# Homework 4 Solution

2020 General Physics II

- In all problems, $(-2)$ for each simple mistakes.

**Problem 1** (5 pts)**.** A particle with charge $3 \times 10^{-6}$ C is traveling with a velocity of $5 \times 10^{6}$ m/s, perpendicular to the magnetic field. A magnitude of the field is 0.05 T. The charged particle experiences a magnetic force which has a magnitude of

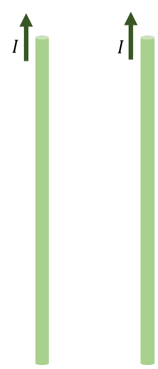
a) 0.075 N   b) 0.75 N   c) 0.045 N   d) 0.45 N

*Solution.* The formula $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ becomes $F = qvB$ since the particle travels perpendicular to the magnetic field. Therefore

$$F = 3 \times 10^{-6}\,\mathrm{C} \times 5 \times 10^{6}\,\mathrm{m/s} \times 0.05\,\mathrm{T} = 0.75\,\mathrm{N}.$$

□

**Problem 2** (5 pts)**.** Two parallel infinite wires are placed as in the below figure. Current is flowing in both wires in the same direction, going upward. Due to the magnetic force, the two wires




a) attract each other

b) repel each other

c) attract for a while and then repel each other

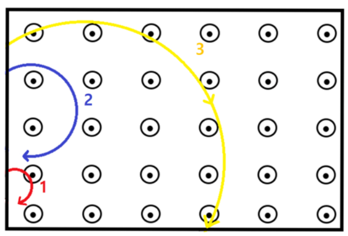
d) remain unmoved

*Solution.* The right wire makes magnetic field pointing out of the paper in the region of the left wire. Hence, the left wire experiences magnetic force pointing right. The left wire makes the field pointing into the paper in the region of the right wire, so the right wire experiences magnetic force to the left. □

**Problem 3** (5 pts)**.** A circuit consists of a light bulb rated as 60 W at 24 V and an ideal power source. The source provides 24 V constantly. A resistance of the light bulb is 9.6 Ω. [O/X]

*Solution.* From the formula $P = V^2/R$ and $V = 24\,\mathrm{V}, P = 60\,\mathrm{W}$, it follows that $R = 9.6\,\Omega$. □

**Problem 4** (5 pts)**.** Three charged particles with same momentum are moving in a uniform magnetic field. All three particles' trajectories are shown the below figure. The magnetic field is pointing out of the paper. Based on the trajectories, which particle has the largest charge?




a) 1 b) 2 c) 3 d) Don't know

*Solution.* The equation of motion is $qvB = mv^2/R$, where $R$ is the radius of the circular trajectory. Hence, we obtain $R = mv/Bq$. Since $p = mv$ and $B$ are same for all particles, $q$ is larger if $R$ is smaller. □

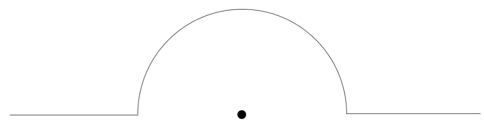
**Problem 5** (15 pts)**.** Consider two infinite wires as follows. One passes through $(-a, 0, 0)$ with the current $2I$ in $+z$ direction, and the other passes through $(a, 0, 0)$ with the current $I$ in $-z$ direction. Find all points where the magnetic field vanishes. Reason why there are no other points than what you have found.

*Solution.* The magnetic field due to an infinite wire with current $I$ at distance $r$ away has the magnitude $\frac{\mu_0 I}{2\pi r}$. Therefore, the condition for the magnetic field to be zero at the point $(x, 0, z)$ is

$$\frac{2}{x+a} - \frac{1}{x-a} = 0. \quad (+5)$$

This is solved by $x = 3a$. (+5) This is the full solution on the plane $y = 0$. What about other points, i.e. points with nonzero $y$-coordinate? The magnetic field has the direction which is determined by the right hand rule: $\mathbf{B} = \frac{\mu_0 \vec{I} \times \hat{\mathbf{r}}}{2\pi r}$. Hence, magnetic fields arising from two wires are not parallel, and hence for all magnitudes, they never vanish. Therefore, this is indeed the full solution. (+5) □

**Problem 6** (15 pts)**.** Consider an infinite wire carrying the current $I$, which is bent in a way that it contains a semicircle of radius $a$. Overall, it looks like a semicircle with its ends connected to semi-infinite wires. Find the magnitude of the magnetic field at the center of the semicircle.



