EE20011 2025 Fall HW 2 Solution

Problem 1. Birth of quantum mechanics (10 points)

Explain the following experiments briefly in the views of the wave and particle duality.

a) Blackbody radiation

A blackbody is an idealized physical body that absorbs all incident electromagnetic radiation, regardless of frequency or angle of incidence. The radiation emitted by a black body in thermal equilibrium with its environment is called black-body radiation.

According to classical wave theory, a blackbody should have ultraviolet catastrophe, where radiation at short wavelengths should approach infinity. However, Max Planck explained blackbody radiation energy is quantized, which means the light sometimes be treated as particles, not just waves.

b) Photoelectric effect

The photoelectric effect is a phenomenon where electromagnetic radiation, such as light, causes electrons to be ejected from a material.

According to classical wave theory, when light shines on metal surface, the energy of the electrons should increase in proportion to the light's intensity. However, in experiments, light shining on metal surface ejects electrons only if the light's frequency is above a threshold, regardless of intensity. Einstein explained that by assuming that light is a stream of particles. This also proved that light has particles' properties.

c) Compton effect

The Compton effect is the quantum theory of scattering of a high-frequency photon through an interaction with a charged particle, usually an electron.

According to classical wave theory, the light should have same frequency and wavelength after collision. However, in experiments, X-rays scattered off electron electrons, wavelength shifted. Compton treated this as a particle collision between a photon and electron, which are collision of the two particles. This experiment proved that light (photon) have momentum just like particles.

d) Davisson and Germer experiment

The Davisson-Germer experiment was an experiment, in which electrons, scattered by the surface of a crystal of nickel metal, displayed a diffraction pattern.

If electrons were purely particles, they should have scattered in all directions after collision. However, in experiments, electrons fired at a nickel crystal displayed diffraction pattern, which is characteristic of waves. De Broglie said that particles have a wavelength of $\lambda = h/p$, which means all matter possesses both particle and wave properties. This experiment directly proved that matter exhibit both particle and wave properties.

Problem 2. De Broglie wavelength (10 points)

Consider an electron has a kinetic energy of 0.1eV.

a) Determined the wavelength of the electron.

Kinetic energy, which is given, can be expresses as $\mathrm{K.E.} = \frac{p^2}{2m}$.

Therefore, $\mathrm{p} = \sqrt{2m \cdot K.E.} \approx 1.708 \times 10^{-25}\ kg \cdot m/s$

Then, the wavelength can be calculated by:

$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\ J/s}{1.708 \times 10^{-25}\ kg \cdot m/s} \approx 3.879\ \mathrm{nm}$

b) Explain why they are using an electron microscope to have higher magnification than optical one in terms of wavelength.

Microscope's resolution is the ability to distinguish two separate points as distinct entities, with a smaller distance indicating better resolution. The resolution can be expressed as : $\mathrm{d} = \frac{\lambda}{2nsin\theta}$, which is proportional to wavelength. This means shorter wavelength allows better resolution. Optical microscopes uses visible light with a wavelength between 400nm and 700nm. The electron's wavelength from problem 2-a) is much shorter than visible lights. Therefore, electron microscopes have higher magnification that optical microscopes.

Problem 3. Momentum of photon and electron (10 points)

a) Find the momentum of a photon in green light of λ = 550nm.

$\mathrm{p} = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34}\ J/s}{550 \times 10^{-9}\ m} \approx 1.205 \times 10^{-27} kg \cdot m/s$

b) If an electron has the same wavelength, determine the electron velocity and momentum.

$\mathrm{p} = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34}\ J/s}{550 \times 10^{-9}\ m} \approx 1.205 \times 10^{-27} kg \cdot m/s$

$$\mathrm{v} = \frac{p}{m} = \frac{1.205 \times 10^{-27} kg \cdot m/s}{9.11 \times 10^{-31} kg} \approx 1.32 \times 10^{3}\ m/s$$

Problem 4. Conjugate variables (10 points)

a) What are the conjugate variables in a wave function?

Conjugate variables of a wave function are as follows:

Momentum – Position

Energy - Time

b) Explain the properties of the variables with the integral transformation relation.

Conjugate variables are mathematically related through a Fourier Transform. Therefore, if one variable exhibits a sharp peak, the other will have a broad spectrum, and vice versa.

Problem 5. Uncertainty principle (15 points)

Consider an electron has a kinetic energy of 0.1eV.

a) The electron's energy is measured with an uncertainty no greater than 0.8eV. Determine the minimum uncertainty in the time over which the measurement is made.

