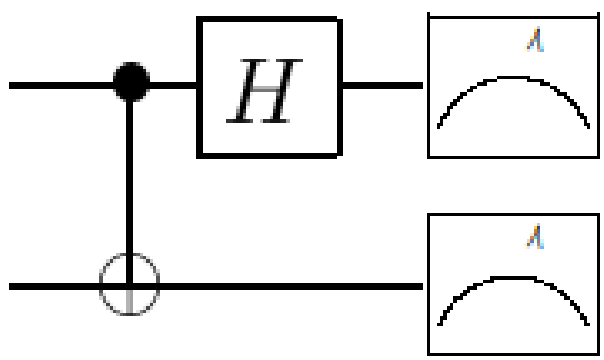
1. Interferometers. Consider this single qubit model of an interferometer, where the goal is to estimate an unknown phase ϕ :

Let the box with ϕ map $|0\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow e^{i\phi}|1\rangle$. (a) Give the states $|\psi_1\rangle, |\psi_2\rangle$, and $|\psi_3\rangle$. (b) What is the probability p of measuring the final qubit to be one? (c) If this is experiment is repeated n times, what is the standard deviation Δp of the value estimated for p from the measurement results? Also give the uncertainty in the resulting estimate for ϕ, $\Delta\phi = \Delta p / |dp/d\phi|$.

4.3 MIT P4 [1/1]

4.3.1 P1

Measurement in the Bell basis Show that the circuit performs a measurement in the basis of the Bell states. Specifically, show that this circuit results in a measurement being performed with four operators $\{M_k\}$ such that $M_k^\dagger M_k$ are the four projectors onto the Bell states.



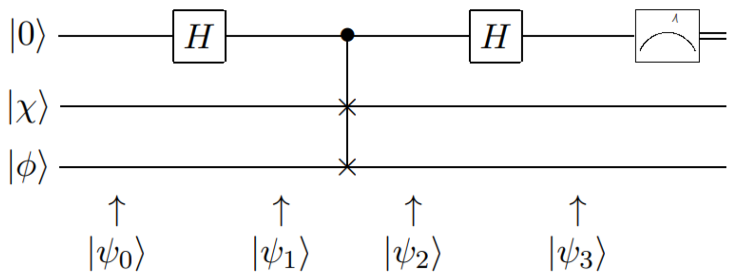
4.4 MIT MidTerm [2/2]

4.4.1 P2

Entanglement distillation by non-projective measurement. Suppose Alice and Bob share the two-qubit state $|\psi_{AB}\rangle = (\sqrt{3}|00\rangle + |11\rangle)/2$. Recall that a quantum measurement is specified by a set of operators $\{M_0, M_1\}$ such that $\sum_k M_k^\dagger M_k = I$. (a) Give a and b such that the quantum measurement outcome from operator $M_0 = a|0\rangle\langle 0| + b|1\rangle\langle 1|$ acting on $|\psi_{AB}\rangle$ produces the post-measurement result $(|00\rangle + |11\rangle)/\sqrt{2}$ with probability 1/4. (15 points) (b) Give an operator M_1 such that $M_0^\dagger M_0 + M_1^\dagger M_1 = I$. With what probability does the corresponding outcome occur, acting on $|\psi_{AB}\rangle$, and what is the post-measurement result? (10 points)

4.4.2 P4

1. Qubit tests. Consider the following three-qubit quantum circuit, in which $|\chi\rangle$ and $|\phi\rangle$ are arbitrary qubits:



(a) Give the intermediate states of the circuit, $|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$. (4 points each)

(b) If the measurement result is zero (ie the top qubit is $|0\rangle$), what is the state of the bottom two qubits? (4 points)

(c) If $|\langle\chi \mid \phi\rangle| = \alpha$, with what probability is the measurement result zero? (10 points)

5 Condensed Matter Physics [0/14]

- Provided by Prof. Liu

6 Sakurai

6.1 Chapter 1

6.1.1 1.2

Prove

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$$

6.1.2 1.4

Suppose a 2×2 matrix X (not necessarily Hermitian, nor unitary) is written as

$$X = a_0 + \sigma \cdot \mathbf{a},$$

where the matrices σ are given in (3.50) and a_0 and $a_{1,2,3}$ are numbers.

1. How are a_0 and $a_k (k = 1, 2, 3)$ related to $\operatorname{tr}(X)$ and $\operatorname{tr}(\sigma_k X)$?
2. Obtain a_0 and a_k in terms of the matrix elements X_{ij}.

6.1.3 1.6

Using the rules of bra-ket algebra, prove or evaluate the following:

1. $\operatorname{tr}(XY)=\operatorname{tr}(YX)$, where X and Y are operators;
2. $(XY)^{\dagger}=Y^{\dagger}X^{\dagger}$, where X and Y are operators;
3. $\exp[if(A)]=?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known;
4. $\sum_{a'}\psi_{a'}^{*}(\mathbf{x}')\,\psi_{a'}(\mathbf{x}'')$, where $\psi_{a'}(\mathbf{x}')=\langle\mathbf{x}'\mid a'\rangle$.

6.1.4 1.7

1. Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'\mid\alpha\rangle,\langle a''\mid\alpha\rangle,\ldots$ and $\langle a'\mid\beta\rangle,\langle a''\mid\beta\rangle,\ldots$ are all known, where $|a'\rangle,|a''\rangle,\ldots$ form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis.
2. We now consider a spin $\frac{1}{2}$ system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|S_z;+\rangle$ and $|S_x;+\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (s_z diagonal) basis.

