

HW#6 Solution

E7.2 A tape recorder with a unity feedback speed control system has the loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K}{s(s+1)(s^2 + 10s + 24)}.$$

- (a) Sketch a root locus for K , and show that the dominant roots are $s = -0.41 \pm j0.384$, when $K = 8$.
- (b) For the dominant roots of part (a), calculate the settling time and overshoot for a step input.

(Ans)

- (a) Write the characteristic equation when $K = 8$

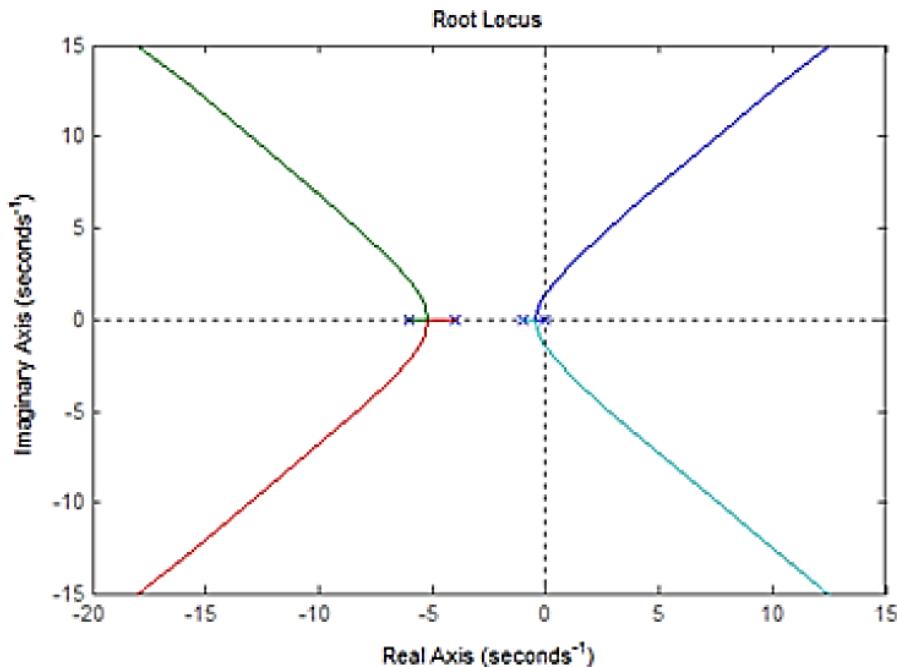
$$s(s+1)(s^2 + 10s + 24) + K = 0$$

$$s^4 + 11s^3 + 34s^2 + 24s + 8 = 0 .$$

The roots of the characteristic equation are

$$s_1 = -5.8472, s_2 = -4.3323 \quad \text{and} \quad s_{3,4} = -0.41 \pm j0.384 .$$

The dominant roots are $-0.41 \pm j0.384$, when $K = 8$



(b) The dominant roots are

$$(s + 0.41 + j0.384)(s + 0.41 - j0.384) = s^2 + 0.82s + 0.315 .$$

From this we determine that

$$\omega_n = 0.561 \quad \text{and} \quad \zeta = 0.73 .$$

Thus, the settling time is

$$T_s = \frac{4}{\zeta \omega_n} = 9.76 \text{ sec} .$$

The percent overshoot is $P.O. = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 3.50\% .$

E7.9 The primary mirror of a large telescope can have a diameter of 10 m and a mosaic of 36 hexagonal segments with the orientation of each segment actively controlled. Suppose this unity feedback system for the mirror segments has the loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K}{s(s^2 + 2s + 5)}.$$

- (a) Find the asymptotes and sketch them in the s -plane.
- (b) Find the angle of departure from the complex poles.
- (c) Determine the gain when two roots lie on the imaginary axis.
- (d) Sketch the root locus.

