Spring 2024 IE 241 – Midterm Practice

1. Five people get on an elevator that stops at five floors. Assuming that each has an equal probability of going to any one floor, find the probability thatthey all get off at different floors.

2. Let A and B be events. Prove the following.

   (a) If A and B are mutually exclusive and $P(A) > 0, P(B) > 0$, then they are not independent.

   (b) If A and B are independent , then $\bar{A}$ and $\bar{B}$ are independent.

   (c) $P(A \cap B) \geq P(A) + P(B) - 1$.

   (d) $P(A \cup B) \leq P(A) + P(B)$.

3. Suppose $P(B) = 0.2$, $P(A|B) = 0.5$, and $P(A \cap \bar{B}) = 0.3$. Find $P(A \cup B)$.

4. An electronic fuse is produced by three production lines in a manufacturing operation. Assume that production line 1 produces 10%, line 2 20%, and line 3 30% defective fuses, respectively. One day a customer received items, tested five fuses, and two failed. What is the probability that the five items were produced on line 3?

5. A die is loaded so that the probability of a face coming up is proportional to the number on that face. The die is rolled with outcome Y.

   (a) Find the probability distribution of Y. (Note that the die is not fair, so for example, $P(Y = 1) = 1/21$).

   (b) Find the probability that Y is less than 3 given that it is an odd number.

   (c) Are the events $A = \{Y < 3\}$ and $B = \{Y \text{is odd}\}$ independent?

   (d) Find the mean μ of Y.

   (e) Find the variance σ^2 of Y.

6. As items come to the end of a production line, an inspector chooses which items are to go through a complete inspection. Ten percent of all items produced are defective. Sixty percent of all defective items go through a complete inspection, and 20% of all good items go through a complete inspection. Given that an item is completely inspected, what is the probability it is defective?

7. Eight tires of different brands are ranked from 1 to 8 (best to worst) according to mileage performance. Assume that four of these tires are chosen at random by a customer. Let Y denote the actual quality rank of the best tire selected by the customer. For example, $(Y = 3)$

means that the best tire among those selected by the customer is actually ranked third among the original eight.

(a) Find the probability distribution of Y.

(b) Find $E[Y(Y+2)+1]$.

8. Suppose that two defective refrigerators have been included in a shipment of four refrigerators. The buyer begins to test the four refrigerators one at a time. Define the random variable Y as the number of tests until the buyer locates both of the defective refrigerators.

   (a) Find the probability distribution of Y.

   (b) Find $E[(Y - \frac{10}{3})^2]$.

9. In a class of 80 students, the professor calls on 1 student chosen at random for a recitation in each class period. There are 32 class periods in a term. Write a formula for the exact probability that a given student is called upon j times during the term.

10. A die is rolled until the first time Y that a six turns up.

    (a) What is the probability distribution for Y? Identify the parameter(s) of the distribution.

    (b) Find $P(Y > 3)$ and describe the event $\{Y > 3\}$ in words.

    (c) Find $P(Y > 6|Y > 3)$ and describe the event $\{Y > 6|Y > 3\}$ in words.

11. A Personnel director has two lists of applicants for jobs. List 1 contains the names of five women and two men, whereas list 2 contains the names of two women and six men. A name is randomly selected from list 1 and added to list 2. A name is then randomly selected from the augmented list 2. Given that the name selected from the augmented list 2 is that of a man, what is the probability that a woman's name was originally selected from list 1?

12. A manufactured lot of buggy whips has 20 items, of which 5 are defective. A random sample of 5 items is chosen to be inspected. Find the probability that the sample contains exactly one defective item

    (a) if the sampling is done with replacement.

    (b) if the sampling is done without replacement.

13. In a Tennis, Federer competes against Nadal in consecutive sets, and the game continues until one player wins three sets. Assume that, for each set, P(Federer wins)=.6, P(Nadal wins)=.4, and the outcomes of different sets are independent. Let Y stand for the number of sets Federer wins.

(a) Identify the elementary outcomes associated with each value of Y.

(b) Obtain the probability distribution Y.

(c) Compute the mean μ and the standard deviation σ for the random variable Y.

