IE241 HW5 solution

4.81 **a** If $\alpha > 0$, $\Gamma(\alpha)$ is defined by $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy$, show that $\Gamma(1) = 1$.
***b** If $\alpha > 1$, integrate by parts to prove that $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.

(a) $\Gamma(1) = \int_0^\infty e^{-y}\,dy = \left[-e^{-y}\right]_0^\infty = 1$

(b) $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \cdot e^{-y}\,dy = \left[-y^{\alpha-1} \cdot e^{-y}\right]_0^\infty + (\alpha-1)\int_0^\infty y^{\alpha-2} \cdot e^{-y}\,dy = (\alpha-1)\Gamma(\alpha-1)$ when $\alpha > 1$

4.89 The operator of a pumping station has observed that demand for water during early afternoon hours has an approximately exponential distribution with mean 100 cfs (cubic feet per second).

a Find the probability that the demand will exceed 200 cfs during the early afternoon on a randomly selected day.
b What water-pumping capacity should the station maintain during early afternoons so that the probability that demand will exceed capacity on a randomly selected day is only .01?

(a) $Y \sim \exp(\frac{1}{100})$, $f(y) = \frac{1}{100}\exp^{-\frac{y}{100}}$

$P(Y > 200) = \frac{1}{100}\int_{200}^\infty \exp^{-\frac{y}{100}} = \left[-\exp^{-\frac{y}{100}}\right]_{200}^\infty = e^{-2} \simeq 0.135$

(b) $P(Y > k) = 0.01 \iff \left[-\exp^{-\frac{y}{100}}\right]_k^\infty = e^{-\frac{k}{100}} \simeq 0.01$

$-\frac{k}{100} = \log^{0.01} \iff k \simeq 461$

4.95 Let Y be an exponentially distributed random variable with mean β. Define a random variable X in the following way: $X = k$ if $k - 1 \le Y < k$ for $k = 1, 2, \ldots$.

a Find $P(X = k)$ for each $k = 1, 2, \ldots$.
b Show that your answer to part (a) can be written as

$$P(X = k) = \left(e^{-1/\beta}\right)^{k-1}\left(1 - e^{-1/\beta}\right), \qquad k = 1, 2, \ldots$$

and that X has a geometric distribution with $p = \left(1 - e^{-1/\beta}\right)$.

$Y \sim \exp(\frac{1}{\beta})$, $f(y) = \frac{1}{\beta}\exp^{-\frac{y}{\beta}}$

(a), (b) $P(X = k) = P(k-1 \le Y < k) = \int_{k-1}^{k} \frac{1}{\beta}\exp^{-\frac{y}{\beta}} = \left[-\exp^{-\frac{y}{\beta}}\right]_{k-1}^{k} = -\exp^{-\frac{k}{\beta}} + \exp^{-\frac{(k-1)}{\beta}}$

$\Rightarrow \exp^{-\frac{(k-1)}{\beta}} - \exp^{-\frac{(k-1)}{\beta}} \cdot \exp^{-\frac{1}{\beta}} = \exp^{-\frac{k-1}{\beta}}(1 - \exp^{-\frac{1}{\beta}}) = (e^{-\frac{1}{\beta}})^{k-1} \cdot (1 - e^{-\frac{1}{\beta}})$

$\sim geo(1 - e^{-1/\beta})$

4.96 Suppose that a random variable Y has a probability density function given by

$$f(y) = \begin{cases} ky^3 e^{-y/2}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the value of k that makes $f(y)$ a density function.
b Does Y have a χ^2 distribution? If so, how many degrees of freedom?
c What are the mean and standard deviation of Y?
d **Applet Exercise** What is the probability that Y lies within 2 standard deviations of its mean?

