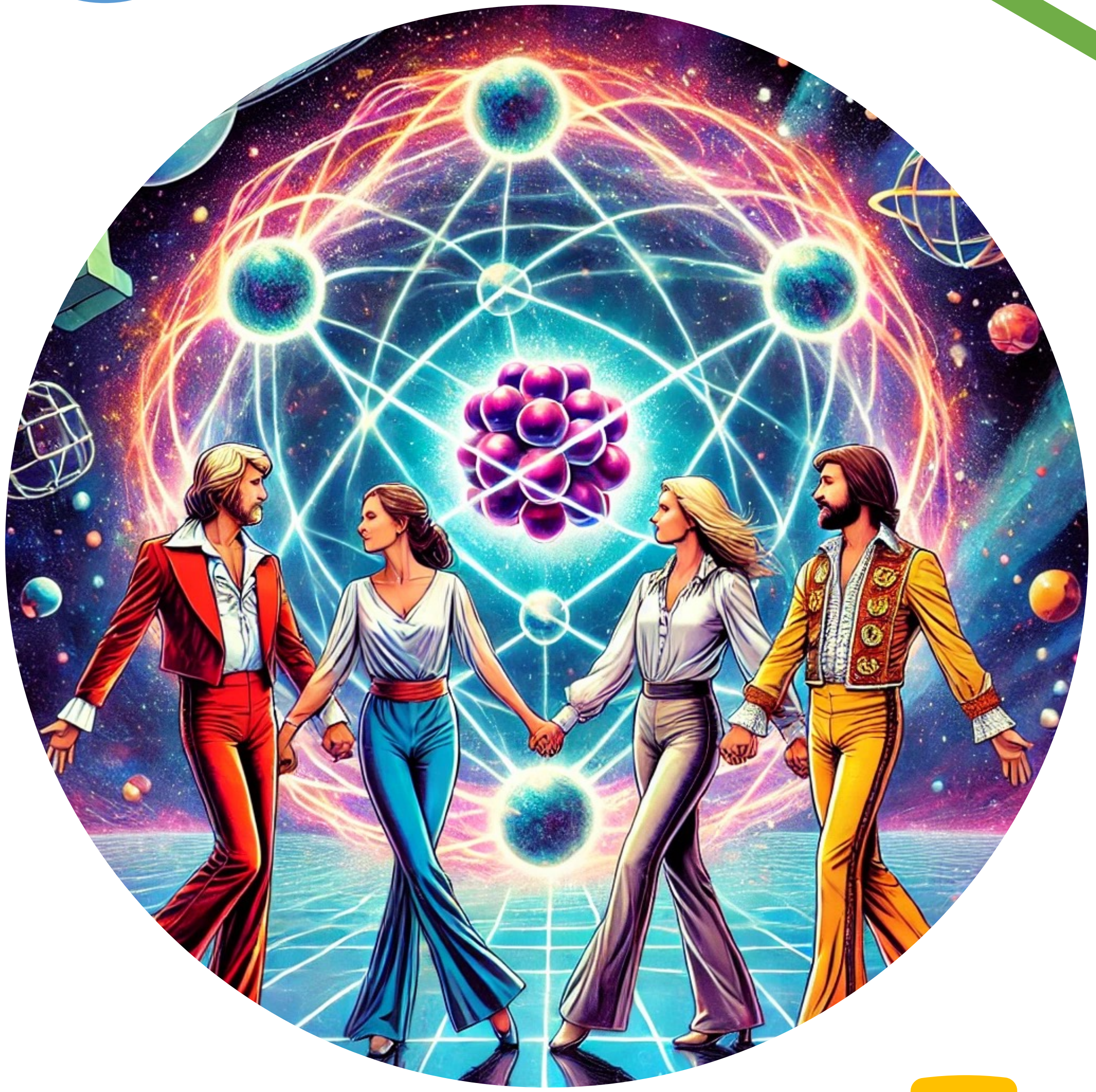


European Research Council

BFKL in N=4

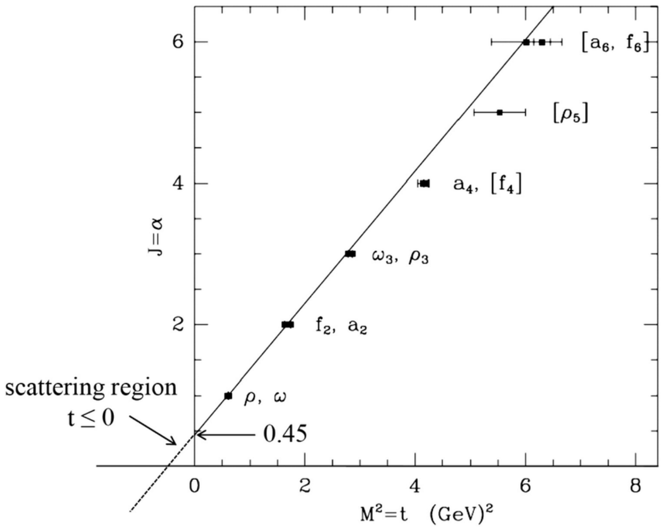
Based on work with Ekhammar, Preti `24

International workshop on exact methods in quantum field theory and string theory
Nanjing, 2024

Regge Trajectories

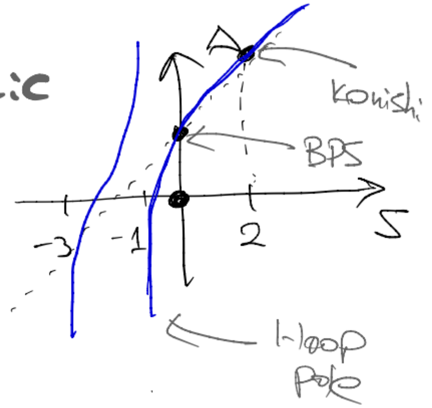
- Regge trajectories $J(m^2)$ key objects in the study of amplitudes.

- They link the spectrum of particles with the high energy asymptotic of amplitudes



Regge Trajectories in $\mathcal{N}=4$ SYM

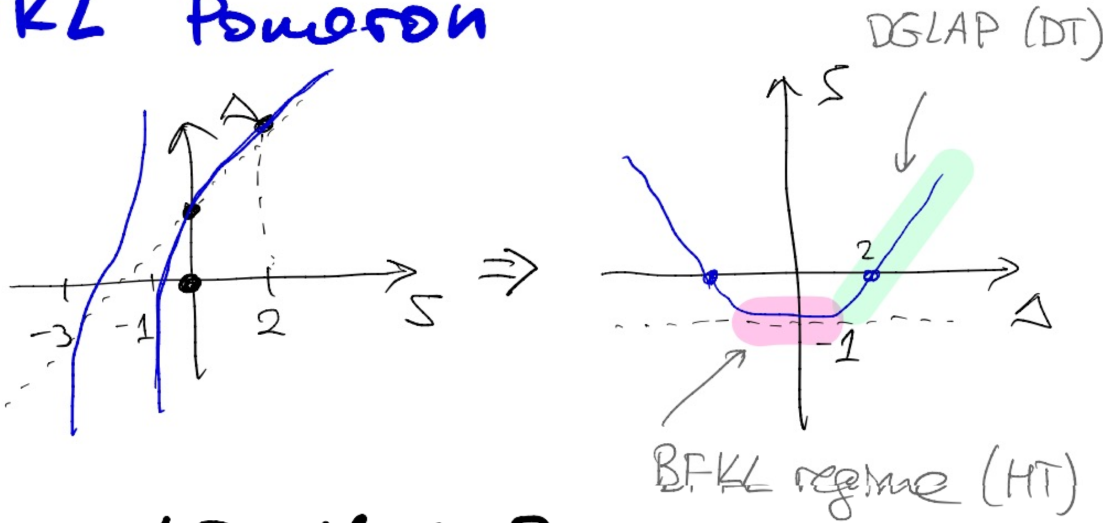
- $\mathcal{N}=4$ is a CFT $\quad m^2 \to \Delta$, $\quad J \to S$
- example: $\mathrm{tr}\, Z D_+^S Z = \mathcal{O}_S \qquad \langle \mathcal{O}_S \bar{\mathcal{O}}_S \rangle = \frac{1}{x^{2\Delta(S)}}$
- tree level $\quad \Delta = S+2$
- 1-loop $\quad \Delta = S + 2 + g^2 \underbrace{H_1(S)}_{\text{Harmonic sum}}$



- Near the pole $\quad \Delta = \# + g^2 \frac{\#}{\omega} + g^4 \frac{\#}{\omega^2} + \dots = $ BFKL Pomeron
- $\frac{1}{\omega} \sim \ln S$ leading logs resummation

BFKL Pomeron

- $\Delta(S) \longrightarrow S(\Delta)$



- LO QCD Pomeron = LO $N=4$ Pomeron

$$S(\Delta) = -1 + g^2\left(\psi\left(\frac{\Delta+1}{2}\right) + \psi\left(\frac{1-\Delta}{2}\right) + 2\gamma\right)$$

polygamma function $\sum_{n=0}^{\infty}\frac{1}{x+n} = \psi(x)$

- Only NLO g^4 known in QCD (8 years to comput

BFKL Pomeron

Jaroszewicz, 1982
Lipatov 1986
Kotikov,Lipatov 2002

- At the LO:

$$\chi(\gamma) = \boxed{2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma)}\,, \quad \Psi(\gamma) = \Gamma'(\gamma)/\Gamma(\gamma)$$

Fadin, Lipatov 1998
Kotikov,Lipatov 2002
Kotikov,Lipatov 2000

- At NLO:

$$\begin{aligned}\delta(\gamma) \;&=\; -\Bigg[\left(\frac{11}{3} - \frac{2n_f}{3N_c}\right)\frac{1}{2}\left(\chi^2(\gamma) - \Psi'(\gamma) + \Psi'(1-\gamma)\right) - \left(\frac{67}{9}\boxed{-\frac{\pi^2}{3}} - \frac{10}{9}\frac{n_f}{N_c}\right)\chi(\gamma \\ &\boxed{-6\zeta(3)} + \frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left(3 + \left(1 + \frac{n_f}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right) \\ &\boxed{-\Psi''(\gamma) - \Psi''(1-\gamma)} - \frac{\pi^3}{\sin(\pi\gamma)} + \boxed{4\phi(\gamma)}\Bigg]\,.\end{aligned}$$

$$\begin{aligned}\phi(\gamma) \;&=\; -\int_0^1 \frac{dx}{1+x}\left(x^{\gamma-1} + x^{-\gamma}\right)\int_x^1 \frac{dt}{t}\ln(1-t) \\ &=\; \sum_{n=0}^{\infty}(-1)^n\left[\frac{\Psi(n+1+\gamma) - \Psi(1)}{(n+\gamma)^2} + \frac{\Psi(n+2-\gamma) - \Psi(1)}{(n+1-\gamma)^2}\right]\end{aligned}$$

N=4 SYM

BFKL at NNLO

Alfimov, Gromov, Kazakov (2014)

