School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2021 Spring

Final

Tuesday, June 15, 2021
9:00–11:20 am

- Be sure to **show all relevant work and reasoning** in your answer sheet. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.
- Please be clear in writing—we can't grade what we can't decipher!
- Don't forget to upload your answer sheet during 11:10-11:20 am through KLMS. The system will be automatically closed at that time. If the system does not work, you should email it to `ee210b_21spring@kaist.ac.kr` by 11:20 am. Late submissions will not be accepted/graded.

<u>Problem 1</u> (10 Points)

a) (5 points) Find the smallest n, the number of samples, for which the Chebyshev inequality yields a guarantee

$$\Pr(|M_n - p| \geq 0.5) \leq 0.05. \tag{1}$$

Assume that $\operatorname{var}(X_i) = v$ for some constant v. State your answer as a function of v.

Since $\mathbb{E}[M_n] = p$ and $\operatorname{var}(M_n) = \frac{v}{n}$, by Chebyshev inequality,

$$\Pr(|M_n - p| \geq 0.5) \leq \frac{\operatorname{var}(M_n)}{0.5^2} = \frac{v}{n \cdot 0.5^2} = 0.05. \tag{2}$$

The required n is $80v$.

b) (5 points) Assume that $n = 10000$. Find an approximate value for the probability

$$\Pr(|M_{10000} - p| \geq 0.5) \tag{3}$$

using the Central Limit Theorem. Assume again that $\operatorname{var}(X_i) = v$ for some constant v. Give your answer in terms of v, and the standard normal CDF $\Phi(\cdot)$.

By CLT, we can approximate

$$\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{nv}} \tag{4}$$

by a standard normal distribution when n is large. Hence,

$$\Pr(|M_{10000} - p| \geq 0.5) = \Pr\left(\left|\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{nv}}\right| \geq \frac{0.5\sqrt{n}}{\sqrt{v}}\right) = 2\left(1 - \Phi\left(\frac{50}{\sqrt{v}}\right)\right). \tag{5}$$

Problem 2 (10 Points)

Consider a biased coin where the coin lands with head with probability equal to $q \in [0, 1]$. The probability of head, q, is sampled from a random variable Q with pdf

$$f_Q(q) = \begin{cases} 6q(1-q), & 0 \le q \le 1 \\ 0, & \text{o.w.} \end{cases} \tag{6}$$

and once it is sampled the value is fixed during the experiments. We flip the coin n times and count the number of heads, K, which is a random variable. Given $K = k$, derive the following estimates of Q:

a) (5 points) Find the MAP estimator, $\hat{q}_{\mathsf{MAP}} = \arg\max_q f_{Q|K}(q|k)$ where $f_{Q|K}(q|k)$ is the conditional pdf of Q given $K = k$..

   Note that for a fixed q, K is distributed by binomial(n, q). Thus, the conditional PMF of K given $Q = q$ is

$$P_{K|Q}(k|q) = \binom{n}{k} q^k (1-q)^{n-k}. \tag{7}$$

   By Bayes' rule, the conditional pdf of Q given $K = k$ is

$$f_{Q|K}(q|k) = \frac{P_{K|Q}(k|q) f_Q(q)}{P_K(k)} \tag{8}$$

   Note that only the numerator depends on q. Thus, we need to find q that maximizes

$$P_{K|Q}(k|q) f_Q(q) = 6\binom{n}{k} q^{k+1}(1-q)^{n-k+1}. \tag{9}$$

   By taking derivatives, we can solve q that satisfies

$$\frac{d[P_{K|Q}(k|q) f_Q(q)]}{dq} = 6\binom{n}{k} q^k (1-q)^{n-k}[(k+1)(1-q) - (n-k+1)q] = 0, \tag{10}$$

   which is

$$\hat{q}_{\mathsf{MAP}} = \frac{k+1}{n+2}. \tag{11}$$

b) (5 points) Find the least mean square estimator, $\hat{q}_{\mathsf{LMS}} = \mathbb{E}[Q|K = k]$.

