EE202B Signals and Systems, Spring 2018 **Name:** ____________________

Final Exam **Student ID No.:** ____________________

June 11, 2018

Time Limit: 150 Minutes Lecturer: Jung-Woo Choi

Grade Table (for teacher use only)

Question:	1	2	3	4	5	6	7	Total
Points:	15	15	15	15	10	25	5	100
Score:								

1. (15 points) Determine whether or not each of the following statement is true or false. Justify your answers.

   (a) (5 points) For a causal & stable continuous-time LTI system, real parts of its poles are less than zero.

   Answer: **True**

   For a causal system, region of convergence is the right half-plane from the location of poles. To make it stable, the RoC should include $\mathcal{R}e\{s\} = 0$ axis. Therefore, all poles should be in the left half-plane, which means that real parts of poles are less than zero.

   (b) (5 points) A system's frequency response $H(j\omega)$ can be uniquely determined, provided that poles, zeros and Region of Convergence (RoC) are specified.

   Answer: **False**

   Only with poles, zeros and RoC, the system's overall gain cannot be specified. For example, the gain G of $H(s) = G\frac{N(s)}{D(s)}$ is unknown.

   (c) (5 points) When a signal $x(t)$ band-limited in $\omega \in [-\omega_M, \omega_M]$ is sampled with Nyquist rate, the original signal $x(t)$ can be perfectly reconstructed from the sampled sequence $x[n]$ using a single zero-order-hold (ZOH) interpolation filter.

   Answer: **False**

   The frequency response of a ZOH interpolation filter is a sinc function, which has nonideal stopband and passband characteristics as compared to the ideal low-pass filter.

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2. (15 points) Let $X(j\omega)$ be the continuous-time Fourier transform (CTFT) of a continuous-time domain signal $x(t)$.

   (a) (5 points) Express the CTFT of the following signal $y(t)$ in terms of $X(j\omega)$.

$$y(t) = x^3(t) + 4 \qquad (1)$$

Answer:

$$x^3(t) + 4 = x(t) \times x(t) \times x(t) + 4 \Longleftrightarrow \frac{1}{(2\pi)^2} X(j\omega) * X(j\omega) * X(j\omega) + 4 \times 2\pi\delta(\omega)$$

(b) (5 points) Suppose that $X(j\omega)$ is band-limited in $\omega \in [-\omega_M, \omega_M]$. What is the Nyquist rate to sample the signal $y(t)$ without aliasing?

Answer: $6\omega_M$

After the three-times convolution of (a), the signal's bandwidth is enlarged to $\pm 3\omega_M$. Therefore, the Nyquist rate is $2 \times 3\omega_M = 6\omega_M$

(c) (5 points) A signal is given by $x(t) = \sin(t)/\pi t$. When this signal $x(t)$ is sampled with a sampling rate $\omega_s = 2$ (rad/sec), evaluate the following sum of the sampled sequence $x[n]$:

$$\sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{\pi}{3}n} =? \tag{2}$$

Answer: $\frac{1}{\pi}$

The Fourier transform of $x(t)$ is

$$x(t) = \sin(Wt)/\pi t \longleftrightarrow X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

In this case W=1. Since the CTFT is bounded in $\omega \in [-1, 1]$, no aliasing occurs when sampled by $\omega_s = 2$.

After the sampling, the CTFT $X(j\omega)$ is scaled by $\frac{1}{T} = \frac{\omega_s}{2\pi} = \frac{1}{\pi}$, and DTFT is given by

$$X_d(e^{j\omega/T}) = \frac{1}{T} X(j\omega)$$

The sum given in the problem is equal to the DTFT of the sampled sequence, and the DTFT is uniform with magnitude $\frac{1}{\pi}$.

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3. (15 points) For frequency responses of three causal LTI systems given below, answer to the following questions.