NG,Levkovich-Maslyk,Sizov
Phys.Rev.Lett. 115 (2015)

$$S = -1 + \sum_{n=1}^{\infty} g^{2n} \left[F_n\left(\tfrac{\Delta-1}{2}\right) + F_n\left(\tfrac{-\Delta-1}{2}\right) \right]$$

$$\begin{aligned}
&\frac{1}{256} F_3 = \\
&-\frac{5S_{-5}}{8} - \frac{S_{-4,1}}{2} + \frac{S_1 S_{-3,1}}{2} + \frac{S_{-3,2}}{2} - \frac{5S_2 S_{-2,1}}{4} \\
&+\frac{S_{-4}S_1}{4} + \frac{S_{-3}S_2}{8} + \frac{3S_{3,-2}}{4} - \frac{3S_{-3,1,1}}{2} - S_1 S_{-2,1,1} \\
&+S_{2,-2,1} + 3S_{-2,1,1,1} - \frac{3S_{-2}S_3}{4} - \frac{S_5}{8} + \frac{S_{-2}S_1 S_2}{4} \\
&+\pi^2 \left[\frac{S_{-2,1}}{8} - \frac{7S_{-3}}{48} - \frac{S_{-2}S_1}{12} + \frac{S_1 S_2}{48} \right] \\
&+\zeta_3 \left[-\frac{7S_{-1,1}}{4} + \frac{7S_{-2}}{8} + \frac{7S_{-1}S_1}{4} - \frac{S_2}{16} \right] \\
&+ \left[2\mathrm{Li}_4 - \frac{\pi^2 \log^2 2}{12} + \frac{\log^4 2}{12} \right] (S_{-1} - S_1) - \pi^4 \left[\frac{2S_{-1}}{45} - \frac{S_1}{96} \right] \\
&+\frac{\log^5 2}{60} - \frac{\pi^2 \log^3 2}{36} - \frac{2\pi^4 \log 2}{45} - \frac{\pi^2 \zeta_3}{24} + \frac{49\zeta_5}{32} - 2\mathrm{Li}_5
\end{aligned}$$

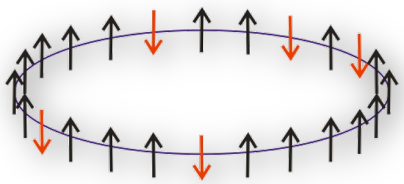
Found from integrability

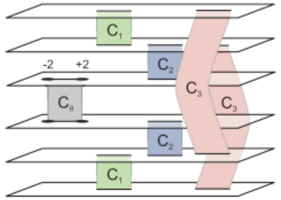
Later confirmed by Caron-Huot, Herran

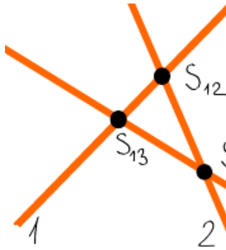
• This talk $\underbrace{N \ldots N}_{L/2\ \text{times}}LO$ for $\mathrm{tr}\, D_+^S Z^{L}$ R-charge

We of course need integrability to do this

Historical path for N=4 SYM







Origins of YM integrability: Lipatov's BFKL Hamiltonian

Perturbative integrability in N=4 SYM

Classical integrability of string σ-model on AdS5×S5, quasiclassics

S-matri
asympto
Bethe Ans
Cusp
dimensi

1993

Lipatov
Faddeev,Korchemsky

2002

Minahan,Zarembo,
Beisert,Kristijanssen,Staudacher

2003-2008

Bena,Polchinski,Roiban
Kazakov,Marshakov, Minahan,
Zarembo, Frolov, Tseytlin
Schafer-Nameki
Beisert,Kazakov,Sakai,Zarembo
NG,Vieira

2004-20

Arutyunov,Frolov
Staudacher,
Janik
Hernandez,
Roiban, Ts
Beisert,Eden,S

$\mathcal{N}=4$ Integrability / Strings in AdS_5

- Integrability $\Rightarrow$ $\Psi(r,\theta,\varphi) = \Psi_\theta(\theta)\,\Psi_\varphi(\varphi)\,\Psi_r(r)$

- String theory $\Psi\Big($  $\Big) = \prod_{i=0}^{\infty} Q_{A_i}(u_i)$

  string configuration   polarization

- 2^8 different Q-functions, 4+4 elementary

  $Q_i(u)$ $i=1\ldots4$ $\longleftrightarrow$ $su(2,2)$ AdS polarizations

  $P_a(u)$ $a=1\ldots4$ $\longleftrightarrow$ $su(4)$ S^5 R-symmetry

  $Q_1 \sim u^{\Delta+S}$ $P_a \sim u^{\pm L}$ at $u\to\infty$

  $Q_2 \sim u^{\Delta-S}$

  $\vdots$

- How do we find the Q-functions?

XXX warm up

$$(*)\ Q(u+i)(u-\tfrac{i}{2})^L + T(u)Q(u) + Q(u-i)(u+\tfrac{i}{2})^L = 0$$

- has 2-solutions Q_1 Q_2
- Wronskian is $Q_1(u+\tfrac{i}{2})Q_2(u-\tfrac{i}{2}) - c.c. = u^L$
- $(*) \Leftrightarrow$ $0 = \begin{vmatrix} Q_1(u+i) & Q_1(u) & Q_1(u-i) \\ Q_1(u+i) & Q_1(u) & Q_1(u-i) \\ Q_2(u+i) & Q_2(u) & Q_2(u-i) \end{vmatrix} = Q_1(u+i)\ldots + Q_1(u)$
- states 1 to 1 with a) Wronskian
  b) Q_a are polynomials
- su(N): Q_i $i=1\ldots N$ Baxter: $(N+1)\times(N+1)$ det
- Nested BA: zeroes of $Q_1, Q_{12}, Q_{123} \ldots$

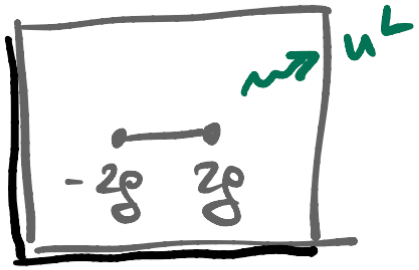
e.g. $Q_{12}(u+\tfrac{i}{2})Q_{13}(u-\tfrac{i}{2}) - Q_{12}(u-\tfrac{i}{2})Q_{13}(u+\tfrac{i}{2}) = Q_{123}$

Spectrum of int. models

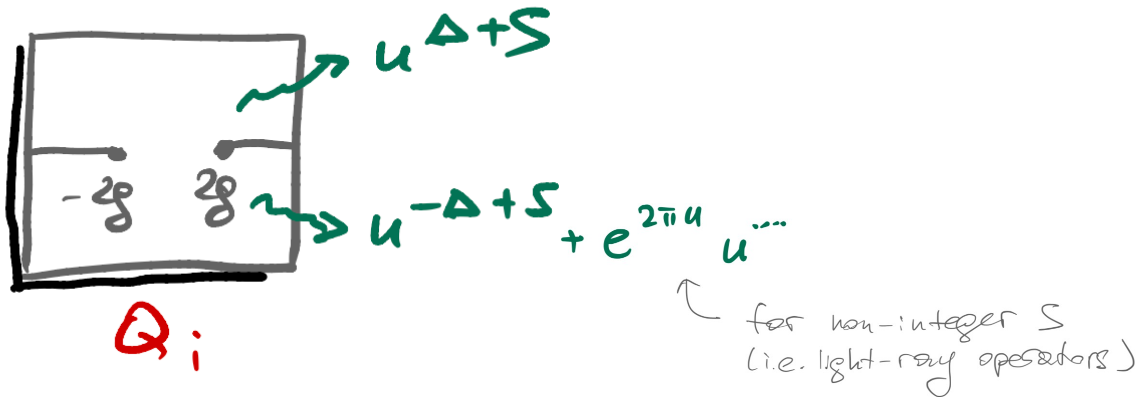
a) QQ-Wronskian relations

b) Analyticity

For N=4:



P_a



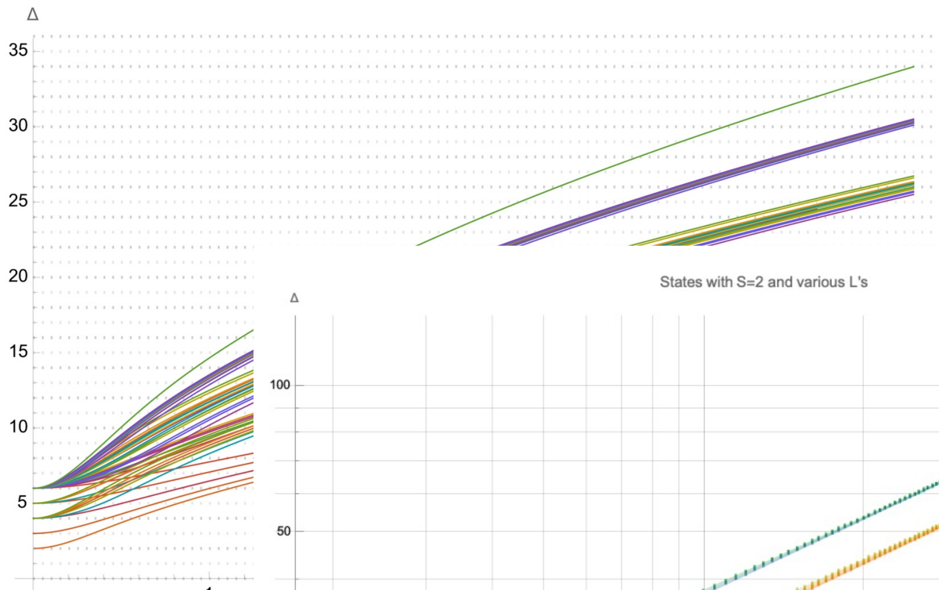
Q_i

claim: a) + b) is in 1-to-1 with local operators

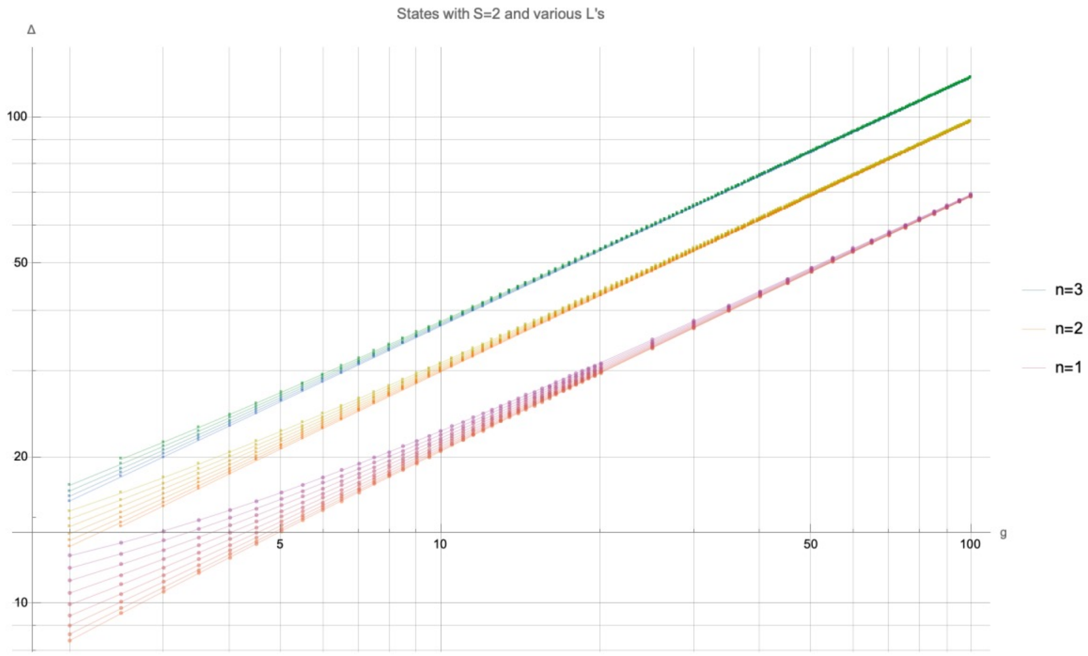
Generalization to: ABJM, β-def, AdS_3 ...

