- Two Functions, Two Variables

$(X_1, X_2) \sim f_X(x_1, x_2)$

(Y_1, Y_2) : Some functions of (X_1, X_2)

$$\begin{cases} Y_1 = u_1(X_1, X_2) \\ Y_2 = u_2(X_1, X_2) \end{cases} \qquad \begin{aligned} X_1 &= v_1(Y_1, Y_2) \\ X_2 &= v_2(Y_1, Y_2) \end{aligned}$$

pdf of Y_1, Y_2

$$: f_Y(y_1, y_2) = f_X\big(v_1(y_1, y_2), v_2(y_1, y_2)\big) \cdot |J|$$

$$J = \det \begin{pmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{pmatrix}$$

Eg) $X_1, X_2 \sim$ iid $\text{Exp}(1)$

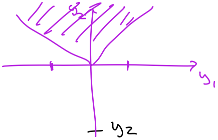
$$f_X(x_1, x_2) = e^{-x_1} e^{-x_2}, \qquad x_1 > 0 \quad x_2 > 0$$

$$\begin{cases} Y_1 = \underline{X_1 - X_2} \\ Y_2 = X_1 + X_2 \end{cases} \Rightarrow \begin{cases} X_1 = \frac{Y_1 + Y_2}{2} \\ X_2 = \frac{Y_2 - Y_1}{2} \end{cases}$$

$$x_1 = \frac{y_1 + y_2}{2}$$

$$x_2 = \frac{y_2 - y_1}{2}$$

$$J = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}$$



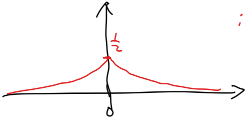
$$f_Y(y_1, y_2) = e^{-\frac{y_1+y_2}{2}} \cdot e^{-\frac{y_2-y_1}{2}} \cdot \frac{1}{2} = \frac{1}{2} e ,$$

$$\left.\begin{array}{l} x_1 > 0 \implies 0 < y_1 + y_2 < \infty \\ x_2 > 0 \implies 0 < y_2 - y_1 < \infty \end{array}\right] \implies \boxed{\begin{array}{l} -y_2 < y_1 < y_2 \\ 0 < y_2 < \infty \end{array}}$$

Find the marginals:

- $f(y_2) = \int_{-y_2}^{y_2} \frac{1}{2} e^{-y_2}\, dy_1 = y_2 e^{-y_2}, \quad y > 0$: Gamma$(2, 1)$

- $$f(y_1) = \begin{cases} \int_{-y_1}^{\infty} \frac{1}{2} e^{-y_2}\, dy_2 = \frac{1}{2} e^{-y_1}, & -\infty < y_1 < 0 \\ \int_{y_1}^{\infty} \frac{1}{2} e^{-y_2}\, dy_2 = \frac{1}{2} e^{-y_1}, & 0 < y_1 < \infty \end{cases}$$



: · Double Exponentials

· Laplace

Eg) $Y_1, Y_2 \sim U(0, 1)$ $\quad 0 < y_1 < 1 \quad 0 < y_2 < 2$

$$\begin{cases} U = Y_1 Y_2 \\ V = Y_2 \end{cases} \qquad \begin{cases} Y_1 = \frac{U}{V} \\ Y_2 = V \end{cases} \qquad \begin{cases} y_1 = \frac{u}{v} \\ y_2 = v \end{cases} \qquad J = \det \begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{pmatrix}$$

$$0 < \frac{u}{v} < 1 \implies 0 < u < v \qquad = \frac{1}{v}$$

$$\boxed{f_{U,V}(u, v) = \frac{1}{v}, \quad 0 < u < v < 1}$$

$$f_U(u) = \int_u^1 \frac{1}{v}\, dv = -\log u. \quad 0 < u < 1.$$

The Method of Moment-Generating Functions

$m_X(t), m_Y(t)$: mgfs of r.v.'s X and Y. If they exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

U: a function of r.v.'s $Y_1, Y_2, \ldots, Y_n$.

1. Find the mgf for U, $m_U(t)$.
2. Compare $m_U(t)$ with other well-known mgfs. If $m_U(t) = m_V(t)$ for all values of t, U and V have identical distribution.

Use this to prove that

- Sum of independent Poisson is Poisson.

$X_i \sim \text{Poisson}(\lambda_i) \quad i = 1, \ldots n$

$X_1 + \cdots + X_n \sim \text{Poisson}(\lambda_1 + \cdots + \lambda_n)$

$$E\left(e^{t(X_1+\cdots+X_n)}\right) = E\left(e^{tX_1}\right)\cdots\cdots E\left(e^{tX_n}\right)$$

$$= \exp\left(e^{\lambda_1(t-1)}\right)\cdots\cdots\exp\left(e^{\lambda_n(t-1)}\right)$$

$$= \exp\left(e^{(\Sigma\lambda_i)(t-1)}\right) \quad : \text{Poisson}(\Sigma\lambda_i)$$

DIY

- Sum of $\underbrace{\text{Binomials}}_{\text{Indep}}$ with same p. is Binomial
- Sum of Indep. Gamma with Same β is Gamma.

: $Z \sim N(0,1)$

$Y = Z^2$

$$E(e^{tY}) = \int_{-\infty}^{\infty} e^{tZ^2}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}Z^2}\,dz = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{1}{2}+t\right)Z^2}\,dz$$

...

DIY

need $t < \frac{1}{2}$

: a part of $N(0, \ldots)$

$$= (1-2t)^{-\frac{1}{2}}$$

$\therefore$ Gamma$\left(\frac{1}{2}, 2\right) = \chi^2(1)$

Super useful for normal-related problems.

$$Y_1 \ldots Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$☆ \quad \bar{Y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\because E(e^{t\bar{Y}}) = E\left(e^{\frac{1}{n}t(Y_1 + \cdots + Y_n)}\right)$$

$$= \underbrace{E\left(e^{\overbrace{\frac{1}{n}t}^{t'}Y_1}\right)}_{\exp\left(\mu t' + \frac{1}{2}\sigma^2 t'^2\right)} \cdots\cdots E\left(e^{\frac{1}{n}tY_n}\right)$$

$$= \left(\exp\left(\frac{1}{n}\mu t + \frac{1}{2}\frac{\sigma^2}{n^2}t^2\right)\right)^n = \exp\left(\mu t + \frac{1}{2}\frac{\sigma^2}{n}t^2\right)$$

- $Y_i \sim N(\mu_i, \sigma_i^2)$ $i=1, \dots n.$ indep.

Show that using mgf method

☆☆ $$\Sigma a_i Y_i \sim N(\Sigma a_i \mu_i, \ \Sigma a_i^2 \sigma_i^2)$$

e.g

$$\left\{ \begin{array}{l} X_1 \dots X_n \overset{iid}{\sim} N(\mu_1, \sigma_1^2) \\ Y_1 \dots Y_m \overset{iid}{\sim} N(\mu_2, \sigma_1^2) \end{array} \right) \text{ indep}$$

Check!! $$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \ \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right).$$