1

(a) $P(X = 2) = \frac{1}{6}$, $P(X = 3) = \frac{1}{3}$, $P(X = 4) = \frac{1}{2}$.

(b) Note that $E(X) = \frac{10}{3}$. Since $E(X^2) = \frac{35}{3}$,

$$E\left[\left(X - \frac{10}{3}\right)^2\right] = V(X) = E(X^2) - E(X)^2 = \frac{5}{9}.$$

2

(a) Since

$$\int_{-1}^{0} \frac{1}{5}\,dx + \int_{0}^{1} \left(\frac{1}{5} + cx\right) dx = \frac{2}{5} + \frac{c}{2} = 1,$$

we get $c = \frac{6}{5}$.

(b) $F(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{5}(1+x) & -1 < x \le 0 \\ \frac{1}{5}(1 + x + 3x^2) & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$

(c) $F(-1) = 0$, $F(0) = \frac{1}{5}$, and $F(1) = 1$.

(d) $P(0 \le X \le \frac{1}{2}) = F(\frac{1}{2}) - F(0) = \frac{9}{20} - \frac{1}{5} = \frac{1}{4}$.

(e)

$$P\left(X > \frac{1}{2} \middle| X > \frac{1}{10}\right) = \frac{P(X > \frac{1}{2})}{P(X > \frac{1}{10})} = \frac{0.55}{0.774} = 0.71.$$

3 (a)

$$P\left(X < \frac{1}{2}, Y > \frac{1}{4}\right) = \int_{\frac{1}{4}}^{1} \int_{0}^{\frac{1}{2}} (x+y)\,dxdy = \frac{21}{64}$$

(b)

$$P(X + Y \le 1) = P(X \le 1 - Y) = \int_{0}^{1} \int_{0}^{1-y} (x+y)\,dxdy = \frac{1}{3}$$

(c)

$$f_X(x) = \int_0^1 (x+y)\,dy = x + \frac{1}{2}, \qquad 0 \le x \le 1$$

Similarly,

$$f_Y(y) = y + \frac{1}{2}, \qquad 0 \le y \le 1.$$

(d) Since

$$P\left(Y \ge \frac{1}{2}\right) = \int_{\frac{1}{2}}^{1} \left(y + \frac{1}{2}\right)\,dy = \frac{5}{8},$$
$$P\left(X \ge \frac{1}{2}, Y \ge \frac{1}{2}\right) = \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} (x+y)\,dxdy = \frac{3}{8},$$

we get $P(X \ge \frac{1}{2} \big| Y \ge \frac{1}{2}) = \frac{3}{5}$.

(e)

$$P\left(X > \frac{3}{4} \middle| Y = \frac{1}{2}\right) = \frac{\int_{\frac{3}{4}}^{1} \left(x + \frac{1}{2}\right)\,dx}{\frac{1}{2} + \frac{1}{2}} = \frac{11}{32}.$$

(f) Since $f(x,y) \ne f_X(x) f_Y(y)$, X and Y are dependent.

4 (a)

$$f_X(x) = \int_0^x 3x\,dy = 3x^2, \qquad 0 \le x \le 1,$$
$$f_Y(y) = \int_y^1 3x\,dx = \frac{3}{2}(1-y^2), \qquad 0 \le y \le 1.$$

(b) No, since $f(x,y) \neq f_X(x)f_Y(y)$.

(c) $f_{X|Y}(x|Y=y) = \frac{2x}{(1-y^2)}$, $y \le x \le 1$.

(d)

$$P\left(X \le \frac{3}{4}\middle| Y = \frac{1}{2}\right) = \frac{\int_{\frac{1}{2}}^{\frac{3}{4}} 2x\,dx}{1-\frac{1}{2^2}} = \frac{5}{12}$$

(e) Since

$$E(X) = \int_0^1 3x^3\,dx = \frac{3}{4},$$
$$E(Y) = \int_0^1 \frac{3}{2}y(1-y^2)\,dy = \frac{3}{8},$$
$$E(XY) = \int_0^1 \int_y^1 3x^2 y\,dx\,dy = \frac{3}{10},$$

we get $Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{3}{160}$.

5 (a)

$$E(T_1) = E\left(\frac{X_1+X_2+X_3}{3}\right) = \frac{1}{3}\sum_{i=1}^{3} E(X_i) = \mu,$$
$$E(T_2) = E\left(\frac{X_1+2X_2+2X_3}{5}\right) = \frac{1}{5}\left(E(X_1)+2E(X_2)+2E(X_3)\right) = \mu,$$

so, $E(T_1) = E(T_2)$.

