[Score table]

Prob.	1					2							3			4	Total
	3	3	3	3	3	5	10	5	5	5	10	10	10	10	10	5	100
Score																	

Problems & Solutions

[Chapter 1 | 15 points]

(True/False) Determine whether or not each of the following statement is true or false. Justify your answers. **[3 pts each, total 12 pts]**

(1-1) $e^{j\pi n/\sqrt{2}}$ is periodic

Suppose that the signal is periodic with period N. Then $e^{j2\pi m/\sqrt{2}} = e^{j2\pi(m+N)/\sqrt{2}} \Rightarrow 2\pi\frac{N}{\sqrt{2}}$ cannot be integer multiple of 2π **Answer: false**

(1-2) $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau) d\tau$ is invertible

Answer: True, invertible $h(t) = e^{-t}u(t)$ → **refer to the solution of prob. (2-6)**

(1-3) If $x(t)$ and $h(t)$ are odd functions, then $y(t) = x(t) * h(t)$ is also an odd function.

Answer: False.

$$y(-t) = \int_{-\infty}^{\infty} x(\tau)h((-t)-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} \underbrace{-x(-\tau)}_{x \text{ is odd}} \times \underbrace{-h(t+\tau)}_{h \text{ is odd}} d\tau \underset{\tau=-\tau'}{=} \int_{-\infty}^{\infty} \underbrace{x(\tau')}_{x \text{ is odd}} \times \underbrace{h(t-\tau')}_{h \text{ is odd}} d\tau' = y(t)$$

$y(t)$ is an even function

(1-4) If $y(t) = h(t) * x(t)$, then $\frac{d}{dt} y(t) = h(t) * \frac{d}{dt} x(t)$

Answer: True

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

$$\frac{d}{dt} y(t) = \int_{-\infty}^{\infty} \left(\frac{d}{dt} x(t-\tau)\right) h(\tau)d\tau = h(t) * \frac{d}{dt} x(t)$$

(1-5) [**3 pts**] Express the magnitude $|z|$ and phase $\measuredangle z$ of $z = \frac{j\omega - 1}{1 + j\omega}$ in terms of ω.

$$|z| = \sqrt{zz^*} = \sqrt{\frac{1+\omega^2}{1+\omega^2}} = 1$$

$$z = \frac{-(1-\omega^2 - 2j\omega)}{1+\omega^2} \Rightarrow \measuredangle z = \tan^{-1}\left(\frac{-2\omega}{1-\omega^2}\right)$$

[Chapter 2 | 45 points]

(2-1) **[5 pts]** For the following pairs of discrete sequences, use the convolution sum to find the response $y[n]$ of the LTI system with impulse response $h[n]$ to the input $x[n]$. Derive the result for both cases: $a > b$ and $a = b$.

$x[n] = e^{-an}u[n]$, $h(t) = e^{-bn}u[n]$ (a, b : positive real scalar)

Refer to the lecture slides 7

(2-2) **[10 pts]** Find the impulse response $h(t)$ of a causal LTI system described by the following differential equation.

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = x(t)$$

- Impulse response to $x(t) = \delta(t)$ = homogeneous solution with $y(0) = 1$

Let's solve $\frac{dy(t)}{dt} + \frac{1}{2}y(t) = 0$ with $y(0) = 1$

Substituting a test function $y(t) = Ye^{st}$ into the differential equation, we have

$(s + \frac{1}{2})Ye^{st} = 0 \Rightarrow s = -\frac{1}{2}$ & from the auxiliary condition $Y = 1$

$\therefore h(t) = e^{-\frac{1}{2}t}u(t)$ ($u(t)$: from condition of initial rest)

System is causal & LTI $\leftrightarrow$ System satisfies the condition of initial rest

($y(t) = 0$ for $t \le t_0$ *if* $x(t) = 0$ for $t \le t_0$)

(2-3) **[5 pts]** Determine the frequency response $H(j\omega)$ of the system described in (2-2).