6.1.5 1.8: Eigenvalue

Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator A.

Under what condition can we conclude that $|i\rangle+|j\rangle$ is also an eigenket of A ? Justify your answer.

6.1.6 1.10

1.10 Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i,S_j]=i\varepsilon_{ijk}\hbar S_k,\quad \{S_i,S_j\}=\left(\frac{\hbar^2}{2}\right)\delta_{ij},$$

where $$\begin{aligned}S_x&=\frac{\hbar}{2}(|+\rangle\langle-|+|-\rangle\langle+|),\quad S_y=\frac{i\hbar}{2}(-|+\rangle\langle-|+|-\rangle\langle+|),\\ S_z&=\frac{\hbar}{2}(|+\rangle\langle+|-|-\rangle\langle-|).\end{aligned}$$

6.1.7 1.12: Braket Notation to Matrix

The Hamiltonian operator for a two-state system is given by

$$H = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|),$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

6.1.8 1.13

A two-state system is characterized by the Hamiltonian

$$H = H_{11}|1\rangle \langle 1 | + H_{22}| 2\rangle \langle 2| + H_{12}[|1\rangle\langle 2| + |2\rangle\langle 1|]$$

where H_{11}, H_{22}, and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and corresponding energy eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$. 1.14 A spin $\frac{1}{2}$ system is known to be in an eigenstate of $\mathbf{S} \cdot \hat{\mathbf{n}}$ with eigenvalue $\hbar/2$, where $\hat{\mathbf{n}}$ is a unit vector lying in the $x z$-plane that makes an angle γ with the positive z-axis.

1. Suppose S_x is measured. What is the probability of getting $+\hbar/2$?
2. Evaluate the dispersion in S_x, that is,

$$\left\langle (S_x - \langle S_x\rangle)^2 \right\rangle$$

(For your own peace of mind check your answers for the special cases $\gamma = 0, \pi/2$, and π.)

6.1.9 1.14

A spin $\frac{1}{2}$ system is known to be in an eigenstate of $\mathbf{S} \cdot \hat{\mathbf{n}}$ with eigenvalue $\hbar/2$, where $\hat{\mathbf{n}}$ is a unit vector lying in the $x z$-plane that makes an angle γ with the positive z-axis.

1. Suppose S_x is measured. What is the probability of getting $+\hbar/2$?
2. Evaluate the dispersion in S_x, that is,

$$\left\langle (S_x - \langle S_x\rangle)^2 \right\rangle$$

(For your own peace of mind check your answers for the special cases $\gamma = 0, \pi/2$, and π.)

6.1.10 1.15

A beam of spin $\frac{1}{2}$ atoms goes through a series of Stern-Gerlach type measurements as follows.

1. The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.
2. The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where s_n is the eigenvalue of the operator $\mathbf{S} \cdot \hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ making an angle β in the \$x z\$-plane with respect to the \$z\$-axis.
3. The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam?

6.1.11 1.18

Two Hermitian operators anticommute:

$$\{A, B\} = AB + BA = 0.$$

Is it possible to have a simultaneous (that is, common) eigenket of A and B ? Prove or illustrate your assertion.

6.1.12 1.19

Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute,

$$[A_1, A_2] \neq 0$$

yet we also know that A_1 and A_2 both commute with the Hamiltonian:

$$[A_1, H] = 0, \quad [A_2, H] = 0$$

Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central-force problem $H = \mathbf{p}^2/2m + V(r)$, with $A_1 \rightarrow L_z, A_2 \rightarrow L_x$

6.1.13 1.26

1. Prove that $(1/\sqrt{2})(1+i\sigma_x)$, where the matrix σ_x is given in (3.50), acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the \$x\$-axis by angle $-\pi/2$. (The minus sign signifies that the rotation is clockwise.)

2. Construct the matrix representation of S_z when the eigenkets of S_y are used as base vectors.

6.1.14 1.28

Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent with the general relation

$$U = \sum_r \left| b^{(r)} \right\rangle \left\langle a^{(r)} \right|$$

6.2 Chapter 2

6.2.1 2.8: Heisenberg Picture

2.8 Consider a free-particle wave packet in one dimension. At $t=0$ it satisfies the minimum uncertainty relation

$$\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle = \frac{\bar{h}^2}{4} \quad (t=0).$$

In addition, we know

$$\langle x \rangle = \langle p \rangle = 0 \quad (t=0).$$

Using the Heisenberg picture, obtain $\left\langle (\Delta x)^2 \right\rangle_t$ as a function of $t (t \geq 0)$ when $\left\langle (\Delta x)^2 \right\rangle_{t=0}$ is given. (Hint: Take advantage of the property of the minimumuncertainty wave packet you worked out in Chapter 1, Problem 1.20.)

6.3 Chapter 3

6.3.1 3.12: State Representation

1. Consider a pure ensemble of identically prepared spin $\frac{1}{2}$ systems. Suppose the expectation values $\langle S_x \rangle$ and $\langle S_z \rangle$ and the sign of $\langle S_y \rangle$ are known. Show how we may determine the state vector. Why is it unnecessary to know the magnitude of $\langle S_y \rangle$?

2. Consider a mixed ensemble of spin $\frac{1}{2}$ systems. Suppose the ensemble averages $[S_x]$, $[S_y]$, and $[S_z]$ are all known. Show how we may construct the 2×2 density matrix that characterizes the ensemble.

6.4 Chapter 4

6.5 Chapter 5