(Ans)

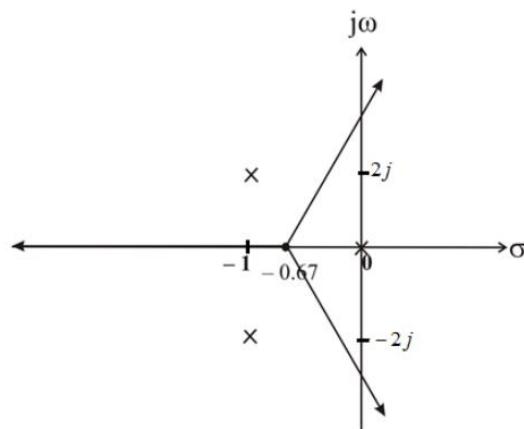
The characteristic equation is

$$1 + K \frac{1}{s(s^2 + 2s + 5)} = 0$$

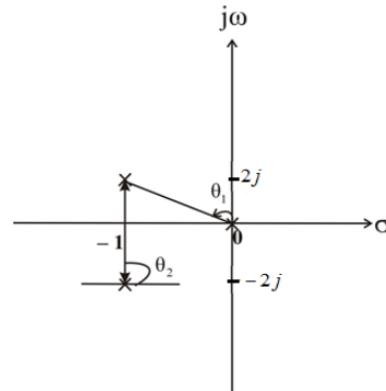
or

$$s^3 + 2s^2 + 5s + K = 0 .$$

- (a) The system has three poles at $s = 0$ and $-1 \pm j2$. The number of asymptotes in $n_p - n_z = 3$ centered at $\sigma_{cent} = -2/3$, and the angles are ϕ_{asympt} at $\pm 60^\circ, 180^\circ$.



(b) The angle of departure, θ_d , is $90^\circ + \theta_d + 116.6^\circ = 180^\circ$, so $\theta_d = -26.6^\circ$.



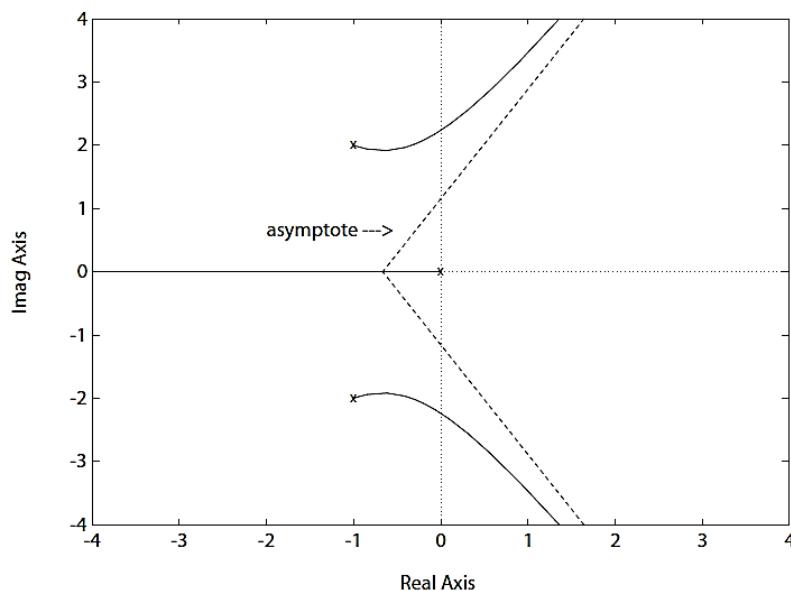
(c) The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 2 & K \\ s & b \\ s^0 & K \end{array}$$

where $b = 5 - K/2$. So, when $K = 10$ the roots lie on the imaginary axis. The auxiliary equation is

$$2s^2 + 10 = 0 \quad \text{which implies } s_{1,2} = \pm j\sqrt{5} .$$

(d) The root locus is shown in Figure E7.9



E7.16 A negative unity feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{Ke^{-sT}}{s + 1},$$

where $T = 0.1$ s. Show that an approximation for the time delay is

$$e^{-sT} \approx \frac{\frac{2}{T} - s}{\frac{2}{T} + s}.$$

Using

$$e^{-0.1s} = \frac{20 - s}{20 + s},$$

obtain the root locus for the system for $K > 0$. Determine the range of K for which the system is stable.

