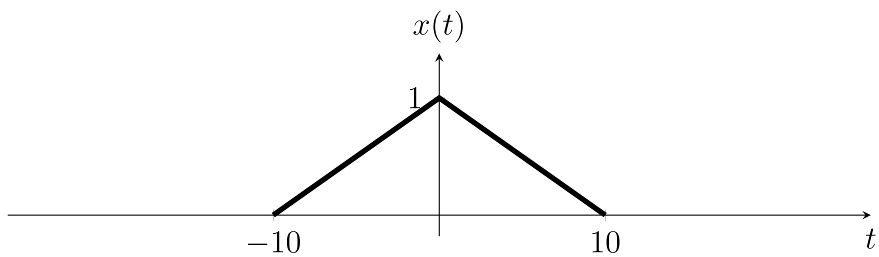
EE202C Signals and Systems, Fall 2020
Dec 16, 2020
Time Limit: 12:45 - 15:30 Lecturer: Jung-Woo Choi

Score Table (for teacher use only)

Question:	1	2	3	4	5	Total
Points:	15	20	25	20	20	100
Score:						

This is an open-book exam. Please provide ALL DERIVATIONS and EXPLANATIONS with your answers. Simply writing down the final answer corresponds to 1 point. Any communication with others during the exam will be regarded as a cheating case.

1. (15 points) Consider a triangular signal $x(t)$ shown below:



(a) (5 points) Find the Fourier transform $X(j\omega)$ of this signal.

Solution 1) Consider a signal $g(t)$ of duration $L = 10$.

$$g(t) = \mathrm{rect}(t/10)$$

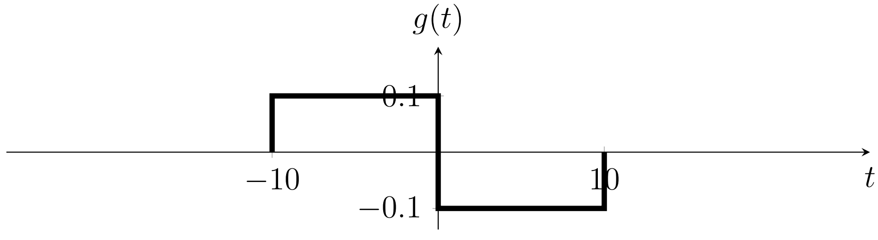
The given signal is obtained by the convolution: $x(t) = (1/10)(g(t) * g(t))$. From the FT of the rectangular signal

$$\mathrm{rect}(t/L) \leftrightarrow L\mathrm{sinc}(Lf) = 2\frac{\sin(\omega L/2)}{\omega}$$

and the convolution-multiplication property of the FT,

$$X(j\omega) = \frac{1}{10}\left(2\frac{\sin(5\omega)}{\omega}\right)^2 = 2\frac{\sin^2(5\omega)}{5\omega^2}$$

Solution 2) The given signal is obtained by integrating the following signal $g(t)$:



This signal consists of two delayed rectangular signals of duration $L = 10$:

$$g(t) = -\frac{1}{10}\text{rect}((t-5)/10) + \frac{1}{10}\text{rect}((t+5)/10)$$

From the FT of the rectangular signal

$$\text{rect}(t/L) \leftrightarrow L\text{sinc}(Lf) = 2\frac{\sin(\omega L/2)}{\omega}$$

and the time shifting property

$$x(t-\tau) \leftrightarrow X(j\omega)e^{-j\omega\tau},$$

the FT of $g(t)$ is given by

$$\begin{aligned} G(j\omega) &= \frac{\sin(5\omega)}{5\omega}\left(e^{j5\omega} - e^{-j5\omega}\right) \\ &= \frac{\sin(5\omega)}{5\omega} \cdot 2j\sin(5\omega) \\ &= 2j\frac{\sin^2(5\omega)}{5\omega} \end{aligned}$$

The integral property of FT states that

$$\int_{-\infty}^{t} g(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}G(j\omega) + \underbrace{\pi G(0)\delta(\omega)}_{\text{dc value}}$$

