IE241: Engineering Statistics I

Midterm Examination

Course Name/Number: ___________________________________

Student Name: _____________________________ Student ID: __________________

By signing below, I affirm that I have not violated any part of the Code of Academic Conduct on this exam, and that this work is honest representation of my learning from the class.

Signature_________________________________ Date _________________________

Problem	**Points**	**Full score**
1. True or False Questions		**10**
2. Flipping Tetrahedron		**10**
3. Moment-Generating Function of Normal Distribution		**15**
4. Conditional Probability		**15**
5. Sequential Coin Toss		**15**
6. Banking Business at Woori-bank		**15**
7. Multivariate Random Variables		**15**
8. Stick Cutting		**15**
Total		**110**

Caution

- Simplify your answer as possible as you can. You have to show your work to get full credit.
- Write your name and student ID on each answer sheet.
- While taking the exams, your two hands must be on the table.
- If you have a question, you can write your question on Zoom chat box. Unless, you should not use your keyboard during final exam. This is for preventing cheating.
- You cannot use laptop calculator, smartphone calculator or tablet calculator.
- You may not get full credit if your solution file has unclear image, so take photo clearly.
- You should submit a single pdf file on KLMS until 12:00 PM. WE DO NOT ACCEPT ANY LATE SUBMISSION.
- If you finish writing your solution before 11:45 AM, you can upload your file and leave the Zoom class earlier. However, before taking a picture, you have to write 'I will submit now.' or '제출하겠습니다.' on Zoom chat box.
- After uploading your solution pdf file on KLMS, you should write 'I upload my answer sheet.' or '제출완료했습니다.' on Zoom chat box and leave the Zoom class.

Discrete Distributions

Distribution	Probability Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$
Hypergeometric	$p(y) = \dfrac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$
Poisson	$p(y) = \dfrac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$

Continuous Distributions

Distribution	Probability Function
Uniform	$f(y) = \dfrac{1}{\theta_2 - \theta_1};\ \theta_1 \le y \le \theta_2$
Normal	$f(y) = \dfrac{1}{\sigma\sqrt{2\pi}}\exp\left[-\left(\dfrac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$
Exponential	$f(y) = \dfrac{1}{\beta}e^{-y/\beta};\quad \beta > 0$ $0 < y < \infty$
Gamma	$f(y) = \left[\dfrac{1}{\Gamma(\alpha)\beta^{\alpha}}\right] y^{\alpha-1}e^{-y/\beta};$ $0 < y < \infty$
Chi-square	$f(y) = \dfrac{(y)^{(v/2)-1}e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$
Beta	$f(y) = \left[\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$

(10 Pts) Problem 1 (True or False Questions)

Check whether the following statements are True or False. For any correct answer, you will get +2 points, and for any wrong answer, you will get -2 points. If you leave it blank, it earns 0 point. You **DO NOT** have to show your work. Just write the answer.

A. $F(x) = \frac{3}{4} + \frac{1}{2\pi}\tan^{-1} x\,, (-\infty < x < \infty)$ is a cumulative distribution function.

B. There exist two random variables X and Y s.t. $P(X \leq x) < P(Y \leq x)$ for all $x \in \mathbb{R}$.

C. If two random variables have an identical distribution, then they have an identical moment-generating function if they exist and vice-versa.

D. If a random variable X is a continuous random variable, then X has continuous and differentiable cumulative distribution function.

E. When the joint probability mass function of (M, N) is $p(m, n) = \frac{1}{2^{m+1}}$ $(m \geq n,\ m = 1,2,3, \dots,\ n = 1,2,3, \dots)$, then M and N are independent.

Solution

A. F

B. T

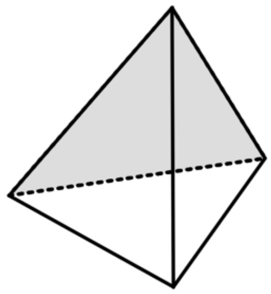
C. T

D. F

E. F

(10 Pts) Problem 2 (Flipping Tetrahedron)

A regular tetrahedron that has one black face and three white faces is on the desk. The current color of the bottom face is white. Jaeyeon selects one edge of the bottom plane at random and flip the tetrahedron on the side. Find the probability that an event that the bottom face is black happens twice in 5 trials.



