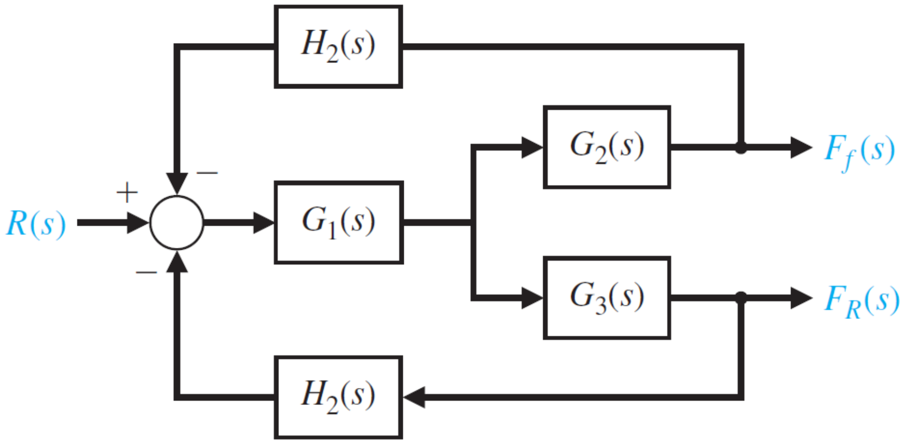
EE381 Control System Engineering

Mid-term Exam. (May 4, 2023)

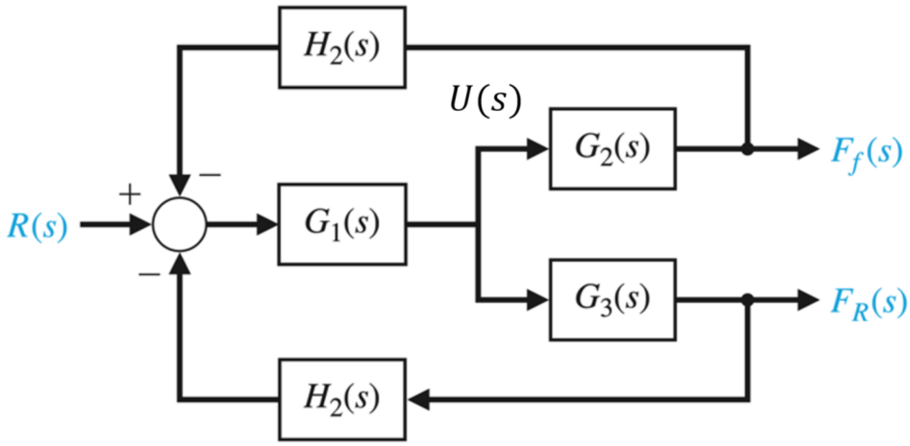
Score (________)

Student ID: ______________________ Name: ________________________

1. A four-wheel antilock automobile braking system uses electronic feedback to control automatically the brake force on each wheel. A block diagram model of a brake control system is shown in the figure below, where $F_f(s)$ and $F_R(s)$ are the braking force of the front and rear wheels, respectively, and $R(s)$ is the desired automobile response on an icy road. Find $F_R(s)/R(s)$. (**10 points**)



(Ans)



Let's define $U(s)$ as shown above.

Then,

$$F_f(s) = G_2(s)U(s) \qquad (1.1)$$
$$F_R(s) = G_3(s)U(s). \qquad (1.2)$$

We can solve the equation as

$$U(s) = \frac{1}{G_2(s)} F_f(s) \qquad (1.3)$$

$$F_R(s) = \frac{G_3(s)}{G_2(s)} F_f(s). \quad (1.4)$$

Also, considering the block diagram in the figure, we can say

$$U(s) = G_1(s)\left[R(s) - H_2(s)F_f(s) - H_2(s)F_R(s)\right] . \qquad (1.5)$$

Putting (1.5) into (1.3):

$$F_f(s) = G_1(s)G_2(s)\left[R(s) - H_2(s)F_f(s) - H_2(s)F_R(s)\right] . \qquad (1.6)$$

Putting (1.4) into (1.6)

$$F_f(s) = G_1(s)G_2(s)[R(s) - H_2(s)F_f(s) - H_2(s)\frac{G_3(s)}{G_2(s)}F_f(s)] . \qquad (1.7)$$

