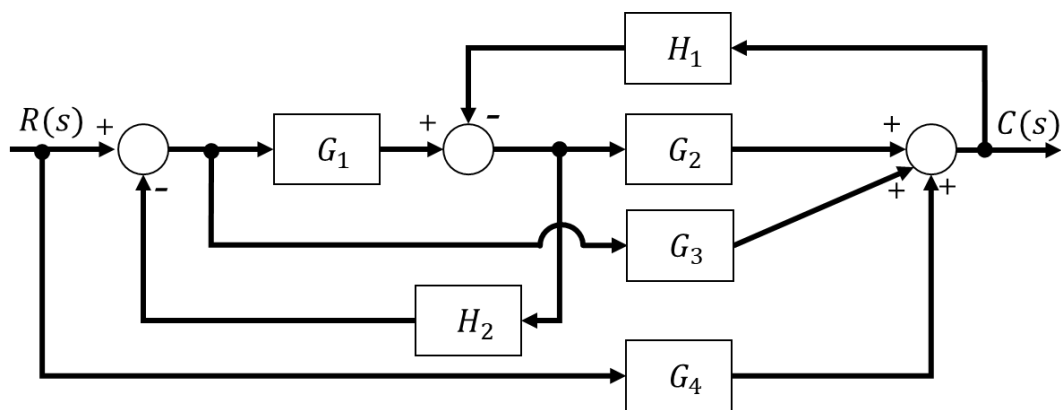# EE381 Control System Engineering
Mid-term Exam. (Apr. 22, 2021)

Score (_____)

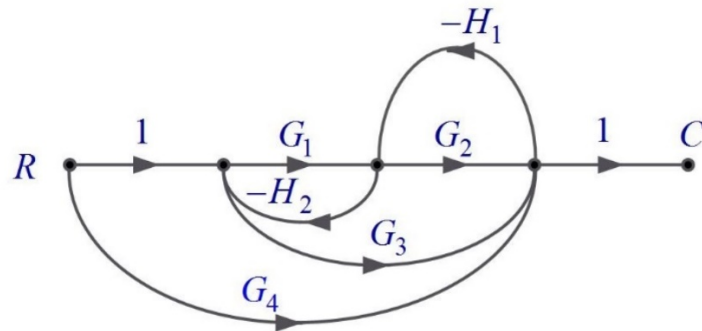Student ID: _____   Name: _____

1. For the following block diagram,
   (a) Draw the corresponding signal flow graph. (**5 points**)
   (b) Find the transfer function $C(s)/R(s)$ using Mason's formula. (**15 points**)

(Ans)
   (a) The corresponding signal flow graph can be drawn as follows:

   (b) There are three forward paths from R to C and three loops, with gains
$$P_1 = G_1 G_2$$
$$P_2 = G_3$$
$$P_3 = G_4$$
$$L_1 = -G_1 H_2$$
$$L_2 = -G_2 H_1$$
$$L_3 = G_3 H_1 H_2$$

All three loops touch, so Δ is 1 minus sum of the loop gains
$$\Delta = 1 - (-G_1 H_2 - G_2 H_1 + G_3 H_1 H_2)$$

$$= 1 + G_1H_2 + G_2H_1 - G_3H_1H_2$$

If all forward paths and all loops touch each other, it may be seen that the cofactors are all unity.

In this problem, the forward paths P_1 and P_2 touch all three loops so the corresponding cofactors Δ_1 and Δ_2 are unity. However, loop L_1 does not touch path P_3 so,
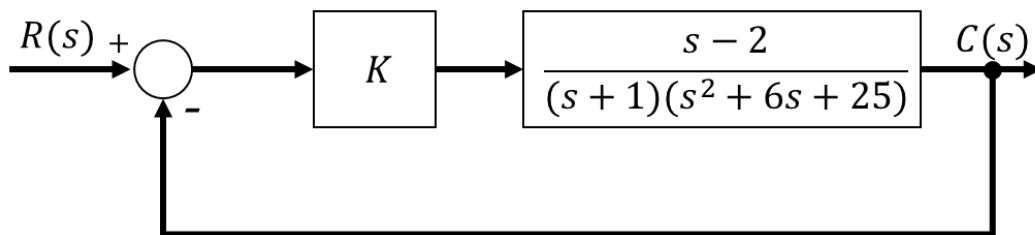
$$\Delta_3 = 1 + G_1H_2$$

