EE381 Control System Engineering
Final Exam Solution (June 15, 2023)

Total score ( )

Student ID: ____________________ Name: ____________________

1. (Ans)

(a) (**5 points**)

$$G(j\omega)=\frac{Ke^{-j\omega}}{j\omega}$$
$$\angle G(j\omega)=-\omega-\frac{\pi}{2}$$

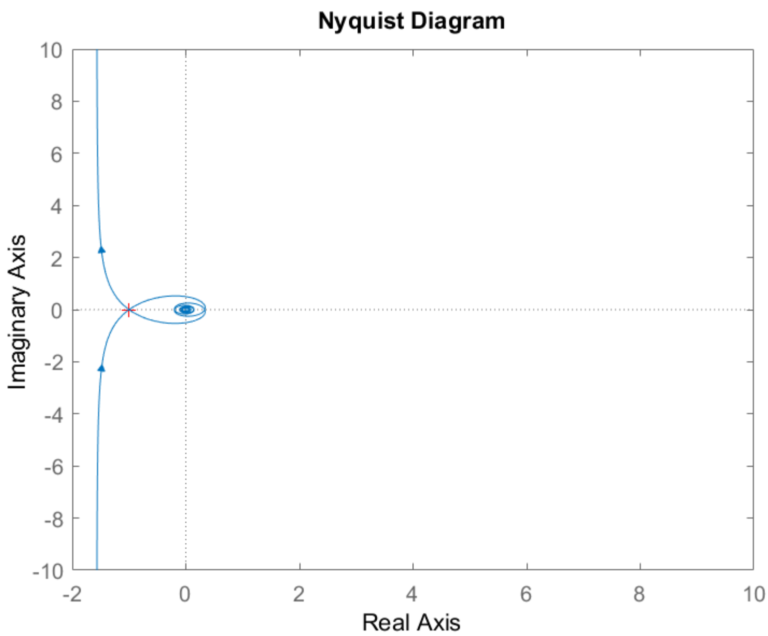
The phase angle became equal to $-180°$ at $\omega=\pi/2$ rad/sec. For stability, the magnitude $|G(j\omega)|$ at $\omega=\pi/2$ must be less than unity. Hence,

$$|G(j\omega)|=\frac{K}{\omega}\ .$$

We require that $K\ <\ \pi/2$ for stability.

(b) (**5 points**)

The Nyquist plot for $K=\pi/2$ is shown as follows:

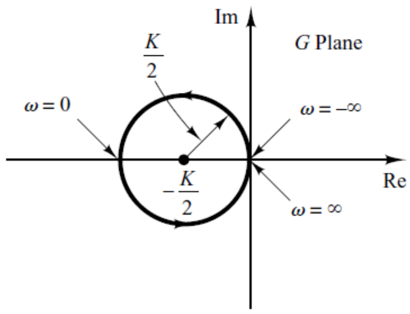


2. **(Ans)** (**10 points**)

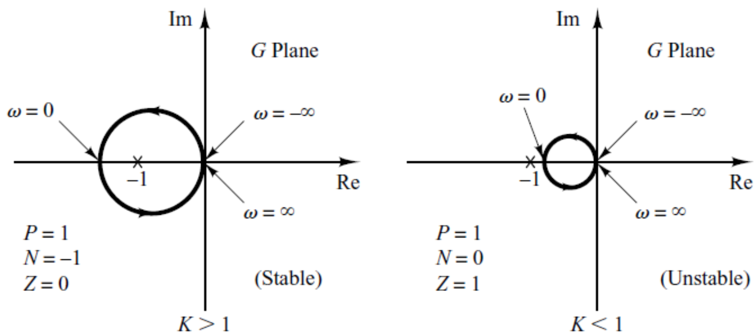
The polar plot of

$$G(j\omega) = \frac{K}{j\omega - 1}$$

is a circle with center at $-K/2$ on the negative real axis and radius $K/2$, as shown in Figure (a). As ω is increased from $-\infty$ to ∞, the $G(j\omega)$ locus makes a counterclockwise rotation. In this system, $P = 1$ because there is one pole of $G(s)$ in the right-half s plane. For the closed-loop system to be stable, Z must be equal to zero. Therefore, $N = Z - P$ must be equal to -1, or there must be one counterclockwise encirclement of the $-1 + j0$ point for stability. (If there is no encirclement of the $-1 + j0$ point, the system is unstable.) Thus, for stability, K must be greater than 1, and $\boldsymbol{K} = \mathbf{1}$ gives the stability limit. Figure (b) shows both stable and unstable cases of $G(j\omega)$ plots.



