School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2022 Spring

Midterm

Thursday, April 21, 2022
1:00–3:00 pm

- Be sure to **show all relevant work and reasoning** in your answer sheet. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.
- Please be clear in writing—we can't grade what we can't decipher!

Problem 1 (10 Points)

Provide answers and reasonings for each the question below.

a) (3 points) In answering a multiple-choice question, suppose that a student knows the answer with probability p and guesses randomly with probability $1-p$. If there are m choices for the question, what is the conditional probability that the student knows the answer given that the question is answered correctly?

b) (3 points) On a game show, you are given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 2, and the host, who knows what is behind each door, opens another door, say No. 3, which has a goat. The host then says to you, "Do you want to pick door No. 1 instead?" Is it to your advantage to switch your choice? What are the probability that the car is behind No. 2 (your original choice) and the probability that the car is behind No.1?

c) (4 points) A drone is missing and it is equally likely to have gone down in one of three possible regions. Let $1 - \alpha_i$ be the probability the drone can be found upon a search of the ith region when the drone is in fact in that region, for $i = 1, 2, 3$. What is the conditional probability that the drone is in the ith region, $i = 1, 2, 3$, given that a search of region 1 is unsuccessful?

<u>Problem 2</u> (10 Points)

Each laptop has a lifetime that is exponentially distributed with parameter λ. The lifetime of laptops are independent of each other. Suppose you have two laptops, which you begin using at the same time. Let L_1 be the lifetime of laptop 1 and L_2 be the lifetime of laptop 2, where L_1 and L_2 are i.i.d. exponential random variables with CDF $F_L(l) = 1 - e^{-\lambda l}$.

Define T_1 as the time of the first failure of any of two laptops, and T_2 as the time of failure for the rest of the two laptops.

a) (3 points) Compute the probability density function (PDF) $f_{T_1}(t_1)$.

b) (3 points) Let $X = T_2 - T_1$. Compute the conditional PDF $f_{X|T_1}(x|t_1)$.

c) (4 points) Compute the PDF $f_{T_2}(t_2)$ and the expectation $\mathbb{E}[T_2]$.

Problem 3 (10 Points)

Assume that your course grade is determined by your midterm score X_1 and your final score X_2. Your scores X_1, X_2 are independent normal random variables with the same mean $\mu < 90$ and the same variance σ^2.

a) (3 points) Assume that the grade is determined by the average score $Z = \frac{X_1}{2} + \frac{X_2}{2}$. You earn grade 'A' if $Z > 90$. What is the probability $\mathbf{P}(A) = \mathbf{P}(Z > 90)$? Write down this probability in terms of the CDF for the standard normal. (Remind that the CDF for the standard normal is $\Phi(y) = \mathbf{P}(Y \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt$.)

b) (3 points) A different student proposes that the better exam is the one that should count and the grades should be based on $M = \max(X_1, X_2)$. What is $\mathbf{P}(A) = \mathbf{P}(M > 90)$ for this case?

c) (4 points) Assume that the mean and the standard deviation are $\mu = 74$ and $\sigma = 16$, respectively. Use the table below for the CDF values of the standard normal to calculate the expected increase in the number of A's awarded by using $M = \max(X_1, X_2)$ instead of $Z = \frac{X_1}{2} + \frac{X_2}{2}$ in a class of 100 students. (You may use $\sqrt{2} \approx 1.41$.)

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \leq y)$, where Y is a standard normal random variable, for y between 0 and 1.99. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When y is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

<u>Problem 4</u> (10 Points)

Bob is a lazy worker and every morning he flips a coin to decide whether or not going to work. If the coin is head, he goes to work, and if it comes up tail he stays home. The coin he has is not necessarily fair, rather it possess a probability of heads equal to q, where the probability q is modeled by a random variable Q with density

$$f_Q(q) = \begin{cases} 2q, & \text{for } 0 \leq q \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that conditioned on Q each coin flip is independent.

a) (3 points) What is the probability that Bob goes to work if he flips the coin once?

b) (3 points) Assume that when Bob goes to work he earns \$10 every day. Define X as the Bob's payout for a month (30 days) if he flips the coin every morning for the next 30 days. Find $\text{var}(X)$.

   Hint: You may use the law of total variance: $\text{var}(X) = \mathbb{E}[\text{var}(X|Y)] + \text{var}(\mathbb{E}[X|Y])$.

c) (4 points) Let event B be that Bob stays home as least once in n days. Find the conditional density of Q given the event B, $f_{Q|B}(q)$.