Energy – Time uncertainty relationship: $\Delta E \cdot \Delta t \geq \frac{\hbar}{2}$

$$\therefore \Delta t \geq \frac{\hbar}{2\Delta E} = \frac{1.054 \times 10^{-34}\, \frac{J}{s}}{2 \times \left(0.8\, eV \times 1.602 \times 10^{-19} \frac{J}{eV}\right)} = 4.11 \times 10^{-16}\, s$$

b) The uncertainty in the position of an electron is no greater than 1.5A. Determine the minimum uncertainty in its momentum.

Position – Momentum uncertainty relationship: $\Delta p \cdot \Delta x \geq \frac{\hbar}{2}$

$$\therefore \Delta p \geq \frac{\hbar}{2\Delta x} = \frac{1.054 \times 10^{-34}\, \frac{J}{s}}{2 \times (1.5 \times 10^{-10}\, m)} = 3.51 \times 10^{-25}\, kg \cdot m/s$$

✓ **The formula in the lecture note differs from the conventional one, so both cases will be accepted as correct.**

Problem 6. Separation of variables (10 points)

Drive the time independent Schrodinger's equation from the time dependent equation. Show the time dependent factor of the wave function becomes $\phi(t) = Ae^{-i\frac{Et}{\hbar}}$

Time dependent Schrodinger's equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t) \cdot \Psi(x,t) = i\hbar\frac{\partial \Psi(x,t)}{\partial t} \quad -(1)$$

When potential is independent of time, that is, V = V(x), it is possible to apply separation of variable.

$$\Psi(x, t) = \psi(x)\cdot\phi(t) - (2)$$

Substitute (2) into (1)

$$-\frac{\hbar^2}{2m}\phi(t)\cdot\frac{d^2\psi(x)}{dx^2} + V(x)\cdot\psi(x)\cdot\phi(t) = i\hbar\psi(x)\cdot\frac{d\phi(t)}{dt} - (3)$$

Divide both sides of (3) by $\psi(x)\cdot\phi(t)$

$$-\frac{1}{\psi(x)}\cdot\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x) = i\hbar\frac{1}{\phi(t)}\cdot\frac{d\phi(t)}{dt} - (4)$$

Left-hand side of (4) is a function of a position and the right-hand side is a function of time. Therefore, it should equal some constant E to make both sides the same.

$$-\frac{1}{\psi(x)}\cdot\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x) = i\hbar\frac{1}{\phi(t)}\cdot\frac{d\phi(t)}{dt} = E - (5)$$

Separate into position-dependent part and time-dependent part:

$$-\frac{1}{\psi(x)}\cdot\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x) = E - (6)$$

$$i\hbar\frac{1}{\phi(t)}\cdot\frac{d\phi(t)}{dt} = E - (7)$$

Multiply $\psi(x)$ on both sides of (6) leads to Time-independent Schrodinger Equation.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\cdot\psi(x) = E\psi(x) - (8)$$

Integration (7) over t leads to

$$\phi(t) = Ae^{-i\frac{Et}{\hbar}} \text{ - } (9)$$

Problem 7. Step potential (10 points)

a) Calculate the penetration depth d of a particle impinging on a potential barrier. The d is defined where the wave function magnitude has decayed to e-1 of its value at x=0. Consider an incident electron that traveling at a velocity of 1 x 105 m/s in a region I in the following figure. Assume Vo(at region II) = 2 T, where T is the kinetic energy of a particle at region I.

Solve Schrodinger's equation in region II

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V_0\cdot\psi(x) = E\psi(x) - (1)$$

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m(V_0-E)}{\hbar^2}\psi(x) - (2)$$

The solution to equation (2) is given by

$$\psi(x) = Ae^{-kx} + Be^{kx}, \quad \text{where } k = \sqrt{\frac{2m(V_0-E)}{\hbar^2}} - (3)$$

If B $\neq$ 0, the solution will diverge as x $\rightarrow$ ∞ which is physically meaningless.

Therefore, B = 0.

$$\therefore \psi(x) = Ae^{-kx} \ -(4)$$

$$d = \frac{1}{k} = \sqrt{\frac{\hbar^2}{2m(V_0-E)}} = 1.16 \text{ nm}$$

b) What is the reflection coefficient R?