(a)



(b)

3. (Ans)

(a) (**5 points**)

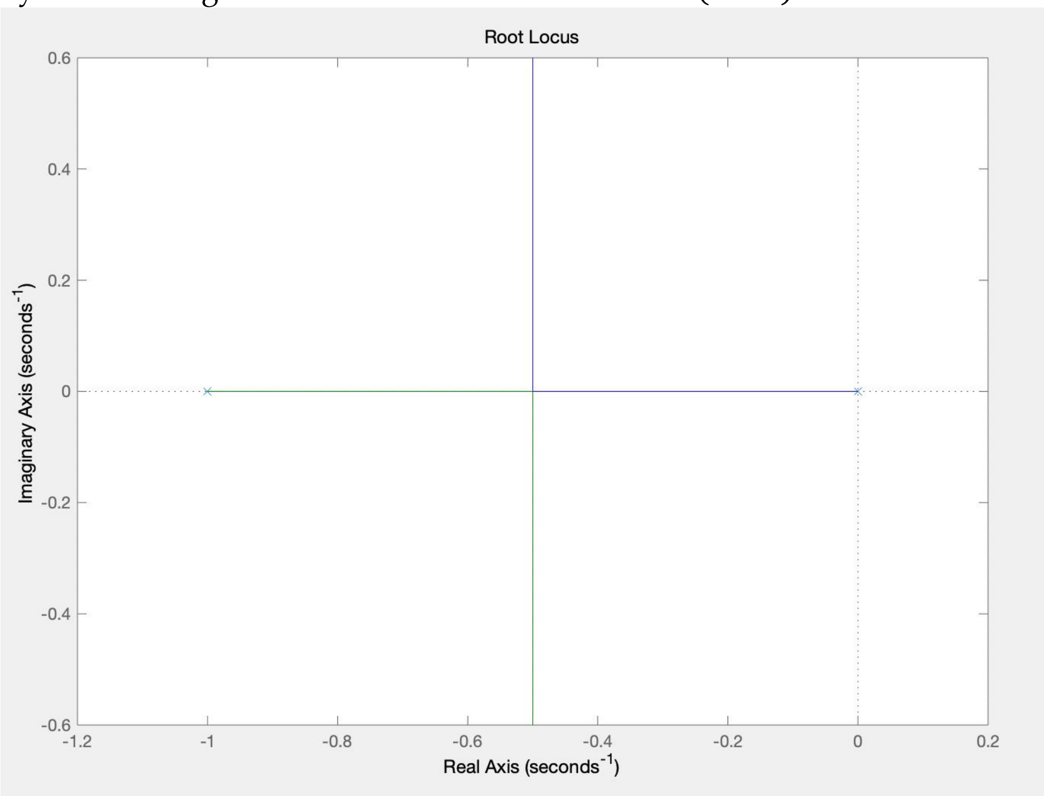
Because the response of the system is underdamped with clear evidence of overshoot and oscillation. Such phenomena cannot occur in first order system.

(b) (**5 points**)

Because the steady state error for step input is zero, type number of the open loop system must be 1 or higher. (i.e. type number 1 or higher) Hence the open loop transfer function must contain an integrator, or pole at the origin.

(c) (**5 points**)

Because the system is the second order and open loop system includes a pole at origin, let $L(s) = K\frac{1}{s(s+2\zeta\omega_n)}$. Then $T_s \approx \frac{4}{\zeta\omega_n} = 8$, hence $\zeta\omega_n = 0.5$, which implies that the other pole is at -1. Breakaway point at -0.5. This can be found by determining where the maximum of $K = -s(s+1)$ occurs.



(d) (**5 points**)

$\phi_m = 55°$. Then $\sin 55° = \frac{9}{11} = \frac{\alpha-1}{\alpha+1}$, hence $\alpha = 10$. Then $-10\log\alpha = -10\ dB$. Then we can find ω_n as follows: $|G(jw)| = \left|\frac{\omega_n^2}{s(s+1)}\right| = \frac{\omega_n^2}{\omega\sqrt{1+\omega^2}} = \frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{10}}$, then $\omega\sqrt{1+\omega^2} = 3\sqrt{10}$, and it directly follows $\omega_m = 3$. Therefore, $p = \omega_m\sqrt{\alpha} = 3\sqrt{10}$, and $z = \frac{3\sqrt{10}}{10}$. Hence the appropriate lead compensator is given by $Gc(s) = \frac{1+\frac{10s}{3\sqrt{10}}}{1+\frac{s}{3\sqrt{10}}}$.

