HW 10 solution

9.3 **(+10pt)**

a. $E\left(\hat{\theta}_1\right) = E(\bar{Y}) - 1/2 = \theta + 1/2 - 1/2 = \theta$. From Section 6.7 , we can find the density function of $\hat{\theta}_2 = Y_{(n)} : g_n(y) = n(y-\theta)^{n-1}, \theta \leq y \leq \theta + 1$. From this, it is easily shown that $E\left(\hat{\theta}_2\right) = E\left(Y_{(n)}\right) - n/(n+1) = \theta$.

b. $V\left(\hat{\theta}_1\right) = V(\bar{Y}) = \sigma^2/n = 1/(12n)$. With the density in part a, $V\left(\hat{\theta}_2\right) = V\left(Y_{(n)}\right) = \frac{n}{(n+2)(n+1)^2}$. Thus, eff $\left(\hat{\theta}_1, \hat{\theta}_2\right) = \frac{12n^2}{(n+2)(n+1)^2}$.

9.4

See Exercises 8.18 and 6.74. Following those, we have that $V\left(\hat{\theta}_1\right) = (n+1)^2 V\left(Y_{(n)}\right) = \frac{n}{n+2}\theta^2$. Similarly, $V\left(\hat{\theta}_2\right) = \left(\frac{n+1}{n}\right)^2 V\left(Y_{(n)}\right) = \frac{1}{n(n+2)}\theta^2$. Thus, the ratio of these variances is as given.

9.15

Referring to Ex. 9.3, since both estimators are unbiased and the variances go to 0 with as n goes to infinity the estimators are consistent.

9.18

Note that this estimator is the pooled sample variance estimator S_p^2 with $n_1 = n_2 = n$. In Ex. 8.133 it was shown that S_p^2 is an unbiased estimator. Also, it was shown that the variance of S_p^2 is $\frac{2\sigma^4}{n_1+n_2-2} = \frac{\sigma^4}{n-1}$. Since this quantity goes to 0 with n, the estimator is consistent.

9.21

Note that this is a generalization of Ex. 9.5. The estimator $\hat{\sigma}^2$ can be written as

$$\hat{\sigma}^2 = \frac{1}{k}\left[\frac{(Y_2-Y_1)^2}{2} + \frac{(Y_4-Y_3)^2}{2} + \frac{(Y_6-Y_5)^2}{2} + \ldots + \frac{(Y_n-Y_{n-1})^2}{2}\right].$$

There are k independent terms in the sum, each with mean σ^2 and variance $2\sigma^4$.

a. From the above, $E\left(\hat{\sigma}^2\right) = \left(k\sigma^2\right)/k = \sigma^2$. So $\hat{\sigma}^2$ is an unbiased estimator.

b. Similarly, $V\left(\hat{\sigma}^2\right) = k\left(2\sigma^4\right)/k^2 = 2\sigma^4/k$. Since $k = n/2, V\left(\hat{\sigma}^2\right)$ goes to 0 with n and $\hat{\sigma}^2$ is a consistent estimator.

9.24 **(+10pt)**

a. From Chapter 6, $\sum_{i=1}^n Y_i^2$ is chi-square with with n degrees of freedom.

b. Note that $E\left(W_n\right)=1$ and $V\left(W_n\right)=2/n$. Thus, as $n \rightarrow \infty, W_n \rightarrow E\left(W_n\right)=1$ in probability.

9.26

a. We have that $P\left(\theta-\varepsilon \leq Y_{(n)} \leq \theta+\varepsilon\right)=F_{(n)}(\theta+\varepsilon)-F_{(n)}(\theta-\varepsilon)$.

- If $\varepsilon>\theta, F_{(n)}(\theta+\varepsilon)=1$ and $F_{(n)}(\theta-\varepsilon)=0$. Thus, $P\left(\theta-\varepsilon \leq Y_{(n)} \leq \theta+\varepsilon\right)=1$.
- If $\varepsilon<\theta, F_{(n)}(\theta+\varepsilon)=1, F_{(n)}(\theta-\varepsilon)=\left(\frac{\theta-\varepsilon}{\theta}\right)^n$. So, $P\left(\theta-\varepsilon \leq Y_{(n)} \leq \theta+\varepsilon\right)=1-\left(\frac{\theta-\varepsilon}{\theta}\right)^n$.

b. The result follows from $\lim_{n \rightarrow \infty} P\left(\theta-\varepsilon \leq Y_{(n)} \leq \theta+\varepsilon\right)=\lim_{n \rightarrow \infty}\left[1-\left(\frac{\theta-\varepsilon}{\theta}\right)^n\right]=1$.

