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- Linear transformations and matrices. [6.1, 6.3, 6.4, 8.1]
- Let A be an $m \times n$ matrix. For any $n \times 1$ matrix $\mathbf{x}$, $A\mathbf{x}$ is an $m \times 1$ matrix. So A can be used to define a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **linear**, if it preserves scalar multiplication and addition, i.e., for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$ and $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.

  In other words, T is linear if it maps a linear combination $\alpha\mathbf{x} + \beta\mathbf{y}$ of vectors $\mathbf{x}$ and $\mathbf{y}$ to a linear combination $\alpha T(\mathbf{x}) + \beta T(\mathbf{y})$ of $T(\mathbf{x})$ and $T(\mathbf{y})$.

  In linear algebra, a linear mapping is called a linear transformation usually.
- Kernel and range. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The **kernel** of T, $\ker(T)$, is the set $\{\mathbf{v} \in \mathbb{R}^n : T(\mathbf{v}) = \mathbf{0}\}$ and the **range** of T, $\operatorname{ran}(T)$, is the set $\{T(\mathbf{v}) : \mathbf{v} \in \mathbb{R}^n\}$. We can show that $\ker(T)$ is a subspace of $\mathbb{R}^n$ and $\operatorname{ran}(T)$ is a subspace of $\mathbb{R}^m$.
- For an $m \times n$ matrix A we associate a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(\mathbf{x}) = A\mathbf{x}$. Then $\ker(T_A)$ is the solution space of $A\mathbf{x} = \mathbf{0}$, or the **null space** of A, $\operatorname{null}(A)$, and $\operatorname{ran}(T_A)$ is the column space of A.
- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the $m \times n$ matrix whose i-th column is $T(\mathbf{e}_i)$ for all $i = 1, 2, \ldots, n$. Then $T = T_A$.
- Questions. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$.
  - If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent, are $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)$ linearly independent?
  - If $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)$ are linearly independent, are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ linearly independent?
- A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto if its range is the entire codomain $\mathbb{R}^m$.
- A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if T maps distinct vectors in $\mathbb{R}^n$ into distinct vectors in $\mathbb{R}^m$.
- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.
  Proof. Suppose T is one-to-one. Since T is linear, we have that $T(\mathbf{0}) = \mathbf{0}$. The fact that T is one-to-one implies that $\ker(T) = \{\mathbf{0}\}$.

  Suppose $\ker(T) = \{\mathbf{0}\}$. We will show that if $\mathbf{x}_1 \neq \mathbf{x}_2$, then $T(\mathbf{x}_1) \neq T(\mathbf{x}_2)$. If $\mathbf{x}_1 \neq \mathbf{x}_2$, then $\mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$, so $\mathbf{x}_1 - \mathbf{x}_2 \notin \ker(T)$. Then, $T(\mathbf{x}_1 - \mathbf{x}_2) = T(\mathbf{x}_1) - T(\mathbf{x}_2) \neq \mathbf{0}$.
- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$.
  - If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent and $\ker(T) = \{\mathbf{0}\}$, then $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)$ are linearly independent.
  - If $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.
- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one if and only if it is onto.
  Proof. Suppose T is one-to-one. Its kernel is $\{\mathbf{0}\}$. The set $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)\}$ is linearly independent. So the range of T has dimension at least n, which implies that the range of T is $\mathbb{R}^n$.

  Suppose T is onto. There are vectors $\mathbf{v}_i$ such that $T(\mathbf{v}_i) = \mathbf{e}_i$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent and is a basis for $\mathbb{R}^n$. Let $\mathbf{x} = \sum_{i=1} a_i\mathbf{v}_i \in \ker(T)$. Then $T(\mathbf{x}) = \sum_{i=1} a_i T(\mathbf{v}_i) = \sum_{i=1} a_i\mathbf{e}_i = \mathbf{0}$, which implies that $a_i = 0$ for all i, and so $\mathbf{x} = \mathbf{0}$.
- If T is a one-to-one linear transformation, then so is T^{-1}.
- Let $T_A : \mathbb{R}^n \to \mathbb{R}^k$, $T_B : \mathbb{R}^k \to \mathbb{R}^m$ be linear transformations. Then $T_B \circ T_A = T_{BA}$ is also a linear transformation.