Some results of QSC

Spectrum of local operators:



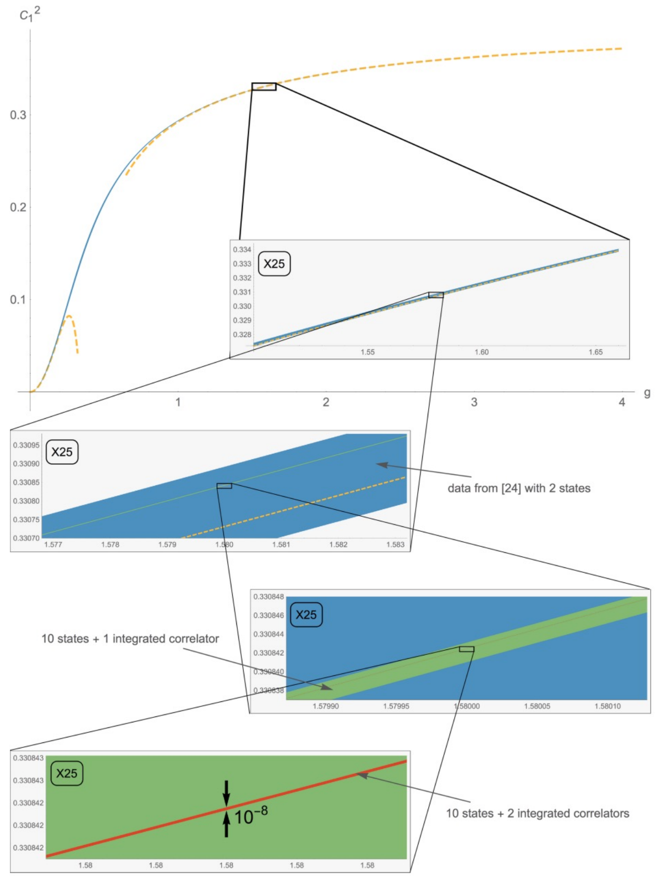
[NG, Julius, N.So



[Ekhammar, NG, Ryan '24] <- See Simon's talk

Structure constants:

[NG, Kazakov, Leurent, Volin '13]



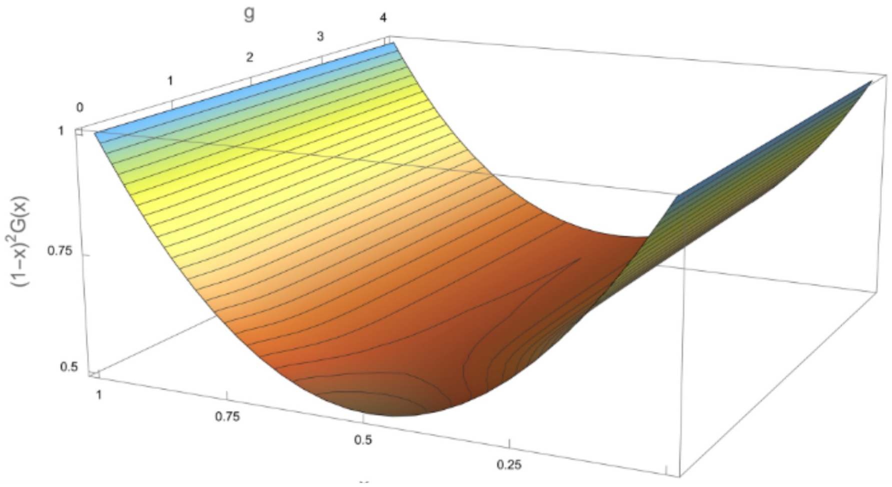
[Cavaglia, NG, Julius, Preti '21]

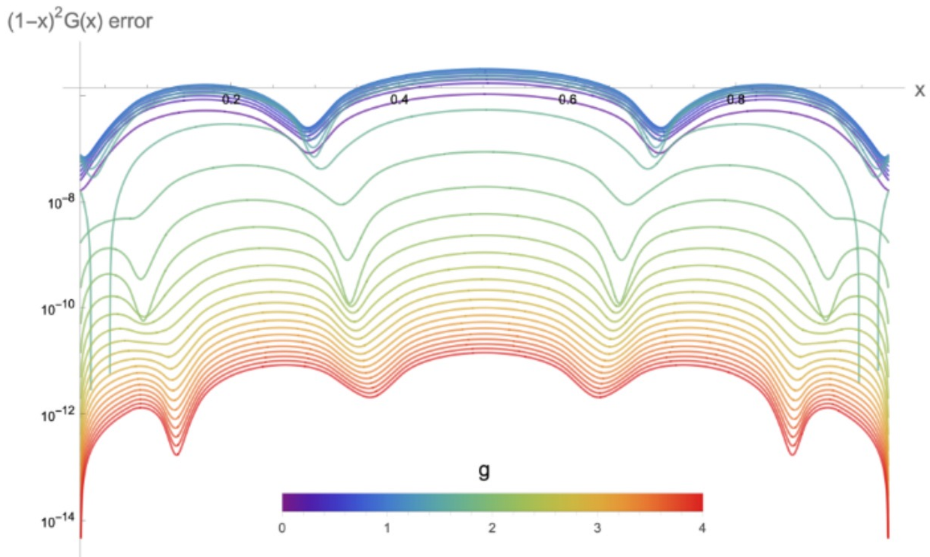
And beyond…

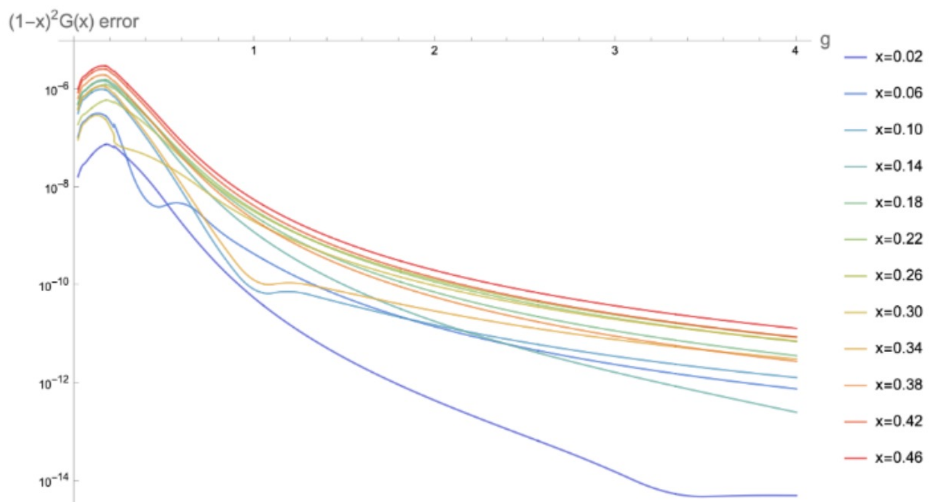
[Basso Georgoudis Sueiro '22]

4 pt correlation functions in N=4

[Cavaglia, NG, Julius, Preti '23]



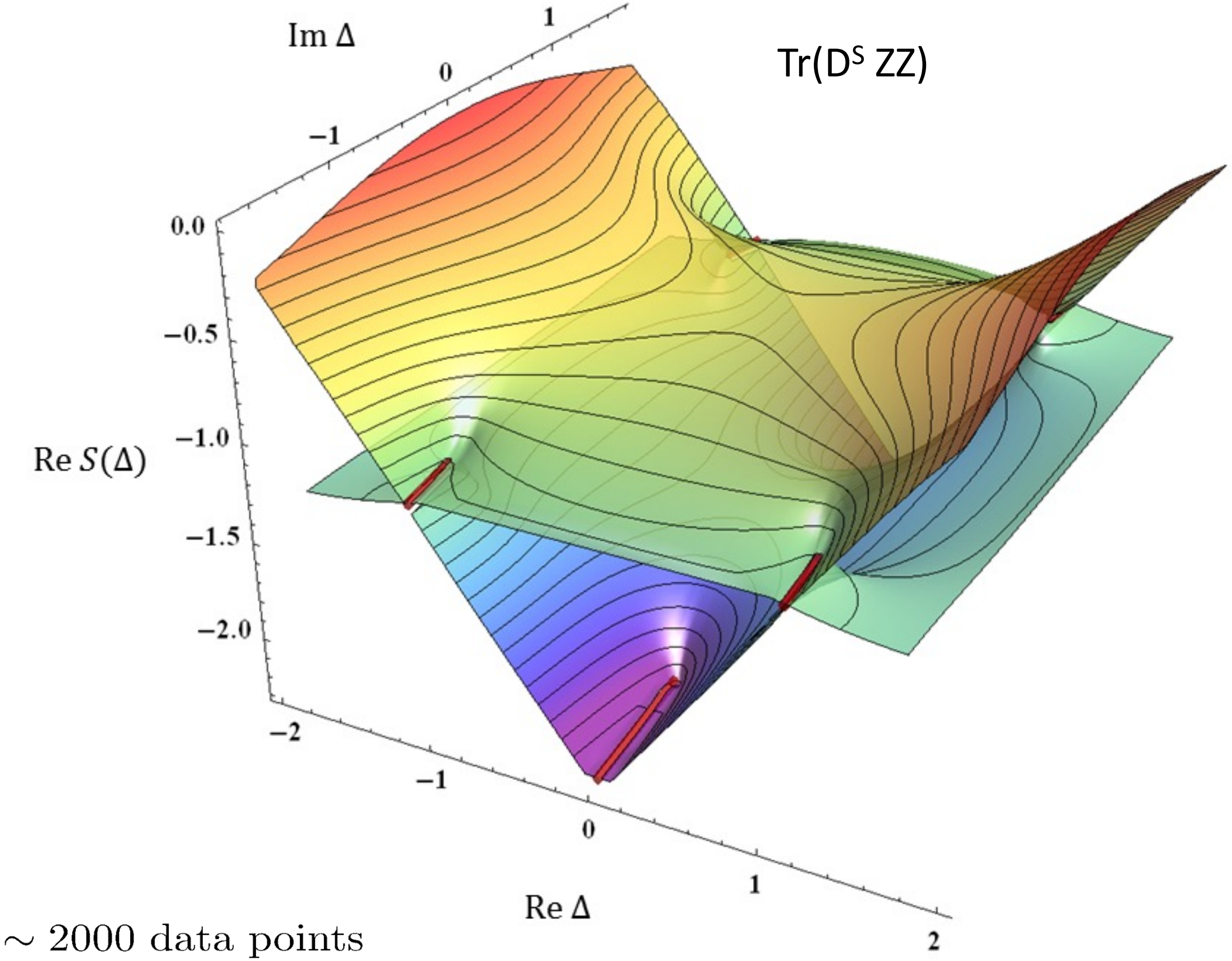




Back to BFKL

N=4 SYM Pomeron Trajectory, non-perturbative:

Gromov,FLM,Sizov 2015

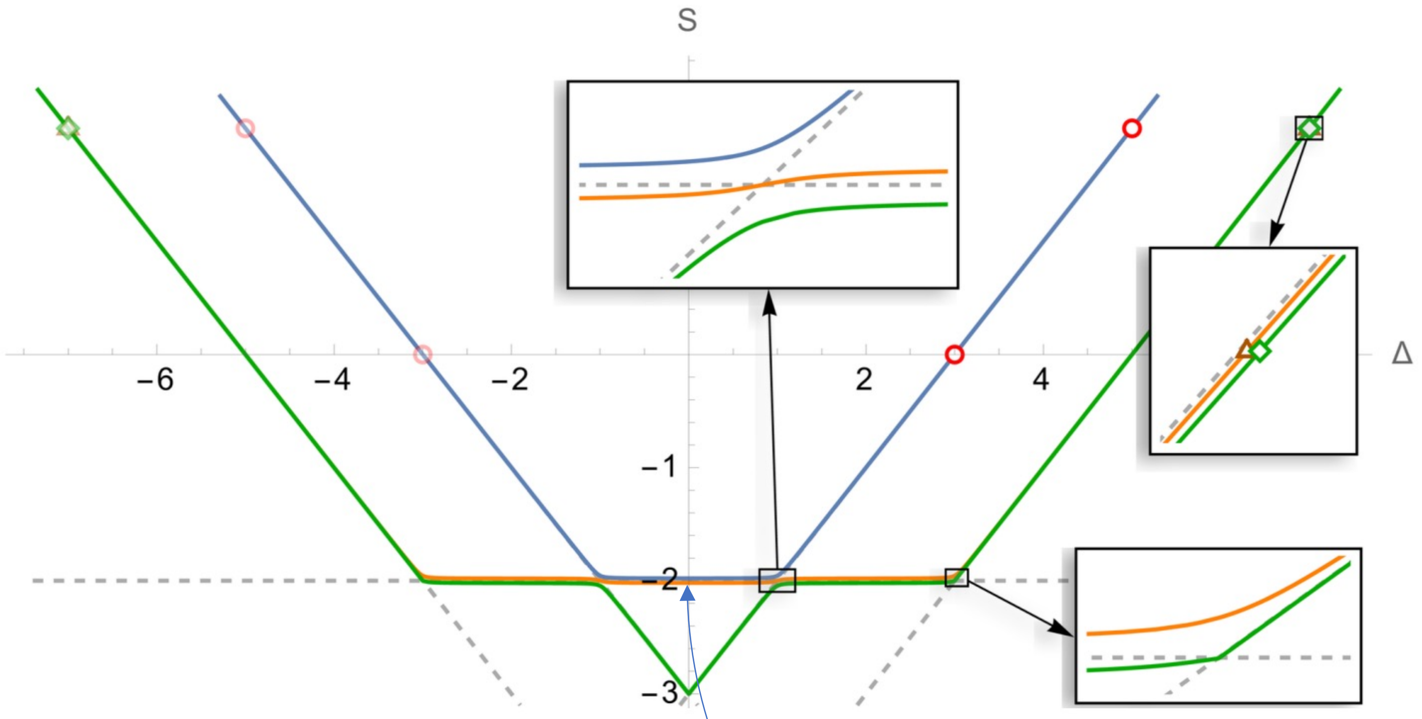


$\sim$ 2000 data points

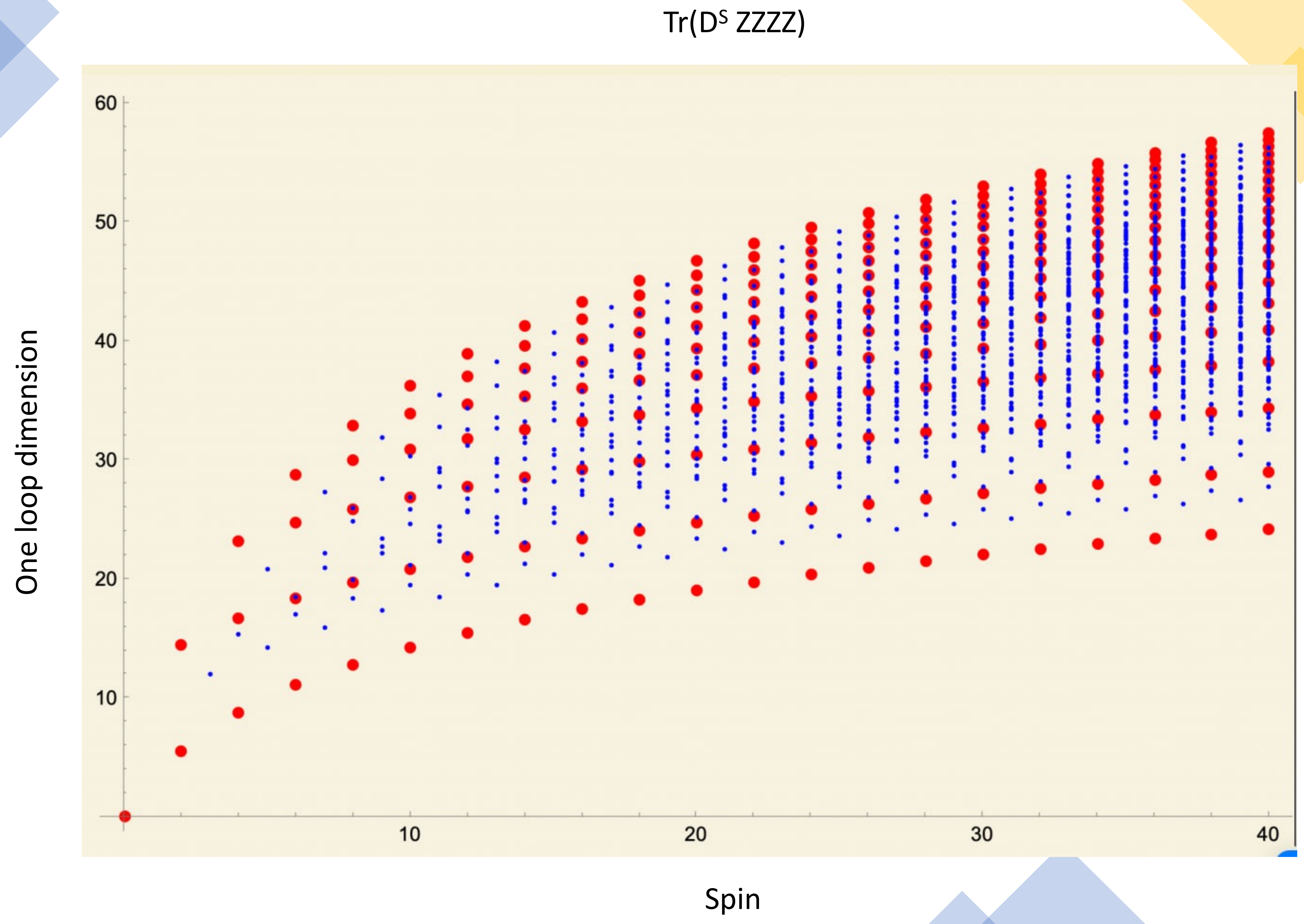
Higher R-charge – new physics?

[Klabbers, Preti, Szecsenyi `23]

Tr(D^S ZZZ)



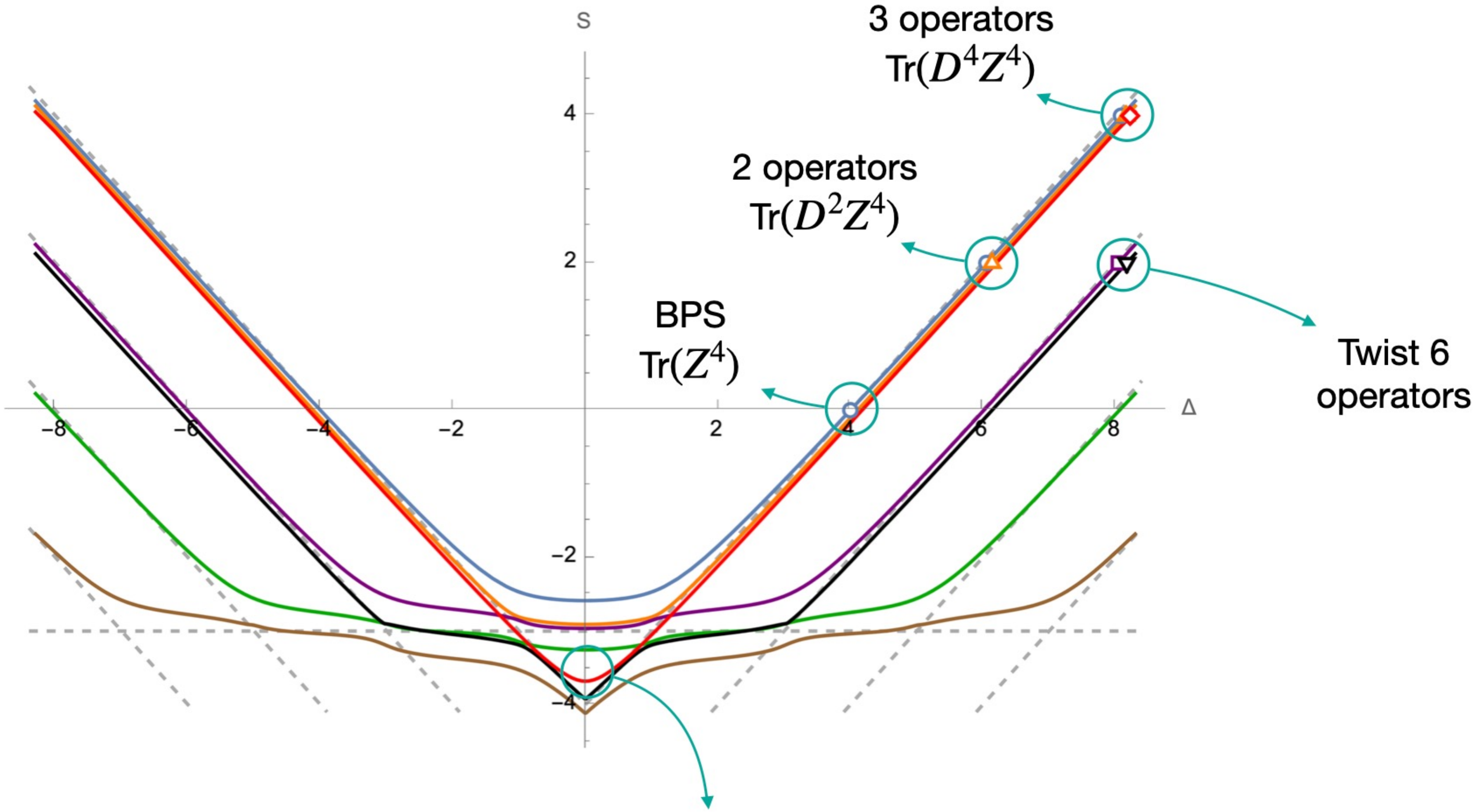
$$\alpha = -2 + 2g + 16\log 2\; g^2 - \frac{2\pi^2}{3} g^3$$



Tr(D^S ZZZZ)
One loop dimension
Spin
60
50
40
30
20
10
10
20
30
40

Tr(D^S ZZZZ)

Coupling
$g = 0.1$

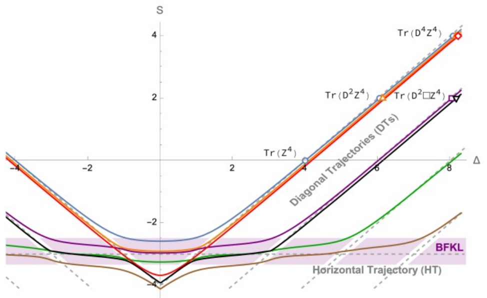


Red trajectory is one of an infinite set of trajectories
that doesn't bend at the branch point

