Notice: Only the questions with the score written have been scored.

1 [20 points]

$$f(x|\theta) = \frac{3x^2}{\theta^3}(0 \le x \le \theta)$$

(a) $E(\hat{\theta}_1) = 4E(X_1)/3 = \frac{4}{3}\int_0^\theta xf(x|\theta)dx = \theta,\ Bias(\hat{\theta}_1) = 0$
$V(\hat{\theta}_1) = \frac{16}{9n}V(X_1) = \frac{16}{9n}\left[\int_0^\theta x^2 f(x|\theta)dx - (\int_0^\theta xf(x|\theta)dx)^2\right] = \theta^2/(15n),$
$MSE(\hat{\theta}_1) = \theta^2/(15n)$. **(+5 points)**

(b) $\hat{\theta}_2 = X_{(n)} = max_i X_i$**(+5 points)**

(c) $F_{X_{(n)}}(x) = P(X_{(n)} \le x) = \prod_{i=1}^n P(X_i \le x) = (x^3/\theta^3)^n,\ 0 \le x \le \theta$
$f_{X_{(n)}}(x) = 3n * x^{3n-1}/\theta^{3n},\ 0 \le x \le \theta$ **(+5 points)**

(d) $E(X_{(n)}) = 3n\theta/(3n+1),\ Bias(X_{(n)}) = -\theta/(3n+1)$
$V(X_{(n)}) = 3n\theta^2/(3n+2)(3n+1)^2$
$MSE(X_{(n)}) = 2\theta^2/(3n+1)(3n+2)$ **(+5 points)**

2 [20 points]

(a) Note that there is no assumption that population distribution is Normal.

$$\text{Since } \frac{\bar{X}-\mu}{s/\sqrt{n}} \sim N(0,1) \quad \text{by CLT,}$$

95% C.I. for μ:

$$\begin{aligned}&\left(\bar{x} - z_{0.025}\cdot\frac{s}{\sqrt{n}},\ \bar{x} + z_{0.025}\cdot\frac{s}{\sqrt{n}}\right)\\ &= \left(530 - 1.96\cdot\frac{70}{\sqrt{48}},\ 530 + 1.96\cdot\frac{70}{\sqrt{48}}\right)\\ &= (510.20, 549.80)\ \textbf{(+5 points)}\end{aligned}$$

The confidence interval (510.20, 549.80) would cover the population mean μ, the mean reaction time, with 95% certainty in terms of relative frequency. If we use random sample, this confidence interval can be generalized to the population.**(+5 points)**

(b) The sample size we need is 753.**(+5 points)**

$$n = \left(\frac{z_{\alpha/2}\cdot\sigma}{E}\right)^2 = \left(\frac{z_{0.025}\cdot 70}{5}\right)^2 = 752.93\textbf{(+5 points)}$$

3 [20 points]

(a) The sample size we need is 307**(+5 points)**.

$$n = p(1-p)\left(\frac{z_{\alpha/2}}{E}\right)^2 = \frac{1}{2}\cdot\frac{1}{2}\left(\frac{z_{0.04}}{0.05}\right)^2 = 306.49\textbf{(+5 points)}$$

(b) The sample size we need is 295**(+5 points)**.

$$n = p(1-p)\left(\frac{z_{\alpha/2}}{E}\right)^2 = 0.4(1-0.4)\left(\frac{z_{0.04}}{0.05}\right)^2 = 294.23\textbf{(+5 points)}$$

4 [20 points]

The estimated standard deviation is

$$S_p = \sqrt{\frac{(n_1-1)\,s_1^2 + (n_2-1)\,s_2^2}{n_1+n_2-2}}\textbf{(+5 points)} = \sqrt{\frac{11\times 3000^2 + 14\times 4000^2}{12+15-2}} = 3594.4$$

Thus, the 95% confidence interval for the difference of population means is

$$\begin{aligned}&\bar{X}_1 - \bar{X}_2 \pm t_{\alpha/2}\,(n_1+n_2-2)\,S_p\textbf{(+5 points)}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\\ &= 41000 - 42500 \pm 2.06\times 3594.4\sqrt{\frac{1}{12}+\frac{1}{15}}\\ &= (-4367.7, 1367.7)\ \textbf{(+5 points)}\end{aligned}$$

where $t_{\alpha/2}\,(n_1+n_2-2) = t_{0.025}\,(25) = 2.06$.
Note that 0 falls in this interval, so there is no difference of the mean salary between males and females.**(+5 points)**

5 (0.515, 0.723)

The confidence interval would cover the true proportion with 95% certainty in terms of relative frequency.

6 The following can be found in the chapter 7 exercise problems:

11. (a) Since $X_{n+1} \sim N(\mu, 1)$ and $\bar{X}_n \sim N(\mu, 1/n)$, $X_{n+1} - \bar{X}_n \sim N(0, 1 + 1/n)$.
(b) By (a),
$$-z_{0.05} < \frac{X_{n+1} - \bar{X}_n}{\sqrt{1 + 1/n}} < z_{0.05}$$
Thus, $X_{n+1} \in (\bar{X}_n - 1.64\sqrt{1 + 1/n}, \bar{X}_n + 1.64\sqrt{1 + 1/n})$.

17. 95% interval: $\bar{X} \pm t_{0.975}(23) \times \frac{6.9576}{\sqrt{24}} = 333.9958 \pm 2.9379$
99% interval: $\bar{X} \pm t_{0.995}(23) \times \frac{6.9576}{\sqrt{24}} = 333.9958 \pm 3.987$

32. [20 points]
Since $X_{n+1} \sim N(\mu, \sigma^2)$ and $\bar{X}_n \sim N(\mu, \sigma^2/n)$,
$X_{n+1} - \bar{X}_n \sim N(0, \sigma^2(1 + 1/n))$.**(+10 points)**
Thus, $E[(X_{n+1} - \bar{X}_n)^2] = V(X_{n+1} - \bar{X}_n) - [E(X_{n+1} - \bar{X}_n)]^2 = \sigma^2(1 + 1/n)$.
(+10 points)

39. (a) 0.00805
(b) Note that
$$\frac{(n-1)s^2}{\chi_{1-\alpha/2}(n-1)} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi_{\alpha/2}(n-1)}.$$
Then, since $\chi_{0.05}(7) = 2.1673$ and $\chi_{0.95}(7) = 14.0671$, we can obtain $\sigma^2 \in (0.0000323, 0.000209)$. Then, $\sigma \in (0.00568, 0.01447)$.

46. Note that
$$\frac{S_y^2/\sigma_2^2}{S_x^2/\sigma_1^2} \sim F(m-1, n-1).$$
Thus, a 95 percent two-sided confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ is
$$\left(F_{0.025}(m-1, n-1)\frac{S_x^2}{S_y^2}, F_{0.975}(m-1, n-1)\frac{S_x^2}{S_y^2}\right).$$
As $s_1^2 = 0.007486$ and $s_2^2 = 0.0622$, the calculated interval is $(0.02867, 0.5805)$.

61. Let calculate MSE to find the preferred estimator. $MSE[d_1, \theta] = 6 + 0 = 6$ and $MSE[d_2, \theta] = 2 + (2)^2 = 6$. Thus, they have the same efficiency.