School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2020 Spring

Final Part II

Thursday, July 2, 2020
1:00–3:15 pm

- Be sure to **show all relevant work and reasoning** in your answer sheet. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.
- Please be clear in writing—we can't grade what we can't decipher!
- Don't forget to upload your answer sheet for
  - Problem 1 by 1:30pm
  - Problem 2 by 2:05pm
  - Problem 3 by 2:40pm
  - Problem 4 by 3:15pm

  through KLMS. The system will be automatically closed at each due time. If the system does not work, you should email it to `ee210b_20spring@kaist.ac.kr` by the due time. Late submissions will not be accepted/graded.

Problem 1 (15 Points)

Let's consider a signal transmission (communication) problem over a noisy channel. Our goal is to transmit a signal X through a noisy channel. The noisy channel adds a Gaussian noise $Z \sim N(0, \sigma^2)$ to the transmitted signal, so that the observed signal at the receiver is $Y = X + Z$. Let's assume that X is independent of Z and X can take value a with probability $p_a > 0$ and value b with probability $p_b > 0$ where $p_a + p_b = 1$ and $b > a$. Conditioned on $X = a$, the observed signal is $Y \sim N(a, \sigma^2)$ and conditioned on $X = b$, the observed signal is $Y \sim N(b, \sigma^2)$.

Given $Y = y$, we want to make a guess of the value for $X \in \{a, b\}$. Let's denote our guess as $G \in \{a, b\}$. Our goal is to minimize $\mathbf{P}(G \neq X)$, the probability that our guess is incorrect.

Let's consider the following strategy for the guessing: Compare two conditional probabilities $\mathbf{P}(X = a|Y = y)$ and $\mathbf{P}(X = b|Y = y)$ and say that our guess is $G = a$ if $\mathbf{P}(X = a|Y = y) \geq \mathbf{P}(X = b|Y = y)$ and that $G = b$ if $\mathbf{P}(X = a|Y = y) < \mathbf{P}(X = b|Y = y)$.

a) (5 points) Consider the case where $p_a = p_b = 1/2$. Specify the range of y such that our guess is $G = a$. (Hint: You may need to find the threshold r such that if $y \leq r$ your guess $G = a$.)

Solution: The range of y such that $\mathbf{P}(X = a|Y = y) \geq \mathbf{P}(X = b|Y = y)$ is

$$\frac{\mathbf{P}(X = a)\mathbf{P}(Y = y|X = a)}{\mathbf{P}(Y = y)} \geq \frac{\mathbf{P}(X = b)\mathbf{P}(Y = y|X = b)}{\mathbf{P}(Y = y)}$$

by Bayes rule.(**1 point**) Since $p_a = p_b = 1/2$, above inequality can be simplified as

$$\mathbf{P}(Y = y|X = a) \geq \mathbf{P}(Y = y|X = b). \qquad \textbf{(1 point)}$$

Since $\mathbf{P}(Y = y|X = a) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-a)^2}{2\sigma^2}}$ and $\mathbf{P}(Y = y|X = b) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-b)^2}{2\sigma^2}}$,(**1 point**) the range of y where $\mathbf{P}(X = a|Y = y) \geq \mathbf{P}(X = b|Y = y)$ becomes

$$y \leq \frac{(b^2 - a^2)}{2(b - a)} = \frac{b + a}{2}. \qquad \textbf{(2 points)}$$

b) (5 points) For the case $p_a = p_b = 1/2$, calculate the probability $\mathbf{P}(G \neq X)$, the probability that our guess is incorrect. Write down this probability in terms of the CDF for the standard normal. (Remind that the CDF for the standard normal is $\Phi(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r} e^{-t^2/2} dt$.)

Solution: When we let $r := \frac{b+a}{2}$,

$$\begin{aligned}
\mathbf{P}(G \neq X) &= \frac{1}{2}\mathbf{P}(G \neq X|X=a) + \frac{1}{2}\mathbf{P}(G \neq X|X=b) && (\textbf{1 point})\\
&= \frac{1}{2}\mathbf{P}(Y \geq r|X=a) + \frac{1}{2}\mathbf{P}(Y \leq r|X=b) && (\textbf{1 point})\\
&= \frac{1}{2}\mathbf{P}\left(\frac{Y-a}{\sigma} \geq \frac{r-a}{\sigma}|X=a\right) + \frac{1}{2}\mathbf{P}\left(\frac{Y-b}{\sigma} \leq \frac{r-b}{\sigma}|X=b\right)\\
& && (\textbf{1 point})\\
&= \frac{1}{2}\left(1 - \mathbf{P}\left(\frac{Y-a}{\sigma} \leq \frac{b-a}{2\sigma}|X=a\right)\right)\\
&\quad + \frac{1}{2}\mathbf{P}\left(\frac{Y-b}{\sigma} \leq \frac{a-b}{2\sigma}|X=b\right)\\
&= \frac{1}{2}\left(1 - \Phi\left(\frac{b-a}{2\sigma}\right)\right) + \frac{1}{2}\left(\Phi\left(\frac{a-b}{2\sigma}\right)\right) && (\textbf{2 point})\\
&= \frac{1}{2}\left(1 - \Phi\left(\frac{b-a}{2\sigma}\right)\right) + \frac{1}{2}\left(1 - \Phi\left(\frac{b-a}{2\sigma}\right)\right)\\
&= 1 - \Phi\left(\frac{b-a}{2\sigma}\right)
\end{aligned}$$

c) (5 points) Consider a general $p_a, p_b > 0$, not necessarily $p_a = p_b = 1/2$. Specify the range of y such that our guess is $G = a$ in terms of p_a and p_b. Again, you may need to find the threshold r such that if $y \leq r$ your guess $G = a$. Describe how r should change as p_a increases (or p_b increases) while satisfying $p_a + p_b = 1$.

Solution: The range of y such that $\mathbf{P}(X = a|Y = y) \geq \mathbf{P}(X = b|Y = y)$ is

$$\frac{\mathbf{P}(X=a)\mathbf{P}(Y=y|X=a)}{\mathbf{P}(Y=y)} \geq \frac{\mathbf{P}(X=b)\mathbf{P}(Y=y|X=b)}{\mathbf{P}(Y=y)}, \qquad (\textbf{1 point})$$

which is equivalent to

$$p_a\mathbf{P}(Y=y|X=a) \geq p_b\mathbf{P}(Y=y|X=b). \qquad (\textbf{1 point})$$

Since $\mathbf{P}(Y = y|X = a) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-a)^2}{2\sigma^2}}$ and $\mathbf{P}(Y = y|X = b) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-b)^2}{2\sigma^2}}$,(**1 point**) the range of y where $\mathbf{P}(X = a|Y = y) \geq \mathbf{P}(X = b|Y = y)$ becomes

$$y \leq \frac{(b^2-a^2) + 2\sigma^2 \ln \frac{p_a}{p_b}}{2(b-a)} = \frac{b+a}{2} + \frac{\sigma^2 \ln \frac{p_a}{p_b}}{b-a}. \qquad (\textbf{1 point})$$

As p_a increases, the threshold $r := \frac{(b^2-a^2)+2\sigma^2 \ln \frac{p_a}{p_b}}{2(b-a)} = \frac{b+a}{2} + \frac{\sigma^2 \ln \frac{p_a}{p_b}}{b-a}$ should move to the right and as p_b increases, the threshold r should move to the left.(**1 point**)

<u>Problem 2</u> (15 Points)

Consider a Bernoulli process $X_1, X_2, X_3, \dots$ with unknown probability of arrival q, i.e., $\mathbb{P}(X_i = 1) = q$ for all i. Define the k-th interarrival time T_k as

$$T_1 = Y_1, \quad T_k = Y_k - Y_{k-1}, \quad k = 2, 3, \dots$$

where Y_k is the time of the k-th arrival. In this problem, we estimate the arrival probability q from observed interarrival times $(t_1, t_2, t_3, \dots)$.

