HW 11 solution

9.75

The density given is a beta density with $\alpha = \beta = \theta$. Thus, $\mu_1' = E(Y) = .5$. Since this doesn't depend on θ, we turn to $\mu_2' = E(Y^2) = \frac{\theta+1}{2(2\theta+1)}$ (see Ex. 4.200). Hence, with $m_2' = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$, the MOM estimator of θ is $\hat{\theta} = \frac{1-2m_2'}{4m_2'-1}$.

9.78

For Y following the given power family distribution,

$$E(Y) = \int_0^3 \alpha y^\alpha 3^{-\alpha} dy = \alpha 3^{-\alpha} \frac{y^{\alpha+1}}{\alpha+1}\bigg|_0^3 = \frac{3\alpha}{\alpha+1}.$$

Thus, the MOM estimator of θ is $\hat{\theta} = \frac{\bar{Y}}{3-\bar{Y}}$.

9.82

The likelihood function is $L(\theta) = \theta^{-n} r^n \left(\prod_{i=1}^{n} y_i\right)^{-1} \exp\left(-\sum_{i=1}^{n} y_i^r/\theta\right)$.

a. By Theorem 9.4 , a sufficient statistic for θ is $\sum_{i=1}^{n} Y_i^r$.

b. The log-likelihood is

$$\ln L(\theta) = -n\ln\theta + n\ln r + (r-1)\ln\left(\prod_{i=1}^{n} y_i\right) - \sum_{i=1}^{n} y_i^r/\theta.$$

By taking a derivative w.r.t. θ and equating to 0 , we find $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i^r$.

c. Note that $\hat{\theta}$ is a function of the sufficient statistic. Since it is easily shown that $E(Y^r) = \theta, \hat{\theta}$ is then unbiased and the MVUE for θ.

9.83

a. The likelihood function is $L(\theta) = (2\theta+1)^{-n}$. Let $\gamma = \gamma(\theta) = 2\theta+1$. Then, the likelihood can be expressed as $L(\gamma) = \gamma^{-n}$. The likelihood is maximized for small values of γ. The smallest value that can safely maximize the likelihood (see Example 9.16) without violating the support is $\hat{\gamma} = Y_{(n)}$. Thus, by the invariance property of MLEs, $\hat{\theta} = \frac{1}{2}\left(Y_{(n)} - 1\right)$.

b. Since $V(Y) = \frac{(2\theta+1)^2}{12}$. By the invariance principle, the MLE is $\left(Y_{(n)}\right)^2/12$.

9.86

First, similar to Example 9.15 , the MLEs of μ_1 and μ_2 are $\hat{\mu}_1 = \bar{X}$ and $\hat{\mu}_2 = \bar{Y}$. To estimate σ^2, the likelihood is

$$L\left(\sigma^2\right) = \frac{1}{(2\pi)^{(m+n)/2}\sigma^{m+n}} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{m}\left(\frac{x_i - \mu_1}{\sigma}\right)^2 - \sum_{i=1}^{n}\left(\frac{y_i - \mu_2}{\sigma}\right)^2\right]\right\}.$$

The log-likelihood is

$$\ln L\left(\sigma^2\right) = K - (m+n)\ln\sigma - \frac{1}{2\sigma^2}\left[\sum_{i=1}^{m}(x_i - \mu_1)^2 - \sum_{i=1}^{n}(y_i - \mu_2)^2\right]$$

By differentiating and setting this quantity equal to 0 , we obtain

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{m}(x_i - \mu_1)^2 - \sum_{i=1}^{n}(y_i - \mu_2)^2}{m+n}.$$

As in Example 9.15, the MLEs of μ_1 and μ_2 can be used in the above to arrive at the MLE for σ^2 :

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{m}\left(X_i - \bar{X}\right)^2 - \sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2}{m+n}.$$

9.93

a. Following the hint, the MLE for θ is $\hat{\theta} = Y_{(1)}$.

b. Since $F(y \mid \theta) = 1 - 2\theta^2 y^{-2}$, the density function for $Y_{(1)}$ is easily found to be

$$g_{(1)}(y) = 2n\theta^{2n}y^{-(2n+1)}, y > \theta.$$

If we consider the distribution of $T = \theta/Y_{(1)}$, the density function of T can be found to be

$$f_T(t) = 2nt^{2n-1}, 0 < t < 1.$$

c. (Similar to Ex. 9.92) Constants a and b can be found to satisfy $P(a < T < b) = 1 - \alpha$ such that $P(T < a) = P(T > b) = \alpha/2$. Using the density function from part b, these are given by $a = (\alpha/2)^{1/(2n)}$ and $b = (1 - \alpha/2)^{1/(2n)}$. So, we have

$$1 - \alpha = P\left(a < \theta/Y_{(1)} < b\right) = P\left(aY_{(1)} < \theta < bY_{(1)}\right).$$

Thus, $\left[(\alpha/2)^{1/(2n)}Y_{(1)}, (1 - \alpha/2)^{1/(2n)}Y_{(1)}\right]$ is a $(1 - \alpha)100\%$ CI for θ.

9.106

2

Following the method used in Ex. 9.65, construct the random variable T such that

$$T = 1 \text{ if } Y_1 = 0 \text{ and } T = 0 \text{ otherwise}$$

Then, $E(T) = P(T = 1) = P(Y_1 = 0) = \exp(-\lambda)$. So, T is unbiased for $\exp(-\lambda)$. Now, we know that $W = \sum_{i=1}^{n} Y_i$ is sufficient for λ, and so it is also sufficient for $\exp(-\lambda)$. Recalling that W has a Poisson distribution with mean $n\lambda$,

$$E(T \mid W = w) = P(T = 1 \mid W = w) = P(Y_1 = 0 \mid W = w) = \frac{P(Y_1 = 0, W = w)}{P(W = w)}$$

$$= \frac{P(Y_1 = 0) P\left(\sum_{i=2}^{n} Y_i = w\right)}{P(W = w)} = \frac{e^{-\lambda}\left(e^{-(n-1)\lambda} \frac{[(n-1)\lambda]^w}{w!}\right)}{e^{-n\lambda} \frac{(n\lambda)^w}{w!}} = \left(1 - \frac{1}{n}\right)^w.$$

Thus, the MVUE is $\left(1 - \frac{1}{n}\right)^{\Sigma Y_i}$. Note that in the above we used the result that $\sum_{i=2}^{n} Y_i$ is Poisson with mean $(n - 1)\lambda$.

9.112

a. (Refer to Section 9.3.) By the Central Limit Theorem, $\frac{\bar{Y} - \lambda}{\sqrt{\lambda/n}}$ converges to a standard normal variable. Also, $\bar{Y}/\lambda$ converges in probability to 1 by the Law of Large Numbers, as does $\sqrt{\bar{Y}/\lambda}$. So, the quantity

$$W_n = \frac{\frac{\bar{Y} - \lambda}{\sqrt{\lambda/n}}}{\sqrt{\bar{Y}/\lambda}} = \frac{\bar{Y} - \lambda}{\sqrt{\bar{Y}/n}}$$

converges to a standard normal distribution.

b. By part a, an approximate $(1 - \alpha)100\%$CI for λ is $\bar{Y} \pm z_{\alpha/2}\sqrt{\bar{Y}/n}$.