(a) $\int_0^{\infty} k \cdot y^3 e^{-\frac{y}{2}} dy = k \int_0^{\infty} y^3 \cdot e^{-\frac{y}{2}} dy$, let $\frac{y}{2} = z$ then

$= 8k \underbrace{\int_0^{\infty} z^3 \cdot e^{-z}}_{\hookrightarrow \text{gamma function}} \cdot (2dz) = 16k \cdot \Gamma(4) = 1$

$\therefore k = \frac{1}{2^4 \times \Gamma(4)} = 1/96$

(b) $Y \sim \chi^2(v)$ if $Y \sim \text{Gamma}(v/2, 2)$

pdf of Gamma (α, β) : $f(y) = \frac{y^{\alpha-1} \cdot e^{-\frac{y}{\beta}}}{\beta^{\alpha} \cdot \Gamma(\alpha)}$ then pdf of $\chi^2(v)$

$\frac{y^{\alpha-1} \cdot e^{-\frac{y}{2}}}{2^{\alpha} \cdot \Gamma(\alpha)}$, if $\alpha = 4$ then same at (a)

$\therefore Y \sim \chi^2(8)$

(c) mean $= 8$, $sd = 4$ (property of χ^2-dist)

(d) $P(|Y-8| < 8) \Leftrightarrow P(0 < Y < 16) \simeq 0.96$ (calculate by R)

4.110 If Y has a probability density function given by

$$f(y) = \begin{cases} 4y^2 e^{-2y}, & y > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

obtain $E(Y)$ and $V(Y)$ by inspection.

(5PtS)

Y has a gamma$(3, 0.5)$, $\therefore E(Y) = 1.5$, $V(Y) = 0.75$ $\Rightarrow$ Not get a full Points !

$E(Y) = \int_0^{\infty} y \cdot f(y) dy = \int_0^{\infty} 4y^3 e^{-2y} dy \overset{z=2y}{=} \frac{1}{4} \int_0^{\infty} z^3 \cdot e^{-z} dz = \frac{1}{4} \times \Gamma(4) = \frac{6}{4} = 1.5$

$E(Y^2) = \int_0^{\infty} y^2 f(y) dy = \int_0^{\infty} 4y^4 \cdot e^{-2y} dy = \frac{1}{8} \int_0^{\infty} z^4 \cdot e^{-z} dz = \frac{1}{8} \times \Gamma(5) = 3$

$V(Y) = E(Y^2) - (E(Y))^2 = 3 - (1.5)^2 = 0.75$

4.129 During an eight-hour shift, the proportion of time Y that a sheet-metal stamping machine is down for maintenance or repairs has a beta distribution with $\alpha = 1$ and $\beta = 2$. That is,

$$f(y) = \begin{cases} 2(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The cost (in hundreds of dollars) of this downtime, due to lost production and cost of maintenance and repair, is given by $C = 10 + 20Y + 4Y^2$. Find the mean and variance of C.

(5PtS)

$E(C) = 10 + 20E(Y) + 4E(Y^2)$. $E(Y) = \int_0^1 2y - 2y^2 dy = \frac{1}{3}$, $E(Y^2) = \int_0^1 (2y^2 - 2y^3) dy = \frac{1}{6}$

$= 10 + \frac{20}{3} + \frac{2}{3} = \frac{52}{3}$

$V(C) = E(C^2) - [E(C)]^2 = E(C^2) - \left(\frac{52}{3}\right)^2$

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

$E(C^2) = E(100+400Y^2+16Y^4+400Y+160Y^3+80Y^2)$

$= 100+400E(Y)+480E(Y^2)+160E(Y^3)+16E(Y^4)$

$E(Y^3) = \int_0^1 (2y^3-2y^4)\,dy = \frac{1}{10}$, $E(Y^4) = \int_0^1 (2y^4-2y^5)\,dy = \frac{1}{15}$ by using it,

$V(C) = 29.96$

4.130 Prove that the variance of a beta-distributed random variable with parameters α and β is (10pts)

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

$f(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot y^{\alpha-1} \cdot (1-y)^{\beta-1}$

$E(Y) = \int_0^1 y f(y)\,dy = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{\alpha} \cdot (1-y)^{\beta-1}\,dy = \underbrace{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 y^{(\alpha+1)-1} \cdot (1-y)^{\beta-1}\,dy}_{\text{pdf of beta}(\alpha+1,\beta)} \times \frac{\Gamma(\alpha+\beta)\cdot\alpha}{\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta}$

$E(Y^2) = \int_0^1 y^2 f(y)\,dy = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{\alpha+1} \cdot (1-y)^{\beta-1}\,dy = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta)} \cdot \int_0^1 y^{(\alpha+2)-1} \cdot (1-y)^{\beta-1}\,dy \times \frac{\Gamma(\alpha+\beta)\,\alpha(\alpha+1)}{\Gamma(\alpha+\beta+2)} =$

$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}$.

