Semidefinite Programming in Ground State Energy Calculation

Goemans-Williamson Algorithm and Bootstrap Method

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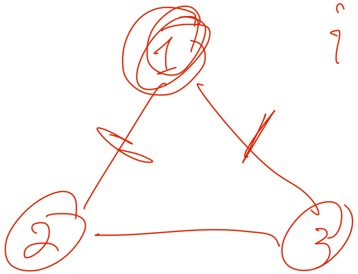
MaxCut Problem

$G = (V, E)$

$$\max \sum_{(i,j)\in E} \frac{1 - z_i z_j}{2}$$

$z_i \in \{-1, 1\}$



Cl_1 Cl_2

1 2, 3

Classical Ising Model

$$\max\left(\sum_{i,j} \frac{1 - z_i z_j}{2}\right) \rightarrow H = -\sum \frac{I - Z_i Z_j}{2}.$$

ground state.

$z_i \in \{\pm 1\}$

NP-Hard

Goemans-Williamson Algorithm

Probabilistic Algorithm whose answer is within 0.878 of the optimal solution.

$$\text{Algo}(G) \geq 0.878 \cdot \text{MAXCUT}(G)$$

Variable Relaxation

$$\max \sum_{i,j} \left(\frac{1 - z_i z_j}{2}\right) \qquad z_i \in \{+1, -1\}$$

$$\vec{x}_i \in \mathbb{R}^{d-1}$$

n is # of vertices in G.

$d = n$.

$$\max \sum_{i,j} \left(\frac{1 - \vec{x}_i \cdot \vec{x}_j}{2}\right) \implies \max \sum_{i,j} \left(\frac{1 - U_{ij}}{2}\right)$$

$$\|\vec{x}_i\|^2 = 1$$

constraint: $U_{ii} = 1$

$$U \succeq 0$$

$$X = [\vec{x}_1 \; \vec{x}_2 \; \cdots \; \vec{x}_n]$$

$$U = X^T X$$

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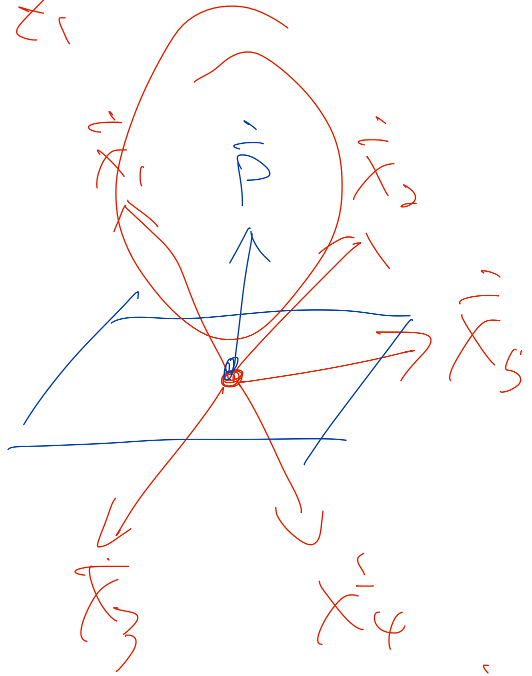
$$\vec{x}_i \in \mathbb{R}^n$$

$$\vec{x}_i = \left(\quad \right)$$

Change of Variable and Semidefinite Programming Formalism

Project back

$\vec{x}_i \to z_i$

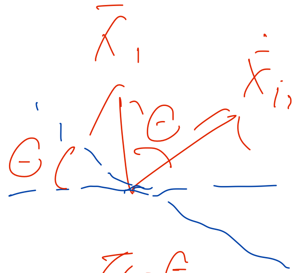


if $\vec{x}_i \cdot \vec{p} \ge 0$ $\quad z_i \to +1$

else $\quad z_j \to -1$

$$\boxed{\frac{1 - z_i z_j}{2}}$$

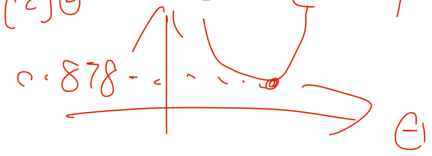
$$\frac{\theta}{\pi} \ge \alpha \cdot \frac{1-\cos\theta}{2}$$



$$\frac{1-\vec{x}_i\cdot\vec{x}_j}{2} = \boxed{\frac{1-\cos\theta}{2}}$$

$$\frac{\theta}{\pi}\cdot 1 + \frac{\pi-\theta}{\pi}\cdot 0 = \boxed{\frac{\theta}{\pi}}$$

$$\frac{\theta}{\pi}\cdot\frac{2}{1-\cos\theta} \Rightarrow \alpha \qquad \theta \in [-\pi, \pi]$$



$$\frac{1-\vec{x}_i\cdot\vec{x}_j}{2} = \frac{1-\cos\theta}{2}$$

Bounding the Ground State

For $p_{i,j}$ $\text{Dacl}_{i,\,\text{Dg}\,G_j}$

$$\frac{1 - z_i z_j}{2} \geq 0.878\,\frac{1 - \vec{x}_i \cdot \vec{x}_j}{2}$$

$$\sum_{i,j} \frac{1 - z_i \cdot z_j}{2} \geq 0.878 \boxed{\sum_{i,j} \frac{1 - \vec{x}_i \cdot \vec{x}_j}{2}}$$

$\vec{x}_i = \begin{pmatrix} \pm 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \to z_i = \pm 1$

$\text{MAXCUT}(G) \leq \text{SDP}(G)$

$\geq 0.878\,\text{MAXCUT}(G)$

$\text{SDP}(G)$

$$\boxed{\langle A | g_0(G) \rangle \geq 0.878 \cdot \text{MAXCUT}(G)}$$

$\text{MAXCUT}(G)$

Bootstrap Quantum Anharmonic Oscillator

$H = \hat{p}^2 + \hat{x}^2 + g\hat{x}^4$

$H|\psi\rangle = E|\psi\rangle$

$\langle\psi|[H, O]|\psi\rangle$

$(E, \langle x^2\rangle)$

min E

SDP

$\langle x^t\rangle = \langle x^{t-1}\rangle + \ldots$

$\langle x^j\rangle = \ldots$

$j > 10$ or $> 10 \ldots$

infin

E and $<x^2>$ as unknown

Recursive Relation

Search for Consistent E and $<x^2>$