EE202 Signals and Systems, Fall 2024
Oct 23, 2024
Time Limit: 13:00 - 15:45 Lecturer: Jung-Woo Choi

Score Table (for teacher use only)

Question:	1	2	3	4	Total
Points:	20	15	25	40	100
Score:					

This is a CLOSED-BOOK exam.
Please provide ALL DERIVATIONS and EXPLANATIONS with your answers.
Any access to online or offline material during the exam will be regarded as a cheating case.

1. (20 points) (T/F) Determine whether the following statements are true/false. Justify your answers.

   (a) (5 points) The system described by $y(t) = e^{-jx(t)}$ is causal

   Answer: **True**
   The output $y(t)$ at time t is determined by the input of the present time.

   (b) (5 points) The system described by $y(t) = x(e^{-t})$ is causal.

   Answer: **False**
   The case $e^{-t} > t$ exists, such that the future information of $x()$ at e^{-t} is needed to calculate the output $y(t)$ of the present time t.

   (c) (5 points) The system represented by $y(t) = u(t) * x(t)$ is BIBO stable.

   Answer: **False**
   The impulse response of this LTI system is $u(t)$, which is not absolutely integrable ($\int_{-\infty}^{\infty} u(t)dt \to \infty$).

   (d) (5 points) The LTI system defined by the following equation has an LTI inverse:
   $y[n] = 2x[n] - 2x[n-1]$

   Answer: **True** (for the below expl.) / **False** (for the input signal with non-zero constant)
   This is the first-order difference operation multiplied by two. The inverse operation of the first-order difference is the running sum operation, given by $f[n] = \frac{1}{2}\sum_{m=-\infty}^{n} y[m]$.

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2. (15 points) Derive the Fourier series coefficients a_k of the following periodic signals: (Hint: Find the fundamental frequency of the signal first.)

   (a) (5 points) $\tilde{x}[n] = \sum_{m=-\infty}^{\infty}(-1)^m \delta[n - mN]$

(Answer) $a_k = (1-(-1)^k)/(2N)$
The fundamental period of this signal is $2N$, and the fundamental frequency is $\Omega_0 = 2\pi/2N = \pi/N$. According to the analysis equation of FS expansion,

$$\begin{aligned} a_k &= \frac{1}{2N}\sum_{n=0}^{2N-1} x[n]e^{-jk\Omega_0 n} \\ &= \frac{1}{2N}\sum_{n=0}^{2N-1} (\delta[n]-\delta[n-N])e^{-jk\Omega_0 n} \\ &= (1-e^{-jk\Omega_0 N})/(2N) = (1-e^{-jk\pi})/(2N) \\ &= (1-(-1)^k)/(2N) \end{aligned}$$

(b) (5 points) $\tilde{x}(t) = \sin(4\pi t) + \cos(2\pi t)$

The fundamental period of $\sin(4\pi t)$ is 0.5 ($T_1 = 2\pi/\omega = 2\pi/(4\pi) = 0.5$). The second term has $T_2 = 1$. The least common multiple of T_1 and T_2 is 1. Therefore, the fundamental frequency is $\omega_0 = 2\pi/T = 2\pi$. According to the Euler's relation,

$$\begin{aligned} \tilde{x}(t) &= \frac{e^{j4\pi t} - e^{-j4\pi t}}{2j} + \frac{e^{j2\pi t} - e^{-j2\pi t}}{2} \\ &= \frac{1}{2j}\left(e^{j2\omega_0 t} - e^{-2\omega_0 t}\right) + \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \end{aligned}$$

(Answer) $a_{-2} = -\frac{1}{2j},\quad a_{-1} = \frac{1}{2},\quad a_1 = \frac{1}{2},\quad a_2 = \frac{1}{2j}$.

(c) (5 points) $\tilde{x}(t) = \cos\left(\frac{\pi t}{3}\right)\tilde{\text{rect}}_6(t/4) + \frac{1}{2}$.
($\tilde{\text{rect}}_T(t/L)$ indicates the rectangular function of duration L and period T.)

