EE202A Signals and Systems, Fall 2023
Oct 16, 2023
Time Limit: 09:00 - 11:45 Lecturer: Jung-Woo Choi

Score Table (for teacher use only)

Question:	1	2	3	4	Total
Points:	15	30	30	25	100
Score:					

This is a CLOSED-BOOK exam.
Please provide ALL DERIVATIONS and EXPLANATIONS with your answers.
Any communication with others during the exam will be regarded as a cheating case.

1. (15 points) (T/F) Determine whether the following statements are true/false. Justify your answers.

   (a) (3 points) The system described by $y(t) = x(t-2) + 1$ is linear.

   Answer: **False**
   For $y_1(t) = x_1(t-2) + 1$ and $y_2(t) = x_2(t-2) + 1$, $y_1(t) + y_2(t) \neq x_1(t-2) + x_2(t-2) + 1$. The constant addition (+1) makes the system nonlinear.

   (b) (3 points) The system described by $y(t) = \int_{-\infty}^{t} e^{(t-\tau)} x(\tau) d\tau$ is stable.

   Answer: **False**
   The integral is equivalent to the convolution $y(t) = (e^t u(t)) * x(t)$. Since the system's output is described by the convolution integral, the system is LTI. The total power of an LTI system's impulse response should be finite to be stable. However, $h(t) = e^t u(t)$ diverges as $t \to \infty$.

   (c) (3 points) The differential and convolution operators are commutative. In other words, the following always holds:

$$\frac{d}{dt}\left(h(t) * g(t)\right) = \frac{dh(t)}{dt} * g(t) = h(t) * \frac{dg(t)}{dt}$$

   Answer: **True**
   First, a time delay operation is commutative with the differential operation, i.e., $\frac{dy(t-\tau)}{dt} =$

$\frac{dy(t)}{dt}\Big|_{t\to t-\tau}$. From two definitions of convolution integral, we have

$$\begin{aligned}\frac{d}{dt}(h(t)*g(t)) &= \frac{d}{dt}\int_{-\infty}^{\infty} h(t-\tau)g(\tau)d\tau \\ &= \int_{-\infty}^{\infty}\left[\frac{d}{dt}h(t-\tau)\right]g(\tau)d\tau = \frac{dh(t)}{dt}*g(t) \\ \frac{d}{dt}(h(t)*g(t)) &= \frac{d}{dt}\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\frac{dg(t-\tau)}{dt}d\tau = h(t)*\frac{dg(t)}{dt}\end{aligned}$$

(d) (3 points) The system represented by $y(t) = \int_{-\infty}^{t} x(\tau)d\tau + 2$ is linear and time-invariant.

(False)

- Linearity

$$y(t) = \int_{-\infty}^{t} a*x_1(\tau) + b*x_2(\tau)d\tau + 2 \neq \left(a\int_{-\infty}^{t} x_1(\tau)d\tau + 2\right) + \left(b\int_{-\infty}^{t} x_2(\tau)d\tau + 2\right)$$

The system is not linear.

- Time invariance

$$\begin{aligned}y_1(t) &= y(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)d\tau + 2 \\ y_2(t) &= \int_{-\infty}^{t} x(\tau - t_0)d\tau + 2 \\ &= \int_{-\infty}^{t-t_0} x(\tau')d\tau' + 2 \\ y_1(t) &= y_2(t)\end{aligned}$$

(e) (3 points) The Fourier series coefficients of the signal $x[n] = \sum_{m=-\infty}^{\infty}(-1)^m\delta[n-mN]$ are $a_k = (1-(-1)^k)/(2N)$.

(True)

The fundamental period of this signal is $2N$, and the fundamental frequency is $\Omega_0 = 2\pi/2N = \pi/N$. According to the analysis equation of FS expansion,

$$\begin{aligned}a_k &= \frac{1}{2N}\sum_{n=0}^{2N-1} x[n]e^{-jk\Omega_0 n} \\ &= \frac{1}{2N}\sum_{n=0}^{2N-1}(\delta[n]-\delta[n-N])e^{-jk\Omega_0 n} \\ &= (1-e^{-jk\Omega_0 N})/(2N) \qquad = (1-e^{-jk\pi})/(2N) \\ &= (1-(-1)^k)/(2N)\end{aligned}$$

2. (30 points) The LTI system described by the following LCCDE satisfies the condition of initial rest.

