1. For the region r>d, the intensity of the electric field is $\frac{3q}{2\pi\varepsilon_0 r^2}$ (O / X)

Solution) Let's use the Gauss's law. Here, we can draw the Gauss surface that has the same concentric center with the two spherical shell as the figure 1. Let's say $r > d$. The Gauss law is

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$$

Because we draw the Gauss surface as a spherical surface, an electric field vector due to the charges of the shells passes through the Gauss surface perpendicularly. In addition, because the Gauss surface is symmetric, each electric vector has the same intensity on this surface. Thus, the integral term becomes/

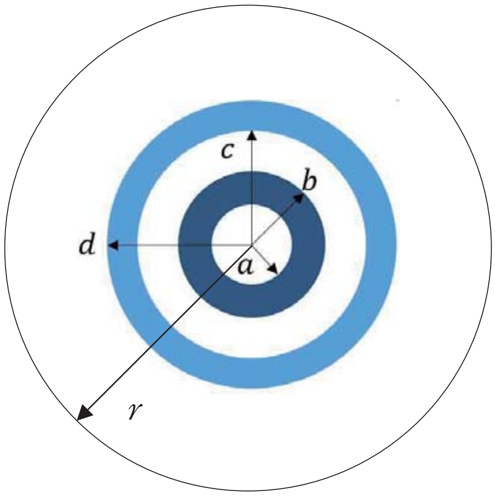

Figure 1. The Gauss surface that hs the radius of r

$$\int E da = E(4\pi r^2) = \frac{q_{enc}}{\varepsilon_0}$$

Here, the total charge inside this Gauss surface is

$$q_{enc} = (+4q) + (+2q) = +6q$$

Thus, the intensity of the electric field for the region $r > d$ is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{+6q}{r^2} = \frac{3q}{2\pi\varepsilon_0 r^2}$$

2. For the region a<r<b, the intensity of the electric field is $\frac{q}{2\pi\varepsilon_0 r}\frac{r^2 - a^2}{b^2 - a^2}$. (O / X) [5 pts]

Solution) Firstly, let's assume that this inner shell has the charge density of $\rho(r) = \rho_0$, as the problem said it has the 'uniformly distributed total positive charge', which means there is no r dependence. We can define this physical quantity as below.

$$\rho_0 = \frac{q}{V} = \frac{+2q}{\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3} = \frac{3q}{2\pi(b^3 - a^3)}$$

Now, let's draw a Gauss surface as we did in the problem 1, but make r be located in the region $a < r < b$. Then, the Gauss law says

$$E(4\pi r^2) = \frac{q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_a^r \rho_0 \, dV = \frac{3q}{2\pi\varepsilon_0(b^3 - a^3)}\int_a^r\int_0^\pi\int_0^{2\pi} r'^2 \sin\theta dr' d\theta d\phi$$

Here, I used the spherical coordinate system to use the volume element $dV = r^2 \sin\theta dr d\theta d\phi$. If we calculate the integral part, we will get answer as below.

$$E(4\pi r^2) = \frac{3q}{2\pi\varepsilon_0(b^3 - a^3)}\frac{4\pi}{3}(r^3 - a^3)$$

$$\therefore E = \frac{q}{2\pi\varepsilon_0 r^2}\frac{r^3 - a^3}{b^3 - a^3}$$

3. Let r=c. What is the surface charge on this surface? [5 pts]

① -2q ② 0 ③ 2q ④ 4q

Solution) Let's use the Gauss low again. But now, let's draw a Gauss surface to be inside the conductor. (You don't need to draw a Gauss surface symmetrically! Just make your Gauss surface to be inside the conductor!) Then,