(b)

$$V(T_1) = \frac{1}{9}\sum_{i=1}^{3} V(X_i) = \frac{\sigma^2}{3},$$
$$V(T_2) = \frac{1}{25}\left(V(X_1)+4V(X_2)+4V(X_3)\right) = \frac{9\sigma^2}{25},$$

so, $V(T_1)$ is smaller than the other.

4.9

$$\begin{aligned}
P(N_1=1,N_2=1) &= \frac{3}{5}\times\frac{2}{4}=\frac{3}{10},\\
P(N_1=1,N_2=2) &= \frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}=\frac{1}{5},\\
P(N_1=1,N_2=3) &= \frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}=\frac{1}{10},\\
P(N_1=2,N_2=1) &= \frac{2}{5}\times\frac{3}{4}\times\frac{2}{3}=\frac{1}{5},\\
P(N_1=2,N_2=2) &= \frac{2}{5}\times\frac{3}{4}\times\frac{1}{3}=\frac{1}{10},\\
P(N_1=3,N_2=1) &= \frac{2}{5}\times\frac{1}{4}=\frac{1}{10},
\end{aligned}$$

and $P(N_1,N_2)=0$ otherwise.

4.10

(a) Since

$$\begin{aligned}
\int_0^2\int_0^1 f(x,y)\,dx\,dy &= \int_0^2\int_0^1 \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\,dx\,dy\\
&= \int_0^2 \frac{6}{7}\left[\frac{1}{3}x^3+\frac{x^2y}{4}\right]_0^1\,dy\\
&= \frac{6}{7}\left[\frac{1}{3}y+\frac{1}{8}y^2\right]_0^2=1,
\end{aligned}$$

it is indeed a joint density function.

(b)

$$f_X(x)=\int_0^2\frac{6}{7}\left(x^2+\frac{xy}{2}\right)\,dy=\frac{12}{7}x^2+\frac{6}{7}x,\qquad 0<x<1$$

(c)

$$\begin{aligned}
P(X>Y) &= \int_0^1\int_y^1\frac{6}{7}\left(x^2+\frac{xy}{2}\right)\,dx\,dy\\
&= \int_0^1\frac{6}{7}\left[\frac{1}{3}x^3+\frac{x^2y}{4}\right]_y^1\,dy\\
&= \int_0^1\frac{6}{7}\left(\frac{1}{3}+\frac{1}{4}y-\frac{7}{12}y^3\right)\,dy=\frac{15}{56}
\end{aligned}$$

4.11

$$F_M(x)=P(M\le x)=P(X_1\le x,\cdots,X_n\le x)=\prod_{i=1}^n P(X_i\le x)=x^n,\qquad 0\le x\le 1$$

By differentiating F_M, we get the probability density function

$$f_M(x)=nx^{n-1},\qquad 0\le x\le 1$$

4.45

(a) $P(X_1 = 0) = \frac{3}{16}$, $P(X_1 = 1) = \frac{1}{8}$, $P(X_1 = 2) = \frac{5}{16}$, $P(X_1 = 3) = \frac{3}{8}$. Also, $P(X_2 = 1) = P(X_2 = 2) = \frac{1}{2}$.

(b)

$$\begin{aligned}
E[X_1] &= 1 \times \frac{1}{8} + 2 \times \frac{5}{16} + 3 \times \frac{3}{8} = \frac{15}{8}, \\
E[X_2] &= \frac{3}{2}, \\
E[X_1^2] &= 1 \times \frac{1}{8} + 4 \times \frac{5}{16} + 9 \times \frac{3}{8} = \frac{19}{4}, \\
E[X_2^2] &= \frac{5}{2}, \\
E[X_1X_2] &= 1 \times \frac{1}{16} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 4 \times \frac{1}{8} + 6 \times \frac{1}{4} = \frac{47}{16}, \\
Var[X_1] &= E[X_1^2] - E[X_1]^2 = \frac{19}{4} - \left(\frac{15}{8}\right)^2 = \frac{79}{64}, \\
Var[X_2] &= E[X_2^2] - E[X_2]^2 = \frac{5}{2} - \frac{9}{4} = \frac{1}{4}, \\
Cov(X_1, X_2) &= E[X_1X_2] - E[X_1]E[X_2] = \frac{1}{8}.
\end{aligned}$$

4.52

$$\begin{aligned}
Cov(X_1 - X_2, X_1 + X_2) &= E[(X_1 - X_2)(X_1 + X_2)] - E[X_1 - X_2]E[X_1 + X_2] \\
&= E[X_1^2 - X_2^2] - (E[X_1] - E[X_2])(E[X_1] + E[X_2]) \\
&= E[X_1^2] - E[X_2^2] - (E[X_1]^2 - E[X_2]^2) \\
&= Var(X_1) - Var(X_2) \\
&= 0
\end{aligned}$$