EE202A Signals and Systems, Fall 2024
18 December, 2024
Time Limit: 13:00 - 15:45 Lecturer: Jung-Woo Choi

Score Table (for teacher use only)

Question:	1	2	3	4	Total
Points:	20	30	20	30	100
Score:					

This is a CLOSED-BOOK exam.
Please provide ALL DERIVATIONS and EXPLANATIONS with your answers.
Any communication with others during the exam will be regarded as a cheating case.

1. (20 points) Consider a signal $x(t) = 2\cos(\omega_0 t)\text{sinc}(\Delta f t)$, where $\omega_0 > 2\pi\Delta f$. For the signal $x(t)$, answer to the following questions.
(note: $\text{sinc}(\theta) = \sin(\pi\theta)/(\pi\theta)$).

(a) (5 points) Derive the CT Fourier transform $X(j\omega)$ of $x(t)$.

Answer) $\frac{1}{\Delta f}\left(\text{rect}\left(\frac{\omega-\omega_0}{2\pi\Delta f}\right) + \text{rect}\left(\frac{\omega+\omega_0}{2\pi\Delta f}\right)\right)$

Solution) From the properties Fourier transform

$$\begin{aligned} \cos(\omega_0 t) &\iff \pi(\delta(\omega-\omega_0) + \delta(\omega-\omega_0)) \\ \Delta f \cdot \text{sinc}(\Delta f t) &\iff \text{rect}\left(\frac{\omega}{2\pi\Delta f}\right) \\ x(t)y(t) &\iff \frac{1}{2\pi}X(j\omega) * Y(j\omega) \end{aligned}$$

The multiplication of two signals in time results in convolution in frequency domain

$$2\cos(\omega_0 t)\text{sinc}(\Delta f t) \iff \frac{1}{\Delta f}\left(\text{rect}\left(\frac{\omega-\omega_0}{2\pi\Delta f}\right) + \text{rect}\left(\frac{\omega+\omega_0}{2\pi\Delta f}\right)\right),$$

where $\text{rect}(\omega/a) = \begin{cases} 1 & \text{if } |\omega| < a/2 \\ 0 & \text{if } |\omega| > a/2 \end{cases}$

(b) (5 points) Determine the group delay of this signal at $\omega = \omega_0$.

Answer) 0
Solution)
The definition of the group delay is the derivative of phase. For the Fourier transform $X(j\omega) = |X(j\omega)|e^{j\phi(\omega)}$, the group delay is given by

$$\left.\frac{d\phi(\omega)}{d\omega}\right|_{\omega=\omega_0}$$

The Fourier transform from Prob.(a) has zero phase around $\omega = \omega_0$, so its derivative and group delay are also zero.

(c) (5 points) What is the Nyquist rate ω_s to sample the signal $x(t)$ without aliasing artifact?

The bandwidth of $X(j\omega)$ is $2\omega_0 + 2\pi\Delta f$. Therefore, $\omega_s = 2\omega_0 + 2\pi\Delta f$.

(d) (5 points) The signal $x(t)$ is fed into an LTI system having the impulse response

$$h(t) = \mathrm{sinc}^2\left(\frac{\omega_0 t}{2\pi}\right).$$

For the output $y(t) = h(t) * x(t)$ from the system, find the Nyquist rate ω_s' to sample $y(t)$ without aliasing artifact.

The Fourier transform of $h(t)$ is given by

$$H(j\omega) = \frac{2\pi}{\omega_0}\mathrm{rect}\left(\frac{\omega}{\omega_0}\right) * \frac{2\pi}{\omega_0}\mathrm{rect}\left(\frac{\omega}{\omega_0}\right)$$

Answer) $2\omega_0$
Solution)
Due to the convolution of two rectangular function of bandwidth ω_0, the bandwidth of $H(j\omega)$ is given by $2\omega_0$. When $H(j\omega)$ is multiplied to $X(j\omega)$, the bandwidth of $Y(j\omega)$ follows the smaller bandwidth ($2\omega_0$).

