IE241 midterm Practice Solution (odd number)

1. $\frac{5!}{5^5}$ $\Rightarrow$ easy to cheak

3. $P(A|B) = \frac{P(A\cap B)}{P(B)}$, ① $P(A\cap B) = P(A|B)P(B) = 0.1$.

$P(A\cap B^c) = P(A) - P(A\cap B) = 0.3$, ② $P(A) = 0.4$

$\therefore P(A\cup B) = P(A) + P(B) - P(A\cap B) = 0.4 + 0.2 - 0.1 = 0.5$

5. (a) $f(y) = P(Y=y) = \frac{y}{21}$

(b) $P(Y<3 \mid Y \text{ is odd}) = \frac{P(\{Y<3\}\cap\{Y \text{ is odd}\})}{P(Y \text{ is odd})} = \frac{P(Y=1)}{P(Y=1,3,5)} = \frac{1/21}{9/21} = \frac{1}{9}$

(c) $P(A)\cdot P(B) = \frac{9}{21}\cdot\frac{9}{21}$, $P(A\cap B) = P(Y=1) = \frac{1}{21}$

$\Rightarrow$ $P(A)P(B) \neq P(A\cap B)$. Hence A, B are not independent!

(d) $E(Y) = \sum_{y=1}^{6} y\cdot\frac{y}{21} = \frac{1}{21}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{21} \simeq 4.33$

(e) $V(Y) = E(Y^2) - \{E(Y)\}^2$. $E(Y^2) = \sum_{y=1}^{6} y^2\cdot\frac{y}{21} = 21$. $V(Y) = 21 - (4.33)^2 \simeq 2.25$

7. (a) $f(y) = \frac{(8-y)(7-y)(6-y)}{420}$, $y = \{1,2,3,4,5\}$

↳ additional explanation.

① when 4 out of 8 are selected, the worst rank among the selected tires is 5

$\Rightarrow$ support of $y = \{1,2,3,4,5\}$

② $P(Y=y) = \frac{{}_{(8-y)}C_3 \times 1}{{}_8C_4}$ $\Rightarrow$ when Choosing 3 tires lower than rank i X select ith rank tire

$\Rightarrow$ 4 out of 8 are selected

$\therefore P(Y=y) = \frac{(8-y)!}{3!\,(5-y)!} \times \frac{1}{{}_8C_4} = \frac{(8-y)(7-y)(6-y)}{420}$

(b) $E(Y(Y+2)+1) = E((Y+1)^2) = \sum_{i=1}^{5} (y+1)^2\cdot\frac{(8-y)(7-y)(6-y)}{420} \simeq 26.29$

9. This Probabilities follows $J \sim B(32, \frac{1}{80})$. Using pdf of binomial dist

$\Rightarrow \binom{32}{j}\cdot\left(\frac{1}{80}\right)^{j}\cdot\left(\frac{79}{80}\right)^{32-j}$

11. List 1 = { 5 womens , 2 mens }
List 2 = { 2 womens , 6 mens } . we need to know

$P(\underbrace{\text{selected women from list 1}}_{\text{called } B} \mid \underbrace{\text{selected man from argumented list 2}}_{\text{called } A})$. $P(A) = \left(\frac{5}{7}\times\frac{6}{9}\right) + \left(\frac{2}{7}\times\frac{7}{9}\right) = \frac{44}{63}$

$P(A\cap B) = \frac{5}{7}\times\frac{6}{9}$. Hence, $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{30}{44} = \frac{15}{22}$

13. (a) $Y = 0, 1, 2, 3$

(b) $P(Y=0) = (0.4)^3 = 0.064$

$P(Y=1) = \binom{3}{1}\cdot(0.6)\cdot(0.4)^3 \simeq 0.115$

$P(Y=2) = \binom{4}{2}\cdot(0.6)^2\cdot(0.4)^3 \simeq 0.138$

$P(Y=3) = (0.6)^3 + \binom{3}{1}\cdot(0.6)^3\cdot(0.4) + \binom{4}{2}\cdot(0.6)^3\cdot(0.4)^2 \simeq 0.683$

(c) $\mu = \sum_{y=0}^{3} y\cdot P(Y=y)$, $\sigma = \sqrt{Var(Y)}$, $Var(Y) = E(y^2) - [E(y)]^2 = \sum_{y=0}^{3} y^2\cdot P(Y=y) - \left[\sum_{y=0}^{3} y\cdot P(Y=y)\right]^2$