The signal $g(t)$ has zero mean value, so the integration of $g(t)$ corresponds to

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}G(j\omega) = 2\frac{\sin^2(5\omega)}{5\omega^2} = X(j\omega)$$

(b) (5 points) The signal $x(t)$ is now filtered by a filter

$$h(t) = \frac{1}{\pi}\frac{\sin\left(\frac{\pi}{5}t\right)}{t}$$

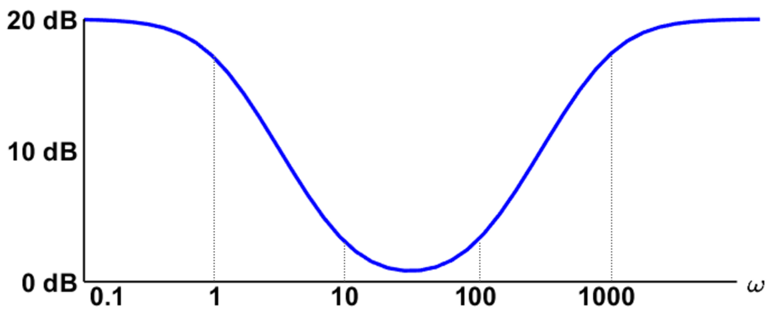
Determine the Nyquist rate ω_s to sample the filtered signal $y(t) = x(t) * h(t)$ without aliasing.

The original signal $X(j\omega)$ has infinite bandwidth. This is filtered by the $h(t) \leftrightarrow H(j\omega) = \text{rect}(\omega/(2\pi/5))$ of bandwidth $2\pi/5$. Therefore, the Nyquist rate is the filter's bandwidth $\omega_s = 2\pi/5$.

(c) (5 points) After the sampling at the Nyquist rate, the signal $y(t)$ is converted to a discrete sequence $y[n]$. Determine the Fourier transform $Y(e^{j\Omega})$ of the sampled sequence $y[n]$.

The conversion to a DT sequence accompanies the amplitude scaling by $1/T (T = 2\pi/\omega_s = 5)$ and frequency scale compression by $(\Omega = \omega T)$. As a result, the sampled sequence becomes $Y(e^{j\Omega}) = 2\frac{\sin^2(\Omega)}{\Omega^2}$.

2. (20 points) A stable LTI system's magnitude response $|H(j\omega)|$ is shown as a Bode plot with its 4 break frequencies. The frequency response is the rational function of $j\omega$, and its impulse response is a real-valued signal.



(a) (10 points) Derive its frequency response $H(j\omega)$ such that its phase change is minimal. In other words, design a 'minimum phase' system that yields the given magnitude response.

From the magnitude response, the possible forms of $H(j\omega)$ are given by

$$10\frac{(j\omega \pm 10)(j\omega \pm 100)}{(j\omega \pm 1)(j\omega \pm 1000)}$$

Since the LTI system is stable, there are four possibilities (any solution among these four cases will be evaluated as a correct answer):

(1) For right-sided signal

Poles should be on the left hand side

$$10\frac{(j\omega \pm 10)(j\omega \pm 100)}{(j\omega + 1)(j\omega + 1000)}$$

To have the minimum phase, the phase drop at $\omega = 1$ should be compensated by the phase change at $\omega = 10$. Likewise, the coefficient at $\omega = 1000$ should have the same sign with that for $\omega = 100$.

$$\therefore \quad H_1(j\omega) = 10\frac{(j\omega + 10)(j\omega + 100)}{(j\omega + 1)(j\omega + 1000)}$$

(2) For left-sided signal

Poles should be on the right half plane. The same signs for $\omega = 1, 10$ and $\omega = 100, 10000$ yields

$$\therefore \quad H_2(j\omega) = 10\frac{(j\omega - 10)(j\omega - 100)}{(j\omega - 1)(j\omega - 1000)}$$