4. (Ans)

(a) **(10 points)**

Using free body diagram, we can write the equation of motion like below.

$$M\ddot{x} + b\dot{x} + kx = b\dot{r} + kr$$

Therefore, by taking Laplace transform, the transfer function $\frac{X(s)}{R(s)}$ is

$$\frac{X(s)}{R(s)} = \frac{bs + k}{Ms^2 + bs + k}\ .$$

Then, using given parameters $M = 2$ kg, $b = 8$ Ns/m and $k = 128$ N/m,

$$\frac{X(s)}{R(s)} = \frac{8s + 128}{2s^2 + 8s + 128} = \frac{4s + 64}{s^2 + 4s + 64}\ .$$

(b) **(10 points)**

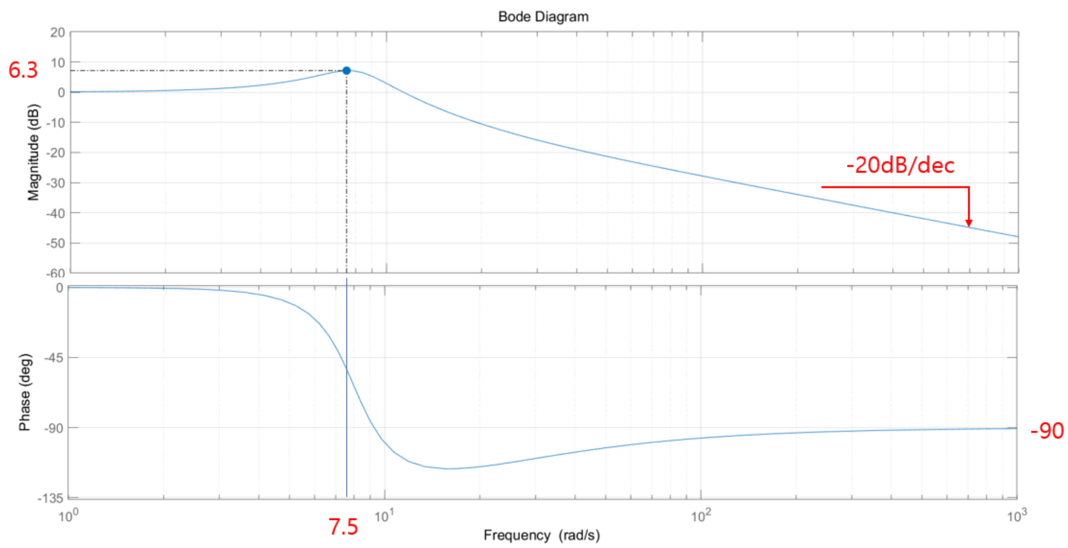
We can think about dominant complex conjugate poles.

Therefore, $\omega_n = \sqrt{k/M} = \sqrt{64} = 8\ rad/s$ and the damping ratio $\zeta = \frac{b/M}{2\omega_n} = 0.25$.

Then, $\omega_r = \omega_n\sqrt{1 - 2\zeta^2} = 8\sqrt{1 - 2(0.25)^2} = 2\sqrt{14} = 7.5\frac{rad}{s}$

$$M_{p\omega} = |G_o(j\omega_r)| = \left(2\zeta\sqrt{1 - \zeta^2}\right)^{-1} = \frac{1}{2 * 0.25\sqrt{1 - (0.25)^2}} = \frac{8}{\sqrt{15}} = \frac{8\sqrt{15}}{15} = 2.07$$

$$= 20\log\left(\frac{8\sqrt{15}}{15}\right) = 6.3\ \ dB$$



[Bode plot of the transfer function]

You should represent DC gain, $M_{p\omega}$, ω_r in the Bode plot.

5. (Ans)

(a) (**5 points**)

$P = 0, N = 1 + 1 = 2,$ so $Z = N + P = 2$. The system is unstable and has two roots in the right-hand s-plane.

(b) (**5 points**)

$N = -1$ (one CCW encirclement) $+\ 1 = 0$, so $Z = 0$. The system is stable.

6. (Ans)

(a) (**5 points**)

The system with initial compensation is depicted in the block diagram representation in Fig. 1(i), which can be converted to a system with cascade compensation as in Fig. 1(ii) by moving the summing point ahead of $G_c(s)$.

