DG Discretization of the Variable Coefficient Poisson Equation

Buğrahan Temür

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Start with the model problem with a tensor-valued continuous variable coefficient $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{x})$

$$-\nabla \cdot (\boldsymbol{\lambda} \nabla u) = f \qquad \text{in } \Omega \ , \tag{1}$$

$$u = g \qquad \text{on } \Gamma^D, \tag{2}$$

$$\nabla u \cdot \boldsymbol{n} = h \qquad \text{on } \Gamma^N. \tag{3}$$

Introduce an auxiliary variable $\boldsymbol{\sigma}$ to acquire two first-order equations

$$-\nabla \cdot (\boldsymbol{\lambda}\boldsymbol{\sigma}) = f \ , \tag{4}$$

$$\boldsymbol{\sigma} = \nabla u \ . \tag{5}$$

Multiply Eq. 5 with the variable coefficient $\boldsymbol{\lambda}$, then multiply both equations with appropriate test functions and integrate over cells

$$\left(v_h, -\nabla \cdot (\boldsymbol{\lambda}\boldsymbol{\sigma}_h)\right)_{\Omega_e} = (v_h, f)_{\Omega_e} \ , \tag{6}$$

$$(\boldsymbol{\tau}_h, \boldsymbol{\lambda}\boldsymbol{\sigma}_h)_{\Omega_e} = (\boldsymbol{\tau}_h, \boldsymbol{\lambda}\nabla u_h)_{\Omega_e} \ . \tag{7}$$

Integrate the LHS of Eq. 6 by parts and introduce the numerical flux $\boldsymbol{\sigma}_h^*$ of the auxiliary variable

$$(\nabla v_h, \boldsymbol{\lambda}\boldsymbol{\sigma}_h)_{\Omega_e} - \left(v_h, \boldsymbol{\sigma}_h^* \boldsymbol{\lambda}^T \boldsymbol{n}\right)_{\partial\Omega_e} = (v_h, f)_{\Omega_e} \ . \tag{8}$$

Integrate the RHS of Eq. 7 by parts and introduce the numerical flux u_h^* of the solution

$$(\boldsymbol{\tau}_h, \boldsymbol{\lambda}\boldsymbol{\sigma}_h)_{\Omega_e} = -\left(\nabla \cdot (\boldsymbol{\tau}_h \boldsymbol{\lambda}), u_h\right)_{\Omega_e} + (\boldsymbol{\tau}_h, \boldsymbol{\lambda} u_h^* \boldsymbol{n})_{\partial\Omega_e} \ . \tag{9}$$

As the next step to get to the primal formulation, integrate this once more by parts to obtain

$$(\boldsymbol{\tau}_h, \boldsymbol{\lambda}\boldsymbol{\sigma}_h)_{\Omega_e} = (\boldsymbol{\tau}_h, \boldsymbol{\lambda}\nabla u_h)_{\Omega_e} - (\boldsymbol{\tau}_h, \boldsymbol{\lambda}\,(u_h - u_h^*)\,\boldsymbol{n})_{\partial\Omega_e} \ . \tag{10}$$

Set $\boldsymbol{\tau}_h = \nabla v_h$ and substitute Eq. 10 for the first term in Eq. 8, which gives the primal form

$$\begin{aligned}(\nabla v_h, \boldsymbol{\lambda}\nabla u_h)_{\Omega_e} &- (\nabla v_h, \boldsymbol{\lambda}\,(u_h - u_h^*)\,\boldsymbol{n})_{\partial\Omega_e} \\ &- \left(v_h, \boldsymbol{\sigma}_h^*\boldsymbol{\lambda}^T\boldsymbol{n}\right)_{\partial\Omega_e} \\ &- (v_h, f)_{\Omega_e} \\ &= 0 \ .\end{aligned} \tag{11}$$

Define the fluxes according to the SIPG method with the average and jump operators

$$u_h^* = \{\{u_h\}\} \ , \tag{12}$$

$$\boldsymbol{\sigma}_h^* = \{\{\nabla u_h\}\} - \tau\,[\![u_h]\!] \ . \tag{13}$$

Substitute the fluxes to finally get

$$\begin{aligned}(\nabla v_h, \boldsymbol{\lambda}\nabla u_h)_{\Omega_e} &- \frac{1}{2}\,(\nabla v_h, \boldsymbol{\lambda}\,[\![u_h]\!])_{\partial\Omega_e} \\ &- \left(v_h, \{\{\nabla u_h\}\}\,\boldsymbol{\lambda}^T\boldsymbol{n}\right)_{\partial\Omega_e} \\ &- \left(v_h, \tau\,[\![u_h]\!]\,\boldsymbol{\lambda}^T\boldsymbol{n}\right)_{\partial\Omega_e} \\ &- (v_h, f)_{\Omega_e} \\ &= 0 \ .\end{aligned} \tag{14}$$