   A. $H(j\omega) = \frac{1}{(j\omega)^2 + 4}$

   B. $h(t) = e^{-t}\sin(t)u(t)$

   C. $H(s) : \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{d^2x(t)}{dt^2}$

(a) (3 points) Which one has the lowest damping? (A, B, C)

(b) (3 points) Which one has the highest break frequency? (A, B, C)

(c) (3 points) Which one is stable ? Multiple choices are allowed. (A, B, C)

(d) (3 points) Which one has a high-pass response? Multiple choices are allowed. (A, B, C)

(e) (3 points) Which one has a low-pass response? Multiple choices are allowed. (A, B, C)

Answer: (a) A (b) A (c) B & C (d) C (e) A & B

Solution:

Standard form of low-pass 2nd order system is given by $H(j\omega) = \frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2}$.

$$A : H(j\omega) = \frac{1}{(j\omega)^2 + 2^2} \rightarrow \text{poles at } \pm 2j, \omega_n = 2, \zeta = 0$$

$$B : h(t) = \frac{1}{2j}\left(e^{(-1+j)t} - e^{(-1-j)t}\right)u(t) \Longleftrightarrow H(j\omega) = \frac{1}{2j}\left(\frac{1}{j\omega+1-j} - \frac{1}{j\omega+1+j}\right)$$

$$H(j\omega) = \frac{1}{(j\omega)^2 + 2(j\omega) + 2}, \text{ poles at } -1 \pm j, \omega_n = \sqrt{2}, \zeta = \frac{1}{\sqrt{2}}$$

$$C : H(j\omega) = \frac{(j\omega)^2}{(j\omega)^2 + 3(j\omega) + 2}$$

$$= \frac{(j\omega)^2}{(j\omega+2)(j\omega+1)}, \text{ poles at } -2, -1, \omega_n = \sqrt{2}, \zeta = \frac{3\sqrt{2}}{4}$$

(a) The smallest daming is from the system A, with $\zeta = 0$.
(b) The highest break frequency case is A, with $\omega_n = 2$
(c) Systems with negative real valued poles are B & C.
(d) C has flat response as $\omega \to \infty$, and its responses vanishes as $\omega \to 0$.
(e) A & B have flat responses as $\omega \to 0$, and their system responses vanish with increasing ω.

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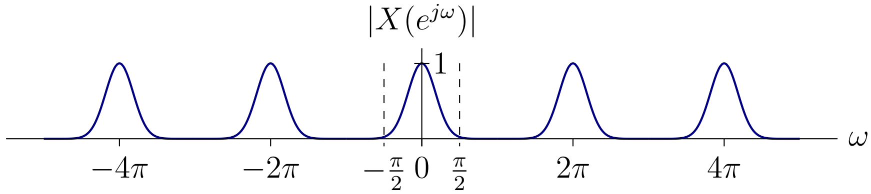
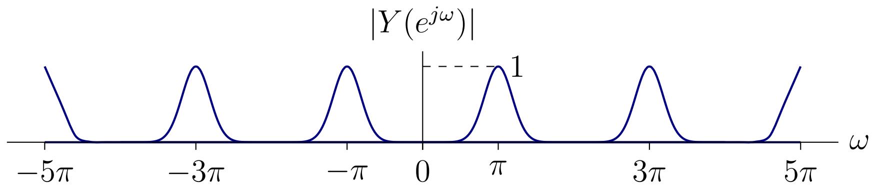
4. (15 points) A discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is shown below:



The spectrum $X(e^{j\omega}) = 0$ for $\frac{\pi}{2} \leq |\omega| \leq \pi$, and the phase response $\measuredangle X(e^{j\omega}) = 0$ for all ω.

(a) (5 points) Express the inverse Fourier transform $y[n]$ of the following spectrum $|Y(e^{j\omega})|$ in terms of $x[n]$. ($\measuredangle Y(e^{j\omega}) = 0$ for all frequency ω).



Answer: $y[n] = (-1)^n x[n]$

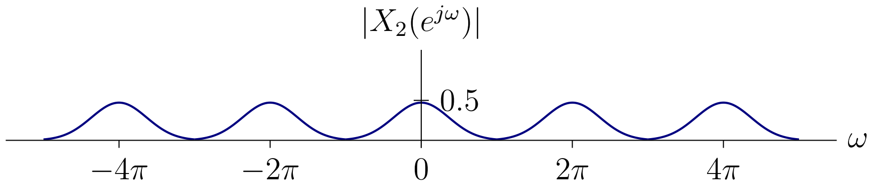
Solution: $Y(e^{j\omega})$ is equal to $X(e^{j\omega})$ shifted by π. From the property of DTFT,

$$X(e^{j(\omega-\omega_0)}) \Longleftrightarrow x[n]e^{j\omega_0 n},$$
$$X(e^{j(\omega-\pi)}) \Longleftrightarrow x[n]e^{j\pi n} = x[n](-1)^n$$

(b) (10 points) Plot the magnitude of DTFT $G(e^{j\omega})$ of a sequence $g[n] = x[2n]$. Also indicate its peak magnitude.