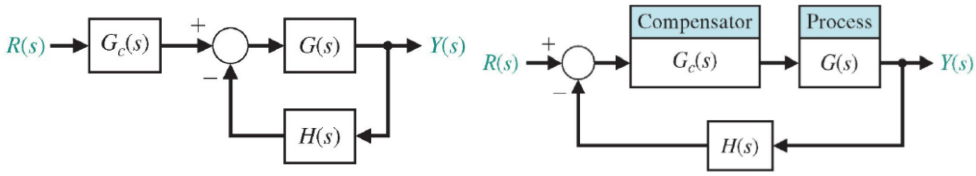


Fig.1 (i) Input compensation (ii) Cascade compensation

Since we want the converted system in Fig. 1(ii) to have unity feedback, then we need to satisfy:

$$\frac{H(s)}{G_c(s)} = 1 \quad \rightarrow G_c(s) = H(s) = \frac{4}{s+10} \ .$$

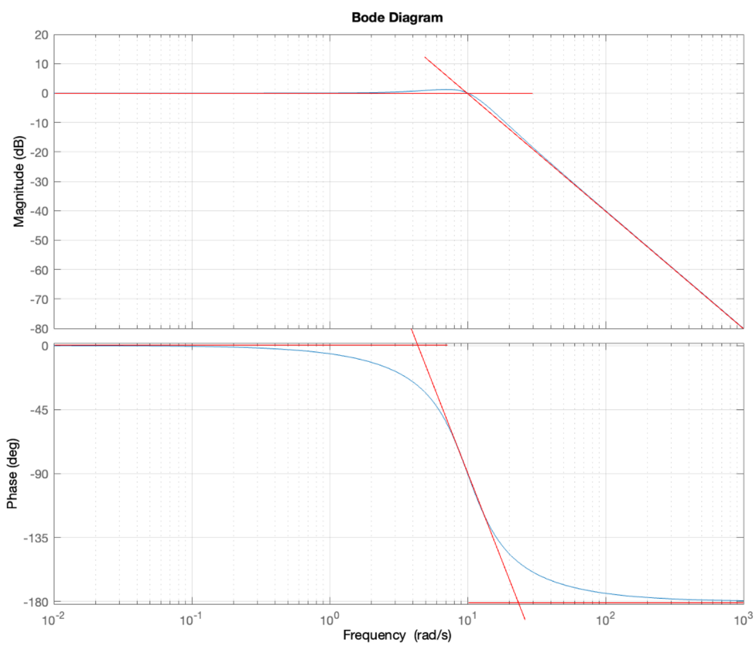
We have $T(s)$ as in the following so it consists of a complex conjugate pole at $\omega_n = 10$ rad/s and $\zeta = 0.5$.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{100}{s^2 + 10s + 100} \quad \rightarrow T(j\omega) = \frac{1}{1 + 0.1j\omega + 0.01(j\omega)^2} \ .$$

(b) (**5 points**)

From this, the Bode plot of $T(s)$ has the following asymptotic components as shown in the figure below.

- A magnitude plot that is 0 dB at $\omega \ll 10$ rad/s and decreases at -40 dB/dec at $\omega \gg 10$ rad/s, which asymptotes meet at 0 dB line at $\omega = 10$ rad/s
- A phase angle plot that is 0 deg at $\omega \ll 10$ rad/s, -180 deg at $\omega \gg 10$ rad/s, and -90 deg at $\omega = 10$ rad/s



7. (Ans)

(a) (**5 points**)

The loop transfer function in the frequency domain becomes:

$$L(j\omega) = \frac{K}{(j\omega t_1 + 1)(j\omega t_2 + 1)(j\omega t_3 + 1)} \ .$$

Taking the magnitude gives:

$$|L(j\omega)| = \frac{|K|}{\sqrt{1 + (\omega t_1)^2}\sqrt{1 + (\omega t_2)^2}\sqrt{1 + (\omega t_3)^2}} \ .$$

And the phase is:

$$\angle L(j\omega) = -\tan^{-1}(\omega t_1) - \tan^{-1}(\omega t_2) - \tan^{-1}(\omega t_3) \ .$$

As ω approaches infinity, the phase of $L(j\omega)$ becomes:

$$\angle L(j\omega) = -\frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} = -\frac{3\pi}{2} \ .$$

(b) (**5 points**)