   To find $\mathbb{E}[Q|K = k]$, we need to first calculate

$$f_{Q|K}(q|k) = \frac{P_{K|Q}(k|q) f_Q(q)}{P_K(k)}. \tag{12}$$

   Note that

$$P_K(k) = 6\binom{n}{k} \int_0^1 q^{k+1}(1-q)^{n-k+1} dq. \tag{13}$$

By using

$$\int_0^1 p^l(1-p)^{m-l}dp = \frac{l!(m-l)!}{(m+1)!} \text{ for } 0 \le l \le m,$$

we get

$$P_K(k) = 6\binom{n}{k}\frac{(k+1)!(n-k+1)!}{(n+3)!}. \tag{14}$$

Thus, the conditional pdf of Q given $K = k$ is

$$f_{Q|K}(q|k) = \frac{P_{K|Q}(k|q)f_Q(q)}{P_K(k)} = \begin{cases} \frac{(n+3)!}{(k+1)!(n-k+1)!}q^{k+1}(1-q)^{n-k+1}, & 0 \le q \le 1 \\ 0, & \text{o.w.} \end{cases} \tag{15}$$

By using the fact that

$$\int_0^1 \frac{(i+j-1)!}{(i-1)!(j-1)!}p^i(1-p)^{j-1}dp = \frac{i}{i+j}.$$

we can calculate

$$\hat{q}_{\mathsf{LMS}} = \mathbb{E}[Q|K=k] = \frac{k+2}{n+4}. \tag{16}$$

Problem 3 (15 Points)

We conduct an elementary experiment (e.g. some physical experiment) independently total N times, where N is a Poisson random variable of mean λ, i.e., $\mathbb{P}(N = n) = \frac{\lambda^n e^{-\lambda}}{n!}$. The outcome of each experiment is sampled from a set $\{a_1, \ldots, a_K\}$, where the probability of getting an outcome a_k is equal to p_k for $1 \leq k \leq K$ where $\sum_{k=1}^{K} p_k = 1$.

a) (3 points) Let N_k denote the number of experiments performed for which the outcome is equal to a_k. Find the PMF for N_k $(1 \leq k \leq K)$. (Hint: no calculation is necessary.)

We can view the experiment as a combination of K Poisson processes where the k-th process has rate $p_k\lambda$ and the combined process has rate λ. At $t = 1$, the total number of experiments is Poisson with mean λ and the k-th process is Poisson with mean $p_k\lambda$. Thus,

$$p_{N_k}(n) = \frac{(\lambda p_k)^n e^{-\lambda p_k}}{n!}. \tag{17}$$

b) (3 points) Find the PMF of $N_1 + N_2$.

By the same argument,

$$p_{N_1+N_2}(n) = \frac{(\lambda(p_1 + p_2))^n e^{-\lambda(p_1+p_2)}}{n!}. \tag{18}$$

c) (3 points) Find the conditional PMF for N_1 given that $N = n$.

Each of the n combined arrivals over $(0, 1]$ is then a_1 with probability p_1. Thus, N_1 is binomial given that $N = n$,

$$p_{N_1|N}(n_1|n) = \binom{n}{n_1}(p_1)^{n_1}(1 - p_1)^{n-n_1}. \tag{19}$$

d) (3 points) Find the conditional PMF for $N_1 + N_2$ given that $N = n$.

Let the sample value of $N_1 + N_2$ be n_{12}. By the same argument as in (c),

$$p_{N_1+N_2|N}(n_{12}|n) = \binom{n}{n_{12}}(p_1 + p_2)^{n_{12}}(1 - p_1 - p_2)^{n-n_{12}}. \tag{20}$$

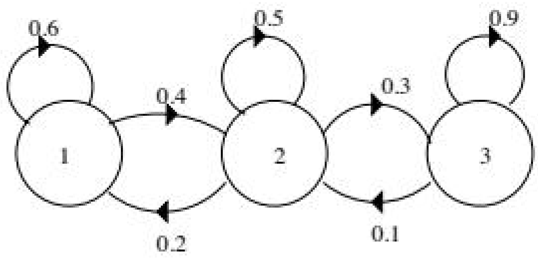
e) (3 points) Find the conditional PMF for N given that $N_1 = n_1$.

