School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2022 Spring

Final Part II

Thursday, July 2, 2020
1:00–3:15 pm

- Be sure to **show all relevant work and reasoning** in your answer sheet. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.
- Please be clear in writing—we can't grade what we can't decipher!
- Don't forget to upload your answer sheet for each due
  - Problem 1: 1:00 pm–1:30 pm
  - Problem 2: 1:35 pm–2:05 pm
  - Problem 3: 2:10 pm–2:40 pm
  - Problem 4: 2:45 pm–3:15 pm

  through KLMS. The system will be automatically closed at each due time. If the system does not work, you should email it to `ee210b_20spring@kaist.ac.kr` by the due time. Late submissions will not be accepted/graded.

Problem 1 (15 Points) Upload your answer by 1:30pm

Let's consider a signal transmission (communication) problem over a noisy channel. Our goal is to transmit a signal X through a noisy channel. The noisy channel adds a Gaussian noise $Z \sim N(0, \sigma^2)$ to the transmitted signal, so that the observed signal at the receiver is $Y = X + Z$. Let's assume that X is independent of Z, and X can take either value a with probability $p_a > 0$ or value b with probability $p_b > 0$ where $p_a + p_b = 1$ and $b > a$. Conditioned on $X = a$, the observed signal is $Y \sim N(a, \sigma^2)$ and conditioned on $X = b$, the observed signal is $Y \sim N(b, \sigma^2)$.

Given $Y = y$, we want to make a guess of the value of the transmitted signal $X \in \{a, b\}$. Let's denote our guess as $G \in \{a, b\}$. Our goal is to minimize $\mathbf{P}(G \neq X)$, the probability that our guess is incorrect.

Let's consider the following strategy for the guessing: Compare two conditional probabilities $\mathbf{P}(X = a|Y = y)$ and $\mathbf{P}(X = b|Y = y)$ and say that our guess is $G = a$ if $\mathbf{P}(X = a|Y = y) \geq \mathbf{P}(X = b|Y = y)$ and that $G = b$ if $\mathbf{P}(X = a|Y = y) < \mathbf{P}(X = b|Y = y)$.

a) (5 points) Consider the case where $p_a = p_b = 1/2$. Specify the range of y such that our guess is $G = a$. (Hint: You may need to find the threshold r such that if $y \leq r$ your guess $G = a$.)

b) (5 points) For the case $p_a = p_b = 1/2$, calculate the probability $\mathbf{P}(G \neq X)$, the probability that our guess is incorrect. Write down this probability in terms of the CDF for the standard normal. (Remind that the CDF for the standard normal is $\Phi(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r} e^{-t^2/2} dt$.)

c) (5 points) Consider a general $p_a, p_b > 0$, not necessarily $p_a = p_b = 1/2$. Specify the range of y such that our guess is $G = a$ in terms of p_a and p_b. Again, you may need to find the threshold r such that if $y \leq r$ your guess $G = a$. Describe how r should change as p_a increases (or p_b decreases) while satisfying $p_a + p_b = 1$.

<u>Problem 2</u> (15 Points) Upload your answer by 2:05pm

Consider a Bernoulli process $X_1, X_2, X_3, \ldots$ with unknown probability of arrival q, i.e., $\mathbb{P}(X_i = 1) = q$ for all i. Define the k-th interarrival time T_k as

$$T_1 = Y_1, \quad T_k = Y_k - Y_{k-1}, \quad k = 2, 3, \ldots$$

where Y_k is the time of the k-th arrival. In this problem, we estimate the arrival probability q from observed interarrival times $(t_1, t_2, t_3, \ldots)$. Assume q is sampled from the random variable Q which is uniformly distributed over $[0, 1]$.

You may find the following integral useful: For any non-negative integer k and m,

$$\int_0^1 q^k (1-q)^m dq = \frac{k!m!}{(k+m+1)!}.$$

a) (5 points) Find the PMF of T_1, $p_{T_1}(t)$.

b) (5 points) Compute the least mean squares (LMS) estimate of Q conditioned on the first arrival time $T_1 = t_1$, i.e., find $\mathbb{E}[Q|T_1 = t_1]$.

c) (5 points) Compute the maximum a posteriori (MAP) estimate of Q given the first k arrival times, $(T_1, T_2, \ldots, T_k) = (t_1, t_2, \ldots, t_k)$, i.e., find $\arg\max_q f_{Q|T_1,\ldots,T_k}(q|t_1, \ldots, t_k)$.

<u>Problem 3</u> (15 Points) Upload your answer by 2:40pm

The voters in a given town arrive at the place of voting according to a Poisson process of rate $\lambda = 100$ voters per hour. The voters independently vote for candidate A and candidate B each with probability 1/2. Assume that the voting starts at time 0 and continues indefinitely.

a) (5 points) Conditioned on that 1000 voters arrived during the first 10 hours of voting, find the probability that candidate A receives n of those votes.

b) (5 points) Let $T_{1,A}$ be the arrival of the first voter who votes for candidate A. Find the pdf of $T_{1,A}$, $f_{T_{1,A}}(t)$.

c) (5 points) Define V_B as the number of voters for candidate B who arrive before the first voter for A. Find the pmf of V_B, $p_{V_B}(k)$.

<u>Problem 4</u> (15 Points) Upload your answer by 3:15pm

In this problem, we want to have a Markov chain that models the spread of a virus. Assume a population of n individuals. At each daybreak (7 am), each individual is either infected or susceptible (not yet infected but capable of being infected by contacts of infected people). Suppose that each pair of people (i, j), $i \neq j$, independently comes into contact with one another during the daytime (7am to 7pm) with probability p. Whenever a susceptible individual comes into contact with an infected individual, the susceptible individual is infected right away. Assume that during overnight (7pm to 7am next day), any individual who has been infected will recover with probability $0 < q < 1$ and return to being susceptible, independently of everything else.

a) (5 points) Suppose that there are m infected individuals at one daybreak (7am). The number of total population is n. What is the pmf of the number of new infected individuals N at the end of the daytime (7pm)?

b) (5 points) Suppose that $n = 2$. Draw a Markov chain with states 0,1,2 (each of which indicates the number of infected individuals among the 2 at each daybreak) to model the spread of the virus.

c) (5 points) Suppose that $n = 2$ and let's assume that the initial state is $X_0 = 1$ (the number of infected individuals at day 0 is equal to 1). Calculate the mean first passage time to the state 0, i.e., $t_1 = \mathbb{E}[\min\{n \geq 0 \text{ such that } X_n = 0\} | X_0 = 1]$ (the expected number of days to have 0 infected individual for the first time.)