# IE 241 Engineering Statistics I – Midterm Exam

1, 2, 5, 6 : Graded by Taemin Park

April 18, 2024 3, 4, 7, 8 : Graded by Hyunwoo Cha Time allowed: 2 hr 45 minutes

**Name:** TA **Student ID:** .

**Instructions:** Show all your work on your paper; You will NOT receive credits if you do not justify your answers.

**1.** (10pts) Suppose that there is a 30% chance of rain tomorrow. If it rains, there is a 10% chance classes will be cancelled. If it does not rain, there is only a 2% chance classes will be cancelled.

(a) (3pts) What is the probability it will rain and classes will be cancelled?

$$P(\text{Rain} \,\&\, \text{Cancel}) = P(\text{rain}) \cdot P(\text{Cancel} \mid \text{rain}) = \frac{3}{10} \times \frac{1}{10} = \frac{3}{100}$$

(+1) (+1) (+1)

(b) (3pts) What is the total probability (whether or not it rains) that classes will be cancelled?

$$P(\text{Cancel}) = P(\text{Cancel} \cap \text{rain}) + P(\text{Cancel} \cap \text{no rain}) \quad (+1)$$

$$= P(\text{rain})\, P(\text{Cancel} \mid \text{rain}) + P(\text{no rain})\, P(\text{Cancel} \mid \text{no rain}) \quad (+1)$$

$$= \frac{30}{100} \times \frac{10}{100} + \frac{70}{100} \times \frac{2}{100} = \frac{44}{1000} \quad (+1)$$

(c) (4pts) If classes are indeed cancelled tomorrow, what is the probability that it is due to rain?

$$P(\text{rain} \mid \text{Cancel}) = \frac{P(\text{rain} \cap \text{cancel})}{P(\text{cancel})} = \frac{\frac{3}{100}}{\frac{44}{1000}} = \frac{15}{22} \simeq 0.6818$$

(+1) (+1) (+2)

**2.** (10pts) Five balls, numbered 1, 2, 3, 4, and 5, are placed in an urn. Two balls are randomly selected from the five without replacement. Let $X$ be the smaller number of the two.

(a) (5pts) List the possible values for $X$ and assign a probability for each value.

| $X$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p_X$ | $\frac{4}{10}$ | $\frac{3}{10}$ | $\frac{2}{10}$ | $\frac{1}{10}$ |

$\Rightarrow$ Listing (+2)

$\Rightarrow$ value (+3)

$\Rightarrow$ no partial for each part

(b) (5pts) Let another random variable $Y$ be 1 if $X = 1$, and be 2 if $X > 1$. Find the correlation between $X$ and $Y$.

$E(X) = 2, \quad V(X) = 1$

$E(Y) = 1.6, \quad V(Y) = 0.24, \quad E(XY) = \frac{2}{5} + \frac{12}{10} + \frac{12}{10} + \frac{8}{10} = 3.6$

$Cov(X,Y) = 3.6 - 3.2 = 0.4$ (+2) with full process

$$corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \text{(+2)} = \frac{0.4}{\sqrt{0.24}} \text{(+1)}$$

calculation mistake (−2)

Deduction factors are written on exam paper.

**3.** (6pts) The time (in hours) a manager takes to interview a job applicant has an exponential distribution with $\beta = 1/2$. The applicants are scheduled at quarter-hour intervals, beginning at 8:00 A.M., and the applicants arrive exactly on time.

(a) (3pts) When the applicant with an 8:15 A.M. appointment arrives at the manager's office, what is the probability that she will have to wait before seeing the manager? (Find a numerical answer)

$\beta = \frac{1}{2}, \quad f(y : \frac{1}{2}) = 2e^{-2y}$. P(she will have to wait) = P(First applicant's interview $> \frac{1}{4}$hr)

$$\therefore P(Y > \tfrac{1}{4}) = 1 - P(Y < \tfrac{1}{4}) = 1 - \int_0^{\frac{1}{4}} 2e^{-2y}\,dy = 1 - \left[-e^{-2y}\right]_0^{\frac{1}{4}} = 1 - (-e^{-\frac{1}{2}} + 1) = e^{-\frac{1}{2}}$$