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$$

Because the Gauss surface is inside the "conductor", there must be no electric field that pass through the Gauss surface. Then, the left-hand side becomes 0. Therefore, the q_{enc} must be 0 too. We know that there is no charge inside the conductor. The conductor only has its charges on its surfaces. Thus, the total q_{enc} can be expressed as

$$q_{enc} = 0 = +2q + q_{r=c}$$

Consequently,

$$q_{r=c} = -2q$$

4. Let r=d. What is the surface charge on this surface? [5 pts]

① 0 ② +2q ③ +4q ④ +6q

Solution) As I mentioned just above, the conductor only can have its charges on its surfaces. The total charge of outer shell is given; $+4q$. Thus,

$$+4q = q_{r=c} + q_{r=d}$$

$$\therefore q_{r=d} = +4q - q_{r=c} = +4q - (-2q) = +6q$$

5. Find an expression for the potential at the point that the electron is located. [15 pts]

Solution) The formula that can give us the electrical potential is

$$V(r)=\frac{q}{4\pi\varepsilon_0 r}$$

Here, $r=\sqrt{s^2+x^2}$. As the annulus carries positive charge Q uniformly, we can define the electrical density as below.

$$\sigma = Q/\pi(b^2-a^2) \;:\; 2 \text{ pts}$$

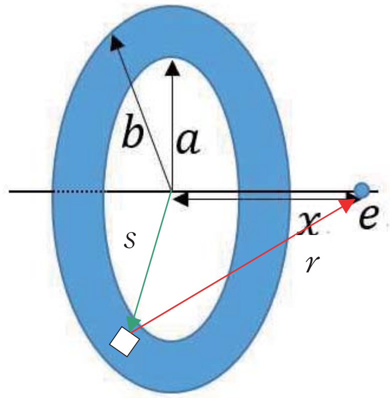

Figure 2. surface element is making an electrical potential at the point that the electron is located.

Thus,

$$dV=\frac{1}{4\pi\varepsilon_0}\frac{\sigma dA}{\sqrt{s^2+x^2}} \;:\; 5 \text{ pts}$$

$$V=\frac{1}{4\pi\varepsilon_0}\int\frac{\sigma dA}{\sqrt{s^2+x^2}}=\frac{\sigma}{4\pi\varepsilon_0}\int_a^b\int_0^{2\pi}\frac{s\,ds\,d\theta}{\sqrt{s^2+x^2}}$$

$$=\frac{1}{4\pi\varepsilon_0}\cdot\left(\frac{Q}{\pi(b^2-a^2)}\right)\cdot 2\pi\cdot\int_a^b\frac{s\,ds}{\sqrt{s^2+x^2}}$$

We can calculate the integral term easily if we set $t^2=s^2+x^2$, $tdt=sds$. In other words, the integration by substitution is a key to calculate this integration. Then,

$$V=\frac{Q}{2\pi\varepsilon_0(b^2-a^2)}\int_{\sqrt{a^2+x^2}}^{\sqrt{b^2+x^2}}dt=\frac{Q}{2\pi\varepsilon_0(b^2-a^2)}\left(\sqrt{b^2+x^2}-\sqrt{a^2+x^2}\right) \;:\; 7 \text{ pts}$$

<u>Of course, the upper end and lower end can be $-\sqrt{b^2+x^2}$ and $-\sqrt{a^2+x^2}$ mathematically. However, we chose plus sign values due to the reason that we want to consider distance between the annulus and the electron as positive number.</u> : 1 pt

6. First, let us locate the electron sufficiently far away from the annulus, but it is still on the axis of it. Evaluate the force vector that the electron feels due to the annulus and interpret this result. [15 pts]

Solution) First, let's derive an electrical field vector due to the annulus before we calculate the force. The relationship between the electrical field and potential in the 3D Cartesian coordinates system is

$$\vec{E}=-\frac{\partial V}{\partial x}\hat{x}-\frac{\partial V}{\partial y}\hat{y}-\frac{\partial V}{\partial z}\hat{z} \;:\; 1 \text{ pt}$$

Because the electron is on the axis of the annulus(I want to set this axis as x-axis. Then, we don't need to consider the others because they don't contribute the motion of the electron. The electric fields of them cancel out.), the electron will also move along the axis. Thus,

$$\vec{E_x}=-\frac{\partial V}{\partial x}\hat{x}=-\frac{\partial}{\partial x}\left[\frac{Q}{2\pi\varepsilon_0(b^2-a^2)}\left(\sqrt{b^2+x^2}-\sqrt{a^2+x^2}\right)\right]\hat{x}$$

$$= \frac{Q}{2\pi\varepsilon_0(b^2-a^2)}\left(\frac{x}{\sqrt{a^2+x^2}}-\frac{x}{\sqrt{b^2+x^2}}\right)\hat{x} \ : \ 2 \ \text{pts}$$