Solution

$\boxed{\text{1st Solution}}$ Use total probability

$$P(\text{White bottom, 5 trials, 2 black}) = P(W,5,2 \mid \text{flip to Black}) \cdot \frac{1}{3} + P(W,5,2 \mid \text{flip to White}) \cdot \frac{2}{3}$$

$$= P(B,4,1) \cdot \frac{1}{3} + P(W,4,2) \cdot \frac{2}{3}$$

(∵ Black to White in probability 1)

$$= P(W,3,1) \cdot \frac{1}{3} + \left\{ P(B,3,1) \cdot \frac{1}{3} + P(W,3,2) \cdot \frac{2}{3} \right\} \cdot \frac{2}{3}$$

$$\vdots$$

$$= \boxed{\frac{10}{27}}$$

$\boxed{\text{2nd Solution}}$ Find the possible cases.

Current bottom		1st trial		2nd		3rd		4th		5th	probability
W	$\xrightarrow{\frac{1}{3}}$	B	$\xrightarrow{1}$	W	$\rightarrow$	B	$\rightarrow$	W	$\xrightarrow{\frac{2}{3}}$	W	$\left(\frac{1}{3}\right)^2 \cdot \frac{2}{3}$
	$\rightarrow$	B	$\rightarrow$	W	$\rightarrow$	W	$\rightarrow$	B	$\rightarrow$	W	$\left(\frac{1}{3}\right)^2 \cdot \frac{2}{3}$
	$\rightarrow$	B	$\rightarrow$	W	$\rightarrow$	W	$\rightarrow$	W	$\rightarrow$	B	$\left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^2$
	$\xrightarrow{\frac{2}{3}}$	W	$\rightarrow$	B	$\rightarrow$	W	$\rightarrow$	B	$\rightarrow$	W	$\left(\frac{1}{3}\right)^2 \cdot \frac{2}{3}$
	$\rightarrow$	W	$\rightarrow$	B	$\rightarrow$	W	$\rightarrow$	W	$\rightarrow$	B	$\left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^2$
	$\rightarrow$	W	$\rightarrow$	W	$\rightarrow$	B	$\rightarrow$	W	$\rightarrow$	B	+) $\left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^2$

$$\boxed{\frac{10}{27}}$$

(15 Pts) Problem 3 (Moment-Generating Function of Normal Distribution)

Let a random variable X follows a normal distribution with mean μ and variance σ^2.

A. Derive a moment-generating function of X.

B. Using the result of A, show that the mean and variance of X are μ and σ^2, respectively.

A 10pt.

$X \sim N(\mu, \sigma^2)$

$$M(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2 - 2\sigma^2 tx}{2\sigma^2}\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2 - 2\mu x + \mu^2 - 2\sigma^2 tx}{2\sigma^2}\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-(\mu+\sigma^2 t))^2 - 2\mu\sigma^2 t - \sigma^4 t^2}{2\sigma^2}\right) dx$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\right) dx}_{=1 \ (\because N(\mu+\sigma^2 t, \sigma^2))}$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

B. 5pt.

$$E[X] = M'(t)\big|_{t=0} = (\mu + \cancel{\sigma^2 t}) \cdot \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\Big|_{t=0} = \mu.$$

$$E[X^2] = M''(t)\big|_{t=0} = (\mu + \cancel{\sigma^2 t})^2 \cdot \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) + \sigma^2 \cdot \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\Big|_{t=0}$$

$$= \mu^2 + \sigma^2$$

$$\therefore V[X] = E[X^2] - \{E[X]\}^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

(15 Pts) Problem 4 (Conditional Probability)

Let $X \sim Uniform(0,1)$ and let $0 < a < b < 1$. Define two random variables Y and Z as

$$Y = \begin{cases} 1 & if\ X < b \\ 0 & otherwise \end{cases}$$

and

$$Z = \begin{cases} 1 & if\ a < X < 1 \\ 0 & otherwise \end{cases}$$

A. Are Y and Z independent? Explain why.

B. Find $P(Y|Z = z)$.

C. Find $P(Z|Y = y)$.

Solution

A. 5 Pts

Consider $P(Y = 0, Z = 0)$. By definition, $P(Y = 0, Z = 0) = P(X \geq b \& X \leq a) = 0$.

But if Y and Z are independent, $P(Y = 0, Z = 0) = P(Y = 0)P(Z = 0) = a(1 - b) > 0$.

Since these two are different, it is dependent.

B. 5 Pts

If $Z = 0$, then $X \leq a$. Thus, Y is always 1 since $X \leq a < b$.

If $Z = 1$, then $a < X < 1$. Thus, $P(Y = 1|Z = 1) = P(X < b|a < X < 1) = \frac{b-a}{1-a}$.

