

- Please submit before the submission deadline.
- Assignment submitted one (1) day after the assignment deadline will be accepted with 20% deduction on corresponding assignment grade.
- Assignment submitted more than one (1) day late will not be accepted.

Problem 1: Selection (50 points)

Algorithm 1 Randomized Selection

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1: function RAND-SELECT(A[1..n], p, q, i)
2:   if  $p = q$  then
3:     return  $A[p]$ 
4:   end if
5:    $r \leftarrow RAND - PARTITION(A, p, q)$ 
6:    $k \leftarrow r - p + 1$ 
7:   if  $i = k$  then
8:     return  $A[r]$ 
9:   end if
10:  if  $i < k$  then
11:    return RAND - SELECT( $A, p, r - 1, i$ )
12:  else
13:    return RAND - SELECT( $A, r + 1, q, i - k$ )
14:  end if
15: end function

```

- (a) Derive the recurrence relations for Algorithm 1 in $T(n)$ for best-, worst-, and average case. (10 points)

$$\begin{aligned}(\text{Best})\ T(n) &= \\(\text{Worst})\ T(n) &= \\(\text{Average})\ T(n) &= \end{aligned}$$

Solution:

- (Best) $T(n) = T(n/2) + O(n) = O(n)$
- (Worst) $T(n) = T(n - 1) + O(n)$
- (Average) $T(n) = \sum_{k=0}^{n-1} X_k(T(\max\{k, n - k - 1\})) + O(n)$
where

$$X_k = \begin{cases} 1, & \text{if } PARTITION \text{ generates a } k : n - k - 1 \text{ split,} \\ 0, & \text{otherwise} \end{cases}$$

Grading criteria:

- 3 points for the best-case
- 3 points for the worst-case
- 4 points for the average case
- -2 points if the average case does not have an explanation.

- (b) What will Algorithm 2 print (see line 10) for the following input? (5 points)

A: [4, 7, 11, 35, 18, 29, 30, 32, 2, 6, 9, 21, 25, 31, 15, 23, 19, 20, 13, 16, 14, 12, 34, 22, 1, 28, 5, 10, 17, 27, 24, 8, 33, 3, 26]
k: 5

Algorithm 2 Worst-case Linear-time Selection