You may find the following integral useful: For any non-negative integer k and m,

$$\int_0^1 q^k (1-q)^m dq = \frac{k!m!}{(k+m+1)!}.$$

Assume q is sampled from the random variable Q which is uniformly distributed over $[0, 1]$.

a) (5 points) Find the PMF of T_1, $p_{T_1}(t)$.

Solution: By the total probability theorem, we have

$$\begin{aligned} p_{T_1}(t) &= \int_0^1 p_{T_1|Q}(t,q) f_Q(q) dq = \int_0^1 q(1-q)^{t-1} dq \qquad (\textbf{2 point}) \\ &= \frac{1!(t-1)!}{(t+1)!} = \frac{1}{(t+1)t} \quad \text{for } t = 1, 2, \dots \qquad (\textbf{3 point}) \end{aligned}$$

b) (5 points) Compute the least mean squares (LMS) estimate of Q conditioned on the first arrival time $T_1 = t_1$, i.e., find $\mathbb{E}[Q|T_1 = t_1]$.

Solution: The LSE estimate of q conditioned on $T_1 = t_1$ is equal to $\mathbb{E}[Q|T_1 = t_1]$, which can be calculated as

$$\begin{aligned} \mathbb{E}[Q|T_1 = t_1] &= \int_0^1 p_{Q|T_1}(q|t_1) q dq \\ &= \int_0^1 \frac{p_{T_1|Q}(t_1|q) f_Q(q)}{p_{T_1}(t)} q dq \qquad (\textbf{2 point}) \\ &= \int_0^1 t_1(t_1+1) q (1-q)^{t_1-1} q dq \\ &= t_1(t_1+1) \int_0^1 q^2 (1-q)^{t_1-1} dq = t_1(t_1+1) \frac{2(t_1-1)!}{(t_1+2)!} = \frac{2}{t_1+2}. \ (\textbf{3 point}) \end{aligned}$$

c) (5 points) Compute the maximum a posteriori (MAP) estimate of Q given the first k arrival times, $(T_1, T_2, \dots, T_k) = (t_1, t_2, \dots, t_k)$, i.e., find $\arg\max_q f_{Q|T_1,\dots,T_k}(q|t_1, \dots, t_k)$.

Solution: The posterior probability of Q given $(T_1, T_2, \dots, T_k) = (t_1, t_2, \dots, t_k)$ is

$$\begin{aligned} f_{Q|T_1,\dots,T_k}(q|t_1,\dots,t_k) &= \frac{f_Q(q) \prod_{i=1}^k p_{T_i|Q}(T_i = t_i | Q = q)}{C} \qquad (\textbf{2 point}) \\ &= \frac{q^k (1-q)^{(\sum_{i=1}^k t_i) - k}}{C} \qquad (\textbf{1 point}) \end{aligned}$$

where $C = \int_0^1 f_Q(q) \prod_{i=1}^k p_{T_i|Q}(T_i = t_i|Q = q)dq$, which does not depend on q. The MAP estimate of Q is the value q that maximizes $f_{Q|T_1,\ldots,T_k}(q|t_1, \ldots, t_k)$. We can find the MAP estimate of Q by finding q that makes the first derivative of $f_{Q|T_1,\ldots,T_k}(q|t_1, \ldots, t_k)$ with respect to q equal to 0, i.e.,

$$kq^{k-1}(1-q)^{(\sum_{i=1}^k t_i)-k} - \left(\sum_{i=1}^k t_i - k\right) q^k (1-q)^{(\sum_{i=1}^k t_i)-k-1} = 0,$$

or equivalently,

$$k(1-q) - \left(\sum_{i=1}^k t_i - k\right) q = 0,$$

which yields the MAP estiamte

$$\hat{q}_{\text{MAP}} = \frac{k}{\sum_{i=1}^k t_i}. \qquad \textbf{(2 point)}$$

Problem 3 (15 Points)

The voters in a given town arrive at the place of voting according to a Poisson process of rate $\lambda = 100$ voters per hour. The voters independently vote for candidate A and candidate B each with probability $1/2$. Assume that the voting starts at time 0 and continues indefinitely.

a) (5 points) Conditioned on that 1000 voters arrived during the first 10 hours of voting, find the probability that candidate A receives n of those votes.

Solution: We can consider the splitting of the original Poisson process with $\lambda = 100$ into two Poisson processes, each of which indicates the votes for candidate A and B, respectively, by splitting arrivals into two streams using independent coin flips of a fair coin. The first poisson process with $\lambda_A = 50$ indicates the arrival of voters who vote for candidate A and the other Poisson process with $\lambda_B = 50$ indicates arrival of voters who vote candidate B. The two Poisson processes are independent.

Since the arrivals are splitted into two processes with probability $1/2$ and $1/2$, conditioned on that 1000 voters arrived during the first 10 hours, the probability that candidate A receives n votes follows the binomial distribution with $n = 1000$ and $p = 1/2$. (**3 point**)
So that the probability is equal to

$$\binom{1000}{n}\left(\frac{1}{2}\right)^n. \qquad (\textbf{2 point})$$

This can also be found by calculating $\frac{\mathbf{P}(N_A(10)=n, N_B(10)=1000-n)}{\mathbf{P}(N_A(10)+N_B(10)=1000)}$ where $N_A(\tau), N_B(\tau)$ is the number of arrivals for the Poisson process for A and B, respectively, for time duration of τ:

$$\frac{\mathbf{P}(N_A(10) = n, N_B(10) = 1000 - n)}{\mathbf{P}(N_A(10) + N_B(10) = 1000)} = \frac{\frac{(50\cdot 10)^n e^{-50\cdot 10}}{n!}\,\frac{(50\cdot 10)^{1000-n} e^{-50\cdot 10}}{(1000-n)!}}{\frac{(100\cdot 10)^{1000} e^{-100\cdot 10}}{1000!}} = \binom{1000}{n}\left(\frac{1}{2}\right)^n.$$

For student who multiplied the conditional term : (**-2 point**)

b) (5 points) Let $T_{1,A}$ be the arrival of the first voter who votes for candidate A. Find the pdf of $T_{1,A}$, $f_{T_{1,A}}(t)$.

Solution: $T_{1,A}$ is the first arrival time of the Poisson process with rate $\lambda_A = 50$. (**3 point**) Its pdf is the exponential(λ_A), i.e.,

$$f_{T_{1,A}}(t) = 50e^{-50t}, \text{ for } t \geq 0. \qquad (\textbf{2 point})$$

c) (5 points) Define V_B as the number of voters for candidate B who arrive before the first voter for A. Find the pmf of V_B.

