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- Solving linear systems and $A = LU$: elementary row/column operation. [3.3, 3.5, 3.7]
- Row elementary matrices
  - Any matrix obtained from I_n by one elementary row operation is called a **row elementary matrix**.
  - $E(i; a)$: Multiply Row i by a nonzero constant a.
  - $E(i, j; a)$: Add a times Row i to Row j.
  - $E(i, j)$: Interchange Row i and Row j.
  - Row elementary matrices are invertible.
  - $E(i; a)^{-1} = E(i; \frac{1}{a})$, $E(i, j; a)^{-1} = E(i, j; -a)$, $E(i, j)^{-1} = E(i, j)$,
- Applying an elementary row operation to a matrix A is the same as multiplying A with the corresponding row elementary matrix from the left.
- Computation of A^{-1}. Let A, B be $n \times n$ matrices and B_i the ith column of B. Then

$$AB = A \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix} = \begin{bmatrix} AB_1 & AB_2 & \cdots & AB_n \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} = I_n$$

  if and only if $AB_i = \mathbf{e}_i$ for all i. The column vector B_i is a solution to $A\mathbf{x} = \mathbf{e}_i$.
- Using the reduced row echelon form to compute A^{-1}. (Gauss-Jordan elimination)

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \quad \longrightarrow \quad \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 8 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 3 & 3 & 8 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 2 & -\frac{3}{2} & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & -\frac{3}{2} & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 2 & -\frac{3}{2} & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{2} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{2} & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{2} & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{4} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{2} & 0 \end{array}\right]$$

$$E(1; 1/2), E(1, 2; -3), E(2, 3), E(2; 1/2), E(3; 1/2), E(3, 2; -1), E(3, 1; -2), E(2, 1; -1)$$

$$E(2, 1; -1)E(3, 1; -2)E(3, 2; -1)E(3; 1/2)E(2; 1/2)E(2, 3)E(1, 2; -3)E(1; 1/2)A = I_3$$

$$A^{-1} = E(2, 1; -1)E(3, 1; -2)E(3, 2; -1)E(3; 1/2)E(2; 1/2)E(2, 3)E(1, 2; -3)E(1; 1/2)$$

  So the last matrix is $\left[\begin{array}{c|c} I_3 & A^{-1} \end{array}\right]$.
- Theorem 3.3.9. If A is an $n \times n$ matrix, then the following statements are equivalent:
  1. The reduced row echelon form of A is I_n.
  2. A is a product of elementary matrices.
  3. A is invertible.
  4. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  5. $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.

6. $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$.

- L, U: $n \times n$, L is lower triangular, U is upper triangular
  - A square matrix $L = (a_{ij})$ is a lower triangular matrix if $a_{ij} = 0$ for all $i < j$.
  - A square matrix $U = (a_{ij})$ is a upper triangular matrix if $a_{ij} = 0$ for all $i > j$.
- LU-decomposition. Express A as a product, $A = LU$.
  - L: lower triangular, U: upper triangular.

Does any square matrix have an LU-decomposition? Some matrices don't have an LU-decomposition. If a square matrix A doesn't have an LU-decomposition, we can find a permutation matrix P such that PA, interchanging rows of A, has an LU-decomposition.

- Applying an elementary row operation to A is equivalent to multiplying the corresponding elementary matrix to A from the left hand side.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 3 & 3 & 8 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 8 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E(1;1/2), E(1,2;-3), E(2,3), E(2;1/2), E(3;1/2)$$

The above matrix doesn't have an LU-decomposition. The above computation tells us that if we interchange Row 2 and Row 3, the resulting matrix has an LU-decomposition. As shown below.

$$A' = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 2 & 2 \\ 3 & 3 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 3 & 3 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$E(1;1/2), E(1,3;-3), E(2;1/2), E(3;1/2)$$

$$E(3;1/2)E(2;1/2)E(1,3;-3)E(1;1/2)A' = U \to A' = [E(3;1/2)E(2;1/2)E(1,3;-3)E(1;1/2)]^{-1}U$$

$$L = [E(3;1/2)E(2;1/2)E(1,3;-3)E(1;1/2)]^{-1} = E(1;2)E(1,3;3)E(2;2)E(3;2)I_3$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Then we have $PA = A' = LU$, where P is a permutation interchanging Row 2 and Row 3.