(b) (3pts) When the applicant with an 8:45 A.M. appointment arrives at the manager's office, how would you find the probability that she will have to wait before seeing the manager? (No need for a numerical answer)

(+) mgf of $Y_i \sim exp(\beta) = (1-\beta t)^{-1}$ $\downarrow$ mgf of $Gamma(\alpha, \beta) = (1-\beta t)^{-\alpha}$ $\Rightarrow$ mgf of $\sum_{i=1}^{n} Y_i = (1-\beta t)^{-n} = Gamma(n, \beta)$

fourth applicant

P(The applicant with an 8:45AM will have to wait)

= P(The total interview time for applicants 1, 2, 3 $> \frac{3}{4}$hr)

Let $Y^* = Y_1 + Y_2 + Y_3$, where $Y_i$: interview time for applicant $i$

Then $Y^* \sim Gamma(3, \frac{1}{2}) \Rightarrow P(Y^* > \frac{3}{4}) = \int_{\frac{3}{4}}^{\infty} f(y^*)\,dy^* \simeq 0.6693$

(No need for a numerical answer)

**4.** (15pts) Let $X$ denote the number of digital cameras sold during a particular week by a certain store. The pmf of $X$ is

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p_X(x)$ | .1 | .2 | .3 | .3 | .1 |

Sixty percent of all customers who purchase these cameras also buy an extended warranty. Let $Y$ denote the number of purchasers during this week who buy an extended warranty.

(a) (5pts) Find the probability $P(X = 4, Y = 2)$.

$Y|X \sim B(X, 0.6)$, $P(X,Y) = P(Y|X)P(X)$.

By using it, $P(X=4, Y=2) = P(Y=2 \mid X=4) \cdot P(X=4)$ (+3)

$P(Y=2 \mid X=4) = \binom{4}{2} \cdot (0.6)^2 (0.4)^2$, $P(X=4) = 0.1$

$\therefore P(X=4, Y=2) = \binom{4}{2}(0.6)^2(0.4)^2(0.1) = 0.03456$ (no need for calculation)

(+2)

(b) (5pts) Find the probability $P(Y = 2)$.

$P(Y=2) = \sum_{i=0}^{4} P(X=i, Y=2) = \sum_{i=2}^{4} P(X=i, Y=2)$ (+3)

$= P(X=2, Y=2) + P(X=3, Y=2) + P(X=4, Y=2)$

by (a)

$= \binom{2}{2}(0.6)^2 \cdot (0.4)^0 (0.3) + \binom{3}{2}(0.6)^2 (0.4)^1 (0.3) + \binom{4}{2}(0.6)^2(0.4)^2(0.1)$

(+2)

$= 0.29216$ (no need for calculation)

(c) (5pts) Are $X$ and $Y$ independent? Give a reason for your answer.

Not independent. (+2)

$P(X=4)P(Y=2) = (0.1) \cdot (0.29216) \neq P(X=4, Y=2)$

or

$P(X=0, Y=2) = 0 \neq P(X=0)P(Y=2)$.

$\Rightarrow$ you only need to see one of many cases like this.

(+3)

**5.** (15pts) Suppose $Y_1$ and $Y_2$ have a bivariate normal distributed with $N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. Let $\sigma_{12} = \rho\sigma_1\sigma_2$.

(a) (5pts) Show that $X := Y_1 - aY_2$ is uncorrelated with $Y_2$ if $a = \sigma_{12}/\sigma_2^2$. Explain why this implies that $X$ and $Y_2$ are independent.

$$\mathrm{cov}(X, Y_2) = \mathrm{cov}(Y_1, Y_2) - a\,\mathrm{var}(Y_2)$$

$$= \sigma_{12} - a\sigma_2^2, \quad \text{when } a = \frac{\sigma_{12}}{\sigma_2^2}$$

$\mathrm{cov} = 0 \iff \mathrm{corr} = 0$ (+3)

$(Y_1, Y_2)$ are jointly normal $\Rightarrow$ $(X, Y_2)$ are jointly normal (+2)