Solution: The pmf of the number of voters for candidate B who arrive before the first voter for A follows the geometric distribution with $p = 1/2$, thus

$$p_{V_B}(k) = (1/2)^k \cdot 1/2 = (1/2)^{k+1}. \qquad (\textbf{5 point})$$

We can also find this by calculating the pmf for V_B conditioned on $T_{1,A} = t_1$ and by using the total probability theory. Note that

$$p_{V_B|T_{1,A}}(k|t_1) = \frac{(50t_1)e^{-50t_1}}{k!}$$

By using the pdf of $T_{1,A}$ which is $f_{T_{1,A}} = 50e^{-50t}$ for $t \geq 0$, we get

$$\begin{aligned}
p_{V_B}(k) &= \int_0^{\infty} p_{V_B|T_{1,A}}(k|t_1) f_{T_{1,A}}(t_1) dt_1 (\textbf{2 point}) \\
&= \int_0^{\infty} \frac{(50t_1)^k e^{-50t_1}}{k!} 50e^{-50t_1} dt_1 (\textbf{1 point}) \\
&= \frac{(50)^{k+1}}{k!} \int_0^{\infty} {t_1}^k e^{-100t_1} dt_1 \\
&= \frac{(50)^{k+1}}{k!} \frac{k!}{100^{k+1}} = (1/2)^{k+1}. (\textbf{2 point})
\end{aligned}$$

Problem 4 (15 Points)

In this problem, we want to have a Markov chain that models the spread of a virus. Assume a population of n individuals. At each daybreak (7 am), each individual is either infected or susceptible (not yet infected but capable of being infected by contacts of infected people). Suppose that each pair of people (i, j), $i \neq j$, independently comes into contact with one another during the daytime (7am to 7pm) with probability p. Whenever an infected individual comes into contact with a susceptible individual, the susceptible individual is infected right away. Assume that during overnight (7pm to 7am next day), any individual who has been infected will recover with probability $0 < q < 1$ and return to being susceptible, independently of everything else.

a) (5 points) Suppose that there are m infected individuals at one daybreak (7am). What is the pmf of the number of new infections N at the end of the daytime (7pm)?

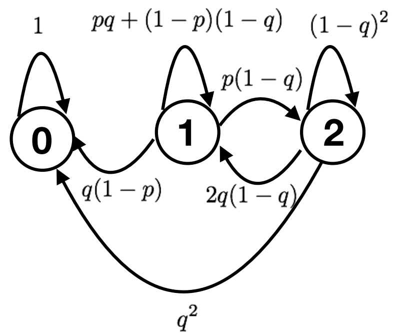
Solution: If m out of n individuals are infected, there are $n - m$ susceptible individuals. Each of these susceptible individuals will be independently infected during the daytime with probability $r = 1 - (1 - p)^m$. Thus, the number of new infections N will be binomial random variable with parameters $n - m$ and r, so that

$$p_N(k) = \binom{n-m}{k} r^k (1-r)^{n-m-k}, \quad k = 0, 1, \ldots, n - m.$$

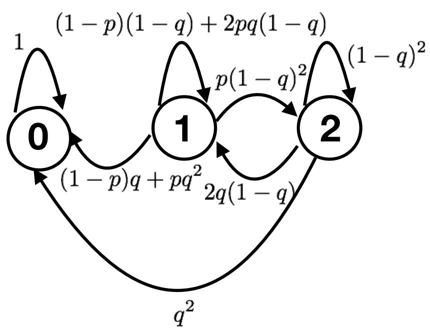
Partial point: $r = 1 - (1 - p)^m$ (3 points),

b) (5 points) Suppose that there are $n = 2$ people. Draw a Markov chain with states 0,1,2 (each of which indicates the number of infected people among the 2 at each daybreak) to model the spread of the virus.

Solution:



Assuming that a new infected individual cannot be recovered on the same day



Assuming that a new infected individual can be recovered on the same day

Partial point: it depends on the figure of the Markov chain you draw and the number of correct transition probabilities.

c) (5 points) Suppose that $n = 2$ and let's assume that the initial state is $X_0 = 1$ (the number of infected individuals at day 0 is equal to 1). Calculate the mean first passage time to the state 0, i.e., $t_1 = \mathbb{E}[\min\{n \geq 0 \text{ such that } X_n = 0\}|X_0 = 1]$ (the expected number of days to have 0 infected individual for the first time.)

Solution: We need to solve the following equations:

$$t_1 = 1 + p_{11}t_1 + p_{12}t_2$$
$$t_2 = 1 + p_{21}t_1 + p_{22}t_2.$$

We can find that

$$t_1 = \frac{1 - p_{22} + p_{12}}{(1 - p_{11})(1 - p_{22}) - p_{12}p_{21}}$$

Partial point: it depends on how you set up the equation to solve the problem. Even though the transition probabilities calculated in problem b) was wrong, I give you full points if the equation for t_1 is appropriate.

Every description of partial points is explained in KLMS.

School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2021 Spring

Final

Tuesday, June 15, 2021
9:00–11:20 am

- Be sure to **show all relevant work and reasoning** in your answer sheet. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.
- Please be clear in writing—we can't grade what we can't decipher!
- Don't forget to upload your answer sheet during 11:10-11:20 am through KLMS. The system will be automatically closed at that time. If the system does not work, you should email it to `ee210b_21spring@kaist.ac.kr` by 11:20 am. Late submissions will not be accepted/graded.

Problem 1 (**10 Points**)

a) (5 points) Find the smallest n, the number of samples, for which the Chebyshev inequality yields a guarantee

$$\Pr(|M_n - p| \geq 0.5) \leq 0.05. \tag{1}$$

Assume that $\mathrm{var}(X_i) = v$ for some constant v. State your answer as a function of v.

Since $\mathbb{E}[M_n] = p$ and $\mathrm{var}(M_n) = \frac{v}{n}$, by Chebyshev inequality,

$$\Pr(|M_n - p| \geq 0.5) \leq \frac{\mathrm{var}(M_n)}{0.5^2} = \frac{v}{n \cdot 0.5^2} = 0.05. \tag{2}$$

The required n is $80v$.

b) (5 points) Assume that $n = 10000$. Find an approximate value for the probability

$$\Pr(|M_{10000} - p| \geq 0.5) \tag{3}$$

using the Central Limit Theorem. Assume again that $\mathrm{var}(X_i) = v$ for some constant v. Give your answer in terms of v, and the standard normal CDF $\Phi(\cdot)$.

By CLT, we can approximate

$$\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{nv}} \tag{4}$$

by a standard normal distribution when n is large. Hence,

$$\Pr(|M_{10000} - p| \geq 0.5) = \Pr\left(\left|\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{nv}}\right| \geq \frac{0.5\sqrt{n}}{\sqrt{v}}\right) = 2\left(1 - \Phi\left(\frac{50}{\sqrt{v}}\right)\right). \tag{5}$$

Problem 2 (10 Points)

Consider a biased coin where the coin lands with head with probability equal to $q \in [0, 1]$. The probability of head, q, is sampled from a random variable Q with pdf

$$f_Q(q) = \begin{cases} 6q(1-q), & 0 \le q \le 1 \\ 0, & \text{o.w.} \end{cases} \tag{6}$$

and once it is sampled the value is fixed during the experiments. We flip the coin n times and count the number of heads, K, which is a random variable. Given $K = k$, derive the following estimates of Q:

a) (5 points) Find the MAP estimator, $\hat{q}_{\mathsf{MAP}} = \arg\max_q f_{Q|K}(q|k)$ where $f_{Q|K}(q|k)$ is the conditional pdf of Q given $K = k$..