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2. (30 points) Consider the mappings given by

$$\begin{aligned} v(t) &= x(t) * h(t), \\ w(t) &= v(-t) * h(t) \\ y(t) &= w(-t). \end{aligned}$$

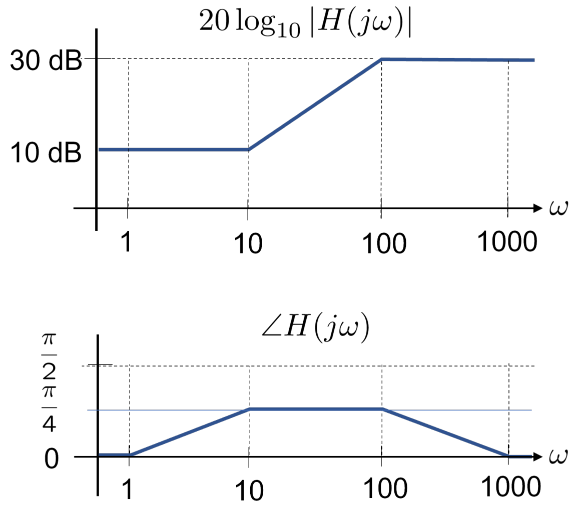
For real-valued signals $x(t)$, $h(t)$, answer to the following questions.

(a) (5 points) Express the Fourier transform $Y(j\omega)$ of $y(t)$ in terms of $H(j\omega)$ and $X(j\omega)$ (Without using $H(-j\omega)$ or $X(-j\omega)$).

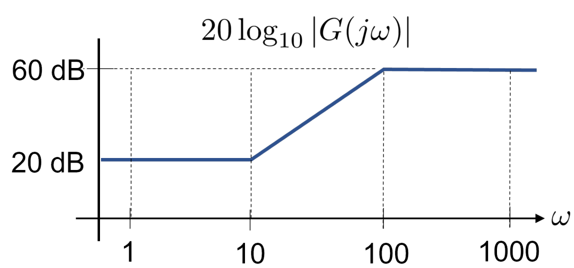
(Answer) $Y(j\omega) = |H(j\omega)|^2 X(j\omega)$
(Solution)
Applying Fourier transform gives

$$\begin{aligned} V(j\omega) &= X(j\omega)H(j\omega) \\ W(j\omega) &= V(-j\omega)H(j\omega) \\ &= X(-j\omega)H(-j\omega)H(j\omega) \\ &= X(-j\omega)H(j\omega)^* H(j\omega) \quad \text{(from conjugate symmetry)} \\ &= X(-j\omega)|H(j\omega)|^2 \\ Y(j\omega) &= W(-j\omega) = X(j\omega)|H(j\omega)|^2 \end{aligned}$$

(b) (5 points) The frequency response $H(j\omega)$ of Prob.(a) is shown in the figure as a Bode plot. Using this response, draw the magnitude response $20\log_{10}|G(j\omega)| = 20\log_{10}\frac{|Y(j\omega)|}{|X(j\omega)|}$ for $Y(j\omega)$ and $X(j\omega)$ of Prob.(a). Specify all the magnitudes at $\omega = 0, 10, 100, 1000$ rad/s on the graph.



From the answer of Prob. (a), $G(j\omega) = |H(j\omega)|^2$. This leads to doubled dB scale magnitude: $20\log_{10}|G(j\omega)| = 20\log_{10}|H(j\omega)|^2 = 2 \times 20\log_{10}|H(j\omega)|$.

