Score Table (for teacher use only)

Question:	1	2	3	4	5	Total
Points:	10	20	20	15	35	100
Score:						

This is a CLOSED-BOOK exam.
Please provide ALL DERIVATIONS and EXPLANATIONS with your answers.
Any communication with others during the exam will be regarded as a cheating case.

1. (10 points) Discrete Fourier transform

   (a) (2 points) Given that $x[n]$ has Fourier transform $X(e^{j\omega})$, express the Fourier transform of the following signal in terms of $X(e^{j\omega})$.

   $$x_2[n] = x[1-n] + x[-1-n]$$

   (Answer)
   $$X_2(j\omega) = (e^{j\omega} + e^{-j\omega})X(e^{-j\omega}) = 2\cos\omega X(e^{-j\omega})$$

   (Solution)

   $$x[1-n] = y[n-1] \text{ and } x[-1-n] = y[n+1], \text{ where } y[n] = x[-n]$$
   $$Y(e^{j\omega}) = X(e^{-j\omega})$$
   $$y[n-1] \Longleftrightarrow e^{-j\omega}Y(e^{j\omega}) = e^{-j\omega}X(e^{-j\omega})$$
   $$y[n+1] \Longleftrightarrow e^{j\omega}Y(e^{j\omega}) = e^{j\omega}X(e^{-j\omega})$$

   Therefore, the discrete FT is given by

   $$(e^{j\omega} + e^{-j\omega})X(e^{-j\omega}) = 2\cos(\omega)X(e^{-j\omega})$$

   (both forms are considered as a correct answer)

   (b) (8 points) Use the properties of Fourier transform to derive A of the following sum

   $$A = \sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n$$

   (Answer)
   $$A = 2$$

(Solution) Let $a = 1/2$ and $x[n] = a^n u[n]$. Then

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} \tag{1}$$

$$= \frac{1}{1 - ae^{-j\omega}} \tag{2}$$

When $Y(e^{j\omega}) = j\frac{dX(e^{j\omega})}{d\omega}$, $y[n] = nx[n]$. Accordingly, the given sum is the discrete FT of $Y(e^{j\omega})$ at $\omega = 0$.

$$Y(e^{j0}) = \sum_{n=0}^{\infty} n \left(\frac{1}{2}\right)^n$$

Taking derivative of $X(e^{j\omega})$ gives

$$Y(e^{j\omega}) = j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1 - ae^{-j\omega}}\right)$$

$$= -j\frac{jae^{-j\omega}}{(1 - ae^{-j\omega})^2}$$

Therefore, the given sum can be derived as $Y(e^{j0}) = a/(1 - a)^2 = 2$

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2. (20 points) Consider a causal LTI system described by the following LCCDE.

$$y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n-1] - \frac{1}{2}x[n-2]$$

(a) (5 points) Find the frequency response $H(e^{j\omega})$ of this system.

(Answer) $H(e^{j\omega}) = e^{-j\omega}\frac{1 - \frac{1}{2}e^{-j\omega}}{(1 + \frac{1}{2}e^{-j\omega})^2}$

(Solution) By applying discrete FT to both sides of LCCDE, we get

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$= \frac{e^{-j\omega} - \frac{1}{2}e^{-2j\omega}}{1 + e^{-j\omega} + \frac{1}{4}e^{-2j\omega}}$$

Factorization of this yields

$$H(e^{j\omega}) = e^{-j\omega}\frac{1 - \frac{1}{2}e^{-j\omega}}{1 + e^{-j\omega} + \frac{1}{4}e^{-2j\omega}}$$

$$= e^{-j\omega}\frac{1 - \frac{1}{2}e^{-j\omega}}{(1 + \frac{1}{2}e^{-j\omega})^2}$$

(b) (10 points) Find the impulse response of this system. ~~In particular, derive the response in the form of $h[n] = B \cdot nC^n u[n]$ and find the constants B and C.~~

(Answer) $B = -4$, $C = -1/2$

(Solution) To simplify the problem, consider a first-order system

$$G(e^{j\omega}) = \frac{1}{1 + \frac{1}{2}e^{-j\omega}},$$

which has an inverse DFT $g[n] = \left(-\frac{1}{2}\right)^n u[n]$. Then, the derivative of $G(e^{j\omega})$ is

$$\frac{dG(e^{j\omega})}{d\omega} = j\frac{\frac{1}{2}e^{-j\omega}}{(1 + \frac{1}{2}e^{-j\omega})^2}.$$

Using this result, $H(e^{j\omega})$ can be rewritten as

$$H(e^{j\omega}) = -2j\frac{dG(e^{j\omega})}{d\omega} + j\frac{dG(e^{j\omega})}{d\omega}e^{-j\omega}$$

