School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2023 Spring

Final

Tuesday, June 13, 2023
9:00–11:30 am

NAME: ______________________________

Student ID: ______________________________

- Don't forget to put your name and student ID.
- **Record all your solutions in this answer booklet. Only this answer booklet will be considered in the grading of your exam.**
- Be sure to show all relevant work and reasoning. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.

Problem	Your score	Max score
1		10
2		10
3		10
4		10
Total		40

Problem 1 (10 Points)

Consider three random variables Θ, X, and Y, with known variances $\mathrm{var}(\Theta)$, $\mathrm{var}(X)$, and $\mathrm{var}(Y)$, and covariances, $\mathrm{cov}(\Theta, X)$, $\mathrm{cov}(\Theta, Y)$, and $\mathrm{cov}(X, Y)$. Assume that $\mathbb{E}[\Theta] = \mathbb{E}[X] = \mathbb{E}[Y] = 0$, $\mathrm{var}(X) > 0$, $\mathrm{var}(Y) > 0$, and $|\rho(X, Y)| \neq 1$. (Remind that $\rho(A, B) = \mathrm{cov}(A, B)/\sqrt{\mathrm{var}(A)\,\mathrm{var}(B)}$ and for any two zero-mean random variables A, B, $\mathrm{cov}(A, B) = \mathbb{E}[AB]$.)

We consider a linear estimator of Θ based on X and Y, in the form of

$$\hat{\Theta} = aX + bY,$$

for some constants a, b. We aim to choose a, b to minimize the mean squared error $\mathbb{E}[(\Theta - \hat{\Theta})^2]$. Find a and b in terms $\mathrm{var}(\Theta)$, $\mathrm{var}(X)$, $\mathrm{var}(Y)$, $\mathrm{cov}(\Theta, X)$, $\mathrm{cov}(\Theta, Y)$, and $\mathrm{cov}(X, Y)$ for the following two cases.

a) (5 points) Find a and b, when X and Y are uncorrelated, i.e., $\mathbb{E}[XY] = 0$.

Answer:

$$a = \frac{\mathbb{E}[\Theta X]}{\mathbb{E}[X^2]} = \frac{\mathrm{cov}(\Theta, X)}{\mathrm{var}(X)}$$
$$b = \frac{\mathbb{E}[\Theta Y]}{\mathbb{E}[Y^2]} = \frac{\mathrm{cov}(\Theta, Y)}{\mathrm{var}(Y)}$$

Reasoning for Problem 1(a): Note that

$$\mathbb{E}[(\Theta - aX - bY)^2] = \mathbb{E}[\Theta^2] + a^2\mathbb{E}[X^2] + b^2\mathbb{E}[Y^2] - 2a\mathbb{E}[\Theta X] - 2b\mathbb{E}[\Theta Y] + 2ab\mathbb{E}[XY].$$

Assuming that X and Y are uncorrelated, we have $\mathbb{E}[XY] = 0$. We differentiate the expression for the mean squared error with respect to a and b, and set the derivatives to zero to obtain

$$a = \frac{\mathbb{E}[\Theta X]}{\mathbb{E}[X^2]} = \frac{\mathrm{cov}(\Theta, X)}{\mathrm{var}(X)},$$
$$b = \frac{\mathbb{E}[\Theta Y]}{\mathbb{E}[Y^2]} = \frac{\mathrm{cov}(\Theta, Y)}{\mathrm{var}(Y)}.$$

b) (5 points) Find a and b for the general case where X and Y are not necessarily uncorrelated.)

Answer:

$$
\begin{aligned}
a &= \frac{\operatorname{var}(Y)\operatorname{cov}(\Theta, X) - \operatorname{cov}(\Theta, Y)\operatorname{cov}(X, Y)}{\operatorname{var}(X)\operatorname{var}(Y) - \operatorname{cov}^2(X, Y)} \\
b &= \frac{\operatorname{var}(X)\operatorname{cov}(\Theta, Y) - \operatorname{cov}(\Theta, X)\operatorname{cov}(X, Y)}{\operatorname{var}(X)\operatorname{var}(Y) - \operatorname{cov}^2(X, Y)}
\end{aligned}
$$

