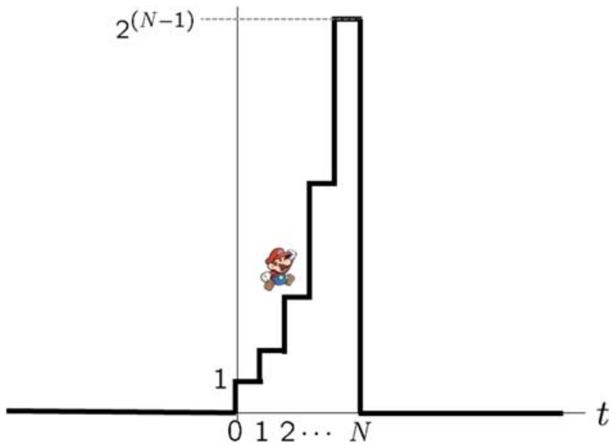
[Score table]

Prob.	1			2			3		4		5				Extra	Total
	5	5	10	5	5	5	20	15	5	5	5	5	5	5	P/F	100
Score																

Problems & Solutions

[Problem 1 – Fourier transform | 10 points]

Mr. Mario is climbing up a staircase of N steps. The n^{th} step has a width of 1 and height 2^n. When the shape of the stairs is denoted as a signal $x(t)$, derive its Fourier transform $X(j\omega)$.



Answer) $X(j\omega) = \dfrac{\sin(\omega/2)}{\omega/2} e^{-j\omega/2} \dfrac{1-\left(2e^{-j\omega}\right)^N}{1-2e^{-j\omega}}$

Consider a step of amplitude 1 and width 1. It is a unit rectangle delayed by the half width ($g(t) = \mathrm{Rect}_1(t-0.5)$), so its Fourier transform is given by

$$G(j\omega) = 2\frac{\sin(\omega/2)}{\omega} e^{-j\omega/2} = \frac{\sin(\omega/2)}{\omega/2} e^{-j\omega/2}.$$

The total signal is a sum of $g(t)$ delayed by n ($n = 0, \cdots, N-1$) and scaled by 2^n.

$$\begin{aligned} X(j\omega) &= \sum_{n=0}^{N-1} G(j\omega) 2^n e^{-j\omega n} \\ &= G(j\omega) \sum_{n=0}^{N-1} \left(2e^{-j\omega}\right)^n \\ &= G(j\omega) \frac{1-\left(2e^{-j\omega}\right)^N}{1-2e^{-j\omega}} \end{aligned}$$

[Problem 2 – Sampling | 15 points]

For a continuous-time signal $x(t)=\dfrac{\sin 2t}{\pi t}$, answer to the following questions:

(a) [**5 pts**] Find the Nyquist rate ω_N to sample the signal $p(t)=x(t)\cos(3t)$ without aliasing.

(b) [**10 pts**] let

$$y(t)=\frac{d}{dt}\left[x^n(\tfrac{1}{2}t-\tau)\right]$$

be a nonlinear signal constructed from $x(t)$ (n: positive integer). Express the **Nyquist rate** ω_N of $y(t)$ in terms of n.

Answer) (a) 10 (b) $2n$

(1) The Fourier transform of $x(t)=\dfrac{\sin 2t}{\pi t}$ is given by

$$X(j\omega)=\begin{cases}1, & |\omega|<2\\ 0, & |\omega|>2\end{cases}$$

and hence, its Fourier transform is bounded within $\omega\in[-\omega_M,\omega_M]$ $(\omega_M=2)$.

The multiplication with $\cos(\omega_0 t)$ yields $P(j\omega)=\dfrac{1}{2}\{X(j(\omega-\omega_0))+X(j(\omega+\omega_0))\}$, so the final spectrum is bounded between $\omega\in[-\omega_M-\omega_0,\omega_0+\omega_M]$. $(\omega_0=3)$

The Nyquist rate is two times of the highest frequency: $\omega_N=2(\omega_M+\omega_0)=10$.

(b) The signal $y(t)$ and $x(t)$ are related by $y(t)=\dfrac{d}{dt}\left[x^n(\tfrac{1}{2}t-\tau)\right]$.

