School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2021 Spring

Final

Tuesday, June 15, 2021
9:00–11:20 am

- Be sure to **show all relevant work and reasoning** in your answer sheet. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.
- Please be clear in writing—we can't grade what we can't decipher!
- Don't forget to upload your answer sheet during 11:10-11:20 am through KLMS. The system will be automatically closed at that time. If the system does not work, you should email it to `ee210b_21spring@kaist.ac.kr` by 11:20 am. Late submissions will not be accepted/graded.

<u>Problem 1</u> (10 Points)

Assume that X_i's are independent and identically distributed random variables with mean p. To estimate p, we consider the sample mean defined by

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

a) (5 points) Find the smallest n, the number of samples, for which the Chebyshev inequality yields a guarantee

$$\mathbb{P}(|M_n - p| \geq 0.5) \leq 0.05. \tag{1}$$

Assume that $\text{var}(X_i) = v$ for some constant v. State your answer as a function of v.

b) (5 points) Assume that $n = 10000$. Find an approximate value for the probability

$$\mathbb{P}(|M_{10000} - p| \geq 0.5) \tag{2}$$

using the Central Limit Theorem. Assume again that $\text{var}(X_i) = v$ for some constant v. Give your answer in terms of v, and the standard normal CDF $\Phi(\cdot)$.

<u>Problem 2</u> (10 Points)

Consider a biased coin where the coin lands with head with probability equal to $q \in [0, 1]$. The probability of head, q, is sampled from a random variable Q with PDF

$$f_Q(q) = \begin{cases} 6q(1-q), & 0 \leq q \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

and once it is sampled the value of Q is fixed during the experiments. We flip the coin n times and count the number of heads, K, which is a random variable. Given $K = k$, derive the following estimates of Q:

a) (5 points) Find the MAP estimator, $\hat{q}_{\mathsf{MAP}} = \arg\max_q f_{Q|K}(q|k)$ where $f_{Q|K}(q|k)$ is the conditional PDF of Q given $K = k$.

b) (5 points) Find the least mean square estimator, $\hat{q}_{\mathsf{LMS}} = \mathbb{E}[Q|K = k]$.

You may use the identity

$$\int_0^1 p^l (1-p)^{m-l} dp = \frac{l!(m-l)!}{(m+1)!} \text{ for } 0 \leq l \leq m,$$

and the fact that

$$\int_0^1 \frac{(i+j-1)!}{(i-1)!(j-1)!} p^i (1-p)^{j-1} dp = \frac{i}{i+j}.$$

<u>Problem 3</u> (15 Points)

We conduct an elementary experiment (e.g. some physical experiment) independently total N times, where N is a Poisson random variable of mean λ, i.e., $\mathbb{P}(N = n) = \frac{\lambda^n e^{-\lambda}}{n!}$. The outcome of each experiment is sampled from a set $\{a_1, \ldots, a_K\}$, where the probability of getting an outcome a_k is equal to p_k for $1 \le k \le K$ where $\sum_{k=1}^{K} p_k = 1$.

a) (3 points) Let N_k denote the number of experiments performed for which the outcome is equal to a_k. Find the PMF for N_k $(1 \le k \le K)$. (Hint: no calculation is necessary.)

b) (3 points) Find the PMF of $N_1 + N_2$.

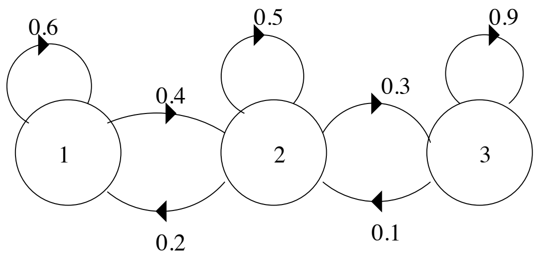
c) (3 points) Find the conditional PMF for N_1 given that $N = n$.

d) (3 points) Find the conditional PMF for $N_1 + N_2$ given that $N = n$.

e) (3 points) Find the conditional PMF for N given that $N_1 = n_1$.

Problem 4 (15 Points)

Consider a Markov chain $\{X_n : n = 0, 1, \ldots, \}$, specified by the following transition diagram.



a) (3 points) Given that the chain starts with $X_0 = 1$, find the probability that $X_2 = 2$.

b) (3 points) Find the steady-state probabilities π_1, π_2, π_3 for the state 1,2, and 3.

c) (3 points) Let $Y_n = X_n - X_{n-1}$. Thus, $Y_n = 1$ indicates that the n-th transition was to the right, $Y_n = 0$ indicates it was a self-transition, and $Y_n = -1$ indicates it was a transition to the left. Find $\lim_{n\to\infty} \mathbb{P}(Y_n = 1)$.

d) (3 points) Given that the n-th transition was a transition to the right ($Y_n = 1$), find the probability that the previous state was state 1. (You can assume that n is large.)

e) (3 points) Suppose that $X_0 = 1$. Let T be defined as the first positive time at which the state is again equal to 1. Show how to find $\mathbb{E}[T]$. (It is enough to write down whatever equations need to be solved; you do not need to actually solve it to produce a numerical answer.)