(b) (5pts) Derive $\mathrm{E}(Y_1|Y_2)$ from (a). (Hint: Use $\mathrm{E}(Y_1|Y_2) = \mathrm{E}(X|Y_2) + a\mathrm{E}(Y_2|Y_2)$).

$$E(Y_1|Y_2) = E(X|Y_2) + aE(Y_2|Y_2)$$

(+2) (+2)

$$= E(X) + aY_2$$

$$= u_1 - au_2 + aY_2 = u_1 + a(Y_2 - u_2) \quad (+1)$$

(c) (5pts) Derive $\mathrm{Var}(Y_1|Y_2)$ from (a).

$$\mathrm{Var}(Y_1|Y_2) = \mathrm{var}(X + aY_2 \mid Y_2)$$

$$= \mathrm{var}(X|Y_2) + a^2\mathrm{var}(Y_2|Y_2) \quad \text{since } \mathrm{cov}(X, Y_2) = 0$$

(+2)

$$= \mathrm{var}(X) \quad (+1)$$

$$= \mathrm{var}(Y_1) + a^2\mathrm{var}(Y_2) - 2a\,\mathrm{cov}(Y_1, Y_2) \quad (+1)$$

$$= \sigma_1^2 + a^2\sigma_2^2 - 2a\sigma_{12}$$

$$= \sigma_1^2 - \frac{\sigma_{12}}{\sigma_2^2} \quad (+1)$$

**6.** (16pts) Suppose that $Y_1, \ldots, Y_n$ are i.i.d. random variables from Poisson($\lambda$).

(a) (3pts) Find the distribution of $T = Y_1 + \cdots + Y_n$ by finding mgf of $T$.

mgf of T: $E(e^{tT}) \overset{(+1)}{=} \prod_{i=1}^{n} E(e^{tY_i}) \overset{(+1)}{=} \exp(n\lambda(e^t - 1))$

$\Rightarrow$ mgf of poisson $(n\lambda)$ (+1)

(b) (3pts) Let $\bar{Y} = T/n$. show that the conditional distribution of $Y_1$ given that $\bar{Y} = m$ is a binomial distribution. Specify the parameters.

$$P(Y_1 = y \mid \bar{Y} = m) = P(Y_1 = y \mid T = nm) = \frac{P(Y_1 = y)\, P(Y_2 + Y_3 \cdots + Y_n = nm - y)}{P(T = nm)} \quad (+1)$$

$$= \cdots = \binom{nm}{y} \cdot \left(\frac{\lambda}{n\lambda}\right)^y \cdot \left(\frac{(n-1)\lambda}{n\lambda}\right)^{nm-y} \sim B\left(nm, \frac{1}{n}\right)$$

(+1) (+1)

(c) (3pts) Using (b), find the conditional mean $\mathrm{E}(Y_1|\bar{Y})$ and conditional variance $\mathrm{Var}(Y_1|\bar{Y})$.

$E(Y_1 \mid \bar{Y}) = nm \times \frac{1}{n} = m = \bar{Y}$ (+1)

$V(Y_1 \mid \bar{Y}) = nm \times \frac{1}{n} \times \frac{n-1}{n} = m\left(1 - \frac{1}{n}\right) = \bar{Y} \cdot \left(1 - \frac{1}{n}\right)$ (+2)

(d) (4pts) Uing (c), show that $\mathrm{E}(S^2|\bar{Y}) = \bar{Y}$.

$$E(S^2 \mid \bar{Y}) = \frac{1}{n-1} E\left(\Sigma Y_i^2 - n\bar{Y}^2 \mid \bar{Y}\right) \quad (+1)$$

$$= \frac{1}{n-1}\left(n E(Y_1^2 \mid \bar{Y}) - n\bar{Y}^2\right) \quad (+1)$$

$$= \frac{1}{n-1}\left(n\left[Var(Y_1 \mid \bar{Y}) + \{E(Y_1 \mid \bar{Y})\}^2\right] - n\bar{Y}^2\right) \quad (+1)$$

$$= \frac{1}{n-1}\left(n\bar{Y}\left(1 - \frac{1}{n}\right) + n\bar{Y}^2 - n\bar{Y}^2\right) = \bar{Y} \quad (+1)$$

