Université Laval
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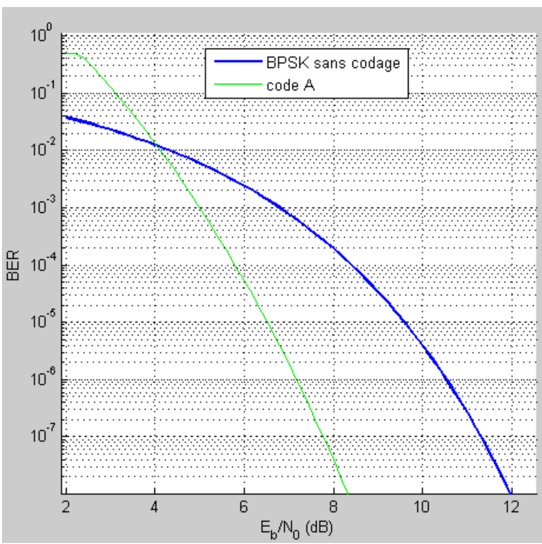
Département de génie électrique
et de génie informatique

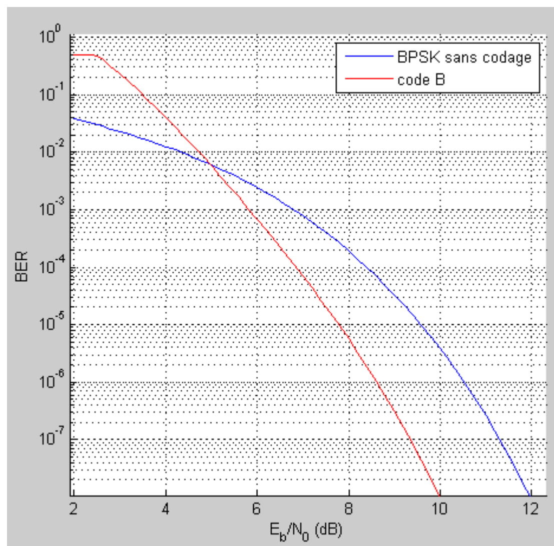
GEL4200: Communications numériques
2016 Examen final

Wednesday 22 April 2016; Time: 11h30 to 13h20
Documentation provided; calculator permitted

Problem 1 (10 points over 100)

The following are bit error rate curves (BER) vs. the ratio Eb/N0 for BPSK transmission. The figure to the left presents a curve for BPSK without a correcting code, and a curve when using convolutional code A. The figure to the right presents a curve for BPSK without a correcting code, and a curve when using ANOTHER convolutional code B.





A. (5 points) The two convolutional codes illustrated have constraint lengths K=4 and K=7. Which code corresponds to which value of K? Justify your response.

B. (5 points) What is the definition of FEC threshold? What is the FEC threshold for each code?

.

Problem 2 (20 points over 100)

The following are equations for the parity bits in a block code.

$$p_1 = m_1 + m_2 + m_4$$
$$p_2 = m_1 + m_3 + m_4$$
$$p_3 = m_1 + m_2 + m_3$$
$$p_4 = m_2 + m_3 + m_4$$

A. (3 points) Find *n*, the code word length, *k*, the data message length, et *r,* the code rate.

B. (8 points) Give the generator matrix for a systematic code.

C. (9 points) Find the minimal distance for the code.

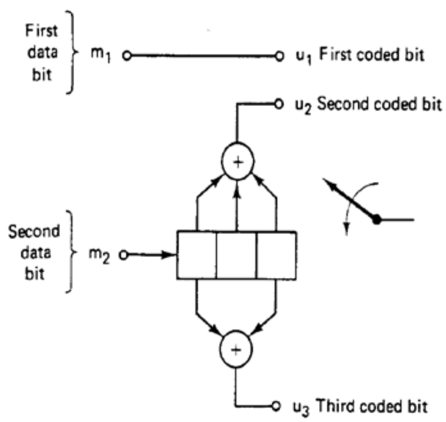
Problem 3 (25 points over 100)

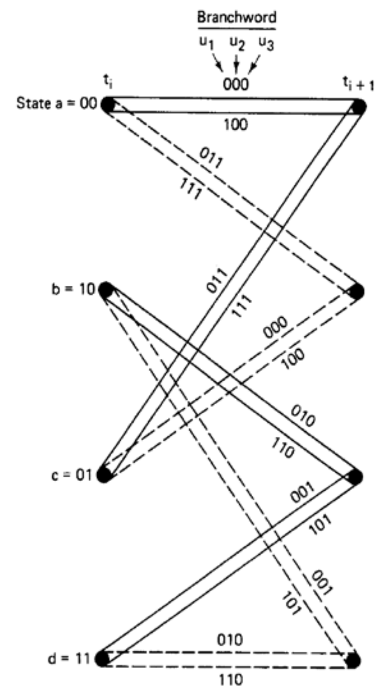
A. (10 points) Why is knowledge of the channel required for the maximum likelihood sequence estimator? How is the channel information exploited?

B. (15 points) Contrast the zero forcing equalizer and the minimum mean square error equalizer. How are they similar? How are they different? When are they equivalent? Which is more effective?

Problem 4 (25 points over 100)

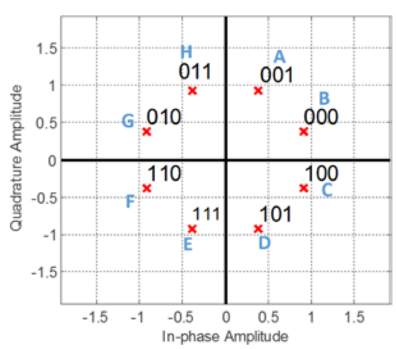
Consider this TCM encoder.