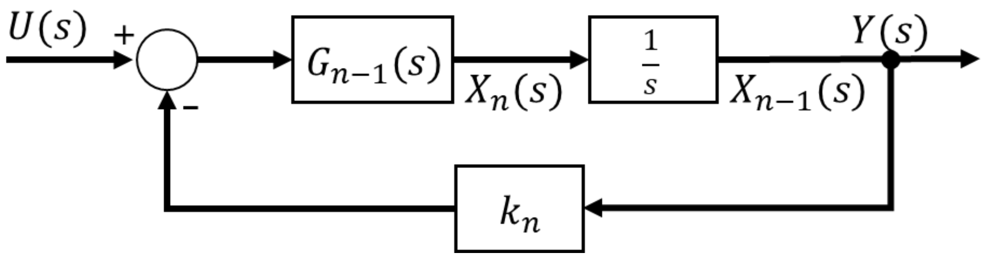
By solving the above equation, we can get

$$F_f(s) = \left[\frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_2(s) + G_1(s)G_3(s)H_2(s)}\right] R(s) .$$

By using (1.4)

$$\frac{F_R(s)}{R(s)} = \frac{G_1(s)G_3(s)}{1 + G_1(s)G_2(s)H_2(s) + G_1(s)G_3(s)H_2(s)} .$$

2. Consider the following control system that has the recursive form with input $U(s)$ and output $Y(s)$, as shown in the figure below:



(a) Rewrite the transfer function G_n in terms of G_{n-1}, k_n, and s, where n is an integer greater than 1. Then, express G_n using only in terms of s and k_i for $i = 1, 2, \cdots, n$. Assume that $G_n(s) = Y(s)/U(s)$ and $G_0 = 1$. (**5 points**)

(b) Consider the state-space equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

Using x_n and x_{n-1} shown in the above figure, formulate the state-space representation of the system with $n = 2$. Then, obtain the transfer function by evaluating the matrix representation of the system. Are the results of (a) and (b) the same for the case $n = 2$? (**5 points**)

(c) For the transfer function obtained in (b), if $k_1 = k_2 = 4$, is the system Bounded Input Bounded Output (BIBO) stable? (**5 points**)

(d) If all $k_i = 1$, how many poles does the system have? Determine whether the system is BIBO stable. Assume that n is the finite integer number greater than 4. (**10 points**)

(Ans)

(a) For the given control system, the transfer function $G_n = \frac{Y(s)}{U(s)}$. We can express G_n in terms of G_{n-1}, k_n, and s as follows:

$$G_n = \frac{Y(s)}{U(s)} = \frac{\frac{G_{n-1}}{s}}{1 + k_n \frac{G_{n-1}}{s}} = \left(k_n + \frac{1}{G_{n-1}} s\right)^{-1}.$$

Using this recursive relationship, we can express G_n as:

$$G_n = \left(k_n + \left(k_{n-1} + \frac{1}{G_{n-2}} s\right) s\right)^{-1} = (k_n + k_{n-1}s + k_{n-2}s^2 + \cdots + s^n)^{-1}.$$

(b) Formulating the state-space representation of the system with $n = 2$:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = [1 \quad 0], \mathbf{D} = 0.$$

The transfer function is obtained by evaluating the matrix representation: $G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$. First, we find $(s\mathbf{I} - \mathbf{A})^{-1}$:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -1 \\ k_2 & s + k_1 \end{bmatrix},$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\det(s\mathbf{I} - \mathbf{A})} \begin{bmatrix} s + k_1 & 1 \\ -k_2 & s \end{bmatrix} = \frac{1}{s^2 + k_1 s + k_2} \begin{bmatrix} s + k_1 & 1 \\ -k_2 & s \end{bmatrix}.$$

Now, we can calculate the transfer function:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = [1 \quad 0] \left(\frac{1}{s^2 + k_1 s + k_2} \begin{bmatrix} s + k_1 & 1 \\ -k_2 & s \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{1}{s^2 + k_1 s + k_2}.$$

(c) By using the result of (b),

$$G(s) = \frac{1}{s^2 + 4s + 4} = \frac{1}{(s + 2)^2}.$$

The system is BIBO stable if and only if all its poles have negative real parts. In this case, the poles of the system are located at $s = -2$. Since the real part of the pole is negative, the system is BIBO stable.