(3,4) For both-sided signal

There are two chances. Two denominator terms for $\omega = 1$ and $\omega = 1000$ should lie on the other side of half plane, respectively, so that the ROC is given by the strip. The minimum phase condition implies that the signs for $\omega = 1, 10$ and $\omega = 100, 10000$ should be the same. Accordingly, we have

$$\therefore \quad H_3(j\omega) = 10\frac{(j\omega - 10)(j\omega + 100)}{(j\omega - 1)(j\omega + 1000)} \quad \text{or} \quad H_4(j\omega) = 10\frac{(j\omega + 10)(j\omega - 100)}{(j\omega + 1)(j\omega - 1000)}.$$

(b) (10 points) Find its impulse response $h(t)$.

Any solution among those shown below will be regarded as a correct answer.
To derive a left-sided or right-sided impulse response, we use the property

$$\begin{aligned}\frac{1}{j\omega+a} &\leftrightarrow e^{-at}u(t) \text{ for } a>0 \text{ and right-sided signal}\\ &\leftrightarrow -e^{-at}u(-t) \text{ for } a<0 \text{ and left-sided signal}\end{aligned}$$

(1) For right-sided signal

$$\begin{aligned}H_1(j\omega) &= 10\frac{(j\omega)^2+110(j\omega)+1000}{(j\omega)^2+1001(j\omega)+1000}\\ &= 10\left(1-891\frac{(j\omega)}{(j\omega+1)(j\omega+1000)}\right)\\ &= 10\left(1+\frac{891}{999}\left[\frac{1}{j\omega+1}-\frac{1000}{j\omega+1000}\right]\right)\\ \therefore h_1(t) &= 10\left(\delta(t)+\frac{33}{37}\left[e^{-t}-1000e^{-1000t}\right]u(t)\right)\end{aligned}$$

(2) For left-sided signal

$$\begin{aligned}H_2(j\omega) &= H_1(j\omega)^*\\ \therefore h_2(t) &= h_1(-t)^*\\ &= 10\left(\delta(t)+\frac{33}{37}\left[e^{t}-1000e^{1000t}\right]u(-t)\right)\end{aligned}$$

(3)

$$\begin{aligned}H_3(j\omega) &= 10\frac{(j\omega+10)(j\omega-100)}{(j\omega+1)(j\omega-1000)}\\ &= 10\frac{(j\omega)^2-90(j\omega)-1000}{(j\omega)^2-999(j\omega)-1000}\\ &= 10\left(1+909\frac{(j\omega)}{(j\omega+1)(j\omega-1000)}\right)\\ &= 10\left(1-\frac{909}{1001}\left[\frac{1}{j\omega+1}+\frac{1000}{j\omega-1000}\right]\right)\\ \therefore h_3(t) &= 10\left(\delta(t)+\frac{909}{1001}\left[e^{-t}u(t)-1000e^{1000t}u(-t)\right]\right)\end{aligned}$$

(4)

$$\begin{aligned}H_4(j\omega) &= H_3(j\omega)^*\\ \therefore h_4(t) &= h_3(-t)^*\\ &= 10\left(\delta(t)+\frac{909}{1001}\left[e^{t}u(-t)-1000e^{-1000t}u(t)\right]\right)\end{aligned}$$

3. (25 points) Consider a CT filter $h(t)$ given by

$$h(t) = \text{sgn}(t)e^{-a|t|}$$

where $\text{sgn}(t) = 1$ for $t > 0$ and $\text{sgn}(t) = -1$ for $t < 0$. The parameter a is positive ($a > 0$).

(a) (10 points) Find the Fourier transform $H(j\omega)$ of $h(t)$.

The signal can be rewritten as

$$h(t) = e^{-at}u(t) - e^{at}u(-t)$$

Using the following properties of FT,

$$\begin{aligned} e^{-at}u(t) &\leftrightarrow \frac{1}{j\omega + a} \\ x(-t) &\leftrightarrow X(-j\omega) \end{aligned}$$

The FT of $h(t)$ is given by

$$\begin{aligned} H(j\omega) &= \frac{1}{j\omega + a} - \frac{1}{-j\omega + a} \\ &= \frac{-2j\omega}{\omega^2 + a^2} \end{aligned}$$

(b) (10 points) Suppose that we want to design a system that has the maximum magnitude response at frequency ω_0. Determine the system parameter a to have the maximum response at $\omega = \omega_0$.

