Problem 1 (10 Points)

a) (5 points)

$$\begin{aligned}
\Pr(\text{error}) &= \Pr(\text{error}|X = -3)\Pr(X = -3) + \Pr(\text{error}|X = 3)\Pr(X = 3) \\
&= \frac{1}{2}\Pr(Y \geq 0|X = -3) + \frac{1}{2}\Pr(Y < 0|X = 3) \\
&= \frac{1}{2}\Pr(N \geq 3) + \frac{1}{2}\Pr(N < -3) \\
&= \Pr(N \geq 3) = \Pr\left(\frac{N-0}{3} \geq \frac{3-0}{3}\right) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587.
\end{aligned}$$

b) (5 points) By symmetry, $\Pr(\text{error}|\text{sent } 0) = \Pr(\text{error}|\text{sent } 1)$.

$$\Pr(\text{error}|\text{sent } 0) = \binom{3}{2}(0.1587)^2(1 - 0.1587) + \binom{3}{3}(0.1587)^3.$$

Problem 2 (10 Points)

a) (5 points) Let A be the event that the machine is functional. By the total probability theorem, the probability of A is given by

$$
\begin{aligned}
\Pr(A) &= \int_0^1 \Pr(A|T=t) f_T(t) dt \\
&= \int_0^1 t\, dt = 1/2.
\end{aligned}
$$

b) (5 points) Let B be the event that the machine is functional on m out of the last n days. Then, using the total probability theorem,

$$
\begin{aligned}
\Pr(B) &= \int_0^1 \Pr(B|T=t) f_T(t) dt \\
&= \int_0^1 \binom{n}{m} q^m (1-q)^{n-m} f_T(t) dt \\
&= \binom{n}{m} \frac{m!(n-m)!}{(n+1)!} = \frac{1}{n+1}.
\end{aligned}
$$

Then, using Bayes rule,

$$
\begin{aligned}
f_{T|B}(q) &= \frac{P(B|Q=q) f_Q(q)}{\Pr(B)} \\
&= \frac{\binom{n}{m} q^m (1-q)^{n-m}}{\binom{n}{m} \frac{m!(n-m)!}{(n+1)!}} = \frac{q^m(1-q)^{n-m}}{\frac{m!(n-m)!}{(n+1)!}}, \quad 0 \le q \le 1, \ n \ge m.
\end{aligned}
$$

Problem 3 (15 Points)

a) (5 points) Find the CDF of U_n, denoted by $F_{U_n}(u)$.

$$\begin{aligned}
F_{U_n}(u) &= \Pr(\max(X_1, \ldots, X_n) \le u) \\
&= \Pr(X_1 \le u, X_2 \le u, \ldots, X_n \le u) \\
&= \Pr(X_1 \le u) \Pr(X_2 \le u) \cdots \Pr(X_n \le u) \\
&= (F_X(u))^n.
\end{aligned}$$

b) (5 points) Find the CDF of L_n, denoted by $F_{L_n}(l)$.

$$\begin{aligned}
F_{L_n}(l) &= \Pr(\min(X_1, \ldots, X_n) \le l) \\
&= 1 - \Pr(\min(X_1, \ldots, X_n) > l) \\
&= 1 - \Pr(X_1 > l, X_2 > l, \ldots, X_n > l) \\
&= 1 - (1 - F_X(l))^n.
\end{aligned}$$

c) (5 points) Find the joint CDF of L_n and U_n, denoted by $F_{L_n,U_n}(l, u)$. Note that by the total probability theorem,

$$\Pr(U_n \le u) = \Pr(L_n > l, U_n \le u) + \Pr(L_n \le l, U_n \le u)$$

It is easier to analyze $\Pr(L_n > l, U_n \le u)$ since

$$\begin{aligned}
\Pr(L_u > l, U_n \le u) &= \Pr(\min(X_1, \ldots, X_n) > l, \max(X_1, \ldots, X_n) \le u) \\
&= \Pr(l < X_1 \le u, i = 1, \ldots, n) \text{ where } l \le u \\
&= \Pr(l < X_1 \le u) \cdots \Pr(l < X_n \le u) \\
&= (F_X(u) - F_X(l))^n.
\end{aligned}$$

When $l \ge u$ we have $\Pr(L_u > l, U_n \le u) = 0$.