   Note that for a fixed q, K is distributed by binomial(n, q). Thus, the conditional PMF of K given $Q = q$ is

$$P_{K|Q}(k|q) = \binom{n}{k} q^k (1-q)^{n-k}. \tag{7}$$

   By Bayes' rule, the conditional pdf of Q given $K = k$ is

$$f_{Q|K}(q|k) = \frac{P_{K|Q}(k|q) f_Q(q)}{P_K(k)} \tag{8}$$

   Note that only the numerator depends on q. Thus, we need to find q that maximizes

$$P_{K|Q}(k|q) f_Q(q) = 6\binom{n}{k} q^{k+1}(1-q)^{n-k+1}. \tag{9}$$

   By taking derivatives, we can solve q that satisfies

$$\frac{d[P_{K|Q}(k|q) f_Q(q)]}{dq} = 6\binom{n}{k} q^k (1-q)^{n-k}[(k+1)(1-q) - (n-k+1)q] = 0, \tag{10}$$

   which is

$$\hat{q}_{\mathsf{MAP}} = \frac{k+1}{n+2}. \tag{11}$$

b) (5 points) Find the least mean square estimator, $\hat{q}_{\mathsf{LMS}} = \mathbb{E}[Q|K = k]$.

   To find $\mathbb{E}[Q|K = k]$, we need to first calculate

$$f_{Q|K}(q|k) = \frac{P_{K|Q}(k|q) f_Q(q)}{P_K(k)}. \tag{12}$$

   Note that

$$P_K(k) = 6\binom{n}{k} \int_0^1 q^{k+1}(1-q)^{n-k+1} dq. \tag{13}$$

By using

$$\int_0^1 p^l(1-p)^{m-l}dp = \frac{l!(m-l)!}{(m+1)!} \text{ for } 0 \le l \le m,$$

we get

$$P_K(k) = 6\binom{n}{k}\frac{(k+1)!(n-k+1)!}{(n+3)!}. \tag{14}$$

Thus, the conditional pdf of Q given $K = k$ is

$$f_{Q|K}(q|k) = \frac{P_{K|Q}(k|q)f_Q(q)}{P_K(k)} = \begin{cases} \frac{(n+3)!}{(k+1)!(n-k+1)!}q^{k+1}(1-q)^{n-k+1}, & 0 \le q \le 1 \\ 0, & \text{o.w.} \end{cases} \tag{15}$$

By using the fact that

$$\int_0^1 \frac{(i+j-1)!}{(i-1)!(j-1)!}p^i(1-p)^{j-1}dp = \frac{i}{i+j}.$$

we can calculate

$$\hat{q}_{\mathsf{LMS}} = \mathbb{E}[Q|K=k] = \frac{k+2}{n+4}. \tag{16}$$

Problem 3 (15 Points)

We conduct an elementary experiment (e.g. some physical experiment) independently total N times, where N is a Poisson random variable of mean λ, i.e., $\mathbb{P}(N = n) = \frac{\lambda^n e^{-\lambda}}{n!}$. The outcome of each experiment is sampled from a set $\{a_1, \ldots, a_K\}$, where the probability of getting an outcome a_k is equal to p_k for $1 \leq k \leq K$ where $\sum_{k=1}^{K} p_k = 1$.

a) (3 points) Let N_k denote the number of experiments performed for which the outcome is equal to a_k. Find the PMF for N_k ($1 \leq k \leq K$). (Hint: no calculation is necessary.)

We can view the experiment as a combination of K Poisson processes where the k-th process has rate $p_k\lambda$ and the combined process has rate λ. At $t = 1$, the total number of experiments is Poisson with mean λ and the k-th process is Poisson with mean $p_k\lambda$. Thus,

$$p_{N_k}(n) = \frac{(\lambda p_k)^n e^{-\lambda p_k}}{n!}. \tag{17}$$

b) (3 points) Find the PMF of $N_1 + N_2$.

By the same argument,

$$p_{N_1+N_2}(n) = \frac{(\lambda(p_1 + p_2))^n e^{-\lambda(p_1+p_2)}}{n!}. \tag{18}$$

c) (3 points) Find the conditional PMF for N_1 given that $N = n$.

Each of the n combined arrivals over $(0, 1]$ is then a_1 with probability p_1. Thus, N_1 is binomial given that $N = n$,

$$p_{N_1|N}(n_1|n) = \binom{n}{n_1}(p_1)^{n_1}(1 - p_1)^{n-n_1}. \tag{19}$$

d) (3 points) Find the conditional PMF for $N_1 + N_2$ given that $N = n$.

Let the sample value of $N_1 + N_2$ be n_{12}. By the same argument as in (c),

$$p_{N_1+N_2|N}(n_{12}|n) = \binom{n}{n_{12}}(p_1 + p_2)^{n_{12}}(1 - p_1 - p_2)^{n-n_{12}}. \tag{20}$$

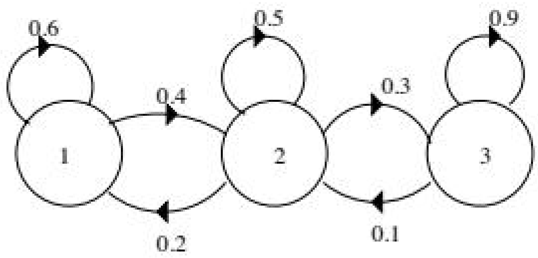
e) (3 points) Find the conditional PMF for N given that $N_1 = n_1$.

Since N is then n_1 plus the number of arrivals from the other processes, and those additional arrivals are Poisson with mean $\lambda(1 - p_1)$, we have

$$p_{N|N_1}(n|n_1) = \frac{(\lambda(1 - p_1))^{n-n_1} e^{-\lambda(1-p_1)}}{(n - n_1)!}, \quad \text{for } n \geq n_1. \tag{21}$$

<u>Problem 4</u> (15 Points)

Consider a Markov chain $\{X_n : n = 0, 1, \dots,\}$, specified by the following transition diagram.



a) (3 points) Given that the chain starts with $X_0 = 1$, find the probability that $X_2 = 2$.

The two-step transition probability is

$$\begin{aligned} r_{12}(2) &= p_{11} \cdot p_{12} + p_{12} \cdot p_{22} \\ &= 0.6 \cdot 0.4 + 0.4 \cdot 0.5 = 0.44. \end{aligned} \tag{22}$$

b) (3 points) Find the steady-state probabilities π_1, π_2, π_3 for the state 1,2, and 3.

We set up the balance equations of a birth-death process and the normalization equation as such:

$$\begin{aligned} \pi_1 p_{12} &= \pi_2 p_{21} \\ \pi_2 p_{23} &= \pi_3 p_{32} \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned} \tag{23}$$

Solving the system of equations yields the following steady-state probabilities:

$$\begin{aligned} \pi_1 &= 1/9, \\ \pi_2 &= 2/9, \\ \pi_3 &= 6/9. \end{aligned} \tag{24}$$

c) (3 points) Let $Y_n = X_n - X_{n-1}$. Thus, $Y_n = 1$ indicates that the n-th transition was to the right, $Y_n = 0$ indicates it was a self-transition, and $Y_n = -1$ indicates it was a transition to the left. Find $\lim_{n\to\infty} \mathbb{P}(Y_n = 1)$.