```

1: function SELECT(A[1..n], k)
2:   if  $n \leq 25$  then
3:     use brute force
4:   else
5:      $m \leftarrow \lfloor n/5 \rfloor$ 
6:     for  $i \leftarrow 1$  to  $m$  do
7:        $B[i] \leftarrow \text{SELECT}(A[5i - 4..5i], 2)$ 
8:     end for
9:      $mom \leftarrow \text{SELECT}(B[1..m], \lfloor m/2 \rfloor)$ 
10:    Print(mom)
11:     $r \leftarrow \text{PARTITION}(A[1..n], mom)$ 
12:  end if
13:  if  $k < r$  then
14:    return  $\text{SELECT}(A[1..r - 1], k)$ 
15:  else if  $k > r$  then
16:    return  $\text{SELECT}(A[r + 1..n], k - r)$ 
17:  else
18:    return mom
19:  end if
20: end function
```

Solution:

19 (or 17 assuming that k in SELECT() is 1-indexed.)

Grading criteria:

5 points if correct, 0 if wrong.

(c) Determine the recurrence relation for SELECT in $T(n)$ when the input array for SELECT is partitioned into sets of three rather than five. (10 points)

$$T(n) \leq T(\underline{\hspace{2cm}}) + T(\underline{\hspace{2cm}}) + O(\underline{\hspace{2cm}})$$

Solution:

$$T(n) \leq T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + O(n)$$

Grading criteria:

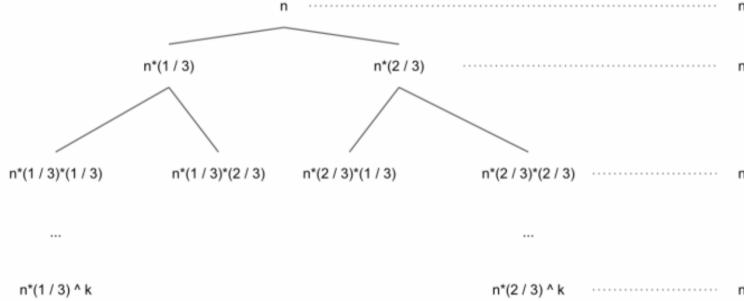
- 4 points for the first blank.
- 4 points for the second blank.
- 2 points for the third blank.

(d) If we partition the groups into sets of 3, does the recurrence relation in (c) meet the condition $T(n) = O(n)$? If so, confirm this using the substitution method. If not, explain why the recurrence relation fails to make $T(n)$ linear. (15 points)

Solution:

The recurrence is not linear.

Proof 1: Using a recursive tree method, the tree can be drawn like below.



The leftmost height of the tree at level k would be $k = \log_3 n$ and the rightmost height of the tree at level k would be $k = \log_{\frac{3}{2}} n$. Getting the bound for the recurrent relation, it becomes

$$n \log_3 n \leq T(n) \leq n \log_{\frac{3}{2}} n$$

Within the same height, the time complexity is $T(n(cn))$, and the lower bound becomes $T(n \log n)$. Therefore, the recurrence relation is not linear.

Proof 2:

To prove that $T(n) = O(n)$, we guess:

$$T(n) \leq \begin{cases} dn_0, & \text{if } n = n_0, \\ d \cdot n, & \text{if } n > n_0 \end{cases}$$

For the base case, we pick $n_0 = 1$ and use the standard assumption that $T(1) = 1 \leq d$. For the inductive hypothesis, we assume that our guess is correct for any $n < k$, and we prove our guess for k . That is, consider d such that for all $n_0 \leq n < k$, $T(n) \leq dn$.

$$\begin{aligned} T(k) &\leq T\left(\frac{k}{3}\right) + T\left(\frac{2k}{3}\right) + cn \leq \frac{dk}{3} + \frac{2dk}{3} + ck \leq dk \\ d + c &\leq d \\ c &\leq 0 \end{aligned}$$

There is no choice of d that works. Thus our guess was wrong, and the recurrence is not linear.

Grading criteria:

- 5 points for the correct conclusion.
- 5 points for writing the inequality correctly.
- 5 points for writing a valid explanation.

(e) If we implement the modifications proposed in (d) to Algorithm 2, what adjustments should we make to the pseudo-code? Please provide the necessary alterations for lines 5, 7, 14, and 16 of Algorithm 2. (10 points)

Solution:

- line 5: $m \leftarrow \lfloor n/3 \rfloor$
- line 7: $B[I] \leftarrow \text{SELECT}(A[3i - 4\dots 3i], 2)$

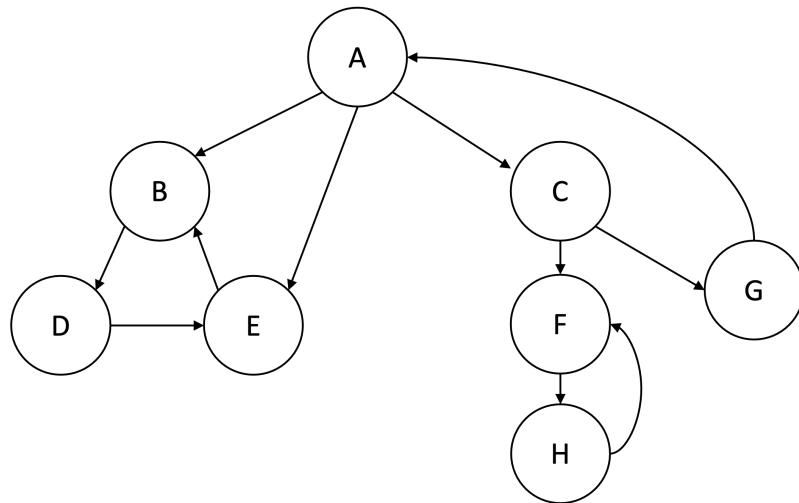
- line 14: nothing to change.
- line 16: nothing to change

Grading criteria:

- 4 points for entirely correct modification to line 5
- 4 points for entirely correct modification to line 7
- 1 points for line 14
- 1 points for line 16

Problem 2: Depth First Search (20 points)

Start the traversal at vertex A and resolve ties by the vertex alphabetical order.



- (a) Traverse the above graph by depth-first search. Give the order in which the vertices were reached for the first time and the order in which the vertices became dead-ends. (12 points)

Solution:

- The order of reaching: A, B, D, E, C, F, H, G
- The order of dead-ends: E, D, B, H, F, G, C, A

Grading criteria:

6 points for each order.

- (b) Classify all edges in the graph. (8 points)

Solution:

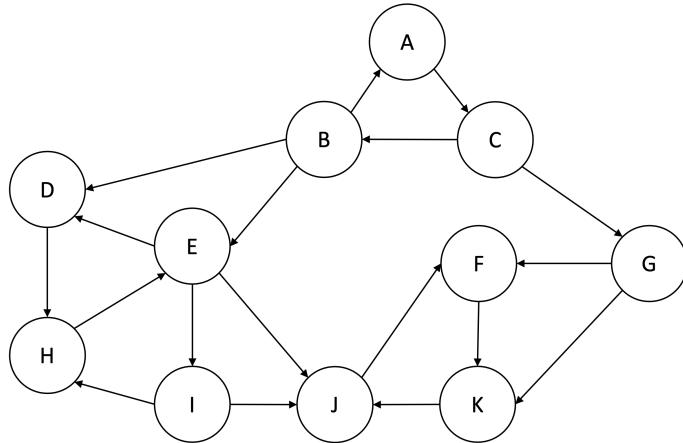
- Tree edges: (A,B), (B,D), (D,E), (A,C), (C,F), (C,G), (F,H)
- Back edges: (E,B), (H,F), (G,A)
- Forward edges: (A,E)
- Cross edges: X

Grading criteria:

2 points for each edge type.

Problem 3: Strongly Connected Components (30 Points)

Consider the directed graph G in the below figure.



- (a) Write all the strongly connected components (if the node v and the node u are SCC components, write in form of $\{v, u\}$). (15 points)

Solution:

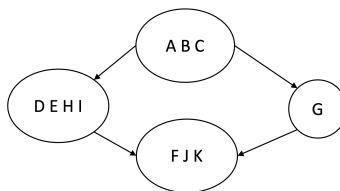
$\{A, B, C\}, \{G\}, \{D, E, H, I\}, \{F, J, K\}$

Grading criteria:

- 15 points for all correct components.
- 12 points for 3 correct components.
- 10 points for 2 correct components.
- 5 points for 1 correct component.

- (b) Draw a meta-graph of G. (8 points)

Solution:



Grading criteria:

- 5 points for vertex only being correct.
- 8 points if vertex and edge are both correct.

- (c) Find the source and sink vertices in the meta-graph of G. (7 points)

Solution:

- source: A, B, C
- sink: F, J, K

Grading criteria:

- 7 points for all correct answer.
- 4 points for only one of the correct answer, source or sink.