From the properties of CTFT : $f(t-\tau)\overset{CTFT}{\Leftrightarrow}F(j\omega)e^{-j\omega\tau}$, $\dfrac{d}{dt}f(t)\overset{CTFT}{\Leftrightarrow}j\omega F(j\omega)$,

the temporal derivative and time delay doesn't change the width of the Fourier transform $F(j\omega)$.
However, the time expansion $x(2t)$ compresses $X(j\omega)$ by factor of two ($\omega_M=1$).

Next, $x^n(t)$ is the (n-1)-times multiplication of $x(t)$ with itself, so its Fourier transform can be expressed as the multiple convolution of $X(j\omega)$ in frequency domain.

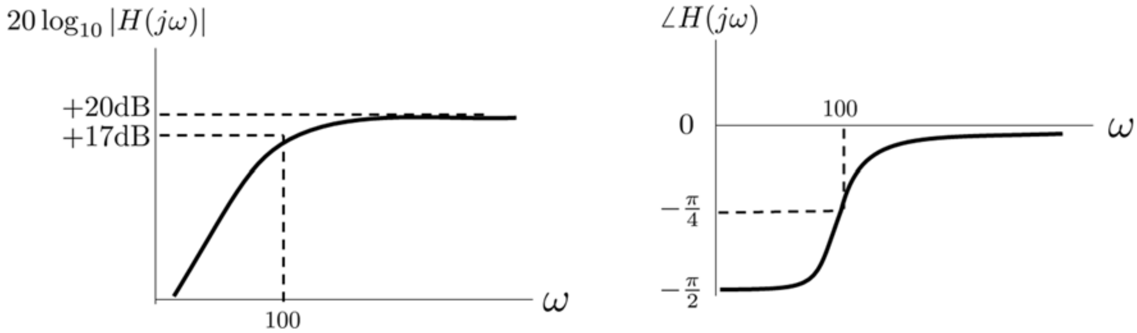
$$x^n(t)\overset{CTFT}{\Leftrightarrow}\underbrace{X(j\omega)*X(j\omega)*\cdots*X(j\omega)}_{\text{convolution }(n-1)\text{ times}}$$

For a band-limited $X(j\omega)$ ($\omega\in[-\omega_M,\omega_M]$), the highest frequency increases by ω_M for each convolution, so the highest frequency after (n-1) times convolution becomes $\omega_{Max} = \omega_M + (n-1)\omega_M = n\omega_M$.

Therefore, the Nyquist rate is $2 \times n\omega_M = 2n$.

[Problem 3 – Characterization | 15 points]

(a) [10 pts] Design a <u>first-order</u> IIR high-pass filter $H(j\omega)$ with the following frequency response.



(b) [5 pts] Find the impulse response $h(t)$ of this filter. Is this filter stable?

Answers:

(a) $H(j\omega) = 10\left(\dfrac{j\omega}{j\omega+100}\right)^* = 10\left(\dfrac{j\omega}{j\omega-100}\right)$

$H(j\omega) = 10\left(\dfrac{j\omega}{j\omega+100}\right)^*$: phase is inverted from typical high-pass filter with break frequency $\omega = 100$