Since N is then n_1 plus the number of arrivals from the other processes, and those additional arrivals are Poisson with mean $\lambda(1 - p_1)$, we have

$$p_{N|N_1}(n|n_1) = \frac{(\lambda(1 - p_1))^{n-n_1} e^{-\lambda(1-p_1)}}{(n - n_1)!}, \quad \text{for } n \geq n_1. \tag{21}$$

Problem 4 (15 Points)

Consider a Markov chain $\{X_n : n = 0, 1, \ldots, \}$, specified by the following transition diagram.



a) (3 points) Given that the chain starts with $X_0 = 1$, find the probability that $X_2 = 2$.

The two-step transition probability is

$$\begin{aligned} r_{12}(2) &= p_{11} \cdot p_{12} + p_{12} \cdot p_{22} \\ &= 0.6 \cdot 0.4 + 0.4 \cdot 0.5 = 0.44. \end{aligned} \tag{22}$$

b) (3 points) Find the steady-state probabilities π_1, π_2, π_3 for the state 1,2, and 3.

We set up the balance equations of a birth-death process and the normalization equation as such:

$$\begin{aligned} \pi_1 p_{12} &= \pi_2 p_{21} \\ \pi_2 p_{23} &= \pi_3 p_{32} \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned} \tag{23}$$

Solving the system of equations yields the following steady-state probabilities:

$$\begin{aligned} \pi_1 &= 1/9, \\ \pi_2 &= 2/9, \\ \pi_3 &= 6/9. \end{aligned} \tag{24}$$

c) (3 points) Let $Y_n = X_n - X_{n-1}$. Thus, $Y_n = 1$ indicates that the n-th transition was to the right, $Y_n = 0$ indicates it was a self-transition, and $Y_n = -1$ indicates it was a transition to the left. Find $\lim_{n\to\infty} \mathbb{P}(Y_n = 1)$.

Using the total probability theorem and steady-state probabilities,

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}(Y_n = 1) &= \sum_{i=1}^{3} \pi_i \cdot \mathbb{P}(Y_n = 1 | X_{n-1} = i) \\ &= \pi_1 p_{12} + \pi_2 p_{23} = 1/9. \end{aligned} \tag{25}$$

d) (3 points) Given that the n-th transition was a transition to the right ($Y_n = 1$), find the probability that the previous state was state 1. (You can assume that n is large.)

Using Bayes' Rule,

$$\begin{aligned}\mathbb{P}(X_{n-1}=1|Y_n=1) &= \frac{\mathbb{P}(X_{n-1}=1)\mathbb{P}(Y_n=1|X_{n-1}=1)}{\sum_{i=1}^{3}\mathbb{P}(X_{n-1}=i)\mathbb{P}(Y_n=1|X_{n-1}=i)} \\ &= \frac{\pi_1 p_{12}}{\pi_1 p_{12}+\pi_2 p_{23}} = 2/5. \end{aligned} \tag{26}$$

e) (3 points) Suppose that $X_0 = 1$. Let T be defined as the first positive time at which the state is again equal to 1. Show how to find $\mathbb{E}[T]$. (It is enough to write down whatever equations need to be solved; you do not need to actually solve it to produce a numerical answer.)

In order to find the mean recurrence time of state 1, the mean first passage times to state 1 are first calculated by solving the following system of equations:

$$\begin{aligned} t_2 &= 1 + p_{22}t_2 + p_{23}t_3 \\ t_3 &= 1 + p_{32}t_2 + p_{33}t_3, \end{aligned} \tag{27}$$

which is

$$\begin{aligned} t_2 &= 1 + 0.5t_2 + 0.3t_3 \\ t_3 &= 1 + 0.1t_2 + 0.9t_3, \end{aligned} \tag{28}$$

The mean recurrence time of state 1 is then given by $t_1^* = 1 + p_{12}t_2$, which is $t_1^* = 1 + 0.4t_2$.

Solving the system of equations yields $t_2 = 20$ and $t_3 = 30$ and $t_1^* = 9$. (no need to get these numbers to get the full credit)