Pre-lab report Module 2

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1. Derive the orthogonality between two different frequency components. Consider both cases $k = l$ and $k \neq l$.

Orthogonality

$$\sum_{n=0}^{N-1} e^{j\Omega_k n} e^{-j\Omega_\ell n} = \begin{cases} N, & k-\ell = 0, \pm N, \pm 2N, \cdots \\ 0, & \text{otherwise} \end{cases}$$

hint: use the power series formula: $\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \frac{1-\alpha^N}{1-\alpha}, & \text{if } \alpha \neq 1 \\ N, & \text{if } \alpha = 1 \end{cases}$

Using hint, we can express $\sum_{n=0}^{N-1} e^{j\Omega_k n} e^{-j\Omega_l n}$ as

$$\sum_{n=0}^{N-1} e^{j(\Omega_k - \Omega_l)n} = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n} = \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}(k-l)}\right)^n = \frac{1-e^{j(k-l)2\pi}}{1-e^{j(k-l)\frac{2\pi}{N}}}$$

Because $e^{j2\pi}$ is always 1, $1-e^{j(k-l)2\pi}$ always becomes 0 regardless of the value $(k-l)$.
Therefore, in case $1-e^{j(k-l)\frac{2\pi}{N}} \neq 0$ (i.e. $k \neq l$), the result value of $\sum_{n=0}^{N-1} e^{j\Omega_k n} e^{-j\Omega_l n}$ is 0. Otherwise(i.e. when $k = l$), result value of $\sum_{n=0}^{N-1} e^{j\Omega_k n} e^{-j\Omega_l n}$ is $\frac{1-\alpha^N}{1-\alpha} = N \ (\because \alpha = 1)$

$$\therefore \sum_{n=0}^{N-1} e^{j\Omega_k n} e^{-j\Omega_l n} = \begin{cases} N, & k-l = 0, \pm N, \pm 2N, \cdots \\ 0, & otherwise \end{cases}$$

2. For a sine signal given below, derive the DFT X[k].

$$x[n] = Asin(\frac{2\pi}{N}ln + \theta)$$

Using Euler's equation,

$$x[n] = A\left(\frac{e^{j(\frac{2\pi}{N}ln+\theta)} - e^{-j(\frac{2\pi}{N}ln+\theta)}}{2j}\right) = \frac{A}{2j}\left(e^{j\theta}e^{j\frac{2\pi}{N}ln} - e^{-j\theta}e^{-j\frac{2\pi}{N}ln}\right)$$

By analysis eqaution,

$$X[k] = \sum_{n=0}^{N-1} \frac{A}{2j}\left(e^{j\theta}e^{j\frac{2\pi}{N}ln} - e^{-j\theta}e^{-j\frac{2\pi}{N}ln}\right)e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{Ae^{j\theta}}{2j}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}ln}e^{-j\frac{2\pi}{N}kn} - \frac{Ae^{-j\theta}}{2j}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}ln}e^{-j\frac{2\pi}{N}kn} = \frac{Ae^{j\theta}}{2j}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(l-k)n} - \frac{Ae^{-j\theta}}{2j}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(l+k)n}$$

By Orthogonality shown in Problem 1, $\sum e^{j\frac{2\pi}{N}(l-k)n}$ is N when $l = k$ and 0 when others. Similar to $\sum e^{-j\frac{2\pi}{N}(l+k)n}$.

$$\therefore X[k] = \frac{ANe^{j\theta}}{2j}\delta[k-l] - \frac{ANe^{-j\theta}}{2j}\delta[k+l]$$

3. Discuss the role of 'power normalization' in the code example 3. Explain why the code provides the proper normalization of power for a windowed signal.

Role: To reduce the spectral leakage, we apply a window function(in this case, hanning function). In the process of applying window function, the power of the signal can be changed from the original signal. Therefore, we need the process "Power normalization" to make the power of windowed signal and original signal statistically same.

Why: Let's see the code

```
xw = xw / np.linalg.norm(np.hanning(N)) * np.sqrt(N)
```

np.hanning(N): Produces hanning window w[n]

np.linalg.norm(np.hanning(N)): Make window's 2-norm $\sqrt{\sum_{n=0}^{N-1}|w[n]|^2}$

np.sqrt(N): Square root of sample size N

The power of original signal is $P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$ and the power of windowed signal is $P_{xw} = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]w[n]|^2$.

To be $P_x = P_{xw}$, let's say $w_{norm}[n] = C \cdot w[n]$. From the equation of $P_{xw} = P_x$,

$$\frac{1}{N}\sum_{n=0}^{N-1}|x[n]\cdot(C\cdot w[n])|^2 = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$$

$$C^2\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2|w[n]|^2 \approx \left(\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2\right)\cdot\left(\frac{1}{N}\sum_{n=0}^{N-1}|w[n]|^2\right)$$

$$C = \sqrt{\frac{N}{\sum_{n=0}^{N-1}|w[n]|^2}} = \frac{\sqrt{N}}{\sqrt{\sum_{n=0}^{N-1}|w[n]|^2}}$$

which is same as power normalization in the code example 3.

4. Explain the role of the following Raspberry Pi configuration in cmdline.txt (Week 1).

```
modules-load=dwc2,g_ether
```

This configuration is a kernal parameter that automatically load two specific kernel modules during boot process.

1. dwc2: This enables Raspberry Pi's USB port to function in "gadget" mode. In gadget mode, the Pi can act as a device itself when connected to another host.

2. g_ether: USB gadget Ethernet module. This module makes the Raspberry Pi emulate a virtual Ethernet device (like a USB network adapter).

These allow PC to connect to the Raspberry Pi (via SSH) using only a USB cable. And computer will recognize the Raspberry Pi as a network device.