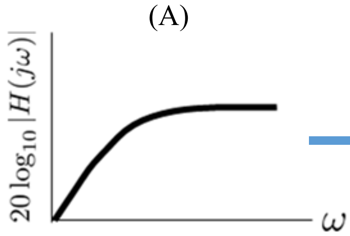
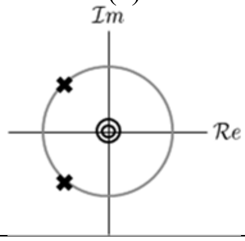
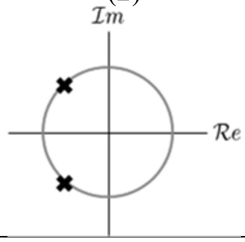
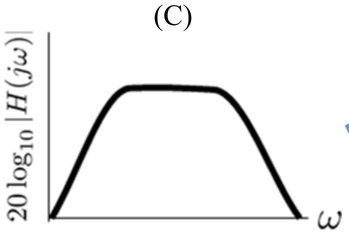
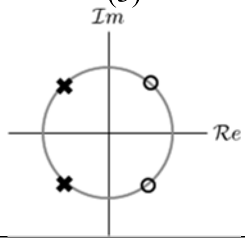
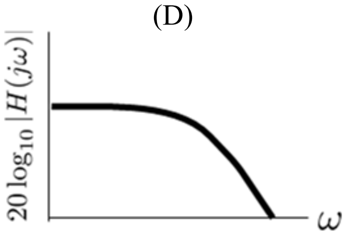
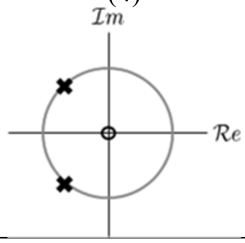
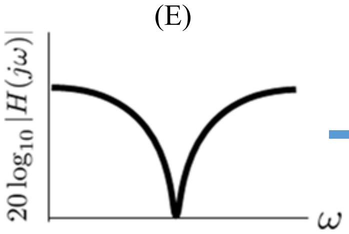
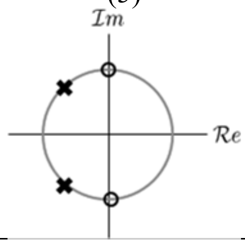
The factor of 10 comes from the magnitude at pass band ($j\omega \to \infty$, $H(j\omega) \sim 20dB$)

(b) $h(t) = 10\left(\delta(t) + 100e^{100t}u(t)\right)$, unstable

(b) $H(j\omega) = 10\left(1 + \dfrac{100}{j\omega - 100}\right)$ → CTFT : $h(t) = 10\left(\delta(t) + 100e^{100t}u(t)\right)$: unstable

[Problem 4 – Laplace transform | 25 points]

(a) [5 pts for each correct pair] Find the matched pairs (e.g., (A)-(1)-(I)) among the given bode plots, pole-zero plots and system responses.

Bode plot	Pole-zero plot	System response $H(s)$
(A) 	(1) 	(I) $\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
(B) 	(2) 	(II) $\dfrac{2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2}$
(C) 	(3) 	(III) $\dfrac{s^2+\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
(D) 	(4) 	(IV) $\dfrac{s^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
(E) 	(5) 	(V) $\dfrac{s^2-2\zeta\omega_n s+\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Highpass: (A)-(1)-(IV), All-pass: (B)-(3)-(V), Band-pass: (C)-(4)-(II),

Low-pass: (D)-(2)-(I), Band-stop: (E)-(5)-(III)

[Problem 5 – z-transform I | 35 points]

1. [20 pts] Consider a causal LTI system that produces output $y[n]=(-\frac{1}{2})^n u[n]+2u[n]$ for input $x[n]=u[n]$.

(a) **[5 pts]** Find impulse response $h[n]$ of the given LTI system.

Answer: $h[n]=3\left(-\frac{1}{2}\right)^n u[n]$

Solution) Z-transform of input & output :

$$X(z)=\frac{1}{1-z^{-1}} \quad , \quad Y(z)=\frac{1}{1+\frac{1}{2}z^{-1}}+\frac{2}{1-z^{-1}}=\frac{3}{(1+\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{3}{1+\frac{1}{2}z^{-1}}$$

$$h[n]=\mathcal{Z}^{-1}\{H(z)\}=3\left(-\frac{1}{2}\right)^n u[n]$$

Alternative) $y[n]$ is a step response, so its impulse response can be obtained by $h[n]=y[n]-y[n-1]$.

$$\begin{aligned} h[n] &= y[n]-y[n-1] \\ &= (-\tfrac{1}{2})^n u[n]+2u[n]-(-\tfrac{1}{2})^{n-1}u[n-1]-2u[n-1] \\ &= (-\tfrac{1}{2})^n u[n]+\underbrace{2\delta[n]+2(-\tfrac{1}{2})^n u[n-1]}_{2\left(-\frac{1}{2}\right)^n u[n]}=3(-\tfrac{1}{2})^n u[n] \end{aligned}$$

(b) **[5 pts]** Determine whether the given system is stable or not.