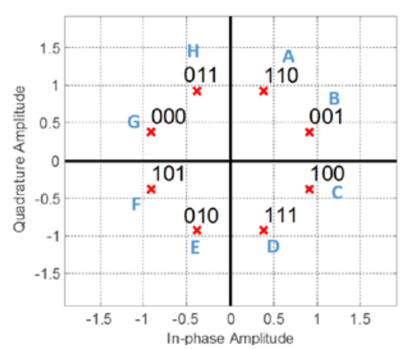


A. (10 points) Assuming that we start at state a, find the encoder output when the input is 00 11 01 10 00

B. (5 points) Choose the best mapping between code words and 8PSK symbols. Justify your response.



Option 1

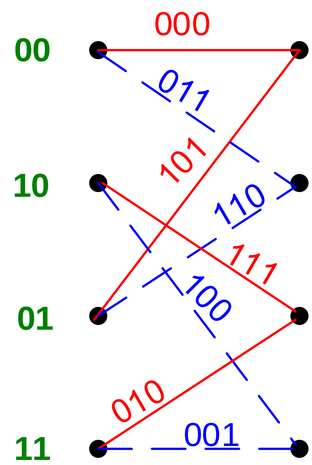


Option 2

C. (10 points) Using the option chosen in part 4B, find the symbol sequence (A, B etc.) transmitted when the input is 00 11 01 10 00.

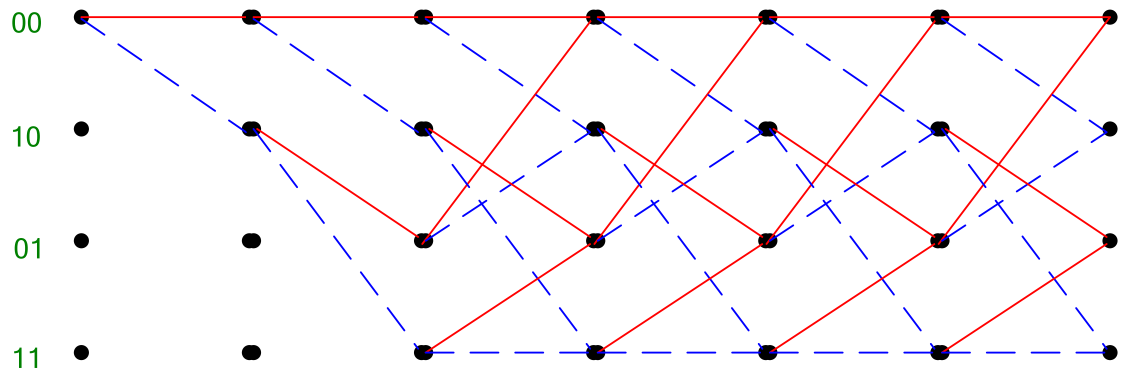
Problème 5 (20 points sur 100)

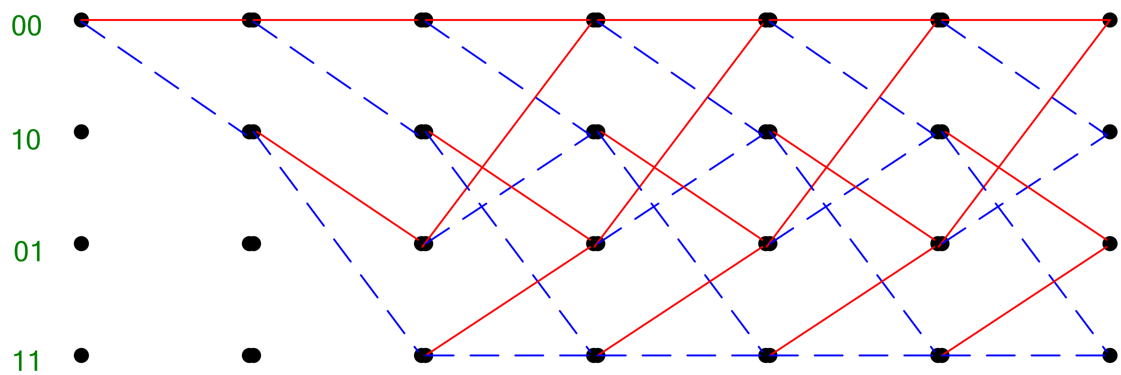
Consider the following encoder trellis for a convolutional code.

