$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$: generalized Factorial

- $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$: Integration by parts

$\Downarrow$

- α : positive integer $\Rightarrow$ $\Gamma(\alpha) = (\alpha-1)!$

- Need a computer to evaluate $\Gamma(\alpha)$.

- Except

$\frac{1}{2}z^2 = x$

part of $N(0,1)$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx = \cdots\cdots = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2}} \cdot \sqrt{2\pi} = \sqrt{\pi}$$

$$\Gamma(\tfrac{5}{2}) = (\tfrac{3}{2})\Gamma(\tfrac{3}{2}) = \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(\tfrac{1}{2}) = \cdots$$

$$\int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot y^{\alpha-1} e^{-y/\beta} dy = 1$$

$\frac{y}{\beta} = x$, use $\Gamma(\alpha)$ definition, etc . . . .

later.

$Y \sim \text{Gamma}(\alpha_1, \beta)$
$X \sim \text{Gamma}(\alpha_2, \beta)$) indep $\Rightarrow X+Y \sim \text{Gamma}(\alpha_1+\alpha_2, \beta)$

Gamma Probability Distribution

$\Gamma \quad \gamma$

$Gam(\alpha, \lambda) = Gamma(\alpha, \frac{1}{\beta})$
shape, rate

$Y \sim Gamma(\alpha, \beta)$: gamma r.v. if
Scale

$$f(y) = \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad 0 \le y < \infty$$

$$\int_0^\infty y^{\alpha-1} e^{-y/\beta} dy = \beta^\alpha \cdot \Gamma(\alpha)$$

for $\alpha > 0$, $\beta > 0$, and where the gamma function is

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy.$$

- $\mu = E(Y) = \alpha\beta$ and $\sigma^2 = V(Y) = \alpha\beta^2$.

Check! Check!

"Go together"

$$\int_0^\infty y \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta} dy$$

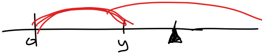
use $\begin{cases} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha \\ \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \alpha(\alpha+1) \end{cases}$

Recall: Poisson(λ) : λ : average no. occurrences in a given time frame.

Poisson rate in 1 unit

$\boxed{\text{Ex } 4.168}$

Y : length of time until the first arrival (bet'n two successive events)



# of Poisson events $\sim$ Poisson(λy)

(y min)

$$P(Y > y) = P(\text{no arrival in the time } \overbrace{[0, y]}^{y})$$

$$= \frac{(\lambda y)^0 e^{-\lambda y}}{0!} = e^{-\lambda y}$$

CDF: $F(y) = 1 - e^{-\lambda y}$

$$f(y) = \lambda e^{-\lambda y} \quad : \text{Exp}\left(\tfrac{1}{\lambda}\right)$$

Ex 4.169

· Calls come at rate 10/hr

· P(more than 15 minutes elapsed bet'n two calls)

Y : length of time (hr) bet'n two calls.

$$\sim \text{Exp}\left(\tfrac{1}{10}\right)$$

$$P(Y > .25\,\text{hr}) = \int_{.25}^{\infty} 10\, e^{-10y}\, dy = .082.$$

$\boxed{\text{Ex 170}}$ Generalize to Gamma

a. Y : waiting time until $\underset{k}{2^{nd}}$ arrival

$P(Y>y) = P($ none or one arrival in $[0, y]$ $)$

$$= \frac{(\lambda y)^0 e^{-\lambda y}}{0!} + \frac{(\lambda y)^1 e^{-\lambda y}}{1!} = (\lambda y + 1) e^{-\lambda y}$$

$F(y) = \vdots$

$f(y) = \lambda^2 y e^{-\lambda y}$: Gamma$(2, \frac{1}{\lambda})$

k

$\boxed{\text{MGF}}$ $\left(\begin{array}{ll} Y_1 \text{ has } & m_1(t) \\ Y_2 \text{ has } & m_2(t) \end{array} \right)$ indep

Then $Y_1 + Y_2$ has mgf $m_1(t) m_2(t)$.

$$E\left(e^{(Y_1+Y_2)t}\right) \underset{\text{indep}}{=} E\left(e^{tY_1}\right) E\left(e^{tY_2}\right) = m_1(t) \cdot m_2(t)$$

MGF of Gamma (α, β)

$y\left(t-\frac{1}{\beta}\right)$

$\left(t-\frac{1}{\beta}\right)^{-1} = \beta^*$

$$\int e^{ty} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} \, dy =$$

$$= \frac{1}{\beta^\alpha} \underbrace{\left[\int \frac{1}{\beta^{*\alpha}\Gamma(\alpha)} \cdot y^{\alpha-1} e^{-y/\beta^*} \, dy\right]}_{=1} \beta^{*\alpha}$$

$$= \left(\frac{\beta^*}{\beta}\right)^\alpha = \underline{(1-\beta t)^{-\alpha}}$$

Use this find $E(Y)$ $E(Y^2)$ ...

