# CH 4. Continuous Random Variables and Their Probability Distributions

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# (Cumulative) Distribution Function (CDF)

Exists for any r.v !!

$$F(y) = F_Y(y) = P(Y \le y), \quad \text{for } -\infty < y < \infty.$$

- $F(-\infty) := \lim_{y \to -\infty} F(y) = 0$
- $F(\infty) := \lim_{y \to \infty} F(y) = 1$
- $F(y)$ is a non-decreasing function of $y$.

$Y$ is continuous if $F(y)$ is continous for $-\infty < y < \infty$.

(e.g.)

| $Y$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(y)$ | .4 | .3 | .2 | .1 |

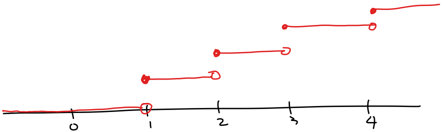
$F(-1) = 0$

$F(1) = .4$ $\quad F(2-) = .4$

$F(2) = .7$

⋮

$F(4) = 1$




- Step ft
- right-continous.

different unless $\boxed{Y \text{ is conti}}$

$$\begin{cases} P(a \le Y < b) = P(Y < b) - P(Y < a) = F(b-) - F(a-) \\ P(a < Y \le b) = P(Y \le b) - P(Y \le a) = F(b) - F(a) \end{cases}$$

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Some continuous RV, if their CDF is

<u>absolutely</u> continous, then it has a density.
(pdf)

We focus on those only !!

# Probability Density Function

$\cancel{f(y) = P(Y=y)}$

$$f(y) = \frac{dF(y)}{dy} = F'(y), \text{ or}$$

$$F(y) = \boxed{\int_{-\infty}^{y} f(t)dt = P(Y \le y)}$$

F : non-decreasing

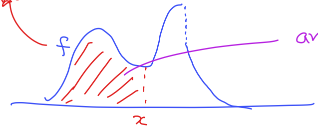
- $\boxed{f(y) \geq 0}$ for any value of $y$.
- $\int_{-\infty}^{\infty} f(y)dy = 1$.

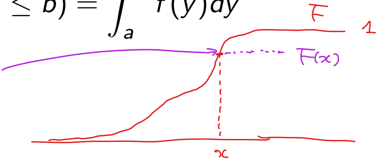
note : f(y) can be > 1
it is NOT a probability !!

Probability can be found by:

$$P(a \leq Y \leq b) = \int_a^b f(y)dy$$

· not bounded ) possible
· not continous )



# Expected Values for Continous r.v.

$$E(Y) = \mu = \int_{-\infty}^{\infty} yf(y)dy,$$

provided that the integral exists.

$g(y)$: a real-valued function of $Y$. Then

$$E(g(Y)) = \int_{-\infty}^{\infty} g(y)f(y)dy$$

$V(Y) = E[(Y-\mu)^2] = \int_{-\infty}^{\infty}(y-\mu)^2 f(y)dy = \sigma^2$: Variance

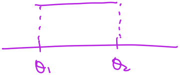
$\sigma = \sqrt{V(Y)}$: Standard deviation

# Uniform Probability Distribution

Flat

$Y$ has a continuous uniform probability distribution on the interval $(\theta_1, \theta_2)$ iff the density of $Y$ is

$$f(y) = \frac{1}{\theta_2 - \theta_1} \quad \theta_1 \le y \le \theta_2,$$




denoted as $\boxed{Y \sim U(\theta_1, \theta_2).}$

Check!!

$$\mu = E(Y) = \frac{\theta_1 + \theta_2}{2} \quad \text{and} \quad \sigma^2 = V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}.$$

Proof of Tchebyshev's inequality ⟶ ( Markov Ineq.)

$\forall k > 0, \qquad P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

LHS $= P(Y \leq \mu - k\sigma$ or $Y \geq \mu + k\sigma)$

$$= \int_{-\infty}^{\mu-k\sigma} 1 \cdot f(y)\,dy + \int_{\mu+k\sigma}^{\infty} f(y)\,dy$$

$y \leq \mu - k\sigma$

$\Leftrightarrow \quad y - \mu \leq -k\sigma$

$\Leftrightarrow \quad \frac{y-\mu}{k\sigma} \leq -1$

$\Leftrightarrow \quad \frac{(y-\mu)^2}{k^2\sigma^2} \geq 1$

$$\leq \int_{-\infty}^{\mu-k\sigma} \frac{(y-\mu)^2}{k^2\sigma^2} f(y)\,dy + \int_{\mu+k\sigma}^{\infty} \frac{(y-\mu)^2}{k^2\sigma^2} f(y)\,dy$$

$$\leq \frac{1}{k^2}\frac{1}{\sigma^2}\int_{-\infty}^{\infty} (y-\mu)^2 f(y)\,dy = \frac{1}{k^2}$$