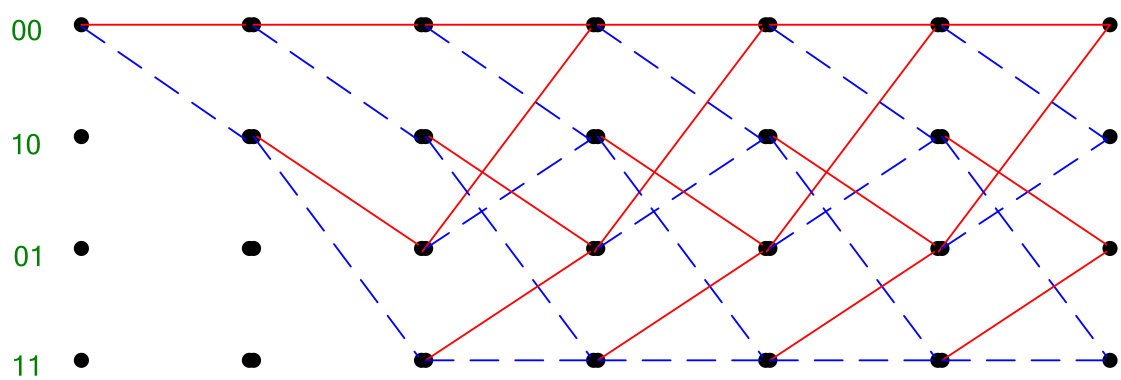


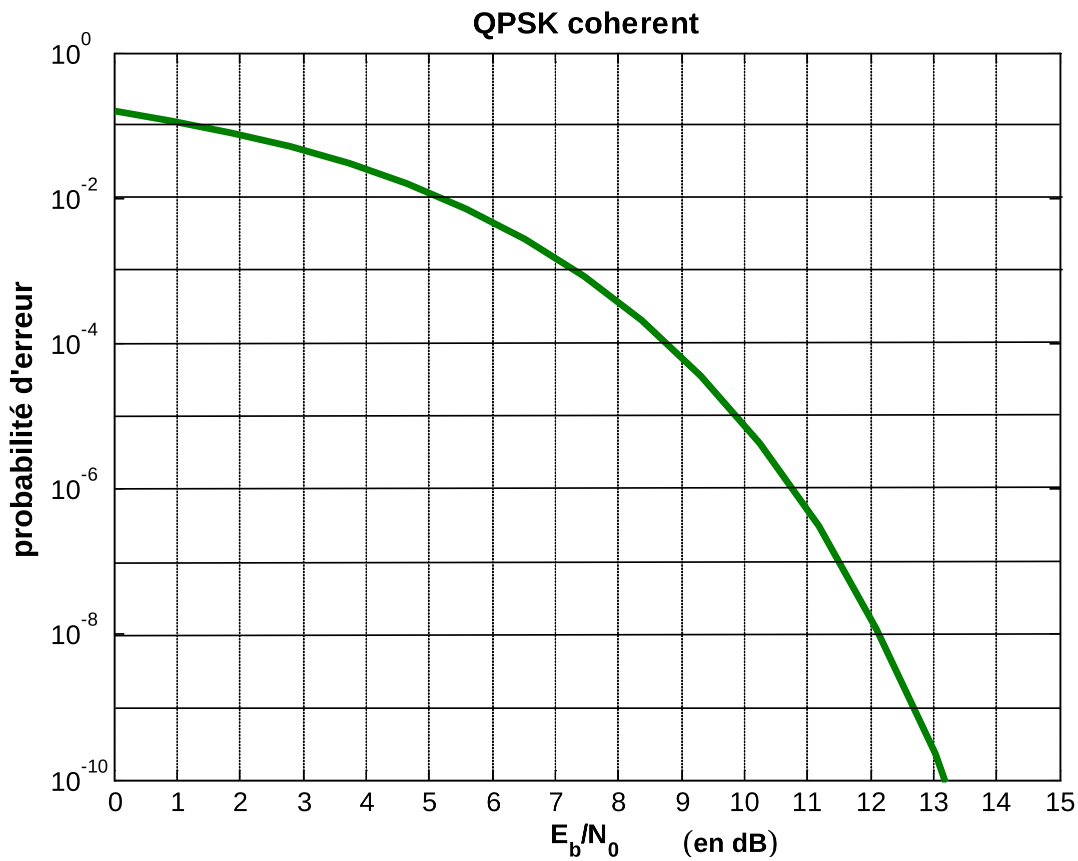
Find the free distance d_f of the convolutional code assuming hard decisions.
How many paths are there at the minimal distance?

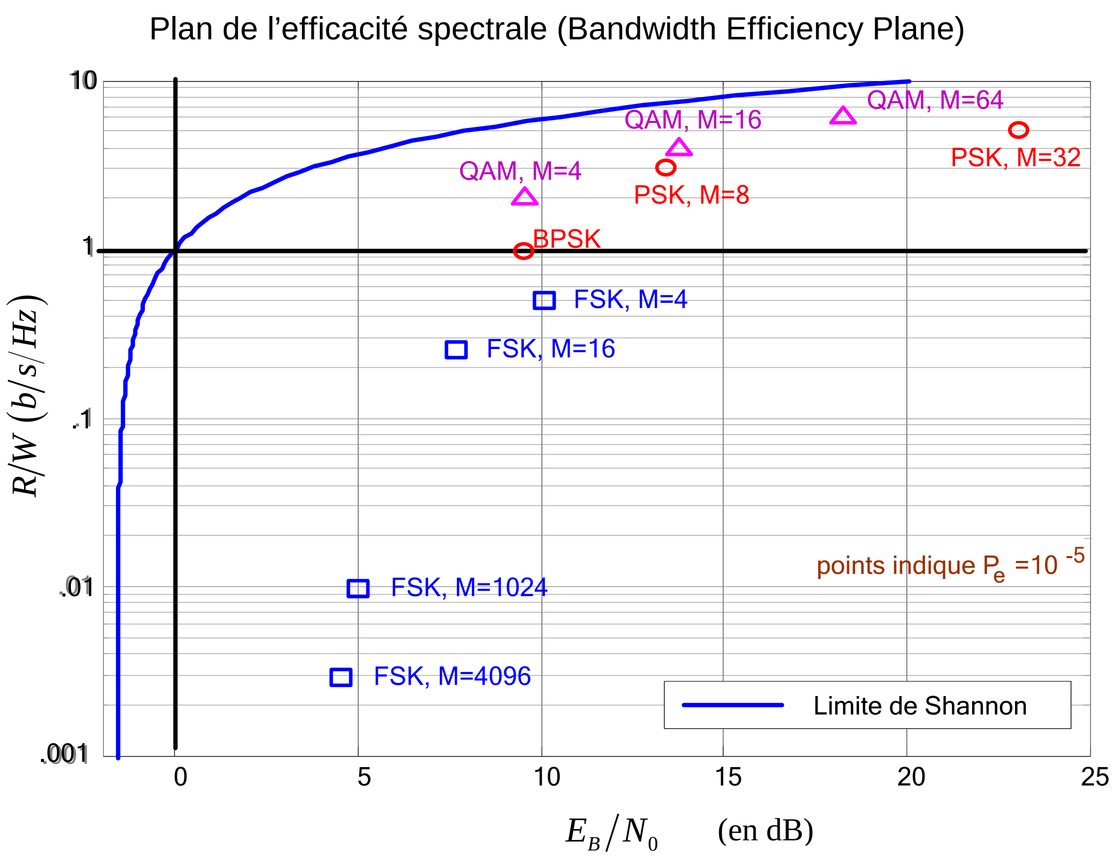
Please use the following sheet of decoder trellises in finding the free distance.
Place these sheets in your blue exam book.