Asymptotic Baxter-Bethe Ansatz

- for local operators Asymptotic Bether Ansatz (ABA) gives us great control over Δ up to g^{2L+4} order

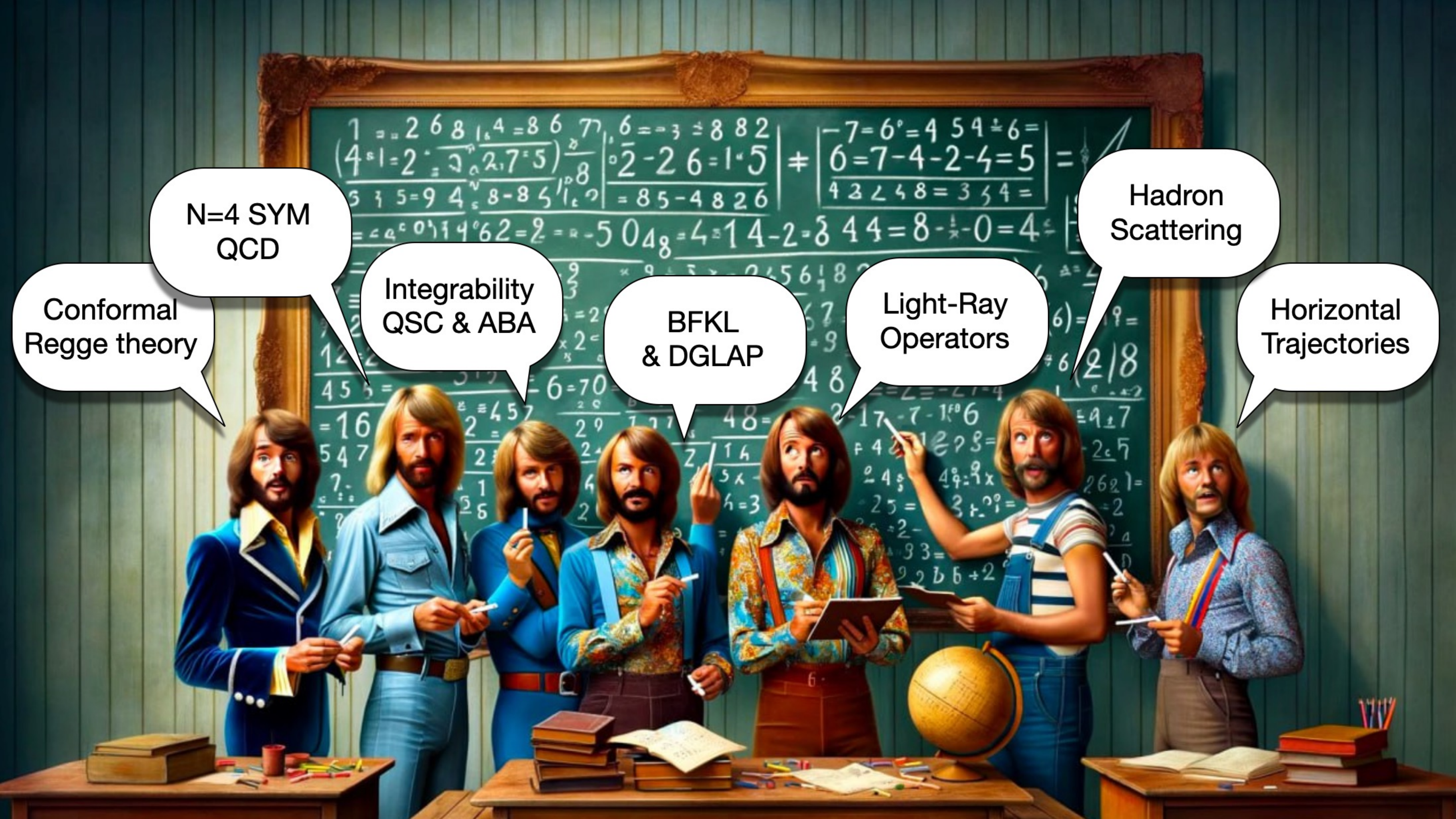
- It enables weak coupling Form-factors for structure constants (Hexagon)



- We found that in BFKL regime a similar equations can be written valid to g^{L+2} order, but with a different particle content

- New feature: massless magnons

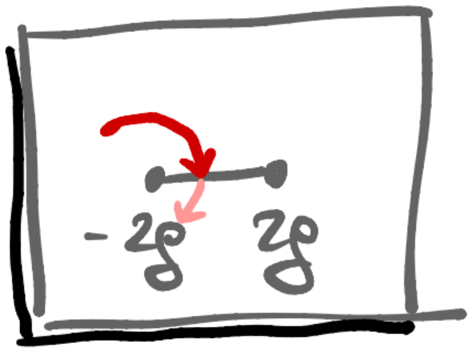
$$E = g \sin P/2 \quad \text{vs} \quad \sqrt{1+16g^2 \sin^2 P/2}$$



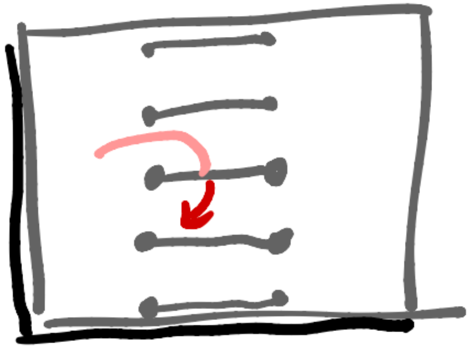
Conformal
Regge theory
N=4 SYM
QCD
Integrability
QSC & ABA
BFKL
& DGLAP
Light-Ray
Operators
Hadron
Scattering
Horizontal
Trajectories

ABA from QSC

$P\mu$ - system



P_a



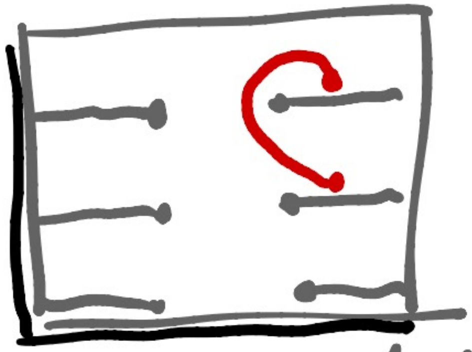
$\tilde{P}_a$

monodromy

$$\tilde{P}_a = \mu_{ab} P^b$$

monodromy

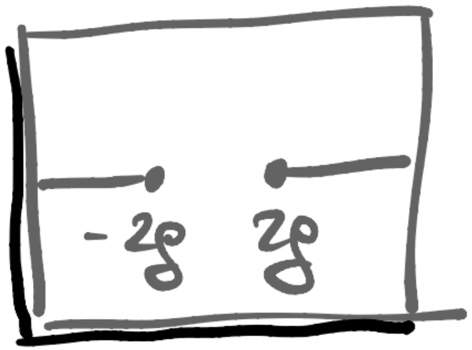
$$\tilde{P}_a = \mu_{ab} P^b$$



long-cut periodicity

ABA from QSC

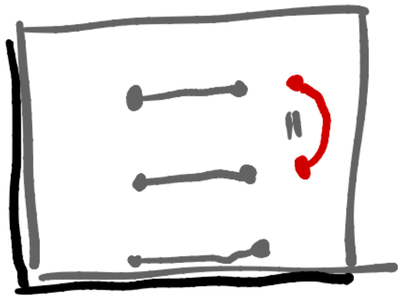
Qω - system



Q_i

$$\tilde{Q}_i = \omega_{ij} \, Q^j$$

monodromy



$$\omega_{ij}(u+i) = \omega_{ij}(u)$$

short-cut periodicity

ABA from QSC

$P\mu \longleftrightarrow Q\omega$

$$P_a = Q_{a i} Q^i$$

rotation $P \leftrightarrow Q$

$$\mu_{ab} = Q_{ai} Q_{bj} \omega^{ij}$$

ABA (local operators) limit : $\omega^{12} \sim g^{-2L-4} \to \infty$

ABBA (BFKL limit) : $\omega^{13} \sim g^{-L-2} \to \infty$

- key simplification

not periodic

$$F = \frac{\mu_{ab}(u+i/2)}{\mu_{ab}(u-i/2)} = \frac{Q^+_{ai} Q^+_{bj} \omega^{ij}(u+i/2)}{Q^-_{ai} Q^-_{bj} \omega^{ij}(u-i/2)}$$

periodic

- $\Rightarrow F = \dfrac{\mu_{ab}(u+i/2)}{\mu_{ab}(u-i/2)} = \dfrac{Q^+ Q^+}{Q^- Q^-}$ ← analytic UHP

- $\overline{\mu_{ab}(u+i/2)} = \pm\mu_{ab}(u-i/2) \Rightarrow \bar{F} = \dfrac{1}{F}$

- $\mu_{ab}(u+i/2) = \tilde{\mu}_{ab}(u-i/2) \Rightarrow \tilde{F} = \dfrac{1}{F}$

- Conclution: whenever one $w_{ij} \to \infty$

  $F, \tilde{F}$ both are functions with one cut

$\Rightarrow F$ = rational function of x

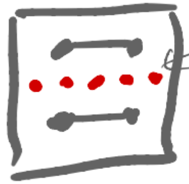
$$x + \frac{1}{x} = \frac{u}{g} \Rightarrow \text{BAE}$$

ABA vs ABBA

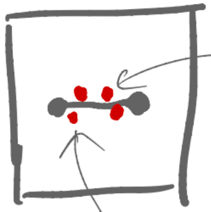
- $\omega_{12} \leftrightarrow \omega_{13}$ same as $\Delta \leftrightarrow S$
- μ_{12} was $\sim u^{\Delta}$, in ABBA $\mu_{12} \sim u^{S} e^{2\pi u}$

change of gluing

- $\mu_{12}(u+i/2)$ roots become magnons



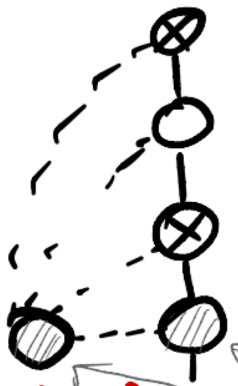
- we found new type of rooms



BES	$\mathfrak{sl}_2$
roots of P_1	no roots
roots of $Q_{1\|1}$	no roots
roots of $Q_{1\|12}$	no roots
roots of $Q_{12\|12}$	S roots

$$\Delta = L + S + 2ig^2 \sum_k \left(\frac{1}{x_k^+} - \frac{1}{x_k^-} \right)$$

ABBA	$\mathfrak{sl}_2$
roots of P_1	no roots
roots of $Q_{1\|1}$	no roots
roots of $Q_{1\|13}$	no roots
roots of $Q_{12\|13}$	no roots



roots of $\mathcal{A}_{12}$ on cut

4-roots

massive magnon

massles magnon

$$S = -L + 1 + ig \sum_{k=1}^{4} \left(z_k - \frac{1}{z_k} \right)$$

$$+ 2ig^2 \sum_k \left(\frac{1}{x_k^+} - \frac{1}{x_k^-} \right)$$

$\mathfrak{sl}_2$ Sector

- $\mathbf{P}_a$: $\mathbf{P}_1(u) = \sigma_0(u)\, x^{-\frac{L}{2}-1}$

massless analog of BES phase

$$\ln \sigma_0(z) = \oint \frac{dx}{2\pi i} \oint \frac{dy}{2\pi i} \frac{1}{z-x} \left(\frac{1}{y-z_i} - \frac{1}{y-1/z_i} \right) \ln \frac{\Gamma(1+iu_x-iu_y)}{\Gamma(1-iu_x+iu_y)}$$

