MAS250 - 2023 Spring Midterm

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1. (20 pts) Use the axioms of probability to show the following two statements.

(a) (10 pts) If $E \subset F$, then $P(E) \le P(F)$.

(b) (10 pts) The probability that exactly one of the events E or F occurs equals $P(E)+P(F)-2P(E \cap F)$.

2. (15 pts) A genetic disease occurs with probability $0 < p < 1$. A test has been developed to detect the presence of the disease, but it is not perfect. The test has a false positive rate of 1% (meaning that 1% of people without the disease will test positive) and a false negative rate of 5% (meaning that 5% of people with the disease will test negative).

(a) (5 pts) If a person tests positive for the disease, what is the probability that he or she actually has the disease?

(b) (5 pts) If a person tests negative for the disease, what is the probability that he or she actually does not have the disease?

(c) (5 pts) Explain which test result, either positive or negative, is more trustworthy (in terms of probabilities of being accurate), when $p = 1/94$.

3. (20 pts) Let N be a Poisson random variable representing the number of event having mean λ, and suppose that each of these events is independently classified as being one of the types 1 and 2 with respective probabilities p and $1-p$. Prove that the number of type 1 and 2 events are independent Poisson random variables with respective means λp and $\lambda(1-p)$.

4. (15 pts) Suppose that a random variable U has a uniform distribution over $[-1, 1]$, and a random variable N has a normal distribution with parameters $(0, 1)$. Assume that the random variables U and N are independent.

(a) (10 pts) Are U and UN independent? Justify your answer.

(b) (5 pts) Compute $\mathrm{Cov}(U, UN)$.

5. (20 pts) Suppose X and Y are independent discrete random variables with the following probability mass functions:

$$P(X = 1) = P(X = -1) = 1/2 \quad \text{and} \quad P(Y = 1) = P(Y = 2) = 1/2$$

(a) (5 pts) Are XY and Y independent? Justify your answer.

(b) (5 pts) Compute $\mathrm{Cov}(XY, Y)$.

(c) (5 pts) Are X^2 and X independent? Justify your answer.

(d) (5 pts) Compute $\mathrm{Cov}(X^2, X)$.

6. (10 pts) Consider a set $\{1, \cdots, n\}$. Suppose that its permutation π is chosen at random among all possible choices. Let N be the number of integer values i in $\{1, \cdots, n\}$ such that the ith number of the permutation is i itself. For example, if $n = 5$ and the permutation π is given by $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 5$, $\pi(4) = 4$, and $\pi(5) = 2$, then $N = 2$. Compute $E[N]$ and $\mathrm{Var}(N)$.

7. (15 pts) Suppose that buses arrive at a given bus stop in accordance with a Poisson process with rate three per hour.

(a) (5 pts) What is the probability there will be at least two buses from 1:00 PM to 1:30 PM?

(b) (5 pts) What are the mean and variance of the waiting time of a person at a bus stop if he or she just missed a bus?

(c) (5 pts) Assuming that the event in part **(a)** occurs, what is the probability that there will be at least three buses from 1:00 PM to 2:00 PM?

8. (15 pts) Let X, Y, Z be a sample from a normal distribution with mean μ and variance σ^2. Construct a random variable by using all X, Y, Z ($e.g.$, $(X+Y+Z)/\sigma^2$ or $XY(Z-\mu)$) that has the following distributions.

(a) (3 pts) Chi-square distribution with 1 degrees of freedom

(b) (3 pts) Chi-square distribution with 2 degrees of freedom

(c) (3 pts) Chi-square distribution with 3 degrees of freedom

(d) (3 pts) t-distribution with 2 degrees of freedom

(e) (3 pts) F-distribution with 2 and 1 degrees of freedom

9. (15 pts) Find $P\{|X| \leq 1\}$, when X has the following moment generating function.

(a) (5 pts) $\phi(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t}$

(b) (10 pts) $\phi(t) = (1-p+pe^t)^n$ for some $p \in (0,1)$ and a positive integer n.

10. (20 pts)

(a) (10 pts) Let $X_1, X_2, \cdots, X_n$ be a sample of values from a population having mean μ and variance σ^2. By using Chebyshev's inequality, derive a lower bound of the probability that the difference between the sample mean and μ is less than a positive constant ϵ.

(b) (10 pts) We plan a survey to estimate the proportion p, $0 < p < 1$, of the population who favors a certain candidate in an upcoming election. Based on **(a)**, how many people should we survey, regardless of the value p, so that our guess (sample mean) has no less than a 0.95 probability of being within 0.02 of the true population proportion p?

11. (15 pts) A real-valued function $\varphi(x)$ is called convex if and only if for any $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$, and $0 \leq t \leq 1$,

$$\varphi\big(tx_1 + (1-t)x_2\big) \leq t\varphi(x_1) + (1-t)\varphi(x_2)$$

(a) (5 pts) Let $X \sim \mathrm{Bernoulli}(p)$, where $0 \leq p \leq 1$. Prove that

$$\varphi(E[X]) \leq E[\varphi(X)].$$

(b) (10 pts) Let X be a discrete random variable such that

$$P(X = x_i) = p_i, \quad i - 1, 2, \cdots, n, \quad \text{and} \quad \sum_{i=1}^{n} p_i = 1,$$

where $n \geq 1$ is an integer. Prove that

$$\varphi(E[X]) \leq E[\varphi(X)].$$