EE210: Probability and Introductory Random Processes

1. (15 points) A lie detector test is 95 percent effective in detecting a lie when a person tell a lie. However, the test also yields a "false detection" result for 1 percent of true statements. That is, if a true statement is tested, then with probability 0.01, the test result will imply he or she tells a lie. If 5 percent of the test statements actually are lies, what is the probability that a statement is a lie given that the test result indicates that the test statement is a lie?

Solution:
L: the event that the tested person tell a lie
A: the event that the test result is that the teset statement is a lie

$$
\begin{aligned}
P[L|A] &= \frac{P[L \cap A]}{P[A]} \\
&= \frac{P[A|L]P[L]}{P[A|L]P[L] + P[A|L^C]P[L^C]} \\
&= \frac{0.95 \cdot 0.05}{0.95 \cdot 0.05 + 0.01 \cdot 0.95} \\
&= \frac{5}{6} = 83.33....\%
\end{aligned}
$$

2. (30 points) The joint probability density function of random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 \le x \le x_0,\ 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (8 points) Find the mean μ_X and the variance σ_X^2.

(b) (8 points) Find the mean μ_Y and the variance σ_Y^2.

(c) (9 points) Find the covariance $\mathrm{Cov}[X,Y]$.

(d) (5 points) Are X and Y uncorrelated?

Solution:

(a)

$$\begin{aligned}\int_0^{x_0}\int_0^1 (x+y)dydx &= \int_0^{x_0}(x+\frac{1}{2})dx \\ &= \frac{x_0^2}{2}+\frac{x_0}{2} = 1 \to x_0 = 1\end{aligned}$$

$$f_X(x) = \int_0^1 (x+y)dy = x+\frac{1}{2}$$

$$f_X(x) = \begin{cases} x+\frac{1}{2} & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}[X] = \int_0^1 (x^2+\frac{x}{2})dx = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$$

$$\mathbb{E}[X^2] = \int_0^1 (x^3+\frac{x^2}{2})dx = \frac{1}{4}+\frac{1}{6} = \frac{5}{12}$$

$$\sigma_X^2 = \frac{60}{144}-\frac{49}{144} = \frac{11}{144}$$

(b) By symmetric,

$$f_Y(y) = \begin{cases} y+\frac{1}{2} & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\sigma_Y^2 = \frac{60}{144}-\frac{49}{144} = \frac{11}{144}$$

(c)

$$\begin{aligned}\mathbb{E}[XY] &= \int_0^1\int_0^1 (x^2y+xy^2)dxdy = \int_0^1 (\frac{y}{3}+\frac{y^2}{2})dy \\ &= \frac{1}{6}+\frac{1}{6} = \frac{1}{3}\end{aligned}$$

$$\mathrm{Cov}[X,Y] = \mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y] = \frac{48}{144}-\frac{49}{144} = -\frac{1}{144}$$

3. (30 points) Suppose we wish to predict the values of a random variable Y by observing the values of another random variable X. The available data suggest that a good prediction model for Y is the linear function that $\hat{Y} = \alpha X + \beta$, which is called *linear regression*. In general, the correlation coefficient between X and Y is neither 1 nor -1, i.e., $|\rho| \neq 1$. For the random variables, $E[X] = \mu_X$, $E[Y] = \mu_Y$, $\mathrm{Var}(X) = \sigma_X^2$, $\mathrm{Var}(Y) = \sigma_Y^2$, and the correlation coefficient between X and Y is ρ.

(a) (15 points) Determine the coefficients α and β in order to minimize the mean-square error

$$e \triangleq E[(Y - \hat{Y})^2].$$

(b) (15 points) Find the minimum mean-square error.