14. There are N cards labeled from 1 to N. One card is randomly selected and let Y be the number on the selected card.

(a) Find the probability distribution of Y.

(b) Find the $E(Y)$ and $V(Y)$.

15. Ten percent of the engines manufactured on an assembly line are defective. Engines are randomly selected one at a time and tested.

(a) What is the probability that the first nondefective engine will be found on the second trial?

(b) What is the probability that the third nondefective engine will be found on or before the fifth trial?

16. Suppose that $Y \sim Poisson(\lambda)$. Calculate $E(e^Y)$.

17. Let

$$f(y) = \frac{3^y e^{-3}}{y!}, \quad y = 0, 1, 2, \ldots,$$

be the probability distribution function of the random variable Y.

(a) Find the moment-generating function of Y.

(b) Find $E(Y)$.

18. Let the distribution of a random variable Y be

$$F(y) = \begin{cases} 0, & y \leq 0 \\ \frac{y}{12}, & 0 < y < 3 \\ \frac{y^2}{36}, & 3 \leq y < 6 \\ 1, & y \geq 6 \end{cases}$$

(a) Find the density function of Y.

(b) Find $P(Y \geq 1 | Y \leq 3)$.

(c) Find the mean.

19. (a) If Y has a geometric distribution with success probability p, show that

$$P(Y = \text{ an odd integer}) = \frac{p}{1-q^2}.$$

(b) If Y is $B(n,p)$, show that

$$E\left(\frac{Y}{n}\right) = p \quad \text{and} \quad E\left[\left(\frac{Y}{n} - p\right)^2\right] = \frac{p(1-p)}{n}.$$

20. The number of people entering the intensive care unit at a hospital on any single day possesses a Poisson distribution with a mean equal to five persons per day.

(a) Let Y be the number of people entering the intensive care unit on a particular day. Given that $Y > 2$, calculate the probability of $Y > 5$.

(b) Let Y be the number of people entering the intensive care unit on a particular day. Calculate the probability of $Y > 1$.

(c) The cost function of using the intensive care unit is given as $C = 3^{Y+1}$. Find the expected cost.

(d) Show that the moment-generating function of $W = 2Y + 1$ is

$$m_W(t) = e^{5e^{2t}+t-5}.$$

(e) Using **(c)**, Find $E(W)$.

21. Let

$$f(y) = \left(\frac{1}{2}\right)^y, \quad y = 1, 2, 3, \ldots,$$

be the probability distribution function of the random variable Y.

(a) Find the moment-generating function of Y.

(b) Find $E(Y)$.

22. Let the probability density function of a random variable Y be

$$f(y) = \begin{cases} y, & 0 \leq y \leq 1 \\ 1, & 1 < y \leq 1.5 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the distribution $F(y)$.

(b) Find $P(.5 \leq Y \leq 1.2)$.

(c) Find $E(Y)$.

23. Let Y be a random variable of the continuous type with p.d.f. $f(y)$, which is positive provided $0 < y < b < \infty$, and is equal to zero elsewhere. Show that

$$E(Y) = \int_0^b [1 - F(y)]dy,$$

where $F(y)$ is the distribution function of Y.

24. A function sometimes associated with continuous nonnegative random variables is the failure rate (or hazard rate) function, which is defined by

$$r(t) = \frac{f(t)}{1 - F(t)}$$

for a density function $f(t)$ with corresponding distribution function $F(t)$. Show that for an exponential density function, $r(t)$ is constant.

25. Suppose that Y has an exponential probability density function with mean $1/\lambda$.

(a) Show that, if $a > 0$ and $b > 0$,

$$P(Y > a + b | Y > a) = P(Y > b).$$

(b) Show that the moment-generating function of Y is $\lambda/(\lambda - t)$ if $t < \lambda$.

26. If events are occurring in time according to a Poisson distribution with mean λt, then the interarrival times between events have an exponential distributions with mean $1/\lambda$. If calls come into a police emergency center at the rate of 10 per hour, what is the probability that more than 15 minutes will elapse between the next two calls?