9.30

Note that Y is beta with $\mu=3/4$ and $\sigma^2=3/5$. Thus, $E(\bar{Y})=3/4$ and $V(\bar{Y})=3/(5n)$. Thus, $V(\bar{Y}) \rightarrow 0$ and $\bar{Y}$ converges in probability to $3/4$.

9.36

Let $X_1, X_2, \ldots, X_n$ be a sequence of Bernoulli trials with success probability p. Thus, it is seen that $Y=\sum_{i=1}^n X_i$. Thus, by the Central Limit Theorem, $U_n=\frac{\hat{p}_n-p}{\sqrt{\frac{pq}{n}}}$ has a limiting standard normal distribution. By Ex. 9.20, it was shown that $\hat{p}_n$ is consistent for p, so it makes sense that $\hat{q}_n$ is consistent for q, and so by Theorem 9.2, $\hat{p}_n \hat{q}_n$ is consistent for pq. Define $W_n=\sqrt{\frac{\hat{p}_n \hat{q}_n}{pq}}$ so that W_n converges in probability to 1 . By Theorem 9.3 , the quantity $\frac{U_n}{W_n}=\frac{\hat{p}_n-p}{\sqrt{\frac{\hat{p}_n \hat{q}_n}{n}}}$ converges to a standard normal variable.

9.37 **(+5pt)**

The likelihood function is $L(p)=p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$. By Theorem 9.4, $\sum_{i=1}^n X_i$ is sufficient for p with $g\left(\sum X_i, p\right)=p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$ and $h(y)=1$.

9.38

For this exercise, the likelihood function is given by

$$L=\frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left[-\frac{\sum_{i=1}^n\left(y_i-\mu\right)^2}{2\sigma^2}\right]=(2\pi)^{-n/2}\sigma^{-n} \exp\left[\frac{-1}{2\sigma^2}\left(\sum_{i=1}^n y_i^2-2\mu n\bar{y}+n\mu^2\right)\right].$$

a. When σ^2 is known, $\bar{Y}$ is sufficient for μ by Theorem 9.4 with

$$g(\bar{y}, \mu)=\exp\left(\frac{2\mu n\bar{y}-n\mu^2}{2\sigma^2}\right) \text{ and } h(\boldsymbol{y})=(2\pi)^{-n/2}\sigma^{-n} \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^n y_i^2\right).$$

b. When μ is known, use Theorem 9.4 with

$$g\left(\sum_{i=1}^n\left(y_i-\mu\right)^2, \sigma^2\right)=\left(\sigma^2\right)^{-n/2} \exp\left[-\frac{\sum_{i=1}^n\left(y_i-\mu\right)^2}{2\sigma^2}\right] \text{ and } h(y)=(2\pi)^{-n/2}.$$

c. When both μ and σ^2 are unknown, the likelihood can be written in terms of the two statistics $U_1 = \sum_{i=1}^n Y_i$ and $U_2 = \sum_{i=1}^n Y_i^2$ with $h(y) = (2\pi)^{-n/2}$. The statistics $\bar{Y}$ and S^2 are also jointly sufficient since they can be written in terms of U_1 and U_2.

9.43

With θ known, the likelihood is $L(\alpha) = \alpha^n \theta^{-n\alpha} \left(\prod_{i=1}^n y_i\right)^{\alpha-1}$. By Theorem 9.4, $U = \prod_{i=1}^n Y_i$ is sufficient for α with $g(u, \alpha) = \alpha^n \theta^{-n\alpha} \left(\prod_{i=1}^n y_i\right)^{\alpha-1}$ and $h(y) = 1$.