Using the total probability theorem and steady-state probabilities,

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}(Y_n = 1) &= \sum_{i=1}^{3} \pi_i \cdot \mathbb{P}(Y_n = 1 | X_{n-1} = i) \\ &= \pi_1 p_{12} + \pi_2 p_{23} = 1/9. \end{aligned} \tag{25}$$

d) (3 points) Given that the n-th transition was a transition to the right ($Y_n = 1$), find the probability that the previous state was state 1. (You can assume that n is large.)

Using Bayes' Rule,

$$\begin{aligned}\mathbb{P}(X_{n-1}=1|Y_n=1) &= \frac{\mathbb{P}(X_{n-1}=1)\mathbb{P}(Y_n=1|X_{n-1}=1)}{\sum_{i=1}^{3}\mathbb{P}(X_{n-1}=i)\mathbb{P}(Y_n=1|X_{n-1}=i)} \\ &= \frac{\pi_1 p_{12}}{\pi_1 p_{12}+\pi_2 p_{23}} = 2/5. \end{aligned} \tag{26}$$

e) (3 points) Suppose that $X_0 = 1$. Let T be defined as the first positive time at which the state is again equal to 1. Show how to find $\mathbb{E}[T]$. (It is enough to write down whatever equations need to be solved; you do not need to actually solve it to produce a numerical answer.)

In order to find the mean recurrence time of state 1, the mean first passage times to state 1 are first calculated by solving the following system of equations:

$$\begin{aligned} t_2 &= 1 + p_{22}t_2 + p_{23}t_3 \\ t_3 &= 1 + p_{32}t_2 + p_{33}t_3, \end{aligned} \tag{27}$$

which is

$$\begin{aligned} t_2 &= 1 + 0.5t_2 + 0.3t_3 \\ t_3 &= 1 + 0.1t_2 + 0.9t_3, \end{aligned} \tag{28}$$

The mean recurrence time of state 1 is then given by $t_1^* = 1 + p_{12}t_2$, which is $t_1^* = 1 + 0.4t_2$.

Solving the system of equations yields $t_2 = 20$ and $t_3 = 30$ and $t_1^* = 9$. (no need to get these numbers to get the full credit)

Final

Thursday, June 16, 2022
1:00–3:00 pm

NAME: ______________________________

Student ID: ______________________________

- Don't forget to put your name and student ID.
- **Record all your solutions in this answer booklet. Only this answer booklet will be considered in the grading of your exam.**
- Be sure to show all relevant work and reasoning. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.

Problem	Your score	Max score
1		15
2		15
3		15
4		15
Total		60

Problem 1 (15 Points)

Consider two discrete random variables X_n and Y_n whose PMFs are defined as

$$p_{X_n}(x) = \begin{cases} 1 - 1/n, & \text{for } x = 0, \\ 1/n, & \text{for } x = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$p_{Y_n}(x) = \begin{cases} 1 - 1/n, & \text{for } x = 0, \\ 1/n, & \text{for } x = n, \\ 0, & \text{otherwise.} \end{cases}$$

a) (2 points) Find the expected value and variance of X_n and Y_n.

Answer:

$\mathbb{E}[X_n] = 1/n, \qquad \text{var}(X_n) = (n-1)/n^2,$

$\mathbb{E}[Y_n] = 1, \qquad \text{var}(Y_n) = n - 1.$

Reasoning for Problem 1(a):

$$\begin{aligned} \mathbb{E}[X_n] &= 0 \cdot (1 - 1/n) + 1 \cdot 1/n = 1/n, \\ \text{var}(X_n) &= (0 - 1/n)^2 \cdot (1 - 1/n) + (1 - 1/n)^2 \cdot (1/n) = (n-1)/n^2, \\ \mathbb{E}[Y_n] &= 0 \cdot (1 - 1/n) + n \cdot 1/n = 1, \\ \text{var}(Y_n) &= (0 - 1)^2 \cdot (1 - 1/n) + (n-1)^2 \cdot (1/n) = n - 1. \end{aligned}$$

b) (5 points) What does the Chebyshev inequality tell us about the convergence of X_n and Y_n?

Answer:

X_n converges to 0 in probability;

We cannot conclude anything about the converge of Y_n through Chebyshev's inequality.

Reasoning for Problem 1(b): Using Chebyshev's inequality, we have

$$\lim_{n\to\infty} \mathbb{P}\left(|X_n - 1/n| \geq \epsilon\right) \leq \lim_{n\to\infty} \frac{n-1}{n^2\epsilon^2} = 0.$$

It follows that X_n converges to 0 in probability.

For Y_n, Chebyshev inequality suggests that

$$\lim_{n\to\infty} \mathbb{P}\left(|Y_n - 1| \geq \epsilon\right) \leq \lim_{n\to\infty} \frac{n-1}{\epsilon^2} = \infty.$$

Thus, we cannot conclude anything about the converge of Y_n through Chebyshev's inequality.

c) (5 points) Is Y_n convergent in probability? If so, to what value? (Hint: A sequence Z_n converges in probability to a number a if for any $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|Z_n - a| \geq \epsilon) = 0$.)

Answer:

Yes, it converges to 0.

Reasoning for Problem 1(c):

For any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|Y_n| \geq \epsilon) \leq \lim_{n\to\infty} \frac{1}{n} = 0,$$

so Y_n converges to zero in probability.

d) (3 points) If a sequence of random variables converges in probability to a value a, does the corresponding sequence of expected values converge to a? Prove or give a counter example.

Answer: A counter example is Y_n. Y_n converges to 0 in probability yet its expectation value is 1 for all n.

Problem 2 (15 Points)

Consider a Bernoulli process $X_1, X_2, X_3, \dots$ with unknown probability of arrival q, i.e., $\mathbb{P}(X_i = 1) = q$ for all i. Define the k-th interarrival time T_k as

$$T_1 = Y_1, \quad T_k = Y_k - Y_{k-1}, \quad k = 2, 3, \dots$$

where Y_k is the time of the k-th arrival. In this problem, we estimate the arrival probability q from observed interarrival times $(t_1, t_2, t_3, \dots)$. Assume q is sampled from the random variable Q which is uniformly distributed over $[0, 1]$.

You may find the following integral useful: For any non-negative integer k and m,

$$\int_0^1 q^k (1-q)^m dq = \frac{k!m!}{(k+m+1)!}.$$

a) (5 points) Find the PMF of T_1.

Answer:

$$p_{T_1}(t) = \frac{1}{(t+1)t} \quad \text{for } t = 1, 2, \dots$$

Reasoning for Problem 2(a): By the total probability theorem, we have

$$p_{T_1}(t) = \int_0^1 p_{T_1|Q}(t,q) f_Q(q) dq = \int_0^1 q(1-q)^{t-1} dq = \frac{1!(t-1)!}{(t+1)!} = \frac{1}{(t+1)t} \quad \text{for } t = 1, 2, \dots$$

b) (5 points) Compute the least mean squares (LMS) estimate of Q conditioned on the first arrival time $T_1 = t_1$.

