Marginal and Conditional Probability Distributions

Just the univariate density, emphasizing that we look one variable only.

Y_1, Y_2: continuous r.v. with $f(y_1, y_2)$

Marginal density of Y_1 (or Y_2)

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 \quad \text{and} \quad f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1.$$

Must not depend on Y_2

Conditional density of Y_1 given $Y_2 = y_2$

$$f(y_1 | y_2) = \frac{f(y_1, y_2)}{f_2(y_2)}$$

: a ft of Y_2 Y_1

provided that $f_2(y_2) > 0$.

$f_{Y_1 Y_2}(y_1, y_2)$

$//$

$f(y_1, y_2) = 24 y_1 y_2 \quad 0 \le y_1 \le 1, \quad 0 \le y_2 \le 1, \quad y_1 + y_2 \le 1$

marginal: $f(y_1) = f_{Y_1}(y_1) = \int_0^{1-y_1} f(y_1, y_2)\, dy_2 = 12 y_1 (y_1 - 1)^2,$ $\frac{\Gamma(5)}{\Gamma(2)\Gamma(3)}$

$0 \le y_1 \le 1$

∵ nothing to do with Y_2.

Beta(2, 3)

Conditional density of $Y_1 | Y_2 = y_2$.

$$f(y_1 | y_2) = f_{Y_1 | Y_2 = y_2}(y_1 | y_2) = \frac{f(y_1, y_2)}{f(y_2)} = \frac{24 y_1 y_2}{12 y_2 (y_2 - 1)^2}$$

r.v.

a fixed value, not a r.v.

$$= \frac{2 y_1}{(y_2 - 1)^2}, \quad 0 \le y_1 \le 1 - y_2$$

∵ a ft of y_1, but also depends on y_2.

(also a ft of y_2).

Check: $\int_0^{1-y_2} \frac{2 y_1}{(y_2 - 1)^2}\, dy_1 = 1$.

$X \sim$ Some distribution of height

$$\underset{m}{\arg\min} E(X-m)^2 = \mu = E(X)$$

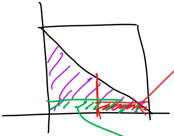
$$\underset{m}{\arg\min} E|X-m| = \text{median of } X$$

$$P(Y_1 > .5 \mid Y_2 = .1)$$

$$= \int_{.5}^{.9} f(y_1 \mid .1)\, dy_1$$

$$P(Y_1 > .5 \mid Y_2 < .1)$$

$$= \frac{P(Y_1 > .5,\ Y_2 < .1)}{P(Y_2 < .1)}$$



$$\iint 24 y_1 y_2 \, dy_1 dy_2 \quad \ldots\ldots$$

Ex 40) $Y_1 \sim B(n_1, p)$ $\quad Y_2 \sim B(n_2, p)$ $\quad$ indep

$$W = Y_1 + Y_2 \sim B(n_1 + n_2, p)$$

$$P(Y_1 = y_1 \mid W = w) = \frac{\binom{n_1}{y_1} p^{y_1} (1-p)^{n_1 - y_1} \binom{n_2}{w - y_1} p^{w - y_1} (1-p)^{n_2 - w + y_1}}{\binom{n_1 + n_2}{w} p^{w} (1-p)^{n_1 + n_2 - w}}$$

$$\overset{\parallel}{\underset{\parallel}{P(Y_1 = y_1, W = w)}} \Big/ P(W = w)$$

$P(Y_1 = y_1, Y_2 = w - y_1) / P(W = w) = \dfrac{\binom{n_1}{y_1}\binom{n_2}{w-y_1}}{\binom{n_1+n_2}{w}}$: HG

- $Y_1 \sim \text{Poisson}(\lambda_1)$ $\quad Y_2 \sim \text{Poisson}(\lambda_2)$ indep

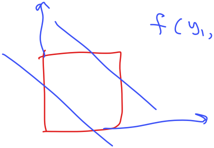
$W = Y_1 + Y_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$