(Hint: This is a decimation process, which is equivalent to the sampling of the compressed CT signal. Note that the compression - expansion property of CTFT does not directly hold for DTFT.)

Solution: the decimation of $x[n]$ by $\times 2$ can be likened to the sampling of compressed CT signal $x_2(t) = x(2t)$. This yields the expansion of CTFT signal $X_2(j\omega) = \frac{1}{2}X(j\omega/2)$. After the sampling of $x_2(t) = x(2t)$, the DTFT of the sampled sequence will be:



with peak amplitude 0.5.

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5. (10 points) Consider a linear phase discrete-time LTI system with frequency response $H(e^{j\omega})$ and real impulse response $h[n]$. The group delay function for such a system is defined as

$$\tau(\omega) = -\frac{d}{d\omega}\measuredangle H(e^{j\omega}) \tag{3}$$

where $\measuredangle H(e^{j\omega})$ has no discontinuities. Suppose that, for this system,

$$|H(e^{j\frac{\pi}{3}})| = 4, \measuredangle H(e^{j0}) = 0, \text{ and } \tau\left(\frac{\pi}{4}\right) = 1. \tag{4}$$

Determine the output $y[n]$ of the system for the input $x[n] = \cos\left(\frac{11\pi}{3}n\right)$.

Answer: $4\cos\left(\frac{\pi}{3}n - \frac{\pi}{3}\right)$

The DT frequency response is periodic with a period of 2π.

Therefore, $x[n] = \cos\left(\frac{11\pi}{3}n\right) = \frac{1}{2}\left(e^{j\frac{(12-1)\pi}{3}n} + e^{-j\frac{(12-1)\pi}{3}n}\right) = \frac{1}{2}\left(e^{-j\frac{\pi}{3}n} + e^{j\frac{\pi}{3}n}\right)$.

On the other hand, $h[n]$ is a real function $\rightarrow H(e^{j\omega})$ is conjugate symmetric.

$\rightarrow |H(e^{j\omega})|$ is even, and $\measuredangle H(e^{j\omega})$ is odd.

Accordingly, at frequency $\omega = \pm\frac{\pi}{3}$, the magnitude response is $|H(e^{\pm j\frac{\pi}{3}})| = 4$.

For a linear phase system, group delay is equivalent to the phase delay (= -ϕ/ω) and constant across all frequency. The slope of phase is equal to one ($\tau\,(\pi/4) = 1$), so the phase of this system is $\phi = -\omega$, and $\measuredangle H(e^{\pm j\frac{\pi}{3}}) = \mp\frac{\pi}{3}$. From these magnitude and phase response of H, the output of the system is given by

$$y[n] = \frac{1}{2}\left(H(e^{-j\frac{\pi}{3}})e^{-j\frac{\pi}{3}n} + H(e^{j\frac{\pi}{3}})e^{j\frac{\pi}{3}n}\right)$$
$$= \frac{1}{2}\left(4e^{j\frac{\pi}{3}}e^{-j\frac{\pi}{3}n} + 4e^{-j\frac{\pi}{3}}e^{j\frac{\pi}{3}n}\right)$$
$$= 4\cos\left(\frac{\pi}{3}n - \frac{\pi}{3}\right)$$

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6. (25 points) A causal DT LTI system is described in terms of the difference equation

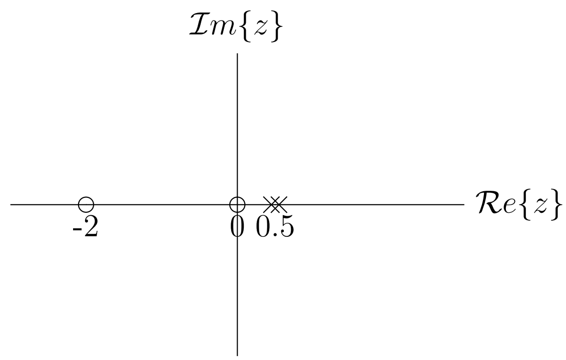
$$4y[n] - 4y[n-1] + y[n-2] = x[n] + 2x[n-1] \tag{5}$$