Answer:

$$\mathbb{E}[Q|T_1 = t_1] = \frac{2}{t_1 + 2}$$

Reasoning for Problem 2(b): The LSE estimate of q conditioned on $T_1 = t_1$ is equal to $\mathbb{E}[Q|T_1 = t_1]$, which can be calculated as

$$\begin{aligned}
\mathbb{E}[Q|T_1 = t_1] &= \int_0^1 f_{Q|T_1}(q|t_1) q dq \\
&= \int_0^1 \frac{p_{T_1|Q}(t_1|q) f_Q(q)}{p_{T_1}(t)} q dq \\
&= \int_0^1 t_1(t_1+1) q(1-q)^{t_1-1} q dq \\
&= t_1(t_1+1) \int_0^1 q^2 (1-q)^{t_1-1} dq = t_1(t_1+1) \frac{2(t_1-1)!}{(t_1+2)!} = \frac{2}{t_1+2}.
\end{aligned}$$

c) (5 points) Compute the maximum a posteriori (MAP) estimate of Q given the first k arrival times, $(T_1, T_2, \ldots, T_k) = (t_1, t_2, \ldots, t_k)$.

Answer:

$$\arg\max_q f_{Q|T_1,\ldots,T_k}(q|t_1, \ldots, t_k) = \frac{k}{\sum_{i=1}^k t_i}$$

Reasoning for Problem 2(c): The posterior probability of Q given $(T_1, T_2, \ldots, T_k) = (t_1, t_2, \ldots, t_k)$ is

$$\begin{aligned} f_{Q|T_1,\ldots,T_k}(q|t_1, \ldots, t_k) &= \frac{f_Q(q) \prod_{i=1}^k p_{T_i|Q}(T_i = t_i|Q = q)}{C} \\ &= \frac{q^k (1-q)^{(\sum_{i=1}^k t_i)-k}}{C} \end{aligned}$$

where $C = \int_0^1 f_Q(q) \prod_{i=1}^k p_{T_i|Q}(T_i = t_i|Q = q) dq$, which does not depend on q. The MAP estimate of Q is the value q that maximizes $f_{Q|T_1,\ldots,T_k}(q|t_1, \ldots, t_k)$. We can find the MAP estimate of Q by finding q that makes the first derivative of $f_{Q|T_1,\ldots,T_k}(q|t_1, \ldots, t_k)$ with respect to q equal to 0, i.e.,

$$kq^{k-1}(1-q)^{(\sum_{i=1}^k t_i)-k} - \left(\sum_{i=1}^k t_i - k \right) q^k (1-q)^{(\sum_{i=1}^k t_i)-k-1} = 0,$$

or equivalently,

$$k(1-q) - \left(\sum_{i=1}^k t_i - k \right) q = 0,$$

which yields the MAP estiamte

$$\hat{q}_{\mathrm{MAP}} = \frac{k}{\sum_{i=1}^k t_i}.$$

Problem 3 (15 Points)

Alice, Bob and Charlie run laps around a track, with the duration of each lap (in hours) being exponentially distributed with parameter $\lambda_A = 21, \lambda_B = 23$ and $\lambda_C = 24$, respectively. Assume that all lap durations are independent.

a) (5 points) Write down the PMF of the total number L of completed laps (3 runners combined) over the first hour.

Answer:

$$p_L(l) = \begin{cases} \frac{68^l e^{-68}}{l!}, & l = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Reasoning for Problem 3(a): Given the problem statement, we can treat Alice, Bob and Charlie's running as 3 independent Poisson processes, where the arrivals correspond to lap completions and the arrival rates indicate the number of laps completed per hour. Since the three processes are independent, we can merge them to create a new process that captures the lap completions of all three runners. This merged process will have arrival rate $\lambda_M = \lambda_A + \lambda_B + \lambda_C = 68$. The total number of completed laps, L, over the first hour is then described by a Poisson PMF with $\lambda_M = 68$ and $\tau = 1$.

b) (5 points) What is the probability that Alice finishes her first lap before any of the others?

Answer:

$$\frac{\lambda_A}{\lambda_A + \lambda_B + \lambda_C} = \frac{21}{68}.$$

Reasoning for Problem 3(b): The event that Alice is the first to finish a lap is the same as the event that the first arrival in the merged process came from Alice's process. This probability is $\frac{\lambda_A}{\lambda_A+\lambda_B+\lambda_C}$.

c) (5 points) Suppose that the runners have been running for a very long time when you arrive at the track. What is the distribution of the duration T of Alice's current lap? (This includes the duration of that lap both before and after the time of your arrival.)

You may use that Erlang distribution for the time Y_k of the k-th arrival in the Poisson process of rate λ is $f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$, $y \geq 0$.

Answer:

$$f_T(t) = \begin{cases} 21^2 t e^{-21t}, & t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Reasoning for Problem 3(c): This is an instance of the random incident paradox, so the duration of Alice's current lap consists of the sum of the duration from the time of your arrival until Alice's next lap completion and the duration from the time of your arrival back to the time of Alice's previous lap completion. This is the sum of 2 independent exponential random variables with parameter $\lambda_A = 21$ (i.e., a second-order Erlang random variable).

Problem 4 (15 Points)

In this problem, we want to have a Markov chain that models the spread of a virus. Assume a population of n individuals. At the beginning of each day (say 7am), each individual is either infected or susceptible (not yet infected but capable of being infected by contacts of infected people). Suppose that each pair of people (i, j), $i \neq j$, independently comes into contact with one another during the daytime (7am to 7pm) with probability p. Whenever a susceptible individual comes into contact with an infected individual, the susceptible individual is infected right away. Assume that during overnight (7pm to 7am next day), any individual who has been infected for at least 24 hours will recover with probability $0 < q < 1$ and return to being susceptible, independently of everything else (i.e., assume that a newly infected individual will spend at least one night without recovery).

a) (5 points) Suppose that there are m infected individuals at one daybreak (7am). The number of total population is n. What is the PMF of the number of new infected individuals N at the end of the daytime (7pm)?

   Answer:

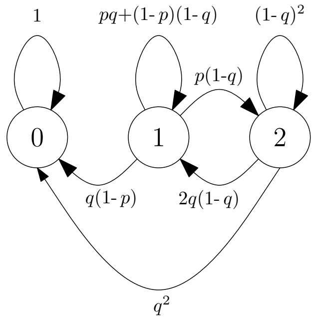
$$p_N(k) = \binom{n-m}{k} r^k (1-r)^{n-m-k}, \quad k = 0, 1, \ldots, n-m.$$

Reasoning for Problem 4(a): If m out of n individuals are infected, there are $n - m$ susceptible individuals. Each of these susceptible individuals will be independently infected during the daytime with probability $r = 1-(1-p)^m$. Thus, the number of new infections N will be binomial random variable with parameters $n - m$ and r, so that

$$p_N(k) = \binom{n-m}{k} r^k (1-r)^{n-m-k}, \quad k = 0, 1, \ldots, n-m.$$

b) (5 points) Suppose that $n = 2$. Draw a Markov chain with states 0,1,2 (each of which indicates the number of infected individuals among the 2 at each daybreak) to model the spread of the virus.

