- Please submit before the submission deadline.
- Assignment submitted one (1) day after the assignment deadline will be accepted with 20% deduction on corresponding assignment grade.
- Assignment submitted more than one (1) day late will not be accepted.

Problem 1: Solve by unrolling (25 points)

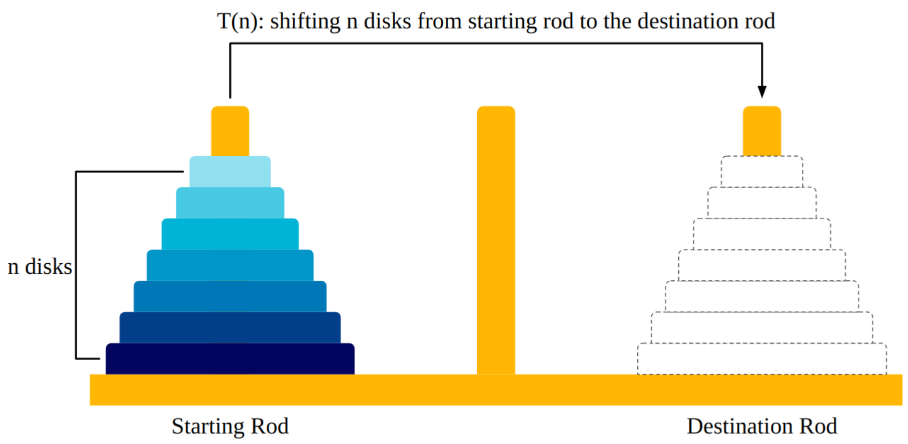

Figure 1: Tower of Hanoi

The Tower of Hanoi is the problem with moving a set of disk from the starting rod to the destination rod. It has several restrictions while moving the disks:

- One disk should be moved at a time.
- A bigger disk cannot be placed on top of a smaller disk.
- A disk can be placed on an empty rod or a disk bigger than the current disk.

(a) (7 points) Assume there are n disks, and $T(n)$ denotes the number of movements for shifting n disks from the starting rod to the destination rod. Find the recurrence relation of the T(n). Briefly explain what each term represents from the recurrence relation.

$$T(n) = 2T(n-1) + 1$$

Explanation of each term:
$T(n-1)$ denotes moving $n-1$ disks to the empty(second) rod.
1 is for moving the largest n^{th} disk to the destination rod.
Another $T(n-1)$ takes to move $n-1$ disks to the destination rod.

(b) (18 points) Solve the recurrence relation by unrolling. You need to unroll at least twice. Derive the non-recursive formula and the complexity of solving the Tower of Hanoi.

$$\begin{aligned}
T(n) &= 2T(n-1) + 1 \\
&= 2[2T((n-1)-1) + 1] + 1 \\
&= 4T(n-2) + 3 \\
&= 4[2T((n-1)-2) + 1] + 3 \\
&= 8T(n-3) + 7
\end{aligned}$$

By unrolling, you get $T(n) = 1+2+4+...+2^{n-1}$. It is a geometry sequence with a common ratio of 2.

$$\begin{aligned} 2T(n) - T(n) &= (2+4+...+2^n) - (1+2+...+2^{n-1}) \\ &= 2^n - 1 \\ T(n)/(2-1) &= 2^n - 1 \end{aligned}$$

Therefore, $T(n) = \Theta(2^n)$.

Grading Criteria:

(a) (+1.5 point) for explaining T(n-1) each
(+1 point) for explaining 1
(+3 point) for correct T(n)

(b) (+2 point) for each unrolling
(-2 point) missing one unroll (sorry)
(+11 point) for solving the geometry sequence
(-2 point) for expanding up to 2^n
(-4 point) missing explanation how to derive the formula
(+3 point) for correct answer
(no deduction) for writing exponential
(2 points) for formula, (1 points) for asymptotic notation

Problem 2: Solve by substitution (30 points)

Prove the following statement using the substitution method.
(Hint: If it is difficult to prove using $T(n) \le c_1 n \log n$, try to use $(n - c_2)$ instead of just n)

$$T(n) = 7T(n/7 + 2024) + n = O(n \log n)$$

Assume $T(k) \le c_1(k - c_2)\log(k - c_2)$ for $k < n$.
We want to prove $T(n) \le c_1(n - c_2)\log(n - c_2)$.

$$T(n) \le 7(c_1(n/7 + 2024 - c_2)\log(n/7 + 2024 - c_2)) + n$$

$$\begin{aligned} &= c_1(n + 14168 - 7c_2)\log\left(\frac{n + 14168 - 7c_2}{7}\right) + n \\ &= c_1(n + 14168 - 7c_2)\log(n + 14168 - 7c_2) - c_1(n + 14168 - 7c_2) + n \end{aligned}$$

$$\le c_1(n + 14168 - 7c_2)\log(n + 14168 - 7c_2),\ (c_1 > 1, n > n_0)$$
$$\le c_1(n - c_2)\log(n - c_2),\ (c2 \ge 14168)$$
$$T(n) = O(n \log n)$$

Grading Criteria:

2. (+3 point) for correct guess and induction hypothesis
(+7 point) for substituting $n/7$ into equation
(+13 point) for reaching into I.H. form
(+5 point) for range of constant
(+2 point) for showing $O(n \log n)$

Problem 3: Master Theorem (45 Points)

Solve the following recurrence relations using the master theorem. Examine $f(n)$ with the condition and specify the case. Then, determine the asymptotic tight bound for each $T(n)$ (assume $T(1) = \Theta(1)$). If the master theorem cannot be used, state the reason.