$$\left.\begin{array}{r} x(t) = e^{j\omega t} \\ y(t) = H(j\omega)e^{j\omega t} \end{array}\right\} \rightarrow H(j\omega)(j\omega + \tfrac{1}{2})e^{j\omega t} = e^{j\omega t}$$

$$\therefore H(e^{j\omega}) = \frac{1}{\frac{1}{2} + j\omega}$$

(2-4) **[5 pts]** Describe the characteristics of a given system: low-pass, high-pass, band-pass ?

$$|H(j\omega)| = \frac{1}{\sqrt{\frac{1}{4} + \omega^2}}$$

→ $|H(j\omega)| \approx 2$ for low frequency (small ω)

→ $|H(j\omega)| \approx \frac{1}{\omega}$ for high frequency (big ω)

$\therefore$ Low-pass

(2-5) [**5 pts**] Determine whether the system's response is stable or not. Justify your answer.

For LTI system, stability is evaluated by convergence of absolute sum of an impulse response.

$h(t) = e^{-\frac{1}{2}t}u(t)$, $\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} e^{-\frac{1}{2}t} dt = -2e^{-\frac{1}{2}t}\Big|_0^{\infty} = 2$ is bounded. → $\therefore$ Stable

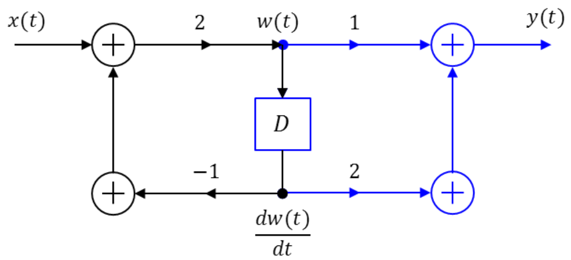
(2-6) **[10 pts]** Find the impulse response $h(t)$ of a causal LTI system described by the following differential equation. Draw the block diagram representing the given system in a direct form II realization, and use the result of (2-2) to find out the response.

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = x(t) + 2\frac{dx(t)}{dt}$$

- Direct form II realization

$$\frac{dw(t)}{dt} + \frac{1}{2}w(t) = x(t) \quad \rightarrow \quad y(t) = w(t) + 2\frac{dw(t)}{dt}$$

$$w(t) = 2\left(-\frac{dw(t)}{dt} + x(t)\right) \qquad y(t) = w(t) + 2\frac{dw(t)}{dt}$$



From Prob. 2-2, $w(t) = e^{-\frac{1}{2}t}u(t) \rightarrow y(t) = e^{-\frac{1}{2}t}u(t) - e^{-\frac{1}{2}t}u(t) + 2e^{-\frac{1}{2}t}\delta(t) = 2\delta(t)$

(2-7) **[10 pts]** Using the result of (2-2) and (2-6), determine whether the system of (2-2) is invertible or not. If it is invertible, describe the differential equation of the inverse system. If it is not, find two input signals to the system that have the same output.

- If a LTI system is invertible, it has a LTI inverse $h^{-1}(t)$ that satisfies $h^{-1}(t) * h(t) = \delta(t)$.

From (2-6), $y(t) = \frac{1}{2}\left(w(t) + 2\frac{dw(t)}{dt}\right) = \left(\frac{1}{2} + \frac{d}{dt}\right)w(t)$ reverts the impulse response of (2-1) to a unit impulse $\delta(t)$. Therefore, the system is invertible and the inverse system is described by

$$y(t) = \left(\frac{1}{2} + \frac{d}{dt}\right)x(t)$$

[Chapter 3 | 30 pts]

(3-1) **[10 pts]** Suppose that the Fourier series coefficients of the signal $\tilde{x}_T(t)$ with fundamental period $T = 4$ (Figure 3-1) is given by a_k .

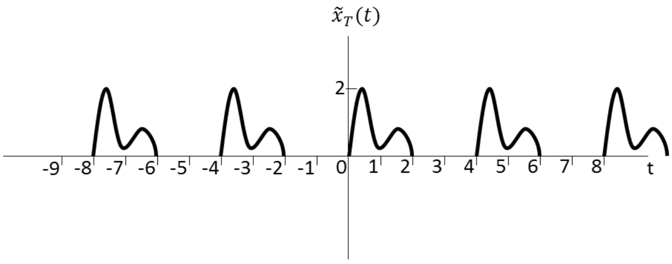


Figure 3-1

Express the Fourier series coefficients b_k of the following signal in terms of a_k when the signal is <u>analyzed with the same fundamental period</u> $T = 4$.