(e) (3pts) Using (d), show that $\mathrm{Var}(S^2) \geq \mathrm{Var}(\bar{Y})$.

$$Var(S^2) = E(Var(S^2 \mid \bar{Y})) + Var(E(S^2 \mid \bar{Y})) \quad (+1)$$

$$\geq Var(E(S^2 \mid \bar{Y})) = Var(\bar{Y})$$

(+1)

since $Var(S^2 \mid \bar{Y}) \geq 0 \Rightarrow E(Var(S^2 \mid \bar{Y})) \geq 0$ (+1)

**7.** (16pts) Let $X$ and $Y$ have a joint density function given by

$$f(x, y) = 3x, \quad 0 \le y \le x \le 1.$$

(a) (4pts) Find $P(X < a)$ for $0 < a < 1$.

$$f(x) = \int_0^x 3x\,dy = [3xy]_0^x = 3x^2$$

$$\therefore P(X<a) = \int_0^a 3x^2\,dx = a^3 \quad (0<a<1) \quad (+4)$$

(b) (4pts) Show that $X$ has a beta distribution. Identify the paremeters.

pdf of Beta $(\alpha, \beta)$ : $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1}$

by (a), $f(x) = 3x^2 = \frac{\Gamma(4)}{\Gamma(3)\cdot\Gamma(1)} \cdot x^{3-1} \cdot (1-x)^{1-1} = 3x^2 \quad (+3)$

$\therefore X \sim \text{Beta}(3, 1) \quad (+1)$

(c) (4pts) Show that given $X = x$, $Y$ has a uniform distribution **and** find the conditional mean of $Y$ given $X = x$.

$$f_{Y|X}(x,y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{3x}{3x^2} = \frac{1}{x} \quad (0 \le y \le x)$$

$\sim \text{Uniform}(0, x) \quad (+2)$

$\therefore E(Y|X=x) = \frac{x}{2}$ (using mean of uniform distribution) $(+2)$

(d) (4pts) Using (b) and (c), find the mean of $Y$.

$$E(Y) = E(E(Y|X)) = E\left(\frac{X}{2}\right) = \frac{1}{2}E(X) = \frac{3}{8} \quad (+4)$$

by (c)

by (b), mean of Beta $(3,1)$ = $\frac{3}{3+1}$

**8.** (12pts) Suppose a random variable $X$ has a probability density function

$$f(x) = ke^{-2(x-1)^2}, \quad -\infty < x < \infty.$$

(a) (3pts) Find $k$.

Let $X \sim N(1, (\frac{1}{2})^2)$, pdf of $X$: $f(x) = (\frac{\pi}{2})^{-\frac{1}{2}} \cdot \exp^{(-2(x-1)^2)}$

$\therefore k = (\frac{\pi}{2})^{-\frac{1}{2}}$ (+3)

(b) (3pts) Find the probability $P(X < 2)$.

$P(X<2) = P(\frac{X-1}{\frac{1}{2}} < \frac{2-1}{\frac{1}{2}})$ (+2)

$= P(Z < 2) \simeq 0.9772$ (by Z-table) (+2)

(c) (3pts) Find the 80% percentile of the distribution of $X$.

$P(X < c) = 0.8$. we need to find $c$

$\Leftrightarrow P(\frac{X-1}{\frac{1}{2}} < \frac{c-1}{\frac{1}{2}}) = P(Z < 2(c-1)) = 0.8$ (+2)

by Z-table $2(c-1) = 0.84$ or $0.85$ $\Rightarrow c = 1.42$ or $1.425$ (+2)

(d) (3pts) Find $\mathrm{E}((X-1)^4)$.

