1. Suppose Y_1, Y_2, Y_3 are from Exp(β). Let $X_1 = Y_{(1)}$, $X_2 = Y_{(2)} - Y_{(1)}$, $X_3 = Y_{(3)} - Y_{(2)}$.
Where $Y_{(i)}$: i th order statistic.

(a) Find joint pdf of X_1, X_2, X_3

$f_Y(y) = \frac{1}{\beta} e^{-\frac{y}{\beta}}$, $(y \geq 0)$ 이다.

Y_1, Y_2, Y_3가 모두 iid exponential random variables 로, X_1, X_2, X_3 을 구하기 전 먼저

Y_1, Y_2, Y_3 의 관점에서 보자.

일반적인 order statistics 에서 joint distribution 은 $f(y_1, \cdots, y_n) = n! \prod_{i=1}^{n} f(y_i)$, $(y_1 \leq \cdots \leq y_n)$ 이다.

이를 적용하면, $f_{Y_{(1)}, Y_{(2)}, Y_{(3)}}(y_1, y_2, y_3) = 3!\left(\frac{1}{\beta} e^{-\frac{y_1}{\beta}}\right)\left(\frac{1}{\beta} e^{-\frac{y_2 - y_1}{\beta}}\right)\left(\frac{1}{\beta} e^{-\frac{y_3 - y_2}{\beta}}\right) = \frac{6}{\beta^3} e^{-\frac{y_3}{\beta}}$, $(0 \leq y_1 \leq y_2 \leq y_3)$ 가 된다.

이제, X_1, X_2, X_3로 바꾸자. $X_1 = Y_{(1)}$, $X_2 = Y_{(2)} - Y_{(1)}$, $X_3 = Y_{(3)} - Y_{(2)}$ 이므로, $x_1 = y_1$, $y_2 = x_1 + x_2$, $y_3 = x_1 + x_2 + x_3$ 를 얻을 수 있다. 변환을 위한 Jacobian은,

$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1$ 이다. 즉, $f_{X_1, X_2, X_3}(x_1, x_2, x_3) = f_{Y_{(1)}, Y_{(2)}, Y_{(3)}}(x_1, x_1 + x_2, x_1 + x_2 + x_3) \times J$ 로,

$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \left(\frac{1}{\beta}\right)^3 \cdot 6 \cdot e^{-\frac{3x_1 + 2x_2 + x_3}{\beta}} = 6\beta^{-3} e^{-\frac{3x_1 + 2x_2 + x_3}{\beta}}$, $(x_1, x_2, x_3 \geq 0)$ 이다.

(b) Find marginal pdf of X_1, X_2, X_3

a)에서 $f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \frac{6}{\beta^3} e^{-\frac{3x_1+2x_2+x_3}{\beta}}$ 을 구하였다.

이를 이용해, marginal pdf를 구하면, 다음과 같다.

$$f_{X_1}(x_1) = \int_0^\infty \int_0^\infty \frac{6}{\beta^3} \exp\left(-\frac{3x_1+2x_2+x_3}{\beta}\right) dx_3 dx_2$$

$$= \int_0^\infty \frac{6}{\beta^3} e^{-\frac{3x_1+2x_2}{\beta}} \int_0^\infty e^{-\frac{x_3}{\beta}} dx_3 \, dx_2 = \frac{6}{\beta^2} e^{-\frac{3x_1}{\beta}} \int_0^\infty e^{-\frac{2x_2}{\beta}} dx_2 = \frac{3}{\beta} e^{-\frac{3x_1}{\beta}}$$

$$f_{X_2}(x_2) = \int_0^\infty \int_0^\infty \frac{6}{\beta^3} \exp\left(-\frac{3x_1+2x_2+x_3}{\beta}\right) dx_3 dx_1 = \int_0^\infty \frac{6}{\beta^2} e^{-\frac{3x_1+2x_2}{\beta}} dx_1 = \frac{2}{\beta} e^{-\frac{2x_2}{\beta}}$$

$$f_{X_3}(x_3) = \int_0^\infty \int_0^\infty \frac{6}{\beta^3} \exp\left(-\frac{3x_1+2x_2+x_3}{\beta}\right) dx_2 dx_1 = \int_0^\infty \frac{3}{\beta^2} e^{-\frac{3x_1+x_3}{\beta}} dx_1 = \frac{1}{\beta} e^{-\frac{x_3}{\beta}}$$

따라서,

$$f_{X_1}(x_1) = \frac{3}{\beta} e^{-\frac{3x_1}{\beta}}, \quad f_{X_2}(x_2) = \frac{2}{\beta} e^{-\frac{2x_2}{\beta}}, \quad f_{X_3}(x_3) = \frac{1}{\beta} e^{-\frac{x_3}{\beta}}$$

(c) Show that X_1, X_2, X_3 are independent

(a) 에 의해 $f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{6}{\beta^3} e^{-\frac{3x_1+2x_2+x_3}{\beta}}$ 이다.

(b) 에 의해 $f_{x_1}(x_1) = \frac{3}{\beta}e^{-\frac{3x_1}{\beta}}$, $f_{x_2}(x_2) = \frac{2}{\beta}e^{-\frac{2x_2}{\beta}}$, $f_{x_3}(x_3) = \frac{1}{\beta}e^{-\frac{x_3}{\beta}}$ 이다.

여기서, $f_{X_1,X_2,X_3}(x_1, x_2, x_3) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot f_{X_3}(x_3)$ 임을 확인할 수 있다.

따라서, X_1, X_2, X_3는 independent 함을 알 수 있다.