EE202C Signals and Systems, Spring 2020
Oct 21, 2020
Time Limit: 12:45 - 15:00 Lecturer: Jung-Woo Choi

Score Table (for teacher use only)

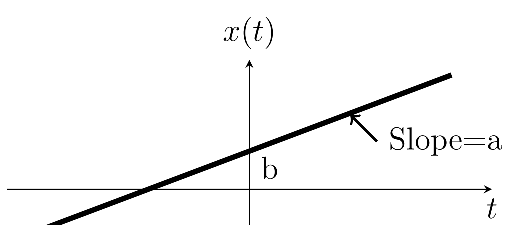
Question:	1	2	3	4	Total
Points:	20	35	25	20	100
Score:					

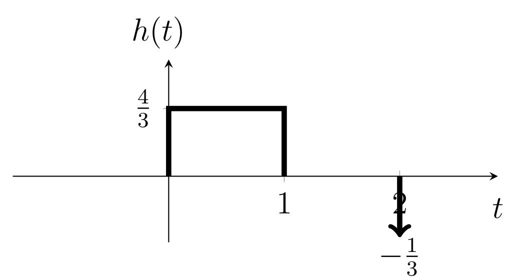
1. (20 points) (a) (10 points) Determine which of the following properties
   (1) Memoryless
   (2) Time invariant
   (3) Linear
   (4) Causal
   (5) Stable
   hold and which do not hold for the given continuous-time system. Justify your answers. In each example, $x(t)$ denotes the system input and $y(t)$ is the system output.

$$y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-2), & t \geq 0 \end{cases}$$

Please refer to the solution of HW 1.

(b) (10 points) For the following pairs of waveforms, use the convolution integral to find the response of $y(t)$ of the LTI system with impulse response $h(t)$ to the input $x(t)$. Sketch your results.





Refer to the solution of HW 2.

2. (35 points) (a) (5 points) Prove that the differential and convolution operators are commutative. In other words, show that

$$\frac{d}{dt}\left(h(t) * g(t)\right) = \frac{dh(t)}{dt} * g(t) = h(t) * \frac{dg(t)}{dt}$$

First, a time delay operation is commutative with the differential operation, i.e., $\frac{dy(t-\tau)}{dt} = \left.\frac{dy(t)}{dt}\right|_{t\to t-\tau}$. From two definitions of convolution integral, we have

$$\begin{aligned}
\frac{d}{dt}(h(t)*g(t)) &= \frac{d}{dt}\int_{-\infty}^{\infty} h(t-\tau)g(\tau)d\tau \\
&= \int_{-\infty}^{\infty}\left[\frac{d}{dt}h(t-\tau)\right]g(\tau)d\tau = \frac{dh(t)}{dt}*g(t) \\
\frac{d}{dt}(h(t)*g(t)) &= \frac{d}{dt}\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\frac{dg(t-\tau)}{dt}d\tau = h(t)*\frac{dg(t)}{dt}
\end{aligned}$$

(b) (5 points) Consider a unit doublet function $\delta_1(t)$ defined as

$$\delta_1(t) = \frac{d\delta(t)}{dt} = \lim_{\Delta\to 0}\frac{1}{\Delta}[\delta(t)-\delta(t-\Delta)].$$

Using the result of (a), show that

$$\frac{dx(t)}{dt} = \delta_1(t)*x(t)$$

From the sifting property, $x(t) = x(t)*\delta(t)$.

$$\frac{dx(t)}{dt} = \frac{d}{dt}(x(t)*\delta(t)) = x(t)*\frac{d\delta(t)}{dt} = x(t)*\delta_1(t)$$

(c) (10 points) The LTI system described by the following LCCDE satisfies the condition of initial rest. Using the result of (b), find the impulse response of this system.

$$\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + x(t)$$

First, separate the given LCCDE into FIR and IIR parts:

$$(FIR):\ w(t) = \frac{dx(t)}{dt} + x(t), \quad (IIR):\ \frac{dy(t)}{dt} + 2y(t) = w(t)$$

Let the impulse responses of the FIR and IIR systems be $h_{FIR}(t)$ and $h_{IIR}(t)$, respectively. From the result of (b) and sifting property of convolution

$$w(t) = \frac{dx(t)}{dt} + x(t) = (\delta_1(t)+\delta(t))*x(t)$$

Therefore, $h_{FIR}(t) = \delta_1(t)+\delta(t)$.