Let's do one more example.

$$B = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 2 & 2 \\ 3 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 3 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$E(1;1/2), E(1,3;-3), E(2;1/2), E(2,3;-1)$$

$$E(2,3;-1)E(2;1/2)E(1,3;-3)E(1;1/2)B = U \to B = [E(2,3;-1)E(2;1/2)E(1,3;-3)E(1;1/2)]^{-1}U$$

$$L = [E(2,3;-1)E(2;1/2)E(1,3;-3)E(1;1/2)]^{-1} = E(1;2)E(1,3;3)E(2;2)E(2,3;1)I_3$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$B = LU$$

Why is it so easy to get the entries of L from the product $L = E(1;2)E(1,3;3)E(2;2)E(2,3;1)$?

- L: lower triangular, U: upper triangular. For any $\mathbf{b}$, linear systems $L\mathbf{x} = \mathbf{b}$, $U\mathbf{x} = \mathbf{b}$ are easy to solve.
- If $A = LU$, $A\mathbf{x} = \mathbf{b}$ is easy to solve as well. Since $A\mathbf{x} = LU\mathbf{x} = L(U\mathbf{x}) = \mathbf{b}$, we solve $A\mathbf{x} = \mathbf{b}$ by two separate steps: $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$.
- Advantage of LU-decomposition.
  Suppose A is a large matrix, say of $10^5 \times 10^5$. You are going to solve $A\mathbf{x} = \mathbf{b}$ for many $\mathbf{b}$, say 10^6. Then you can save costs by using LU-decomposition of A. First compute L and U, $A = LU$, and for various $\mathbf{b}$, solve $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$.
- In considering the complexity of an algorithm, we often count arithmetic operations with floating-point numbers to perform. The word **flop** is a short form of **floating-point operation** meaning *arithmetic operation with floating-point numbers.* An ordinary PC does 10 gigaflops (10^{10} flops) per second.
- Costs of solving $A\mathbf{x} = \mathbf{b}$. The augmented matrix, $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ is of $n \times (n+1)$.
  - Gaussian elimination, backsubstitution: $\frac{2}{3}n^3$, n^2
  - Gauss-Jordan elimination: $\frac{2}{3}n^3$, n^2
  - LU-decomposition, $L\mathbf{y} = \mathbf{b}$, $U\mathbf{x} = \mathbf{y}$: $\frac{2}{3}n^3$, n^2, n^2
  - A^{-1} by reducing $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ to $\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$: $2n^3$
  - $A^{-1}\mathbf{b}$: $2n^3$
- The geometry of linear systems
- A: $m \times n$ matrix, $A : \mathbb{R}^n \to \mathbb{R}^m$, a linear transformation.

$$A : \mathbf{x} \mapsto \mathbf{Ax}$$

  $A\mathbf{x} = x_1A_1 + \cdots + x_nA_n$, a linear combination of columns of A.
- Homogeneous system, $A\mathbf{x} = \mathbf{0}$.
  - $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace: closed under the addition and the scalar multiplication.
  - The set of all vectors in $\mathbb{R}^n$ orthogonal to the rows of A.

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{bmatrix}$$

- Nonhomogeneous system, $A\mathbf{x} = \mathbf{b}$.
  - A particular solution and a general solution
  - Structure of solutions
    For any $\mathbf{b} \in \mathbb{R}^m$ satisfying $A\mathbf{x}_0 = \mathbf{b}$ for some $\mathbf{x}_0 \in \mathbb{R}^n$, we have

$$\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\} = \mathbf{x}_0 + \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} = \{\mathbf{x}_0 + \mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ satisfying } A\mathbf{x} = \mathbf{0}\}.$$

    Proof. Let $W = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$. We can show that $\mathbf{x}_0 + W \subset \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$ and $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\} \subset \mathbf{x}_0 + W$.