00
10
01
11

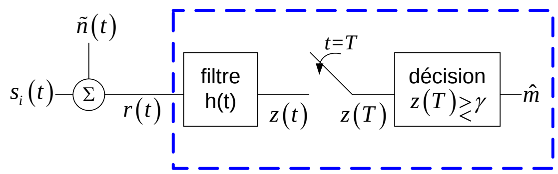

00
10
01
11


00
10
01
11


QPSK coherent
probabilité d'erreur
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
E_b/N_0 (en dB)


Plan de l'efficacité spectrale (Bandwidth Efficiency Plane)
$R/W\ (b/s/Hz)$
10
1
.1
.01
.001
0 5 10 15 20 25
E_B/N_0 (en dB)
QAM, M=64
QAM, M=16
QAM, M=4
PSK, M=32
PSK, M=8
BPSK
FSK, M=4
FSK, M=16
FSK, M=1024
FSK, M=4096
points indique $P_e = 10^{-5}$
Limite de Shannon

Récepteur d'échantillonnage



MAP: i qui maximise $p(z|s_i)\,p(s_i)$

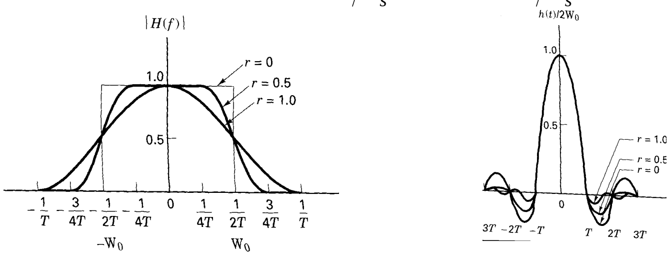
i qui minimise $\left\|\mathbf{r}-\mathbf{s}_i\right\|^2 - N_0 \ln P(\mathbf{s}_i)$

$P(\mathbf{s}_i) =$ probabilité a priori de symbole $\mathbf{s}_i$

ML: i qui maximise $p(z|s_i)$

i qui minimise $\left\|\mathbf{r}-\mathbf{s}_i\right\|^2$

Raised cosine $v(t) = \dfrac{\sin(\pi t/T_s)}{\pi t/T_s}\dfrac{\cos(r\pi t/T_s)}{1-4r^2t^2/T_s^2}$



Énergie moyenne

$$E_{moy} = \frac{1}{M}\sum_{i=1}^{M}\left\|\mathbf{s}_i\right\|^2 = \frac{1}{M}\sum_{i=1}^{M}\left[\text{énergie du signal } i\right]$$

Énergie par bit v. énergie par symbole

$$E_b \log_2 M = E_s$$

QAM $\eta = \log_2 M$ †

Conversion de l'espace I/Q vers espace du signal

$$\left(\tilde{a}_n^I, \tilde{a}_n^Q\right) = \sqrt{\frac{M\cdot E_s}{\sum_{i=1}^{M}\left[\left(a_n^I\right)^2+\left(a_n^Q\right)^2\right]}}\left(a_n^I, a_n^Q\right)$$

coordonnées, espace du signal

coordonnées, espace I/Q

cas rectangulaire (carrée) $M=L^2$

$$P_e = 2\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3\log_2 M}{(M-1)}\frac{E_b}{N_0}}\right) \qquad d_{\min} = \sqrt{\frac{6\log_2 L}{L^2-1}}$$

Borne d'union

$$P_e \approx \frac{2K}{M}Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right) = \frac{2K}{M}Q\left(d_{\min}\sqrt{\frac{E_b}{N_0}}\right)$$

K est le nombre des paires des signaux séparés par la distance minimale D_{min}

Distance minimale dans l'espace du signal

$$D_{\min} = \min_{i\neq k}\left\|\mathbf{s}_i - \mathbf{s}_k\right\| \text{ et } d_{\min} = \frac{D_{\min}}{\sqrt{2E_b}}$$

$$P_e(BPSK) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$P_e(OOK) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Perte par rapport à QPSK

