Corrigé Examen Final

MAT-2910 Analyse numérique pour l'ingénieur

Hiver 2016

Question 1

a)

$$
\begin{array}{cccccccc}
0 & -2 & -33 & & & & \\
 & & & 31 & & & \\
1 & -1 & -2 & & -15 & & \\
 & & & 1 & & 5 & \\
2 & 0 & -1 & & 0 & & 0 \\
 & & & 1 & & 5 & \\
3 & 1 & 0 & & 15 & & \\
 & & & 31 & & & \\
4 & 2 & 31 & & & &
\end{array}
$$

b)

$$P(x) = -33 + 31(x+2) - 15(x+1)(x+2) + 5x(x+1)(x+2)$$

$$f(0.5) \approx P(0.5) = -2.375$$

c)

$$|E(x)| = \left| \frac{f^{(5)}\big(\xi(x)\big)}{5!} x(x-1)(x-2)(x+1)(x+2) \right|$$

$$
\begin{array}{ccccc}
\left|\frac{0.025}{5!}x(x-1)(x-2)(x+1)(x+2)\right| & \leq & |E(x)| & \leq & \left|\frac{0.05}{5!}x(x-1)(x-2)(x+1)(x+2)\right| \\
\left|\frac{0.025}{120}(-3.28125)\right| & \leq & |E(1.5)| & \leq & \left|\frac{0.05}{120}(-3.28125)\right| \\
0.00068359375 & \leq & |E(1.5)| & \leq & 0.0013671875
\end{array}
$$

C'est impossible car l'erreur est au moins $0.68359375 \cdot 10^{-3}$.

Question 2

a)

$$S_0'(x) = a + b(x+1) + \frac{3}{16}(x+1)^2$$
$$S_1'(x) = 1 + \frac{3}{4}(x-1) - \frac{3}{8}(x-1)^2$$

$$S_0''(x) = b + \frac{3}{8}(x+1)$$
$$S_1''(x) = \frac{3}{4} - \frac{3}{4}(x-1)$$

$$b + \frac{3}{4} = S_0''(1) = S_1''(1) = \frac{3}{4} \iff b = 0$$

$$a + \frac{3}{4} = S_0'(1) = S_1'(1) = 1 \iff a = \frac{1}{4}$$

b)

$$S(0) = S_0(0) = \frac{1}{4} + \frac{3}{48} = \frac{5}{16}$$

Question 3

$$f(x) = x^4 + 6x + 1$$

a)

$$I \approx \frac{0.5}{3}\big(f(0) + 4f(0.5) + f(1)\big) = \frac{101}{24}$$

b)

$$I \approx \frac{0.25}{3}\big(f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\big) = \frac{252031}{60000}$$

c) i)

$$E(x) = -\frac{(b-a)}{180}f''''(\eta)h^4 = -\frac{1}{180}(24)(0.25)^4 = -\frac{1}{1920}$$

ii) La quadrature de Gauss-Legendre à 3 points est exacte pour les polynômes de degré 5 et moins. Puisque f(x) est un polynôme de degré 4, l'approximation sera exacte.

Question 4

a)

$$f''(0) \approx \frac{f(0.1) - 2f(0) + f(-0.1)}{(0.1)^2} = 84$$

b)

$$|E(h)| = \left| -\frac{h^2}{24} f^{(4)}(\eta) \right| = |60h^2\eta| < 60|h|^3 < 10^{-5} \iff h < \sqrt[3]{\frac{10^{-5}}{60}}$$

c) i) Cette formule est d'ordre 7.

ii)

$$E(h) = \frac{f^{(5)}(\eta)}{5!} h^7 = 12h^7$$

Question 5

a)

$$f(t, y_n) = 3t_n^2 y_n \;\&\; t_0 = 0 \;\&\; y_0 = 1$$

$$\hat{y} = y_n + hf(t_n, y_n) \implies y_{n+1} = y_n + \frac{h}{2}\big(f(t_n, y_n) + f(t_n + h, \hat{y})\big)$$

$h = 0.2$:

$$\hat{y} = (1) + (0.2)(0) = 1 \implies y_1 = (1) + \frac{(0.2)}{2}\big((0) + (0.12)\big) = 1.012$$

$h = 0.1$:

$$\hat{y} = (1) + (0.1)(0) = 1 \implies y_1 = (1) + \frac{(0.1)}{2}\big((0) + (0.03)\big) = 1.0015$$

$$\begin{aligned} \hat{y} &= (1.0015) + (0.1)(0.030045) = 1.0045045 \\ \implies y_2 &= (1.0015) + \frac{(0.1)}{2}\big((0.030045) + (0.12054054)\big) \\ &= 1.009029277 \end{aligned}$$

erreurs :

$$E(0.2) = \left| e^{0.2^3} - 1.012 \right| \approx 0.003967914$$

$$E(0.1) = \left| e^{0.2^3} - 1.009029277 \right| \approx 0.000997191$$

b) Puisque la méthode d'Euler modifiée est d'ordre et que $\frac{0.2}{0.1} = 2$, on s'attend à ce que $\frac{E(0.2)}{E(0.1)} \approx 2^2 = 4$.

$$\frac{E(0.2)}{E(0.1)} \approx \frac{0.003967914}{0.000997191} \approx 3.979091267$$

On observe effectivement le fait attendu.

c) On utilise l'extrapolation de Richardson.

$$\frac{2^n Q\left(\frac{h}{2}\right) - Q(h)}{2^n - 1} = \frac{2^2(1.009029277) - (1.012)}{2^2 - 1} = 1.008039036$$