- $\mathbf{Q}_{a|i}$: $\left(\left(u+\frac{i}{2}\right)^2 - \theta_2^2\right) Q_{1|\beta}(u+i) + \left(\left(u+\frac{i}{2}\right)^2 - \theta_1^2\right) Q_{1|\beta}(u-i)$

$$= \left(2u^2 - \tfrac{1}{4}\left(\Delta^2 + 4\theta_1^2 + 4\theta_2^2 + 1\right)\right) Q_{1|\beta}(u)$$

$g\left(z_1 + \frac{1}{z_1}\right)$

$$\frac{(-1)^{-1/4}\, 2^{1-\Delta}\, \frac{e^{\frac{3i\pi\Delta}{4}}}{1+e^{i\pi\Delta}}\, \sqrt{\pi}\, \Gamma\left(1-\frac{\Delta}{2}\right) \Gamma\left(-iu+i\theta_1+\frac{1}{2}\right) \Gamma\left(-2i\theta_2\right)}{\Gamma\left(-iu-i\theta_2+\frac{1}{2}\right) \Gamma\left(-\frac{\Delta}{2}-i\theta_1-i\theta_2+\frac{1}{2}\right) \Gamma\left(-\frac{\Delta}{2}+i\theta_1-i\theta_2+\frac{1}{2}\right) \Gamma\left(i\theta_1+i\theta_2+1\right)}$$

$${}_3F_2\left(iu+i\theta_2+\frac{1}{2}, -\frac{\Delta}{2}+i\theta_1+i\theta_2+\frac{1}{2}, \frac{\Delta}{2}+i\theta_1+i\theta_2+\frac{1}{2}; i\theta_1+i\theta_2+1, 2i\theta_2+1; 1\right)$$

For $\mathrm{tr}\, D_+^S Z^L$

only 4 massless roots

$$(i z_k)^{2L+4} = \frac{1}{G_{kk}^{\frac{1+\Delta}{2}}\, G_{kk}^{\frac{1-\Delta}{2}}}\; \frac{\sigma_0^2(-z_k)}{\sigma_0^2(-1/z_k)} \prod_{\substack{n\neq 0\\ j=1..4}} \frac{z_k^{[2n]} - z_j}{z_k^{[2n]} - 1/z_j} \prod_{e=1,2} \frac{G_{ke}^{\frac{1+\Delta}{2}}\, G_{ke}^{\frac{1-\Delta}{2}}}{G_{ke}^{1}}$$

$$G_{ke}^{\delta} = \frac{\Gamma(\delta + i\theta_k + i\theta_e)}{\Gamma(\delta - i\theta_k - i\theta_e)}\,\frac{\Gamma(\delta + i\theta_k - i\theta_e)}{\Gamma(\delta - i\theta_k + i\theta_e)}$$

$$S = 1 - L + ig \sum_{k=1}^{4} \left(z_k - \frac{1}{z_k}\right) \leftarrow \text{„Energy"}$$

$g \to 0$ r.h.s. $\to 1$

$$(i z_1)^{L+2} = (i z_2)^{L+2} = \pm 1 \quad \Rightarrow \quad z_k = -i\, e^{\frac{\pi i n_k}{L+2}}, \quad n_k \to 1 \ldots L+1$$

$\sim L^2/4$ states

L	states
2	13
3	13, 24
4	13, 15, 24, 35
5	13, 15, 24, 26, 35, 46

perturbatively

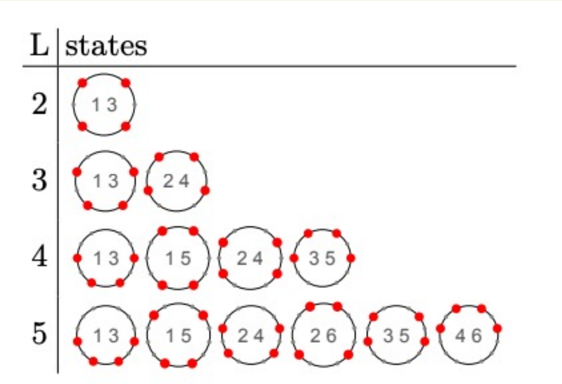
$$z_k = z_k^0 + \sum g^n a_{k,n}$$

Examples of the states

L=2:

$$-4\, g^2\, \chi(\Delta) + O(g^4)$$

$$\psi\left(\tfrac{1-\Delta}{2}\right) + \psi\left(\tfrac{1+\Delta}{2}\right) + 2\gamma \equiv \chi,$$



L=3:

$$2\, g - 4\, g^2\, \chi(\Delta) - \frac{2\, \pi^2\, g^3}{3} + g^4 \left(24\, \chi''(\Delta) + \frac{4}{3}\, \pi^2\, \chi(\Delta) + 28\, \zeta(3)\right) + O(g^5)$$

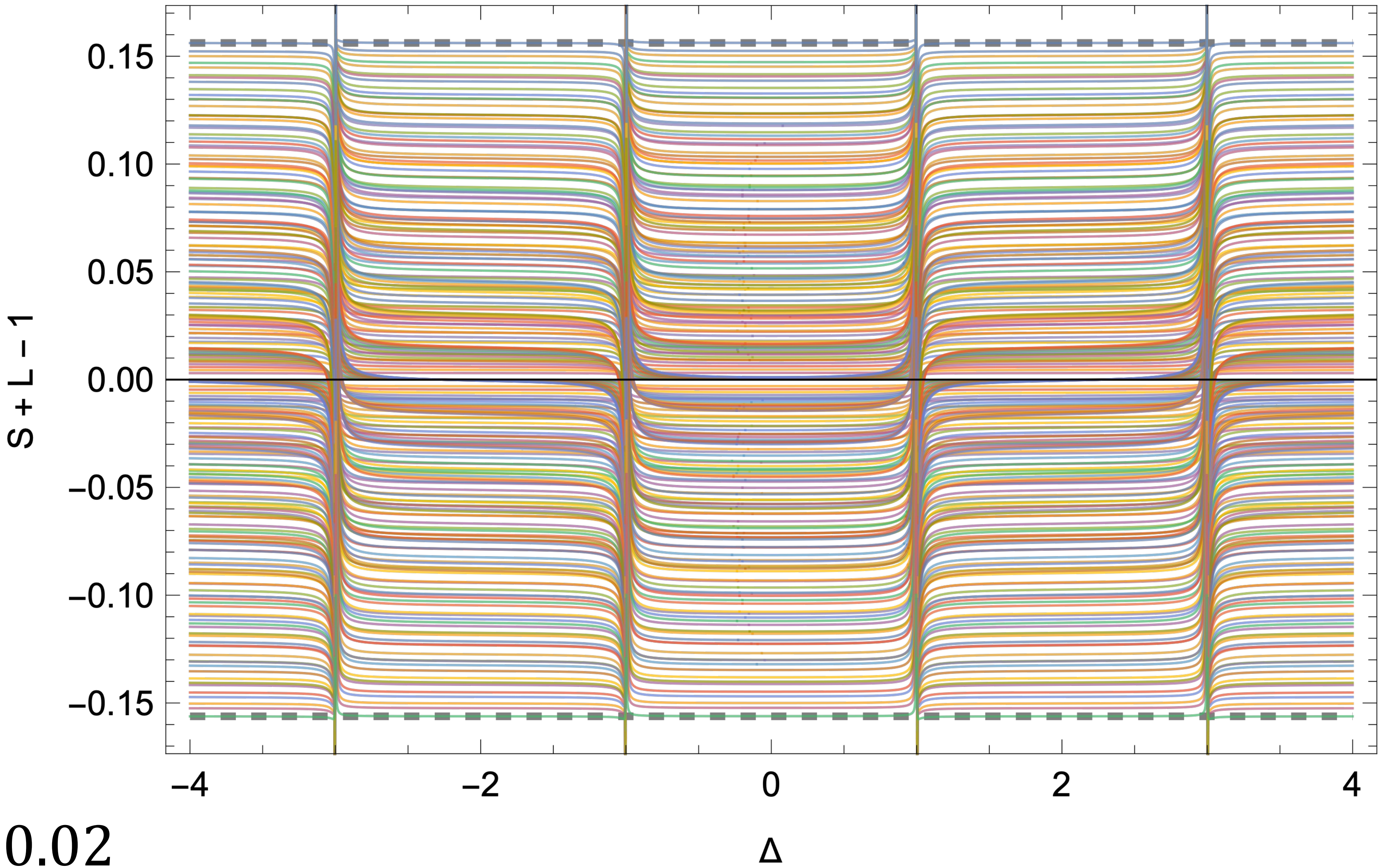
L=4:

$$2\sqrt{3}\, g - \frac{10}{3}\, g^2\, \chi(\Delta) - \frac{g^3\, (\chi(\Delta)^2 + 2\,\pi^2)}{\sqrt{3}} + \frac{4}{81}\, g^4\, \left(369\, \chi''(\Delta) + 8\, \chi(\Delta)^3 + 18\, \pi^2\, \chi(\Delta) + 531\, \zeta(3)\right) +$$
$$\frac{g^5\, \left(5\, \chi(\Delta)\, \left(9360\, \chi''(\Delta) - 115\, \chi(\Delta)^3 + 324\, \pi^2\, \chi(\Delta) + 3312\, \zeta(3)\right) + 2376\, \pi^4\right)}{1620\, \sqrt{3}} + O(g^6)$$

L=5:

