HW #4 Solution

3.145

$P(Y=y) = \binom{n}{y} p^y q^{n-y}$ for $y = 0, \cdots, n$ (+2) (5pt)

$m(t) = \sum_{y=0}^{n} \binom{n}{y} (pe^t)^y q^{n-y} = (pe^t + q)^n$ by binomial thm (+3)

3.146

$\frac{d}{dt} m(t) = n(pe^t+q)^{n-1} pe^t$ (+1), $m'(0) = np = E(Y)$ (+1)

$\frac{d}{dt} m'(t) = n(n-1)(pe^t+q)^{n-2}(pe^t)^2 + n(pe^t+q)^{n-1} pe^t$. (+1) $m''(0) = np^2(n-1) + np$ (+1)

$V(Y) = np^2(n-1) + np - (np)^2 = np(1-p)$ (+1) (5pt)

3.147

$P(Y=y) = pq^{y-1}$, $y = 1, \cdots$, $m(t) = \sum_{y=1}^{\infty} pe^{ty} q^{y-1} = \sum_{y=1}^{\infty} pe^t (e^t q)^{y-1}$

$= pe^t \sum_{y=1}^{\infty} (qe^t)^{y-1} = \frac{pe^t}{1-qe^t}$

3.148

$m'(t) = \frac{pe^t}{(1-qe^t)^2}$, $m'(0) = \frac{1}{p} = E[Y]$, $m''(t) = \frac{(1-qe^t)^2 pe^t - 2pe^t(1-qe^t)(-qe^t)}{(1-qe^t)^4}$

$m''(0) = \frac{1+q}{p^2} \Rightarrow V(Y) = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$

3.152

$\lambda = 6 \Rightarrow \mu = 6,\ \sigma = \sqrt{6} \Rightarrow P(|Y-\mu| \le 2\sigma) \simeq P(1.1 \le Y \le 10.9)$

$= P(2 \le Y \le 10) = 0.94$

3.155 (10pt)

a. $m'(t) = \frac{1}{6}e^{t} + \frac{4}{6}e^{2t} + \frac{9}{6}e^{3t}$, $m'(0) = \frac{7}{3} = E[Y]$ (+3)

b. $m''(t) = \frac{1}{6}e^{t} + \frac{8}{6}e^{2t} + \frac{27}{6}e^{3t}$, $m''(0) = 6$, $V(Y) = 6 - \frac{49}{9} = \frac{5}{9}$ (+3)

c. $m(t) = \sum_y e^{ty} P(Y=y) \rightarrow \underbrace{P(Y=0)}_{0} + e^{t}\underbrace{P(Y=1)}_{\frac{1}{6}} + e^{2t}\underbrace{P(Y=2)}_{\frac{2}{6}} + e^{3t}\underbrace{P(Y=3)}_{\frac{3}{6}} + \underbrace{\cdots}_{0}$

$P(Y=y) = \begin{cases} 1/6 & y=1 \\ 2/6 & y=2 \\ 3/6 & y=3 \end{cases}$ (+4)

3.169

a. $E[Y] = -1 \cdot \frac{1}{18} + 0 \cdot \frac{16}{18} + 1 \cdot \frac{1}{18} = 0$. $E[Y^2] = \frac{1}{18} + 0 \cdot \frac{16}{18} + \frac{1}{18} = \frac{1}{9}$

$\Rightarrow V[Y] = \frac{1}{9} \Rightarrow \sigma[Y] = \frac{1}{3}$

b. $P(|Y-0| \geq 1) = P(Y=-1) + P(Y=1) = \frac{1}{9}$

by Tchebysheff's thm, $P(|Y-\mu| \geq 3\sigma) \leq \frac{1}{3^2} = \frac{1}{9}$ → upper bound

3.177 (5pt)

$C = 50 + 3Y$, $E[C] = E[50+3Y] = 50 + 3E[Y] = 80$ (+1), $V[C] = 9V[Y] = 90$ (+1)

$\Rightarrow \sigma[C] = \sqrt{90}$. By Tchebysheff thm, $P(|C-80| < 2\sqrt{90}) \geq 0.75$,

$\Rightarrow C \in (80 - 2\sqrt{90}, 80 + 2\sqrt{90})$ or $(61.03, 98.97)$ (+3)

3.200

$(q+pe^{t})^n = \left[q + p(1 + t + \frac{t^2}{2!} + \cdots)\right]^n = \left[1 + pt + p\frac{t^2}{2!} + p\frac{t^3}{3!} + \cdots\right]^n$

$\xrightarrow{\text{expanding multinomial}}$ $\underbrace{1^n + (np)t \cdot 1^{n-1} + [n(n-1)p^2 + np]\frac{t^2}{2!} \cdot 1^{n-2}}$ $+ \cdots$

1st, 2nd moments.