$P(Y_1 = y_1 \mid W = w) = \ldots \ldots = \binom{w}{y_1}\left(\dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^{y_1}\left(\dfrac{\lambda_2}{\lambda_1+\lambda_2}\right)^{w-y_1}$

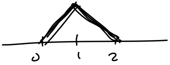
: Binomial$\left(w, \dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)$

In general, the distribution of $Y_1 + Y_2$, we can find the CDF of $W = Y_1 + Y_2$ using $f(y_1, y_2)$

$P(Y_1 + Y_2 \le u) = \ldots\ldots$



e.g. $Y_1 \sim \text{Unif}(0,1)$, $Y_2 \sim \text{Unif}(0,1)$ } indep



$$P(Y_1 + Y_2 \le u) = \int_0^u \int_0^{u-y_2} 1 \, dy_1 \, dy_2 = \begin{cases} 1 - \frac{1}{2}(2-u)^2 & 1 \le u \le 2 \\ \frac{1}{2}u^2 & 0 \le u \le 1 \end{cases}$$

Independent Random Variables

$f(y_1, y_2) = \begin{cases} 2-u & 1 \le u \le 2 \\ u & 0 \le u \le 1 \end{cases}$

Y_1 and Y_2: independent if and only if $F(y_1, y_2) = F_1(y_1)F_2(y_2)$.
Equivalently,

- Discrete: $p(y_1, y_2) = p_1(y_1)p_2(y_2)$.
- Continuous: $f(y_1, y_2) = f_1(y_1)f_2(y_2)$. Joint = product of marginals.
- $f(y_1, y_2) = g(y_1)h(y_2)$ where $g(y_1)$ is a nonnegative function of y_1 alone and $h(y_2)$ is a nonnegative function of y_2 alone.

not unique $\quad g(y_1) \geq 0$

Eg. $f(y_1, y_2) = \frac{1}{8} y_1 e^{-(y_1+y_2)/2}$ $\quad 0 < y_1 < \infty$, $0 < y_2 < \infty$

$\therefore \begin{cases} Y_1 \sim \text{Gamma}(2, 2) = \chi^2(4) \\ Y_2 \sim \text{Exp}(2) \end{cases}$

Bayesian

$Y|\lambda \sim \text{Poisson}(\lambda)$. $\underbrace{\lambda \sim \text{Exp}(1)}$ → prior

$$f(y, \lambda) = f(y|\lambda) f(\lambda)$$

$$= \frac{e^{-\lambda} \lambda^{y}}{y!} \cdot e^{-\lambda} = \frac{e^{-2\lambda} \lambda^{y}}{y!}, \quad \lambda > 0, \quad y = 0, 1, 2, \ldots$$

$$\underline{f(y)} = \int_0^\infty f(y, \lambda)\, d\lambda = \int_0^\infty \frac{1}{y!} \underbrace{\lambda^{y+1-1} e^{-\lambda/\frac{1}{2}}}_{\text{Gamma}(y+1, \frac{1}{2})} d\lambda = \ldots\ldots$$

$$= \left(\frac{1}{2}\right)^{y+1}, \quad y = 0, 1, \ldots$$

: Geometric

$\underline{f(\lambda|y)}$: posterior density

Expected Value of a Function of Random Variables

There are k variables

Joint density

$$E[g(Y_1, Y_2, \ldots, Y_k)] = \int_{y_k} \cdots \int_{y_2} \int_{y_1} g(y_1, y_2, \ldots, y_k) \underbrace{f(y_1, y_2, \ldots, y_k)} dy_1 dy_2 \cdots dy_k.$$

any ft of $Y_1, \ldots Y_k$

a and b: constants

- $E(ag(Y_1, Y_2) + b) = aE[g(Y_1, Y_2)] + b$
- $E[\sum_{i=1}^{k} g_i(Y_1, Y_2)] = \sum_{i=1}^{k} E[g_i(Y_1, Y_2)]$
- $E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)]$ if Y_1 and Y_2 are independent.

$\not\Rightarrow$

indep.

$E(Y_1 Y_2) = E(Y_1) E(Y_2)$ if $Y_1 \perp Y_2$