IE241 HW3 solution

3.50 A missile protection system consists of n radar sets operating independently, each with a probability of .9 of detecting a missile entering a zone that is covered by all of the units.

a If $n = 5$ and a missile enters the zone, what is the probability that exactly four sets detect the missile? At least one set?

b How large must n be if we require that the probability of detecting a missile that enters the zone be .999?

a) $P(4 \text{ sets detect the missile}) = P(4 \text{ detect } \& \ 1 \text{ fail}) = 5 \times (0.9)^4 \times (0.1) = 0.32805$

$P(\text{at least one set}) = 1 - P(5 \text{ fail}) = 1 - (0.1)^5 = 0.99999$

b) $1 - (0.1)^n \geq 0.999$, if $n=3$ $1-(0.1)^3 = 0.999$. Hence $n \geq 3$

(10pts)

3.55 Suppose that Y is a binomial random variable with $n > 2$ trials and success probability p. Use the technique presented in Theorem 3.7 and the fact that $E\{Y(Y-1)(Y-2)\} = E(Y^3) - 3E(Y^2) + 2E(Y)$ to derive $E(Y^3)$.

$Y \sim B(n, p)$, $n > 2$

$$E(Y(Y-1)(Y-2)) = \sum_{y=3}^{n} y(y-1)(y-2) \cdot \binom{n}{y} \cdot p^y \cdot (1-p)^{n-y}$$

using $\binom{n}{y} = \frac{n!}{(n-y)!\,y!}$

$$= \sum_{y=3}^{n} \frac{n!}{(n-y)!\,(y-3)!} \cdot p^y \cdot (1-p)^{n-y}$$

, Let $z = y-3$ then

$$= \sum_{z=0}^{n-3} \frac{n!}{(n-3-z)!\,z!} \cdot p^{z+3} \cdot (1-p)^{n-3-z}$$

$$= \sum_{z=0}^{n-3} p^3 \cdot n \times (n-1)(n-2) \cdot \frac{(n-3)!}{(n-3-z)!\,z!} \cdot p^z \cdot (1-p)^{n-3-z}$$

$$= p^3 \cdot n(n-1)(n-2) \sum_{z=0}^{n-3} \frac{(n-3)!}{(n-3-z)!\,z!} \cdot p^z \cdot (1-p)^{n-3-z}$$

pmf of $B(n-3, p)$
Hence $\sum_{z=0}^{n-3} \text{pmf} = 1$

$$= p^3 n(n-1)(n-2)$$

$$\therefore E(Y^3) = E(Y(Y-1)(Y-2)) + 3E(Y^2) - 2E(Y)$$

$$= p^3 n(n-1)(n-2) + 3\left(n(n-1)p^2 + np\right) - 2np$$

$$= np\left(p^2(n-1)(n-2) + 3p(n-1) + 1\right)$$

3.62 Goranson and Hall (1980) explain that the probability of detecting a crack in an airplane wing is the product of p_1, the probability of inspecting a plane with a wing crack; p_2, the probability of inspecting the detail in which the crack is located; and p_3, the probability of detecting the damage.

a What assumptions justify the multiplication of these probabilities? $\Rightarrow$ each inspecting events are independent!

b Suppose $p_1 = .9$, $p_2 = .8$, and $p_3 = .5$ for a certain fleet of planes. If three planes are inspected from this fleet, find the probability that a wing crack will be detected on at least one of them.

$P(\text{crack will be detected}) = P_1 \times P_2 \times P_3 = 0.36$

$P(\text{Detected at least } 1) = 1 - P(\text{Not detected crack})$

$= 1 - (0.64)^3 \simeq 73.7\%$

3.66 Suppose that Y is a random variable with a geometric distribution. Show that

a $\sum_y p(y) = \sum_{y=1}^{\infty} q^{y-1}p = 1.$

b $\dfrac{p(y)}{p(y-1)} = q$, for $y = 2, 3, \dots$. This ratio is less than 1, implying that the geometric probabilities are monotonically decreasing as a function of y. If Y has a geometric distribution, what value of Y is the most likely (has the highest probability)?

a) $\sum_{y=1}^{\infty} q^{y-1}p = p \cdot \underbrace{\sum_{y=1}^{\infty} q^{y-1}}_{=\frac{1}{1-q}} = \dfrac{p}{1-q} = p \cdot \dfrac{1}{1-q} = p \cdot \dfrac{1}{p} = 1$ (infinite geometric series)

b) $\dfrac{P(y)}{P(y-1)} = \dfrac{P(1-p)^{y-1}}{P(1-p)^{y-2}} = \dfrac{1}{(1-p)^{-1}} = q$ $\Rightarrow$ monotone decreasing as a function of y

If $Y \sim geo(P)$, support of $Y = \{\underset{\to min}{1}, 2, \cdots \cdots\}$

$\therefore$ $Y=1$ is the most likely has the highest probability !