Answer (without reasoning):



c) (5 points) Suppose that $n = 2$ and let's assume that the initial state is $X_0 = 1$ (the number of infected individuals at day 0 is equal to 1). Calculate the mean first passage time to the state 0, i.e., $t_1 = \mathbb{E}[\min\{n \geq 0 \text{ such that } X_n = 0\}|X_0 = 1]$ (the expected number of days to have 0 infected individual for the first time.)

Answer:

$$t_1 = \frac{1-(1-q)^2 + p(1-q)}{(1-pq-(1-p)(1-q))(1-(1-q)^2) - 2pq(1-q)^2}.$$

Reasoning for Problem 4(c): We need to solve the following equations:

$$\begin{aligned} t_1 &= 1 + p_{11}t_1 + p_{12}t_2 \\ t_2 &= 1 + p_{21}t_1 + p_{22}t_2. \end{aligned}$$

We can find that

$$t_1 = \frac{1 - p_{22} + p_{12}}{(1-p_{11})(1-p_{22}) - p_{12}p_{21}}$$

By plugging in the transition probabilities,

$$t_1 = \frac{1-(1-q)^2 + p(1-q)}{(1-pq-(1-p)(1-q))(1-(1-q)^2) - 2pq(1-q)^2}.$$

School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2023 Spring

Final

Tuesday, June 13, 2023
9:00–11:30 am

NAME: ____________________________

Student ID: ____________________________

- Don't forget to put your name and student ID.
- **Record all your solutions in this answer booklet. Only this answer booklet will be considered in the grading of your exam.**
- Be sure to show all relevant work and reasoning. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.

Problem	Your score	Max score
1		10
2		10
3		10
4		10
Total		40

Problem 1 (10 Points)

Consider three random variables Θ, X, and Y, with known variances $\text{var}(\Theta)$, $\text{var}(X)$, and $\text{var}(Y)$, and covariances, $\text{cov}(\Theta, X)$, $\text{cov}(\Theta, Y)$, and $\text{cov}(X, Y)$. Assume that $\mathbb{E}[\Theta] = \mathbb{E}[X] = \mathbb{E}[Y] = 0$, $\text{var}(X) > 0$, $\text{var}(Y) > 0$, and $|\rho(X, Y)| \neq 1$. (Remind that $\rho(A, B) = \text{cov}(A, B)/\sqrt{\text{var}(A)\,\text{var}(B)}$ and for any two zero-mean random variables A, B, $\text{cov}(A, B) = \mathbb{E}[AB]$.)

We consider a linear estimator of Θ based on X and Y, in the form of

$$\hat{\Theta} = aX + bY,$$

for some constants a, b. We aim to choose a, b to minimize the mean squared error $\mathbb{E}[(\Theta - \hat{\Theta})^2]$. Find a and b in terms $\text{var}(\Theta)$, $\text{var}(X)$, $\text{var}(Y)$, $\text{cov}(\Theta, X)$, $\text{cov}(\Theta, Y)$, and $\text{cov}(X, Y)$ for the following two cases.

a) (5 points) Find a and b, when X and Y are uncorrelated, i.e., $\mathbb{E}[XY] = 0$.

Answer:

$$a = \frac{\mathbb{E}[\Theta X]}{\mathbb{E}[X^2]} = \frac{\text{cov}(\Theta, X)}{\text{var}(X)}$$
$$b = \frac{\mathbb{E}[\Theta Y]}{\mathbb{E}[Y^2]} = \frac{\text{cov}(\Theta, Y)}{\text{var}(Y)}$$

Reasoning for Problem 1(a): Note that

$$\mathbb{E}[(\Theta - aX - bY)^2] = \mathbb{E}[\Theta^2] + a^2\mathbb{E}[X^2] + b^2\mathbb{E}[Y^2] - 2a\mathbb{E}[\Theta X] - 2b\mathbb{E}[\Theta Y] + 2ab\mathbb{E}[XY].$$

Assuming that X and Y are uncorrelated, we have $\mathbb{E}[XY] = 0$. We differentiate the expression for the mean squared error with respect to a and b, and set the derivatives to zero to obtain

$$a = \frac{\mathbb{E}[\Theta X]}{\mathbb{E}[X^2]} = \frac{\text{cov}(\Theta, X)}{\text{var}(X)},$$
$$b = \frac{\mathbb{E}[\Theta Y]}{\mathbb{E}[Y^2]} = \frac{\text{cov}(\Theta, Y)}{\text{var}(Y)}.$$

b) (5 points) Find a and b for the general case where X and Y are not necessarily uncorrelated.)

Answer:

$$a = \frac{\operatorname{var}(Y)\operatorname{cov}(\Theta, X) - \operatorname{cov}(\Theta, Y)\operatorname{cov}(X, Y)}{\operatorname{var}(X)\operatorname{var}(Y) - \operatorname{cov}^2(X, Y)}$$
$$b = \frac{\operatorname{var}(X)\operatorname{cov}(\Theta, Y) - \operatorname{cov}(\Theta, X)\operatorname{cov}(X, Y)}{\operatorname{var}(X)\operatorname{var}(Y) - \operatorname{cov}^2(X, Y)}$$

Reasoning for Problem 1(b): If X and Y are uncorrelated, we similarly set the derivatives of the mean squared error to zero We obtain and then solve a system of two linear equations

$$a\mathbb{E}[X^2] + b\mathbb{E}[XY] = \mathbb{E}[\Theta X]$$
$$a\mathbb{E}[XY] + b\mathbb{E}[Y^2] = \mathbb{E}[\Theta Y]$$

in the unknowns a and b, whose solution is

$$a = \frac{\operatorname{var}(Y)\operatorname{cov}(\Theta, X) - \operatorname{cov}(\Theta, Y)\operatorname{cov}(X, Y)}{\operatorname{var}(X)\operatorname{var}(Y) - \operatorname{cov}^2(X, Y)}$$
$$b = \frac{\operatorname{var}(X)\operatorname{cov}(\Theta, Y) - \operatorname{cov}(\Theta, X)\operatorname{cov}(X, Y)}{\operatorname{var}(X)\operatorname{var}(Y) - \operatorname{cov}^2(X, Y)}.$$

<u>Problem 2</u> (10 Points)

Assume that X_i's are independent and identically distributed random variables with mean p. To estimate p, we consider the sample mean defined by

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

a) (5 points) Find the smallest n, the number of samples, for which the Chebyshev inequality yields a guarantee

$$\mathbb{P}(|M_n - p| \geq 0.1) \leq 0.05.$$

Assume that $\mathrm{var}(X_i) = v$ for some constant v. State your answer as a function of v.

Answer:

$$n = 2000v$$

Reasoning for Problem 2(a): Since $\mathbb{E}[M_n] = p$ and $\mathrm{var}(M_n) = \frac{v}{n}$, by Chebyshev inequality,

$$\Pr(|M_n - p| \geq 0.1) \leq \frac{\mathrm{var}(M_n)}{0.1^2} = \frac{v}{n \cdot 0.01} = 0.05.$$

The required n is $80v$.

b) (5 points) Assume that $n = 10,000$. Find an approximate value for the probability

$$\mathbb{P}(|M_{10000} - p| \geq 0.1)$$

using the Central Limit Theorem. Assume again that $\text{var}(X_i) = v$ for some constant v. Give your answer in terms of v, and the standard normal CDF $\Phi(\cdot)$.

