Chapter 2. Probability

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Definitions

Experiments: a process or an action that has an outcome that is now known beforehand.

- Flip a coin 3 times = $\{HHH, \cdots TTT\}$
- Flip " until first H. " $\{H, TH, TTH, \cdots\}$
- Wait for the bus $\{x : x \geq 0\}$

Sample space (S): the set of all possible (elementary) outcomes.

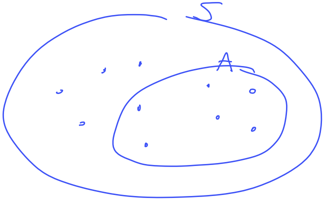
Weight height measuring $\Rightarrow S = \{(x, y) \mid x \geq 0, y \geq 0\}$

Event: a subset of S , $A, B, C, \ldots \subset S$ $\quad \phi \subset S$

Once an experiment is done, we know whether an event has occurred or not.

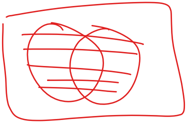
Set Operations

A^c,



$A, B \subset S$

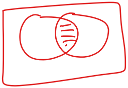
$A \cup B$: the event that A or B or both occur.



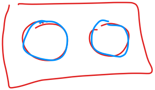
simultaneously

$A \cap B$: the event that both A and B occur at the same time.

$A - B = A \cap B^c$



Disjoint (Mutually exclusive) events: $A \cap B = \emptyset$



$\neq$ Independent.

Probability of an event

"nice"

Given an experiment and the sample space S, for each event $A \subset S$, a probability is an "assigned" value that measures the chance of occurrence. "numerical measure of uncertainty"

Axioms of probability: Kolmogorov

- $P(A) \geq 0$
- $P(S) = 1$
- $A_1, A_2, \ldots$: sequence of disjoint sets

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

finite version : $A_1, \ldots, A_m$, disjoint,

$$P(\bigcup_{1}^{m} A_i) = \sum_{1}^{m} P(A_i)$$

Probability

Many properties can be derived based on the axioms.

$A_1 \cdots A_m = \phi$ all empty sets

disjoint, $\bigcup^m A_i = \phi$

Axiom ③, LHS $= P(\phi)$

RHS $= \sum_i^m P(\phi) = P(\phi) + \cdots + P(\phi) = m P(\phi)$

LHS = RHS $\Rightarrow P(\phi) = 0$

- $P(\emptyset) = 0$
- $P(A^c) = 1 - P(A)$ DIY
- $P(A) \leq 1$ → from and axiom ① $1 - P(A) \geq 0 \Rightarrow P(A) \leq 1$
- $P(A) \leq P(B)$ if $A \subset B$ DIY
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$P(A \cap B^c) + P(B)$ Ax ③

$= P(A) - P(A \cap B) + P(B)$

· $A \cup B = \underbrace{(A \cap B^c) \cup B}_{\text{disjoint}}$

· $A = \underbrace{(A \cap B) \cup (A \cap B^c)}_{\text{disjoint}}$

Probabilistic Statements

- How likely will an event occur?
- 20% of raining ← Experiment = observe ~~to~~morrow's weather record rain or not
  $S = \{ \text{rain}, \text{norain} \}$
- ~~Most likely to be~~ at home by 7pm
  80%
- We have a good chance of winning the championship this year.

We interpret probability as the **long-run** stabilized **relative frequency**.

- repeat the experiment as many as possible and observe the proportion of times the event occurs. As the number of trials increases, the relative frequency tends to stabi- lize.
- The stabilized value can be served as Probability, which is unknown but can be esti- mated accurately by repeating the experiment many times.

– Frequentist "

Counting Rules

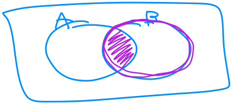
- $S = \{E_1, \dots, E_N\}$
- Elementary outcomes are equally likely. $P(\{E_i\}) = 1/N$
- $\forall A \subset S$, $P(A) = (\text{no. of elements})/N$.

Product Rule

Permutation

Combination

Conditional Probability



For $A, B \subset S$ with $P(B) > 0$,

Given information, we find revised probability

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

is the conditional probability of A given that B has happened.

"B acts like new Sample space"

- Revised probability due to new information
- $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$.