$E(Y^k)$ ... . .

- We can show

$Y_1 \sim \chi^2(\nu_1)$, $Y_2 \sim \chi^2(\nu_2) \rightarrow (1-2t)^{-\nu_2/2}$

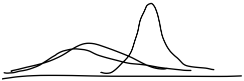
$m_1(t) = (1-2t)^{-\nu_1/2}$

$Y_1 + Y_2 \sim \chi^2(\nu_1+\nu_2)$

mgf of $Y_1+Y_2 = (1-2t)^{-\frac{\nu_1+\nu_2}{2}}$: mgf of $\chi^2(\nu_1+\nu_2)$

Normal case

$$\left[\begin{array}{l} Y_1 \sim N(\mu_1, \sigma_1^2) \\ Y_2 \sim N(\mu_2, \sigma_2^2) \end{array}\right\} \text{indep}$$



$$\underline{Y_1 + Y_2} \sim N(\mu_1 + \mu_2, \ \sigma_1^2 + \sigma_2^2)$$

mgf of $Y_1 + Y_2$: $\exp\left(\mu_1 t + \frac{1}{2}\sigma_1^2 t^2\right) \times \exp\left(\mu_2 t + \frac{1}{2}\sigma_2^2 t^2\right)$

$$= \exp\left((\mu_1 + \mu_2)t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)t^2\right)$$

; Normal !!

$$N(\mu_1 + \mu_2, \ \sigma_1^2 + \sigma_2^2)$$

Gamma Probability Distribution

$Y \sim Gamma(\alpha, \beta)$ is the waiting time for the αth Poisson event with $\lambda = 1/\beta$.

Chi-squared **Distribution**

nu (integer)

$Y \sim \chi^2(\nu)$ if $Y \sim Gamma(\nu/2, 2)$

"Chi"

$Gama(3, 2) = \chi^2(6)$

ν: degrees of freedom = "# of independent components"

$E(Y) = \nu$

$V(Y) = 2\nu$

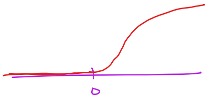
$Z \sim N(0,1)$ then $Z^2 \sim \chi^2(1)$

$\underbrace{Z_1, \cdots Z_\nu \sim N(0,1)}_{\text{Indep}}$ then $\sum_1^\nu Z_i^2 \sim \chi^2(\nu)$

$V(\sum_1^\nu Z_i^2) = \cdots\cdots$

$E(\sum_1^\nu Z_i^2) = \nu \cdot E(Z_i^2) = \nu$

Exponential Probability Distribution



$F(y) = \int_0^y \frac{1}{\beta} e^{-x/\beta}\, dx = \underline{1 - e^{-y/\beta}}$ $(y \geq 0)$

$F(y) = 0 \quad y \leq 0$

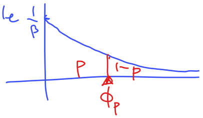
· non-decreasing

$\lim_{y \to -\infty} F(y) = 0$

$\lim_{y \to \infty} F(y) = 1$

$Y \sim Exp(\beta)$ if $Y \sim Gamma(1, \beta)$

p^{th} quantile



$\phi_p = -\beta \cdot \log(1-p)$

$$\boxed{f(y) = \frac{1}{\beta} e^{-y/\beta},} \quad 0 \leq y < \infty$$

- Modeling waiting time for the first Poisson event
- $\mu = E(Y) = \beta$ and $\sigma^2 = V(Y) = \beta^2$
- Memoryless property

$\Leftrightarrow \; P(Y > a+b \mid Y > a) = P(Y > b)$

$\Leftrightarrow \; \frac{1 - F(a+b)}{1 - F(a)} = 1 - F(b)$ Check!