Modified on Monday 21$^{\text{st}}$ September, 2020, 11:25:4511:25

Sep 21, 2020

- Determinant, inverse, Cramer's rule. [4.1, 4.2, 4.3]
- Determinant of 2×2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

  If the coefficient matrix of the system $u = ax + by, v = cx + dy$ is invertible then the solution is

$$x = \frac{du - bv}{ad - bc} = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{av - cu}{ad - bc} = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

- Determinant. Let $A = (a_{ij})$ be an $n \times n$ matrix.

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1,\pi_1} a_{2,\pi_2} \cdots a_{n,\pi_n}$$

- Permutation of $\{1, 2, \ldots, n\}$: $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$.
  - a rearrangement of $\{1, 2, \ldots, n\}$.
  - a bijection $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, $i \mapsto \pi_i$. In this case, we can write π in two line notation.

$$\pi = \begin{bmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}$$

  Let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. An inversion in π is a pair (i, j) with $i < j$ such that i appears to the right of j in $\pi_1\pi_2 \cdots \pi_n$. How many inversions does $\pi = 23154$ have? Which permutation in S_n has the least number of inversions? The greatest number of inversions? A permutation is called an even permutation if it has even number of inversions; an odd permutation, otherwise. The sign of a permutation π, $\operatorname{sgn}(\pi)$, is defined to be 1, if it is even; -1, otherwise.

- Example of S_n: $S_3 = \{123, 132, 213, 231, 312, 321\}$. How many inversions each permutation in S_3 have?
- Minor and cofactor of a square matrix $A = (a_{ij})$.

  The **minor** of entry a_{ij} (or the ij-minor of A), M_{ij}, is the determinant of the matrix obtained from A by deleting the i-th row and j-th column. The **cofactor** of a_{ij} (or the ij-cofactor of A), C_{ij}, is $(-1)^{i+j} M_{ij}$. The matrix $C = (C_{ij})$ is called the matrix of cofactors of A. The **adjugate** of A, denoted by $\operatorname{adj}(A)$, is the transpose of C.

- Cofactor expansion. Let A be $n \times n$ matrix. For any i and j,

$$\det(A) = \sum_{k=1}^{n} a_{kj} C_{kj} = \sum_{k=1}^{n} a_{ik} C_{ik}.$$

  The first identity is the cofactor expansion along the jth column and the latter is the cofactor expansion along ith row.

- Properties of determinant. Let $A = (a_{ij})$ be an $n \times n$ matrix.
  - The determinant function $\det : (\mathbb{R}^n)^n = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is multilinear (i.e., linear separately in each variable).

$$\det(\begin{bmatrix} A_1 + B_1 & A_2 & \cdots & A_n \end{bmatrix}) = \det(\begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}) + \det(\begin{bmatrix} B_1 & A_2 & \cdots & A_n \end{bmatrix})$$

$$\det(\begin{bmatrix} kA_1 & A_2 & \cdots & A_n \end{bmatrix}) = k \det(\begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix})$$

  - If two columns of A are switched, then the determinant changes by a factor -1.
  - If two columns of A are the same, then $\det(A) = 0$.
  - If a multiple of one column of A is added to another column, then the determinant does not change.
  - $\det(A)$ is nonzero if and only if A is invertible.
  - $\det(A) = \det(A^T)$.
  - $\det(AB) = \det(A)\det(B)$.
- If A is invertible, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

  Proof. We show that $A \operatorname{adj}(A) = \det(A) I_n$. The ij-entry of $A \operatorname{adj}(A)$ is $\sum_{k=1}^{n} a_{ik} C_{jk}$, which is clearly $\det(A)$ for $i = j$. For $i \neq j$, consider the matrix obtained from A by replacing the j-row with the i-th row.

- Cramer's rule. If A is an invertible matrix, then we can solve $A\mathbf{x} = \mathbf{b}$ by $\mathbf{x} = A^{-1}\mathbf{b}$, which gives the solutions $\mathbf{x} = (x_1, x_2, \ldots, x_n)$,

$$x_i = \frac{\det(A_i)}{\det(A)},$$

  where A_i is obtained from A by replacing the i-th column with $\mathbf{b}$.

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

- How can we compute the determinant efficiently? How many terms are there in the expression below?

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1,\pi_1} a_{2,\pi_2} \cdots a_{n,\pi_n}$$

  The Stirling formula for the factorial.

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

- Recall elementary row operations and elementary matrices. $E(i;k), E(i,j;c), E(i,j)$. What are the determinants of elementary matrices?

$$\det(E(i;k)) = k, \quad \det(E(i,j;c)) = 1, \quad \det(E(i,j)) = -1$$

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 3 & 3 & 8 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 8 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} = B$$

$$E(1;1/2), E(1,2;-3), E(2,3)$$

$$E(2,3)E(1,2;-3)E(1;1/2)A = B$$

$$\det(A) = \frac{\det(B)}{\det(E(2,3))\det(E(1,2;-3))\det(E(1;1/2))} = \frac{4}{-\frac{1}{2}} = -8$$

- Let A be an $n \times n$ matrix. The determinant of a matrix can be computed efficiently by using elementary row operations.

$$\det(A) = \begin{vmatrix} 2 & 2 & 4 \\ 3 & 3 & 8 \\ 0 & 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 8 \\ 0 & 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{vmatrix} = -8$$