The $e^{-j\omega}$ term is only a single sample delay. Using the DFT property

$$-jng[n] \iff \frac{dG(e^{j\omega})}{d\omega},$$

$$h[n] = -2ng[n] + (n-1)g[n-1]$$
$$= -2n\left(-\frac{1}{2}\right)^n u[n] + (n-1)\left(-\frac{1}{2}\right)^{n-1} u[n-1]$$
$$= -2n\left(-\frac{1}{2}\right)^n u[n] - 2(n-1)\left(-\frac{1}{2}\right)^n u[n-1]$$
$$= -2n\left(-\frac{1}{2}\right)^n u[n] - 2(n-1)\left(-\frac{1}{2}\right)^n (u[n] - \delta[n])$$
$$= (-4n+2)\left(-\frac{1}{2}\right)^n u[n] - 2\delta[n]$$

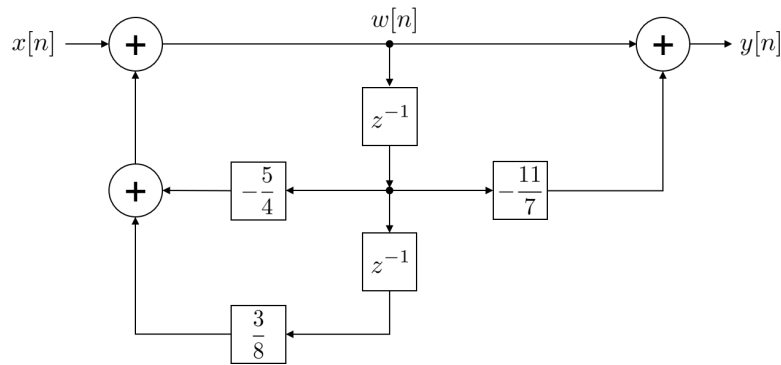(c) (5 points) Determine whether the LTI inverse of this system is causal or not.

(Answer) non-causal

(Solution) Due to the $e^{-j\omega}$ term in $H(e^{j\omega})$, its LTI inverse includes $e^{j\omega}$ term, which makes the inverse system non-causal.

$$H_{inv}(e^{j\omega}) = e^{j\omega}\frac{1 + e^{-j\omega} + \frac{1}{4}e^{-2j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$
$$= e^{j\omega}\left(1 + \frac{\frac{3}{2}e^{-j\omega} + \frac{1}{4}e^{-2j\omega}}{1 - \frac{1}{2}e^{-j\omega}}\right)$$
$$= e^{j\omega}\left(1 + e^{-j\omega}\left[-\frac{1}{2} + \frac{2}{1 - \frac{1}{2}e^{-j\omega}}\right]\right)$$

$$h_{inv}[n] = \delta[n+1] - \frac{1}{2}\delta[n] + \left(\frac{1}{2}\right)^{n-1}u[n]$$
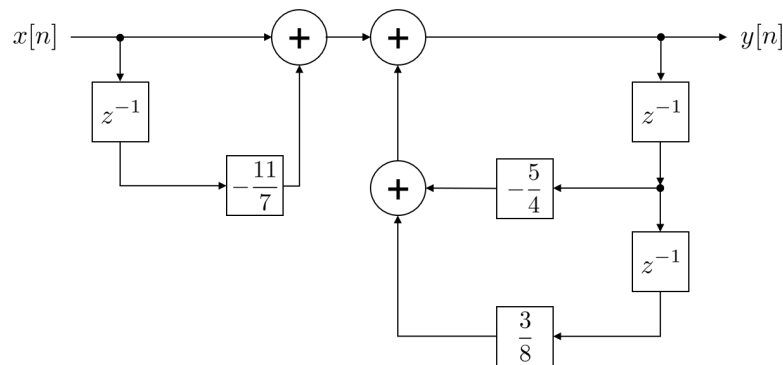
3. (20 points) [ST]

The input $x[n]$ and output $y[n]$ of a causal LTI system are related through the block diagram representation as shown in the figure.

(a) (5 points) Determine the difference equation relating $y[n]$ and $x[n]$.

(Answer) $y[n] + \frac{5}{4}y[n-1] - \frac{3}{8}y[n-2] = x[n] - \frac{11}{7}x[n-1]$
(Solution)

Changing the Direct form II representation to the Direct form I yields

$$y[n] = x[n] - \frac{11}{7}x[n-1] - \frac{5}{4}y[n-1] + \frac{3}{8}y[n-2]$$
$$\Rightarrow y[n] + \frac{5}{4}y[n-1] - \frac{3}{8}y[n-2] = x[n] - \frac{11}{7}x[n-1]$$

(b) (5 points) Determine the system function $H(z)$ for this causal LTI system.