(a) (5 points) Derive the frequency response $H(e^{j\omega})$ of the system.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$
$$= \frac{1+2e^{-j\omega}}{4(1 - e^{-j\omega} + \frac{1}{4}e^{-2j\omega})}$$
$$= \frac{1+2e^{-j\omega}}{4(1 - \frac{1}{2}e^{-j\omega})^2}$$

(b) (5 points) Draw the pole-zero plot (Nyquist plot) of this system. Indicate the locations of poles and zeros of this system. (Use ◎ or xx to indicate duplicated zeros or poles.)

Answer:



Solution:

The system response $H(z)$ is given by

$$H(z) = \frac{1+2z^{-1}}{4(1-\frac{1}{2}z^{-1})^2}$$
$$= z\frac{z+2}{4(z-\frac{1}{2})^2}$$

There are two zeros at $z_{zero} = 0, -2$ and duplicated poles at $z_{pole} = \frac{1}{2}$.

(c) (5 points) Determine whether the system is stable or not. Justify your answer.

Answer: Stable

Solution: Since all poles of the system response $H(z)$ is inside of unit circle ($|z_{pole}| < 1$), the RoC of the causal LTI system is the exterior region from $|z| > z_{pole} = \frac{1}{2}$, which includes the unit circle.

(d) (10 points) Determine the impulse response of this system.

Answer: $h[n] = \frac{1}{4}\left(\frac{1}{2}\right)^n nu[n] + \left(\frac{1}{2}\right)^n (n-1)u[n-1]$

Solution:
The frequency response can be rewritten as

$$\begin{aligned} H(e^{j\omega}) &= \frac{1+2e^{-j\omega}}{4(1-\frac{1}{2}e^{-j\omega})^2} \\ &= \frac{1}{4(1-\frac{1}{2}e^{-j\omega})^2} + \frac{e^{-j\omega}}{2(1-\frac{1}{2}e^{-j\omega})^2} \end{aligned}$$

From the properties of Fourier transform (Quiz session 2)

$$\begin{aligned} X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})^2} &\Longleftrightarrow x[n] = (n+1)a^n u[n] \\ X_1(e^{j\omega})X_2(e^{j\omega}) &\Longleftrightarrow x_1[n] * x_2[n] \\ e^{-j\omega n_0}X(e^{j\omega}) &\Longleftrightarrow x[n-n_0], \end{aligned}$$

the inverse Fourier transform of $H(e^{j\omega})$ is given by

$$\begin{aligned} h[n] &= \frac{1}{4}\left(\frac{1}{2}\right)^n (n+1)u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} nu[n-1] \\ &= \frac{1}{4}\left(\frac{1}{2}\right)^n (n+1)u[n] + \left(\frac{1}{2}\right)^n nu[n-1], \end{aligned}$$

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7. (5 points) The Matlab code for filtering an audio signal is shown below. Which of the following commands best suits for (1) & (2)?

   A. (1) y = dfft(b,a,fs); (2) y = dfilter(bd,ad,x);

   B. (1) y = fft(b,a,fs); (2) y = filter(bd,ad,x);

   C. (1) y = con2dis(b,a,fs); (2) y = dfilter(bd,ad,x);

   D. (1) y = bilinear(b,a,fs); (2) y = filter(bd,ad,x);

```
[x_, fs]= audioread('music.mp3');
x = x_(:,1);

N = length(x);
t = (0:N-1)/fs;
```

```
f = (0:N-1)/N*fs;  f=f(1:fix(N/2));
w = 2*pi*f;

omega_n = 100*2*pi;
zeta    = 1;

b = [1/omega_n 10];          % numerator coefficients
a = [1/omega_n 1];           % denominator coefficients

% analog to digital conversion
[bd,ad] =  ( 1 ) ; % analog-to-digital conversion
y= ( 2 );             % filtering

%% play audio signal
ha= audioplayer(y,fs);

play(ha)
```

Answer: D. (CT filter coefficients are transformed to DT coefficients by 'bilinear' function, and then actual filtering occurs with 'filter' command)