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} = x(t) \tag{1}$$

Find the impulse response of this system through the following steps.

(a) (5 points) Find the impulse response of the LTI system satisfying the condition of initial rest and the following LCCDE.

$$\frac{dw(t)}{dt} + 2w(t) = x(t) \tag{2}$$

The impulse response of the IIR system can be obtained by solving the homogeneous equation with an inhomogeneous auxiliary condition:

$$\begin{aligned} \frac{dh_1(t)}{dt} + 2h_1(t) = 0, \quad h_1(0^+) = 1 \\ h_1(t) = Ae^{st} \longrightarrow (s+2)Ae^{st} = 0, \quad s = -2 \\ h_1(0^+) = Ae^{-2\cdot 0^+} = 1, \quad A = 1 \\ \therefore h_1(t) = e^{-2t}u(t) \end{aligned}$$

(b) (5 points) Derive the LCCDE for the LTI inverse of the system defined by Eq. (2).

Let's take the derivative of $h_1(t)$.
$\frac{dh_1(t)}{dt} = -2e^{-2t}u(t) + e^{-2t}\delta(t) = -2e^{-2t}u(t) + \delta(t)$
Therefore, $\frac{dh_1(t)}{dt} + 2h_1(t) = \delta(t)$.
From this result, we can find an LTI inverse satisfying $h_1(t) * h_1^{-1}(t) = \delta(t)$ as
$y(t) = \frac{dw(t)}{dt} + 2w(t)$

(c) (10 points) Express $y(t)$ of Eq. (1) in terms of $w(t)$ of Eq. (2). From this relation, derive the impulse response $h(t)$ of Eq. (1).
(Hint: if necessary, use the relation between unit impulse and unit step signal $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$)

We can simply get Eq. (1) by substituting

$$\frac{dy(t)}{dt} = w(t) \tag{3}$$

into Eq. (2). Therefore, the system described by Eq. (1) can be regarded as the serial interconnection of two systems defined by Eq. (2) and Eq. (3).
To impulse response of the second system can be derived as

$$\begin{aligned} \frac{dh_2(t)}{dt} &= \delta(t) \\ h_2(t) &= \int_{-\infty}^{t} \delta(\tau)d\tau = u(t) \end{aligned}$$

The impulse response of the total system is the convolution of two systems' impulse responses.

That is,

$$
\begin{aligned}
h(t) &= h_1(t) * h_2(t) = e^{-2t}u(t) * u(t) \\
&= \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)u(t-\tau)d\tau \\
&= \begin{cases} \int_0^t e^{-2\tau}d\tau, & \text{for } t > 0 \\ 0, & \text{for } t < 0 \end{cases} \\
&= \frac{1}{2}\left(1 - e^{-2t}\right)u(t)
\end{aligned}
$$

(d) (5 points) Determine whether the system is stable or not. Justify your answer.

The stability of an LTI system of Eq. (1) can be determined by the finiteness of the integral of $|h(t)|$. The integral of $|h(t)|$ is

$$
\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} \frac{1}{2}\left(1 - e^{-2t}\right)u(t)dt \\
&= \frac{1}{2}\left(\infty + \frac{1}{2}\right) \to \infty
\end{aligned}
$$

Therefore, the system is not stable.

(e) (5 points) Find out the frequency response $H(j\omega)$ of this system. Hint: use the eigenfunction $x(t) = e^{j\omega t}$ to find out the response $y(t) = H(j\omega)x(t)$, and determine the eigenvalue $H(j\omega)$.