Answer: stable

Solution) In $H(z)=\frac{Y(z)}{X(z)}=\frac{3}{1+\frac{1}{2}z^{-1}}$, a single pole $z=-\frac{1}{2}$ exists inside the unit circle. Since the system is causal, ROC is outside from the pole location and hence includes the unit circle.

Alternative) From BIBO stability, $\sum_{n=-\infty}^{\infty}|h[n]|<\infty$ should be satisfied.

$$\sum_{n=-\infty}^{\infty}|h[n]|=\sum_{n=-\infty}^{\infty}3(\tfrac{1}{2})^n u[n]=\sum_{n=0}^{\infty}3(\tfrac{1}{2})^n=3\frac{1}{1-\frac{1}{2}}=6<\infty$$

(c) **[5 pts]** Describe the frequency-response characteristic (e.g., high-pass/ low-pass/ band-pass/ all-pass) of $g[n]$

given by

$$g[n]=\begin{cases} h[n/2] & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}, \text{ where } h[n] \text{ is the impulse response of prob. 1(a).}$$

Answer: band-pass system

Solution) (1) A single pole of $H(z)$ is located at $z=-\frac{1}{2}$, so the maximum occurs at the highest DT frequency ($\omega=\angle z=\pm\pi$). Therefore, original h[n] is a high-pass filter.

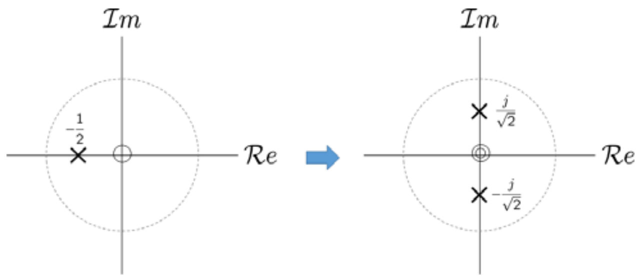
(2) From the scaling property of z-transform, expansion of a sequence ($h[n/2]$) in time is equivalent to the compression of frequency spectrum. So the peak moves to $\pm\frac{\pi}{2}$ and $g[n]$ becomes a band pass filter

(d) **[5 pts]** Draw the pole-zero plot of $g[n]$.

Answer:

Solution) $H(z)=\dfrac{Y(z)}{X(z)}=\dfrac{3}{1+\frac{1}{2}z^{-1}}=\dfrac{3z}{z+\frac{1}{2}}$: single pole at $z=-\frac{1}{2}$ and single zero at $z=0$

$G(z)$ is a compressed along the frequency axis (in angle of z) = $H(z^2)$, $H(z^2)=\dfrac{3z^2}{z^2+\frac{1}{2}}$



2. [15 pts] Consider a general form of system response, described by poles p_m and zeros q_ℓ.

$$H(z)=H_0\frac{\prod_{\ell=1}^{L}(1-q_\ell z^{-1})}{\prod_{m=1}^{M}(1-p_m z^{-1})}=H_0\frac{(1-q_1z^{-1})(1-q_2z^{-1})\times\cdots\times(1-q_Lz^{-1})}{(1-p_1z^{-1})(1-p_2z^{-1})\times\cdots\times(1-p_Mz^{-1})} \quad \text{--- (Eq. 6)}$$

When both poles and zeros are inside the unit circle ($|p_m|,|q_\ell|<1$ for all m,ℓ), the system is called a "minimum phase system".

(a) **[5 pts]** A Maclaurin series of a logarithmic function is described as

$$\log\left(\frac{1}{1-x}\right)=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots=\sum_{n=1}^{\infty}\frac{x^n}{n}, \qquad \text{for } |x|<1 .$$

Using the Maclaurin series, express $\log H(z)$ in terms of poles p_m, zeros q_ℓ, and z.

(b) **[10 pts]** Show that the sequence $c[n]$, defined as the inverse DT Fourier transform of $\log H(e^{j\omega})$, is **causal** for the minimum phase system $H(z)$. In other words, show that

$$c[n] = \mathcal{IDTFT}\left\{\log H(z)\big|_{z=e^{j\omega}}\right\} = \frac{1}{2\pi}\int_0^{2\pi}\left[\log H(e^{j\omega})\right]e^{j\omega n}d\omega$$

is causal for a minimum phase system $H(z)$.