$$\begin{aligned}
&-2\left((-1)^{2/7}\left(1+(-1)^{4/7}\right)\left(1+(-1)^{6/7}\right)\right) g - \frac{4}{7}\, g^2 \left((-1)^{3/7}\left(1 - \sqrt[7]{-1} + (-1)^{2/7} - 4\,(-1)^{4/7} + (-1)^{6/7}\right) \mathrm{chi}(\Delta)\right) + \\
&\frac{2}{147}\, \sqrt[7]{-1}\, \left(1 + \sqrt[7]{-1}\right) g^3 \left(7\left(-2 + 5\,\sqrt[7]{-1} - (-1)^{2/7} - 3\,(-1)^{3/7} + 2\,(-1)^{5/7}\right)\pi^2 - 18\,\sqrt[7]{-1}\,\left((-1)^{2/7} - 1\right) \mathrm{chi}(\Delta)^2\right) - \\
&\frac{1}{1029} \times 4\, g^4 \left((-1)^{6/7}\left(1 + \sqrt[7]{-1}\right)\left(98\left(-7 + 43\,\sqrt[7]{-1} - 50\,(-1)^{2/7} + 40\,(-1)^{3/7} - 25\,(-1)^{4/7} + 10\,(-1)^{5/7}\right) \mathrm{chi}''(\Delta) + \right.\right. \\
&\quad 32\left(-1 + \sqrt[7]{-1} - 2\,(-1)^{2/7} + (-1)^{3/7} - (-1)^{4/7} + (-1)^{5/7}\right) \mathrm{chi}(\Delta)^3 + 7\left(-7 + 21\,\sqrt[7]{-1} - 28\,(-1)^{2/7} + 24\,(-1)^{3/7} - 14\,(-1)^{4/7} + 4\,(-1)^{5/7}\right)\pi^2\, \mathrm{chi}(\Delta) + \\
&\quad \left.\left. 49\left(-11 + 101\,\sqrt[7]{-1} - 112\,(-1)^{2/7} + 80\,(-1)^{3/7} - 56\,(-1)^{4/7} + 32\,(-1)^{5/7}\right)\zeta(3)\right)\right) - \frac{1}{108045} \times 2\, g^5 \\
&\left(\sqrt[7]{-1}\, \left(1 + \sqrt[7]{-1}\right)\left(5880\, \left((-1)^{2/7} - 1\right)\, \mathrm{chi}(\Delta)\, \left(5\left(5 - 11\,\sqrt[7]{-1} + 5\,(-1)^{2/7}\right)\mathrm{chi}''(\Delta) + \left(1 - 34\,\sqrt[7]{-1} + (-1)^{2/7}\right)\zeta(3)\right) - 150\left(-25 - 2\,\sqrt[7]{-1} + 2\,(-1)^{3/7} + 25\,(-1)^{4/7}\right)\right.\right. \\
&\quad \left.\left. \mathrm{chi}(\Delta)^4 - 210\left(-16 - 63\,\sqrt[7]{-1} + 63\,(-1)^{3/7} + 16\,(-1)^{4/7}\right)\pi^2\, \mathrm{chi}(\Delta)^2 + 49\left(-119 + 346\,\sqrt[7]{-1} - 35\,(-1)^{2/7} - 276\,(-1)^{3/7} + 49\,(-1)^{4/7} + 70\,(-1)^{5/7}\right)\pi^4\right)\right) - \\
&\frac{1}{756315} \times 4\, g^6 \left(\left(1 + \sqrt[7]{-1}\right)\left(4410\, \sqrt[7]{-1}\, \left((-1)^{2/7} - 1\right)\, \mathrm{chi}(\Delta)^2\, \left(8\left(17 - 15\,\sqrt[7]{-1} + 17\,(-1)^{2/7}\right)\mathrm{chi}''(\Delta) + \left(113 + 26\,\sqrt[7]{-1} + 113\,(-1)^{2/7}\right)\zeta(3)\right) + \right.\right. \\
&\quad 343\, \left(42\left(370 - 495\,\sqrt[7]{-1} + 389\,(-1)^{2/7} - 185\,(-1)^{3/7} - 19\,(-1)^{4/7} + 125\,(-1)^{5/7}\right)\mathrm{chi}^{(4)}(\Delta) + 10\,\sqrt[7]{-1}\,\left(-95 + 118\,\sqrt[7]{-1} - 118\,(-1)^{3/7} + 95\,(-1)^{4/7}\right)\right. \\
&\qquad \pi^2\, \mathrm{chi}''(\Delta) + 12\,285\, (-1)^{5/7}\, \zeta(5) + 17\,136\, (-1)^{4/7}\, \zeta(5) - 35\,595\, (-1)^{3/7}\, \zeta(5) + 54\,054\, (-1)^{2/7}\, \zeta(5) - 83\,475\, \sqrt[7]{-1}\, \zeta(5) + 71\,190\, \zeta(5) + \\
&\qquad \left. 920\, (-1)^{5/7}\, \pi^2\, \zeta(3) - 340\, (-1)^{4/7}\, \pi^2\, \zeta(3) - 630\, (-1)^{3/7}\, \pi^2\, \zeta(3) + 1600\, (-1)^{2/7}\, \pi^2\, \zeta(3) - 2180\, \sqrt[7]{-1}\, \pi^2\, \zeta(3) + 1260\, \pi^2\, \zeta(3)\right) + \\
&\quad 288\, \sqrt[7]{-1}\, \left(-5 - 27\,\sqrt[7]{-1} + 27\,(-1)^{3/7} + 5\,(-1)^{4/7}\right)\mathrm{chi}(\Delta)^5 + 70\,\sqrt[7]{-1}\, \left(-323 + 10\,\sqrt[7]{-1} - 10\,(-1)^{3/7} + 323\,(-1)^{4/7}\right)\pi^2\, \mathrm{chi}(\Delta)^3 + \\
&\quad \left.\left. 49\left(588 - 1409\,\sqrt[7]{-1} + 1386\,(-1)^{2/7} - 294\,(-1)^{3/7} - 798\,(-1)^{4/7} + 821\,(-1)^{5/7}\right)\pi^4\, \mathrm{chi}(\Delta)\right)\right) + O(g^7)
\end{aligned}$$

Pushing to large twist L=30



$+4g\left(\cos\frac{\pi}{L+2}+\cos\frac{3\pi}{L+2}\right)$

$-4g\left(\cos\frac{\pi}{L+2}+\cos\frac{3\pi}{L+2}\right)$

225 Trajec tories

$g = 0.02$

Can we penetrate from BFKL regime to DGLAP?

[Beccaria '07] [Kotikov, Lipatov, Rej, Staudacher, Velizhanin '07]

$$\frac{\gamma_2^{\mathrm{ABA}}(M)}{2} = 4\,S_1(\frac{M}{2})\,. \qquad \frac{\gamma_4^{\mathrm{ABA}}(M)}{4} = -2S_3 - 4S_1S_2\,,$$

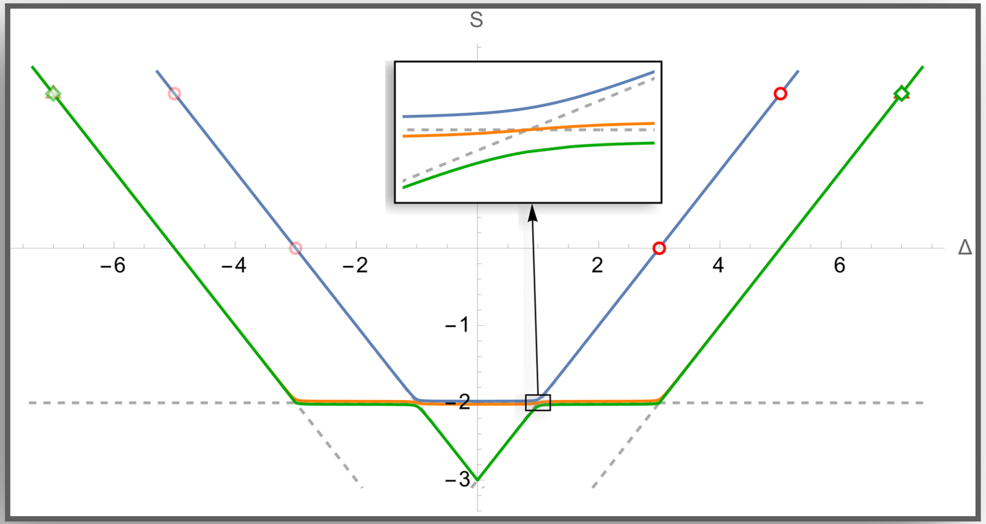
$$\frac{\gamma_6^{\mathrm{ABA}}(M)}{8} = 2S_2S_3 + S_5 + 4S_{3,2} + 4S_{4,1} - 8S_{3,1,1} + S_1\left(4S_2^2 + 2S_4 + 8S_{3,1}\right)$$

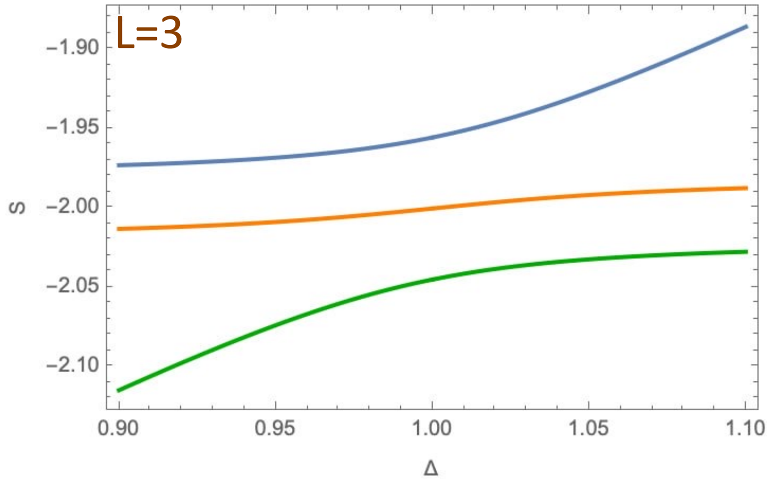
$$\begin{aligned}\frac{\gamma_8^{\mathrm{ABA}}(M)}{16} = {} & S_1^3\left(\frac{40}{3}S_4 - \frac{32}{3}S_{3,1}\right) + S_1^2\left(20S_5 - 40S_{3,2} - 56S_{4,1} + 64S_{3,1,1}\right) \\ & +S_1\Big(7S_6 + 8S_{2,4} - 24S_{3,3} - 56S_{4,2} - 40S_{5,1} - 24S_{2,2,2} - 16S_{2,3,1} \\ & +88S_{3,1,2} + 88S_{3,2,1} + 120S_{4,1,1} - 192S_{3,1,1,1} - 8\zeta(3)S_3\Big) - \frac{56}{3}S_3S_4 \\ & -\frac{107}{6}S_7 + 3S_{2,5} + \frac{41}{3}S_{3,4} + \frac{1}{3}S_{4,3} - 17S_{5,2} - \frac{20}{3}S_{6,1} - 4S_{2,2,3} \\ & -8S_{2,3,2} - 4S_{2,4,1} + \frac{104}{3}S_{3,1,3} + 52S_{3,2,2} + \frac{88}{3}S_{3,3,1} + 60S_{4,1,2} \\ & +60S_{4,2,1} + 40S_{5,1,1} + 8S_{2,3,1,1} - 120S_{3,1,1,2} - 120S_{3,1,2,1} \\ & -120S_{3,2,1,1} - 128S_{4,1,1,1} + 256S_{3,1,1,1,1}.\end{aligned}$$

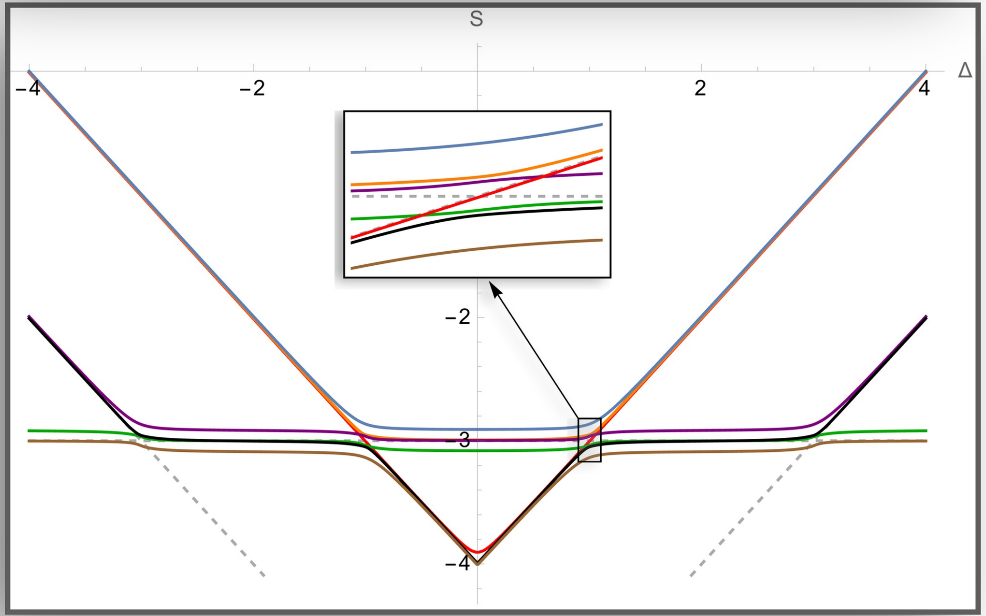
(It is interesting to note that the double-logarithmic behavior of these states is different from the twist-two ones (3.10).