Substituting $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)x(t)$ to LCCDE gives

$$
\begin{aligned}
j\omega(j\omega+2)H(j\omega)e^{j\omega t} &= e^{j\omega t} \\
\therefore H(j\omega) &= \frac{1}{j\omega(j\omega+2)}
\end{aligned}
$$

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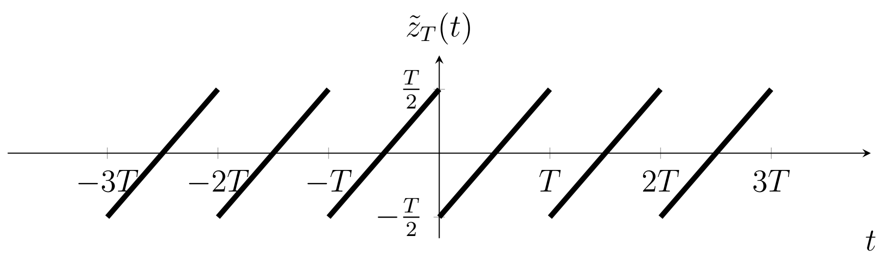
3. (30 points) [ST]

(a) (10 points) Determine the Fourier series coefficients of the following signal $\tilde{z}_T(t)$ with period T.



(Solution) The FS coefficients b_k of $\tilde{z}_T(t)$ are given by

(a) For $k \neq 0$,

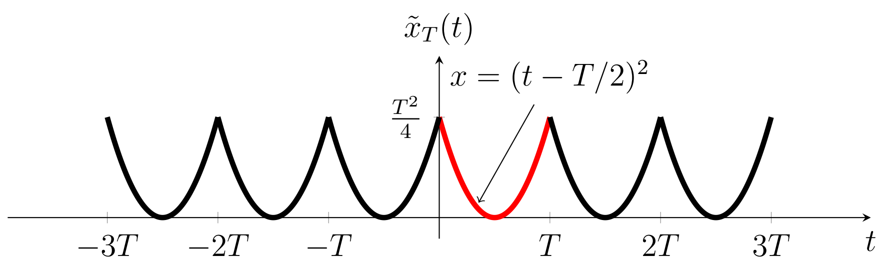
$$\begin{aligned} b_k &= \frac{1}{T}\int_0^T \tilde{z}_T(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_0^T (t-0.5T)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\frac{1}{-jk\omega_0}\left(te^{-jk\omega_0 t}\Big|_0^T - \int_0^T e^{-jk\omega_0 t}dt \right) \\ &= \frac{1}{T}\frac{1}{-jk\omega_0}T = \frac{jT}{2\pi k} \end{aligned}$$

(b) For $k = 0$,

$$b_k = \frac{1}{T}\int_0^T \tilde{z}_T(t)dt = 0$$

Note that the result can be directly derived from the integration of constant signals with amplitude 1. Using the integration property,

(b) (10 points) Derive the Fourier series coefficients of the following signal $\tilde{x}_T(t)$.



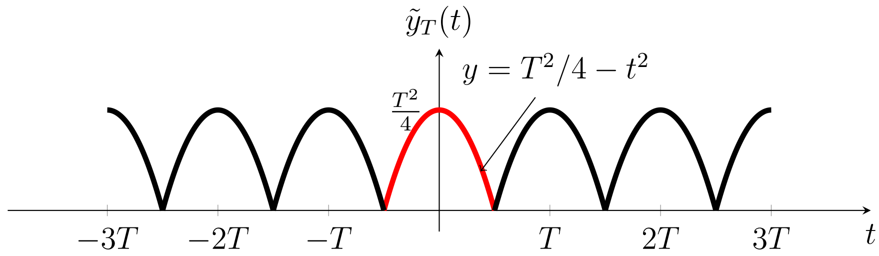
(Solution) Since $\frac{d\tilde{x}_T(t)}{dt} = 2\tilde{z}_T(t)$, we have $\tilde{x}_T(t) = 2\int_{-\infty}^t \tilde{z}_T(\tau)d\tau + C$. From the integration property of FS, the FS coefficients a_k of $\tilde{x}_T(t)$ is given by

$$\begin{aligned} a_k &= \frac{1}{+jk\omega_0}b_k \\ &= \frac{2}{(jk\omega_0)^2} = \frac{T^2}{2\pi^2 k^2} \text{ (for } k \neq 0) \end{aligned}$$

For $k = 0$, a_k can be derived by the average of the signal over a single period. That is

$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T \left(t - \frac{T}{2}\right)^2 dt \\ &= \frac{1}{T}\frac{1}{3}2\left(\frac{T}{2}\right)^3 = \frac{T^2}{12} \end{aligned}$$

(c) (10 points) Derive the Fourier series coefficients of the following signal.