Substituting values from above all three steps into the gain formula yields the closed loop transfer function as follows:

$$T = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta}$$

So,

$$T = \frac{G_1G_2 + G_3 + G_4(1 + G_1H_2)}{1 + G_1H_2 + G_2H_1 - G_3H_1H_2}.$$

2. Consider the closed-loop system shown in the following figure. Determine the range of K for stability. Assume that $K > 0$. (**20 points**)

(Ans)

The closed transfer function $C(s)/R(s)$ is

$$\frac{C(s)}{R(s)} = \frac{K(s - 2)}{(s + 1)(s^2 + 6s + 25) + K(s - 2)} = \frac{K(s - 2)}{s^3 + 7s^2 + (K + 31)s + 25 - 2K}$$

For stability, the denominator of this last equation must be a stable polynomial. For the characteristic equation

$$q(s) = s^3 + 7s^2 + (K + 31)s + 25 - 2K = 0$$

the Routh array becomes as follows:

s^3	1	$31 + K$
s^2	7	$25 - 2K$
s^1	$\dfrac{192 + 9K}{7}$	0
s^0	$25 - 2K$	

Since K is assumed to be positive, for stability, we require
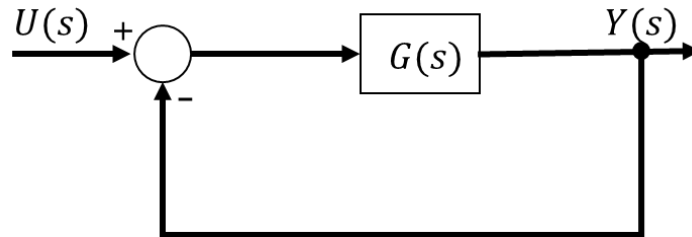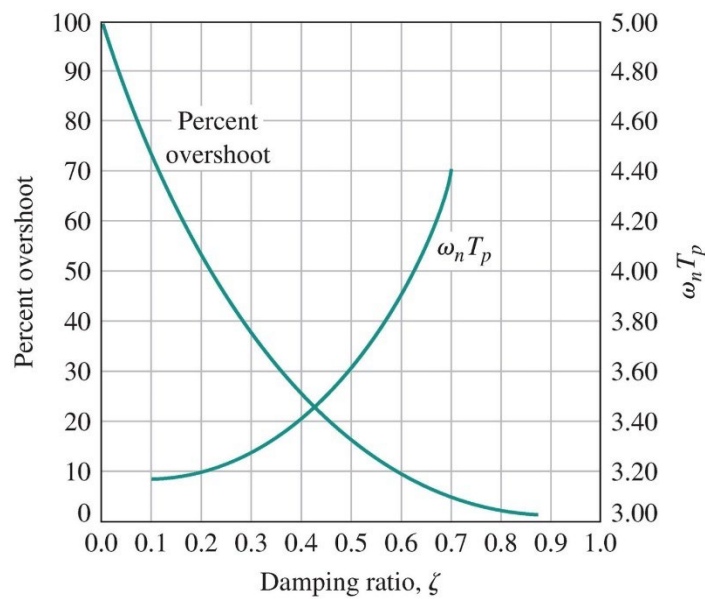
$$0 < K < 12.5.$$

3. In the system below, we have

$$G(s) = \frac{K}{s(s+2)}$$

where $K > 0$.

(a) What value of K should we choose so that the step response of the closed-loop system has a percent overshoot of 5%? (**5 points**)

(b) What is the corresponding settling time (2% criterion) for this K? (**10 points**)

Please use the following graph regarding damping ratio vs. percent overshoot.

(Ans)

The closed-loop system has the transfer function

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{K}{s^2 + 2s + K}$$

which is a standard second order system with the undamped natural frequency $\omega_n = \sqrt{K}$ and the damping ratio $\zeta = 1/\sqrt{K}$.

To have P.O. of 5%, we need $\zeta = 0.7$ from the figure. From $\frac{1}{\sqrt{K}} = 0.7$, we have $K = 2$.

For this particular K, the settling time is $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1} = 4$ s.