15. (a) $(0.9)\times(0.1)$

(b) $\binom{4}{2}\cdot(0.9)^3\cdot(0.1)^2 + \binom{3}{1}\cdot(0.9)^3\cdot(0.1) + (0.9)^3$

17. (a) $M_y(t) = E(e^{ty}) = \sum_{y=0}^{\infty} e^{ty}\cdot\frac{3^y\cdot e^{-3}}{y!} = e^{(3e^t-3)}\cdot\sum_{y=0}^{\infty}\cdot\frac{(3\cdot e^t)^y\cdot e^{-3e^t}}{y!} = e^{3(e^t-1)}$

pdf of poisson$(3e^t)$

(b) $M_y'(0) = \mu = 3\cdot e^0\cdot e^{3\cdot 0} = 3$

19. (a) $P(Y=y) = p\cdot(1-p)^{y-1}$

infinite geometric series

$P(Y=1,3,5\cdots) = p\cdot(1-p)^0 + p\cdot(1-p)^2 + p\cdot(1-p)^4\cdots = \frac{p}{1-(1-p)^2} = \frac{p}{1-q^2}$ (similar to 3.80 in HW3)

(b) $E\left(\frac{Y}{n}\right) = \sum_{y=0}^{n}\frac{y}{n}\cdot\frac{n!}{y!(n-y)!}\cdot p^y\cdot(1-p)^{n-y} = \sum_{y=1}^{n}\frac{y}{n}\cdot\frac{n!}{y!\cdot(n-y)!}\cdot p^y\cdot(1-p)^{n-y}$

$B(n-1, p)$

$= p\sum_{y=1}^{n}\frac{(n-1)!}{(y-1)!(n-y)!}\cdot p^{y-1}\cdot(1-p)^{n-y} = p$

or you can use $E\left(\frac{Y}{n}\right) = \frac{1}{n}\cdot E(Y) = p$ (need to show $E(Y) = np$)

$E\left(\left(\frac{Y}{n}-p\right)^2\right) = \sum_{y=0}^{n}\left(\frac{y}{n}-p\right)^2\cdot\frac{n!}{y!(n-y)!}\cdot p^y\cdot(1-p)^{n-y}$

$= \sum_{y=0}^{n}\frac{y^2}{n^2}\cdot\frac{n!}{y!(n-y)!}\cdot p^y q^{n-y} + -2\sum_{y=0}^{n} p\cdot\underbrace{\frac{y}{n}\cdot\frac{n!}{y!(n-y)!}\cdot p^y\cdot(1-p)^{n-y}}_{=p} + \sum_{y=0}^{n} p^2\cdot\underbrace{\frac{n!}{y!(n-y)!}\cdot p^y\cdot(1-p)^{n-y}}_{1}$

$= \sum_{y=0}^{n}\frac{y^2}{n^2}\cdot\frac{n!}{y!(n-y)!}\cdot p^y q^{n-y} - p^2$

decompose $\frac{1}{n} + \frac{y-1}{n}$

$$= \sum_{y=1}^{n} \left(\frac{y}{n}\right) \cdot \frac{(n-1)!}{(y-1)!(n-y)!} \cdot p^y \cdot q^{n-y} - p^2$$

$$= p \cdot \sum_{y=1}^{n} \left(\frac{1}{n}\right) \cdot \frac{(n-1)!}{(y-1)!(n-y)!} \cdot p^{y-1} \cdot q^{n-y} + \left(\frac{n-1}{n}\right) p^2 \sum_{y=2}^{n} \frac{(n-2)!}{(y-2)! \cdot (n-y)!} \cdot p^{y-2} \cdot q^{n-y} - p^2$$

$$= \frac{p}{n} + \left(\frac{n-1}{n}\right) \cdot p^2 - p^2 = \frac{p - p^2}{n} = \frac{p(1-p)}{n}$$

21. (a) $M(t) = E(e^{tY}) = \sum_{y=1}^{\infty} e^{ty} \cdot \left(\frac{1}{2}\right)^y = \sum_{y=1}^{\infty} \left(\frac{e^t}{2}\right)^y = \frac{\frac{e^t}{2}}{1 - \frac{e^t}{2}} = \frac{e^t}{2-e^t}$

(b) $E(Y) = M'(0) = \frac{e^0(2-e^0) + e^0(2-e^0)}{(2-e^0)^2} = 2$

23. $E(Y) = \int_0^b y \cdot f(y)\,dy = bF(b) - \int_0^b F(y)\,dy = b - \int_0^b F(y)\,dy = \int_0^b [1-F(y)]\,dy$