We first express $L(j\omega)$ in the form of real and imaginary parts. By substituting s with $j\omega$ and multiplying numerator and denominator by complex conjugate, we have:

$$L(j\omega) = \frac{K(1 - \omega^2 t_1)(1 - \omega^2 t_2)(1 - \omega^2 t_3) - j\omega K(t_1 + t_2 + t_3 - \omega^2 t_1 t_2 t_3)}{(1 + \omega^2 t_1^2)(1 + \omega^2 t_2^2)(1 + \omega^2 t_3^2)} \ .$$

Grouping real and imaginary parts yields:

$$K \cdot \frac{\left(1 - \omega^2(t_1 t_2 + t_2 t_3 + t_3 t_1)\right) - j\omega(t_1 + t_2 + t_3 - \omega^2 t_1 t_2 t_3)}{(1 + \omega^2 t_1^2)(1 + \omega^2 t_2^2)(1 + \omega^2 t_3^2)} \ .$$

Thus, the real part of $L(j\omega)$ is given by:

$$\Re(L) = K \cdot \frac{1 - \omega^2(t_1 t_2 + t_2 t_3 + t_3 t_1)}{(1 + \omega^2 t_1^2)(1 + \omega^2 t_2^2)(1 + \omega^2 t_3^2)} \ .$$

And the imaginary part of $L(j\omega)$ is given by:

$$\Im(L) = K \cdot \frac{-\omega(t_1 + t_2 + t_3 - \omega^2 t_1 t_2 t_3)}{(1 + \omega^2 t_1^2)(1 + \omega^2 t_2^2)(1 + \omega^2 t_3^2)} \ .$$

Setting the imaginary part equal to zero and solving for ω gives us:

$$\omega = \sqrt{\frac{t_1 + t_2 + t_3}{t_1 t_2 t_3}} \ .$$

This is the frequency at which the imaginary part of $L(j\omega)$ is zero.

(c) (**5 points**)

To find the value of K where $L(j\omega) = -1$, we assume that $t = t_1 = t_2 = t_3$.

We know from part (b) that the imaginary part is zero for $\omega = \sqrt{\frac{3}{t^2}}$.

Substituting ω_2 with $\frac{3}{t^2}$ in the real part of $L(j\omega)$, we have:

$$\Re(L) = K \cdot \frac{1 - \left(\frac{3}{t^2}\right)(t^2 + t^2 + t^2)}{\left(1 + \left(\frac{3}{t^2}\right)t^2\right)^3} .$$

Simplifying this expression yields:

$$\Re(L) = K \cdot \frac{1 - 9}{(1+3)^3}$$
$$\Re(L) = K \cdot \frac{-8}{64} = -\frac{K}{8} .$$

We want $L(j\omega) = -1$, so we set $-\frac{K}{8} = -1$ and solve for K:

$$-\frac{K}{8} = -1 \rightarrow K = 8 .$$

Thus, the value of K where $L(j\omega) = -1$ for the system, when $t = t_1 = t_2 = t_3$, is $K = 8$.

(d) (**5 points**)

To draw the Nyquist plot, we need to analyze the behavior of $L(s)$ for different frequencies. The given system has $K = 20$. From the results of part (a), we know that for $\omega \rightarrow \infty$, the magnitude part of $L(j\omega)$ is given by $\frac{|K|}{\sqrt{1+(\omega t)^2}^3}$. The Nyquist plot is shown in the image below:

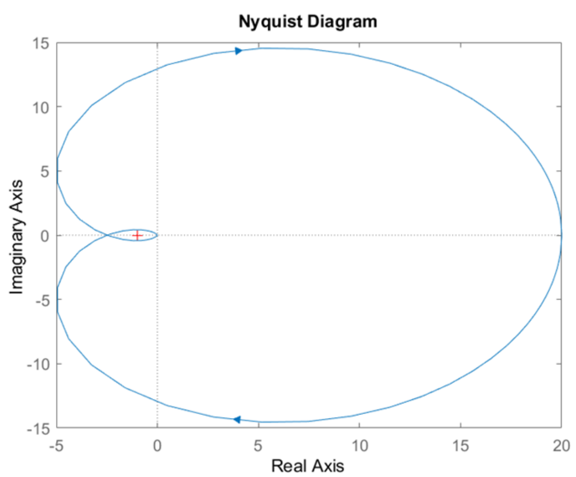


Figure 1: Nyquist Plot of $L(s)$

As seen from the Nyquist plot, the locus of $L(s)$ encircles the point (-1, 0) twice in the clockwise direction. According to the Nyquist criterion, each encirclement represents a right-half plane pole. Since there are two encirclements, it indicates that the system has two right-half plane poles. Therefore, the system is unstable.