$$d_{\min} = \sqrt{x}\sqrt{2} \qquad \text{perte} = -10\log_{10} x$$

Pour une modulation orthogonale

$$P_e(bit) = P_b = P_e(symbol)\frac{M/2}{M-1}$$

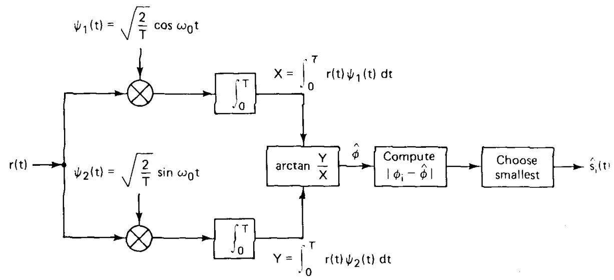
Pour une modulation non-orthogonale avec codage de gray

$$P_e(bit) = P_b = \frac{P_e(symbol)}{\log_2 M}$$

Efficacité spectrale

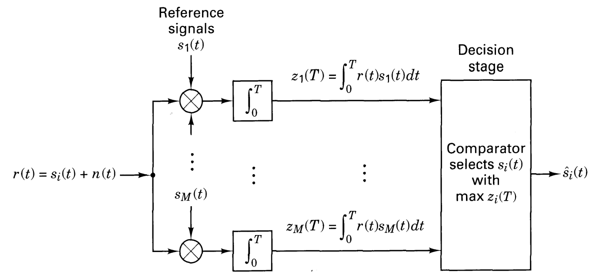
$$\eta = \frac{R_b}{W} = \frac{1}{T_b}\frac{1}{W}\text{ bits/s/Hz}$$

MPSK cohérent $\eta = \log_2 M$ †



$$P_e(M) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin\frac{\pi}{M}\right) = 2Q\left(\sqrt{\frac{2E_b \log_2 M}{N_0}} \sin\frac{\pi}{M}\right)$$

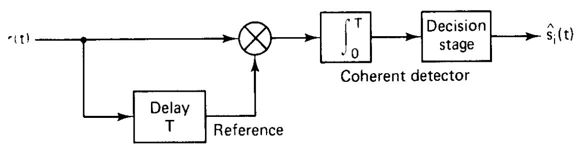
MFSK cohérent $\eta = \dfrac{2\log_2 M}{M+1}$ †



$$P_e = (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right) = (M-1)Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$

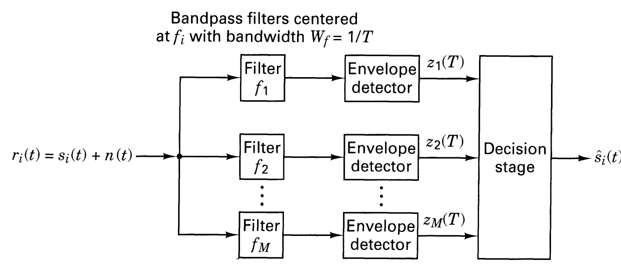
Séparation minimale $1/2T_s$

DPSK incohérent $P_e = \dfrac{1}{2} e^{-E_b/N_0}$



~1 dB de perte entre DPSK et BPSK

MFSK incohérent $\eta = \dfrac{\log_2 M}{M}$ †



$$P_e(BFSK) = \frac{1}{2} e^{-E_b/2N_0}$$

~1 dB de perte BFSK cohérente vs. incohérente

Séparation minimale $1/T_s$

Loi de Shannon

$$C = W \log_2(1 + SNR) \qquad SNR = \frac{E_b}{N_0}\eta$$

$$\frac{E_b}{N_0} = \frac{W}{C}\left(2^{C/W} - 1\right) \qquad \frac{C}{W} \to 0 \quad \Rightarrow \quad \frac{E_b}{N_0} \to -1.6dB$$

Relations trigonométriques

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
0	1	0	0
$\pi/8$	.85	.38	.41
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$
$\pi/2$	0	1	∞

$$\tan(y) = x \Leftrightarrow y = \arctan x + k\pi$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin(\alpha+\beta) = \sin\alpha \sin\beta + \cos\alpha\cos\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Processus Gram Schmidt

$$\psi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t) \text{ où } E_1 = \int_0^T s_1^2(t)\,dt$$

$$\theta_2(t) = s_2(t) - \langle s_2(t), \psi_1(t)\rangle \psi_1(t)$$

$$E_2 = \int_0^T \theta_2^2(t)\,dt \qquad \psi_2(t) = \frac{\theta_2(t)}{\sqrt{E_2}}$$

$$i. \quad \theta_i(t) = s_i(t) - \sum_{k=1}^{i-1} \langle s_i(t), \psi_k(t)\rangle \psi_k(t)$$

$$E_i = \int_0^T \theta_i^2(t)\,dt \qquad \psi_i(t) = \frac{\theta_i(t)}{\sqrt{E_i}}$$

† en supposant une impulsion Nyquist idéale

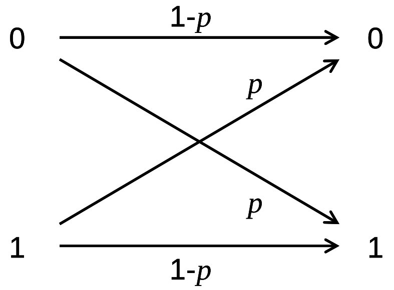
Corrélation croisée

$$z_{ij} \triangleq \int_0^T s_j(t) s_i(t)\, dt$$

Matrices de Hadamard

$$H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & \bar{H}_n \end{bmatrix}$$

Le canal binaire symétrique (BSC)



BPSK avec AWGN: $p = Q\left(\sqrt{2E_b/N_0}\right)$

Distance de Hamming

d(**u**,**v**) = # de positions de bits avec des valeurs différents dans les deux vecteurs **u** et **v**

Distance minimale

$$\min_{i,j} d\left(\mathbf{u}_j, \mathbf{v}_j\right) = \min_{j>2} \mathrm{w}\left(\mathbf{u}_j\right)$$

