

Final practice.

1.

(a) $\int_0^1 e^{t \cdot (-2\ln y)} dy = \int_0^1 (e^{\ln y})^{-2t} dy = \int_0^1 y^{-2t} dy = \frac{1}{1-2t} [y^{1-2t}]_0^1$
 $= \frac{1}{1-2t} \text{ for } t < \frac{1}{2}$

(b) $\frac{1}{1-2t} = \left(\frac{1}{\frac{1}{2}-t}\right)^1 \Rightarrow \text{pdf of Gamma}(1, \frac{1}{2})$

3.

$$F_{Y_{(n)}}(y) = P(Y_{(n)} \leq y) = P(Y_1 \leq y, Y_2 \leq y, \dots, Y_n \leq y) = (F(y))^n$$

$$f_{Y_{(n)}}(y) = \frac{d}{dy} (F(y))^n = \underbrace{n(F(y))^{n-1}} \cdot f(y) \quad (\text{a})$$

$$\text{for } \alpha = \beta = 2, f(y) = 6y(1-y), 0 \leq y \leq 1 \quad \& \quad F(y) = 3y^2 - 2y^3$$

$$\Rightarrow f_{Y_{(n)}}(y) = 6^n [3y^2 - 2y^3]^{n-1} y(1-y), 0 \leq y \leq 1.$$

5. \rightarrow Same as 1.

7.

(a) $\frac{n-1}{6^2} S^2 \sim \chi^2_{n-1}, E\left[\frac{n-1}{6^2} S^2\right] = n-1 \Rightarrow E[S^2] = 6^2, V\left[\frac{n-1}{6^2} S^2\right] = 2(n-1)$
 $V[S^2] = \frac{26^4}{n-1}$

(b) $Y_1 - Y_2 \sim N(0, 26^2), \frac{Y_1 - Y_2}{\sqrt{26^2}} \sim N(0, 1) \Rightarrow \frac{(Y_1 - Y_2)^2}{26^2} \sim \chi^2_1$

(c) $\frac{Y_3 - Y_4}{\sqrt{26^2}} \sim Z \Rightarrow \frac{Y_3 - Y_4}{\sqrt{(Y_1 - Y_2)^2}} \sim T \text{ with df = 1} \quad (\oplus \text{ Independence})$

(d) $\frac{(Y_3 - Y_4)^2}{26^2} \sim \chi^2_1, \Rightarrow W \sim F \text{ with (1,1) df}$

9.

$$U = Y_1 + Y_2, \quad 0 \leq U \leq 1 \quad \Rightarrow \quad \text{for } 0 \leq M \leq 1, \quad F_U(M) = \int_0^M \int_0^{M-y_2} 2 dy_1 dy_2 \\ F_U(M) = P(Y_1 + Y_2 \leq M) = M^2 \\ \therefore f_U(M) = 2M \quad \text{for } 0 \leq M \leq 1$$

11.

(a)

$$P(Y=r) \rightarrow r-1 \text{ failure for } n-1 \text{ trial \& success for } n \text{th trial} \\ \Rightarrow \binom{n-1}{r-1} p^{r-1} \cdot (1-p)^{n-r} \cdot p = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \quad n \geq r$$

(b)

$$P(W_1=k \mid \sum_i W_i = m) = \frac{P(W_1=k) \cdot P(\sum_i W_i = m-k)}{P(\sum_i W_i = m)} = \frac{P(1-p)^{k-1} \cdot \binom{m-k-1}{r-2} p^{r-1} (1-p)^{m-k-r+1}}{\binom{m-1}{r-1} p^r (1-p)^{m-r}} \\ = \frac{\binom{m-k-1}{r-2}}{\binom{m-1}{r-1}}$$

13.

$$(a) \quad \left(\frac{Y_1 + Y_2 + Y_3}{\sqrt{13}} \right)^2$$

$$(b) \quad \frac{Y_1^2}{\left\{ \left(\frac{Y_2}{2} \right)^2 + \left(\frac{Y_3}{3} \right)^2 \right\} / 2}$$

$$(c) \quad \frac{Y_1}{\sqrt{\left\{ \left(\frac{Y_2}{2} \right)^2 + \left(\frac{Y_3}{3} \right)^2 \right\} / 2}}$$

15.

$$(a) E[Y] = \int_0^\theta y f(y|\alpha, \theta) dy = \frac{\alpha}{\alpha+1} \theta, \quad \bar{Y} = \frac{\alpha}{\alpha+1} \theta \Rightarrow \alpha = \frac{\bar{Y}}{\theta - \bar{Y}}$$

(b)

$$\begin{aligned} L(\alpha|y) &= \prod_{i=1}^n \frac{\alpha y_i^{\alpha-1}}{\theta^\alpha}, \quad \frac{d}{d\alpha} \ln L(\alpha|y) = \frac{1}{\alpha} \sum_{i=1}^n (\ln \theta + (\alpha-1) \ln y_i - \alpha \ln \theta) \\ &= \sum_{i=1}^n \left(\frac{1}{\alpha} + \ln y_i - \ln \theta \right) = \frac{n}{\alpha} + \sum \ln y_i - n \ln \theta = 0 \end{aligned}$$

$$\text{MLE of } \alpha = \frac{n}{n \ln \theta - \sum \ln y_i}$$

$$(c) \text{ MLE of } E[Y] = \text{MLE of } \frac{\alpha}{\alpha+1} \theta \rightarrow \frac{\text{MLE of } \alpha}{\text{MLE of } \alpha+1} \cdot \theta$$

