Modified on Wednesday 4th November, 2020, 12:01:0912:01

Nov 4, 2020

- Norms of vectors and matrices
- Vector space: addition and scalar multiplication
- Definition. Let V be a vector space. A norm in V is a real-valued function defined on V, $\mathbf{v} \mapsto \|\mathbf{v}\|$, satisfying
  - for all nonzero vector $\mathbf{v}$, $\|\mathbf{v}\| > 0$,
  - for all scalar c and vector $\mathbf{v}$, $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$,
  - for all vectors $\mathbf{v}$ and $\mathbf{w}$, $\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.
- Various norms in $\mathbb{R}^n$. $\mathbf{v} = (v_i)$.
  - ℓ^1 norm, 1-norm, $\|\mathbf{v}\|_1 = |v_1| + |v_2| + \cdots + |v_n|$. $\Diamond$
  - ℓ^2 norm, Euclidean norm, $\|\mathbf{v}\|_2 = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}$. $\circ$
  - ℓ^∞ norm, max norm, $\|\mathbf{v}\|_\infty = \max(|v_1|, |v_2|, \ldots, |v_n|)$. $\Box$
  - For $1 \leq p \leq \infty$, ℓ^p norm, $\|\mathbf{v}\|_p = (|v_1|^p + |v_2|^p + \cdots + |v_n|^p)^{1/p}$.
- Find the minimum of $\|\mathbf{v}\|_p$ on the hyperplane $\mathbf{a}^T\mathbf{v} = 1$. The solution depends on p. So choosing a suitable norm is important in application.
- Inner product and angles. Recall the standard inner product of $\mathbb{R}^n$, i.e., the dot product.

$$\mathbf{v} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{v} = \|\mathbf{v}\|^2$$

- Norms and inner products of functions
- Any symmetric positive definite matrix S determines an inner product and a norm. $\langle \mathbf{v}, \mathbf{w} \rangle_S = \mathbf{v}^T S \mathbf{w}$, $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle_S$.
- A sequence of vectors $\mathbf{x}_i$ in $\mathbb{R}^n$ is called a Cauchy sequence, if for any $\epsilon > 0$, there exists N such that $\|\mathbf{x}_k - \mathbf{x}_l\| < \epsilon$ for all $k, l \geq N$.
- Does any Cauchy sequence converge in $\mathbb{R}^n$? In fact, yes it does.
- A vector space is called a normed vector space, if it has a norm. A normed vector space is called *complete*, if every Cauchy sequence has a limit in the space, i.e., if every Cauchy sequence in the space converges in the space.
- A complete normed vector space is called a *Banach space.*
- A Banach space with the norm induced from an inner product is called a *Hilbert space.*
- Generalizations of $\mathbb{R}^n$
  - Let V be the vector space of all sequences $\mathbf{v} = (v_i)_{i \geq 1}$ of the form

$$v_i = 0, \quad \forall i > N$$

    for some positive integer N. V is a vector space with infinite dimension. V with ℓ^2 norm is not complete. The sequence $\mathbf{v}_n = (1, \frac{1}{2}, \ldots, \frac{1}{2^n}, 0, \ldots)$ for $n = 0, 1, 2, \ldots$ is a Cauchy sequence in ℓ^2 norm, but it doesn't have a limit in the space.
    The sequence $\mathbf{u}_n = (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, \ldots)$ for $n = 1, 2, \ldots$ is a Cauchy sequence in ℓ^1 norm — not in ℓ^2 norm — but it doesn't have a limit in the space.

- Let W be the vector space of all sequences $\mathbf{v} = (v_i)_{i\geq 1}$ satisfying

$$\|\mathbf{v}\|^2 = \sum_{i\geq 1} v_i^2 < \infty.$$

  W is a vector space with infinite dimension. W with ℓ^2 norm is complete. W is a Hilbert space with an inner product $\langle (a_i), (b_i)\rangle = \sum_{i\geq 1} a_i b_i$.

Note that $V \subset W$.

- Norms of matrices
- Matrix norm. A mapping $A \mapsto \|A\|$ is called a *matrix norm*, if it satisfies the following.
  - For any nonzero matrix A, $\|A\| > 0$
  - For any scalar c and matrix A, $\|cA\| = |c|\|A\|$
  - For any matrices A and B, $\|A + B\| \leq \|A\| + \|B\|$
- Recall norms for matrices that have been introduced. Let A be an $m \times n$ matrix of rank r.
  - Spectral norm, $\|A\|_2 = \max_{\|\mathbf{x}\|_2 \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sigma_1$
  - Frobenius norm, $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{\mathrm{tr}(A^T A)} = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$
  - Nuclear norm, $\|A\|_N = \sigma_1 + \cdots + \sigma_r$

  Show that these norms satisfy $\|AB\| \leq \|A\|\|B\|$.
- Vector norms induce matrix norms
  - ℓ^2 norm induces $\|A\|_2$, $\|A\|_2 = \max_{\|\mathbf{x}\|_2 \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sigma_1$
  - ℓ^1 norm induces, $\|A\|_1$, $\|A\|_1 = \max_{\|\mathbf{x}\|_1 \neq 0} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1}$, the largest ℓ^1 norm of the columns of A
  - ℓ^∞ norm induces, $\|A\|_\infty$, $\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty \neq 0} \frac{\|A\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty}$, the largest ℓ^1 norm of the rows of A

    Proof. Let $A_1, \ldots, A_n$ be rows of A. $A\mathbf{x} = \begin{bmatrix} A_1\mathbf{x} & \cdots & A_n\mathbf{x}\end{bmatrix}^T$.

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty \neq 0} \frac{\|A\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \max_i |A_i\mathbf{x}| = \max_i \|A_i\|_1$$

  Show that these norms satisfy $\|AB\| \leq \|A\|\|B\|$.
- Notice that $\|A\|_\infty = \|A^T\|_1$. We can show that $\|A\|_2^2 \leq \|A\|_1\|A\|_\infty$.

$$\sigma_1^2\|\mathbf{v}\|_1 = \|A^T A\mathbf{v}\|_1 \leq \|A^T\|_1\|A\mathbf{v}\|_1 \leq \|A^T\|_1\|A\|_1\|\mathbf{v}\|_1 = \|A\|_\infty\|A\|_1\|\mathbf{v}\|_1$$

- Consider the following mappings.
  - $A = (a_{ij}) \mapsto \max_{i,j} |a_{ij}|$
  - $A \mapsto \max_\lambda |\lambda|$, where maximum is taken among all eigenvalues of A.

  Do these define norms?