(Ans)

The expansion for e^{-Ts} is

$$e^{-Ts} = 1 - Ts + \frac{(Ts)^2}{2!} - \dots$$

If $(Ts) \ll 1$, then

$$e^{-Ts} \approx 1 - Ts = \frac{a + bs}{c + ds},$$

where a , b , c and d are constants to be determined. Using long division, we expand $(a + bs)/(c + ds)$ and match as many coefficients as possible. In this case, we determined that $a = c = (2/T)$ and also that $b = -d = -1$.

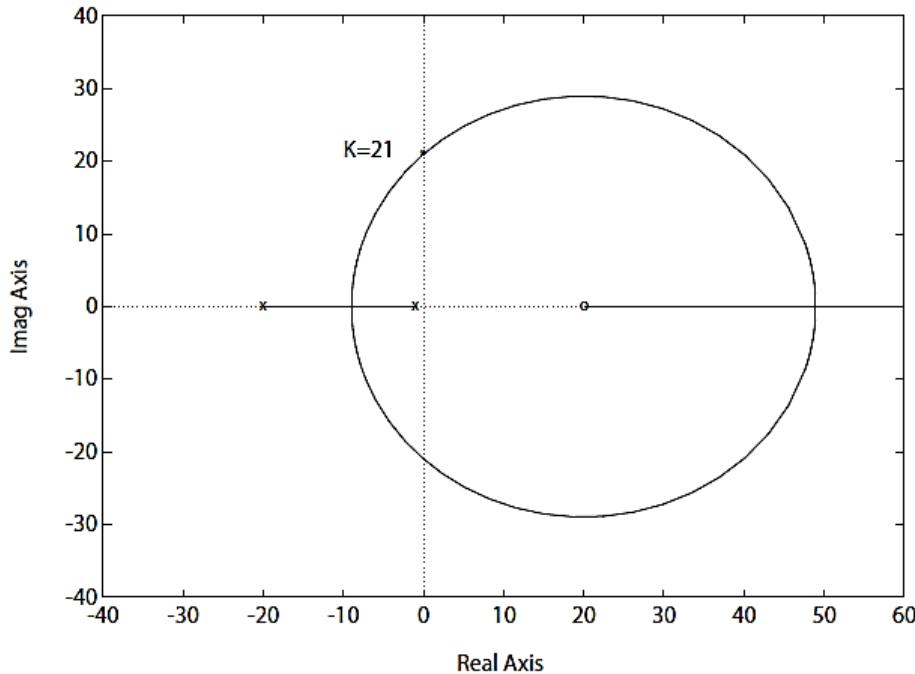
In this case, with $T = 0.1$, we have

$$e^{-Ts} = \frac{20 - s}{20 + s} = -\frac{(s - 20)}{(s + 20)}.$$

So, the characteristic equation is

$$1 + \frac{-K(s - 20)}{(s + 1)(s + 20)},$$

and the root locus is shown in the figure below.



Using a Routh-Hurwitz analysis with the characteristic polynomial

$$s^2 + (21 - K)s + 20 + 20K = 0,$$

we determine that the system is stable for $-1 < K < 21$.

P7.33 The Bell-Boeing V-22 Osprey Tiltrotor is both an airplane and a helicopter. Its advantage is the ability to rotate its engines to 90° from a vertical position for takeoffs and landings as shown in Figure P7.33(a), and then to switch the engines to a horizontal position for cruising as an airplane [20]. The altitude control system in the helicopter mode is shown in Figure P7.33(b). (a) Determine the root locus as K varies and determine the range of K for a stable system. (b) For $K = 280$, find the actual $y(t)$ for a unit step input $r(t)$ and the percentage overshoot and settling time (with a 2% criterion). (c) When $K = 280$ and $r(t) = 0$, find $y(t)$ for a unit step disturbance, $T_d(s) = 1/s$.