3.70 An oil prospector will drill a succession of holes in a given area to find a productive well. The probability that he is successful on a given trial is .2.

a What is the probability that the third hole drilled is the first to yield a productive well?

b If the prospector can afford to drill at most ten wells, what is the probability that he will fail to find a productive well?

a) $(0.8)^2 \times 0.2 \simeq 0.128$

b) $(0.8)^{10} \simeq 0.107$

$\rightarrow$ very intuitive...

3.79 How many times would you expect to toss a balanced coin in order to obtain the first head?

$X \sim geo(\frac{1}{2})$, $E(X) = \sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^{x-1} \cdot \left(\frac{1}{2}\right) = \sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^{x} = \dfrac{1}{1-\frac{1}{2}} = 2$

3.80 Two people took turns tossing a fair die until one of them tossed a 6. Person A tossed first, B second, A third, and so on. Given that person B threw the first 6, what is the probability that B obtained the first 6 on her second toss (that is, on the fourth toss overall)? (5pts)

$\rightarrow$ $P(\text{fourth toss} \mid \text{B threw the first 6})$, Let $X = \#$ of tosses until finish (= 6 appear)

$P(\text{B threw the first 6}) = P(X=2,4,6,\cdots) = \frac{5}{6} \times \frac{1}{6} + \left(\frac{5}{6}\right)^3 \times \frac{1}{6} + \left(\frac{5}{6}\right)^5 \times \frac{1}{6} \cdots$

$= \frac{1}{6}\left(\frac{5}{6} + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^5 \cdots\right) = \frac{1}{6}\left(\dfrac{5/6}{1-(5/6)^2}\right) = \dfrac{5}{11}$. Then

$P(\text{fourth toss} \mid \text{B threw the first 6}) = \dfrac{P(\text{B threw the first 6} \mid \text{fourth toss}) \times P(\text{fourth toss})}{P(\text{B threw the first 6})}$ $\leftarrow$ $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

$= \dfrac{\frac{1}{6} \times \left(\frac{5}{6}\right)^3}{\frac{5}{11}} = \dfrac{275}{1296}$

***3.85** Find $E[Y(Y-1)]$ for a geometric random variable Y by finding $d^2/dq^2\left(\sum_{y=1}^{\infty} q^y\right)$. Use this result to find the variance of Y. (5pts)

$$\frac{d^2}{dq^2}\left(\sum_{y=1}^{\infty} q^y\right) = \frac{d}{dq}\left(\sum_{y=1}^{\infty} y q^{y-1}\right) = \sum_{y=1}^{\infty} y(y-1) q^{y-2} = \frac{2}{(1-q)^3} = \frac{2}{p^3}$$

$$E(Y(Y-1)) = \sum_{y=1}^{\infty} y(y-1) q^{y-1}\cdot p = qp\cdot \sum_{y=1}^{\infty} y(y-1)q^{y-2} = \frac{2q}{p^2}$$

$E(Y(Y-1)) = E(Y^2) - E(Y) = \frac{2q}{p^2}$, we already know $E(Y) = \frac{1}{p}$

then $E(Y^2) = \frac{2q}{p^2} + \frac{1}{p}$

$$\text{var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q+p-1}{p^2} = \frac{q}{p^2}$$

***3.88** If Y is a geometric random variable, define $Y^* = Y - 1$. If Y is interpreted as the number of the trial on which the first success occurs, then Y^* can be interpreted as the number of failures *before* the first success. If $Y^* = Y - 1$, $P(Y^* = y) = P(Y - 1 = y) = P(Y = y + 1)$ for $y = 0, 1, 2, \ldots$. Show that

$$P(Y^* = y) = q^y p, \qquad y = 0, 1, 2, \ldots.$$

The probability distribution of Y^* is sometimes used by actuaries as a model for the distribution of the number of insurance claims made in a specific time period.

$$P(Y^* = y) = P(Y = y+1) = q^{y+1-1}\cdot p = q^y \cdot p$$

3.96 The telephone lines serving an airline reservation office are all busy about 60% of the time.

a If you are calling this office, what is the probability that you will complete your call on the first try? The second try? The third try?

b If you and a friend must both complete calls to this office, what is the probability that a total of four tries will be necessary for both of you to get through?

a) 1st = 0.4 2st = 0.6×0.4 3rd = $(0.6)^2 \times 0.4$

b)

1st	2st	3st	4st
O	X	X	O
X	O	X	O
X	X	O	O

$\Rightarrow 0.4 \times (0.6)^2 \times 0.4 \times 3 \simeq 0.1728$

$\Rightarrow$ exactly same $NB(2, 0.4)$

3.118 Five cards are dealt at random and without replacement from a standard deck of 52 cards. What is the probability that the hand contains all 4 aces if it is known that it contains at least 3 aces? (5pts)

Let X: # of aces in the hand, X follows hypergeometric distribution

$$P(X=4 \mid X\geq 3) = \frac{P(X\geq 3 \mid X=4)\, P(X=4)}{P(X\geq 3)} = \frac{P(X=4)}{P(X\geq 3)} = \frac{P(X=4)}{P(X=3)+P(X=4)}$$