Answer:

$$2\left(1 - \Phi\left(\frac{10}{\sqrt{v}}\right)\right)$$

Reasoning for Problem 2(b):

By CLT, we can approximate

$$\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{nv}}$$

by a standard normal distribution when n is large. Hence,

$$\Pr(|M_{10000} - p| \geq 0.5) = \Pr\left(\left|\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{nv}}\right| \geq \frac{0.1\sqrt{n}}{\sqrt{v}}\right) = 2\left(1 - \Phi\left(\frac{10}{\sqrt{v}}\right)\right).$$

Problem 3 **(10 Points)**

In this problem, we consider Poisson processes. Remind that for Poisson process with rate λ, the probability distribution for the first arrival time T_1 (and also the inter-arrival time $T_k = Y_k - Y_{k-1}$, $k \geq 2$, where Y_k is the k-th arrival time) follows the exponential distribution with rate λ, i.e., $f_{T_1}(t) = \lambda e^{-\lambda t}$, for $t \geq 0$ and $\mathbb{E}[T_1] = 1/\lambda$.

a) (5 points) Consider two independent Poisson processes with rates λ_1 and λ_2, respectively. Let X_1 be the first arrival time in the first process, and X_2 be the first arrival time in the second process. Find the expected value of $\max\{X_1, X_2\}$.

Answer:

$$\mathbb{E}[\max\{X_1, X_2\}] = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{1}{\lambda_1}$$

Reasoning for Problem 3(a):

Let $T = \min\{X_1, X_2\}$ be the first time when one of the processes registers an arrival. Let $S = \max\{X_1, X_2\} - T$ be the additional time until both have registered an arrival. Since the merged process is Poisson with rate $\lambda_1 + \lambda_2$, we have

$$\mathbb{E}[T] = \frac{1}{\lambda_1 + \lambda_2}.$$

Concerning S, there are two cases to consider:

(i) The first arrival comes from the first process, which happens with probability $\lambda_1/(\lambda_1 + \lambda_2)$. We then have to wait for an arrival from the second process, which takes $1/\lambda_2$ time on the average.

(ii) The first arrival comes from the second process, which happens with probability $\lambda_2/(\lambda_1 + \lambda_2)$. We then have to wait for an arrival from the first process, which takes $1/\lambda_1$ time on the average.

Putting everything together, we obtain

$$\begin{aligned} E[\max\{X_1, X_2\}] &= \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{1}{\lambda_1} \\ &= \frac{1}{\lambda_1 + \lambda_2}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right). \end{aligned}$$

b) (5 points) Consider two independent Poisson processes with rates λ_1 and λ_2, respectively. Let Y be the first arrival time in the first process and Z be the second arrival time in the second process. Find the expected value of $\max\{Y, Z\}$. (Hint: you may write down your answer in terms of $\mathbb{E}[\max\{X_1, X_2\}]$, defined in (a). You don't need to specify what $\mathbb{E}[\max\{X_1, X_2\}]$ is in terms of λ_1 and/or λ_2.)

Answer:

$$\mathbb{E}[\max\{Y, Z\}] = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{2}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\mathbb{E}[\max\{X_1, X_2\}].$$

Reasoning for Problem 3(b): As in the previous problem, when we define T be the first time when one of the processes registers an arrival, since the merged process is Poisson with rate $\lambda_1 + \lambda_2$, we have $\mathbb{E}[T] = 1/(\lambda_1 + \lambda_2)$.

Again, we need to consider two cases:

(i) The arrival at time T comes from the first process; this happens with probability $\lambda_1/(\lambda_1+\lambda_2)$. In this case, we have to wait an additional time until the second process registers two arrivals, whose expectation is $1/\lambda_2$.

(ii) The first arrival comes from the second process, which happens with probability $\lambda_2/(\lambda_1+\lambda_2)$. In this case, the additional time S we have to wait is the time until each of the two processes registers an arrival. This is the maximum of two independent exponential random variables and, according to the result of (a), we have

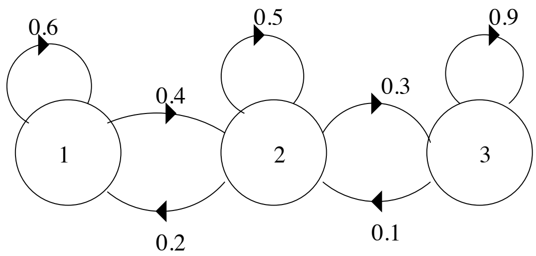
$$\mathbb{E}[S] = \frac{1}{\lambda_1 + \lambda_2}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right).$$

Putting everything together, we obtain

$$\begin{aligned}\mathbb{E}[\max\{Y, Z\}] &= \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{2}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\mathbb{E}[S] \\ &= \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{2}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{1}{\lambda_1 + \lambda_2}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right).\end{aligned}$$

Problem 4 (10 Points)

Consider a Markov chain $\{X_n : n = 0, 1, \ldots, \}$, specified by the following transition diagram.



a) (3 points) Find the steady-state probabilities π_1, π_2, π_3 for the state 1,2, and 3.

Answer:

$$\pi_1 = 1/9 \qquad \pi_2 = 2/9 \qquad \pi_3 = 6/9$$

Reasoning for Problem 4(a): We set up the balance equations of a birth-death process and the normalization equation as such:

$$\begin{aligned} \pi_1 p_{12} &= \pi_2 p_{21} \\ \pi_2 p_{23} &= \pi_3 p_{32} \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

Solving the system of equations yields the following steady-state probabilities:

$$\begin{aligned} \pi_1 &= 1/9, \\ \pi_2 &= 2/9, \\ \pi_3 &= 6/9. \end{aligned}$$

b) (3 points) Let $Y_n = X_n - X_{n-1}$. Thus, $Y_n = 1$ indicates that the n-th transition was to the right, $Y_n = 0$ indicates it was a self-transition, and $Y_n = -1$ indicates it was a transition to the left. Find $\lim_{n\to\infty} \mathbb{P}(Y_n = 1)$.

Answer:

$$\lim_{n\to\infty} \mathbb{P}(Y_n = 1) = 1/9$$

Reasoning for Problem 4(b): Using the total probability theorem and steady-state probabilities,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(Y_n = 1) &= \sum_{i=1}^{3} \pi_i \cdot \mathbb{P}(Y_n = 1 | X_{n-1} = i) \\ &= \pi_1 p_{12} + \pi_2 p_{23} = 1/9.\end{aligned}$$

c) (4 points) Given that the n-th transition was a transition to the right ($Y_n = 1$), find the probability that the previous state was state 1. (You can assume that n is large.)

Answer:

$$\mathbb{P}(X_{n-1} = 1 | Y_n = 1) = 2/5$$

Reasoning for Problem 4(c): Using Bayes' Rule,

$$\begin{aligned}\mathbb{P}(X_{n-1} = 1 | Y_n = 1) &= \frac{\mathbb{P}(X_{n-1} = 1)\mathbb{P}(Y_n = 1 | X_{n-1} = 1)}{\sum_{i=1}^{3} \mathbb{P}(X_{n-1} = i)\mathbb{P}(Y_n = 1 | X_{n-1} = i)} \\ &= \frac{\pi_1 p_{12}}{\pi_1 p_{12} + \pi_2 p_{23}} = 2/5.\end{aligned}$$