School of Electrical Engineering, KAIST
EE210 Probability and Introductory Random Processes
2022 Spring

Midterm

Thursday, April 21, 2022
1:00–3:00 pm

NAME: ______________________________

Student ID: ______________________________

- Don't forget to put your name and student ID.
- **Record all your solutions in this answer booklet. Only this answer booklet will be considered in the grading of your exam.**
- Be sure to show all relevant work and reasoning. A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning.

Problem	Your score	Max score
1		10
2		10
3		10
4		10
Total		40

Problem 1 (10 Points) Provide answers and reasoning for each the question below.

a) (3 points) In answering a multiple-choice question, suppose that a student knows the answer with probability p and guesses randomly with probability $1-p$. If there are m choices for the question, what is the conditional probability that the student knows the answer given that the question is answered correctly?

Answer:

$$\frac{mp}{mp-p+1}$$

Reasoning for Problem 1(a): If there are m choices for the question, what is the conditional probability that the student knows the answer given that the question is answered correctly?

Let E be the event that the student knows the answer, and let F be the event that the student answers the question correctly. Then,

$$\mathbf{P}(F) = \mathbf{P}(F|E)\mathbf{P}(E) + \mathbf{P}(F|E^c)\mathbf{P}(E^c) = p + \frac{1-p}{m}$$

and so

$$\mathbf{P}(E|F) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)} = \frac{p}{p+(1-p)/m} = \frac{mp}{mp-p+1}.$$

b) (3 points) On a game show, you are given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 2, and the host, who knows what is behind each door, opens another door, say No. 3, which has a goat. The host then says to you, "Do you want to pick door No. 1 instead?" Is it to your advantage to switch your choice? What is the probability of winning if you switch?

Answer:

It is advantageous to switch the choice. The probability of winning is then $\frac{2}{3}$.

Reasoning for Problem 1(b): If the car is behind the initially chosen door (probability 1/3), you do not win. If it is not (probability 2/3), and given that another door without a car has been opened for you, you will get to the winning door once you switch. Thus, the probability of winning is 2/3 if you switch.

c) (4 points) A drone is missing and it is equally likely to have gone down in one of three possible regions. Let $1-\alpha_i$ be the probability the drone can be found upon a search of the ith region when the drone is in fact in that region, for $i = 1, 2, 3$. What is the conditional probability that the drone is in the ith region, $i = 1, 2, 3$, given that a search of region 1 is unsuccessful?

Let X be the region the drone is in. Let Y be the indicator that the search in region 1 is successful (i.e., $Y = 1$, successful; $Y = 0$, unsuccessful).

Answer:

$$\begin{aligned}
\mathbf{P}(X=1|Y=0) &= \frac{\mathbf{P}(Y=0|X=1)\mathbf{P}(X=1)}{\mathbf{P}(Y=0)} = \frac{\alpha_1/3}{\alpha_1/3+2/3} = \frac{\alpha_1}{\alpha_1+2},\\
\mathbf{P}(X=2|Y=0) &= \frac{\mathbf{P}(Y=0|X=2)\mathbf{P}(X=2)}{\mathbf{P}(Y=0)} = \frac{1/3}{\alpha_1/3+2/3} = \frac{1}{\alpha_1+2},\\
\mathbf{P}(X=3|Y=0) &= \frac{\mathbf{P}(Y=0|X=3)\mathbf{P}(X=3)}{\mathbf{P}(Y=0)} = \frac{1/3}{\alpha_1/3+2/3} = \frac{1}{\alpha_1+2}.
\end{aligned}$$

Reasoning for Problem 1(c): Let X be the region the drone is in. Let Y be the indicator that the search in region 1 is successful (i.e., $Y = 1$, successful; $Y = 0$, unsuccessful). Then,

$$\mathbf{P}(Y=0) = \sum_{i=1}^{3} \mathbf{P}(Y=0|X=i)\mathbf{P}(X=i) = \frac{\alpha_1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{\alpha_1}{3} + \frac{2}{3},$$

so

$$\begin{aligned}
\mathbf{P}(X=1|Y=0) &= \frac{\mathbf{P}(Y=0|X=1)\mathbf{P}(X=1)}{\mathbf{P}(Y=0)} = \frac{\alpha_1/3}{\alpha_1/3+2/3} = \frac{\alpha_1}{\alpha_1+2},\\
\mathbf{P}(X=2|Y=0) &= \frac{\mathbf{P}(Y=0|X=2)\mathbf{P}(X=2)}{\mathbf{P}(Y=0)} = \frac{1/3}{\alpha_1/3+2/3} = \frac{1}{\alpha_1+2},\\
\mathbf{P}(X=3|Y=0) &= \frac{\mathbf{P}(Y=0|X=3)\mathbf{P}(X=3)}{\mathbf{P}(Y=0)} = \frac{1/3}{\alpha_1/3+2/3} = \frac{1}{\alpha_1+2}.
\end{aligned}$$

Problem 2 (10 Points)

Each laptop has a lifetime that is exponentially distributed with parameter λ. The lifetime of laptops are independent of each other. Suppose you have two laptops, which you begin using at the same time. Let L_1 be the lifetime of laptop 1 and L_2 be the lifetime of laptop 2, where L_1 and L_2 are independent and identically distributed exponential random variables with CDF $F_L(l) = 1 - e^{-\lambda l}$, $l \geq 0$.