4. Consider two systems given by the following transfer functions:
$$H_1(s) = \frac{80}{(s+10)(s^2+4s+8)}, \qquad H_2(s) = \frac{37}{(s+1)(s^2+12s+37)}$$
Suppose the unit step input is given. For each of the above two systems,
(a) Estimate the settling time (2% criterion) of the step response. Use $\ln(50) \approx 4$. (**10 points**)
(b) Find the final value $y(\infty)$ without computing the output $y(t)$ explicitly. (**5 points**)

(Ans)
(a)
$H_1(s)$ has three poles: $p_1 = -10, p_{2,3} = -2 \pm j2$, where $p_{2,3}$ are dominant poles. Thus, its step response can be approximated by a second order system
$$H_1(s) \approx \frac{8}{(s^2+4s+8)}$$
with the damping ratio $\zeta = 1/\sqrt{2}$ and $\omega_n = 2\sqrt{2}$. Thus, $T_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{2} = 2\ s$.
$H_2(s)$ has three poles: $p_1 = -1, p_{2,3} = -6 \pm j1$, where p_1 is the dominant pole. Thus, its step response can be approximated by a first order system
$$H_2(s) \approx \frac{1}{s+1}$$
with the time constant $T = 1$. Thus, $T_s \approx 4T = 4\ s$.
The step response is
$$Y(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} \to y(t) = 1 - e^{-t}.$$
The settling time T_s should satisfy $e^{-T_s} = 0.02$. Then $T_s = -\ln(0.02) = \ln(50) = 4\ s$.

(b) Both systems are stable; hence $y(\infty)$ exists and is given by (by the final value theorem)
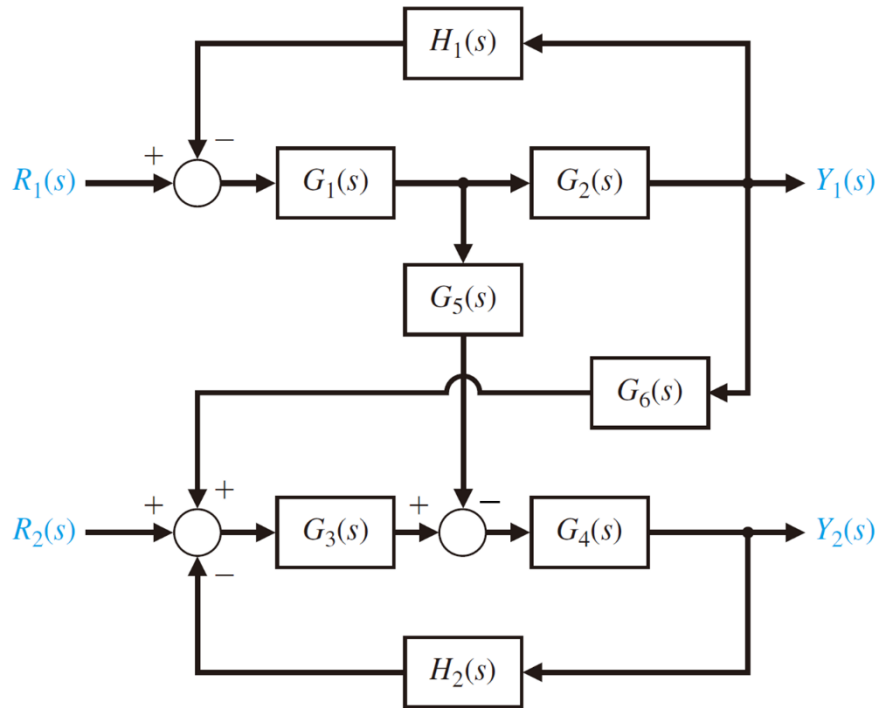For system $H_1(s)$: $\lim_{t\to\infty} y_1(t) = \lim_{s\to 0} sY_1(s) = 1$
For system $H_2(s)$: $\lim_{t\to\infty} y_2(t) = \lim_{s\to 0} sY_2(s) = 1$