$\sigma^2 = E(Y^2) - \{E(Y)\}^2 = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

4.133 The proportion of time per day that all checkout counters in a supermarket are busy is a random variable Y with a density function given by

$$f(y) = \begin{cases} cy^2(1-y)^4, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the value of c that makes $f(y)$ a probability density function.

b Find $E(Y)$. (Use what you have learned about the beta-type distribution. Compare your answers to those obtained in Exercise 4.28.)

c Calculate the standard deviation of Y.

(a) $c = 105$ (see pdf of beta (α,β) : 4.130), $f(y) = \text{beta}(3,5)$

(b) by 4.130 $E(Y) = \frac{\alpha}{\alpha+\beta} \Rightarrow \frac{3}{8}$

(c) $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{15}{8^2\cdot 9} = \frac{5}{192}$, $\sigma = \sqrt{5/192}$

4.136 Suppose that the waiting time for the first customer to enter a retail shop after 9:00 A.M. is a random variable Y with an exponential density function given by (5pts)

$$f(y) = \begin{cases} \left(\frac{1}{\theta}\right) e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the moment-generating function for Y.

b Use the answer from part (a) to find $E(Y)$ and $V(Y)$.

(a) $m_y(t) = E(e^{tY}) = \int_0^\infty \left(\frac{1}{\theta}\right) e^{(t-\frac{1}{\theta})y}\,dy = \left[\frac{1}{\theta}\cdot\left(\frac{\theta}{\theta t-1}\right) e^{(t-\frac{1}{\theta})y}\right]_0^\infty = \frac{1}{1-\theta t}$ $(t < \frac{1}{\theta})$

(b) $E(Y) = m_y'(0) = \theta$, $V(Y) = m_y''(0) - (m_y'(0))^2 = \theta^2$

4.140 Identify the distributions of the random variables with the following moment-generating functions:

a $m(t) = (1 - 4t)^{-2}$. $\rightarrow$ Gamma(2,4)

b $m(t) = 1/(1 - 3.2t)$. $\rightarrow$ exp(3.2)

c $m(t) = e^{-5t+6t^2}$. $\rightarrow$ $N(-5, 12)$

4.144 Consider a random variable Y with density function given by

$$f(y) = ke^{-y^2/2}, \qquad -\infty < y < \infty.$$

a Find k.

b Find the moment-generating function of Y.

c Find $E(Y)$ and $V(Y)$.

(a) $k = \frac{1}{\sqrt{2\pi}}$, $Y \sim N(0,1)$

(b) $m_y(t) = \exp^{(\mu t + \frac{1}{2}\sigma^2 t^2)}$ $\rightarrow$ see lecture note (3/21)

(c) $E(Y) = 0$, $V(Y) = 1$

4.172 Calls for dial-in connections to a computer center arrive at an average rate of four per minute. The calls follow a Poisson distribution. If a call arrives at the beginning of a one-minute interval, what is the probability that a second call will not arrive in the next 20 seconds?

Y = time between calls $\sim$ exp(4)

$\therefore P(Y > \frac{1}{5}) = \int_{\frac{1}{5}}^{\infty} 4e^{-4y}\,dy = e^{-\frac{4}{5}}$

***4.179** A retail grocer has a daily demand Y for a certain food sold by the pound, where Y (measured in hundreds of pounds) has a probability density function given by (5pts)

y^3

$$f(y) = \begin{cases} 3y^2, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(She cannot stock over 100 pounds.) The grocer wants to order $100k$ pounds of food. She buys the food at 6¢ per pound and sells it at 10¢ per pound. What value of k will maximize her expected daily profit?

$$\text{profit} = \begin{cases} 4 \cdot 100k & (Y \ge k) \\ 1000Y - 600k & (Y < k) \end{cases}$$