- Linear transformations in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ (Details in the next lecture)
  - $\mathbb{R}$: $T : \mathbb{R} \to \mathbb{R}$, $x \mapsto ax$.
  - $\mathbb{R}^2$: Reflection, orthogonal projection, rotation
  - $\mathbb{R}^3$: Reflection, orthogonal projection, rotation
- Reflection in $\mathbb{R}^n$
- Projection in $\mathbb{R}^n$
- Rotation in $\mathbb{R}^n$
- When we say $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis, we mean an ordered basis, i.e., the order of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is fixed.
- $\mathbb{R}^n$ with an ordered basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. The coordinate matrix $[\mathbf{x}]_{\mathcal{B}}$
- $\mathbb{R}^n$ with basis $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, the standard basis. The coordinate matrix $[\mathbf{x}]_{\mathcal{S}} = [\mathbf{x}] = \mathbf{x}$
- $T : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbb{R}^n$ with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, $[T]_{\mathcal{B}}$, matrix for T with respect to the basis $\mathcal{B}$. The i-th column of $[T]_{\mathcal{B}}$ is $[T(\mathbf{v}_i)]_{\mathcal{B}}$.
- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Choose a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$ and a basis $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ for $\mathbb{R}^m$. We define $[T]_{\mathcal{B}',\mathcal{B}}$, the matrix for T with respect to bases $\mathcal{B}$ and $\mathcal{B}'$.
- $T : (\mathbb{R}^n, \mathcal{B}) \to (\mathbb{R}^m, \mathcal{B}')$, $[T]_{\mathcal{B}',\mathcal{B}}$, the matrix for T with respect to bases $\mathcal{B}$ and $\mathcal{B}'$.

$$[T(\mathbf{x})]_{\mathcal{B}'} = [T]_{\mathcal{B}',\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

  The i-th column of $[T]_{\mathcal{B}',\mathcal{B}}$ is $[T(\mathbf{v}_i)]_{\mathcal{B}'}$.
- $T_A : (\mathbb{R}^n, \mathcal{S}) \to (\mathbb{R}^m, \mathcal{S}')$, $[T_A]_{\mathcal{S}',\mathcal{S}} = A$, the matrix for T_A with respect to bases $\mathcal{S}$ and $\mathcal{S}'$.
- $\mathrm{id} : (\mathbb{R}^n, \mathcal{S}) \to (\mathbb{R}^n, \mathcal{B})$, $[\mathrm{id}]_{\mathcal{B},\mathcal{S}}$, the matrix for the identity mapping, id, with respect to bases $\mathcal{S}$ and $\mathcal{B}$.

$$[\mathrm{id}(\mathbf{x})]_{\mathcal{B}} = [\mathrm{id}]_{\mathcal{B},\mathcal{S}}[\mathbf{x}]_{\mathcal{S}} \quad \to \quad [\mathbf{x}]_{\mathcal{B}} = [\mathrm{id}]_{\mathcal{B},\mathcal{S}}[\mathbf{x}]_{\mathcal{S}}$$

- $\mathrm{id} : (\mathbb{R}^n, \mathcal{B}) \to (\mathbb{R}^n, \mathcal{S})$, $[\mathrm{id}]_{\mathcal{S},\mathcal{B}}$, the matrix for the identity mapping, id, with respect to bases $\mathcal{B}$ and $\mathcal{S}$.

$$[\mathrm{id}(\mathbf{x})]_{\mathcal{S}} = [\mathrm{id}]_{\mathcal{S},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} \quad \to \quad [\mathbf{x}]_{\mathcal{S}} = [\mathrm{id}]_{\mathcal{S},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

  The i-th column of $[\mathrm{id}]_{\mathcal{S},\mathcal{B}}$ is $[\mathbf{v}_i]_{\mathcal{S}} = \mathbf{v}_i$.
- The relation between $[T]_S : (\mathbb{R}^n, \mathcal{S}) \to (\mathbb{R}^n, \mathcal{S})$ and $[T]_{\mathcal{B}} : (\mathbb{R}^n, \mathcal{B}) \to (\mathbb{R}^n, \mathcal{B})$.

$$[T(\mathbf{x})]_{\mathcal{S}} = [T]_{\mathcal{S}}[\mathbf{x}]_{\mathcal{S}} = [\mathrm{id}]_{\mathcal{S},\mathcal{B}}[T]_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B},\mathcal{S}}[\mathbf{x}]_{\mathcal{S}}$$

$$[T]_{\mathcal{S}} = [\mathrm{id}]_{\mathcal{S},\mathcal{B}}[T]_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B},\mathcal{S}}$$

$$[T]_{\mathcal{B}} = [\mathrm{id}]_{\mathcal{B},\mathcal{S}}[T]_{\mathcal{S}}[\mathrm{id}]_{\mathcal{S},\mathcal{B}}$$