(Solution) Since $\tilde{y}_T(t) = \frac{T^2}{4} - \tilde{x}_T(t - \frac{T}{2})$, the FS coefficients g_k of $\tilde{y}_T(t)$ are given by

$$\begin{aligned} g_k &= -a_k e^{-jk\omega_0 \frac{1}{2}T} = -a_k e^{-jk\frac{2\pi}{T}(\frac{1}{2}T)} = -a_k(-1)^k \\ &= \frac{T^2}{2\pi^2 k^2}(-1)^{k+1}, \quad \text{for } k \neq 0 \\ g_0 &= \frac{T^2}{4} - a_0 \\ &= \frac{T^2}{6} \end{aligned}$$

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4. (25 points) Let $\tilde{x}[n]$ be a periodic signal with period N and Fourier coefficients a_k. Assume that N is even.

(a) (5 points) Express the Fourier series coefficients b_k of $|\tilde{x}[n]|^2$ in terms of a_k.

(Solution) Since $|\tilde{x}[n]|^2 = \tilde{x}[n] \times \tilde{x}[n]^*$, we can use the multiplication-convolution property of FS. Also, the FS coefficients of the conjugated sequence $\tilde{x}[n]^*$ are given by a^*_{-k} (conjugation and flip) from the conjugation property of FS.

$$\tilde{x}[n]\tilde{x}[n]^* \Leftrightarrow a_k * a^*_{-k}$$

$$b_k = a_k * a^*_{-k} = a^*_{-k} * a_k = \sum_{m=<N>} a^*_{-m} a_{k-m} = \sum_{m=<N>} a^*_m a_{k+m}$$

(b) (5 points) If the coefficients a_k are real, is it guaranteed that the coefficients b_k are also real? Justify your answer.

(Solution) If a_k are real, $\tilde{x}[n]^* = \tilde{x}[-n]$ (conjugate symmetry). Therefore, the squared signal becomes $\tilde{y}[n] = |\tilde{x}[n]|^2 = \tilde{x}[n]\tilde{x}[-n]$. Because $\tilde{y}[n]$ is symmetric ($\tilde{y}[-n] = \tilde{y}[n]$) and real ($\tilde{y}[n]^* = \tilde{y}[n]$), its FS coefficients b_k are real and even.

(c) (5 points) Derive the Fourier Series coefficients of $\tilde{x}[n] - \tilde{x}[n - \frac{N}{2}]$.

(Solution) From the time delay property of FS, $\tilde{x}[n - \frac{N}{2}] \Longleftrightarrow a_k e^{-jk\Omega_0 \frac{N}{2}}$. Accordingly,

$$\begin{aligned} \tilde{x}[n] - \tilde{x}[n - \tfrac{N}{2}] &\Longleftrightarrow a_k - a_k e^{-jk\Omega_0 \frac{N}{2}} \\ &= a_k - a_k e^{-jk\left(\frac{2\pi}{N}\right)\frac{N}{2}} \\ &= a_k(1 - (-1)^k) \end{aligned}$$

(d) (10 points) Derive the Fourier Series coefficients of $\tilde{y}[n] = \begin{cases} \tilde{x}[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$. Assume that the fundamental period of $\tilde{y}[n]$ is still N.

(Solution)

$$\begin{aligned} y[n] &= \frac{1}{2}\left(x[n] + (-1)^n x[n]\right) \\ &= \frac{1}{2}\left(x[n] + e^{\pm j\pi n} x[n]\right) \\ &= \frac{1}{2}\left(x[n] + e^{j\left(\frac{N}{2}+mN\right)\frac{2\pi}{N}n} x[n]\right) \text{ for arbitrary integer } m. \end{aligned}$$

$$y[n] \Longleftrightarrow \frac{1}{2}(a_k + a_{k+\frac{N}{2}+mN}) \text{ (from the frequency shift property of FS)}$$

The answer without mN or with $\pm N/2$ is also correct.

[End of Problem]