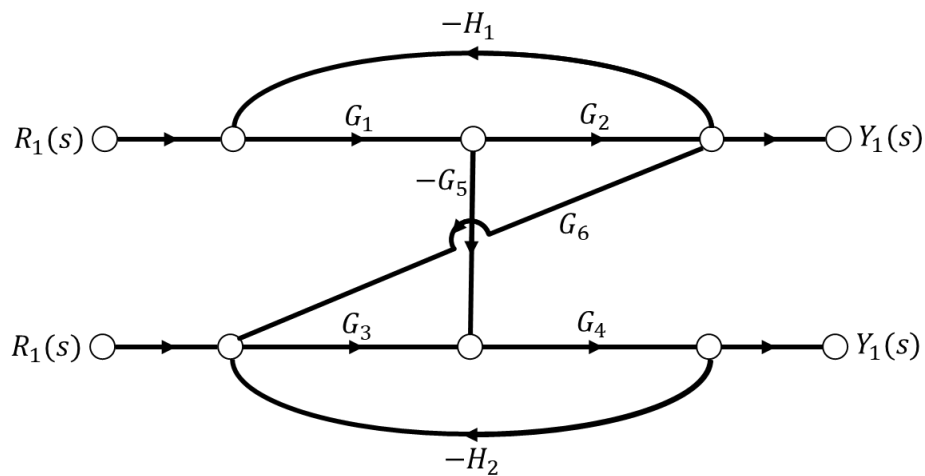5. A system has a block diagram as shown in the figure below.
(a) Determine the transfer function $T_{12}(s) = Y_2(s)/R_1(s)$. (**10 points**)

(b) It is desired to decouple $Y_2(s)$ from $R_1(s)$. Select $G_5(s)$ in terms of the other $G_i(s)$ to achieve decoupling. (**5 points**)

(Ans)

 (a) The signal flow graph can be drawn as follows:

There are two forward paths from $R_1(s)$ to $Y_2(s)$:

$P_1:\ -G_1G_5G_4$

$P_2:\ G_1G_2G_6G_3G_4.$

There are two loops:

$L_1:\ -G_1G_2H_1$

$L_2:\ -G_3G_4H_2.$

So, from the Mason's gain formula,

$$\Delta = 1 - (L_1 + L_2) + (L_1L_2) = 1 - (-G_1G_2H_1 - G_3G_4H_2) + G_1G_2H_1G_3G_4H_2$$
$$= (1 + G_1G_2H_1)(1 + G_3G_4H_2)$$

For P_1 and P_2, there are no non-touching loops, so their gain is 1, i.e., $\Delta_1 = \Delta_2 = 1$.
Then, the closed-loop transfer function from $R_1(s)$ to $Y_2(s)$ is

$$T_{12}(s) = \frac{Y_2(s)}{R_1(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{-G_1G_5G_4 \cdot 1 + G_1G_2G_6G_3G_4 \cdot 1}{\Delta}$$

$$= \frac{-G_1G_5G_4 + G_1G_2G_6G_3G_4}{(1 + G_1G_2H_1)(1 + G_3G_4H_2)}.$$

(b) To decouple $Y_2(s)$ from $R_1(s)$, the numerator of $T_{12}(s)$ should be zero. Thus, $G_5 = G_2G_3G_6$.

6. Consider the state-space equation

$$\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{b}u(t) = \begin{bmatrix} -2 & 5 \\ 0 & 3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 4 \\ 0 \end{bmatrix}u(t)$$

$$y(t) = \mathbf{cx}(t) + du(t) = [7 \quad 8]\mathbf{x}(t) + 1.5u(t)$$

(a) When there is no input, i.e., $u(t) = 0$, obtain the characteristic equation for the state $\mathbf{x}(t)$ and discuss the stability of the state $\mathbf{x}(t)$. (**5 points**)
(b) Solve the overall system transfer function and discuss the BIBO stability of this system. (**5 points**)
(c) Is there a difference in stability between (a) and (b)? Discuss why. (**5 points**)

(Ans)

(a) $q(s) = (s + 2)(s - 3)$. The pole locations are -2, +3. The system state is not stable since one pole is located on the righthand s-plane.

(b) $(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+2 & -5 \\ 0 & s-3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+2} & \frac{5}{(s+2)(s-3)} \\ 0 & \frac{1}{s-3} \end{bmatrix}$

$$T(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d = [7 \quad 8]\begin{bmatrix} \frac{1}{s+2} & \frac{5}{(s+2)(s-3)} \\ 0 & \frac{1}{s-3} \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix} + 1.5$$

$$= [7 \quad 8]\begin{bmatrix} \frac{4}{s+2} \\ 0 \end{bmatrix} + 1.5 = \frac{1.5s + 31}{s + 2}$$

There is only one pole at -2. So the system is BIBO stable.

(c) The state itself is not stable due to the pole at +3. But the overall system is stable due to the unstable pole cancellation.