25. $Y \sim \exp(\lambda)$, $f(y) = \lambda \cdot e^{-\lambda y}$,

(a) $P(Y > a+b \mid Y > a) = \frac{P(Y>a+b, Y>a)}{P(Y>a)} = \frac{P(Y>a+b)}{P(Y>a)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b} = P(Y>b)$

$\Rightarrow$ called 'memoryless property'

(b) $M_Y(t) = \int_0^\infty e^{ty} \cdot \lambda e^{-\lambda y}\,dy = \int_0^\infty \lambda \cdot e^{(t-\lambda)y}\,dy$ (finite only $t<\lambda$)

$$= \frac{\lambda}{t-\lambda} \cdot \left[e^{(t-\lambda)y}\right]_0^\infty = \frac{\lambda}{\lambda - t}$$

27. (a) Let $A = \{g(Y) \geq k\}$, $A^c = \{g(Y) < k\}$

$$E(g(Y)) = \int_A g(y) \cdot f(y)\,dy + \int_{A^c} g(y) \cdot f(y)\,dy$$

$$\geq \int_A g(y) \cdot f(y)\,dy \geq \int_A k\,f(y)\,dy = k \cdot P(g(Y) \geq k)$$

$\therefore E(g(Y)) \geq k \cdot P(g(Y) \geq k) \iff P(g(Y) \geq k) \leq \frac{E(g(Y))}{k}$ (Markov's inequality)

(b) Using (a), $P((Y-\mu)^2 \geq \varepsilon^2) \leq \frac{E((Y-\mu)^2)}{\varepsilon^2} \iff P(|Y-\mu| \geq \varepsilon) \leq \frac{Var(Y)}{\varepsilon^2}$

29. $\text{cost}(C(y)) = 100 \times \left(\frac{1}{2}\right)^y$ $\quad f(y) = \frac{e^{-4} \cdot 4^y}{y!}$

$E(C(y)) = \sum_{y=0}^{\infty} C(y) \cdot f(y)$, $f(y)$ follows Poisson(4)

$E(C(y)) = \sum_{y=0}^{\infty} 100 \times \left(\frac{1}{2}\right)^y \frac{e^{-4} \cdot 4^y}{y!} = 100 e^{-2} \sum_{y=0}^{\infty} \frac{e^{-2} \cdot 2^y}{y!} = \frac{100}{e^2}$

31. (a) Let $X \sim B(n,p)$

$E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \cdot \frac{n!}{x!(n-x)!} \cdot p^x \cdot (1-p)^{n-x}$

$= \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} (pe^t)^x \cdot (1-p)^{n-x}$

$= (1+p(e^t-1))^n$. If $p=0.8$, $n=5$ then exactly same to mgf of Y

$\therefore Y \sim B(5, 0.8)$, easy to check $E(Y)=4$, $Var(Y)=0.8$,

$P(\mu-2\sigma < Y < \mu+2\sigma) = P(4-2\cdot\sqrt{0.8} < Y < 4+2\cdot\sqrt{0.8}) = P(Y=3)+P(Y=4)+P(Y=5)$

$\simeq 0.942$

(b) $Y \sim$ Poisson(4) (see 17.(a)), $E(Y)=4$, $Var(Y)=4=\sigma^2$

$P(\mu-2\sigma < Y < \mu+2\sigma) = P(0<Y<8) = \sum_{i=1}^{7} P(Y=i) \simeq 0.931$

(c) $P(|Y-\mu| < 2\sigma) = 1 - P(|Y-\mu| \geq 2\sigma)$

by 27-(b), $P(|Y-\mu| \geq 2\sigma) \leq \frac{Var(Y)}{4\sigma^2} = \frac{1}{4}$ $\quad (Var(Y) = \sigma^2)$

$P(|Y-\mu|<2\sigma) = 1 - P(|Y-\mu| \geq 2\sigma) \geq 1-\frac{1}{4} = 0.75$

$\therefore$ lower bound is 0.75

33. (a) $Y \sim B(5, 0.8)$ (see 31.(a))

$m_Y(t) = E(e^{ty}) = \sum \left(1 + ty + \frac{(ty)^2}{2!} + \cdots\right) f(y)$

$= 1 + t \cdot E(y) + \frac{t^2}{2!} \cdot E(y^2) + \cdots$

$\mu_2' = m''(0) = E(y^2) = Var(y) + E(y)^2 = np(1-p) + n^2p^2 = np(1-p+np)$

(b) $m_Y(t) = e^{3t+8t^2}$ then $Y \sim N(3, 4^2)$

$P(-1<Y<9) = P\left(\frac{-1-3}{4} < z < \frac{9-3}{4}\right) = P(-1 < z < 1.5) \simeq 0.77$

(c) We need to calculate $E(A) = \pi E(Y^2)$

$m_Y(t) = \frac{e^{2t}-1}{2t}$, $m_Y'(t) = \frac{2e^{2t}\cdot 2t - 2(e^{2t}-1)}{(2t)^2} = \frac{2(e^{2t}-1)+1}{2t^2}$

$m_Y''(0) = \lim_{h\to 0} \frac{m_Y'(h) - m_Y'(-h)}{2h} = \frac{(2h-1)e^{2h} + (2h+1)e^{-2h}}{4h^3} = \frac{4}{3}$ (by L'Hopital's rule)

$\therefore E(A) = \pi E(Y^2) = \frac{4}{3}\pi$

35. (a) $E(Y) = \sum_{y=0}^{2} y \cdot P(Y=y) = 0 + \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$

$E(Y^2) = \sum_{y=0}^{2} y^2 \cdot P(Y=y) = 0 + \frac{3}{8} + \frac{4}{8} = \frac{7}{8}$, $Var(Y) = \{E(Y)\}^2 + E(Y^2) = \left(\frac{5}{8}\right)^2 + \frac{7}{8}$

(b) $m(t) = E(e^{tY}) = \sum_{y=0}^{2} e^{ty} \cdot P(Y=y) = \frac{1}{2} + \frac{3}{8}\cdot e^{t} + \frac{1}{8}\cdot e^{2t}$

(c) $m'(0) = \frac{3}{8} + \frac{1}{4} = \frac{5}{8} = E(Y)$, $m''(0) = \frac{3}{8} + \frac{1}{2} = \frac{7}{8} = E(Y^2)$

37. $\mu = \int_0^\infty y f(y)\,dy = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty y \cdot e^{-\frac{y^2}{2}}\,dy$. Let $\frac{y^2}{2} = z$ then $y \cdot dy = dz$

$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-z}\,dz = \sqrt{\frac{2}{\pi}}$

Let $z = \frac{y^2}{2}$

$\int_0^\infty y^2 \cdot f(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty y^2 \cdot e^{-\frac{y^2}{2}}\,dy = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{2} \cdot z^{\frac{1}{2}} \cdot e^{-z}\,dz = \frac{2}{\sqrt{\pi}} \cdot \Gamma\left(\frac{3}{2}\right)$

$\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

$= 1 = E(Y^2)$

$V(Y) = E(Y^2) - E^2(Y) = 1 - \frac{2}{\pi}$

39. $P(Y \geq 400) = P\left(Z \geq \frac{400-\mu}{25}\right) = 0.1$. using z-table,

$\frac{400-\mu}{25} \simeq 1.28$, $\therefore \mu \simeq 368$

41. (a) $Y \sim geo\left(\frac{5}{6}\right)$, $f(y) = \left(\frac{1}{6}\right)^{y-1} \cdot \frac{5}{6}$

$P(Y \geq 3) = \frac{1}{36}$ (easy to check)

(b) $E(2^{Y-1}) = \sum_{y=1}^{\infty} 2^{y-1} \left(\frac{1}{6}\right)^{y-1} \cdot \frac{5}{6} = \frac{5}{4} \cdot \sum_{y=1}^{\infty} \left(\frac{1}{3}\right)^{y-1} \cdot \frac{2}{3} = \frac{5}{4}$

43. (a) $m(t) = \int_{-\infty}^{\infty} e^{ty}\cdot\left(\frac{1}{2}\right)\cdot e^{-|y|}\,dy = \int_{0}^{\infty} e^{ty}\cdot\frac{1}{2}\cdot e^{-y}\,dy + \int_{-\infty}^{0} e^{ty}\cdot\frac{1}{2}\cdot e^{y}\,dy$

$= \frac{1}{2}\int_{0}^{\infty} \underbrace{e^{y(t-1)}}_{\text{finite } t<1}\,dy + \frac{1}{2}\int_{-\infty}^{0} \underbrace{e^{y(t+1)}}_{\text{finite } t>-1}\,dy = \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t}\right)$

$\therefore m_Y(t) = \frac{1}{1-t^2} \quad (-1<t<1)$

(b) $m(t) = \frac{1}{1-t^2}$, $m'(t) = \frac{2t}{(1-t^2)^2}$ Using it, $E(Y) = m'(0) = 0$

$m''(t) = \frac{2(\cancel{1-t^2}) + 4t^2\cdot 2(\cancel{1-t^2})}{(1-t^2)^{\cancel{4}\,3}} = \frac{8t^2+2}{(1-t^2)^3}$, $E(Y^2) = m''(0) = 2$

$\therefore \operatorname{var}(Y) = E(Y^2) - [E(Y)]^2 = 2$

45. (a) $m_Y(t) = E(e^{ty}) = \Sigma\left(1 + (ty) + \frac{(ty)^2}{2!} + \cdots\right) f(y)$ (33-(a))

$= 1 + t\cdot E(y) + \frac{t^2}{2!}\cdot E(y^2) + \cdots$

$m_Y(t) = 1 + (t^1\cdot 2!\cdot 2^1) + \left(\frac{t^2}{2!}\cdot 3!\cdot 2^2\right) + \left(\frac{t^3}{3!}\cdot 4!\cdot 2^3\right)\cdots +$

$= 1 + 2\cdot(2t) + 3\cdot(2t)^2 + 4\cdot(2t)^3 + \cdots + (m+1)\cdot(2t)^m$

(b) Pdf of Gamma(α,β) : $f(y) = \frac{y^{\alpha-1}\cdot e^{-\frac{y}{\beta}}}{\beta^{\alpha}\cdot\Gamma(\alpha)}$, $0 \le y < \infty$

$E(e^{ty}) = \int_{0}^{\infty} e^{ty}\cdot\frac{y^{\alpha-1}\cdot e^{-\frac{y}{\beta}}}{\beta^{\alpha}\cdot\Gamma(\alpha)} = \frac{1}{\beta^{\alpha}\cdot\Gamma(\alpha)}\int_{0}^{\infty} y^{\alpha-1}\cdot e^{(t-\frac{1}{\beta})y}\,dy$

if $\frac{1}{\beta} > t$ Let $z = \left(\frac{1}{\beta} - t\right)y$ then

$E(e^{ty}) = \frac{1}{\beta^{\alpha}\cdot\Gamma(\alpha)}\int_{0}^{\infty} \underbrace{\left(\frac{z\beta}{1-\beta t}\right)^{\alpha-1}\cdot e^{-z}\left(\frac{\beta}{1-\beta t}\right)}_{z^{\alpha-1}\cdot\beta^{\alpha}\cdot\left(\frac{1}{1-\beta t}\right)^{\alpha}\cdot e^{-z}} dz = \frac{(1-\beta t)^{-\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty} z^{\alpha-1}\cdot e^{-z}\,dz$

and gamma function $\Gamma(\alpha) = \int_{0}^{\infty} z^{\alpha-1}\cdot e^{-z}\,dz$

$\therefore m_Y(t) = (1-\beta t)^{-\alpha}$, $m'_Y(t) = \alpha\beta\cdot(1-\beta t)^{-\alpha-1}$. Then

$m_Y'(0) = \alpha\beta$, $m_Y''(0) = \alpha(\alpha+1)\beta^2$, $m_Y'''(0) = \alpha(\alpha+1)(\alpha+2)\beta^3$

$\Leftrightarrow \alpha = 2$, $\beta = 2$

47. (a) out of midterm range

(b) $P(|Y-\mu|<\varepsilon) \geq 1 - \frac{Var(Y)}{\varepsilon^2}$ (by Tchebysheff's inequality)

$Y \sim \chi^2_5$, then $\mu=5$, $\sigma^2=10$

$P(|Y-5|<\varepsilon) \geq 1 - \frac{10}{\varepsilon^2} = 0.9$, $\varepsilon = 10$

$\therefore P(-5<Y<15) = P(0<Y<15) \geq 0.9$

49 $Y|P \sim B(10, P)$, $P \sim Beta(1,4)$

$E(Y) = E(E(Y|P)) = E(10p) = 10 \times E(P) = 2$

$Var(Y) = E(Var(Y|P)) + Var(E(Y|P)) = E(10p(1-p)) + Var(10p)$

$= 10[E(P) - \{Var(P) + (E(P))^2\}] + 100 Var(P)$

$= 10[\frac{1}{5} - (\frac{2}{75} + \frac{1}{25})] + \frac{8}{3} = 4$

51. $Y \sim unif(0,1)$, $X|Y \sim unif(0,Y)$

(a) $f_{X,Y}(x,y) = f(x|y) \cdot f(y) = \frac{1}{y} \cdot 1$, $0 \leq y \leq 1$, $0 \leq x \leq y$

(b) $f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy = -\log^x$ $(0 < x \leq 1)$