Reasoning for Problem 1(b): If X and Y are uncorrelated, we similarly set the derivatives of the mean squared error to zero We obtain and then solve a system of two linear equations

$$
\begin{aligned}
a\mathbb{E}[X^2] + b\mathbb{E}[XY] &= \mathbb{E}[\Theta X] \\
a\mathbb{E}[XY] + b\mathbb{E}[Y^2] &= \mathbb{E}[\Theta Y]
\end{aligned}
$$

in the unknowns a and b, whose solution is

$$
\begin{aligned}
a &= \frac{\operatorname{var}(Y)\operatorname{cov}(\Theta, X) - \operatorname{cov}(\Theta, Y)\operatorname{cov}(X, Y)}{\operatorname{var}(X)\operatorname{var}(Y) - \operatorname{cov}^2(X, Y)} \\
b &= \frac{\operatorname{var}(X)\operatorname{cov}(\Theta, Y) - \operatorname{cov}(\Theta, X)\operatorname{cov}(X, Y)}{\operatorname{var}(X)\operatorname{var}(Y) - \operatorname{cov}^2(X, Y)}.
\end{aligned}
$$

Problem 2 **(10 Points)**

Assume that X_i's are independent and identically distributed random variables with mean p. To estimate p, we consider the sample mean defined by

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

a) (5 points) Find the smallest n, the number of samples, for which the Chebyshev inequality yields a guarantee

$$\mathbb{P}(|M_n - p| \geq 0.1) \leq 0.05.$$

Assume that $\mathrm{var}(X_i) = v$ for some constant v. State your answer as a function of v.

Answer:

$$n = 2000v$$

Reasoning for Problem 2(a): Since $\mathbb{E}[M_n] = p$ and $\mathrm{var}(M_n) = \frac{v}{n}$, by Chebyshev inequality,

$$\Pr(|M_n - p| \geq 0.1) \leq \frac{\mathrm{var}(M_n)}{0.1^2} = \frac{v}{n \cdot 0.01} = 0.05.$$

The required n is $80v$.

b) (5 points) Assume that $n = 10,000$. Find an approximate value for the probability

$$\mathbb{P}(|M_{10000} - p| \geq 0.1)$$

using the Central Limit Theorem. Assume again that $\text{var}(X_i) = v$ for some constant v. Give your answer in terms of v, and the standard normal CDF $\Phi(\cdot)$.

Answer:

$$2\left(1 - \Phi\left(\frac{10}{\sqrt{v}}\right)\right)$$

Reasoning for Problem 2(b):

By CLT, we can approximate

$$\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{nv}}$$

by a standard normal distribution when n is large. Hence,

$$\Pr(|M_{10000} - p| \geq 0.5) = \Pr\left(\left|\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{nv}}\right| \geq \frac{0.1\sqrt{n}}{\sqrt{v}}\right) = 2\left(1 - \Phi\left(\frac{10}{\sqrt{v}}\right)\right).$$

Problem 3 (10 Points)

In this problem, we consider Poisson processes. Remind that for Poisson process with rate λ, the probability distribution for the first arrival time T_1 (and also the inter-arrival time $T_k = Y_k - Y_{k-1}$, $k \geq 2$, where Y_k is the k-th arrival time) follows the exponential distribution with rate λ, i.e., $f_{T_1}(t) = \lambda e^{-\lambda t}$, for $t \geq 0$ and $\mathbb{E}[T_1] = 1/\lambda$.

a) (5 points) Consider two independent Poisson processes with rates λ_1 and λ_2, respectively. Let X_1 be the first arrival time in the first process, and X_2 be the first arrival time in the second process. Find the expected value of $\max\{X_1, X_2\}$.

Answer:

$$\mathbb{E}[\max\{X_1, X_2\}] = \frac{1}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{1}{\lambda_1}$$

Reasoning for Problem 3(a):

Let $T = \min\{X_1, X_2\}$ be the first time when one of the processes registers an arrival. Let $S = \max\{X_1, X_2\} - T$ be the additional time until both have registered an arrival. Since the merged process is Poisson with rate $\lambda_1 + \lambda_2$, we have

$$\mathbb{E}[T] = \frac{1}{\lambda_1+\lambda_2}.$$

Concerning S, there are two cases to consider:

(i) The first arrival comes from the first process, which happens with probability $\lambda_1/(\lambda_1+\lambda_2)$. We then have to wait for an arrival from the second process, which takes $1/\lambda_2$ time on the average.

(ii) The first arrival comes from the second process, which happens with probability $\lambda_2/(\lambda_1+\lambda_2)$. We then have to wait for an arrival from the first process, which takes $1/\lambda_1$ time on the average.

Putting everything together, we obtain

$$\begin{aligned} E[\max\{X_1, X_2\}] &= \frac{1}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{1}{\lambda_1} \\ &= \frac{1}{\lambda_1+\lambda_2}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right). \end{aligned}$$

b) (5 points) Consider two independent Poisson processes with rates λ_1 and λ_2, respectively. Let Y be the first arrival time in the first process and Z be the second arrival time in the second process. Find the expected value of $\max\{Y, Z\}$. (Hint: you may write down your answer in terms of $\mathbb{E}[\max\{X_1, X_2\}]$, defined in (a). You don't need to specify what $\mathbb{E}[\max\{X_1, X_2\}]$ is in terms of λ_1 and/or λ_2.)