Also, $P(Y = 0|Z = 1) = P(X \geq b|a < X < 1) = \frac{1-b}{1-a}$.

$$P(Y = y|Z = 0) = I_{\{y=1\}}, y \in \{0,1\}$$

$$P(Y = y|Z = 1) = \left(\frac{1-b}{1-a}\right)^{1-y} \left(\frac{b-a}{1-a}\right)^{y}, y \in \{0,1\}$$

C. 5 Pts

Similar to **B**, we can obtain the answer.

$$P(Z = z|Y = 0) = I_{\{z=1\}}, z \in \{0,1\}$$

$$P(Z = z|Y = 1) = \left(\frac{a}{b}\right)^{1-z} \left(\frac{b-a}{b}\right)^{z}, z \in \{0,1\}$$

(15 Pts) Problem 5 (Sequential Coin Toss)

Hyeonah tosses a coin 10 times in a row. Let head denote 'H' and the tail denote 'T'. Let X be the number of times that the previous coin toss result and current coin toss result are different. For example, if your coin toss result is 'HHHHTTTTHT', then $X = 3$.

A. Find the probability mass function of X. Is there any special name for the distribution of X?

B. Derive $\mathbb{E}[X]$ from the definition of expectation.

Solution

A. 10 Pts

After first toss, at each coin toss, the probability that X increases 1 is $\frac{1}{2}$. This means that X is the sum of 9 independent Bernoulli random variables with probability $\frac{1}{2}$. Thus, the probability mass function of X is

$$P(X = x) = \binom{9}{x}\left(\frac{1}{2}\right)^{9-x}\left(\frac{1}{2}\right)^{x}, x = 0,1, \dots, 9,$$

which is binomial distribution with $n = 9, p = \frac{1}{2}$.

B. 5 Pts

By the definition of expectation,

$$\mathbb{E}[X]$$

$$= \sum_{x=0}^{9} x \binom{9}{x}\left(\frac{1}{2}\right)^{9-x}\left(\frac{1}{2}\right)^{x}$$

$$= \sum_{x=0}^{9} x \frac{9!}{x!\,(9-x)!}\left(\frac{1}{2}\right)^{9-x}\left(\frac{1}{2}\right)^{x}$$

$$= \sum_{x=1}^{9} \frac{9!}{(x-1)!\,(9-x)!}\left(\frac{1}{2}\right)^{9-x}\left(\frac{1}{2}\right)^{x}$$

$$= 9 * \frac{1}{2} * \sum_{y=0}^{8} \frac{8!}{y!\,(8-y)!}\left(\frac{1}{2}\right)^{8-y}\left(\frac{1}{2}\right)^{y}$$

$$= \frac{9}{2}$$

(15 Pts) Problem 6 (Banking Business in Woori-bank)

Customers arrive at the Woori-bank to do their banking business. Customers' business is handled at Woori-bank counters. There are two bank counters on the Woori-bank, counter A and counter B. Assume that the process time of each customer business is independent and follows exponential distribution with mean μ.

A. Two customers, Junyoung and Chihyeon, enter each counter at the same time. What is the probability that Junyoung finishes early than Chihyeon?
B. After one of two customers finishes his banking business, another customer Huijae immediately enter the empty counter. Note that we do not know who finishes early. What is the probability that Huijae finishes her banking business early than other customer?

Solution

A. 5 Pts

Let two random variables X and Y denote the process time of Junyoung and Chihyeon, respectively. Then

$$P(X < Y) = \int_0^\infty \int_x^\infty \frac{1}{\mu} e^{-\frac{x}{\mu}} \frac{1}{\mu} e^{-\frac{y}{\mu}} dy\, dx = \int_0^\infty \frac{1}{\mu} e^{-\frac{x}{\mu}} \left(\int_x^\infty \frac{1}{\mu} e^{-\frac{y}{\mu}} dy \right) dx$$
$$= \int_0^\infty \frac{1}{\mu} e^{-\frac{x}{\mu}} \left(-e^{-\frac{y}{\mu}} \right)_x^\infty dx = \int_0^\infty \frac{1}{\mu} e^{-\frac{2x}{\mu}} dx = \frac{1}{2}$$

B. 10 Pts

Let a random variable Z denotes the process time of Huijae. Without loss of generality, we can assume that Junyoung finishes early than Chihyeon, which means $X = x < Y$. By the memoryless property, we know $P(Y > x + y | Y > x) = P(Y > y)$. Thus, the probability that Huijae finishes early than Chihyeon is

$$P(X + Z > Y | X = x, Y > x) = P(Z > Y - x | Y > x) = P(Z > W) = \frac{1}{2}$$

where a random variable $W = Y - X$ denotes additional process time of Chihyeon, which is an exponential distribution with mean μ.

(15 Pts) Problem 7 (Multivariate Random Variables)

Solve the following problems.