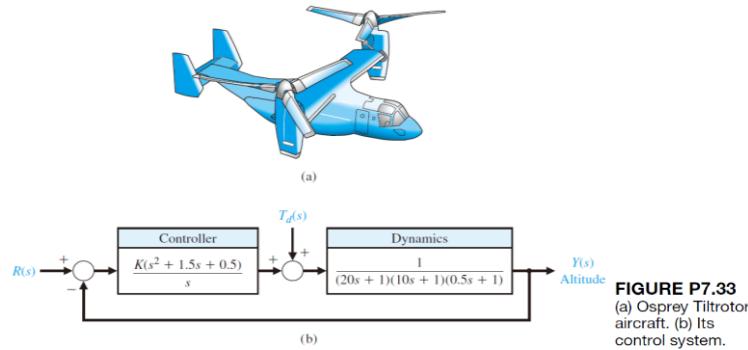


FIGURE P7.33
(a) Osprey Tiltrotor aircraft. (b) Its control system.

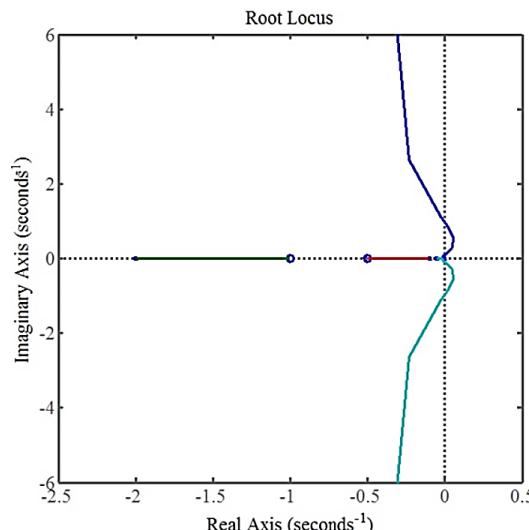
(Ans)

$$(a) L(s) = \frac{K(s^2 + 1.5s + 0.5)}{s} \times \frac{1}{(20s+1)(10s+1)(0.5s+1)}$$

Characteristic equation is

$$1 + L(s) = 1 + \frac{K(s^2 + 1.5s + 0.5)}{s(20s+1)(10s+1)(0.5s+1)}.$$

Plot root locus of the system as K varies



Determine the range of K for a stable system

$$s(20s + 1)(10s + 1)(0.5s + 1) + K(s^2 + 1.5s + 0.5) = 0$$

$$\begin{array}{c|cccc} s^4 & 100 & 30.5 + K & 0.5K \\ s^3 & 215 & 1 + 1.5K & 0 \\ s^2 & \frac{215(30.5 + K) - 100(1 + 1.5K)}{215} & \varepsilon & 0.5K \\ s & \frac{(1 + 1.5K)\varepsilon - (107.5K)}{215} & & \\ s^0 & \frac{\varepsilon}{0.5K} & & \end{array}$$

$$\frac{(215)(30.5 + K) - 100(1 + 1.5K)}{215} > 0 \rightarrow K > -99.35 \dots (1)$$

$$\frac{(1 + 1.5K) \left(\frac{(215)(30.5 + K) - 100(1 + 1.5K)}{215} \right) - 107.5K}{\frac{(215)(30.5 + K) - 100(1 + 1.5K)}{215}} > 0$$