Define T_1 as the time of the first failure of any of two laptops, and let T_2 be the time of failure for the other laptop.

a) (3 points) Compute the probability density function (PDF) of T_1, $f_{T_1}(t)$.

Answer:

$$f_{T_1}(t) = 2\lambda e^{-2\lambda t}, \quad t \geq 0.$$

Reasoning for Problem 2(a): To derive the distribution of T_1, we first find the CDF $F_{T_1}(t)$, and then differentiate it to find the PDF $f_{T_1}(t)$.

$$\begin{aligned} F_{T_1}(t) &= \mathbf{P}(\min(L_1, L_2) < t) \\ &= 1 - \mathbf{P}(\min(L_1, L_2) \geq t) \\ &= 1 - \mathbf{P}(L_1 \geq t)\mathbf{P}(L_2 \geq t) \\ &= 1 - (1 - F_L(t))^2 \\ &= 1 - e^{-2\lambda t}, \quad t \geq 0. \end{aligned}$$

Differentiating $F_{T_1}(t)$ with respect to t yields:

$$f_{T_1}(t) = 2\lambda e^{-2\lambda t}, \quad t \geq 0.$$

b) (3 points) Let $X = T_2 - T_1$. Compute the conditional PDF $f_{X|T_1}(x|t)$.

Answer:

$$f_{X|T_1}(x|t_1) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

Reasoning for Problem 2(b): Conditioned on the time of the first laptop failure, the time until the other laptop fails is exponentially distributed by the memoryless property. Thus,

$$f_{X|T_1}(x|t_1) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

c) (4 points) Compute the PDF $f_{T_2}(t)$, and the expectation $\mathbb{E}[T_2]$.

Answer:

$$f_{T_2}(t) = 2\lambda e^{-\lambda t}(1 - e^{-\lambda t}), \quad t \geq 0.$$

$$\mathbb{E}[T_2] = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}.$$

Reasoning for Problem 2(c): The time of the second failure T_2 is equal to $T_1 + X$. Since X and T_1 are independent, we convolve the densities found in a) and b) to determine $f_{T_2}(t)$.

$$\begin{aligned} f_{T_2}(t) &= \int_0^\infty f_{T_1}(\tau) f_X(t-\tau) d\tau \\ &= \int_0^t 2\lambda^2 e^{-2\lambda\tau} e^{-\lambda(t-\tau)} d\tau \\ &= 2\lambda e^{-\lambda t} \int_0^t \lambda e^{-\lambda\tau} d\tau \\ &= 2\lambda e^{-\lambda t}(1 - e^{-\lambda t}), \quad t \geq 0. \end{aligned}$$

An equivalent method for solving this is to note that $T_2 = \max(L_1, L_2)$ and

$$F_{T_2}(t) = \mathbf{P}(L_1 \leq t)\mathbf{P}(L_2 \leq t) = F_L(t)^2 = 1 - 2e^{-\lambda t} + e^{-2\lambda t}, \quad t \geq 0.$$

Finally, for the above density we obtain that

$$\mathbb{E}[T_2] = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}.$$

Problem 3 (10 Points)

Assume that your course grade is determined by your midterm score X_1 and your final score X_2. Your scores X_1, X_2 are independent normal random variables with the same mean $\mu < 90$ and the same variance σ^2.

a) (3 points) Assume that the grade is determined by the average score $Z = \frac{X_1}{2} + \frac{X_2}{2}$. You earn grade 'A' if $Z > 90$. What is the probability $\mathbf{P}(A) = \mathbf{P}(Z > 90)$? Write down this probability in terms of the CDF for the standard normal. (Remind that the CDF for the standard normal is $\Phi(y) = \mathbf{P}(Y \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt$.)

   Answer:

   $\mathbf{P}(A) = \mathbf{P}(Z > 90) = 1 - \Phi\left(\frac{90-\mu}{\sigma/\sqrt{2}}\right)$

   Reasoning for Problem 3(a): Note that Z is also a normal random variable with mean μ and variance $\sigma^2/2$. By standardizing Z,

$$\mathbb{P}(Z > 90) = 1 - \mathbb{P}(Z \leq 90) = 1 - \mathbb{P}\left(\frac{Z-\mu}{\sigma/\sqrt{2}} \leq \frac{90-\mu}{\sigma/\sqrt{2}}\right) = 1 - \Phi\left(\frac{90-\mu}{\sigma/\sqrt{2}}\right)$$

b) (3 points) A different student proposes that the better exam is the one that should count and the grades should be based on $M = \max(X_1, X_2)$. What is $\mathbf{P}(A) = \mathbf{P}(M > 90)$ for this case?