(d) If all $k_i = 1$, the transfer function becomes:

$$G(s) = \frac{1}{s^n + s^{n-1} + \cdots + s + 1}.$$

By using the identity $(s^n + \cdots + s + 1)(s - 1) = s^{n+1} - 1$, we get $s^n + \cdots + s + 1 = \frac{s^{n+1} - 1}{s - 1}$, thus:

$$G(s) = \frac{s - 1}{s^{n+1} - 1}.$$

For a BIBO stable system, all poles should lie in the left half of the complex plane. Using the above equation, the poles of the system are the solutions of $s^{n+1} - 1 = 0$, which lie on a circle of radius 1 centered at the origin in the complex plane. As n is a finite integer, there will be n poles on the circle except 1. Since the poles are not in the left half of the complex plane, the system is not BIBO stable.

3. Consider the single-input, single-output system described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \mathbf{C}\mathbf{x}(t) = [2 \quad 1]\mathbf{x}(t)$$

Assume that the input is a linear combination of the states, that is,

$$u(t) = -\mathbf{K}\mathbf{x}(t) + r(t)$$

where $r(t)$ is the reference input. The matrix $\mathbf{K} = [K_1 \quad K_2]$ is known as the gain matrix. The design process involves finding $\mathbf{K}$ so that the eigenvalues of the closed-loop system are at desired locations in the left-half plane.

(a) Compute the characteristic polynomial associated with the closed-loop system from $r(t)$ to $y(t)$. (**5 points**)

(b) Determine ranges of $\mathbf{K}$ so that the closed-loop eigenvalues are in the left-half plane. (**5 points**)

(Ans)

Computing the closed-loop system yields

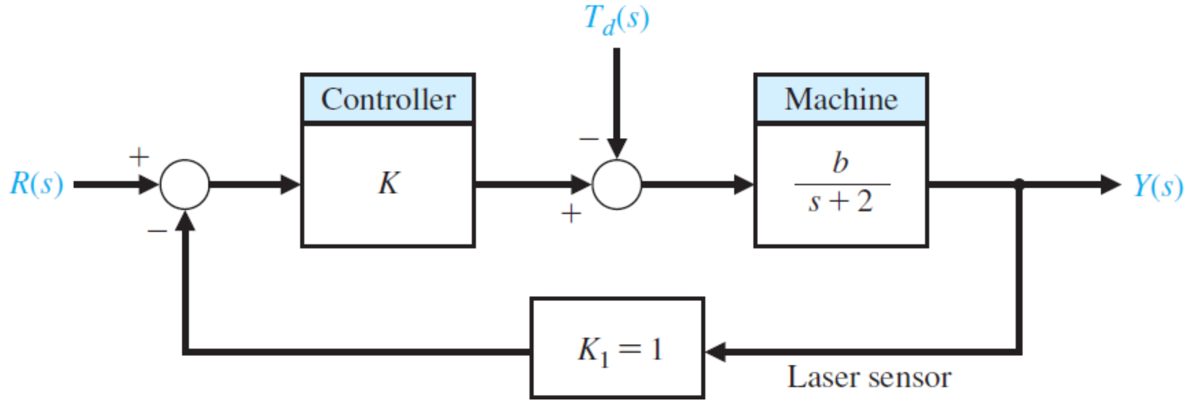
$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} -1 & 1 \\ -K_1 & -K_2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{and } \mathbf{C} = [2 \quad 1] .$$

(a) The characteristic polynomial is

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})| = s^2 + (K_2 + 1)s + K_1 + K_2 = 0 \quad .$$

(b) The roots are in the left-half plane whenever $K_2 + 1 > 0$ and $K_1 + K_2 > 0$.