Answer:

$$\mathbb{E}[\max\{Y, Z\}] = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{2}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\mathbb{E}[\max\{X_1, X_2\}].$$

Reasoning for Problem 3(b): As in the previous problem, when we define T be the first time when one of the processes registers an arrival, since the merged process is Poisson with rate $\lambda_1 + \lambda_2$, we have $\mathbb{E}[T] = 1/(\lambda_1 + \lambda_2)$.

Again, we need to consider two cases:

(i) The arrival at time T comes from the first process; this happens with probability $\lambda_1/(\lambda_1+\lambda_2)$. In this case, we have to wait an additional time until the second process registers two arrivals, whose expectation is $1/\lambda_2$.

(ii) The first arrival comes from the second process, which happens with probability $\lambda_2/(\lambda_1+\lambda_2)$. In this case, the additional time S we have to wait is the time until each of the two processes registers an arrival. This is the maximum of two independent exponential random variables and, according to the result of (a), we have

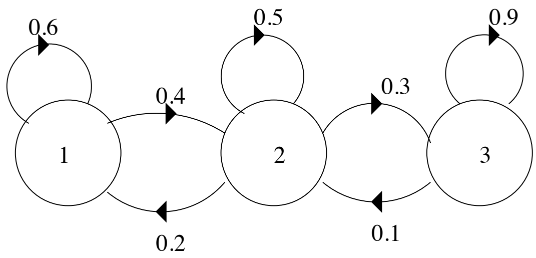
$$\mathbb{E}[S] = \frac{1}{\lambda_1 + \lambda_2}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right).$$

Putting everything together, we obtain

$$\begin{aligned}
\mathbb{E}[\max\{Y, Z\}] &= \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{2}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\mathbb{E}[S] \\
&= \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{2}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{1}{\lambda_1 + \lambda_2}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right).
\end{aligned}$$

Problem 4 (10 Points)

Consider a Markov chain $\{X_n : n = 0, 1, \ldots, \}$, specified by the following transition diagram.



a) (3 points) Find the steady-state probabilities π_1, π_2, π_3 for the state 1,2, and 3.

Answer:

$$\pi_1 = 1/9 \qquad \pi_2 = 2/9 \qquad \pi_3 = 6/9$$

Reasoning for Problem 4(a): We set up the balance equations of a birth-death process and the normalization equation as such:

$$\begin{aligned} \pi_1 p_{12} &= \pi_2 p_{21} \\ \pi_2 p_{23} &= \pi_3 p_{32} \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

Solving the system of equations yields the following steady-state probabilities:

$$\begin{aligned} \pi_1 &= 1/9, \\ \pi_2 &= 2/9, \\ \pi_3 &= 6/9. \end{aligned}$$

b) (3 points) Let $Y_n = X_n - X_{n-1}$. Thus, $Y_n = 1$ indicates that the n-th transition was to the right, $Y_n = 0$ indicates it was a self-transition, and $Y_n = -1$ indicates it was a transition to the left. Find $\lim_{n\to\infty} \mathbb{P}(Y_n = 1)$.

Answer:

$$\lim_{n\to\infty} \mathbb{P}(Y_n = 1) = 1/9$$

Reasoning for Problem 4(b): Using the total probability theorem and steady-state probabilities,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(Y_n = 1) &= \sum_{i=1}^{3} \pi_i \cdot \mathbb{P}(Y_n = 1 | X_{n-1} = i) \\ &= \pi_1 p_{12} + \pi_2 p_{23} = 1/9.\end{aligned}$$

c) (4 points) Given that the n-th transition was a transition to the right ($Y_n = 1$), find the probability that the previous state was state 1. (You can assume that n is large.)

Answer:

$$\mathbb{P}(X_{n-1} = 1 | Y_n = 1) = 2/5$$

Reasoning for Problem 4(c): Using Bayes' Rule,

$$\begin{aligned}\mathbb{P}(X_{n-1} = 1 | Y_n = 1) &= \frac{\mathbb{P}(X_{n-1} = 1)\mathbb{P}(Y_n = 1 | X_{n-1} = 1)}{\sum_{i=1}^{3} \mathbb{P}(X_{n-1} = i)\mathbb{P}(Y_n = 1 | X_{n-1} = i)} \\ &= \frac{\pi_1 p_{12}}{\pi_1 p_{12} + \pi_2 p_{23}} = 2/5.\end{aligned}$$