HW 8 solution

7.11 **(+5 pt)**

$P(|\bar{Y}-\mu| \leq 2)=P(-1.5 \leq Z \leq 1.5)=1-2P(Z>1.5)=1-2(.0668)=.8664.$

7.19 **(+5 pt)**

Given that $s^2=.065$ and $n=10$, suppose $\sigma^2=.04$. The probability of observing a value of s^2 that is as extreme or more so is given by

$$P\left(S^2 \geq .065\right)=P\left(9S^2/.04 \geq 9(.065)/.04\right)=P\left(9S^2/.04 \geq 14.925\right)=.10$$

Thus, it is fairly unlikely, so this casts some doubt that $\sigma^2=.04$.

7.30

a. $E(Z)=0, E\left(Z^2\right)=V(Z)+[E(Z)]^2=1.$

b. This is very similar to Ex. 5.86, part a. Using that result, it is clear that

i. $E(T)=0$

ii. $V(T)=E\left(T^2\right)=vE\left(Z^2/Y\right)=v/(v-2), v>2.$

7.37 **(+10 pt)**

a. By Theorem 7.2, χ^2 with 5 degrees of freedom.

b. By Theorem 7.3, χ^2 with 4 degrees of freedom (recall that $\sigma^2=1$).

c. Since Y_6^2 is distributed as χ^2 with 1 degrees of freedom, and $\sum_{i=1}^5\left(Y_i-\bar{Y}\right)^2$ and Y_6^2 are independent, the distribution of $W+U$ is χ^2 with $4+1=5$ degrees of freedom.

7.38

a. By Definition 7.2, t-distribution with 5 degrees of freedom.

b. By Definition 7.2, t-distribution with 4 degrees of freedom.

c. $\bar{Y}$ follows a normal distribution with $\mu=0, \sigma^2=1/5$. So, $\sqrt{5}\bar{Y}$ is standard normal and $(\sqrt{5}\bar{Y})^2$ is chi-square with 1 degree of freedom. Therefore, $5\bar{Y}^2+Y_6^2$ has a chi-square distribution with 2 degrees of freedom (the two random variables are independent). Now, the quotient

$$2\left(5\bar{Y}^2+Y_6^2\right)/U=\left[\left(5\bar{Y}^2+Y_6^2\right)/2\right] \div [U/4]$$

has an F-distribution with 2 numerator and 4 denominator degrees of freedom. Note: we have assumed that $\bar{Y}$ and U are independent (as in Theorem 7.3).

7.39

a. Note that for $i = 1, 2, \ldots, k$, the $\bar{X}_i$ have independent a normal distributions with mean μ_i and variance σ/n_i. Since $\hat{\theta}$, a linear combination of independent normal random variables, by Theorem $6.3 \hat{\theta}$ has a normal distribution with mean given by

$$E(\hat{\theta}) = E\left(c_1 \bar{X}_1 + \ldots + c_k \bar{X}_k\right) = \sum_{i=1}^{k} c_i \mu_i$$

and variance

$$V(\hat{\theta}) = V\left(c_1 \bar{X}_1 + \ldots + c_k \bar{X}_k\right) = \sigma^2 \sum_{i=1}^{k} c_i^2 / n_i^2.$$

b. For $i = 1, 2, \ldots, k$, $(n_i - 1)\, S_i^2/\sigma^2$ follows a chi-square distribution with $n_i - 1$ degrees of freedom. In addition, since the S_i^2 are independent,

$$\frac{\text{SSE}}{\sigma^2} = \sum_{i=1}^{k} (n_i - 1)\, S_i^2/\sigma^2$$

is a sum of independent chi-square variables. Thus, the above quantity is also distributed as chi-square with degrees of freedom $\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - k$. c. From part a, we have that $\frac{\hat{\theta} - \theta}{\sigma \sqrt{\sum_{i=1}^{k} c_i^2 / n_i^2}}$ has a standard normal distribution. Therefore, by Definition 7.2, a random variable constructed as

$$\frac{\hat{\theta} - \theta}{\sigma \sqrt{\sum_{i=1}^{k} c_i^2 / n_i^2}} \Big/ \sqrt{\frac{\sum_{i=1}^{k} (n_i - 1)\, S_i^2/\sigma^2}{\sum_{i=1}^{k} n_i - k}} = \frac{\hat{\theta} - \theta}{\sqrt{\text{MSE} \sum_{i=1}^{k} c_i^2 / n_i^2}}$$

has the t-distribution with $\sum_{i=1}^{k} n_i - k$ degrees of freedom. Here, we are assuming that $\hat{\theta}$ and SSE are independent (similar to $\bar{Y}$ and S^2 as in Theorem 7.3).