4. The block diagram of a machine-tool control system is shown in the figure below.



(a) Determine the transfer function $T(s) = Y(s)/R(s)$. (**5 points**)
(b) Determine the sensitivity S_b^T. (**5 points**)
(c) Select K when $1 \le K \le 50$ so that the effects of the disturbance and S_b^T are minimized. (**5 points**)

(Ans)

(a) The closed-loop transfer function is

$$T(s) = \frac{Kb}{s + Kb + 2} .$$

(b) The sensitivity is determined to be

$$S_b^T = \frac{\partial T}{\partial b} \cdot \frac{b}{T} = \frac{s+2}{s + Kb + 2} .$$

(c) The transfer function from $T_d(s)$ to $Y(s)$ is

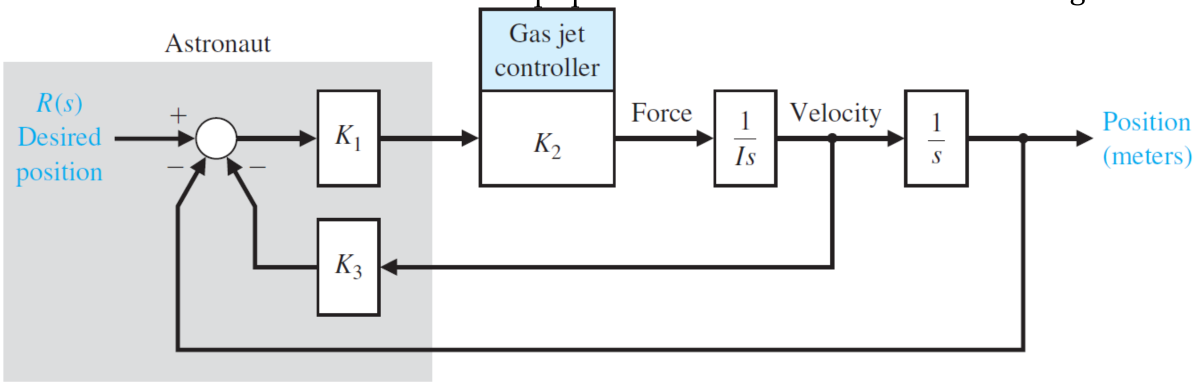
$$\frac{Y(s)}{T_d(s)} = \frac{b}{s + Kb + 2} .$$

So, choose K as large as possible, to make $Y(s)/T_d(s)$ as "small" as possible. Thus, select

$$K = 50 .$$

This also minimizes S_b^T at low frequencies.

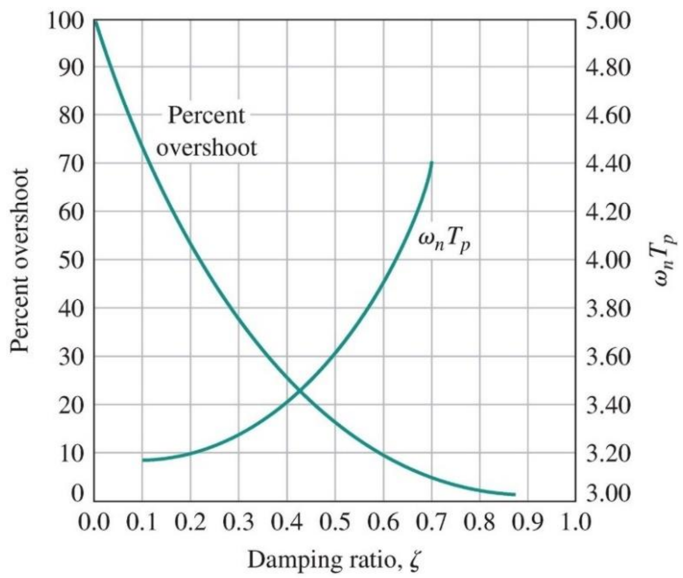
5. An astronaut took the first untethered walk in space using the gas-jet propulsion device. The controller can be represented by a gain K_2, as shown in the figure below. The moment of inertia of the equipment and astronaut is $I = 25$ kg m^2.



(a) Determine the necessary gain K_3 to maintain a steady-state error equal to 1.0 cm when the input is a unit ramp. (**5 points**)

(b) With this gain K_3, determine the necessary gain K_1K_2 in order to restrict the percent overshoot to $P.O. \leq 10\%$. (**5 points**)

Please use the following graph regarding damping ratio vs. percent overshoot.