*Solution.* Using Biot-Savart law, the magnetic field at the center of a circular wire with current $I$ and radius $r$ is given by

$$B = \int dB = \int \frac{\mu_0 I dl}{4\pi r^2} = \frac{\mu_0 I}{4r}, \quad (+10)$$

where the integration of $dl$ went through half the circumference. What about the semi-infinite wires? They do not contribute to the magnetic field since $d\vec{l}$ and current are parallel. (+5) □

**Problem 7** (15 pts)**.** Consider two straight wires with current $I$ as follows. The first wire has infinite length and passes through the origin whose current is in $+z$ direction. The second wire has a length of $2L$ and passes through the point $(r, 0, 0)$ whose current is in $+y$ direction. The second wire is placed symmetrically with respect to $xz$ plane. Find the net force and torque of the second wire due to the Lorentz force.

*Solution.* The magnetic field due to an infinite wire is given by $B = \frac{\mu_0 I}{2\pi r}$, where $r$ is the distance from wire. On the second wire, at $(r, 0, y)$, the magnetic field has the magnitude of $\frac{\mu_0 I}{2\pi\sqrt{r^2+y^2}}$. Since the force on a wire is given by $\vec{F} = I\vec{l} \times \vec{B}$, the magnitude of force per unit length is given by

$$\vec{f} = \frac{\mu_0 I^2}{\sqrt{r^2+y^2}} \sin\theta \hat{z} = \frac{\mu_0 I^2 y}{2\pi(r^2+y^2)} \hat{z} \quad (+3)$$

where $\theta$ is the angle between the magnetic field and the direction of the current. Since the contribution at $y$ and $-y$ cancels, the net force is zero. (+2) The torque is given by $\vec{\tau} = \vec{r} \times \vec{F}$, and hence the torque per unit length $\lambda$ is given as

$$\vec{\lambda} = y\hat{y} \times \frac{\mu_0 I^2 y}{2\pi(r^2+y^2)} \hat{z} = \frac{\mu_0 I^2 y^2}{2\pi(r^2+y^2)} \hat{x}$$

Then, the torque $\vec{\tau}$ is given as

$$\vec{\tau} = \frac{\mu_0 I^2}{2\pi} \int_{-L}^{L} \frac{y^2}{r^2+y^2} dy \hat{x} = \frac{\mu_0 I^2}{\pi} \int_0^L \frac{y^2}{r^2+y^2} dy \hat{x}. \quad (+5)$$

Note that

$$\int_0^L \frac{y^2}{r^2+y^2}dy = \int_0^L \left(1 - \frac{y^2}{r^2+y^2}\right) dy = L - \int_0^L \frac{r^2}{r^2+y^2}dy = L - r\Big[\theta\Big]_0^{\tan^{-1}(L/r)}$$
$$= L - r\tan^{-1}(L/r)$$

where $y = r\tan\theta$ was used. Thus, the torque is given by

$$\vec{\tau} = \frac{\mu_0 I^2}{\pi}\left(L - r\tan^{-1}\left(\frac{L}{r}\right)\right)\hat{x}. \quad (+5)$$

□

- Since the net force is zero, the torque is same for arbitrary reference points.
- $(-2)$ for not writing the torque as a vector.

**Problem 8** (15 pts)**.** Consider an electron constrained to move on the plane $z = 0$. The magnetic field on the plane is given by $-B\hat{\mathbf{z}}$ on the upper half plane $\mathbb{H}^+ = \{(x, y, 0) \in \mathbb{R}^3 | y > 0\}$, and $B\hat{\mathbf{z}}$ on the lower half plane $\mathbb{H}^- = \{(x, y, 0) \in \mathbb{R}^3 | y < 0\}$. The electron has an initial velocity $(v, 0, 0)$, and starts the motion in $\mathbb{H}^+$ with distance $r > 0$ from the $x$-axis.

a) For what values of $r$ would the electron propagate to the $+x$ direction, i.e. reach $x \to \infty$ as $t \to \infty$?

b) For $r = mv/eB$, calculate the mean propagation speed of the electron averaged over a sufficiently long time and find the ratio with the expected propagation speed if there were no magnetic field.

In this problem, assume that $B > 0$ and $v > 0$.

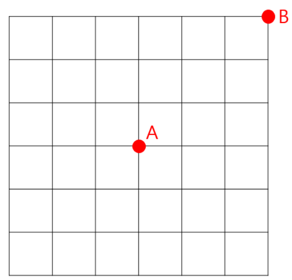
*Solution.* a) When the circular trajectory due to the magnetic field has contact with the $x$-axis before rotating $\pi$, it repeats the same motion with opposite centripetal force, hence propagating to the positive x direction. For this to happen, $r$ must be smaller than the diameter of the circular trajectory. $(+3)$ The radius of the circular motion is given by $evB = mv^2/r$, which gives the radius $mv/eB$. Hence, $r < 2mv/eB$ allows electron to propagate to the positive $x$-axis direction. $(+2)$

b) When $r = mv/eB$, the trajectory of the electron is repeated half-circles. $(+4)$ The angular frequency of the repeated circular motion is $\omega = v/r = eB/m$, which gives the period $T = 2\pi m/eB$. The average speed of the positive $x$-axis propagation is $2r/(T/2) = \frac{2mv/eB}{\pi m/eB} = 2v/\pi$. $(+4)$ Hence, the ratio with respect to the velocity when there was no magnetic field is $\frac{2}{\pi}$. $(+2)$

□

- In b), we also accept $v$ and 1 as an answer, since the magnetic field does not change the speed. Although we intended to find the mean propagation 'velocity', we noticed that there were ambiguities in the problem statement.