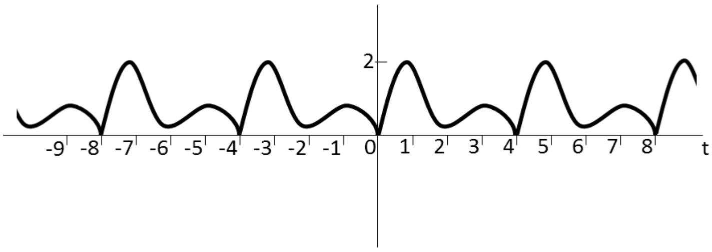
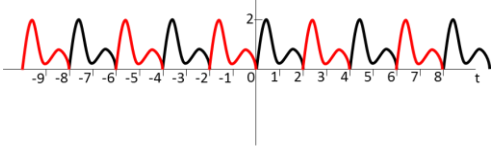


Figure 3-2

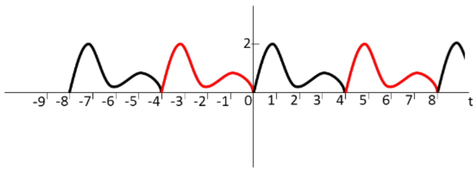
(1) Addition of shifted signal to fill the gap: $x(t) + x(t-2)$



$$a_k \rightarrow a_k + a_k e^{-jk\frac{2\pi}{T}\tau} (\tau : \text{time delay}) = a_k + a_k e^{-jk\pi} = a_k + (-1)^k a_k$$

(2) Stretching in time with respect to $t = 0$

$$x_1(t) = x(\tfrac{1}{2}t) + x(\tfrac{1}{2}t - 2)$$



Time stretching does not change the Fourier series coefficient, but just change the fundamental frequency → $\omega_0' = \omega_0 / 2$

$\therefore b_k = a_k + (-1)^k a_k$ when analyzed by $\omega_0' = \omega_0 / 2$

$b_k = 2a_{2k}$ when analyzed by ω_0

(3-2) [10 pts] Suppose that a signal $\tilde{x}(t)$ of period 2 is convolved with the signal shown in Figure 3-3 ($\tilde{y}(t) = \frac{1}{2}t^2$ for $t \in [-1,1]$). When Fourier series coefficients of $\tilde{x}(t)$ are given by a_k, express Fourier series coefficients c_k of $\boxed{\frac{d}{dt}\tilde{z}(t) = \frac{d}{dt}\left(\tilde{y}(t) *_T \tilde{x}(t)\right)}$ using a_k.

($*_T$: periodic convolution or convolution over single period $\tilde{y}(t) *_T \tilde{x}(t) = \int_T \tilde{x}(\tau)\tilde{y}(t-\tau)d\tau$)

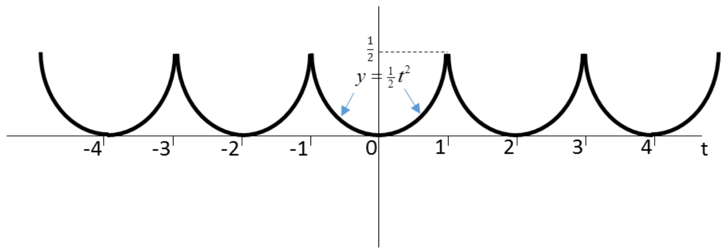
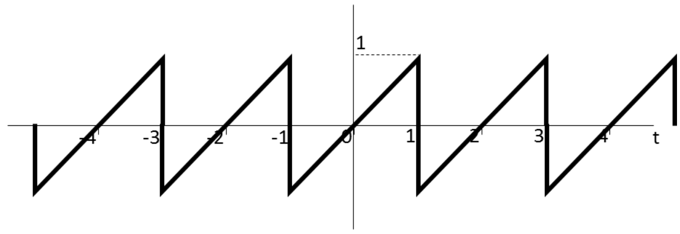


Figure 3-3

$$\tfrac{d}{dt}\tilde{z}(t)=\left(\tfrac{d}{dt}\tilde{y}(t)\right)*\tilde{x}(t) \quad \text{-- (Problem 1-4)}$$

The derivative of $y(t)$ is the following sawtooth signal of period $T=2$.