Sol 1) $\left(\frac{X-1}{\frac{1}{2}}\right)^2 \sim \chi^2(1)$ $\Rightarrow E(4(X-1)^2) = 1$, $V(4(X-1)^2) = 2$

$\Rightarrow E((X-1)^2) = \frac{1}{4}$, $V((X-1)^2) = \frac{1}{8}$

$\therefore E((X-1)^4) = V((X-1)^2) + [E((X-1)^2)]^2 = \frac{3}{16}$

Sol 2) Let $Y = X-1$, $Y \sim N(0, (\frac{1}{2})^2)$. we need to calculate $E(Y^4)$

using mgf $m_Y(t)$. $E(Y^4) = m_Y''''(0) = \frac{3}{16}$

Sol 3) Directly calculate $\int_{-\infty}^{\infty} (x-1)^4 f(x)\,dx$. Sol 1 or 2 or 3 (+2)

$\Rightarrow$ No matter how you calculate it, the answer should be correct. (+1)

| Distribution | PDF or PMF | Mean | Variance | Moment-Generating Function |
| --- | --- | --- | --- | --- |
| **Continuous distributions** | | | | |
| Uniform | $f(y) = \dfrac{1}{\theta_2 - \theta_1}$ | $\dfrac{\theta_1 + \theta_2}{2}$ | $\dfrac{(\theta_2 - \theta_1)^2}{12}$ | $\dfrac{e^{t\theta_2} - e^{t\theta_1}}{t\,(\theta_2 - \theta_1)}$ |
| Normal | $f(y) = \dfrac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left[-\left(\dfrac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ | $\mu$ | $\sigma^2$ | $\exp\left(\mu t + \dfrac{t^2\sigma^2}{2}\right)$ |
| Exponential | $f(y) = \frac{1}{\beta} e^{-y/\beta}$ | $\beta$ | $\beta^2$ | $\left(1 - \beta t\right)^{-1}$ |
| Gamma | $f(y) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}$ | $\alpha\beta$ | $\alpha\beta^2$ | $\left(1 - \beta t\right)^{-\alpha}$ |
| Chi-square | $f(y) = \dfrac{(y)^{(\nu/2)-1} e^{-y/2}}{2^{\nu/2}\Gamma(\nu/2)}$ | $\nu$ | $2\nu$ | $(1 - 2t)^{-\nu/2}$ |
| Beta | $f(y) = \left[\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1}$ | $\dfrac{\alpha}{\alpha+\beta}$ | $\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | does not exist in closed form |
| Bivariate Normal | $f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} Q\right)$, $Q = \left(\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right)$ | | | |
| **Discrete distributions** | | | | |
| Binomial | $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$ | $np$ | $np(1-p)$ | $[pe^t + (1-p)]^n$ |
| Geometric | $p(y) = p(1-p)^{y-1}$ | $\dfrac{1}{p}$ | $\dfrac{1-p}{p^2}$ | $\dfrac{pe^t}{1-(1-p)e^t}$ |
| Hypergeometric | $p(y) = \dfrac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$ | $\dfrac{nr}{N}$ | $n\left(\dfrac{r}{N}\right)\left(\dfrac{N-r}{N}\right)\left(\dfrac{N-n}{N-1}\right)$ | |
| Poisson | $p(y) = \dfrac{\lambda^y e^{-\lambda}}{y!}$ | $\lambda$ | $\lambda$ | $\exp[\lambda(e^t - 1)]$ |
| Negative binomial | $p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$ | $\dfrac{r}{p}$ | $\dfrac{r(1-p)}{p^2}$ | $\left[\dfrac{pe^t}{1-(1-p)e^t}\right]^r$ |
| Multinomial | $p(y_1, \ldots, y_k) = \frac{n!}{y_1! y_2! \cdots y_k!} p_1^{y_1} \cdots p_k^{y_k}$ | | | |

## Standard Normal Probabilities

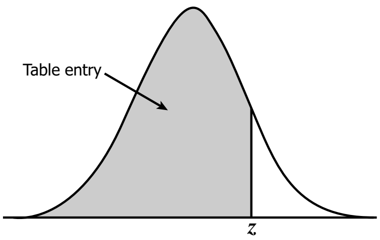


Table entry for $z$ is the area under the standard normal curve to the left of $z$.

| $z$ | **.00** | **.01** | **.02** | **.03** | **.04** | **.05** | **.06** | **.07** | **.08** | **.09** |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |