HW 7 solution

6.10 The total time from arrival to completion of service at a fast-food outlet, Y_1, and the time spent waiting in line before arriving at the service window, Y_2, were given in Exercise 5.15 with joint density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \le y_2 \le y_1 < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window. Find

a the probability density function for U.
b $E(U)$ and $V(U)$. Compare your answers with the results of Exercise 5.108.

a. $F(u) = P(Y_1 - Y_2 \le u) = \int_0^{\infty}\int_{y_2}^{u+y_2} f(y_1, y_2)\, dy_1\, dy_2 = 1 - e^{-u}$

$\therefore F'(u) = f(u) = e^{-u} \quad (u \ge 0)$

b. $E(U) = \int_0^{\infty} u \cdot e^{-u}\, du = 1$

$V(U) = \int_0^{\infty} u^2 \cdot e^{-u}\, du - E(U)^2 = 1$

$\rightarrow U \sim \exp(1)$ then $E(U) = 1$, $V(U) = 1$

6.12 Suppose that Y has a gamma distribution with parameters α and β and that $c > 0$ is a constant.

a Derive the density function of $U = cY$.
b Identify the density of U as one of the types we studied in Chapter 4. Be sure to identify any parameter values.
c The parameters α and β of a gamma-distributed random variable are, respectively, "shape" and "scale" parameters. How do the scale and shape parameters for U compare to those for Y?

(a), (b)

$f_Y(y) = \frac{1}{\Gamma(\alpha)\cdot \beta^{\alpha}} \cdot y^{\alpha-1} \cdot e^{-y/\beta}$

$F_U(U) = P(cY \le u) = P(Y \le u/c)$, $f(u) = F'(u/c) = \frac{1}{c} f_Y\left(\frac{u}{c}\right) = \frac{1}{c} \cdot \frac{1}{\Gamma(\alpha)\cdot \beta^{\alpha}} \cdot \left(\frac{u}{c}\right)^{\alpha-1} \cdot \exp\left(-\frac{u}{c\beta}\right)$

$\therefore f_U(u) = \frac{1}{\Gamma(\alpha)(c\beta)^{\alpha}} \cdot u^{\alpha-1} \cdot \exp\left(-\frac{u}{c\beta}\right) \quad (u \ge 0) \sim \text{gamma}(\alpha, c\beta)$

(c) The shape parameter is the same and the scale parameter is c times beta

6.17 A member of the power family of distributions has a distribution function given by

(5pts)

$$F(y) = \begin{cases} 0, & y < 0, \\ \left(\frac{y}{\theta}\right)^{\alpha}, & 0 \le y \le \theta, \\ 1, & y > \theta, \end{cases}$$

where $\alpha, \theta > 0$.

a Find the density function.
b For fixed values of α and θ, find a transformation $G(U)$ so that $G(U)$ has a distribution function of F when U possesses a uniform (0, 1) distribution.
c Given that a random sample of size 5 from a uniform distribution on the interval (0, 1) yielded the values .2700, .6901, .1413, .1523, and .3609, use the transformation derived in part (b) to give values associated with a random variable with a power family distribution with $\alpha = 2, \theta = 4$.

a. $f(y) = \frac{\alpha}{\theta} \cdot \left(\frac{y}{\theta}\right)^{\alpha-1}$ $(0 \le y \le \theta)$ → easy to cheak

b. $Y = G(U)$, $U \sim \text{unif}(0,1)$ → $f_U(u) = 1$

$f_Y(y) = \underbrace{f_U(G^{-1}(y))}_{=1} \cdot \left|\frac{d}{dy} G^{-1}(y)\right| = \frac{\alpha}{\theta^\alpha} \cdot y^{\alpha-1}$ (result of (a)), $0 \le y \le \theta$, $0 \le u \le 1$

G

$= \frac{d}{dy} G^{-1}(y) = \frac{\alpha}{\theta^\alpha} \cdot y^{\alpha-1}$

$\therefore G^{-1}(y) = y^\alpha / \theta^\alpha = u \iff Y = G(U) = \theta \cdot U^{\frac{1}{\alpha}}$

c. $Y = 4\sqrt{U}$, Calculate by yourself.

6.28 Let Y have a uniform $(0, 1)$ distribution. Show that $U = -2\ln(Y)$ has an exponential distribution with mean 2.

$F_U(u) = P(-2\ln(Y) \le u) = P(Y \ge e^{-\frac{1}{2}u}) = 1 - F_Y(e^{-\frac{1}{2}u})$

$\therefore f_U(u) = (F_U(u))' = \frac{1}{2} e^{-\frac{1}{2}u}$ $(u \ge 0)$. Pdf of exponential dist with beta $= 2$

$\therefore E(U) = 2$

6.31 The joint distribution for the length of life of two different types of components operating in a system was given in Exercise 5.18 by

(5pts)

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

The relative efficiency of the two types of components is measured by $U = Y_2/Y_1$. Find the probability density function for U.

Let $U = Y_2/Y_1$, $V = Y_1$ ⇒ $Y_1 = V$, $Y_2 = UV$

$f_{U,V}(u,v) = f_{Y_1,Y_2}(\underset{\overset{\parallel}{V}}{y_1}, \underset{\overset{\parallel}{UV}}{y_2}) \cdot |J|$, $|J| = \begin{vmatrix} 1 & 0 \\ U & V \end{vmatrix} = V$

$= \frac{1}{8} \cdot v^2 \cdot e^{-u(1+v)/2}$

calculate yourself

$f(U) = \int f_{V,U}(v,u)\, dv = \int_0^\infty \frac{1}{8} v^2 \cdot e^{-\frac{u(1+v)}{2}}\, dv = \cdots = \frac{2}{(1+u)^3}$

6.43 Refer to Exercise 6.41. Let $Y_1, Y_2, \ldots, Y_n$ be independent, normal random variables, each with mean μ and variance σ^2.

a Find the density function of $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

b If $\sigma^2 = 16$ and $n = 25$, what is the probability that the sample mean, $\overline{Y}$, takes on a value that is within one unit of the population mean, μ? That is, find $P(|\overline{Y} - \mu| \le 1)$.

c If $\sigma^2 = 16$, find $P(|\overline{Y} - \mu| \le 1)$ if $n = 36$, $n = 64$, and $n = 81$. Interpret the results of your calculations.

a. $\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$

$\Rightarrow$ we learned exactly the same example using mgf in class.

b. $P(|\bar{Y}-\mu| \le 1) \Leftrightarrow P(|Z| \le \frac{5}{4}) \simeq 0.7888$

c. $P(|\bar{Y}-\mu| \le 1) \Leftrightarrow P(|Z| \le \frac{\sqrt{n}}{6}) = P(|Z| \le \frac{\sqrt{n}}{4})$

$n = 36 \Rightarrow P(|Z| \le 1.5) \simeq 0.8664$
$n = 64 \Rightarrow P(|Z| \le 2) \simeq 0.9544$
$n = 81 \Rightarrow P(|Z| \le 2.25) \simeq 0.9756$

$\rightarrow$ n gets large, probability also large

6.46 Suppose that Y has a gamma distribution with $\alpha = n/2$ for some positive integer n and β equal to some specified value. Use the method of moment-generating functions to show that $W = 2Y/\beta$ has a χ^2 distribution with n degrees of freedom.

$M_Y(t) = (1-\beta t)^{-\frac{n}{2}}$

mgf of $W = M_W(t) = E(e^{tW}) = E(e^{\frac{2tY}{\beta}})$

$= (1-\beta\frac{2t}{\beta})^{-\frac{n}{2}} = (1-2t)^{-\frac{n}{2}} \Rightarrow$ mgf of $\chi^2(n)$

6.60 Suppose that $W = Y_1 + Y_2$ where Y_1 and Y_2 are independent. If W has a χ^2 distribution with ν degrees of freedom and W_1 has a χ^2 distribution with $\nu_1 < \nu$ degrees of freedom, show that Y_2 has a χ^2 distribution with $\nu - \nu_1$ degrees of freedom.

$M_W(t) = M_{Y_1}(t) \cdot M_{Y_2}(t)$, $M_{Y_2}(t) = \frac{M_W(t)}{M_{Y_1}(t)} = \frac{(1-2t)^{-\frac{\nu}{2}}}{(1-2t)^{-\frac{\nu_1}{2}}} = (1-2t)^{-\frac{(\nu-\nu_1)}{2}}$

$\therefore Y_2 \sim \chi^2(\nu-\nu_1)$

***6.62** Let Y_1 and Y_2 be independent normal random variables, each with mean 0 and variance σ^2. Define $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$. Show that U_1 and U_2 are independent normal random variables, each with mean 0 and variance $2\sigma^2$. [*Hint:* If (U_1, U_2) has a joint moment-generating function $m(t_1, t_2)$, then U_1 and U_2 are independent if and only if $m(t_1, t_2) = m_{U_1}(t_1)m_{U_2}(t_2)$.]

$M_{U_1,U_2}(t_1, t_2) = E(e^{t_1U_1 + t_2U_2}) = E(e^{t_1(Y_1+Y_2) + t_2(Y_1-Y_2)})$

$= M_{Y_1}(t_1+t_2) \cdot M_{Y_2}(t_1-t_2)$, mgf of $M_{Y_1}(t) = \exp^{(\mu t + \frac{t^2\sigma^2}{2})}$

mgf of $N(0, 2\sigma^2)$

$\Rightarrow \exp(\frac{\sigma^2}{2}(t_1+t_2)^2) \cdot \exp(\frac{\sigma^2}{2}(t_1-t_2)^2) = \exp^{(\sigma^2 t_1^2)} \cdot \exp^{(\sigma^2 \cdot t_2^2)} = M_{U_1}(t_1) \cdot M_{U_2}(t_2)$

6.65 Let Z_1 and Z_2 be independent standard normal random variables and $U_1 = Z_1$ and $U_2 = Z_1 + Z_2$.

a Derive the joint density of U_1 and U_2.

b Use Theorem 5.12 to give $E(U_1)$, $E(U_2)$, $V(U_1)$, $V(U_2)$, and $\text{Cov}(U_1, U_2)$.

c Are U_1 and U_2 independent? Why?

d Refer to Section 5.10. Show that U_1 and U_2 have a bivariate normal distribution. Identify all the parameters of the appropriate bivariate normal distribution.

(10pts)

a. $f_{U_1,U_2}(u_1,u_2) = f_{Z_1,Z_2}(\overset{u_1}{z_1}, \overset{u_2-u_1}{z_2}) \cdot |J|$, $|J| = 1$

$\Rightarrow \frac{1}{2\pi} \cdot \exp \frac{-(u_1^2 + (u_2-u_1)^2)}{2}$

b. $E(U_1) = 0$, $E(U_2) = E(Z_1+Z_2) = 0$, $V(U_1) = 1$, $V(U_2) = V(Z_1+Z_2) = 2$

$\text{Cov}(U_1,U_2) = \text{Cov}(Z_1, Z_1+Z_2) = \text{Var}(Z_1) + \text{Cov}(Z_1,Z_2) = 1$

c. by (b), $\text{Cov}(U_1,U_2) \neq 0$ Hence U_1, U_2 is not independent.

d. see pdf of bivariate normal distribution, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$\Rightarrow \sigma_1^2 = 1, \sigma_2^2 = 2, \rho = \frac{1}{\sqrt{2}}$

***6.68** Let Y_1 and Y_2 have joint density function

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 8y_1y_2, & 0 \le y_1 < y_2 \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $U_1 = Y_1/Y_2$ and $U_2 = Y_2$.

(5pts)

a Derive the joint density function for (U_1, U_2).

b Show that U_1 and U_2 are independent.

a. $f_{U_1,U_2}(u_1,u_2) = f_{Y_1,Y_2}(y_1,y_2)\,|J| = 8(u_1u_2)\cdot u_2 \cdot |u_2| = 8u_1u_2^3$

$(0 \le u_1, u_2 \le 1)$

b. We need to show $f_{U_1}(u_1)\cdot f_{U_2}(u_2) = f_{U_1,U_2}(u_1,u_2)$

$f_{U_1}(u_1) = \int_0^1 8u_1u_2^3\,du_2 = 2u_1$

$f_{U_2}(u_2) = \int_0^1 8u_1u_2^3\,du_1 = 4u_2^3$,

$\therefore f_{U_1}(u_1)\cdot f_{U_2}(u_2) = f_{U_1,U_2}(u_1,u_2)$

Hence U_1 and U_2 are independent.

***6.71** Suppose that Y_1 and Y_2 are independent exponentially distributed random variables, both with mean β, and define $U_1 = Y_1 + Y_2$ and $U_2 = Y_1/Y_2$.

(5pts)

a Show that the joint density of (U_1, U_2) is

$$f_{U_1,U_2}(u_1, u_2) = \begin{cases} \frac{1}{\beta^2} u_1 e^{-u_1/\beta} \frac{1}{(1+u_2)^2}, & 0 < u_1,\ 0 < u_2, \\ 0, & \text{otherwise.} \end{cases}$$

b Are U_1 and U_2 are independent? Why?

$$\frac{u_1 u_2}{1+u_2} \qquad \frac{u_1}{1+u_2}$$

a. $f_{U_1,U_2}(u_1,u_2) = f_{Y_1,Y_2}(y_1,y_2)\,|J| = f_{Y_1}(y_1)\cdot f_{Y_2}(y_2)\cdot|J|$

$|J| = \left|\frac{-u_1}{(1+u_2)^2}\right|$, $f_{U_1,U_2}(u_1,u_2) = \frac{1}{\beta^2}\cdot\exp\left(-\frac{u_1}{\beta}\right)\cdot\frac{u_1}{(1+u_2)^2}$, $u_1,u_2>0$

b. $f_{U_2}(u_2) = \int_0^\infty \frac{1}{\beta^2}\cdot\exp\left(-\frac{u_1}{\beta}\right)\cdot\frac{u_1}{(1+u_2)^2}\,du_1 = \frac{1}{\beta^2}e^{-\frac{u_1}{\beta}}\cdot u_1$

$f_{U_1}(u_1) = \int_0^\infty \frac{1}{\beta^2}\cdot\exp\left(-\frac{u_1}{\beta}\right)\cdot\frac{u_1}{(1+u_2)^2}\,du_2 = \frac{1}{(1+u_2)^2}$

$\therefore f_{U_1}(u_1)f_{U_2}(u_2) = f_{U_1,U_2}(u_1,u_2)$, U_1 and U_2 are independent.

6.79 Refer to Exercise 6.77. If $Y_1, Y_2, \ldots, Y_n$ are independent, uniformly distributed random variables on the interval $[0, \theta]$, show that $U = Y_{(1)}/Y_{(n)}$ and $Y_{(n)}$ are independent.

Let $V = Y_{(n)}$, $f_{U,V}(u,v) = f_{Y_{(1)},Y_{(n)}}(uv, v)\,|J|$

$$f_{Y_{(k)},Y_{(\ell)}}(y_k, y_\ell) = \frac{n!}{(k-1)!\,(\ell-1-k)!\,(n-\ell)!}\cdot[F(y_k)]^{k-1} f(y_k)\cdot[F(y_\ell)-F(y_k)]^{\ell-1-k}\cdot f(y_\ell)\cdot[1-F(y_\ell)]^{n-\ell}$$

(see lecture note)

$\therefore f_{Y_{(1)},Y_{(n)}}(uv, v)\,|J| = n(n-1)\left(\frac{1}{\theta}\right)^n(v-uv)^{n-2}\cdot|v| = n(n-1)\left(\frac{1}{\theta}\right)^n(1-u)^{n-2}\cdot v^{n-2}$

$(0\le u\le 1,\ 0\le v\le\theta)$

$f_U(u)\cdot f_V(v) = f_{U,V}(u,v)$ (U and V are independent)

6.80 Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables, each with a beta distribution, with $\alpha = \beta = 2$. Find (5pts)

a the probability distribution function of $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$.

b the density function of $Y_{(n)}$.

c $E(Y_{(n)})$ when $n = 2$.

$f(y) = 6\cdot y\cdot(1-y)$, $F(y) = 3y^2-2y^3$ $(0\le y\le 1)$

a. $P(Y_{(n)}\le y) = P(Y_1, Y_2, \cdots Y_n \le y) = [P(Y_1\le y)]^n = [F(y)]^n$
$= [3y^2-2y^3]^n$

b. $f_{(n)}(y) = n\,F(y)^{n-1}\cdot f(y) = 6ny[3y^2-2y^3]^{n-1}\cdot(1-y)$

c. $f_{(2)}(y) = 12y(3y^2-2y^3)$, $\int_0^1 y\cdot 12y(3y^2-2y^3)(1-y)\,dy$

$= \frac{36}{5} - 10 + \frac{24}{7} \simeq 0.6286$

6.85 Let Y_1 and Y_2 be independent and uniformly distributed over the interval $(0, 1)$. Find $P(2Y_{(1)} < Y_{(2)})$.

$f_{Y_{(1)},Y_{(2)}}(y_1, y_2) = 2, \quad (0 \le y_1 \le y_2 \le 1)$

$\therefore P(2Y_{(1)} < Y_{(2)}) = \int_0^{\frac{1}{2}} \int_{2y_1}^{1} 2\, dy_2\, dy_1 = \frac{1}{2}$

6.87 The opening prices per share Y_1 and Y_2 of two similar stocks are independent random variables, each with a density function given by

$$f(y) = \begin{cases} (1/2)e^{-(1/2)(y-4)}, & y \ge 4, \\ 0, & \text{elsewhere.} \end{cases}$$

On a given morning, an investor is going to buy shares of whichever stock is less expensive. Find the

a probability density function for the price per share that the investor will pay.
b expected cost per share that the investor will pay.

a. $f_{(1)}(y) = n[1 - F(y)]^{n-1} f(y) = e^{-(y-4)} \quad (y \ge 4)$

b. $\int_4^{\infty} y \cdot e^{-(y-4)}\, dy = 5$

***6.103** Let Y_1 and Y_2 be independent, standard normal random variables. Find the probability density function of $U = Y_1/Y_2$.

Let $V = Y_2$, $f_{U,V}(u, v) = f_{Y_1,Y_2}(uv, v)\,|J| = \frac{1}{2\pi} e^{-\frac{v^2(1+u^2)}{2}} |v|$

$\therefore f_U(u) = \int_0^{\infty} \frac{v}{\pi} \exp\left(-\frac{v^2(1+u^2)}{2}\right) dv = \frac{1}{\pi(1+u^2)} \quad (-\infty < u < \infty)$