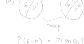
Midterm 1

1.
(a) Population: collection of whether or not experiencing bipolar symptoms in a 12-month period for all US adults.
Sample: collection of whether or not experiencing bipolar symptoms in a 12-month period for the selected 30 US adults.
(b) Parameter: population proportion of experiencing bipolar symptoms in a 12-month period for all US adults.
Statistic: sample proportion of experiencing bipolar symptoms in a 12-month period for the selected 30 US adults.
(c) Variable: whether or not experiencing bipolar symptoms in a 12-month period, Categorical variable
(Or the number of adults experiencing bipolar symptoms in a 12-month period, Quantitative)

2. Box 1 Box 2

(a)



box index

$S = \{G_1G_2,\ G_1D_2,\ D_1G_2,\ D_1D_2\}$

$X = \quad 0 \qquad 1 \qquad 1 \qquad 2$

$$P(X=0) = P(G_1G_2) \overset{\text{indep}}{=} P(G_1)P(G_2) = \frac{15}{20}\cdot\frac{24}{30} = \frac{3}{5} = 0.6$$

$$P(X=1) = P(G_1D_2) + P(D_1G_2) = P(G_1D_2) + P(D_1G_2)$$

$$= \frac{15}{20}\cdot\frac{6}{30} + \frac{5}{20}\cdot\frac{24}{30} = \frac{7}{20} = 0.35$$

$$P(X=2) = P(D_1D_2) = \frac{5}{20}\cdot\frac{6}{30} = \frac{1}{20} = 0.05$$

(b) $E(X) = 0\cdot(0.6) + 1\cdot(0.35) + 2\cdot(0.05) = 0.45$

(c) $E(X^2) = 0^2\cdot(0.6) + 1^2(0.35) + 2^2(0.05) = 0.55$

$$V(X) = E(X^2) - (E(X))^2 = 0.55 - 0.45^2$$

$$= 0.3475$$

3. X_1 = # of defectives from line I $\sim B(5, p)$
X_2 = 〃 line II $\sim B(5, p)$ > indep

$\Rightarrow X_1 + X_2 \sim B(10, p)$

$$P(X=2 \mid X_1+X_2=4) = \frac{P(X_1=2, X_1+X_2=4)}{P(X_1+X_2=4)} \overset{\text{indep}}{=} \frac{P(X_1=2)P(X_2=2)}{P(X_1+X_2=4)}$$

$$= \frac{\binom{5}{2}p^2(1-p)^3\binom{5}{2}p^2(1-p)^3}{\binom{10}{4}p^4(1-p)^6} = \frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}}$$

$= 0.476$

4. (a) X : the life length of a resistor (in thousands of hours)

$$P(X>5) = \int_5^\infty \frac{2x e^{-x^2/10}}{10}\,dx = -e^{-x^2/10}\Big|_5^\infty = e^{-2.5}$$

$= 0.082$

(b) Y = # of resistors that burn out prior to 5000 hours out of three resistors

$\sim B(3, 0.918)$ from (a)

$$P(Y=1) = \binom{3}{1} \cdot 0.918 \cdot (0.082)^2$$

$= 0.0186$

5. (a) By uniqueness of mgf.

X: The amount of night sleep time of a Korean adult

$\sim N(7, 2^2)$

(b) $P(X>10) = P(Z > \frac{10-7}{2}) = P(Z > 1.5)$

$= 0.0668$

(c) $P(X < w) = 0.015$

$\frac{w-7}{2} = -2.17$, $w = 2.66$

(d) $\bar{X} \sim N(7, (\frac{2}{3})^2)$

$P(\bar{X} > 6) = P(Z > \frac{6-7}{2/3}) = P(Z > -1.5)$

$= 0.9332$

(e) Y: # of groups that have a mean sleep less than 6 hrs

$p = 1 - 0.9332 = 0.0668$ by (d)

$\sim B(100, 0.0668) \approx N(6.68, 6.2338)$

$P(Y > 10) = P(Y \geq 11) = P(Y > 10.5)$

$\approx P(Z > \frac{10.5 - 6.68}{\sqrt{6.2338}}) = P(Z > 1.53) = 0.0631$

6. Distribution shape: roughly symmetric

(1&2: hard to tell, 3 might be skewed to the right)

Center: $1 < 2 < 3 < 4 < 5$

Variation: $1 \approx 2 < 3 < 4 < 5$

(or <)

Outliers: one or a few outliers are found in 1, 3, and 4

7. (a) X: waiting time $\sim Exp(2)$

$E(X) = 1/2, \quad V(X) = 1/4$

(b) P(at least one more customer waiting)

$= 1 - P$(no customers arriving in three minutes)

Y: # of customers arriving in three minutes.

$\sim Poisson(2\times 3)$

$= 1 - P(Y=0)$

$= 1 - \dfrac{e^{-6}\cdot 6^0}{0!} = 1 - e^{-6} = 0.9975$

8. (a) $1 = \int_0^1 \int_0^y k(1-y)\,dx\,dy = \int_0^1 ky(1-y)\,dy$

$= k\left(\frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^1 = \frac{k}{6}$

$k = 6$

(b) $f_X(x) = \int_x^1 6(1-y)\,dy = 6\left(y - \frac{1}{2}y^2\right)\Big|_x^1 = 3(1-x)^2, \quad 0<x<1$

$f_Y(y) = \int_0^y 6(1-y)\,dx = 6y(1-y), \quad 0<y<1$

(c) $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{6(1-y)}{6y(1-y)}$

$= \dfrac{1}{y}, \quad 0 < x < y$

$P(X \le \frac{1}{3} | Y = \frac{1}{2}) = \int_0^{\frac{1}{3}} 2 \, dx$

$= \dfrac{2}{3}$

(d) $E(XY) = \int_0^1 \int_0^y 6xy(1-y)\,dx\,dy$

$= \int_0^1 3y^3(1-y)\,dy = \dfrac{3}{20}$

$E(X) = \int_0^1 3x(1-x)^2\,dx = \dfrac{1}{4}$

$E(Y) = \int_0^1 6y^2(1-y)\,dy = \dfrac{1}{2}$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$= \dfrac{3}{20} - \dfrac{1}{4} \cdot \dfrac{1}{2} = \dfrac{1}{40}$

(e) $Cov(X,Y) \neq 0 \Rightarrow$ not independent

or $f(x,y) \neq f_X(x) f_Y(y)$

9.(a) $\frac{X-\mu}{\sigma} \sim N(0,1) \implies \left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi_1^2$

Similarly, $\left(\frac{Y-\mu}{\sigma}\right)^2 \sim \chi_1^2$

$$\underbrace{\frac{(X-\mu)^2}{\sigma^2}}_{\chi_1^2} + \underbrace{\frac{(Y-\mu)^2}{\sigma^2}}_{\chi_1^2} \sim \chi_2^2$$

indep

(b) $V = \frac{\sqrt{2}(U-\mu)}{\sqrt{(X-\mu)^2+(Y-\mu)^2}}$

$\frac{U-\mu}{\sigma} \sim N(0,1)$

$$= \frac{\frac{U-\mu}{\sigma} \sim N(0,1)}{\sqrt{\underbrace{\left\{\left(\frac{X-\mu}{\sigma}\right)^2 + \left(\frac{Y-\mu}{\sigma}\right)^2\right\}}_{\chi_2^2}/2}} \sim t_2$$

indep

(c) $U = V^2 = \frac{\chi_1^2\text{-rv}/1}{\chi_2^2/2} \sim F_{1,2}$

indep