School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2022 Spring

Final

Thursday, June 16, 2022
1:00–3:00 pm

NAME: ______________________________

Student ID: ______________________________

- Don't forget to put your name and student ID.
- **Record all your solutions in this answer booklet. Only this answer booklet will be considered in the grading of your exam.**
- Be sure to show all relevant work and reasoning. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.

Problem	Your score	Max score
1		15
2		15
3		15
4		15
Total		

Problem 1 (15 Points)

Consider two discrete random variables X_n and Y_n whose PMFs are defined as

$$p_{X_n}(x) = \begin{cases} 1 - 1/n, & \text{for } x = 0, \\ 1/n, & \text{for } x = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$p_{Y_n}(x) = \begin{cases} 1 - 1/n, & \text{for } x = 0, \\ 1/n, & \text{for } x = n, \\ 0, & \text{otherwise.} \end{cases}$$

a) (2 points) Find the expected value and variance of X_n and Y_n.

Answer:

$\mathbb{E}[X_n] =$ $\quad\quad\quad\quad$ $\mathrm{var}(X_n) =$

$\mathbb{E}[Y_n] =$ $\quad\quad\quad\quad$ $\mathrm{var}(Y_n) =$

Reasoning for Problem 1(a):

b) (5 points) What does the Chebyshev inequality tell us about the convergence of X_n and Y_n? (Hint: Consider $\lim_{n\to\infty} \mathbb{P}(|X_n - \mathbb{E}[X_n]| \geq \epsilon)$ and $\lim_{n\to\infty} \mathbb{P}(|Y_n - \mathbb{E}[Y_n]| \geq \epsilon)$.)

Answer:

Reasoning for Problem 1(b):

c) (5 points) Is Y_n convergent in probability? If so, to what value? (Hint: A sequence Z_n converges in probability to a number a if for any $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|Z_n - a| \geq \epsilon) = 0$.)

Answer:

Reasoning for Problem 1(c):

d) (3 points) If a sequence of random variables converges in probability to a value a, does the corresponding sequence of expected values converge to a? Prove or give a counter example.

Answer:

Reasoning for Problem 1(d):

Problem 2 (15 Points)

Consider a Bernoulli process $X_1, X_2, X_3, \ldots$ with unknown probability of arrival q, i.e., $\mathbb{P}(X_i = 1) = q$ for all i. Define the k-th interarrival time T_k as

$$T_1 = Y_1, \quad T_k = Y_k - Y_{k-1}, \quad k = 2, 3, \ldots$$

where Y_k is the time of the k-th arrival. In this problem, we estimate the arrival probability q from observed interarrival times $(t_1, t_2, t_3, \ldots)$. Assume q is sampled from the random variable Q which is uniformly distributed over $[0, 1]$ and fixed across the experiment.

You may find the following integral useful: For any non-negative integer k and m,

$$\int_0^1 q^k (1-q)^m dq = \frac{k!m!}{(k+m+1)!}.$$

a) (5 points) Find the PMF of T_1.

Answer: $p_{T_1}(t) =$

Reasoning for Problem 2(a):

b) (5 points) Compute the least mean squares (LMS) estimate of Q conditioned on the first arrival time $T_1 = t_1$.

Answer: $\mathbb{E}[Q | T_1 = t_1] =$

Reasoning for Problem 2(b):

c) (5 points) Compute the maximum a posteriori (MAP) estimate of Q given the first k arrival times, $(T_1, T_2, \ldots, T_k) = (t_1, t_2, \ldots, t_k)$.

Answer:

$\arg\max_q f_{Q|T_1,\ldots,T_k}(q|t_1, \ldots, t_k) =$

Reasoning for Problem 2(c):

<u>Problem 3</u> (15 Points)

Alice, Bob and Charlie run laps around a track, with the duration of each lap (in hours) being exponentially distributed with parameter $\lambda_A = 21, \lambda_B = 23$ and $\lambda_C = 24$, respectively. Assume that all lap durations are independent.

a) (5 points) Write down the PMF of the total number L of completed laps (3 runners combined) over the first hour.

Answer:

$p_L(l) =$

Reasoning for Problem 3(a):

b) (5 points) What is the probability that Alice finishes her first lap before any of the others?

Answer:

Reasoning for Problem 3(b):

c) (5 points) Suppose that the runners have been running for a very long time when you arrive at the track. What is the distribution of the duration T of Alice's current lap? (This includes the duration of that lap both before and after the time of your arrival.)

You may use the fact that the distribution for the k-th arrival time Y_k of the Poisson process with rate λ is $f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$, $y \geq 0$.

Answer:

$f_T(t) =$

Reasoning for Problem 3(c):

<u>Problem 4</u> (15 Points)

In this problem, we want to have a Markov chain that models the spread of a virus. Assume a population of n individuals. At the beginning of each day (say 7am), each individual is either infected or susceptible (not yet infected but capable of being infected by contacts of infected people). Suppose that each pair of people (i, j), $i \neq j$, independently comes into contact with one another during the daytime (7am to 7pm) with probability p. Whenever a susceptible individual comes into contact with an infected individual, the susceptible individual is infected right away. Assume that during overnight (7pm to 7am next day), any individual who has been infected for at least 24 hours will recover with probability $0 < q < 1$ and return to being susceptible, independently of everything else (i.e., assume that a newly infected individual will spend at least one night without recovery).

a) (5 points) Suppose that there are m infected individuals at one daybreak (7am). The number of total population is n. What is the PMF of the number of new infected individuals N at the end of the daytime (7pm)?

Answer:

$p_N(k) =$

Reasoning for Problem 4(a):

b) (5 points) Suppose that $n = 2$. Draw a Markov chain with states 0, 1, 2 (each of which indicates the number of infected individuals among $n = 2$ at each daybreak) to model the spread of the virus.

Answer (without reasoning):

c) (5 points) Suppose that $n = 2$ and let's assume that the initial state is $X_0 = 1$ (the number of infected individuals at day 0 is equal to 1). Calculate the mean first passage time to the state 0, i.e., $t_1 = \mathbb{E}[\min\{n \geq 0 \text{ such that } X_n = 0\} | X_0 = 1]$ (the expected number of days to have 0 infected individual for the first time.)

Answer:

$t_1 =$

Reasoning for Problem 4(c):