(Ans)

(a) The closed-loop transfer function is

$$T(s) = \frac{K_1K_2}{Is^2 + K_1K_2K_3 s + K_1K_2} \quad .$$

The steady-state tracking error for a ramp input is

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\big(1 - T(s)\big)R(s) = \lim_{s\to 0} s\big(1 - T(s)\big)\frac{1}{s^2}$$
$$= \lim_{s\to 0} \frac{Is + K_1K_2K_3}{Is^2 + K_1K_2K_3 s + K_1K_2} = K_3 \quad .$$

But we desire $e_{ss} = 0.01\text{m}$, so $K_3 = 0.01$.

(b) For *P.O.* = 10%, we have $\xi = 0.6$. Also,

$$2\xi\omega_n = \frac{0.01K_1K_2}{25}$$

and

$$\omega_n^2 = \frac{K_1K_2}{25} \quad .$$

Thus, solving for K_1K_2 yields $K_1K_2 = 36 \times 10^4$.

6. A feedback control system has a characteristic equation with positive K:

$$s^3 + (1+K)s^2 + 10s + (5+15K) = 0.$$

(a) What is the maximum value K can assume before the system becomes unstable? (**5 points**)

(b) When K is equal to the maximum value, the system oscillates. Determine the frequency of oscillation. (**5 points**)

(Ans)

(a) Given

$$s^3 + (1+K)s^2 + 10s + (5+15K) = 0.$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 10 \\ s^2 & 1+K & 5+15K \\ s^1 & b & \\ s^0 & 5+15K & \end{array}$$

where

$$b = \frac{(1+K)10 - (5+15K)}{1+K} = \frac{5-5K}{1+K} \ .$$

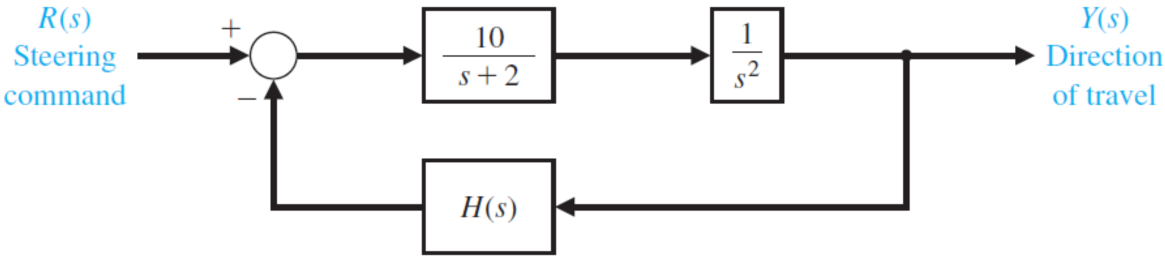
Given that $K > 0$, we determine that the system is stable when $5 - 5K > 0$ or $0 < K < 1$. So, the maximum value of K is 1.

(b) When $K = 1$, the s^2 row yields the auxiliary equation

$$2s^2 + 20 = 0 \ .$$

The roots are $s = \pm j\sqrt{10}$. So, the system frequency of oscillation is $\sqrt{10}$ rads/sec.

7. An automatically guided vehicle on Mars is represented by the system in the figure below. The system has a steerable wheel in both the front and back of the vehicle, and the design requires that $H(s) = Ks + 10$.



(a) Determine the value of K required for stability. (**5 points**)
(b) Determine the value of K when one root of the characteristic equation is equal to $s = -1$. (**5 points**)

(Ans)

(a) The closed-loop characteristic equation is

$$1 + \frac{10H(s)}{s^2(s+2)} = 1 + \frac{10(Ks+10)}{s^2(s+2)} = 0$$

or

$$s^3 + 2s^2 + 10Ks + 100 = 0 \ .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 10K \\ s^2 & 2 & 100 \\ s^1 & 10K-50 & \\ s^0 & 100 & \end{array}$$

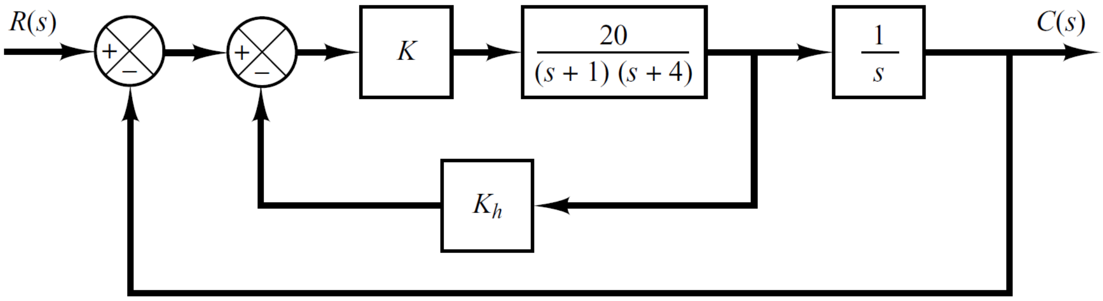
For stability, we require $K > 5$.

