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- Subspaces, linear combination, linear independence, basis. [3.4, 7.1, 7.2]
- A nonempty subset V of $\mathbb{R}^n$ is said to be **closed under scalar multiplication**, if $a\mathbf{x} \in V$ for any scalar $a \in \mathbb{R}$ and any vector $\mathbf{x} \in V$, and said to be **closed under addition**, if $\mathbf{x} + \mathbf{y} \in V$ for any vectors $\mathbf{x}, \mathbf{y} \in V$.
- In the last lecture we have seen a subset $W = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ of $\mathbb{R}^n$. This set is closed under scalar multiplication and addition.
- *Definition.* A nonempty subset of $\mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$, if it is closed under scalar multiplication and addition.
- Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s$ be vectors in $\mathbb{R}^n$ and $c_1, c_2, \dots, c_s$ scalars. The expression

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s$$

  is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s$ with coefficients $c_1, c_2, \dots, c_s$.
- The subspace spanned by vectors. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s$ be vectors in $\mathbb{R}^n$. Define $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s\}$ to be the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s$.

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s : c_1, c_2, \dots, c_s \in \mathbb{R}\}$$

  We can easily show that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s\}$ is a subspace.
- For any $m \times n$ matrix A, the set $W = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, which is called the **solution space** of the homogeneous system $A\mathbf{x} = \mathbf{0}$.
- *Definition.* A nonempty set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s\}$ is said to be **linearly independent** if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s = \mathbf{0}$ implies $c_1 = c_2 = \cdots = c_s = 0$; otherwise, **linearly dependent**. In other words, S is linearly dependent if there are some scalars $c_1, c_2, \dots, c_s$, not all zeros (at least one nonzero), satisfying $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s = \mathbf{0}$.
- Now we can look at the homogeneous system $A\mathbf{x} = \mathbf{0}$ again and give a meaning to its solutions. Let $A_1, A_2, \dots, A_n$ be columns of A and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. The matrix product $A\mathbf{x} = x_1A_1 + \cdots + x_nA_n$ is a linear combination of columns of A with coefficients $x_1, x_2, \dots, x_n$. So we have the following theorem.
- *Theorem.* A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the columns of A are linearly independent.
- Theorem 3.4.9. If A is an $n \times n$ matrix, then the following statements are equivalent:
  1. The reduced row echelon form of A is I_n.
  2. A is a product of elementary matrices.
  3. A is invertible.
  4. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  5. $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.
  6. $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$.
  7. The column vectors of A are linearly independent.
  8. The row vectors of A are linearly independent.
- *Theorem.* A set of more than m vectors in $\mathbb{R}^m$ is linearly dependent.

- *Theorem.* A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ with more unknowns than equations has infinitely many solutions. In other words, for an $m \times n$ matrix A with $m < n$, $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

  Proof. Follows from Gaussian elimination. There is at least one free variable.

- Consider the following matrix A with columns $A_1, \ldots, A_4$. Note that $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

$$A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{bmatrix}$$

  The subspace of $\mathbb{R}^3$ spanned by the columns of A, $\operatorname{span}\{A_1, A_2, A_3, A_4\}$, is called the column space of A. Though this space is spanned by four column vectors of A, it can be spanned by two, A_1 and A_2. The reduction occurs because the four columns are linearly dependent.

- *Definition.* A set of vectors S in a subspace V of $\mathbb{R}^n$ is said to be a **basis** for V if it is linearly independent and spans V.
- In the above matrix A, the column space has a basis, for example, $\{A_1, A_2\}$ or $\{A_1, A_3\}$.
- The standard unit vectors in $\mathbb{R}^n$, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, is a basis of $\mathbb{R}^n$, called the **standard basis** of $\mathbb{R}^n$.
- *Theorem.* If V is a nonzero subspace of $\mathbb{R}^n$, then there is a basis for V with at most n vectors.

  Proof. We construct a basis for V. Choose a nonzero vector $\mathbf{v}_1 \in V$. Let $\mathcal{B}_1 = \{\mathbf{v}_1\}$ and $V_1 = \operatorname{span}(\mathcal{B}_1)$. If $V_1 \neq V$, choose $\mathbf{v}_2 \in V \setminus V_1$ and let $\mathcal{B}_2 = \mathcal{B}_1 \cup \{\mathbf{v}_2\}$. Go on until $V_k = V$. We will not encounter the case $k = n + 1$, since a linearly independent set in $\mathbb{R}^n$ can contain at most n vectors.

- *Theorem.* If $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for a subspace V of $\mathbb{R}^n$, then $|\mathcal{B}_1| = |\mathcal{B}_2|$, i.e., they have the same number of elements.

  Proof. Let $\mathcal{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $\mathcal{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p\}$. Suppose $m < p$. There is an $m \times p$ matrix $A = (a_{ij})$ such that

$$\mathbf{w}_j = \sum_{i=1}^{m} a_{ij}\mathbf{v}_i, \quad j = 1, \ldots, p$$

  Since $m < p$, this system has a nontrivial solution. Choose a nontrivial solution $\mathbf{c} = (c_i)$ to $A\mathbf{x} = \mathbf{0}$. Then

$$\sum_{j=1}^{p} c_j \mathbf{w}_j = \sum_{j=1}^{p} c_j \left(\sum_{i=1}^{m} a_{ij}\mathbf{v}_i \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{p} a_{ij} c_j \right) \mathbf{v}_i = \mathbf{0},$$

  which implies that a basis $\mathcal{B}_2$ is linearly dependent, a contradiction.

- *Definition.* For a nonzero subspace V of $\mathbb{R}^n$, the **dimension** of V, $\dim(V)$, is the number of elements in a basis for V. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be 0.
- Properties of bases
- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ is a basis for V, then every $\mathbf{v} \in V$ can be expressed in exactly one way as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$.
- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ spans V, then a basis for V can be obtained by removing appropriate vectors from S.
- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ is a linearly independent subset of V, then a basis for V can be obtained by adding appropriate vectors from V to S.
- If V and W are subspaces of $\mathbb{R}^n$, and if V is a subspace of W, then $0 \leq \dim(V) \leq \dim(W) \leq n$. Moreover, $V = W$ if and only if $\dim(V) = \dim(W)$.

- What will be a typical dimension we encounter?

  Small dimensions: $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, ...

  Large dimensions: $\mathbb{R}^{10^3}$, $\mathbb{R}^{10^4}$, $\mathbb{R}^{10^5}$, $\mathbb{R}^{10^6}$, ...

  If we deal with digital data, dimension is usually very big.
- The set of all $m \times n$ matrices is a vector space as well. It is similar to $\mathbb{R}^{mn}$.
- A basis for the vector space of all $m \times n$ matrices.

  Define $m \times n$ matrices E_{pq} for $p = 1, \ldots, m$ and $q = 1, \ldots, n$, by $E_{pq} = (a_{ij})$ with

$$a_{ij} = \begin{cases} 1, & (i,j) = (p,q), \\ 0, & (i,j) \neq (p,q). \end{cases}$$

  $\{E_{pq}\}$ is a basis for the vector space of all $m \times n$ matrices.