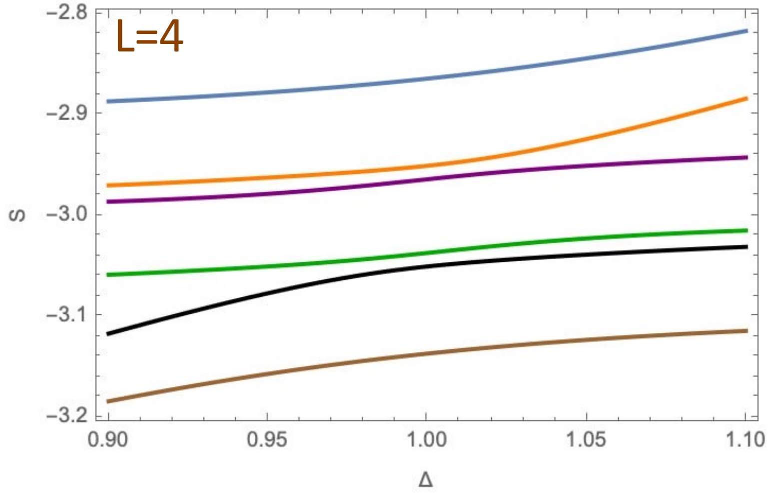
$$\gamma^{\mathrm{ABA}} = -8\frac{g^2}{\omega}\left(\frac{1}{1-t} - \zeta(2)\frac{1+3\,t^2}{(1-t)^2}\omega^2 + \ldots\right) + \ldots\,. \qquad t = \frac{g^2}{\omega^2}. \qquad M = 2\,(\omega - 1)$$

Can we penetrate from BFKL regime to DGLAP?









Can we penetrate from BFKL regime to DGLAP?

- Zoom near the edge: $\Omega \equiv \frac{S+2}{g}, D \equiv \frac{\Delta-1}{g}$

- The horizontal trajectories become: $\Omega_{1,2} \simeq \pm 2 - \frac{8}{D} + \frac{96}{D^3} + \mathcal{O}\left(\frac{1}{D^4}\right)$

- Riemann surface reduces to cubic poly (fixed by $\Omega_{1,2}$)

$$(\Omega - 2)(\Omega + 2)(\Omega - D) - 16\Omega = 0$$

- Solving for Δ

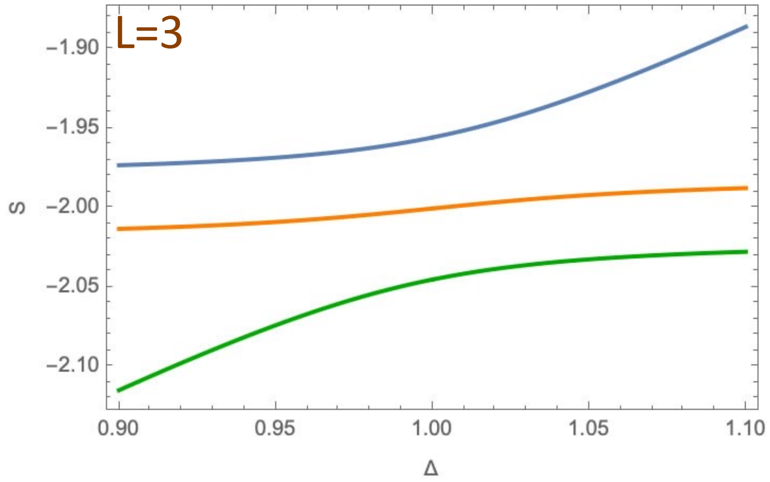
$$\Delta - 1 - \omega = -\frac{16\omega g^2}{\omega^2 - 4g^2} \simeq -\frac{16g^2}{\omega} - \frac{64g^4}{\omega^3} - \frac{256g^6}{\omega^5} - \frac{1024g^8}{\omega^7} \quad \omega = S + 2$$

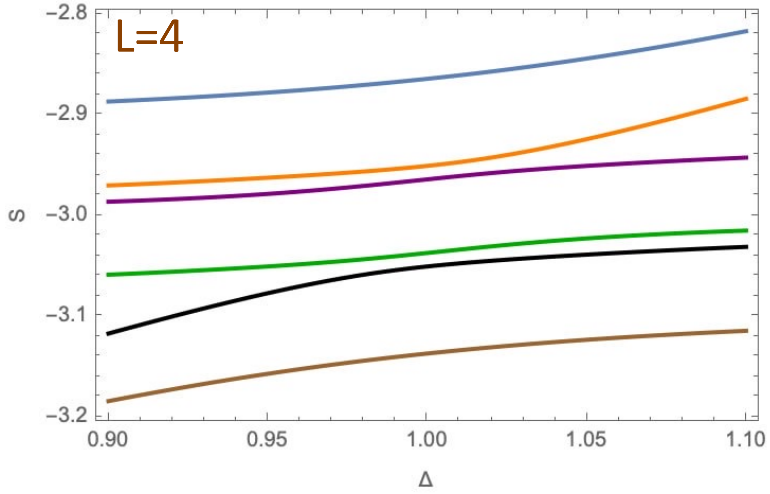
- Perfect agreement with explicit 4-loop for local operators!

[Beccaria '07] [Kotikov, Lipatov, Rej, Staudacher, Velizhanin '07]

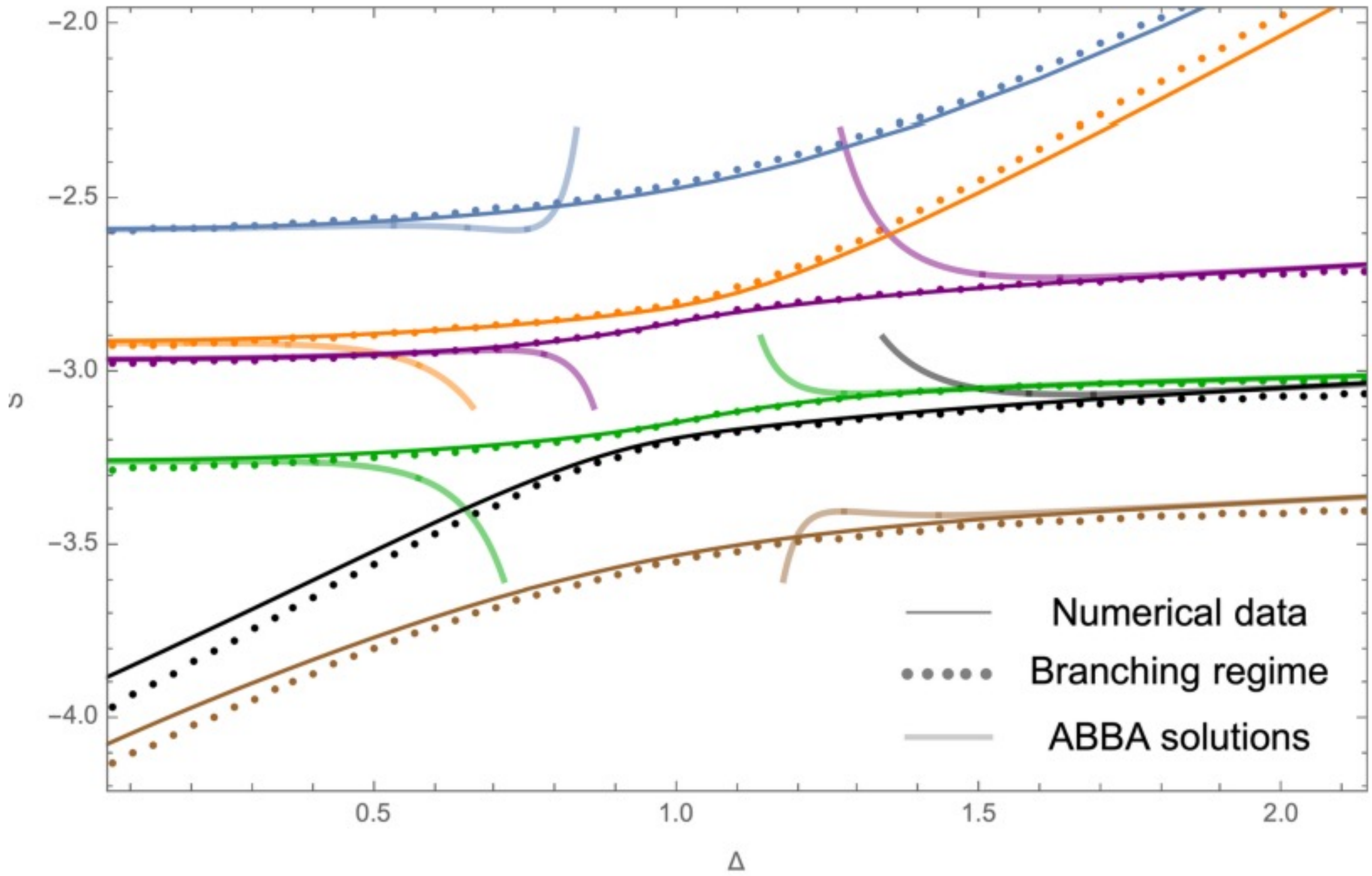
- Repeat the same procedure fo. r $L = 4$

$$(D^2 - 36)\Omega^4 + (192 - 12D^2)\Omega^2 - 2D\Omega^5 + 48D\Omega^3 - 128D\Omega + \Omega^6 - 256 = 0$$





Can we penetrate from BFKL regime to DGLAP?



Spooky observation

- L=3 branching pole

$$\Delta - 1 - \omega = -\frac{16\omega g^2}{\omega^2 - 4g^2} \simeq \boxed{-\frac{16g^2}{\omega}} - \frac{64g^4}{\omega^3} - \frac{256g^6}{\omega^5} - \frac{1024g^8}{\omega^7}$$

[Beccaria `07]
[Kotikov, Lipatov, Rej, Staudacher, Velizhani `07]

[Korchemsky, Kotanski, Manashov `04]

- L=4 two trajectories turn (have pole)

$$\Delta_{\pm}^{L=4} - 1 \simeq \omega - 4g^2 \frac{3 \pm \sqrt{5}}{\omega} - 16g^4 \frac{25 \pm 11\sqrt{5}}{5\omega^3} + \dots$$

- In general:

$$\Delta_L \simeq \frac{B_n g^2}{\omega}$$

- # of trajectories with pole at $s = -L + 1$ match # of operators $\mathrm{tr}\, D_+^2 Z^L$
- The pole B_n is essentially the 1-loop anomalous dimension of this operator $B_n = 32 - 2\gamma_n^{(1)}$ ← checked for $L = 2 \dots 11$

- We don't know why, but this suggest a new ABA type of regime near the poles too!

Open questions

- Derivation from perturbation theory?

  Which mechanism gives odd powers of g? Dipole evolution, Light-Ray operators?

- Massless mode physics? AdS_3? (in progress with Simon and Bogdan)

- Evolution kernel at higher order from data?

- Is there some physics at $L \to \infty$ as Landau-Lifshitz for usual case?

- Simplification of QSC near the corner points where BFKL meets DGLAP? Counting?

- DGLAP at non-integer spin?

- Accessing lower Horizontal Trajectories $S \sim -L + 1 - 2n, n \in \mathbb{N}$?

- Generalise to all states in other sectors and remove parity restrictions? (in progress with Simon and Micio)

- Hexagons? SoV?

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