(b) The desired characteristic polynomial is

$$(s^2 + as + b)(s + 1) = s^3 + (a+1)s^2 + (a+b)s + b = 0 \ .$$

Equating coefficients with the actual characteristic equation we can solve for $a,\ b$ and K, yielding $a = 1, b = 100$ and $K = 10.1$.

8. Consider the servo system with tachometer feedback shown in the figure below. Determine the ranges of stability for K and K_h. (Note that K_h must be positive; x-axis for K; y-axis for K_h) (**10 points**)



(Ans)

From the block diagram, we have

$$\frac{C(s)}{R(s)} = \frac{20K}{s^3 + 5s^2 + (4 + 20KK_h)s + 20K} .$$

The stability of this system is determined by the characteristic polynomial. The Routh array of the characteristic equation

$$s^3 + 5s^2 + (4 + 20KK_h)s + 20K = 0$$

is

$$\begin{array}{c|cc} s^3 & 1 & 4 + 20KK_h \\ s^2 & 5 & 20K \\ s^1 & 4 + 20KK_h - 4K & 0 \\ s^0 & 20K & \end{array}$$

For stability, we require

$$4 + 20KK_h - 4K > 0, \quad 20K > 0$$

or

$$5KK_h > K - 1, \quad K > 0 .$$

The stable region in the $K - K_h$ plane is the region that satisfies these two inequalities. Figure 1 shows the stable region in the $K - K_h$ plane. If a point in the $K - K_h$ plane (that is, a combination of K and K_h value) lies in the shaded region, then the system is stable. Conversely, if a point in the $K - K_h$ plane lies in the nonshaded region, the system is unstable. The dividing curve is defined by $5KK_h = K - 1$. (Any point above this dividing curve corresponds to a stable combination of K and K_h.)

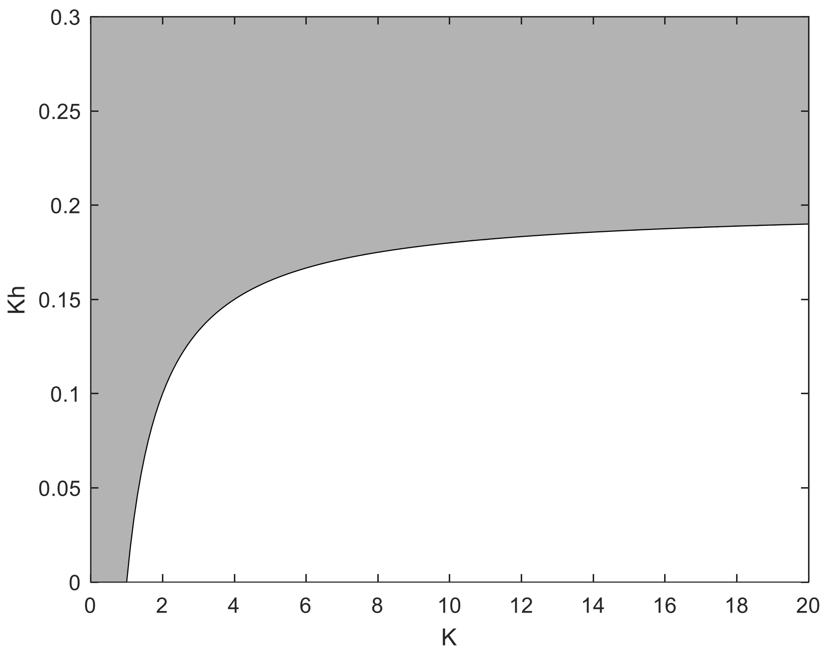


Figure 1. The stable region in the $K - K_h$ plane.