$P(X=x) = \binom{4}{x}\cdot\binom{52-4}{5-x} \Big/ \binom{52}{5}$ ($N=52$, $r=4$ in standard deck & draw 5 cards)

$\therefore P(X=4 \mid X\geq 3) = 0.0105$

***3.120** The sizes of animal populations are often estimated by using a capture–tag–recapture method. In this method k animals are captured, tagged, and then released into the population. Some time later n animals are captured, and Y, the number of tagged animals among the n, is noted. The probabilities associated with Y are a function of N, the number of animals in the population, so the observed value of Y contains information on this unknown N. Suppose that $k = 4$ animals are tagged and then released. A sample of $n = 3$ animals is then selected at random from the same population. Find $P(Y = 1)$ as a function of N. What value of N will *maximize* $P(Y = 1)$?

(5pts)

Total = N, tagged = K $\Rightarrow$ Not tagged = $N-K$ (capture - tag)

$Y \Rightarrow$ when n animals are recaptured, the number of tagged animals.

$\therefore P(Y=1) = \binom{N-4}{2}\binom{4}{1} \Big/ \binom{N}{3}$ ~ Hypergeometric dist.

$= \dfrac{12(N-4)(N-5)}{N(N-1)(N-2)}$.

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# This code is written by R

func_prob = function(N){   # function of p(y =1)
  p_y = (12*(N-4)*(N-5)) / (N*(N-1)*(N-2))
return(p_y)}

prob = c()
for(i in c(5:1000)){
   prob[i] = func_prob(i)
   }
#which.max(prob) # largest N
cat("The value of N that maximizes the probability is:", which.max(prob), "\n")
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The value of N that maximizes the probability is: 11
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3.124 Approximately 4% of silicon wafers produced by a manufacturer have fewer than two large flaws. If Y, the number of flaws per wafer, has a Poisson distribution, what proportion of the wafers have more than five large flaws? [*Hint:* Use Table 3, Appendix 3.]

$\rightarrow P(Y \le 2) \simeq 0.04$ by Appendix, $\lambda = 6.6$

$Y \sim \text{poisson}(\lambda)$, $f(y) = \dfrac{e^{-\lambda} \cdot \lambda^y}{y!}$, $\lambda = 1.6$ then $f(y) = \dfrac{e^{-6.6} \cdot (6.6)^y}{y!}$

$P(Y>5) = 1 - \sum_{i=0}^{5} P(Y=i) \simeq 0.645$

3.130 A parking lot has two entrances. Cars arrive at entrance I according to a Poisson distribution at an average of three per hour and at entrance II according to a Poisson distribution at an average of four per hour. What is the probability that a total of three cars will arrive at the parking lot in a given hour? (Assume that the numbers of cars arriving at the two entrances are independent.)

(5pts)

entrances 1 $X_1 \sim \text{Poisson}(3)$
〃 2 $X_2 \sim \text{Poisson}(4)$ $\Rightarrow$ Let $Y = X_1 + X_2$

by independence

$M_Y(t) = E(e^{ty}) = E(e^{t(X_1+X_2)}) = E(e^{tX_1} \cdot e^{tX_2}) = \prod_{i=1}^{2} E(e^{tX_i})$

$= \sum_{x=0}^{\infty} e^{tx} \cdot \dfrac{e^{-\lambda} \cdot \lambda^x}{x!} = \sum_{x=0}^{\infty} \dfrac{e^{-\lambda} \cdot (\lambda e^t)^x}{x!} = e^{\lambda(e^t-1)} \sum_{x=0}^{\infty} \dfrac{e^{-\lambda e^t} \cdot (\lambda e^t)^x}{x!} = e^{\lambda(e^t-1)}$

$\prod_{i=1}^{2} E(e^{tX_i}) = e^{7(e^t-1)} \sim \text{Poisson}(7) \Rightarrow \therefore P(Y=3) = \dfrac{e^{-7} \cdot 7^3}{3!} \simeq 0.0521$

3.136 Increased research and discussion have focused on the number of illnesses involving the organism *Escherichia coli* (10257:H7), which causes a breakdown of red blood cells and intestinal hemorrhages in its victims (http://www.hsus.org/ace/11831, March 24, 2004). Sporadic outbreaks of *E.coli* have occurred in Colorado at a rate of approximately 2.4 per 100,000 for a period of two years.

a If this rate has not changed and if 100,000 cases from Colorado are reviewed for this year, what is the probability that at least 5 cases of *E.coli* will be observed?

b If 100,000 cases from Colorado are reviewed for this year and the number of *E.coli* cases exceeded 5, would you suspect that the state's mean *E.coli* rate has changed? Explain.

$\lambda = 2.4$

a) $P(Y \geq 5) = 1 - \sum_{i=0}^{4} P(Y=i) \approx 0.096$

b) $P(Y>5) = 1 - \sum_{i=0}^{5} (Y=i) \approx 0.036$

Yes. Changed. Beacuse probability is vary low.