9.54

Again, using the indicator notation, the density is

$$f(y \mid \alpha, \theta) = \alpha \theta^{-\alpha} y^{\alpha-1} I_{0,\theta}(y).$$

The likelihood function is

$$L(\alpha, \theta) = \alpha^n \theta^{-n\alpha} \left(\prod_{i=1}^n y_i\right)^{\alpha-1} \prod_{i=1}^n I_{0,\theta}\left(y_i\right) = \alpha^n \theta^{-n\alpha} \left(\prod_{i=1}^n y_i\right)^{\alpha-1} I_{0,\theta}\left(y_{(n)}\right).$$

Theorem 9.4 is satisfied with $g\left(\prod_{i=1}^n y_i, y_{(n)}, \alpha, \theta\right) = \alpha^n \theta^{-n\alpha} \left(\prod_{i=1}^n y_i\right)^{\alpha-1} I_{0,\theta}\left(y_{(n)}\right), h(y) = 1$ so that $\left(\prod_{i=1}^n Y_i, Y_{(n)}\right)$ is jointly sufficient for α and θ.

9.60

a. The density can be expressed as $f(y \mid \theta) = \theta \exp[(\theta - 1)\ln y]$. Thus, the density has exponential form and $-\sum_{i=1}^n \ln y_i$ is sufficient for θ.

b. Let $W = -\ln Y$. The distribution function for W is

$$F_W(w) = P(W \leq w) = P(-\ln Y \leq w) = 1 - P\left(Y \leq e^{-w}\right) = 1 - \int_0^{e^{-w}} \theta y^{\theta-1} dy = 1 - e^{-\theta w}, w > 0.$$

This is the exponential distribution function with mean $1/\theta$.

c. For the transformation $U = 2\theta$W, the distribution function for U is

$$F_U(u) = P(U \leq u) = P(2\theta W \leq u) = P\left(W \leq \frac{u}{2\theta}\right) = F_W\left(\frac{u}{2\theta}\right) = 1 - e^{-u/2}, u > 0.$$

Note that this is the exponential distribution with mean 2 , but this is equivalent to the chi-square distribution with 2 degrees of freedom. Therefore, by property of independent chi-square variables, $2\theta \sum_{i=1}^n W_i$ is chi-square with $2n$ degrees of freedom.

d. From Ex. 4.112, the expression for the expected value of the reciprocal of a chi-square variable is given. Thus, it follows that $E\left[\left(2\theta \sum_{i=1}^n W_i\right)^{-1}\right] = \frac{1}{2n-2} = \frac{1}{2(n-1)}$.

e. From part d, $\frac{n-1}{\sum_{i=1}^{n} W_i} = \frac{n-1}{-\sum_{i=1}^{n} \ln Y_i}$ is unbiased and thus the MVUE for θ.

9.63 **(+10pt)**

a. The distribution function for Y is $F(y) = y^3/\theta^3, 0 \le y \le \theta$. So, the density function for $Y_{(n)}$ is $f_{(n)}(y) = n[F(y)]^{n-1} f(y) = 3ny^{3n-1}/\theta^{3n}, 0 \le y \le \theta$.

b. From part a, it can be shown that $E\left(Y_{(n)}\right) = \frac{3n}{3n+1}\theta$. Since $Y_{(n)}$ is sufficient for θ, $\frac{3n+1}{3n} Y_{(n)}$ is the MVUE for θ.

9.65

a. $E(T) = P(T = 1) = P\left(Y_1 = 1, Y_2 = 0\right) = P\left(Y_1 = 1\right) P\left(Y_2 = 0\right) = p(1-p)$.

b.

$$\begin{aligned} P(T = 1 \mid W = W) &= \frac{P\left(Y_1 = 1, Y_2 = 0, W = w\right)}{P(W = w)} = \frac{P\left(Y_1 = 1, Y_2 = 0, \sum_{i=3}^{n} Y_i = W - 1\right)}{P(W = w)} \\ &= \frac{P\left(Y_1 = 1\right) P\left(Y_2 = 0\right) P\left(\sum_{i=3}^{n} Y_i = W - 1\right)}{P(W = w)} = \frac{p(1-p)\binom{n-2}{w-1} p^{w-1}(1-p)^{n-(w-1)}}{\binom{n}{W} p^{w}(1-p)^{n-w}} \\ &= \frac{W(n-w)}{n(n-1)}. \end{aligned}$$

c. $E(T \mid W) = P(T = 1 \mid W) = \frac{W}{n}\left(\frac{n-W}{n-1}\right) = \left(\frac{n}{n-1}\right) \frac{W}{n}\left(1 - \frac{W}{n}\right)$. Since T is unbiased by part (a) above and W is sufficient for p and so also for $p(1-p)$, $n\bar{Y}(1-\bar{Y})/(n-1)$ is the MVUE for $p(1-p)$.