By combining the results,

$$\begin{aligned}
F_{L_n,U_n}(l, u) &= \Pr(L_n \le l, U_n \le u) \\
&= \Pr(U_n \le u) - Pr(L_n > l, U_n \le u) \\
&= \begin{cases} (F_X(u))^n - (F_X(u) - F_X(l))^n & \text{for } l \le u \\ (F_X(u))^n & \text{for } l > u \end{cases}
\end{aligned}$$

Problem 4 (15 Points)

a) (5 points) What is the probability that Bob goes to work if he flips the coin once?

Let X_i be the outcome of a coin toss on the i-th trial, where $X_i = 1$ if the coin lands head, and $X_i = 0$ if the coin lands tail. By the total probability theorem

$$
\begin{aligned}
\Pr(X_i = 1) &= \int_0^1 P(X_i = 1 | Q = q) f(q) dq \\
&= \int_0^1 q * 2q dq \\
&= \frac{2}{3}.
\end{aligned}
$$

b) (5 points) Assume that when Bob goes to work he earns \$10 every day. Define X as the Bob's payout if he flips the coin every morning for the next 30 days. Find $\mathrm{var}(X)$.

Note that total payout $X = 10(X_1 + \cdots + X_{30})$ where X_i is the outcome of a coin toss on the i-th day and $X_i = 1$ if the coin lands head, and $X_i = 0$ if the coin lands tail. By the law of total variance,

$$
\begin{aligned}
\mathrm{var}(X) &= 100 \, \mathrm{var}(X_1 + \cdots + X_{30}) \\
&= 100 \left(\mathbb{E}[\mathrm{var}(X_1 + \cdots + X_{30} | Q)] + \mathrm{var}(\mathbb{E}[X_1 + \cdots + X_{30} | Q]) \right)
\end{aligned}
$$

Note that conditioned on $Q = q$, $X_1, \ldots, X_{30}$ are independent. Thus, $\mathrm{var}(X_1 + \cdots + X_{30} | Q) = \mathrm{var}(X_1 | Q) + \cdots \mathrm{var}(X_{30} | Q)$. And since X_i is a Bernoulli random variable with probability Q, its variance is $\mathrm{var}(X_i | Q) = Q(1 - Q)$. So,

$$
\begin{aligned}
\mathrm{var}(X) &= 100 \left(30 \mathbb{E}[Q(1 - Q)] + \mathrm{var}(30Q) \right) \\
&= 100 \left(30 \mathbb{E}[Q] - 30 \mathbb{E}[Q^2] + 900 \mathrm{var}(Q) \right).
\end{aligned}
$$

Note that $\mathbb{E}[Q] = \int_0^1 2q^2 = 2/3$, $\mathbb{E}[Q^2] = \int_0^1 2q^3 dq = 1/2$ and $\mathrm{var}(Q) = \mathbb{E}[Q^2] - (\mathbb{E}[Q])^2 = 1/2 - (2/3)^2 = 1/18$. Thus,

$$
\mathrm{var}(X) = 100 \times 30 \left(\frac{2}{3} - \frac{1}{2} \right) + 100 \times 900 \times \frac{1}{18} = 5500.
$$

c) (5 points) Let event B be that Bob stays home as least once in n days. Find the conditional density of Q given the event B, $f_{Q|B}(q)$. By Bayes Rule

$$
\begin{aligned}
f_{Q|B}(q) &= \frac{P(B | Q = q) f_Q(q)}{\int_0^1 P(B | Q = q) f_Q(q) dq} \\
&= \frac{(1 - q^k) 2q}{\int_0^1 (1 - q^k) 2q dq} \\
&= \frac{2q(1 - q^k)}{1 - 2/(k + 2)}, \quad \text{for } 0 \le q \le 1.
\end{aligned}
$$