The fundamental period of $\cos\left(\frac{\pi t}{3}\right)$ and $\tilde{\text{rect}}_6(t/4)$ is 6.
So, their common period is 6 and the fundamental frequency is $\omega_0 = \frac{2\pi}{6} = \pi/3$.
From the properties of Fourier series,

$$\begin{aligned} \tilde{\text{rect}}_6(t/4) &\overset{\mathcal{FS}}{\Longleftrightarrow} \frac{2}{3}\text{sinc}\left(\frac{2}{3}k\right) \\ \cos\left(\frac{\pi t}{3}\right) = \cos(\omega_0 t) &\overset{\mathcal{FS}}{\Longleftrightarrow} \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] \\ \frac{1}{2} &\overset{\mathcal{FS}}{\Longleftrightarrow} \frac{1}{2}\delta[k] \end{aligned}$$

Multiplication in time yields convolution in frequency domain. Therefore, the answer is

$$a_k = \frac{1}{3}\text{sinc}\left(\frac{2}{3}(k-1)\right) + \frac{1}{3}\text{sinc}\left(\frac{2}{3}(k+1)\right) + \frac{1}{2}\delta[k]$$

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3. (25 points) For a periodic signal $\tilde{x}[n]$ with Fourier series coefficients a_k, derive the Fourier series coefficients of the following signals. The fundamental period of $\tilde{x}[n]$ is an even integer N.

(a) (10 points) $\tilde{y}[n] = \begin{cases} \tilde{x}[n], & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$.
Assume that the fundamental period of $\tilde{y}[n]$ is also N.

(Solution)

$$\begin{aligned} \tilde{y}[n] &= \frac{1}{2}\left(\tilde{x}[n] - (-1)^n \tilde{x}[n]\right) \\ &= \frac{1}{2}\left(\tilde{x}[n] - e^{\pm j\pi n}\tilde{x}[n]\right) \\ &= \frac{1}{2}\left(\tilde{x}[n] - e^{j\left(\frac{N}{2}+mN\right)\frac{2\pi}{N}n}\tilde{x}[n]\right) \text{ for arbitrary integer } m. \end{aligned}$$

$$\tilde{y}[n] \Longleftrightarrow \frac{1}{2}(a_k - a_{k+\frac{N}{2}+mN}) \text{ (from the frequency shift property of FS)}$$

The answer without mN or with $\pm N/2$ is also correct.

(b) (10 points) $\tilde{z}[n] = \sum_{m=0}^{N-1} \tilde{x}[m]\tilde{x}^*[m-n]$.

(Answer) $N|a_k|^2$
(Solution) Let's define a signal $\tilde{x}_2[n] = \tilde{x}^*[-n]$ with FS coefficients c_k.
From the conjugation and time reversal properties of FS, $c_k = a_k^*$.
In terms of $\tilde{x}_2[n]$, the given formula can be rewritten as

$$\begin{aligned} \tilde{z}[n] &= \sum_{m=0}^{N-1} \tilde{x}[m]\tilde{x}_2[n-m] \\ &= \tilde{x}[n] * \tilde{x}_2[n], \end{aligned}$$

which is a convolution sum of $\tilde{x}[n]$ and $\tilde{x}_2[n]$.
The convolution-multiplication property of FS yields: $b_k = Na_kc_k = Na_ka_k^* = N|a_k|^2$.

(c) (5 points) $\tilde{g}[n] = \sum_{m=0}^{N-1} \tilde{x}^*[m]\tilde{x}[m+n]$.

(Answer) $N|a_k|^2$
(Solution) From the previous problem, $\tilde{g}[n] = \tilde{z}[-n]^*$.
The conjugation & flip in time domain corresponds to the conjugation in FS coefficients.
However, the FS expansion b_k of $\tilde{z}[n]$ is a real-valued function ($N|a_k|^2$).
Accordingly, the FS expansion remains the same.