$E(\text{Profit}) = E(400k \cdot I(Y \ge k)) + E((1000y - 600k) \cdot I(y < k))$

$= 400k \cdot P(Y \ge k) + E(y \cdot I(y < k)) - 600k\, P(y < k)$

where I : indicator function. eg. $I(x \ge c) = \begin{cases} 1 & (x \ge c) \\ 0 & (\text{otherwise}) \end{cases}$

$$= 400k \cdot \int_k^1 3y^2\,dy + 1000\int_0^k y\cdot 3y^2\,dy - 600k\int_0^k 3y^2 dy$$

$$= 400k - 250k^4 \iff \text{To maximize profit, } k = (0.4)^{1/3}$$

4.184 Let Y denote a random variable with probability density function given by

$$f(y) = (1/2)e^{-|y|}, \qquad -\infty < y < \infty.$$

Find the moment-generating function of Y and use it to find $E(Y)$.

(5pts)
$\Rightarrow$ exactly same midterm practice 43

$$m(t) = \int_{-\infty}^{\infty} e^{ty}\cdot\left(\tfrac{1}{2}\right)\cdot e^{-|y|}dy = \int_0^{\infty} e^{ty}\cdot\tfrac{1}{2}\cdot e^{-y}dy + \int_{-\infty}^{0} e^{ty}\cdot\tfrac{1}{2}\cdot e^{y}dy$$

$$= \frac{1}{2}\int_0^{\infty} \underbrace{e^{y(t-1)}}_{\text{finite } t<1} dy + \frac{1}{2}\int_{-\infty}^{0} \underbrace{e^{y(t+1)}}_{\text{finite } t>-1} dy = \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t}\right)$$

$$\therefore m_Y(t) = \frac{1}{1-t^2} \quad (-1<t<1)$$

$$m(t) = \frac{1}{1-t^2} \quad m'(t) = \frac{2t}{(1-t^2)^2} \quad \text{using it, } E(Y) = m'(0) = 0$$

*4.193** Because

$$P(Y \le y | Y \ge c) = \frac{F(y) - F(c)}{1 - F(c)}$$

has the properties of a distribution function, its derivative will have the properties of a probability density function. This derivative is given by

$$\frac{f(y)}{1 - F(c)}, \qquad y \ge c.$$

We can thus find the expected value of Y, given that Y is greater than c, by using

$$E(Y|Y \ge c) = \frac{1}{1 - F(c)}\int_c^{\infty} y f(y)\,dy.$$

If Y, the length of life of an electronic component, has an exponential distribution with mean 100 hours, find the expected value of Y, given that this component already has been in use for 50 hours.

$$Y \sim \exp(1/100), \quad f(y) = \frac{1}{100}\exp^{-\frac{1}{100}y}, \quad F(y) = 1 - \exp^{-\frac{1}{100}y}$$

$$E(Y|Y\ge 50) = e^{\frac{1}{2}}\cdot\int_{50}^{\infty} y\cdot f(y)dy = e^{\frac{1}{2}}\cdot\underbrace{\int_{50}^{\infty}\frac{y}{100}\exp^{-\frac{y}{100}}dy}_{} = 150$$

$$\Rightarrow 50\times e^{-\frac{1}{2}} + 100\times e^{-\frac{1}{2}}$$

***4.198** **The Markov Inequality** Let $g(Y)$ be a function of the continuous random variable Y, with $E(|g(Y)|) < \infty$. Show that, for every positive constant k,

$$P(|g(Y)| \le k) \ge 1 - \frac{E(|g(Y)|)}{k}.$$

[*Note:* This inequality also holds for discrete random variables, with an obvious adaptation in the proof.]

Let $A = \{|g(y)| > k\}$, $A^c = \{|g(y)| \le k\}$

$$E(|g(Y)|) = \int_A |g(y)| f(y)\,dy + \int_{A^c} |g(y)| \cdot f(y)\,dy$$

$$\ge \int_A |g(y)| f(y)\,dy > \int_A k f(y)\,dy = k \cdot P(|g(Y)| > k)$$

$$\therefore E(|g(Y)| \ge k \cdot P(|g(Y)| \ge k) \iff P(|g(Y)| \ge k) \le \frac{E(|g(Y)|)}{k}$$

$$\therefore \underbrace{1 - P(|g(Y)| \ge k)}_{P(|g(Y)| \le k)} \ge 1 - \frac{E(|g(Y)|)}{k}$$