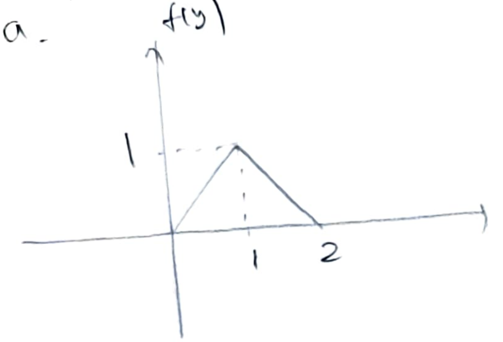
3.211

- stocks 2 item $\rightarrow$ $P(Y \geq 2) \cdot 2.4 - 2 = 0.4$
- " 3 " $\rightarrow$ $P(Y=2) \cdot 2.4 + P(Y \geq 3) \cdot 3.6 - 3 = 0.48$
- " 4 " $\rightarrow$ $P(Y=2) \cdot 2.4 + P(Y=3) \cdot 3.6 + P(Y=4) \cdot 0.48 - 4$

$= 0.08$

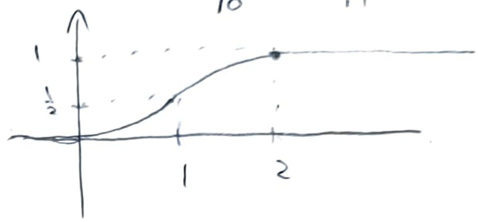
4.14

a.



b. $0 < y < 1$, $F(y) = \int_0^y t\,dt = \frac{y^2}{2}$

$1 \leq y < 2$, $F(y) = \int_0^1 t\,dt + \int_1^y (2-t)\,dt = 2y - \frac{y^2}{2} - 1$



c.

$F(1.2) - F(0.8) = 0.36$

d.

$$P(Y > 1.5 \mid Y > 1) = \frac{P(Y > 1.5)}{P(Y > 1)} = \frac{0.125}{0.5} = 0.25$$

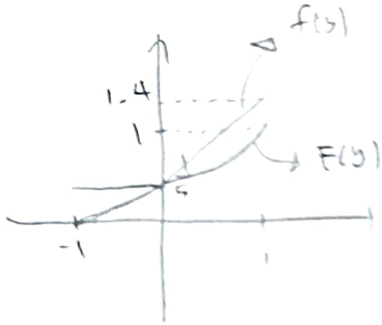
4.18

a. $\int_{-1}^{0} 0.2\,dy + \int_0^1 (0.2 + cy)\,dy = 0.4 + c/2 = 1$, $c = 1.2$

b.

$$F(y) = \begin{cases} 0 & y \leq -1 \\ \frac{1}{5}(1+y) & -1 < y \leq 0 \\ \frac{1}{5}(1+y+3y^2) & 0 < y \leq 1 \\ 1 & y > 1 \end{cases}$$

c.



d. 0, 0.2, 1 , e. $F(0.5) - F(0) = 0.45 - 0.2 = 0.25$

f. $P(Y > 0.5 \mid Y > 0.1) = \frac{P(Y > 0.5)}{P(Y > 0.1)} = \frac{1 - F(0.5)}{1 - F(0.1)} = \frac{0.55}{0.774} \approx 0.71$

4.34

$$E[Y] = \int_0^\infty y f(y)dy = \int_0^\infty \int_0^y dt\, f(y)dy = \int_0^\infty \int_t^\infty f(y)dy\,dt = \int_0^\infty 1-F(t)\,dt$$

$0<t<y<\infty$

$$= \int_0^\infty 1-F(y)\,dy$$

4.35 (5pt)

$$E[(Y-a)^2] = E[\{(Y-\mu)+(\mu-a)\}^2] = E[(Y-\mu)^2] - 2E[(Y-\mu)(\mu-a)] + (\mu-a)^2$$

$$= \sigma^2 + (\mu-a)^2 \text{ where } \mu = E[Y] \quad (+3)$$

Since $E[(Y-\mu)(\mu-a)] = (\mu-a)\{E[Y]-\mu\} = 0$ $\quad\therefore E[Y]=a$ minimizes.

(+2)

4.51

Y: cycle time, $Y \sim U(50, 70)$

$$P(Y>65 \mid Y>55) = \frac{P(Y>65)}{P(Y>55)} = \frac{1}{3}$$

4.75

Y: volume filled, $Y \sim N(\mu, 0.3^2)$. $P(Y>8) = 0.01$

$P(Z>2.33) = 0.01$, $\therefore 2.33 = \dfrac{8-\mu}{0.3}$, $\mu = 7.301$

4.78

$\ln f(y) = -\ln(\sigma\sqrt{2\pi}) - \dfrac{(y-\mu)^2}{2\sigma^2}$, $\dfrac{d}{dy}\ln f(y) = 0$ when $y=\mu$. $\left(\dfrac{d^2}{dy^2}\ln f(y) < 0\right)$

$y=k$ maximizes $f(y)$ $\Leftrightarrow$ $y=k$ maximizes $\ln f(y)$ since $\ln$: increase

$\Rightarrow f(\mu) = 1/(\sigma\sqrt{2\pi})$

4.80 (5pt)

$A = L \cdot W = 3Y^2$, $E[A] = 3E[Y^2] = 3(E[Y]^2 + \sigma(Y)^2) = 3(\mu^2+\sigma^2)$

+1 +3 +1