Modified on Wednesday 16$^{\text{th}}$ September, 2020, 12:47:0012:47

Sep 16, 2020

- Special linear transformations. Orthogonal matrices. Projections and reflections. [3.6, 6.2]
- Diagonal matrices. A square matrix $D = (a_{ij})$ is called a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$. $D = \mathrm{diag}(d_1, \ldots, d_n)$ for $a_{ii} = d_i$.
- D^k and D^{-1} for a diagonal matrix D.
- Lower triangular, upper triangular, strictly lower triangular, strictly upper triangular
  - The set of all $n \times n$ lower (resp. upper) triangular matrices form a subspace of the set of all $n \times n$ matrices.
  - The product of two $n \times n$ lower (resp. upper) triangular matrices is lower (resp. upper) triangular.
- A square matrix A is called symmetric if $A^T = A$, i.e., $a_{ij} = a_{ji}$ for all i and j, and skew-symmetric if $A^T = -A$.
- Examples of symmetric matrices
  - Hessian matrix from calculus
  - adjacency matrix of a undirected graph from graph theory
- For a $n \times m$ matrix A, AA^T and A^TA are both square and symmetric.
- If A is a square matrix, then A, AA^T, and A^TA are either all invertible or singular.
- For an $n \times n$ matrix A with $A^k = \mathbf{0}$,

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}.$$

- Linear transformations in $\mathbb{R}^2, \mathbb{R}^3$. Let $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$ be an ordered basis for $\mathbb{R}^2$.
  - reflection

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

  - orthogonal projection of $\mathbf{a}$ along $\mathbf{b}$

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  - rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Orthogonal basis, orthonormal basis. $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$: a basis for $\mathbb{R}^n$. $\mathcal{B}$ is called an orthogonal basis, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$. It is called an orthonormal basis, if it is an orthogonal basis and $\|\mathbf{v}_i\| = 1$ for all i.
- Orthogonal matrices. A square matrix P is called an orthogonal matrix if $P^T = P^{-1}$.
- The columns of an $n \times n$ orthogonal matrix form an orthonormal basis for $\mathbb{R}^n$.
- A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is called an orthogonal linear operator, if $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.

- Theorem. let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. The following are equivalent:
  - $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.
  - $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

  Proof. One direction is obvious. The other direction follows from below.

$$
\begin{aligned}
T(\mathbf{x}) \cdot T(\mathbf{y}) &= \frac{1}{4}\left(\|T(\mathbf{x}) + T(\mathbf{y})\|^2 - \|T(\mathbf{x}) - T(\mathbf{y})\|^2\right) \\
&= \frac{1}{4}\left(\|T(\mathbf{x} + \mathbf{y})\|^2 - \|T(\mathbf{x} - \mathbf{y})\|^2\right) \\
&= \frac{1}{4}\left(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2\right) \\
&= \mathbf{x} \cdot \mathbf{y}
\end{aligned}
$$

- Theorem. If A is an $m \times n$ matrix, then the following statements are equivalent.
  - $A^T A = I$
  - $||A\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
  - $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.
  - The column vectors of A are orthonormal.
- If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal linear operator, then T is either a rotation about the origin or a reflection about a line through the origin.
- Linear transformations in $\mathbb{R}^n$
  - Reflection
  - Rotation
  - Orthogonal projection