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- Low rank matrices: changes in A^{-1} from changes in A
- Recall that any matrix of rank k is a sum of k rank one matrices. Any $m \times n$ matrix of rank k can be expressed as UV^T, where U is an $m \times k$ matrix and V is an $n \times k$ matrix, both with full column rank.
- If we perturb A with a low rank matrix, say a matrix of rank k with $k < n$? Will it remain invertible?
- For any $n \times 1$ column vectors $\mathbf{u}$ and $\mathbf{v}$, $M = I_n - \mathbf{u}\mathbf{v}^T$ is invertible, with inverse

$$M^{-1} = I_n + \frac{\mathbf{u}\mathbf{v}^T}{1 - \mathbf{v}^T\mathbf{u}}.$$

M is invertible if and only if $1 - \mathbf{v}^T\mathbf{u} \neq 0$. This can be proved by a direct multiplication, i.e.,

$$\left(I_n - \mathbf{u}\mathbf{v}^T\right)\left(I_n + \frac{\mathbf{u}\mathbf{v}^T}{1 - \mathbf{v}^T\mathbf{u}}\right) = I_n,$$

since $(\mathbf{u}\mathbf{v}^T)(\mathbf{u}\mathbf{v}^T) = \mathbf{u}(\mathbf{v}^T\mathbf{u})\mathbf{v}^T = (\mathbf{v}^T\mathbf{u})\mathbf{u}\mathbf{v}^T$.

- Consider an $n+1$ by $n+1$ matrix E,

$$E = \begin{bmatrix} I_n & \mathbf{u} \\ \mathbf{v}^T & 1 \end{bmatrix},$$

with $\det E = 1 - \mathbf{v}^T\mathbf{u}$. E is invertible if and only if $1 - \mathbf{v}^T\mathbf{u} \neq 0$. We can show that, if $\det E \neq 0$,

$$E^{-1} = \frac{1}{\det E}\begin{bmatrix} (\det E)I_n + \mathbf{u}\mathbf{v}^T & -\mathbf{u} \\ -\mathbf{v}^T & 1 \end{bmatrix},$$

and

$$E^{-1} = \begin{bmatrix} M^{-1} & -M^{-1}\mathbf{u} \\ -\mathbf{v}^T M^{-1} & 1 + \mathbf{v}^T M^{-1}\mathbf{u} \end{bmatrix}.$$

Comparing the $(1,1)$ entry of E^{-1} from two expressions, we have

$$M^{-1} = I_n + \frac{\mathbf{u}\mathbf{v}^T}{1 - \mathbf{v}^T\mathbf{u}}.$$

How can we get the above two expressions for E^{-1}?

$$\begin{bmatrix} I_n & \mathbf{0} \\ -\mathbf{v}^T & 1 \end{bmatrix} E = \begin{bmatrix} I_n & \mathbf{0} \\ -\mathbf{v}^T & 1 \end{bmatrix}\begin{bmatrix} I_n & \mathbf{u} \\ \mathbf{v}^T & 1 \end{bmatrix} = \begin{bmatrix} I_n & \mathbf{u} \\ \mathbf{0}^T & 1 - \mathbf{v}^T\mathbf{u} \end{bmatrix},$$

$$E^{-1} = \begin{bmatrix} I_n & \mathbf{u} \\ \mathbf{0}^T & 1 - \mathbf{v}^T\mathbf{u} \end{bmatrix}^{-1}\begin{bmatrix} I_n & \mathbf{0} \\ -\mathbf{v}^T & 1 \end{bmatrix} = \begin{bmatrix} I_n & -\frac{\mathbf{u}}{1-\mathbf{v}^T\mathbf{u}} \\ \mathbf{0}^T & \frac{1}{1-\mathbf{v}^T\mathbf{u}} \end{bmatrix}\begin{bmatrix} I_n & \mathbf{0} \\ -\mathbf{v}^T & 1 \end{bmatrix} = \frac{1}{\det E}\begin{bmatrix} (\det E)I_n + \mathbf{u}\mathbf{v}^T & -\mathbf{u} \\ -\mathbf{v}^T & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_n & -\mathbf{u} \\ \mathbf{0}^T & 1 \end{bmatrix} E = \begin{bmatrix} I_n & -\mathbf{u} \\ \mathbf{0}^T & 1 \end{bmatrix}\begin{bmatrix} I_n & \mathbf{u} \\ \mathbf{v}^T & 1 \end{bmatrix} = \begin{bmatrix} I_n - \mathbf{u}\mathbf{v}^T & \mathbf{0} \\ \mathbf{v}^T & 1 \end{bmatrix},$$

$$E^{-1} = \begin{bmatrix} I_n - \mathbf{u}\mathbf{v}^T & \mathbf{0} \\ \mathbf{v}^T & 1 \end{bmatrix}^{-1}\begin{bmatrix} I_n & -\mathbf{u} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} M^{-1} & \mathbf{0} \\ -\mathbf{v}^T M^{-1} & 1 \end{bmatrix}\begin{bmatrix} I_n & -\mathbf{u} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} M^{-1} & -M^{-1}\mathbf{u} \\ -\mathbf{v}^T M^{-1} & 1 + \mathbf{v}^T M^{-1}\mathbf{u} \end{bmatrix}$$