7.52

Let $\bar{Y}$ denote the average resistance for the 25 resistors. With $\mu = 200$ and $\sigma = 10$ ohms,

a. $P(199 \leq \bar{Y} \leq 202) \approx P(-.5 \leq Z \leq 1) = .5328$.

b. Let $X =$ total resistance of the 25 resistors. Then,

$$P(X \leq 5100) = \mathrm{P}(\bar{Y} \leq 204) \approx P(Z \leq 2) = .9772.$$

7.58

For $W_i = X_i - Y_i$, we have that $E(W_i) = E(X_i) - E(Y_i) = \mu_1 - \mu_2$ and $V(W_i) = V(X_i) - V(Y_i) = \sigma_1^2 + \sigma_2^2$ since X_i and Y_i are independent. Thus, $\bar{W} = \frac{1}{n} \sum_{i=1}^{n} W_i = \frac{1}{n} \sum_{i=1}^{n} (X_i - Y_i) = \bar{X} - \bar{Y}$ so

$E(\bar{W}) = \mu_1 - \mu_2$, and $V(\bar{W}) = (\sigma_1^2 + \sigma_2^2)/n$. Thus, since the W_i are independent,

$$U_n = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}} = \frac{\bar{W} - E(\bar{W})}{\sqrt{V(\bar{W})}}$$

satisfies the conditions of Theorem 7.4 and has a limiting standard normal distribution.

7.62 **(+5 pt)**

Let Y_i represent the time required to process the i^{th} person's order, $i = 1, 2, \ldots, 100$. We have that $\mu = 2.5$ minutes and $\sigma = 2$ minutes. So, since 4 hours = 240 minutes,

$$P\left(\sum_{i=1}^{100} Y_i > 240\right) = P(\bar{Y} > 2.4) \approx P\left(Z > \frac{\sqrt{100}(2.4 - 2.5)}{2}\right) = P(Z > -.5) = .6915.$$

7.79

a. Using the normal approximation:

$$P(Y \geq 2) = P(Y \geq 1.5) = P\left(Z \geq \frac{1.5 - 2.5}{\sqrt{25(1)(.9)}}\right) = P(Z \geq -.67) = .7486.$$

b. Using the exact binomial probability:

$$P(Y \geq 2) = 1 - P(Y \leq 1) = 1 - .271 = .729.$$

7.84

a. $E\left(\frac{Y_1}{n_1} - \frac{Y_2}{n_2}\right) = \frac{E(Y_1)}{n_1} - \frac{E(Y_2)}{n_2} = \frac{m_1 p_1}{n_1} - \frac{n_2 p_2}{n_2} = p_1 - p_2.$

b. $V\left(\frac{Y_1}{n_1} - \frac{Y_2}{n_2}\right) = \frac{V(Y_1)}{n_1^2} + \frac{V(Y_2)}{n_2^2} = \frac{n_1 p_1 q_1}{n_1^2} + \frac{n_2 p_2 q_2}{n_2^2} = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}.$

7.95

a. Since $\mu = 0$ and by Definition 2, $T = \frac{\bar{Y}}{S/\sqrt{10}}$ has a t-distribution with 9 degrees of freedom. Also, $T^2 = \frac{\bar{Y}^2}{S^2/10} = \frac{10\bar{Y}^2}{S^2}$ has an F-distribution with 1 numerator and 9 denominator degrees of freedom (see Ex. 7.33).

b. By Definition 3, $T^{-2} = \frac{S^2}{10\bar{Y}^2}$ has an F-distribution with 9 numerator and 1 denominator degrees of freedom (see Ex. 7.29).

c. With 9 numerator and 1 denominator degrees of freedom, $F_{.05} = 240.5$. Thus,

$$.95 = P\left(\frac{S^2}{10\bar{Y}^2} < 240.5\right) = P\left(\frac{S^2}{\bar{Y}^2} < 2405\right) = P\left(-49.04 < \frac{S}{\bar{Y}} < 49.04\right),$$

so $c = 49.04$.

7.96

Note that Y has a beta distribution with $\alpha = 3$ and $\beta = 1$. So, $\mu = 3/4$ and $\sigma^2 = 3/80$. By the Central Limit Theorem, $P(\bar{Y} > .7) \approx P\left(Z > \frac{7-75}{\sqrt{.0375/40}}\right) = P(Z > -1.63) = .9484$.

7.104 **(+10 pt)**

The mgf for Y_n is given by

$$m_{Y_n}(t) = \left[1 - p + pe^t\right]^n.$$

Let $p = \lambda/n$ and this becomes

$$m_{Y_n}(t) = \left[1 - \frac{\lambda}{n} + \frac{\lambda}{n}e^t\right]^n = \left[1 + \frac{1}{n}\left(\lambda e^t - 1\right)\right]^n.$$

As $n \to \infty$, this is $\exp\left(\lambda e^t - 1\right)$, the mgf for the Poisson with mean λ.

7.105

Let $Y = \#$ of people that suffer an adverse reaction. Then, Y is binomial with $n = 1000$ and $p = .001$. Using the result in Ex. 7.104, we let $\lambda = 1000(.001) = 1$ and evaluate

$$P(Y \geq 2) = 1 - P(Y \leq 1) \approx 1 - .736 = .264,$$

using the Poisson table in Appendix 3.