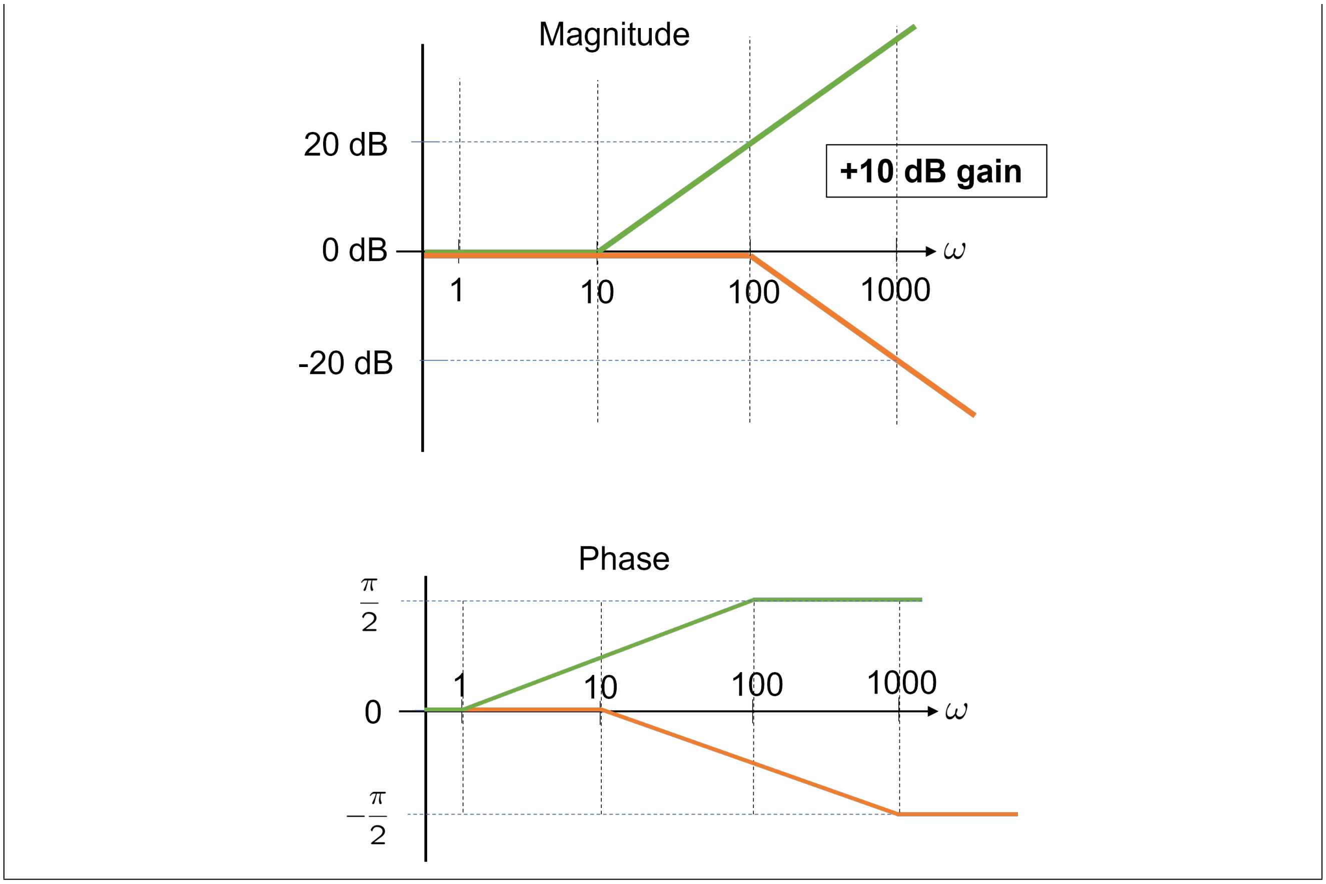


(c) (5 points) When the system represented by $H(j\omega)$ shown above is a causal LTI system, express its frequency response in terms of $j\omega$.

The frequency response can be decomposed into two first-order systems with 10 dB extra gain. That is,

$$H(j\omega) = AH_1(j\omega)H_2(j\omega) = 10\sqrt{10}\frac{10+j\omega}{100+j\omega},$$

$$H_1(j\omega) = \frac{10+j\omega}{10},\ H_2(j\omega) = \frac{100}{100+j\omega},\ A = \sqrt{10}.$$



(d) (5 points) Derive the impulse response $h(t)$ of the system with the frequency response $H(j\omega)$.

$$H(j\omega) = 10\sqrt{10}\left(1 - \frac{90}{100 + j\omega}\right)$$
$$h(t) = 10\sqrt{10}(\delta(t) - 90e^{-100t}u(t))$$

(e) (5 points) Determine the stability of the system described by $H(j\omega)$.

Answer) stable
Solution) The only pole of $H(j\omega)$ is on the LHP ($j\omega = -100$). The impulse response $h(t)$ converges.

(f) (5 points) Determine the causality of the system described by $G(j\omega)$

Answer) non-causal
Solution) Since $G(j\omega) = |H(j\omega)|^2$, the frequency response is real and even. Therefore, its impulse response $g(t)$ is also real and even (symmetric) function. The symmetric impulse response with respect to $t = 0$ includes non-zero response for $t < 0$. Accordingly, the system is not causal.

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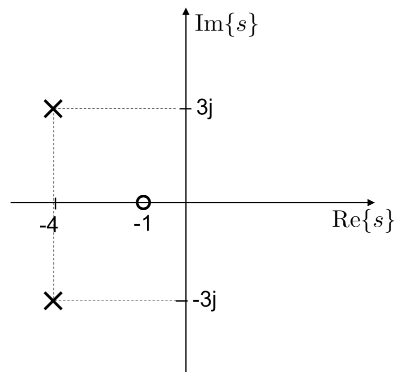
3. (20 points) [Questions for questions] The pole-zero map of an causal LTI system with two poles and single zero is shown below.



When the system satisfies the following LCCDE, answer to the questions.

$$m\frac{d^2y(t)}{dt^2} + n\frac{dy(t)}{dt} + y(t) = K\frac{d^2x(t)}{dt^2} + \ell\frac{dx(t)}{dt} + x(t)$$

(a) (10 points) Determine the values of m, n, K, and ℓ from the pole-zero map.

Answer) $K = 0,\ \ell = 1,\ m = \frac{1}{25},\ n = \frac{8}{25}$
Solution) Applying Laplace transform on both sides of LCCDE yields

$$H(s) = \frac{Ks^2 + \ell s + 1}{ms^2 + ns + 1} \tag{1}$$

On the other hand, the pole-zero map shows one zero at $s = -1$ and two poles at $s = -4 \pm 3j$, which gives the system response

$$H(s) = A\frac{s+1}{(s+4-3j)(s+4+3j)} = A\frac{s+1}{s^2+8s+25} \tag{2}$$

Comparing two system responses show that

$$K = 0,\ \ell = 1,\ A = 25,\ m = \frac{1}{25},\ n = \frac{8}{25}$$

(b) (5 points) Determine the value of the impulse response $h(t)$ of this system at time $t = 0^+$.

Answer) $h(0^+) = 25$
Solution) From the initial value theorem for a causal $h(t)$,

$$\begin{aligned} h(0+) &= \lim_{s\to\infty} sH(s) \\ &= \lim_{s\to\infty} A\frac{s^2+s}{s^2+8s+25} \\ &= A \end{aligned}$$

(c) (5 points) Determine the region of convergence of this system.

Answer) $\text{Re}\{s\} > -4$
Solution) For the causal system, the ROC is given by the right half plane from the rightmost pole. The pole are located at $s = -4 \pm 3j$, so the ROC is given by $\text{Re}\{s\} > -4$.

(Continues on the next page!)

4. (30 points) [Questions for questions] Consider a microphone and loudspeaker installed in a room. The signals played from the loudspeaker and recorded by the microphone are given by $x[n]$ and $y[n]$, respectively.

$$x[n] = 2^n u[n], \quad y[n] = 2^{n+1}u[n] - 0.5^n u[n]$$

Assume no aliasing was induced during the sampling process, and the sound propagation in the room satisfies linearity, time-invariance, and causality.

(a) (5 points) Derive the system function $H(z) = Y(z)/X(z)$.

(Answer) $H(z) = \frac{1+0.5z^{-1}-2z^{-2}}{1-0.5z^{-1}}$
(Solution)

$$\begin{aligned} X(z) &= \frac{1}{1-2z^{-1}}, \\ Y(z) &= \frac{2}{1-2z^{-1}} - \frac{1}{1-0.5z^{-1}} \\ &= \frac{1+z^{-1}}{(1-2z^{-1})(1-0.5z^{-1})} \\ H(z) &= \frac{1+z^{-1}}{1-0.5z^{-1}} \end{aligned}$$

(b) (5 points) Derive the impulse response of this system.

(Answer)
(Solution) From the partial fraction expansion

$$\begin{aligned} H(z) &= 1 + \frac{1.5z^{-1}}{1-0.5z^{-1}} \\ &= -2 + \frac{3}{1-0.5z^{-1}} \end{aligned}$$

The inverse z-transform yields

$$\begin{aligned} h[n] &= \delta[n] + 1.5 \cdot 0.5^{n-1}u[n-1] \\ &= -2\delta[n] + 3 \cdot 0.5^n u[n] \end{aligned}$$

(any answers among these two will be considered correct.)

(c) (5 points) Determine the region of convergence of this system. Specify whether $|z| = \infty$ and $|z| = 0$ belong to the ROC.

$$H(z) = \frac{z+1}{z-0.5}$$

The pole of this system is at $z = 0.5$. The ROC is the exterior region from the outermost pole $z = 0.5$.

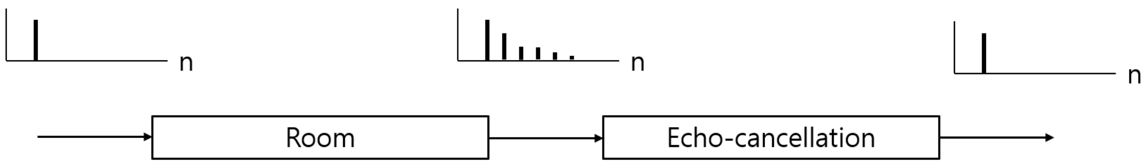
$$\text{ROC} : |z| > 0.5$$

The impulse response is causal, and hence, the ROC includes $|z| = \infty$ but not $|z| = 0$.

(d) (5 points) Is this system stable?

(Answer) Yes
The ROC includes the unit circle ($|z| = 1$), so the system is stable.

(e) (5 points) Among the pulses in the impulse $h[n]$, the first-arriving pulse is called direct sound whereas the other pulses are referred to as echoes. Design a causal echo-cancellation system that eliminates all the echoes from the impulse response, and describe the system response $H_E(z)$ of the echo-cancellation system. The amplitude of direct sound should not change.



The direct sound in the given $h[n]$ is $7\delta[n]$. An echo-cancellation system $h_E[n]$ should satisfy $h[n] * h_E[n] = \delta[n]$. Therefore, we can regard the echo-cancellation system as an LTI inverse of $h[n]$.
The LTI inverse $h[n]$ can be found by inverting $H(z)$ in z-domain. That is

$$H^{-1}(z) = \frac{1 - 0.5z^{-1}}{1 + z^{-1}} \tag{3}$$

(f) (5 points) Determine the stability of the echo-cancellation system.

Since the pole of the echo-cancellation system is at z=-1, the ROC does not include the unit circle. Accordingly, the echo-cancellation system is unstable.

[End of Problem]

Appendix - Formulas of Signals and Systems

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \Re e\{X(e^{j\omega})\} = \Re e\{X(e^{-j\omega})\} \\ \Im m\{X(e^{j\omega})\} = -\Im m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ [$x[n]$ real] $x_o[n] = \mathcal{O}d\{x[n]\}$ [$x[n]$ real]	$\Re e\{X(e^{j\omega})\}$ $j\Im m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	z_0R
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\lvert a\rvert R$ = the set of points $\{\lvert a\rvert z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1-z^{-1})X(z)$	At least the intersection of R and $\lvert z\rvert > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}}X(z)$	At least the intersection of R and $\lvert z\rvert > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem: If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$			

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos \omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$