- Let's go one step further. Let U and V be $n \times k$ matrices of rank k. Then $M = I_n - UV^T$ is invertible if and only if $\det(I_k - V^TU) \neq 0$. The inverse M^{-1} is

$$M^{-1} = I_n + U(I_k - V^TU)^{-1}V^T,$$

which can be proved by

$$\begin{aligned}(I_n - UV^T)[I_n + U(I_k - V^TU)^{-1}V^T] &= I_n - UV^T + (I_n - UV^T)U(I_k - V^TU)^{-1}V^T \\ &= I_n - UV^T + U(I_k - V^TU)(I_k - V^TU)^{-1}V^T \\ &= I_n - UV^T + UV^T.\end{aligned}$$

- Consider an $n+k$ by $n+k$ matrix E,

$$E = \begin{bmatrix} I_n & U \\ V^T & I_k \end{bmatrix},$$

with $\det E = \det(I_k - V^TU)$. E is invertible if and only if $\det E = \det(I_k - V^TU) \neq 0$.

- Exercise. Prove that $\det E = \det(I_n - UV^T) = \det(I_k - V^TU)$.
- Now let's look at the general case. Let A be an $n \times n$ invertible matrix and U, V be $n \times k$ matrices of rank k. Then $M = A - UV^T$ is invertible if and only if $\det(I_k - V^TA^{-1}U) \neq 0$.

Sherman-Morrison-Woodbury formula:

$$M^{-1} = (A - UV^T)^{-1} = A^{-1} + A^{-1}U(I_k - V^TA^{-1}U)^{-1}V^TA^{-1}$$

This formula can be induced from $(A^{-1}M)^{-1} = (I_n - A^{-1}UV^T)^{-1}$. A direct computation proves the formula.

$$\begin{aligned}(A - UV^T)\left(A^{-1} + A^{-1}U(I_k - V^TA^{-1}U)^{-1}V^TA^{-1}\right) &= I_n - UV^TA^{-1} + U(I_k - V^TA^{-1}U)^{-1}V^TA^{-1} \\ &\quad - UV^TA^{-1}U(I_k - V^TA^{-1}U)^{-1}V^TA^{-1} \\ &= I_n - UV^TA^{-1} + UV^TA^{-1}(I_n - UV^TA^{-1})^{-1} \\ &\quad - UV^TA^{-1}(I_n - UV^TA^{-1})^{-1}UV^TA^{-1} \\ &= I_n - UV^TA^{-1} + UV^TA^{-1}(I_n - UV^TA^{-1})^{-1}(I_n - UV^TA^{-1}) \\ &= I_n - UV^TA^{-1} + UV^TA^{-1}\end{aligned}$$

We can also work with an $n+k$ by $n+k$ matrix E,

$$E = \begin{bmatrix} A & U \\ V^T & I_k \end{bmatrix}.$$

E is invertible if and only if $A - UV^T$ is invertible, equivalently, $I_k - V^TA^{-1}U$ is invertible.

- Updating least squares

Consider the normal equation $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$ of $A\mathbf{x} = \mathbf{b}$. Add one more equation $\mathbf{r} = b_{m+1}$. The normal equation is modified.

$$\begin{bmatrix} A^T & \mathbf{r}^T \end{bmatrix} \begin{bmatrix} A \\ \mathbf{r} \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} A^T & \mathbf{r}^T \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ b_{m+1} \end{bmatrix}$$

$$\left[A^TA + \mathbf{r}^T\mathbf{r}\right]\hat{\mathbf{x}} = A^T\mathbf{b} + \mathbf{r}^Tb_{m+1}$$

$$\left[A^TA + \mathbf{r}^T\mathbf{r}\right]^{-1} = (A^TA)^{-1} - c(A^TA)^{-1}\mathbf{r}^T\mathbf{r}(A^TA)^{-1}, \quad c = \frac{1}{1 + \mathbf{r}(A^TA)^{-1}\mathbf{r}^T}$$

Note that $(A^TA)^{-1}\mathbf{r}^T$ can be computed by solving $(A^TA)\mathbf{y} = \mathbf{r}^{\mathbf{T}}$.

$$\left[A^TA + \mathbf{r}^T\mathbf{r}\right]^{-1} = (A^TA)^{-1} - \frac{\mathbf{y}\mathbf{y}^T}{1 + \mathbf{r}\mathbf{y}}$$

- The derivative of A^{-1}

  Find the change in A^{-1} when A changes to $B = A + \Delta A$.

$$B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$$

  Let $A = A(t)$ be a matrix that changes with the time t.

$$\frac{\Delta A^{-1}}{\Delta t} = \frac{(A + \Delta A)^{-1} - A^{-1}}{\Delta t} = -(A + \Delta A)^{-1}\frac{\Delta A}{\Delta t}A^{-1}$$

$$\frac{dA^{-1}}{dt} = -A^{-1}\frac{dA}{dt}A^{-1}$$

- Exercise. Let $f(t) = A(t)^2$, $A(t)$ is an $n \times n$ matrix for any t in an interval I. Find $\dfrac{df}{dt}$.

$$\frac{\Delta A^2}{\Delta t} = \frac{(A(t + \Delta t))^2 - A(t)^2}{\Delta t} = \frac{(A + \Delta A)^2 - A(t)^2}{\Delta t} = \frac{A\Delta A + \Delta A A + (\Delta A)^2}{\Delta t} \to A\frac{dA}{dt} + \frac{dA}{dt}A$$