(Answer) $H(z) = \frac{1 - \frac{11}{7}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 + \frac{3}{2}z^{-1})}$

(Solution) Applying $z-$transform on both sides gives

$$Y(z)\left(1 + \frac{5}{4}z^{-1} - \frac{3}{8}z^{-2}\right) = X(z)\left(1 - \frac{11}{7}z^{-1}\right)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{1 - \frac{11}{7}z^{-1}}{1 + \frac{5}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{1 - \frac{11}{7}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 + \frac{3}{2}z^{-1})}$$

(c) (5 points) Determine the poles and ROC of $H(z)$

(Solution) From the solution of the above problem, the roots of the denominator terms are at

$$p_1 = \frac{1}{4}, \quad p_2 = -\frac{3}{2}$$

Since the given system is causal, the ROC is outside from these poles. The outermost pole is p_2, so ROC is given by

$$|z| > \frac{3}{2}$$

(d) (5 points) Determine the BIBO stability of this system.

(Solution) The ROC does not include the unit circle, so the system is <u>not BIBO stable</u>.

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4. (15 points) [ST] Consider an LTI system with its transfer function $H(s)$.
   The output of the system that corresponds to the input $x(t) = e^{-t}u(t)$ is $y(t) = K(e^{-3t}u(t) + e^t u(-t))$, where K is a constant value.

   (a) (5 points) Derive the Laplace transform $Y(s)$ of $y(t)$ and region of convergence (ROC).

   (Answer) $Y(s) = \frac{-4K}{(s+3)(s-1)}$, ROC: $-3 < \text{Re}\{s\} < 1$.
   (Solution) Laplace transform of the first and second terms are given by

   $$Y_1(s) = \frac{K}{s+3}, \quad \text{ROC}: -3 < \text{Re}\{s\}$$

   $$Y_2(s) = -\frac{K}{s-1}, \quad \text{ROC}: \text{Re}\{s\} < -1$$

   Therefore, $Y(s)$ can be derived as

   $$Y(s) = Y_1(s) + Y_2(s) = K\left(\frac{1}{s+3} - \frac{1}{s-1}\right), \quad \text{ROC}: -3 < \text{Re}\{s\} < 1$$

   $$Y(s) = \frac{-4K}{(s+3)(s-1)}, \quad \text{ROC}: -3 < \text{Re}\{s\} < 1$$

   (b) (5 points) Determine the impulse response $h(t)$ of the system. Assume that the ROC of the system is identical to the ROC of $Y(s)$.

(Answer) $h(t) = -2K(e^{-3t}u(t) + e^t u(-t))$
(Solution) Dividing $Y(s)$ by the Laplace transform $X(s) = \frac{1}{s+1}$ of the input gives

$$H(s) = \frac{Y(s)}{X(s)} = -4K\frac{s+1}{(s+3)(s-1)}$$
$$= -2K\left(\frac{1}{s+3} + \frac{1}{s-1}\right)$$

Since the region of convergence of $H(s)$ is the same as that of $Y(s)$, the impulse response is a double-sided signal given by

$$h(t) = -2K(e^{-3t}u(t) - e^t u(-t))$$

(c) (5 points) The output of the system that corresponds to the input $x(t) = 1$ is $y(t) = \frac{8}{3}$. Determine the value of K.

(Answer) $K = 2$
(Solution) The constant signal $x(t) = 1$ is the zero-frequency signal, and the zero-frequency response of this system is given by

$$H(j0) = -4K\frac{j0+1}{(j0+3)(j0-1)} = \frac{4K}{3}$$

From $y(t) = \frac{8}{3}$, the zero frequency response should satisfy $\frac{4K}{3} = \frac{8}{3}$. Therefore, $K = 2$.

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5. (35 points) [2019 Final Problem] Consider an LTI system with the condition of initial rest. The system satisfies the following differential equation.

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 2\frac{d^2x(t)}{dt^2}$$

(a) (5 points) Determine the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$.

$$\left((j\omega)^2 + 2(j\omega) + 2\right)Y(j\omega) = 2(j\omega)^2 X(j\omega)$$
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2(j\omega)^2}{(j\omega)^2 + 2(j\omega) + 2}$$
$$= \frac{2(j\omega)^2}{(j\omega + 1 + j)(j\omega + 1 - j)}$$