On the other hand, the impulse response of the IIR system can be obtained by solving the homogeneous equation with inhomogeneous auxiliary condition:

$$\begin{aligned}
\frac{dh_{IIR}(t)}{dt} + 2h_{IIR}(t) = 0, &\quad h(0^+) = 1 \\
h_{IIR}(t) = Ae^{st} \longrightarrow (s+2)Ae^{st} = 0, &\quad s = -2 \\
h_{IIR}(0^+) = Ae^{-2\cdot 0^+} = 1, &\quad A = 1 \\
\therefore h_{IIR}(t) = e^{-2t}u(t)&
\end{aligned}$$

Combining $h_{FIR}(t)$ and $h_{IIR}(t)$ gives

$$\begin{aligned} h(t) = h_{FIR}(t) * h_{IIR}(t) &= (\delta_1(t) + \delta(t)) * e^{-2t}u(t) \\ &= \frac{d}{dt}e^{-2t}u(t) + e^{-2t}u(t) \\ &= -2e^{-2t}u(t) + e^{-2t}\delta(t) + e^{-2t}u(t) \\ &= -e^{-2t}u(t) + e^{-2t}\delta(t) \\ &= -e^{-2t}u(t) + \delta(t). \end{aligned}$$

(d) (5 points) Determine whether the system is stable or not. Justify your answer.

The stability of a LTI system can be determined by the finiteness of the integral of $|h(t)|$. From the result of (3), the integral of $|h(t)|$ is finite.

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} e^{-2t}u(t)dt + \int_{-\infty}^{\infty} \delta(t)dt \\ &= \frac{1}{2} + 1 = \frac{3}{2} < \infty \end{aligned}$$

Therefore, the system is BIBO stable.

(e) (5 points) Find out the frequency response $H(j\omega)$ of this system. Hint: use the eigenfunction $x(t) = e^{j\omega t}$ to find out the response $y(t) = H(j\omega)x(t)$, and determine the eigenvalue $H(j\omega)$.

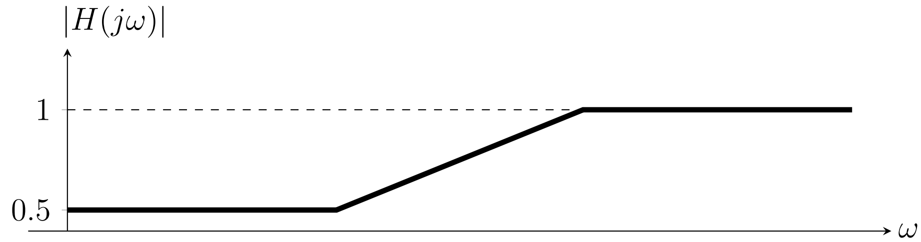
Substituting $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)x(t)$ to LCCDE gives

$$\begin{aligned} (j\omega + 2)H(j\omega)e^{j\omega t} &= (j\omega + 1)e^{j\omega t} \\ \therefore H(j\omega) &= \frac{j\omega + 1}{j\omega + 2} \end{aligned}$$

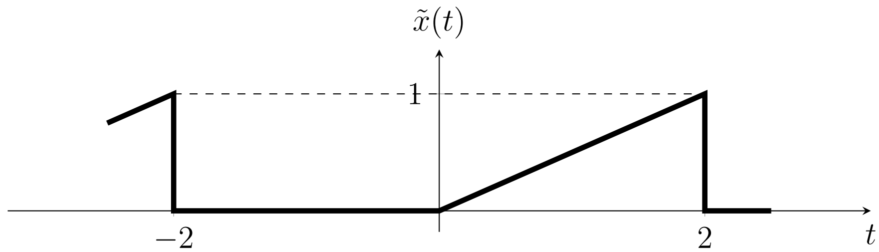
(f) (5 points) Plot the magnitude response $|H(j\omega)|$ with respect to frequency $\omega \geq 0$. Also mark the value of $|H(j\omega)|$ for $\omega = 0$ and ∞.

$$\begin{aligned} |H(j\omega)| &= \frac{|j\omega + 1|}{|j\omega + 2|} \\ &= \frac{\sqrt{\omega^2 + 1^2}}{\sqrt{\omega^2 + 2^2}} \end{aligned}$$

As $\omega \to 0$, $|H(j\omega)| \to \frac{1}{2}$
As $\omega \to \infty$, $|H(j\omega)| \to 1$ (both 1 and 2^2 are negligible compared to ω^2)



3. (25 points) A periodic signal $\tilde{x}(t)$ with period 4 is shown below.
   (a) (10 points) Derive the Fourier series coefficients a_k.