Hence, the electrical force vector that the electron feels due to the annulus is

$$\vec{F} = -e\vec{E} = \frac{-eQ}{2\pi\varepsilon_0(b^2-a^2)}\left(\frac{x}{\sqrt{a^2+x^2}}-\frac{x}{\sqrt{b^2+x^2}}\right)\hat{x}$$

$$= \frac{-eQ}{2\pi\varepsilon_0(b^2-a^2)}\left(\frac{1}{\sqrt{1+\left(\frac{a}{x}\right)^2}}-\frac{1}{\sqrt{1+\left(\frac{b}{x}\right)^2}}\right)\hat{x} \ : \ 3 \ \text{pts}$$

If x is sufficiently large enough, then we can use the Taylor expansion as below.

$$\frac{1}{\sqrt{1+A}} = (1+A)^{-1/2} = 1-\frac{1}{2}A+\frac{3}{8}A^2+\cdots \ \text{(When } A \text{ is small enough)}$$

Consequently,

$$\vec{F} = \frac{-eQ}{2\pi\varepsilon_0(b^2-a^2)}\left(1-\frac{1}{2}\left(\frac{a}{x}\right)^2-1+\frac{1}{2}\left(\frac{b^2}{x}\right)\right)\hat{x} = \frac{-eQ}{2\pi\varepsilon_0(b^2-a^2)}\left(\frac{b^2-a^2}{2x^2}\right)\hat{x}$$

$$= \frac{-eQ}{4\pi\varepsilon_0 x^2}\hat{x} \ : \ 4 \ \text{pts}$$

This result can be interpreted as a point charge Q is exerting an electric force. This interpretation is acceptable, as far away from the annulus(x is sufficiently large enough), the annulus can be regarded as a point. : 5 pts. For example, if you are very far away from the Earth, you will see the Earth as just a Pale blue dot even though the shape of the Earth is very complicated.

7. Now, let us locate the electron sufficiently close to the center of the annulus so that we can consider a distance between them is almost zero. The electron is still on the axis of it. Evaluate the electric force vector that the electron feels due to the annulus. [15 pts]

Solution) Let's use the result of the electric force vector.

$$\vec{F} = -e\vec{E} = \frac{-eQ}{2\pi\varepsilon_0(b^2-a^2)}\left(\frac{x}{\sqrt{a^2+x^2}}-\frac{x}{\sqrt{b^2+x^2}}\right)\hat{x}$$

In this case, x is sufficiently small enough. Thus, x^2 term inside the root can be treated as 0. : 2 pts Hence,

$$\vec{F} = \frac{-eQ}{2\pi\varepsilon_0(b^2-a^2)}\left(\frac{x}{\sqrt{a^2}}-\frac{x}{\sqrt{b^2}}\right)\hat{x} = \frac{-eQ}{2\pi\varepsilon_0(b^2-a^2)}\left(\frac{b-a}{ab}\right)x\hat{x}$$

$$= \frac{-eQ}{2\pi\varepsilon_0 ab(a+b)}x\hat{x} \ : \ 13 \ \text{pts}$$

8. In this case, find the physical quantity that characterizes the motion of the electron. [15 pts]

Solution) Due to the Newton's second law,

$$\vec{F} = m\vec{a} \Rightarrow \frac{-eQ}{2\pi\varepsilon_0 ab(a+b)} x\hat{x} = m_e \frac{d^2x}{dt^2}\hat{x} \ : \ 3 \ \text{pts}$$

Here, m_e is the mass of an electron. Let's divide both side by m_e. Then, the equation becomes

$$\frac{d^2x}{dt^2} = -\frac{eQ}{2\pi\varepsilon_0 m_e ab(a+b)} x = -\omega^2 x$$

This is the equation of harmonic oscillation. : 5 pts

From this, we can figure out the physical quantity ω that can characterize this motion.

