EE321 Final Exam

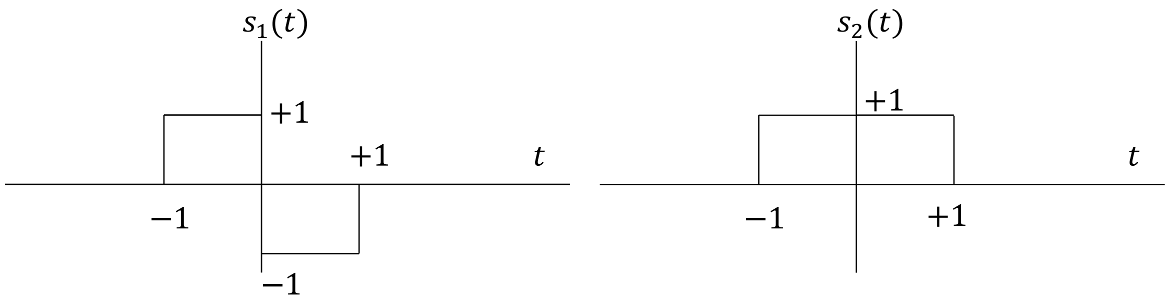
2024 Fall Semester

Do not forget to write your **names** and **student IDs**

Justify all of your answers

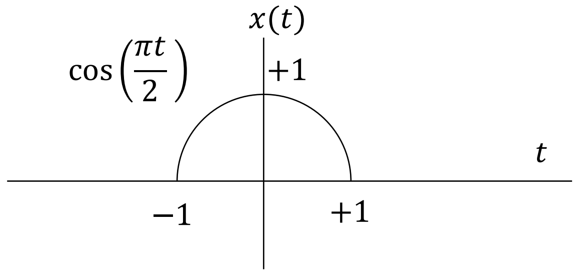
4 Problems

1. (25) Consider the two periodic signal $s_1(t)$ and $s_2(t)$ with a period of 2 seconds. The figures below depct the signals $s_1(t)$ and $s_2(t)$ over only one period.



a. (10) Show the signal $s_1(t)$ and $s_2(t)$ are orthogonal each other and find the correponding orthonormal signals $\phi_1(t)$ and $\phi_2(t)$.

b. (15) Consider the periodic signal $x(t)$ shown in the following figure. Note the signal is depicted over only one period, i.e., 2 seconds.



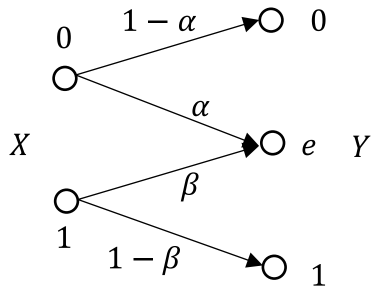
Then, express the periodic signal $x(t)$ as a weighted sum of the orthonormal signals $\phi_1(t)$ and $\phi_2(t)$ to minimize the squared error, ϵ^2 between $x(t)$ and the weighted sum, i.e.,

$$\epsilon^2 = \int_{-1}^{+1} |x(t) - a_1\phi_1(t) - a_2\phi_2(t)|^2 \, dt.$$

The minimum is achieved when

$$a_i = \int_{-1}^{+1} x(t)\phi_i(t)dt$$

2. (30) The following figure illustrates a communication channel in which $X \in \{0,1\}$ and $Y \in \{0, e, 1\}$ represent the input and output of the channel, respectively.



a. (15) For the input probability distribution, $\Pr(X=0)$ and $\Pr(X=1)$, express the mutual information $I(X;Y)$ in terms of the entropy of X, denoted by $H(X)$.

b. (5) Consider the optimal estimate of X, denoted by $\hat{X}$. Then, when $\Pr(X=0) = \Pr(X=1)$, what is the probability of bit error? That is, what is $\Pr(\hat{X} \neq X)$?

c. (10) When $\alpha = \beta$, what is the capacity of the channel?

3. (15) Consider a system has a noisy observation of x expressed as $y = x + n$ where $x \in \{-1, +1\}$, and n is a Gaussian random variable with zero mean and variance σ^2. The system makes its decision on x based on the observation y in such a way that $\hat{x} = +1$ when $y > k$, and $\hat{x} = -1$ when $y \leq k$.

a. (5) Find the conditional probability density function $f_Y(y|x=+1)$ and $f_Y(y|x=-1)$.

b. (10) Express the optimal k minimizing the decision error probability, $\Pr(\hat{x} \neq x)$ when $\Pr(x=+1) = \pi$.

Hint: the probability density function of Gaussian random variable is expressed as

$$f_N(n) = \frac{1}{\sqrt{2\sigma^2}} e^{-\frac{n^2}{2\sigma^2}}$$

4. (30) Consider a BPSK system:

$$s(t) = \begin{cases} A_c \cos(2\pi f_c t), & 0 \le t \le T, \text{if a 1 is sent} \\ A_c \cos(2\pi f_c t + \pi), & 0 \le t \le T, \text{if a 0 is sent} \end{cases}$$

which transmits data at a rate of 1.75M BPSK symbols per second. Assuming $N_0 = 1.26 \times 10^{-20}$W/Hz and the total loss of signal from the transmitter to the receiver is 144dB. Use the approximation of Q-function: $Q(x) \approx e^{-\frac{x^2}{2}}$ in finding your answers to the following questions.

a. (5) The transmission with the BPSK modulation can be understood as data communication over a binary symmetric channel. Then, find the channel capacity when the transmit power, P_T is set to 10 Watts.

b. (10) Suppose that we have random binary bit generator which produces binary source S following a probability distribution:

$$P_S(1) = 0.3 \text{ and } P_S(0) = 0.7.$$

What is the entropy of the data source symbol S?

c. (15) The random binary bit generator defined in Problem 4.a produces 1M binary symbols per second, and the binary data is compressed as small as possible before transmitted with the BPSK modulation. Then, what is the theoretical minimum transmit power, P_T to transmit the binary data over the binary symmetric channel in Problem 4.a? Use the following table for solving Problem 4.c.

p	$H(p)$	p	$H(p)$
0.00561	0.05	0.12730	0.55
0.01299	0.10	0.14610	0.60
0.02154	0.15	0.16666	0.65
0.03112	0.20	0.18930	0.70
0.04169	0.25	0.21450	0.75
0.05324	0.30	0.24300	0.80
0.06579	0.35	0.27604	0.85
0.07938	0.40	0.31602	0.90
0.09410	0.45	0.36913	0.95
0.11003	0.50	0.5	1.0