$$\rightarrow K < 0.48 \quad \text{or} \quad K > 136.5 \dots (2)$$

$$0.5K > 0 \rightarrow K > 0 \dots (3)$$

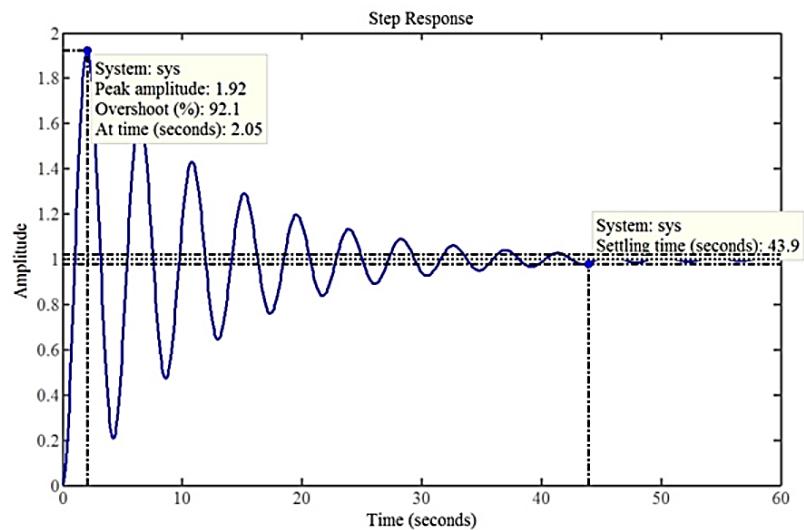
Therefore, the range of K for stable condition lies between

$$0 < K < 0.48 \quad \text{or} \quad K > 136.5 .$$

(b) When $K = 280$

$$Y(s)$$

$$\begin{aligned} &= \frac{L(s)}{1 + L(s)} R(s) \\ &= \frac{\frac{K(s^2 + 1.5s + 0.5)}{s(20s + 1)(10s + 1)(0.5s + 1)}}{1 + \frac{K(s^2 + 1.5s + 0.5)}{s(20s + 1)(10s + 1)(0.5s + 1)}} R(s) \\ &= \frac{280s^2 + 420s + 140}{100s^4 + 215s^3 + 310.5s^2 + 421s + 140} R(s). \end{aligned}$$



The step response has $P.O. = 92\%$ and $T_s = 43.9$ sec .

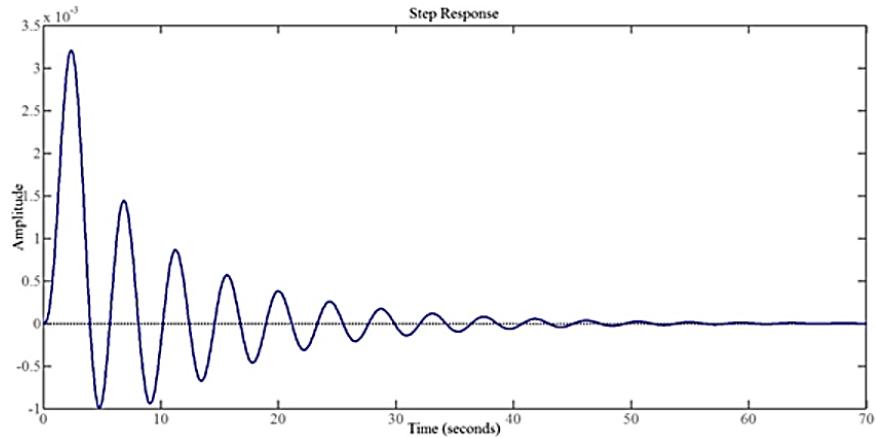
(c) When $r(t) = 0$

$$Y(s) = \frac{1}{1 + \frac{(20s+1)(10s+1)(0.5s+1)}{s(20s+1)(10s+1)(0.5s+1)}} T_d(s) .$$

When $K = 280$,

$$Y(s) = \frac{s}{100s^4 + 215s^3 + 310.5s^2 + 421s + 140} \times \frac{1}{s} .$$

Thus, plot of $y(t)$ for a unit step disturbance is



AP7.14 Consider the unity feedback control system shown in Figure AP7.14. Design a PID controller using Ziegler–Nichols methods. Determine the unit step

response and the unit disturbance response. What is the maximum percent overshoot and settling time for the unit step input?