27. (a) Let $g(Y)$ be a nonnegative function of the random variable Y with a probability density function $f(y)$. Show that

$$P(g(Y) \geq k) \leq \frac{E(g(Y))}{k}.$$

(b) In (a), by taking $g(Y) = (Y - \mu)^2$ and $k = \epsilon^2$ where $\epsilon > 0$, prove that

$$P(|Y - \mu| \geq \epsilon) \leq \frac{V(Y)}{\epsilon^2}.$$

28. If $Y \sim N(\mu, \sigma^2)$, show that $E(|Y - \mu|) = \sigma\sqrt{2/\pi}$.

29. A store owner has overstocked a certain item and decides to use the following promotion to decrease the supply. The item has a marked priced of \$100. For each customer purchasing the

item during a particular day, the owner will reduce the price by a factor of one-half. Thus, the first customer will pay \$50 for the item, the second will pay \$25, and so on. Suppose that the number of customers who purchase the item during the day has a Poisson distribution with mean 4. Find the expected cost of the item at the end of the day.

30. If Y has a geometric distribution with success probability p,

    (a) Show that the moment-generating function of Y is $m_Y(t) = pe^t/(1 - qe^t)$ for $t < -\ln q$ where $q = 1 - p$.

    (b) Using (a), show that the moment-generating function of $W = Y - 1$ is $m_W(t) = p/(1 - qe^t)$ for $t < -\ln q$.

    (c) Using **(b)**, show that $E(W) = q/p$.

    (d) Using **(b)**, show that $V(W) = q/p^2$.

31. (a) When the moment generating function of Y is $\left(.2 + .8e^t\right)^5$, calculate the exact probability of $P(\mu - 2\sigma < Y < \mu + 2\sigma)$ using the distribution of Y where $\mu = E(Y)$ and $\sigma^2 = V(Y)$.

    (b) When the moment generating function of Y is $e^{4(e^t - 1)}$, calculate the exact probability of $P(\mu - 2\sigma < Y < \mu + 2\sigma)$ using the distribution of Y.

    (c) When the distribution of Y is unknown, find a lower bound of $P(\mu - 2\sigma < Y < \mu + 2\sigma)$.

32. Let the probability density function of a random variable Y be

$$f(y) = \begin{cases} cye^{-y^2}, & y > 0 \\ 0, & y \leq 0. \end{cases}$$

    (a) Find c.

    (b) Find the distribution function $F(y)$ of Y.

    (c) Find 50th percentile of Y.

33. Answer the following questions regarding moment generating function, $m(t)$, of Y.

    (a) If $m_Y(t) = (.2 + .8e^t)^5$, find the second moment μ_2' of Y.

    (b) If $m_Y(t) = e^{3t + 8t^2}$, find $P(-1 < Y < 9)$.

    (c) A circle of radius Y has area $A = \pi Y^2$. Let the moment generating function of Y is given as $m_Y(t) = (e^{2t} - 1)/2t$ where $t \neq 0$. Then, what is the mean of the area of the circle?

34. If $Y \sim N(\mu, \sigma^2)$, show that $E(|Y - \mu|) = \sigma\sqrt{2/\pi}$.

35. Let Y be a discrete random variable that takes on values 0, 1, 2 with probabilities $\frac{1}{2}, \frac{3}{8}, \frac{1}{8}$, respectively.

   (a) Find the mean μ and the variance σ^2 of Y.

   (b) Find the moment-generating function of Y, $m(t)$.

   (c) Verify that $E(Y) = m'(0)$ and that $E(Y^2) = m''(0)$.

36. Assume the number of policies an insurance salesman sells during a month is a random variable having the Poisson distribution with $\lambda = 9$.

   (a) With what probability can we assert, according to Tchebyshev's theorem, that he will sell at most 15 or at least 3 policies during any given month?

   (b) Without using Tchebyshev's theorem, how would you calculate the *exact* probability that he will sell at most 15 or at least 3 policies during any given month? Would this probability be larger or smaller than, or equal to your answer in (a)? (You don't need to calculate the actual probability for this question (b)).

37. Let Y be a random variable with the probability density function

$$f(y) = \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}}, \quad 0 \le y < \infty.$$

   Find the mean and variance of Y.

38. The number of arrivals N at a supermarket checkout counter in the time interval from 0 to t follows a Poisson distribution with mean λt. Let T denote the length of time until the second arrival. Find the density function for T. (Show the whole process of deriving $F(t)$ and $f(t)$.)

