School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2021 Spring

Midterm

Tuesday, April 20, 2021
9:00–11:20 am

- Be sure to **show all relevant work and reasoning** in your answer sheet. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.
- Please be clear in writing—we can't grade what we can't decipher!
- Don't forget to upload your answer sheet during 11:10-11:20 am through KLMS. The system will be automatically closed at that time. If the system does not work, you should email it to `ee210b_21spring@kaist.ac.kr` by 11:20 am. Late submissions will not be accepted/graded.

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \leq y)$, where Y is a standard normal random variable, for y between 0 and 1.99. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When y is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

Problem 1 (10 Points)

Consider the communication of binary-valued messages using a noisy channel from Alice to Bob. Specifically, Alice wants to send one of two symbols, either 0 or 1, to Bob, with equal probabilities. Alice and Bob are connected by a noisy channel, and Alice can send some numerical value X through the channel. Due to the noise in the channel, Bob receives some random output $Y = X + N$ where the random variable N represents the noise that is independent of X and distributed by a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 9$.

a) (5 points) Suppose that Alice encodes the symbol 0 with the value $X = -3$ and the symbol 1 with the value $X = 3$. Bob then decodes the message according to the following rule:

- If $Y \geq 0$, then conclude that the symbol 1 was sent.
- If $Y < 0$, then conclude that the symbol 0 was sent.

Calculate the probability that Bob gets the wrong guess of the message from this scheme. Remind that the probability that Alice sends symbol 0 (or 1) is 1/2. Reduce your calculations to a single numerical value by using the standard normal table in page 2.

b) (5 points) Next, consider the following modification of the previous communication scheme. Now, Alice and Bob use the communication channel three times, and Alice sends $(X_1, X_2, X_3) = (-3, -3, -3)$ for symbol 0 and $(X_1, X_2, X_3) = (3, 3, 3)$ for symbol 1. Bob receives $Y_i = X_i + N_i$ for $i = 1, 2, 3$ where each N_i is a normal random variable with mean $\mu = 0$ and variance $\sigma^2 = 9$. Assume that each N_i is independent of each other and independent of X_i's. Each Y_i is decoded according to the rule in a), and Bob uses a majority voting rule to determine which symbol was sent:

- If two or more components of (Y_1, Y_2, Y_3) are greater than or equal to 0, then conclude that symbol 1 was sent
- If two or more components of (Y_1, Y_2, Y_3) are less than 0, then conclude that symbol 0 was sent

What is the probability that Bob gets the wrong guess of the message from this scheme? Just write down the formula (you don't need to calculate the value of the probability).

Problem 2 (10 Points)

Let T be a random variable, uniformly distributed between 0 and 1. On any given day, a particular computer is functional with probability T. Given the value of T, the status of the computer on different day is independent.

a) (5 points) Find the probability that the computer is functional on a particular day.

b) (5 points) We are told that the computer was functional on m out of the last n days. Find the conditional PDF of T.

   You may use the identity

$$\int_0^1 p^k(1-p)^{n-k}dp = \frac{k!(n-k)!}{(n+1)!} \text{ for } 0 \leq k \leq n.$$

<u>Problem 3</u> (15 Points) Random variables $X_1, X_2, \ldots, X_n$ are independent and identically distributed; each X_i has CDF $F_X(x)$. Let's consider

$$\begin{aligned} L_n &= \min(X_1, \ldots, X_n), \\ U_n &= \max(X_1, \ldots, X_n). \end{aligned}$$

In terms of $F_X(x)$:

a) (5 points) Find the CDF of U_n, denoted by $F_{U_n}(u)$.

b) (5 points) Find the CDF of L_n, denoted by $F_{L_n}(l)$.

c) (5 points) Find the joint CDF of L_n and U_n, denoted by $F_{L_n,U_n}(l, u)$.

Problem 4 (15 Points)

Bob is a lazy worker and every morning he flips a coin to decide whether or not going to work. If the coin is head, he goes to work, and if it comes up tail he stays home. The coin he has is not necessarily fair, rather it possess a probability of heads equal to q, where the probability q is modeled by a random variable Q with density

$$f_Q(q) = \begin{cases} 2q, & \text{for } 0 \leq q \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that conditioned on Q each coin flip is independent.

a) (5 points) What is the probability that Bob goes to work if he flips the coin once?

b) (5 points) Assume that when Bob goes to work he earns \$10 every day. Define X as the Bob's payout for a month (30 days) if he flips the coin every morning for the next 30 days. Find $\text{var}(X)$.

   Hint: You may use the law of total variance: $\text{var}(X) = \mathbb{E}[\text{var}(X|Y)] + \text{var}(\mathbb{E}[X|Y])$.

c) (5 points) Let event B be that Bob stays home as least once in n days. Find the conditional density of Q given the event B, $f_{Q|B}(q)$.