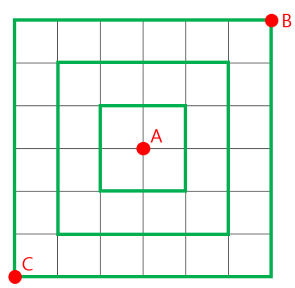
**Problem 9** (20 pts)**.** For $n \in \mathbb{N}$, let $C_n$ be an electric circuit on a 2D square grid of size $2n \times 2n$ where every edge carries the resistance $1\,\Omega$. Let $A$ be the point at the center of the grid and $B$ be a point at one corner of the grid. The below figure shows the grid when $n = 3$.




Find the limit $\lim_{n\to\infty} R_n$, where $R_n$ is the effective resistance between $A$ and $B$.

Hint: You may use the following fact: Given a finite circuit, if the resistance of some resistors increases (decreases, respectively), then the effective resistance of any two points of the circuit does not decrease (does not increase, respectively).

*Solution.* Modify the circuit $C_n$ where every 'concentric' edges have zero resistance. (+10) For example, for $n = 3$ the green lines have zero resistance in the below figure.




From the hint, the resistance $R'_n$ between $A$ and $B$ in the modified circuit should be not greater than $R_n$. However, we can explicitly compute $R'_n$ as

$$R'_n = \frac{1}{4} + \frac{1}{12} + \cdots + \frac{1}{8n-4} = \sum_{k=1}^{n} \frac{1}{8k-4} \quad (+5)$$

after contracting all green edges. (It is a combination of series connections and parallel connections.) Since $R'_n \to \infty$ as $n \to \infty$, we have

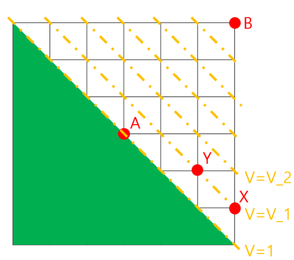
$$\lim_{n\to\infty} R_n = \infty. \quad (+5)$$

□

- (Partial point) at least 5 points for trying to modify the given circuit to use the hint.
- (Alternative solutions from students' answer) There are other ways to modify the circuit. Two examples are shown in the below figure.


In the left figure, green lines have zero resistance and other lines have resistance $1\,\Omega$. (One may think that the green lines originally have infinite resistance and then modified to zero resistance.) In the right figure, blue lines have resistance $1\,\Omega$ and other lines have zero resistance.

- From the students' answer, we found out many wrong arguments. We collect frequently occurred wrong arguments:
  - Some students found the effective resistance between $C$ and $B$ instead. Please read the problem carefully.
  - Some students argued that the concentric circles form equipotential surfaces. This is not true, since $B$ is not equivalent with other three corners. We do not have 4-fold rotation symmetry.
  - One cannot naively ignore the '3rd-quadrant' part of the circuit. Indeed, ignoring these parts amounts to replacing the resistance to $\infty$, but from the hint this does not decrease the effective resistance. Hence one ends up with the inequality in opposite direction.
  - The current does not flow like $(1/2^n)I_0$ at '$n$-th node'. There is no symmetry that guarantees this.
  - It is not true that $R_{CB} = R_{CA} + R_{AB}$. They are not in the series connection!
  - This is the most subtle wrong argument. One student modified the circuit as in the figure and let the resistance of the green triangle part to zero.



Then, he argued that the yellow lines form equipotential surfaces. However, this is not true. Indeed, suppose this is true. Let the voltage at $A$ and $B$ be $V_A = 1, V_B = 0$, and also let the voltage on two yellow lines be $V_1$ and $V_2$. By the current conservation at $X$ and $Y$, one obtains

$$2(1 - V_1) = V_1 - V_2, \quad 2(1 - V_1) = 2(V_1 - V_2).$$

These equations have solution $V_1 = V_2 = 1$. Repeating this argument, one obtains $V_B = 1$ which is contradiction.

Most of the wrong arguments get 5 points if the answer is correct, or 0 points if the answer is wrong.

**Remark.** The statement in the hint is called **Rayleigh's monotonicity law**. Although this fact is intuitive, the proof is not trivial. It is also interesting to consider this problem in different dimensions. For 1D, it is obvious that $R_n = n \to \infty$. Surprisingly, in 3D, it turns out that $R_n$ does not diverge! We leave this to an even more advanced exercise.