From the analysis equation of Fourier series expansion

$$a_k = \frac{1}{4}\int_0^2 \frac{1}{2} t e^{-jk\omega_0 t} dt$$

($T = 4$ and $\omega_0 = 2\pi/4 = \pi/2$).
We use the integration by parts, given by

$$\int uv' = uv - \int u'v.$$

Substituting $u = \frac{1}{2}t$ and $v' = e^{-jk\omega_0 t}$ yields

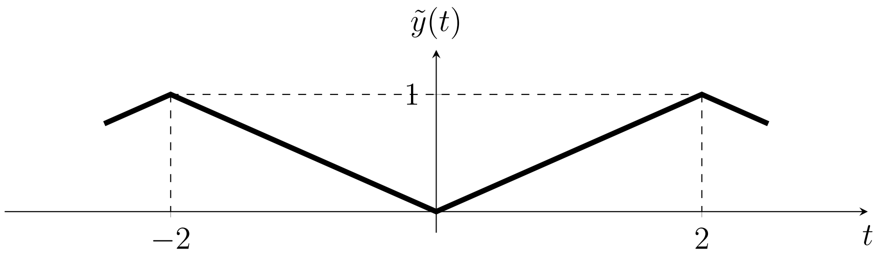
$$\begin{aligned} a_k &= \frac{1}{8} t \cdot \left.\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right|_0^2 - \frac{1}{8}\int_0^2 \frac{e^{-jk\omega_0 t}}{-jk\omega_0} dt \\ &= \frac{e^{-2jk\omega_0}}{-4jk\omega_0} + \left(\frac{e^{-2jk\omega_0} - 1}{8(k\omega_0)^2}\right) \\ &= -\frac{e^{-jk\pi}}{2j(k\pi)} + \frac{e^{-jk\pi/2}}{j(k\pi)^2}\sin(k\pi/2) \\ &= -\frac{(-1)^k}{2j(k\pi)} + \frac{(-j)^k}{j(k\pi)^2}\sin(k\pi/2) \\ &= j(-1)^k\left[\frac{1}{2(k\pi)} - \frac{j^k}{(k\pi)^2}\sin(k\pi/2)\right] \quad \text{for } k \neq 0. \end{aligned}$$

For $k = 0$, the mean value of $\tilde{x}(t)$ is equal to a_0. Therefore,

$$a_k = \begin{cases} j(-1)^k\left[\frac{1}{2(k\pi)} - \frac{j^k}{(k\pi)^2}\sin(k\pi/2)\right] & \text{for } k \neq 0. \\ \frac{1}{4} & \text{for } k = 0. \end{cases}$$

[Note by professor]
The solution in the v1.0 was wrong because the given signal is not the integrated version of the periodic rectangular function. As we integrate the periodic rectangular function over many periods, the integrated function continuously increases and cannot be a periodic triangular signal.

(b) (5 points) Express the Fourier series coefficients b_k of $\tilde{y}(t)$ shown below in terms of a_k of (a).



The signal $\tilde{y}(t)$ is two times of the even part of $\tilde{x}(t)$.

$$\begin{aligned} \tilde{y}(t) &= 2 \cdot \frac{\tilde{x}(t) + \tilde{x}(-t)}{2} \\ b_k &= a_k + a_{-k} \quad (= a_k + a_k^* = 2\mathbf{Re}\{a_k\}) \end{aligned}$$

(c) (10 points) For the coefficients b_k of (b), derive the following sum:

$$\sum_{k=0}^{\infty} |\mathbf{Re}\{b_k\}|^2$$

Here, $\mathbf{Re}\{\cdot\}$ denotes the real operator.