A. Kanghoon and Haewon make an appointment at the Daejeon station. Arrival time of each one follows uniform distribution from 0 to 60 minutes. Anyone who arrives first wait 30 minutes at most. What is the probability that Kanghoon and Haewon meet?

B. Random vector $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ has uniform distribution inside the unit disc. This means that its joint probability density function is:

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} \frac{1}{\pi} & if\ x_1^2 + x_2^2 \leq 1 \\ 0 & otherwise \end{cases}$$

Are X_1 and X_2 independent? Why or why not? Find the marginal probability density function of X_1.

Solution

A. 6 Pts

Let two random variables X and Y are the arrival time of Kanghoon and Haewon, respectively. Note that X and Y follow $Uniform(0,60)$. Thus,

$$\begin{aligned} P(|X-Y| < 30) &= P(X - 30 < Y < X + 30) \\ &= \int_0^{30}\int_0^{x+30} \frac{1}{60} * \frac{1}{60} dy\, dx + \int_{30}^{60}\int_{x-30}^{60} \frac{1}{60} * \frac{1}{60} dy\, dx \\ &= \frac{1}{3600}\left(\frac{1}{2} * (30^2 - 0^2) + 30 * (30 - 0)\right) \\ &+ \frac{1}{3600}\left(90 * (60 - 30) - \frac{1}{2} * (60^2 - 30^2)\right) = \frac{3}{4} \end{aligned}$$

B. 9 Pts

Let's find the marginal probability density function of X_1 first.

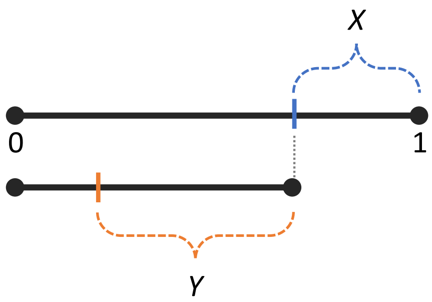
$$f_{X_1}(x_1) = \int f_{X_1,X_2}(x_1, x_2) dx_2 = \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \frac{1}{\pi} dx_2 = \frac{2}{\pi}\sqrt{1 - x_1^2}, -1 \leq x_1 \leq 1.$$

Similarly, $f_{X_2}(x_2) = \frac{2}{\pi}\sqrt{1 - x_2^2}, -1 \leq x_2 \leq 1$

Since $f_{X_1,X_2}(x_1, x_2) \neq f_{X_1}(x_1) f_{X_2}(x_2)$, X_1 and X_2 are dependent.

(15 Pts) Problem 8 (Stick Cutting)

There is a 1m stick. Yohan cuts the stick at random and take the shorter part. Kyuree takes the longer part of the stick. She cuts it at random and take the longer part. Let two random variables X and Y be the length of Yohan's stick and Kyuree's stick, respectively.



A. Find the joint probability density function of (X, Y).

B. Find the marginal probability density function of Y.

Solution

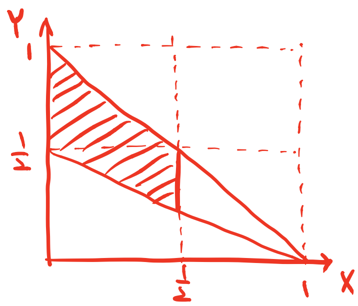
A. 7pt.

$X \sim \mathrm{Unif}(0, \frac{1}{2})$ $\quad f_X(x) = 2 \quad (0 \le x \le \frac{1}{2})$

$\{Y \mid X = x\} \sim \mathrm{Unif}(\frac{1-x}{2}, 1-x)$ $\quad f_Y(y \mid x) = \frac{2}{1-x} \quad (\frac{1-x}{2} \le y \le x, \ 0 \le x \le \frac{1}{2})$

$\therefore f_{X,Y}(x, y) = f_Y(y \mid x) \cdot f_X(x) = \boxed{\frac{4}{1-x} \quad (\frac{1-x}{2} \le y \le x, \ 0 \le x \le \frac{1}{2})}$

B. 8pt



$f_Y(y) = \int_x f_{X,Y}(x, y)\, dx$

$$= \begin{cases} \frac{1}{4} \le y \le \frac{1}{2} & \int_{1-2y}^{\frac{1}{2}} \frac{4}{1-x}\, dx = 4\ln(4y) \\ \frac{1}{2} \le y \le 1 & \int_0^{1-y} \frac{4}{1-x}\, dx = -4\ln(y) \end{cases}$$

$$\therefore \boxed{f_Y(y) = \begin{cases} 4\ln(4y) & (\frac{1}{4} \le y \le \frac{1}{2}) \\ -4\ln(y) & (\frac{1}{2} \le y \le 1) \end{cases}}$$