Solution to the Schrodinger's equation in each region:

$$\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}, \ k_1 = \sqrt{\frac{2mE}{\hbar^2}} - (1)$$

$$\psi_{II}(x) = Ce^{-k_2x}, \ k_2 = \sqrt{\frac{2m(V_0-E)}{\hbar^2}} - (2)$$

A wavefunction and its derivative should be continuous near the finite potential.

$$\psi_I(0) = \psi_{II}(0) \ - (3)$$

$$\frac{d\psi_I(x)}{dx} = \frac{d\psi_{II}(x)}{dx}, \quad \text{at } x = 0 - (4)$$

Solve (3), (4)

$$A = \frac{C}{2}\left(1 + i\frac{k_2}{k_1}\right), \ B = \frac{C}{2}\left(1 - i\frac{k_2}{k_1}\right) \ \text{-} (5)$$

$$\therefore R = \frac{|B|^2}{|A|^2} = 1 - (6)$$

Problem 8. Particle in a box with an infinite potential (15 points)

8. (a) $E_n = \frac{n^2h^2}{8mL^2}(n = 1,2,3,\ldots)$ (+3 pts)

n=1 : $E_1 = \frac{h^2}{8mL^2}$ = 0.00668 eV (+1 pts)

n=2 : $E_2 = \frac{4h^2}{8mL^2}$ = 0.0267 eV (+1 pts)

n=3: $E_3 = \frac{9h^2}{8mL^2}$ = 0.0601 eV (+1 pts)

(b) $P_n(x) = |\psi(x)|^2 = \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)(n = 1,2,3,\ldots)$ (+3 pts)

n=1 : $P_1\left(\frac{L}{2}\right) = \frac{2}{L}\sin^2\left(\frac{\frac{\pi L}{2}}{L}\right) = \frac{2}{L}$ $(+0.5\ pts)$ $=$ $2.67 \times 10^8 \ m^{-1}$(+0.5 pts)

n=2 : $P_2\left(\frac{L}{2}\right) = \frac{2}{L}\sin^2\left(\frac{\frac{\pi 2L}{2}}{L}\right) = 0$ (+1 pts) n=3 : $P_3\left(\frac{L}{2}\right)=\frac{2}{L}\sin^2\left(\frac{\frac{\pi 3L}{2}}{L}\right)=\frac{2}{L}(+0.5\ pts)\ =\ 2.67\ \times 10^{8}\ m^{-1}$ (+0.5 pts)

(c) $<x>=\int_0^L \psi^* \hat{x}\psi dx=\frac{2}{L}\int_0^L x\sin^2\left(\frac{n\pi x}{L}\right)dx$=L/2(+1 pts)=37.5 Å (+2 pts)

Problem 9. Energy bandgap of semiconductor and photon energy (10 points)

9. (a) $E=hf\quad f\_min=\frac{E}{h}=3.433\ \times 10^{14}\ Hz$ (+2.5 pts)

$\lambda=\frac{c}{f}=874\ nm$ (+2.5 pts)

(b) $E=hf\quad f\_min=\frac{E}{h}=2.708\ \times 10^{14}\ Hz$ (+2.5 pts)

$\lambda=\frac{c}{f}=1108\ nm$ (+2.5 pts)

Problem 10. Energy gap and Fermi-level (10 points)

10. (a) $f(E)=\frac{1}{1+\exp(\frac{E-E_F}{kT})}$ (kT=0.0259 eV) (+1 pts)

$f(E_1)=\frac{1}{1+\exp(\frac{E_1-E_F}{kT})}=9.323\ \times 10^{-6}$ (+2 pts)

1- f(E_2)=1-$\frac{1}{1+\exp(\frac{E_2-E_F}{kT})}$=exp$\left(\frac{E_2-E_F}{kT}\right)=1.779\ \times 10^{-14}$ (+2 pts)

(b) $f(E)=\frac{1}{1+\exp(\frac{E-E_F}{kT})}$ (kT=0.0259 eV) (+1 pts)

$f(E_1)=\frac{1}{1+\exp(\frac{E_1-E_F}{kT})}=8.452\ \times 10^{-13}$ (+2 pts)

1- $f(E_2)=1-\frac{1}{1+\exp\left(\frac{E_2-E_F}{kT}\right)}=\exp\left(\frac{E_2-E_F}{kT}\right)=1.962\ \times 10^{-7}$(+2 pts)

Problem 11. Fermi-Dirac distribution (10 points)

11. (a) $f(E)=\frac{1}{1+\exp(\frac{E-E_F}{kT})}$ (kT=0.0259 eV) (+1 pts)

$f(E_F)=\frac{1}{1+\exp(\frac{E_F-E_F}{kT})}=\frac{1}{2}$ (+2 pts)

(b) $f(E)=\frac{1}{1+\exp(\frac{E-E_F}{kT})}$ (kT=0.0259 eV) (+1 pts)

$f(E_C+kT)=\frac{1}{1+\exp(\frac{E_C+kT-E_C}{kT})}$=0.269 (+2 pts)

(c) $f(E_C + kT) = 1 - f(E_C + kT)$=0.5 (+1 pts)

exp($(E_C + kT - E_F)/kT$) =1 (+1 pts)

$E_F = E_C + kT$ (+2 pts)