Master Theorem Let $a \geq 1$ and $b \geq 1$ be constants, let $f(n)$ be a non-negative function, and let $T(n)$ be defined on the non-negative integers by the recurrence

$T(n) = aT(n/b) + f(n)$

where we interpret n/b to be either $\lfloor \frac{n}{b} \rfloor$ or $\lceil \frac{n}{b} \rceil$. Then $T(n)$ has the following asymptotic bounds:

Case 1: If $f(n) = O(n^{\log_b a-\epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
Case 2: If $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some constant $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
Case 3: If $f(n) = \Omega(n^{\log_b a+\epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

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a) (7 Points) $T_1(n) = T_1(2n/5) + 1$

$$\begin{aligned} f(n) &= 1 \\ n^{\log_b a} &= n^{\log_{5/2} 1} = 1 \\ f(n) &= 1 = \Theta(1 \times \log^0 n), \text{ Case 2 (k=0)} \\ T(n) &= \Theta(\log n) \end{aligned}$$

b) (7 Points) $T_2(n) = 24T_2(n/3) + n^3$

$$\begin{aligned} f(n) &= n^3 \\ n^{\log_b a} &= n^{\log_3 24} = n^{2.8927} \\ f(n) &= n^3 = \Omega(n^{2.89+\epsilon}), \text{ Case 3} \\ 24f(n^3/27) &\leq cf(n^3) \text{for } c = 8/9 \leq 1 \\ T(n) &= \Theta(n^3) \end{aligned}$$

c) (7 Points) $T_3(n) = 8T_3(n/2) + n^2 \log n$

$$\begin{aligned} f(n) &= n^2 \log n \\ n^{\log_b a} &= n^{\log_2 8} = n^3 \\ f(n) &= n^2 \log n = O(n^{3-\epsilon}), \text{ Case 1} \\ T(n) &= \Theta(n^3) \end{aligned}$$

Depending on how the meaning of polynomially smaller is defined, the below answers are also accepted (In the exam, the recurrence relation would fall into the clear case).
Accepted Answer:

- State that master theorem cannot be used since it falls in between case 1 and case 2.
  $f(x) \leq g(x)$ for some polynomial g, and $\log n$ is not polynomial different.
- Use L'Hopital's rule to find the polynomial difference.
  Extend polynomially smaller as $f(x) \leq g(x)$ if $df(x) \leq dg(x)$.

d) (10 Points) $T_4(n) = 900T_4(n/30) + n^2$

$$\begin{aligned}
f(n) &= n^2 \\
n^{\log_b a} &= n^{\log_{30} 900} = n^2 \\
f(n) &= n^2 = \Theta(n^2), \text{ Case 2 (k=0)} \\
T(n) &= \Theta(n^2 \log n)
\end{aligned}$$

e) (14 Points) $T_5(n) = 2T_5(\sqrt[3]{n}) + \log_3 n$ (Hint: Use a substitution $S(m) = T(3^m) = T(n)$)

$$\begin{aligned}
T_5(n) &= 2T_5(\sqrt[3]{n}) + \log_{3]} n \\
T_5(3^m) &= 2T_5(3^{m/3}) + \log_3 3^m \\
S(m) &= 2S(m/3) + \log_3 m \\
f(m) &= \log_3 m \\
m^{\log_b a} &= m^{\log_3 2}(\log_3 2 \approx 0.63) \\
f(m) &= \log_3 m = \Omega(m^{\log_3 2+\epsilon}), \text{ Case 3} \\
2f(m/3) &\leq 3cf(m/3) \text{ for } c = 2/3 \leq 1 \\
S(m) &= \Theta(m) \\
T_5(n) &= \Theta(\log_3 n)
\end{aligned}$$

Grading Criteria:

(a-c) (+3 point) for no error when applying master's theorem
(−2 point) for no regularity condition for (b)
(+1 point) for $f(n)$
(+1 point) for the case (or show through asymptotic notation)
(+2 point) for correct answer

(d) (+4 point) for no error when applying master's theorem
(+2 point) for $f(n)$
(+2 point) for the case (or show through asymptotic notation)
(+2 point) for correct answer

(e) (+4 point) for no error when applying master's theorem
(+2 point) for $f(n)$
(+2 point) for the case (or show through asymptotic notation)
(+4 point) for regularity condition
(−4 point) for no regularity condition
(−2 point) for wrong c
(+2 point) for correct answer