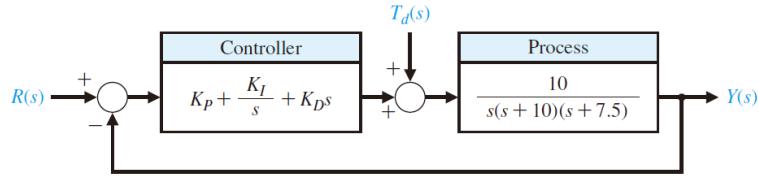


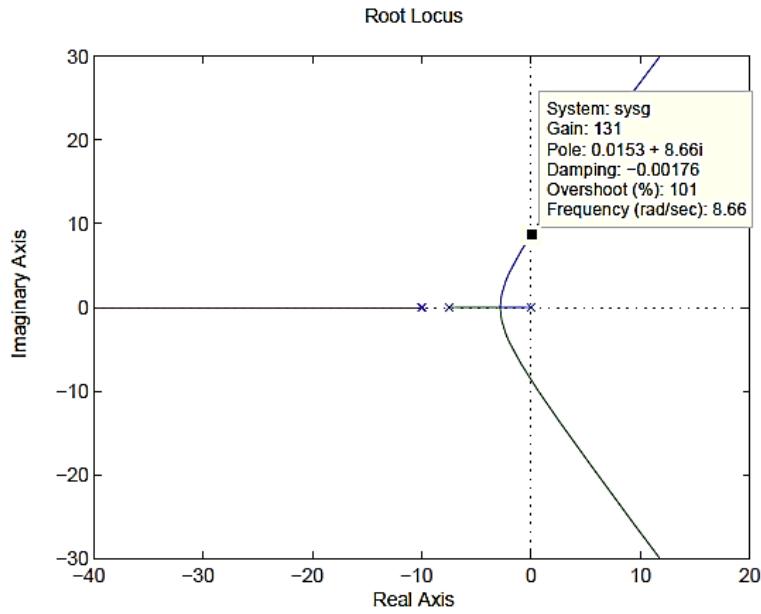
FIGURE AP7.14
Unity feedback loop
with PID controller.

(Ans)

Set $K_I = K_D = 0$.

$$\frac{Y(s)}{R(s)} = \frac{\frac{10}{s(s+10)(s+7.5)}}{1 + K_P \frac{10}{s(s+10)(s+7.5)}} .$$

The root locus of the uncompensated transfer function is



Characteristic equation is

$$s(s + 10)(s + 7.5) + 10K_p = 0$$

$$s^3 + 17.5s^2 + 75s + 10K_p = 0$$

The Routh array is

s^3	1	75
s^2	17.5	$10K_p$
s	$17.5 \times 75 - 10K_p$	0
s^0	17.5	0
	$10K_p$	

The system to be stable when

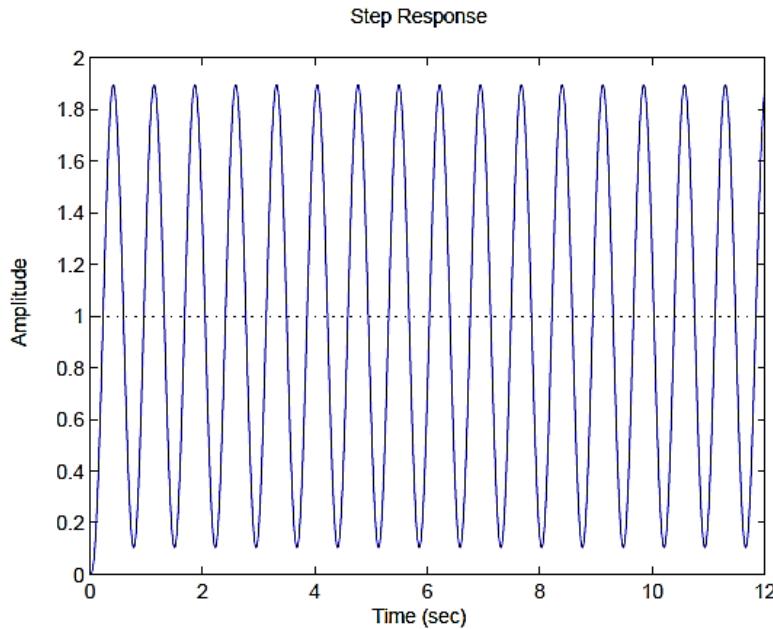
$$\frac{17.5 \times 75 - 10K_p}{17.5} > 0 \text{ and } 10K_p > 0$$