$$\omega = \sqrt{\frac{eQ}{2\pi\varepsilon_0 m_e ab(a+b)}} \ : \ 7 \ \text{pts}$$

(Advanced Problem!!) 9. Let's assume that there is a proton instead of the electron. Find the speed of the proton when it goes sufficiently far away from the annulus. Here, the proton starts its motion from the point that is sufficiently close to the center of the annulus. [20 pts]

Solution) We can write the equation of motion as below.

$$\vec{F} = m_p\vec{a} = m_p\frac{d^2x}{dt^2}\hat{x} = \frac{eQ}{2\pi\varepsilon_0(b^2-a^2)}\left(\frac{x}{\sqrt{a^2+x^2}} - \frac{x}{\sqrt{b^2+x^2}}\right)\hat{x}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{eQ}{2\pi\varepsilon_0 m_p(b^2-a^2)}\left(\frac{x}{\sqrt{a^2+x^2}} - \frac{x}{\sqrt{b^2+x^2}}\right) \ : \ 3 \ \text{pts}$$

Here, m_p is the mass of a proton. To derive the speed of the proton, we can us the chain rule as below.

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\frac{dx}{dt} = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \ : \ 2 \ \text{pts}$$

Then, the equation of motion becomes

$$v\frac{dv}{dx} = \frac{eQ}{2\pi\varepsilon_0 m_p(b^2-a^2)}\left(\frac{x}{\sqrt{a^2+x^2}} - \frac{x}{\sqrt{b^2+x^2}}\right) \ : \ 2 \ \text{pts}$$

Let's multiply both side by dx and integrate. Then, we can calculate the speed of the proton when it is far away form the annulus. If x is large and large enough, the electric force is almost zero so that the proton will move at a constant speed. Let's say this speed is v_∞ : 4 pts. If we use the condition given in the problem, then,

$$\int_0^{v_\infty} v\,dv = \frac{eQ}{2\pi\varepsilon_0 m_p(b^2-a^2)}\int_0^{x_\infty}\frac{x}{\sqrt{a^2+x^2}} - \frac{x}{\sqrt{b^2+x^2}}dx$$

$$\Rightarrow \frac{1}{2}v_\infty^2 = \frac{eQ}{2\pi\varepsilon_0 m_p (b^2 - a^2)}\left(\sqrt{a^2 + x_\infty^2} - a - \sqrt{b^2 + x_\infty^2} + b\right)$$

But here, x_∞ is so large that $\sqrt{a^2 + x_\infty^2}$ and $\sqrt{b^2 + x_\infty^2}$ can be treated as the same value, which means $\sqrt{a^2 + x_\infty^2} \approx x_\infty$ and $\sqrt{b^2 + x_\infty^2} \approx x_\infty$

Technically,

$$\sqrt{x_\infty^2 + a^2} = x_\infty\left(1 + \frac{a^2}{x_\infty^2}\right)^{1/2} = x_\infty + \frac{a^2}{2x_\infty} + O\left(\frac{1}{x_\infty^2}\right)$$

In the same way,

$$\sqrt{x_\infty^2 + b^2} = x_\infty\left(1 + \frac{b^2}{x_\infty^2}\right)^{1/2} = x_\infty + \frac{b^2}{2x_\infty} + O\left(\frac{1}{x_\infty^2}\right)$$

Thus,

$$\sqrt{x_\infty^2 + a^2} - \sqrt{x_\infty^2 + b^2} = O\left(\frac{1}{x_\infty}\right)$$

Consequently, this term also vanishes. This is the reason we can say the underlined sentence without feeling guilty. : 2 pts

Then,

$$\frac{1}{2}v_\infty^2 = \frac{eQ}{2\pi\varepsilon_0 m_p (b^2 - a^2)}(x_\infty - a - x_\infty + b) = \frac{eQ(b-a)}{2\pi\varepsilon_0 m_p (b^2 - a^2)}$$

$$\Rightarrow v_\infty = \sqrt{\frac{eQ}{\pi\varepsilon_0 m_p (a+b)}}$$: 5 pts