The signal $\tilde{y}(t)$ is real and even. So, its Fourier expansion coefficient is also real and even. Because b_k is real-valued, total power sum of its real part is equal to the total power sum of b_k.

$$\sum_{k=0}^{\infty} |\mathbf{Re}\{b_k\}|^2 = \sum_{k=0}^{\infty} |b_k|^2$$

Since b_k is even, the following relation holds:

$$\sum_{k=0}^{\infty} |b_k|^2 = \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} |b_k|^2 + |b_0|^2\right) \tag{1}$$

According to the Parseval's relation, we have

$$\frac{1}{T}\int_T |\tilde{y}(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |b_k|^2$$

The average power in time is given by

$$\frac{1}{2}\int_0^2 \left(\frac{t}{2}\right)^2 dt = \frac{1}{3}$$

From the result of (a) and (b), $b_0 = 2a_0 = 1/2$.
Substituting these values to Eq. 1 gives

$$\frac{1}{2}\left(\sum_{k=-\infty}^{\infty} |b_k|^2 + |b_0|^2\right) = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2^2}\right) = \frac{7}{24}$$

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4. (20 points) Consider a periodic continuous-time (CT) signal $\tilde{x}_3(t)$ with fundamental period 3. Let $\tilde{y}_5(t)$ be another periodic CT signal with fundamental period 5. The Fourier series coefficients of $\tilde{x}_3(t)$ and $\tilde{y}_5(t)$ when they are analyzed by their own fundamental frequencies are given by a_k and b_k, respectively.

   Two signals are then added, yielding a new signal $\tilde{z}(t)$.

$$\tilde{z}(t) = \tilde{x}_3(t) + \tilde{y}_5(t) \tag{2}$$

   (a) (5 points) Find the fundamental frequency ω_0 of $\tilde{z}(t)$ ($\omega_0 = 2\pi/T,\ T$: period).

The period of the added signal should be a least common multiple of two signals' periods. The least common multiple of 3 and 5 is 15. Therefore, the fundamental frequency of $\tilde{z}(t)$ is

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{15}$$

   (b) (15 points) Express the Fourier series coefficients c_k of $\tilde{z}(t)$ in terms of a_k and b_k.

The FS synthesis equations for the signals $\tilde{x}_3(t)$ and $\tilde{y}_5(t)$ are given by

$$\begin{aligned} \tilde{x}_3(t) &= \sum_{m=-\infty}^{\infty} a_m e^{jm\omega_3 t} \\ \tilde{y}_5(t) &= \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_5 t}, \end{aligned}$$

where fundamental frequencies are given by

$$\omega_3 = \frac{2\pi}{3} = 5\omega_0, \quad \omega_5 = \frac{2\pi}{5} = 3\omega_0.$$

When this signal is analyzed by the new fundamental frequency $\omega_0 = 2\pi/15$, the analysis equation yields

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T (\tilde{x}_3(t)+\tilde{y}_5(t))e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_0^T \left[\sum_{m=-\infty}^{\infty} a_m e^{jm\omega_3 t} + \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_5 t}\right] e^{-jk\omega_0 t}dt \\ &= \sum_{m=-\infty}^{\infty} \frac{1}{T}\int_0^T a_m e^{j(5m-k)\omega_0 t}dt + \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_0^T b_n e^{j(3n-k)\omega_0 t}dt \\ &= \sum_{m=-\infty}^{\infty} a_m \delta_{5m,k} + \sum_{n=-\infty}^{\infty} b_n \delta_{3n,k} \end{aligned}$$

For $k = 5m \neq 15\ell$, only a_m is nonzero, giving $c_{5m} = a_m$. $(m, \ell :$ integers)
For $k = 3n \neq 15\ell$, only b_n is nonzero, giving $c_{3n} = b_n$.
For $k = 15\ell$, both $a_{3\ell}$ and $b_{5\ell}$ are nonzero, giving $c_{15\ell} = a_{3\ell} + b_{5\ell}$.
Otherwise, c_k is zero, because $\delta_{5m,k} = \delta_{3n,k} = 0$.

Appendix - Formulas of Signals and Systems

- Basic formula

	Continuous time	Discrete time
Convolution	$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$	$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
Sifting property	$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$	$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
Running integral (sum)	$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t)$	$y[n] = \sum_{k=-\infty}^{n} x[k] = u[n] * x[n]$
Fourier series (Analysis)	$a_k = \frac{1}{T}\int_0^T \tilde{x}_T(t)e^{-jk\omega_0 t}dt$	$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}_N[n]e^{-jk\omega_0 n}$
Fourier series (Synthesis)	$\tilde{x}_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\tilde{x}_N[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$

- Properties of Fourier Series

  (1) Continuous time

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

(2) Discrete time

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x[n] \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$$