$$\rightarrow 0 < K_p < 131.25 .$$

Thus $K_u = 131.25$

$$Y(s) = \frac{1312.5}{s^3 + 17.7s^2 + 75s + 1312.5} R(s) .$$

The step response at the ultimate gain $K_u = 131$.



By analyzing the figure, we can determine $T_u = 0.72$ sec .

Thus,

$$K_P = 0.6K_u = 78.75$$

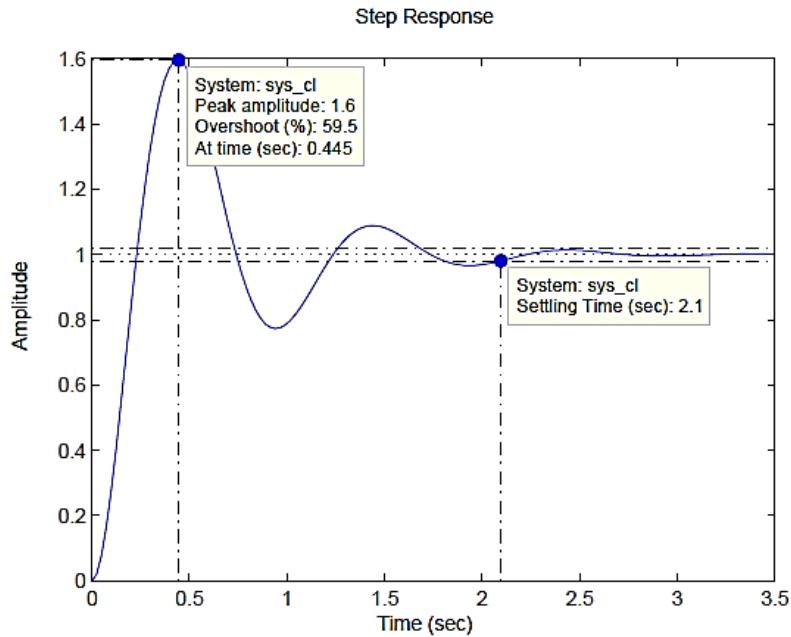
$$K_I = \frac{1.2K_u}{T_u} = 218.75$$

$$K_D = 0.6K_u T_u / 8 = 7.0875$$

i) For unit step input

$$\begin{aligned} Y(s) &= \frac{\frac{10(K_P + \frac{K_I}{s} + K_D s)}{s(s+10)(s+7.5)}}{1 + \frac{10(K_P + \frac{K_I}{s} + K_D s)}{s(s+10)(s+7.5)}} R(s) \\ &= \frac{70.875s^2 + 787.5s + 2187.5}{s^4 + 17.5s^3 + (75 + 70.875)s^2 + 787.5s + 2187.5} \cdot \frac{1}{s} . \end{aligned}$$

The step response with the Ziegler-Nichols tuned PID controller is



By analyzing the figure, $P.O. = 58.9\%$ and $T_s = 2.09$ sec.

ii) Also for unit disturbance response,

$$\begin{aligned}
 Y(s) &= \frac{10}{s(s+10)(s+7.5)} T_d(s) \\
 &\quad \frac{10(K_P + \frac{K_I}{s} + K_D s)}{1 + \frac{10(K_P + \frac{K_I}{s} + K_D s)}{s(s+10)(s+7.5)}} \\
 &= \frac{10s}{s^4 + 17.5s^3 + (75 + 70.875)s^2 + 787.5s + 2187.5} \cdot \frac{1}{s}.
 \end{aligned}$$

The disturbance response with the Ziegler-Nichols tuned PID controller is