39. The lifetime of a certain kind of energy cell is a random variable having a normal distribution with a standard deviation of 25 hours. Find the mean of this distribution if the probability that an energy cell will last 400 hours is .10.

40. Let a random variable Y have a uniform distribution on $[0, 1]$.

   (a) Calculate the moment generating function of $X = -2\ln(Y)$. (Hint: $Y = e^{\ln(Y)}$)

   (b) Show that X has a Gamma distribution. Identify α and β of the Gamma distribution.

41. Consider the following game: a player throws a fair die repeatedly until he rolls a 2, 3, 4, 5, or 6. In other words, the player continues to throw the die as long as he rolls 1s. Let Y be the number of throws needed to obtain the first non-1.

   (a) What is the probability that the player tosses the die at least three times?

(b) If the player is paid 2^{Y-1} dollars, what is the expected amount paid to the player?

42. The joint density function of Y_1 and Y_2 i s given by

$$f(y_1, y_2) = \begin{cases} \frac{1}{8}(y_1^2 - y_2^2)e^{-y_1}, & 0 \le y_1 < \infty, \quad -y_1 < y_2 < y_1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Derive the marginal densities of Y_1 and Y_2.

(b) Are Y_1 and Y_2 independent?

(c) Derive the conditional density of Y_2 given $Y_1 = y_1$.

(d) Find $P(Y_2 \le 1 | Y_1 = 2)$.

43. Let Y denote a random variable with probability density function given by

$$f(y) = (1/2)e^{-|y|}, \quad -\infty < y < \infty.$$

(a) Show that the moment-generating function of Y is $1/(1-t^2)$ where $-1 < t < 1$.

(b) Find $E(Y)$ and $V(Y)$ using (a).

44. The joint density of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \frac{1}{4}e^{-y_1/2}, \quad 0 \le y_2 \le y_1 < \infty.$$

Find $Cov(Y_1, Y_2)$.

45. Let Y be a random variable such that m-th moments are

$$\mu'_m = E(Y^m) = (m+1)!2^m, \quad m = 1, 2, 3 \ldots$$

(a) Show that the moment generating function $m_Y(t)$ of Y is

$$m_Y(t) = 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + 5(2t)^4 + \cdots$$

(b) Show that Y has a Gamma distribution. Identify α and β of the Gamma distribution.

46. Grade point average (GPA) of a large population of college students follows a normal distribution with mean 2.4, and standard deviation of 0.8, find the 95% percentile of the GPA.

47. (a) Suppose Y follows a χ^2 distribution with degree of freedom $\nu = 5$, find $E(\sqrt{Y})$.

(b) Suppose Y follows a χ^2_5 distribution, use the Tchebysheff's inequality to build an interval that covers Y with at least 90% of probability.

48. The joint density function of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 1/2, & \text{if } |y_1| + |y_2| \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Derive the marginal densities of Y_1 and Y_2

(b) Are Y_1 and Y_2 independent?

(c) Calculate $Cov(Y_1, Y_2)$.

(d) Calculate $var(Y_1 + 2Y_2)$.

(e) Derive the conditional density of Y_2 given $Y_1 = y_1$ and find $E(Y_2|Y_1 = y_1)$.

49. A quality control plan for an assembly line involves sampling randomly $n = 10$ finished items and report the number of defective items Y. The defective rate p varies every day and follow the following density, $f(p) = 4(1-p)^3$ for $0 \leq p \leq 1$. Find the mean and variance of Y.

50. The pair of random variables (X, Y) has the following joint probability distribution.

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        |       X
        | 1     2     3
--------------------------
      2 | 1/12  1/6   1/12
  Y   3 | 1/6   0     1/6
      4 |  0    1/3   0
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(a) Show that X and Y are dependent.

(b) Give a probability distribution table (such as the one on the previous page) for random variables U and V that have the same marginals as X and Y but are independent.

51. A point Y is chosen at random from [0, 1]. A second point X is then chosen from the interval $[0, Y]$.

(a) Find the joint density of X and Y.

(b) Find the marginal density for X .