Probabilité d'erreur de bit p

Probabilité d'avoir plus que t **erreurs** de bits parmi un block de N bits

$$\sum_{k=t+1}^{N} \binom{N}{k} p^k (1-p)^{N-k} \approx \binom{N}{t+1} p^{t+1} (1-p)^{N-t-1}$$

$$\binom{N}{k} \equiv \frac{N!}{k!(N-k)!}$$

Codes en bloc

m = message à encoder, **u** = mot de code généré

$$\mathbf{G} = \left[\mathbf{P} \mid \mathbf{I_k}\right] \qquad \mathbf{U} = \mathbf{mG}$$

$$\mathbf{H} = \left[\mathbf{I_{n\text{-}k}} \mid \mathbf{P}^T\right] \qquad \mathbf{S} = \mathbf{rH}^T$$

t = # d'erreurs qui peuvent être corrigés

$$t = \left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor$$

Code Hamming (n,k)=(2^m-1,2^m-1-m)

Tableau Standard

- Première rangé – mots de codes valides
- Première colonne – erreurs corrigibles
- Tous les 2^n mots de codes possibles sont inclus dans la table
- Il n'y a pas de répétition des mots de code

Corriger une erreur

1. Détecter l'erreur $\mathbf{S} = \mathbf{rH^T} \neq 0 \Rightarrow$ erreur v
2. Identifier la rangé avec $\mathbf{e_j H^T} = \mathbf{rH^T}$

   i.e. le syndrome identifie le coset
3. Corriger l'erreur en calculant $\mathbf{U} = \mathbf{r} + \mathbf{e_j}$

(le mot de code dans la colonne de tableau standard où on trouve)

Codes convolutifs

Exemple: $k=1,\ n=2,\ K=3$

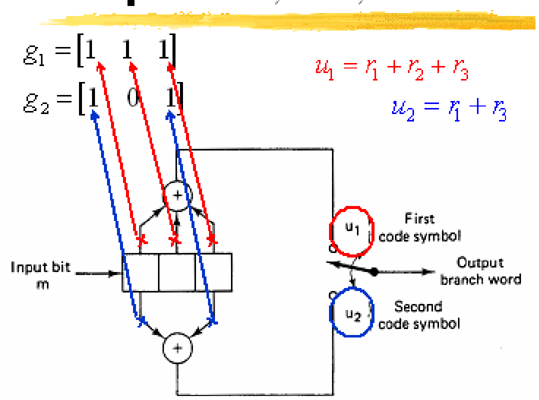
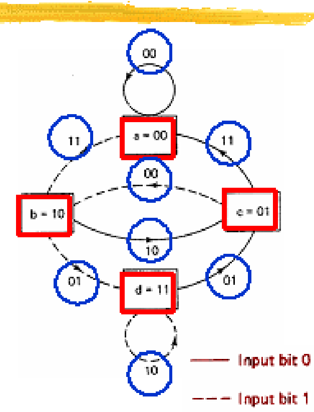


Diagramme de l'état

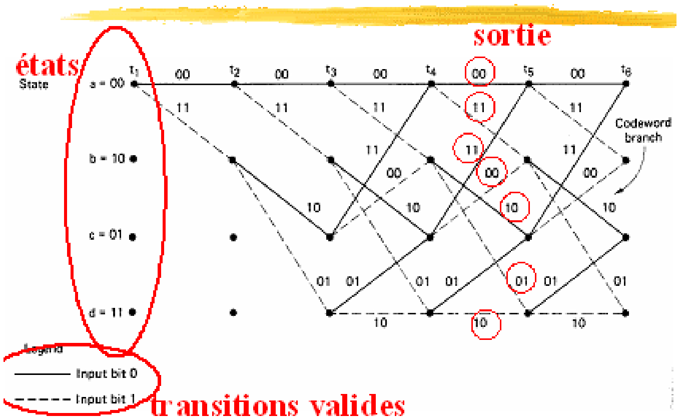
- Fixer les 2^{K-1} états
- Établir les transitions valides
- Générer les codes pour chaque transition