The given magnitude response $|H(j\omega)|$ decays out for both $\omega \to 0,\ \infty$ and has the maximum value at the break frequency. To find the frequency of maximum magnitude, we can take the derivative of $H(j\omega)$ with respect to ω.

$$\begin{aligned} \frac{dH(j\omega)}{dt} &= \frac{-2j}{\omega^2 + a^2} - \frac{-4j\omega^2}{(\omega^2 + a^2)^2} \\ &= \frac{2j(\omega^2 - a^2)}{(\omega^2 + a^2)^2} \end{aligned}$$

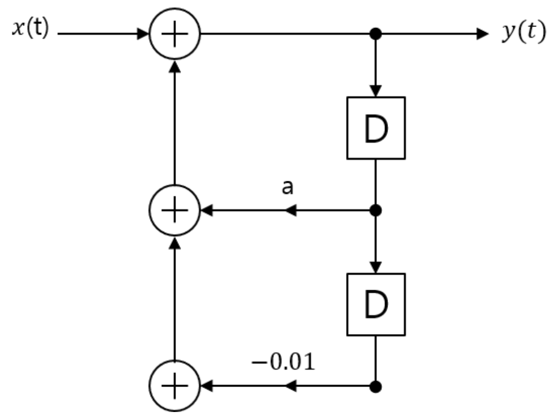
This derivative becomes zero at $\omega = a$. Therefore, a should be equal to ω_0 ($a = \pm\omega_0$ is okay).

(c) (5 points) Derive the Laplace transform of this system and determine the region of convergence (ROC).

$$\begin{aligned} H(s) &= \frac{1}{s + a} - \frac{1}{-s + a} \\ &= \frac{2s}{s^2 - a^2} \end{aligned}$$

Since the impulse response is both-sided, its ROC is given by the strip between two poles. (ROC: $-a < Re\{s\} < a$).

4. (20 points) A continuous-time, causal LTI system's block diagram is shown below.



(a) (10 points) Draw the pole-zero plot of this system for $a = -0.1$ (mark the locations of poles and zeros, and specify the region of convergence).

The LCCDE of the given system can be written as

$$y(t) = x(t) - 0.01\frac{d^2y(t)}{dt^2} + a\frac{dy(t)}{dt}$$
$$\frac{d^2y(t)}{dt^2} - 100a\frac{dy(t)}{dt} + 100y(t) = 100x(t)$$

For $a = -0.1$, the system response is given by

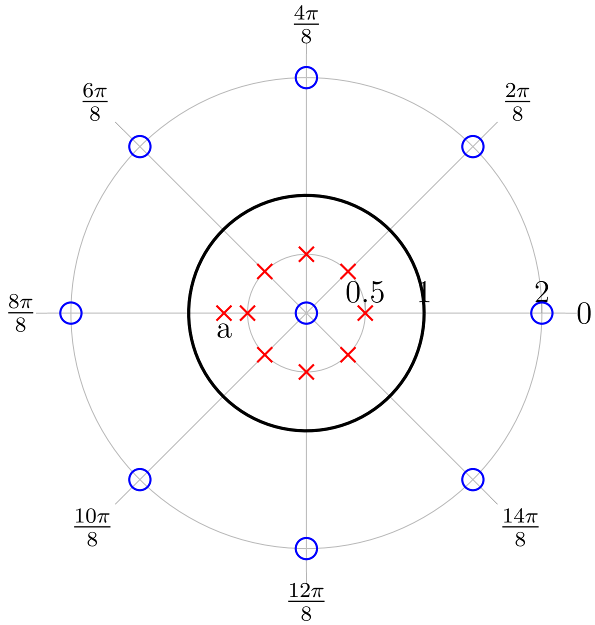
$$\begin{aligned} H(s) &= \frac{100}{s^2 + 10s + 100} \\ &= \frac{100}{(s + 5 + 5\sqrt{3}j)(s + 5 - 5\sqrt{3}j)} \end{aligned}$$

which has two poles at $s = -5 \pm 5\sqrt{3}j$. Since the system is causal, the ROC is the right half plane from $Re\{s\} > -5$.