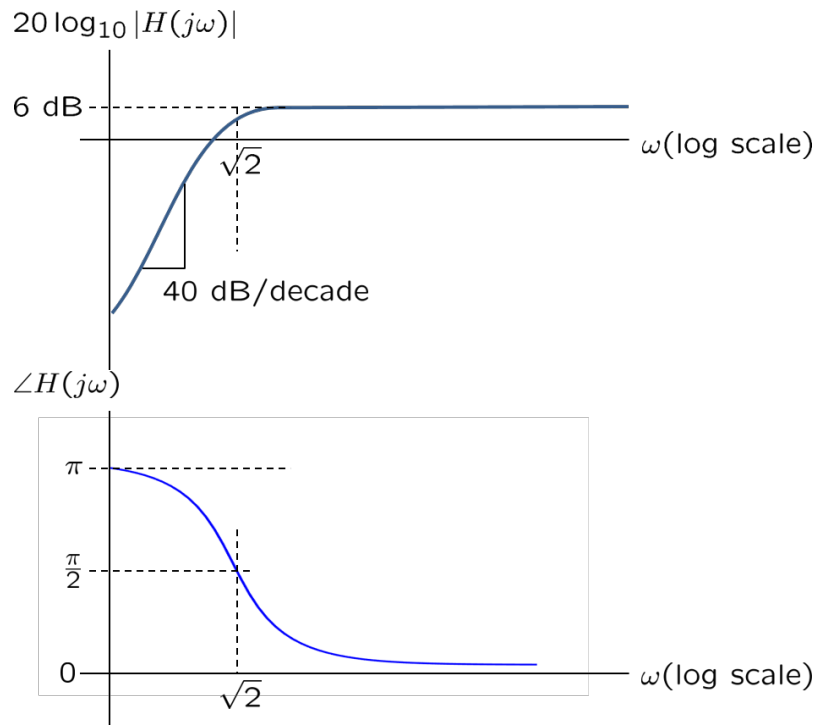(b) (5 points) Determine whether this system is (a) underdamped (b) critically-damped (c) over-damped.

Answer: (a) underdamped
The system has two complex roots and damping is less than 1 ($\zeta = 1/\sqrt{2} < 1$).

(c) (10 points) Draw the bode plot of the frequency response $H(j\omega)$ derived in the problem (a). Show both magnitude and phase responses.
(Specify (1) the break frequency, (2) slopes of the magnitude (in dB/decade) curve, (3) magnitude (in dB) at the pass-band, and (4) phases at $\omega = 0$, ∞, break frequency.)

> The denominator of the frequency response has the standard form of the second-order system $((j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1)$. Therefore, the break frequency ω_n is given by $\sqrt{2}$.
> The numerator has the second-order zeros with increasing slope. Therefore, the system has a typical 2nd-order high-pass response of 20dB/decade slope.
> At the pass-band($\omega \gg \sqrt{2}$), $|H(j\omega)| \approx 2$. Accordingly, the pass-band magnitude is 6dB.
> For the phase, the numerator's phase begins from π ($-1 = e^{j\pi}$). As the frequency increases, the phase is compensated by the denominator's phase and approaches to 0.

(d) (10 points) A signal $x(t) = \cos(0.01t)\cos(\sqrt{2}t)$ is fed into the system. Estimate the delay of its envelope $\cos(0.01t)$ at the output $y(t)$ of this system. (Hint: if necessary, you can use $\frac{d\tan^{-1}g(x)}{dx} = \frac{1}{1+g(x)^2} \cdot \frac{dg(x)}{dx}$)

> This signal includes the envelope function modulated by the carrier frequency $\omega_c = \sqrt{2}$. The delay of the envelope can be obtained by the group delay at ω_c. The group delay of this system is given by
>
> $$\tau_g = -\frac{d\phi}{d\omega}.$$
>
> The phase of this system is
>
> $$\phi = \pi - \tan^{-1}\left(\frac{2\omega}{2 - \omega^2}\right) = \pi - \tan^{-1}g(\omega).$$

Therefore, the group delay is given by

$$
\begin{aligned}
\tau_g(\omega) &= -\frac{d\phi}{d\omega} \\
&= \frac{d\tan^{-1}g(\omega)}{d\omega} \\
&= \frac{1}{1+g(\omega)^2} \cdot \frac{dg(\omega)}{d\omega} \\
&= \frac{1}{1+g(\omega)^2} \cdot \frac{4+2\omega^2}{(2-\omega^2)^2} \\
&= \frac{4+2\omega^2}{(2-\omega^2)^2 + (2\omega)^2},
\end{aligned}
$$

which becomes $\tau_g = 1$ at $\omega_c = \sqrt{2}$.

(e) (5 points) In problem (d), what is the Nyquist rate to sample the output $y(t)$ without aliasing?

(Answer)
$$
\omega = 2\sqrt{2} + 0.02
$$

(Solution) A cosine function's Fourier transform is given by two unit impulses. When two cosine signals are multiplied, their Fourier transforms are convolved in frequency domain. As a result, the highest frequency component exists at the sum of two cosine functions' frequencies ($\sqrt{2} + 0.01$). In addition, the given system has a high-pass characteristic, which does not suppress the highest frequency component. Therefore, the bandwidth of the signal $Y(j\omega)$ is $2(\sqrt{2} + 0.01)$.

[End of Problem]