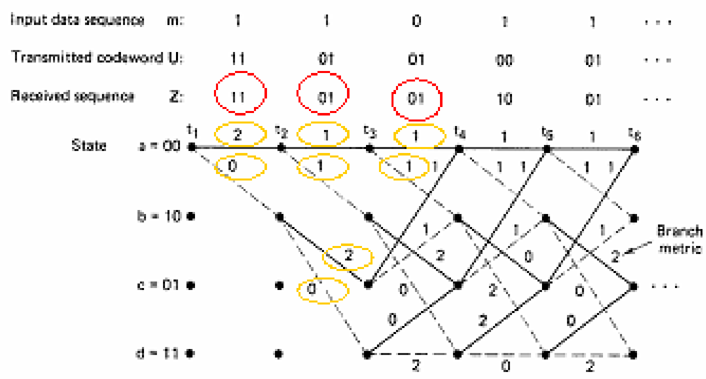
$$g_1 = [1 \quad 1 \quad 1]$$

$$g_2 = [1 \quad 0 \quad 1]$$

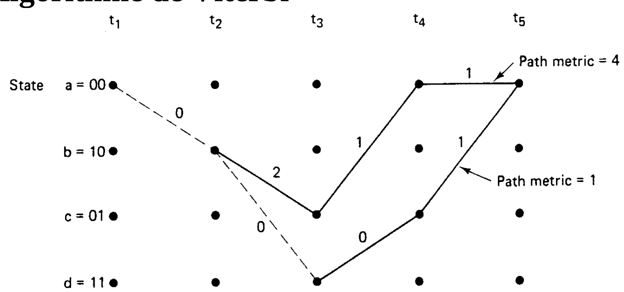
Treillis



Algorithme de Viterbi



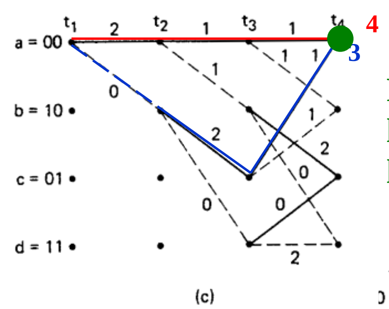
Algorithme de Viterbi



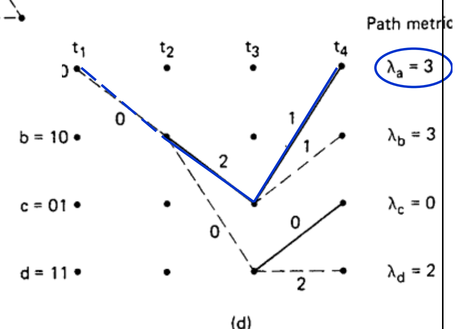
Deux métriques $dist(z_i, u_i)$

- Distance de Hamming
  - Pour les décisions fermes
  - $dist(\underline{z}_i, \underline{u}_i) = \#$ de bits differents
- Distance euclidienne
  - Pour les décisions souples
  - $dist(\underline{z}_i, \underline{u}_i) = \underline{z}_i^T \cdot \underline{u}_i = \sqrt{(z_{i,1} - u_{i,1})^2 + \cdots + (z_{i,n} - u_{i,n})^2}$

Algorithme de Viterbi



Focus:
le point de terminaison – le chemin vient d'ou?



Chapitre 7

Gain de codage: $10\log_{10} d_f^2 / d^2_{\text{min,sans codage}}$

Borne supérieur de gain de codage (en dB)

$10\log_{10} rd_f$ $\quad$ r = taux de codage = k/n

Valeurs typiques

- Décisions
  - Ferme
  - Souples avec 3 bits de quantification
- Longueur de contraint : $3 \le K \le 9$
- Taux de code : $r \ge 1/3$
- Chemin maximale : $h \le 5K$

Distance libre = distance minimale $= d_f$

- Codes linéaires
  - distance équivalent a la distance entre la séquence de zéros et n'importe quelle autre séquence
- Procédure
  1. Commence en état a
  2. Finir en état a
  3. Trajet le plus court ⇒ longueur = distance libre

t = # d'erreurs qui peuvent être corrigés

$$t = \left\lfloor \frac{d_f - 1}{2} \right\rfloor$$

TCM

Taux de codage = 1/*n*

- distance locale

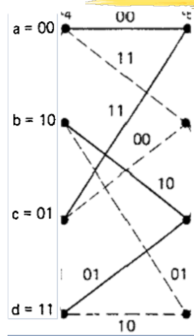
$$dist\left(\underline{u}_1, \underline{v}_1\right) = \sqrt{\left(u_{1_x} - v_{1_x}\right)^2 + \left(u_{1_y} - v_{1_y}\right)^2}$$

- Définition de la distance globale
  (distance entre séquences,
  ou *SED square Euclidean distance*)

$$dist^2\left(U,V\right) = dist^2\left(\underline{u}_1, \underline{v}_1\right) + dist^2\left(\underline{u}_2, \underline{v}_2\right) + dist^2\left(\underline{u}_3, \underline{v}_3\right) + \cdots$$

pour le TCM et les décisions souples

Exemple

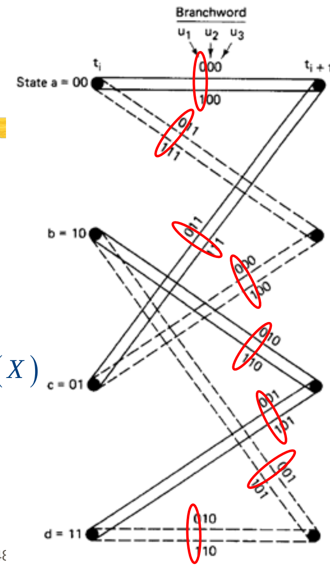


$u_1 = m_1$
(pas codé)

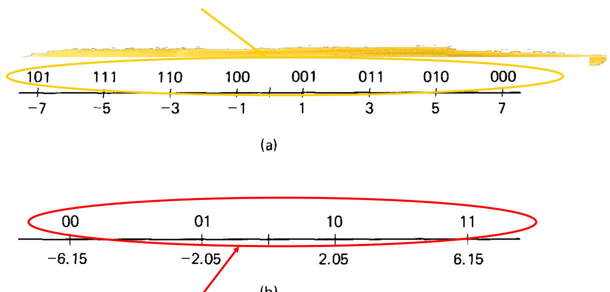
$u_2(X) = m_2(X)g_1(X)$
$u_3 = m_2(X)g_2(X)$
codage convolutif

