School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2023 Spring

Final

Tuesday, June 13, 2023
9:00–11:30 am

NAME: ______________________

Student ID: ______________________

- Don't forget to put your name and student ID.
- **Record all your solutions in this answer booklet. Only this answer booklet will be considered in the grading of your exam.**
- Be sure to show all relevant work and reasoning. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.

Problem	Your score	Max score
1		10
2		10
3		10
4		10
Total		40

Problem 1 **(10 Points)**

Consider three random variables Θ, X, and Y, with known variances $\mathrm{var}(\Theta)$, $\mathrm{var}(X)$, and $\mathrm{var}(Y)$, and covariances, $\mathrm{cov}(\Theta, X)$, $\mathrm{cov}(\Theta, Y)$, and $\mathrm{cov}(X, Y)$. Assume that $\mathbb{E}[\Theta] = \mathbb{E}[X] = \mathbb{E}[Y] = 0$, $\mathrm{var}(X) > 0$, $\mathrm{var}(Y) > 0$, and $|\rho(X, Y)| \neq 1$. (Remind that $\rho(A, B) = \mathrm{cov}(A, B)/\sqrt{\mathrm{var}(A)\,\mathrm{var}(B)}$ and for any two zero-mean random variables A, B, $\mathrm{cov}(A, B) = \mathbb{E}[AB]$.)

We consider a linear estimator of Θ based on X and Y, in the form of

$$\hat{\Theta} = aX + bY,$$

for some constants a, b. We aim to choose a, b to minimize the mean squared error $\mathbb{E}[(\Theta - \hat{\Theta})^2]$. Find a and b in terms $\mathrm{var}(\Theta)$, $\mathrm{var}(X)$, $\mathrm{var}(Y)$, $\mathrm{cov}(\Theta, X)$, $\mathrm{cov}(\Theta, Y)$, and $\mathrm{cov}(X, Y)$ for the following two cases.

a) (5 points) Find a and b, when X and Y are uncorrelated, i.e., $\mathbb{E}[XY] = 0$.

Answer:

$a =$ $b =$

Reasoning for Problem 1(a):

b) (5 points) Find a and b for the general case where X and Y are not necessarily uncorrelated.

Answer:

$$a =$$

$$b =$$

Reasoning for Problem 1(b):

<u>Problem 2</u> (10 Points)

Assume that X_i's are independent and identically distributed random variables with mean p. To estimate p, we consider the sample mean defined by

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

a) (5 points) Find the smallest n, the number of samples, for which the Chebyshev inequality yields a guarantee

$$\mathbb{P}(|M_n - p| \geq 0.1) \leq 0.05.$$

Assume that $\text{var}(X_i) = v$ for some constant v. State your answer as a function of v.

Answer:

$$n =$$

Reasoning for Problem 2(a):

b) (5 points) Assume that $n = 10,000$. Find an approximate value for the probability

$$\mathbb{P}(|M_{10,000} - p| \geq 0.1)$$

using the Central Limit Theorem. Assume again that $\text{var}(X_i) = v$ for some constant v. Give your answer in terms of v, and the standard normal CDF $\Phi(\cdot)$.

Answer:

$$\mathbb{P}(|M_{10000} - p| \geq 0.1) \approx$$

Reasoning for Problem 2(b):

Problem 3 **(10 Points)**

In this problem, we consider Poisson processes. Remind that for a Poisson process with rate λ, the probability distribution for the first arrival time T_1 (and also the inter-arrival time $T_k = Y_k - Y_{k-1}$, $k \geq 2$, where Y_k is the k-th arrival time) follows the exponential distribution with rate λ, i.e., $f_{T_1}(t) = \lambda e^{-\lambda t}$, for $t \geq 0$ and $\mathbb{E}[T_1] = 1/\lambda$.

a) (5 points) Consider two independent Poisson processes with rates λ_1 and λ_2, respectively. Let X_1 be the first arrival time in the first process, and X_2 be the first arrival time in the second process. Find the expected value of $\max\{X_1, X_2\}$.

Answer:

$$\mathbb{E}[\max\{X_1, X_2\}] =$$

Reasoning for Problem 3(a):

b) (5 points) Consider two independent Poisson processes with rates λ_1 and λ_2, respectively. Let Y be the first arrival time in the first process and Z be the second arrival time in the second process. Find the expected value of $\max\{Y, Z\}$. (Hint: you may write down your answer in terms of $\mathbb{E}[\max\{X_1, X_2\}]$, defined in (a). You don't need to specify what $\mathbb{E}[\max\{X_1, X_2\}]$ is in terms of λ_1 and/or λ_2.)

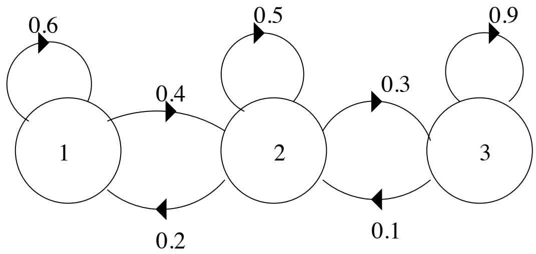
Answer:

$$\mathbb{E}[\max\{Y, Z\}] =$$

Reasoning for Problem 3(b):

Problem 4 (10 Points)

Consider a Markov chain $\{X_n : n = 0, 1, \ldots, \}$, specified by the following transition diagram.



a) (3 points) Find the steady-state probabilities π_1, π_2, π_3 for the states 1,2, and 3.

Answer:

$\pi_1 =$ $\pi_2 =$ $\pi_3 =$

Reasoning for Problem 4(a):

b) (3 points) Let $Y_n = X_n - X_{n-1}$. Thus, $Y_n = 1$ indicates that the n-th transition was to the right, $Y_n = 0$ indicates it was a self-transition, and $Y_n = -1$ indicates it was a transition to the left. Find $\lim_{n\to\infty} \mathbb{P}(Y_n = 1)$.

Answer:

$$\lim_{n\to\infty} \mathbb{P}(Y_n = 1) =$$

Reasoning for Problem 4(b):

c) (4 points) Given that the n-th transition was a transition to the right ($Y_n = 1$), find the probability that the previous state was state 1. (You can assume that n is large.)

Answer:

$$\mathbb{P}(X_{n-1} = 1 | Y_n = 1) =$$

Reasoning for Problem 4(c):