(b) (10 points) Find the value of a that makes this system critically damped.

From the LCCDE, $\omega_n = 10$ and $2\zeta\omega_n = -100a$. Therefore, for critically damped system ($\zeta = 1$), $a = -0.2$.

5. (20 points) A pole-zero plot of a causal DT LTI system is shown below. The real-valued parameter a satisfies $-1 < a < 0$.



(a) (10 points) The system's frequency response $H(e^{j\Omega})$ at zero frequency is given by $H(e^{j0}) = 2$. Plot the magnitude response $|H(e^{j\Omega})|$ in dB scale for the frequency range $-\pi \le \Omega \le \pi$. Also mark the maximum and minimum values of the magnitude response.

The evenly spaced poles (p_n) and zeros (z_n) satisfy the reciprocal relation: $p_n = 1/z_n^*$. Therefore, evenly spaced poles and zeros exhibits all-pass characteristic with unknown gain factor.

On the other hand, the pole at a and the zero at 0 yield the first-order response with unknown gain. Therefore, the total response of the system can be written as

$$H(z) = G \cdot \frac{1}{1 - az^{-1}} \cdot H_{all}(z) \tag{1}$$

where G represents the unknown (complex-valued) gain of the system, and the all-pass response $H_{all}(z)$ satisfies $|H_{all}(z)| = 1$. The standard form of the all-pass response with unit gain is given by

$$H_{all}(z) = \prod_n \frac{z^{-1} - p_n^*}{1 - p_n z^{-1}}$$

At the zero frequency, $z = e^{j\Omega} = 1$. In addition, the poles and zeros of the given all-pass system are also symmetric with respect to the real axis. Accordingly, for any pole p_n, there will be a complex-conjugated pole p_n^* (for the poles on the real axis, the complex conjugation equals to the original pole itself). This results in poles and zeros cancelled out at the zero frequency such that

$$H_{all}(e^{j0}) = \prod_n \frac{1 - p_n^*}{1 - p_n} = 1 \tag{2}$$

Using this result and from the zero frequency response, we can determine the unknown gain

G.

$$\begin{aligned} H(e^{j0}) &= G \cdot \frac{1}{1-a} \cdot H_{all}(e^{j0}) \\ &= G \cdot \frac{1}{1-a} = 2 \\ \therefore G &= 2(1-a) \end{aligned}$$

Consequently, the magnitude response can be written as

$$|H(e^{j\Omega})| = 2(1-a) \cdot \left| \frac{1}{1-ae^{-j\Omega}} \right| \tag{3}$$

For negative a, the response shows the high-pass characteristic, which has a minimum $2(1-a)/(1-a) = 2$ at $\Omega = 0$ and maximum $2(1-a)/(1+a)$ at $\Omega = \pi$.

(b) (10 points) Express the initial value of the impulse response $h[0]$ in terms of a.

From the initial value theorem, $h[0] = \lim_{z\to\infty} H(z)$.
As $z \to \infty$, its inverse $z^{-1} \to 0$. Under this condition, the system response becomes

$$\begin{aligned} \lim_{z\to\infty} H(z) &= \lim_{z\to\infty} 2(1-a)\frac{1}{1-az^{-1}} \prod_n \frac{z^{-1}-p_n^*}{1-p_n z^{-1}} \\ &= 2(1-a)\prod_n(-p_n^*) \end{aligned}$$

Since $p_n = \frac{e^{j2\pi n/8}}{2}$, $\prod_{n=0}^{7}(-p_n^*) = -\frac{1}{2^8} = -\frac{1}{256}$.

$$\therefore h[0] = \frac{a-1}{128}$$