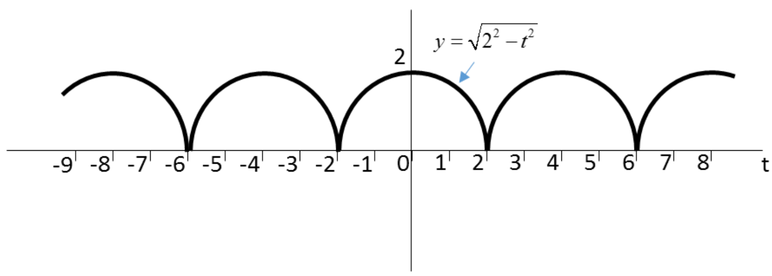
From HW 3.22(a), Fourier series coefficients b_k are given by

$$b_k=\begin{cases}0 & k=0\\ \dfrac{j(-1)^k}{k\pi} & k\neq 0\end{cases}$$

The periodic convolution of two signals is equal to Ta_kb_k.

$$\text{Therefore,}\quad c_k=\begin{cases}0 & k=0\\ 2a_k\dfrac{j(-1)^k}{k\pi} & k\neq 0\end{cases}$$

(3-3) [**10 pts**] Find the sum of Fourier series coefficients $\sum_{k=1}^{\infty}|a_k|^2$ of a signal shown below. (Hint, use the Parseval's relation & $a_0=\frac{1}{T}\int_T \tilde{x}(t)dt$)



Circular signal : for a single period $t^2+y(t)^2=r^2 \;\rightarrow\; y(t)=\sqrt{r^2-t^2}$ (r : radius)

From Parseval's relation, $\dfrac{1}{T}\int_T |x(t)|^2\,dt=\sum_{k=-\infty}^{\infty}|a_k|^2$

$$\sum_{k=-\infty}^{\infty}|a_k|^2 = |a_0|^2 + \sum_{k=1}^{\infty}|a_k|^2 + \sum_{k=-\infty}^{-1}|a_k|^2$$

$$= |a_0|^2 + \sum_{k=1}^{\infty}|a_k|^2 + \sum_{k=1}^{\infty}|a_{-k}|^2$$

For a real-valued signal, $|a_{-k}| = |a_k^*| = |a_k|$ (conjugate symmetry)

$$\sum_{k=-\infty}^{\infty}|a_k|^2 = |a_0|^2 + 2\sum_{k=1}^{\infty}|a_k|^2 \qquad \text{-- (1)}$$

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \frac{1}{2r}\int_{-r}^{r}(r^2 - t^2)dt = \frac{1}{2r}\left[2r^3 - \frac{2}{3}r^3\right] = \frac{2}{3}r^2$$

$$a_0 = \frac{1}{T}\int_T x(t)dt \quad \text{: average area over single period}$$

$$\rightarrow\ a_0 = \frac{\pi r^2}{2}\frac{1}{2r} = \frac{\pi r}{4} \quad \text{(period: } 2r = 4\text{)}$$

From (1), $\displaystyle \frac{2}{3}r^2 = 2\sum_{k=1}^{\infty}|a_k|^2 + \left(\frac{\pi r}{4}\right)^2$

$$\sum_{k=1}^{\infty}|a_k|^2 = \left(\frac{1}{3} - \frac{\pi^2}{32}\right)r^2 = \left(\frac{4}{3} - \frac{\pi^2}{8}\right)$$

[Chapter 4 | 5 pts]

(4-1) Derive the Fourier transform of the impulse response of (2-2) using the analysis equation $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$. Compare your result with the frequency response of (2-3).

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}t}u(t)e^{-j\omega t}dt$$

$$= \int_0^{\infty} e^{-(\frac{1}{2}+j\omega)t}dt \qquad \rightarrow \text{Same as the result of (2-3)}$$

$$= \left.\frac{e^{-(\frac{1}{2}+j\omega)t}}{-\frac{1}{2} - j\omega}\right|_0^{\infty} = \frac{1}{\frac{1}{2} + j\omega}$$