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4. (40 points) A system satisfying the condition of initial rest is described by the following LCCDE:

$$y[n] - \frac{1}{2}y[n-1] = x[n] - 2x[n-1] + x[n-2]$$

(a) (10 points) Derive the impulse response of this system.

(1) We can separate the system into FIR and IIR parts:

$$w[n] = x[n] - 2x[n-1] + x[n-2] \quad \text{(FIR)}$$
$$y[n] - \frac{1}{2}y[n-1] = w[n] \quad \text{(IIR)}$$

These two systems are serially interconnected. Their individual impulse responses are given by

$$h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$$
$$g[n] = \left(\frac{1}{2}\right)^n u[n]$$

(2) The convolution of two impulse responses is given by

$$\begin{aligned} h[n] * g[n] &= \left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{2}\right)^{n-1} u[n-1] + \left(\frac{1}{2}\right)^{n-2} u[n-2] \\ &= \left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{2}\right)^{n-1} u[n-1] + 2\left(\frac{1}{2}\right)^{n-1} u[n-2] \\ &= \left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{2}\right)^{n-1} \delta[n-1] \\ &= \left(\frac{1}{2}\right)^n u[n] - 2\delta[n-1] \text{(sifting property)} \end{aligned}$$

Any of the above equations are considered as correct answers.

(b) (5 points) Find the minimum value of the impulse response.

From the result of (a), the minimum value of $\frac{1}{2} - 2 = -1.5$ is obtained when $n = 1$.

(c) (5 points) Determine whether the system is stable or not. Justify your answer.

The impulse response is a converging power series subtracted by a finite number. Since the impulse response is absolutely summable, the system is BIBO stable.

(d) (10 points) Derive the frequency response of this system.

Substituting $x[n] = X(e^{j\omega})e^{j\omega n}$ and $y[n] = Y(e^{j\omega})e^{j\omega n}$, we get

$$Y(e^{j\omega})e^{j\omega n}\left(1 - \frac{1}{2}e^{-j\omega}\right) = X(e^{j\omega})e^{j\omega n}\left(1 - 2e^{-j\omega} + e^{-j2\omega}\right)$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}} \\ &= \frac{(1 - e^{-j\omega})^2}{1 - \frac{1}{2}e^{-j\omega}} \end{aligned}$$

(e) (5 points) Find two values of the frequency response at the lowest and highest frequencies, respectively. (Here, the lowest frequency denotes $\omega = 0$).

For the DT system, the highest frequency is when $\omega = \pm\pi$. Therefore, $H(e^{j0}) = 0$ for the lowest frequency, and $H(e^{j\pi}) = 2^2/(3/2) = 8/3$ at highest frequency.

(f) (5 points) Suppose that the input to the system is given by $x[n] = (-1)^n$. What is the output from the system?

Answer: $\frac{8}{3}(-1)^n$
The input can be rewritten as $x[n] = (-1)^n = e^{j\pi n}$, which is the complex exponential sequence of frequency π. From (e), the eigenvalue (frequency response) of the system at this frequency is 8/3. Accordingly, the output is the input signal amplified by 8/3 times.

[End of Problems.]

Appendix - Formulas of Signals and Systems

- Basic formula

	Continuous time	Discrete time
Convolution	$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$	$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
Sifting property	$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$	$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
Running integral (sum)	$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t)$	$y[n] = \sum_{k=-\infty}^{n} x[k] = u[n] * x[n]$
Fourier series (Analysis)	$a_k = \frac{1}{T}\int_0^T \tilde{x}_T(t)e^{-jk\omega_0 t}dt$	$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}_N[n]e^{-jk\omega_0 n}$
Fourier series (Synthesis)	$\tilde{x}_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\tilde{x}_N[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$

- Properties of Fourier Series

  (1) Continuous time

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t - \tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

(2) Discrete time

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n - r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n - 1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [\text{x[n] real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [\text{x[n] real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$