   Answer:

   $\mathbf{P}(A) = \mathbf{P}(M > 90) = 1 - \left(\Phi\left(\frac{90-\mu}{\sigma}\right)\right)^2$

   Reasoning for Problem 3(b):

   Note that

$$\begin{aligned}
\mathbf{P}(M > 90) &= 1 - \mathbf{P}(M \leq 90) = 1 - \mathbf{P}(X_1 \leq 90, X_2 \leq 90) \\
&= 1 - \mathbf{P}(X_1 \leq 90)\mathbf{P}(X_2 \leq 90) = 1 - \left(\mathbf{P}\left(\frac{X_1-\mu}{\sigma} \leq \frac{90-\mu}{\sigma}\right)\right)^2 \\
&= 1 - \left(\Phi\left(\frac{90-\mu}{\sigma}\right)\right)^2
\end{aligned}$$

c) (4 points) Assume that the mean and the standard deviation are $\mu = 74$ and $\sigma = 16$, respectively. Use the table below for the CDF values of the standard normal to calculate the expected increase in the number of A's awarded by using $M = \max(X_1, X_2)$ instead of $Z = \frac{X_1}{2} + \frac{X_2}{2}$ in a class of 100 students.

You do not need to calculate the exact number, but write down your answer in terms of the CDF values found in the table below (not in terms of $\Phi(\cdot)$).

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \le y)$, where Y is a standard normal random variable, for y between 0 and 1.99. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When y is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

Answer:

Expected increase in the number of A's$=100 * (0.9207 - 0.8413^2)(\approx 21)$.

Reasoning for Problem 3(c):

For $\mu = 74$ and $\sigma = 16$, $\mathbb{P}(Z > 90) = 1 - \Phi(\sqrt{2}) \approx 1 - 0.9207 = 0.0793$, and $\mathbb{P}(M > 90) = 1 - (\Phi(1))^2 \approx 1 - 0.8413^2 = 0.2922$. Therefore, the expected number of A's for a class of size 100 increases by 29-8=21, approximately.

Problem 4 (10 Points)

Provide answers and reasoning for each of the independent questions below.

a) (5 points) The random variables X and Y are described by a joint PDF which is constant within the unit-area parallelogram specified by vertices (0,0), (0,1), (1,2) and (1,1). Find the variance of $X + Y$.

Answer:

$\text{var}(X + Y) = \frac{5}{12}$

Reasoning for Problem 4(a):

Given a value x of X, the random variable Y is uniformly distributed in the interval $[x, x + 1]$, and the random variable $X + Y$ is uniformly distributed in the interval $[2x, 2x + 1]$. Therefore, $\mathbb{E}[X + Y|X] = 0.5 + 2X$ and $\text{var}(X + Y|X) = \frac{1}{12}$. Thus,

$$\text{var}(X + Y) = \text{var}(0.5 + 2X) + \mathbb{E}[1/12] = 4\text{var}(X) + \mathbb{E}[1/12] = \frac{5}{12}.$$

b) (5 points) You roll a fair six-sided die, and then you flip a fair coin the number of times shown by the die. Find the expected value and the variance of the number of heads, H, obtained.

(Hint: For a discrete random variable uniformly distributed over $\{a, a+1, \ldots, a+n-1\}$, the variance is equal to $\frac{n^2-1}{12}$.)

Answer:

$\mathbb{E}[H] = \frac{7}{4}$

$\text{var}(H) = \frac{77}{48}$

Reasoning for Problem 4(b):

Let X_i be independent Bernoulli random variables that are equal to 1 if the i-th flip results in heads. Let N be the number of coin flips. Then, we have $H = \sum_{i=1}^{N} X_i$. We have $\mathbb{E}[X_i] = 1/2$, $\text{var}(X_i) = 1/4$, $\mathbb{E}[N] = \sum_{i=1}^{6} \frac{1}{6} i = 7/2$, and $\text{var} = \sum_{i=1}^{6} \frac{1}{6}(i - 7/2)^2 = \frac{6^2-1}{12} = 35/12$. (The last inequality is obtained from the formula for the variance of a discrete uniform random variable.)

Therefore, the expected number of heads is

$$\mathbb{E}[H] = \mathbb{E}[X_i] \cdot \mathbb{E}[N] = \frac{7}{4},$$

and the variance is

$$\begin{aligned}
\text{var}(H) =& \mathbb{E}[\text{var}(H|N)] + \text{var}(\mathbb{E}[H|N]) \\
=& \mathbb{E}[N \text{var}(X_i)] + \text{var}(N \mathbb{E}[X_i]) \\
=& \text{var}(X_i)\mathbb{E}[N] + (\mathbb{E}[X_i])^2 \text{var}(N) = \frac{1}{4} \cdot \frac{7}{2} + \frac{1}{4} \cdot \frac{35}{12} = \frac{77}{48}.
\end{aligned}$$