$g_1 = [1 \quad 1 \quad 1]$
$g_2 = [1 \quad 0 \quad 1]$

GEL10280/6448



Correspondence pour TCM



Correspondence de Gray

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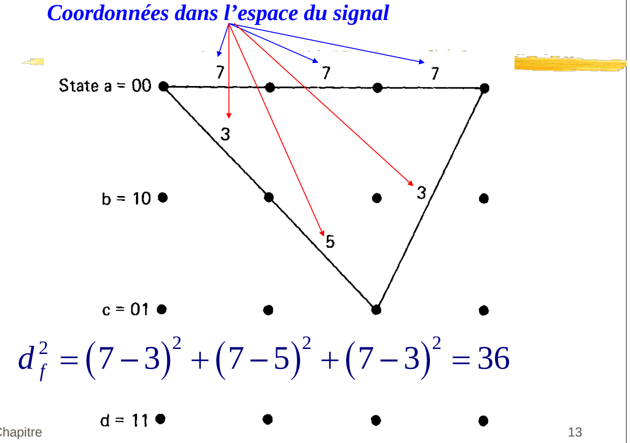
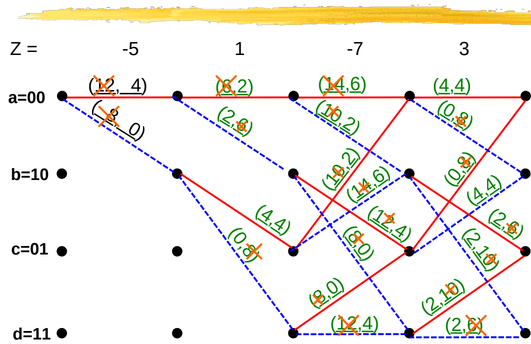
Coordonnées dans l'espace du signal



$$d_f^2 = (7-3)^2 + (7-5)^2 + (7-3)^2 = 36$$

Chapitre 13

Calculer les distances locales



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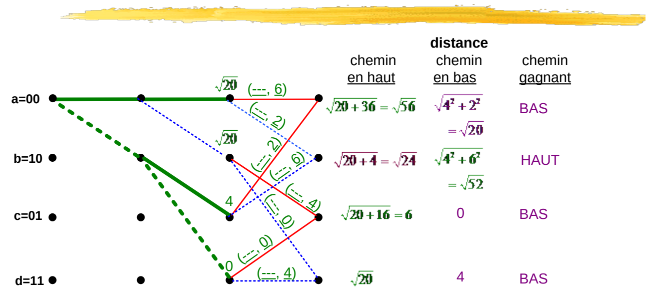
Calculer les distances globales *t*=3



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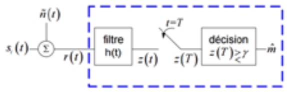
Récepteur ML

- Sans ISI
  - examiner UN intervalle du symbole
  - choisir le plus proche

i qui minimise $\|\mathbf{r}-\mathbf{s}_i\|^2$



- Avec ISI
  - examiner une SÉQUENCE de symboles
  - séquence aussi longue que le mémoire du canal

$$\{i(k)\}_{k=-L}^{0} \text{ qui minimise } \sqrt{\|\mathbf{r}(-LT)-\mathbf{s}_i\|^2 \cdots + \cdots \|\mathbf{r}(-kT)-\mathbf{s}_i\| \cdots + \cdots \|\mathbf{r}(0)-\mathbf{s}_i\|}$$

seule décision retenue

Critère zéro ISI

- Entrée de l'égaliseur p_c (après le canal)
- Sortie

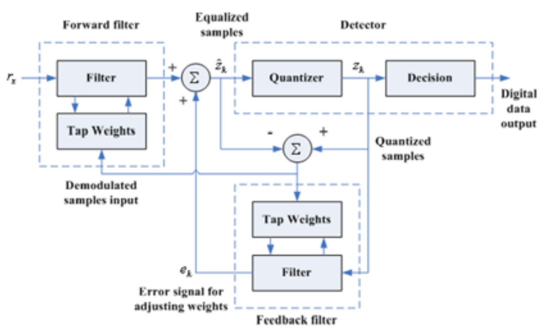
$$p_{eq}(t) = \sum_{n=-N}^{N} \alpha_n p_c(t-n\Delta)$$

- Forcer zero ISI

$$p_{eq}(mT) = \sum_{n=-N}^{N} \alpha_n p_c[(m-n)T]$$

$$= \begin{cases} 1, & m=0 \\ 0, & m \neq 0 \end{cases} \qquad m = 0, \pm 1, \pm 2, \ldots, \pm N$$

Egaliseur « decision feedback »



Chercher $\{\alpha\}$

$$[A] = [P_c]^{-1}[P_{eq}] = [P_c]^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \text{ middle column of } [P_c]^{-1}$$

Critère MMSE

$$E\left\{[z(t)-d(t)]^2\right\} = \text{minimum}$$

$$z(t) = \sum_{n=-N}^{N} \alpha_n y(t-n\Delta)$$

Solution MMSE

$$E\left\{[z(t)-d(t)]^2\right\} = \text{minimum}$$

$$[R_{yy}][A]_{\text{opt}} = [R_{yd}]$$

$$[A]_{\text{opt}} = [R_{yy}]^{-1}[R_{yd}]$$

Bruit AWGN

- Largueur de bande $B=1/2T$ pour le signal

$$\sigma_N^2 = \frac{N_0}{2T}\sum_{j=-N}^{N} \alpha_j^2$$

- BPSK $\quad Q\left(\sqrt{\frac{1}{\sum_j \alpha_j^2}\frac{2E_b}{N_0}}\right)$ perte en rapport signal-à-bruit

DMT modulation par porteuse

- n_i bits dans la constellation pour porteuse i
- Taux de transmission $\quad R_b = \sum_{i=1}^{N} n_i W_i$ bits/s
- Puissance totale divisé
  - Puissance P_j $\quad \sum_j P_j = P$
- Trouver les $\{P_j\}$ qui maximise R_b

OFDM Temps et fréquence