(a) The logarithm of $H(z)$ can be written as

$$\begin{aligned}\log H(z) &= \log\left\{H_0\frac{\prod_{\ell=1}^{L}(1-q_\ell z^{-1})}{\prod_{m=1}^{M}(1-p_m z^{-1})}\right\}\\ &= \log H_0 + \sum_{\ell=1}^{L}\log\left(1-q_\ell z^{-1}\right) - \sum_{m=1}^{M}\log(1-p_m z^{-1})\\ &= \log H_0 - \sum_{\ell=1}^{L}\log\left(\frac{1}{1-q_\ell z^{-1}}\right) + \sum_{m=1}^{M}\log\left(\frac{1}{1-p_m z^{-1}}\right) \quad \cdots (A1)\end{aligned}$$

Note that, if the system is a minimum phase system, then all poles and zeros are inside the unit circle ($\left|q_\ell z^{-1}\right|, \left|p_m z^{-1}\right| < 1$). Therefore, the Maclaurin series always exists. From the Maclaurin series, each term in the parenthesis of (A1) can be expanded to

$$\begin{aligned}\log\left(\frac{1}{1-q_\ell z^{-1}}\right) &= \sum_{n=1}^{\infty}\frac{\left(q_\ell z^{-1}\right)^n}{n} = \sum_{n=1}^{\infty}\left(\frac{q_\ell^n}{n}\right)z^{-n}\\ &= \mathcal{Z}\left\{\frac{q_\ell^n}{n}u[n-1]\right\} \qquad (\mathcal{Z}: z\text{-transform})\end{aligned},$$

which is the z-transform of a sequence $q_\ell[n] = \frac{q_\ell^n}{n}u[n-1]$. Therefore, equation A1 can be arranged to

$$\log H(z) = \log H_0 + \sum_{n=1}^{\infty}\left(-\sum_{\ell}\frac{q_\ell^n}{n} + \sum_{m}\frac{p_m^n}{n}\right)z^{-n} \quad \cdots \quad (A2)$$

(b) For $z = e^{j\omega}$, the z-transform is equivalent to the DTFT, and hence, (A2) can be expressed as

$$\log H(z) = \log H_0 + \sum_{n=1}^{\infty}\left(-\sum_{\ell}\frac{q_\ell^n}{n} + \sum_{m}\frac{p_m^n}{n}\right)z^{-n}$$

$$\log H(e^{j\omega}) = \mathcal{DTFT}\left\{(\log H_0)\delta[n] - \sum_{\ell=1}^{L}q_\ell[n] + \sum_{m=1}^{M}p_m[n]\right\}$$

where $q_\ell[n] = \frac{q_\ell^n}{n}u[n-1]$ and $p_m[n] = \frac{p_m^n}{n}u[n-1]$.

Consequently, the inverse DTFT of $\log H(e^{j\omega})$ is given by

$$c[n] = \mathcal{IDTFT}\left\{\log H(e^{j\omega})\right\}$$
$$= (\log H_0)\delta[n] - \sum_{\ell=1}^{L} q_\ell[n] + \sum_{m=1}^{M} p_m[n].$$

Because all $q_\ell[n]$ and $p_m[n]$ are right sided, causal sequences, the sequence $c[n]$ is causal.

[Extra problem – Matlab | ?? points]

A discrete-time sequence with N=100 samples is transformed to spectrum data using fft function of Matlab. When the sampling rate of the sequence was $f_s = 200\,\text{Hz}$, what is the frequency (in Hz) of the 51$^{\text{st}}$ data sample (beginning from the index 1) in the spectrum data?

Answer) The frequency interval between samples: $\Delta f = \frac{f_s}{N} = 2\,\text{Hz}$. The frequency of the n^{th} sample is given by $f = \Delta f\,(n-1), \quad (n = 1, \cdots N)$. Therefore, the 51$^{\text{st}}$ sample represents the spectrum at $f = 50 \times 2 = 100$ Hz (Nyquist frequency = $f_s / 2$ ($\neq$ Nyquist rate))