Solution:

$$\begin{aligned} e = \mathbb{E}[(Y-\hat{Y})^2] &= \mathbb{E}[Y^2 - 2Y\hat{Y} + \hat{Y}^2] \\ &= \mathbb{E}[Y^2 - 2Y(\alpha X + \beta) + (\alpha X + \beta)^2] \\ &= \mathbb{E}[Y^2] - 2\alpha\mathbb{E}[XY] - 2\beta\mu_Y + \alpha^2\mathbb{E}[X^2] + 2\alpha\beta\mu_X + \beta^2 \end{aligned}$$

(a) Using partial derivative of e, we find α and β to minimize e as

$$\frac{\partial e}{\partial \alpha} = -2\mathbb{E}[XY] + 2\alpha\mathbb{E}[X^2] + 2\beta\mu_X = 0$$
$$\frac{\partial e}{\partial \beta} = -2\mu_Y + 2\alpha\mu_X + 2\beta = 0$$

$$\therefore -\mathbb{E}[XY] + \alpha\mathbb{E}[X^2] + \mu_X\mu_Y - \alpha\mu_X^2 = 0$$

$$\Rightarrow \quad \alpha^* = \frac{\mathbb{E}[XY] - \mu_X\mu_Y}{\sigma_X^2} = \frac{\mathrm{Cov}[X,Y]}{\sigma_X^2} = \rho\frac{\sigma_Y}{\sigma_X}$$
$$\beta^* = \mu_Y - \alpha\mu_X = \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}\mu_X$$

(b) From the result of (a),

$$\hat{Y} = \rho\frac{\sigma_Y}{\sigma_X}X + \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}\mu_X = \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X) + \mu_Y$$

Therefore, minimum mean-square error can be obtain as follow:

$$e = \mathbb{E}[Y^2] - 2\mu_Y^2 - \rho^2\sigma_Y^2 + \mu_Y^2 = \sigma_Y^2 - \rho^2\sigma_Y^2 = \sigma_Y^2(1-\rho^2)$$

4. (25 points) X and Y are identically distributed random variables with $E[X] = E[Y] = 0$ and covariance $\text{Cov}[X, Y] = 3$ and correlation coefficient $\rho_{X,Y} = \frac{1}{2}$. For nonzero constants a and b, $U = aX$ and $V = bY$.

(a) (8 points) Find $\text{Cov}[U, V]$.

(b) (8 points) Find the correlation coefficient $\rho_{U,V}$.

(c) (9 points) Let $W = U + V$. For what values of a and b are X and W uncorrelated?

Solution:

(a) Since X and Y have zero expected value, $\text{Cov}[X, Y] = \mathbb{E}[XY] = 3$, $\mathbb{E}[U] = a\mathbb{E}[X] = 0$ and $\mathbb{E}[V] = b\mathbb{E}[Y] = 0$. It follows that

$$\begin{aligned}\text{Cov}[U, V] &= \mathbb{E}[UV] \\ &= \mathbb{E}[abXY] \\ &= ab\mathbb{E}[XY] = ab\text{Cov}[X, Y] = 3ab.\end{aligned}$$

(b) We start by observing that $\text{Var}(U) = a^2\text{Var}(X)$ and $\text{Var}(V) = b^2\text{Var}(Y)$. It follows that

$$\begin{aligned}\rho_{U,V} &= \frac{\text{Cov}[U, V]}{\sqrt{\text{Var}(U)\text{Var}(V)}} \\ &= \frac{ab\text{Cov}[X, Y]}{\sqrt{a^2\text{Var}(X)b^2\text{Var}(Y)}} = \frac{ab}{\sqrt{a^2b^2}}\rho_{X,Y} = \frac{1}{2}\frac{ab}{|ab|}.\end{aligned}$$

Note that $ab/|ab|$ is 1 if a and b have the same sign or is -1 if they have opposite signs.

(c) Since $\mathbb{E}[X] = 0$,

$$\begin{aligned}\text{Cov}[X, W] &= \mathbb{E}[XW] - \mathbb{E}[X]\mathbb{E}[W] \\ &= \mathbb{E}[XW] \\ &= \mathbb{E}[X(aX + bY)] \\ &= a\mathbb{E}[X^2] + b\mathbb{E}[XY] \\ &= a\text{Var}(X) + b\text{Cov}[X, Y]. \qquad (1)\end{aligned}$$

Since X and Y are identically distributed, $\text{Var}(X) = \text{Var}(Y)$ and

$$\frac{1}{2} = \rho_{X,Y} = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\text{Cov}[X, Y]}{\text{Var}(X)} = \frac{3}{\text{Var}(X)}.$$

This implies $\text{Var}(X) = 6$. From (1), $\text{Cov}[X, W] = 6a + 3b$. Thus X and W are uncorrelated if $6a + 3b = 0$, or $b = -2a$.