17.

$$(a) E[\bar{Y}] = E[Y_i] = \int_0^\infty \frac{1}{\theta+1} y e^{-y/(\theta+1)} dy = \theta+1 \neq \theta$$

$$(b) \text{MSE}(\bar{Y}) = V(\bar{Y}) + (\text{Bias}(\bar{Y}))^2 = \frac{V(Y_i)}{n} + (\theta+1-\theta)^2 = \frac{(\theta+1)^2}{n} + 1$$

$\therefore f(y) = \frac{1}{\theta+1} e^{-y/(\theta+1)}$ $\rightarrow Y \sim \text{Exp}(1/\theta+1)$

$$(c) E[\bar{Y}-1] = \theta \Rightarrow \bar{Y}-1$$

$$(d) V[\bar{Y}-1] = V[\bar{Y}] = \frac{(\theta+1)^2}{n}$$

19.

$$\begin{aligned} (a) P(Y-\theta \leq y) &= P(Y \leq y+\theta) = \int_0^{y+\theta} e^{-x/\theta} dx = \left[-e^{-x/\theta} \right]_0^{y+\theta} \\ &= 1 - e^{-y} = F_{Y-\theta}(y), \quad y > 0. \quad \therefore f_{Y-\theta}(y) = e^{-y}, \quad y > 0 \end{aligned}$$

$Y-\theta \sim \text{Exp}(1) \Rightarrow \text{Pivotal quantity.}$

$$(b) P(Y-\theta \leq b) = 0.975, \quad b = -\ln(0.025) \approx 3.6889$$

$$P(Y-\theta \geq a) = 0.025, \quad a = -\ln(0.975) \approx 0.0253$$

$$\theta \in [Y-3.6889, Y-0.0253]$$

21.

$$\hat{P}_1 = \frac{300}{500} = 0.6, \quad \hat{P}_2 = \frac{64}{100} = 0.64, \quad \hat{P}_1 - \hat{P}_2 = -0.04$$

$$SE = \sqrt{\frac{0.6 \cdot 0.4}{500} + \frac{0.64 \cdot 0.36}{100}} \approx 0.0528, \text{ margin of error: } 2 \cdot SE \approx 0.1056$$

$$-0.04 \pm 0.1056 : [-0.1456, 0.0656]$$

$$\text{for margin of error 0.05, } SE = 0.025 = \sqrt{\frac{0.6 \cdot 0.4}{n}} \Rightarrow n \approx 168$$

23.

$$Y_i \sim N(0, b^2)$$

$$f(y|b^2) = \left(\frac{1}{\sqrt{2\pi}b}\right)^n \exp\left(-\frac{1}{2b^2} \sum_{i=1}^n Y_i^2\right) = \underbrace{\theta(T(Y))}_{\text{sufficient statistic}} \cdot \underbrace{h(y)}_1$$

$$\sum_{i=1}^n Y_i^2$$

→ sufficient statistic for b^2 .

$$E[\sum_{i=1}^n Y_i^2] = n \cdot E[Y_i^2] = n[V(Y_i) + E(Y_i)^2] = nb^2, E[\frac{1}{n} \sum_{i=1}^n Y_i^2] = b^2$$

$$\Rightarrow \frac{1}{n} \sum_{i=1}^n Y_i^2 : \text{MVUE}$$

$$S_n^2 = \frac{1}{n-1} \sum (Y_i - \bar{Y})^2 = \frac{n}{n-1} \left(\frac{1}{n} \sum Y_i^2 - \bar{Y}_n^2 \right) \xrightarrow{P} E[Y^2] - (E[Y])^2 = V(Y)$$

$$\text{Since } Y = \ln x \text{ is continuous, } \sqrt{S_n^2} = \sqrt{\frac{1}{n-1} \sum (Y_i - \bar{Y})^2} \xrightarrow{P} \sqrt{b^2} = b$$

25.

$$L(\theta) = \prod_{i=1}^n f(Y_i|\theta) = \prod_{i=1}^n \frac{1}{\theta^2} Y_i e^{-Y_i/\theta}, \quad l(\theta) = \sum_{i=1}^n \left(\ln \frac{1}{\theta^2} + \ln Y_i - \frac{Y_i}{\theta} \right)$$

$$\frac{\partial l(\theta)}{\partial \theta} = -\frac{2n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n Y_i = 0, \quad \hat{\theta} = \frac{1}{2n} \sum_{i=1}^n Y_i$$

$$E[\hat{\theta}] = E\left[\frac{1}{2n} \sum_{i=1}^n Y_i\right] = \frac{1}{2n} \cdot 2n\theta = \theta, \quad V[\hat{\theta}] = \frac{1}{4n^2} \cdot 2n\theta^2 = \frac{\theta^2}{2n}$$

by $Y_i \sim \text{Gamma with } k=2$

$$V(Y) = 2\theta^2. \quad \text{MLE of } V(Y) = 2 \cdot \left(\frac{1}{2n} \sum_{i=1}^n Y_i \right)^2$$