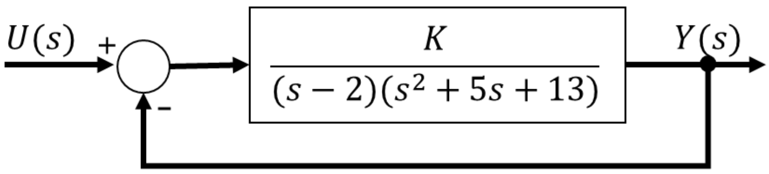
EE381 Control System Engineering

Final Exam Solution (June 17, 2021)

Total score (100)

Student ID: ____________________ Name: ________________________

1. (**20 points**) For the control system shown in the following figure:



(a) Sketch the root loci of the system using the following procedure. (**10 points**)

① Find all the open-loop zeros and poles.

② Determine the segments of the real axis on the root locus.

③ Find the centroid and angles of asymptotes for the diverging branches.

④ Determine the points at which the locus crosses the imaginary axis.

⑤ Find the breakaway point on the real axis (if any).

(b) Justify the shape of root loci. (**5 points**):

(Hint: You can use the relationship $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}$.)

(c) Determine the range of gain K for stability. (**5 points**)

(Ans)

(a)

① Poles: $s = 2,\ s = -2.5 \pm j\sqrt{6.75} = -2.5 \pm j2.6,$ Zeros: none

② On the real axis between $s = 2$ and $s = -\infty$

③ Centroid = $\frac{2-2.5-2.5}{3} = -1,$

Angle of asymptotes = $\pm 180° \frac{2k+1}{3} = 60°, -60°, 180°$

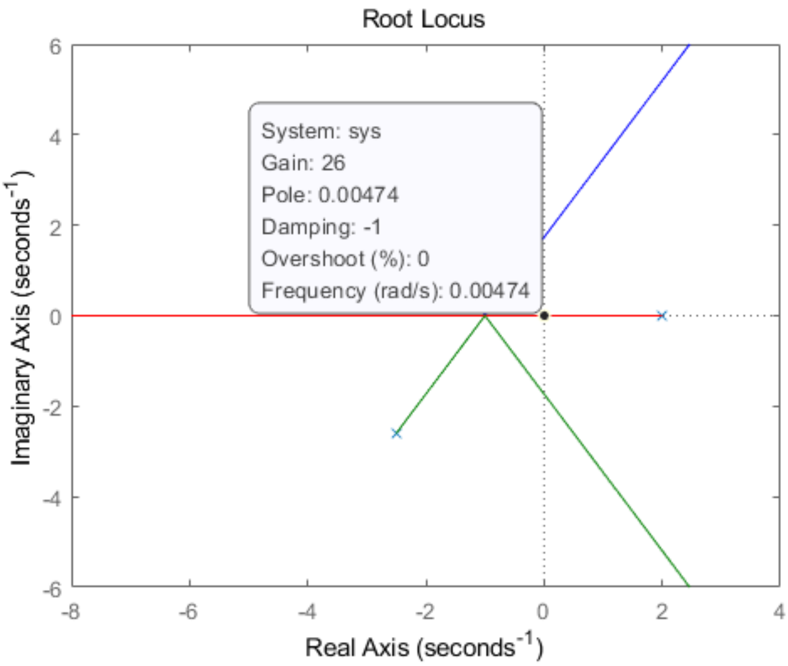
④ $s^3 + 3s^2 + 3s - 26 + K = 0 \rightarrow K - 3\omega^2 - 26 + j(-\omega^3 + 3\omega) = 0$

$$\omega = 0 \text{ or } \pm\sqrt{3} \rightarrow K = 26 \text{ or } 35$$

Crosses imaginary axis at $\pm\sqrt{3}j$

⑤ Breakaway point: $K = -s^3 - 3s^2 - 3s + 26,$

$$\frac{dK}{ds} = -3s^2 - 6s - 3 = -3(s+1)^2 = 0 \rightarrow s = -1$$



(b) By checking the angle constraints

The fact that root-locus branches consist of straight lines can be verified as follows:

Since the angle condition is

$$\angle \frac{K}{(s-2)(s+2.5+j\sqrt{6.75})(s+2.5-j\sqrt{6.75})} = \pm 180^\circ\ (2k\ +\ 1)$$

we have

$$-\angle s-2 - \angle s+2.5+j\sqrt{6.75} - \angle s+2.5-j\sqrt{6.75} = \pm 180^\circ(2k+1)$$

By substituting $s = \sigma + j\omega$ into this last equation,

$$\angle \sigma-2+j\omega + \angle \sigma+2.5+j\omega+j\sqrt{6.75} + \angle \sigma+2.5+j\omega-j\sqrt{6.75} = \pm 180^\circ(2k+1)$$

or

$$\angle \sigma+2.5+j(\omega+\sqrt{6.75}) + \angle \sigma+2.5+j(\omega-\sqrt{6.75}) = -\angle \sigma-2+j\omega \pm 180^\circ(2k+1)$$

which can be rewritten as

$$\tan^{-1}\left(\frac{\omega+\sqrt{6.75}}{\sigma+2.5}\right) + \tan^{-1}\left(\frac{\omega-\sqrt{6.75}}{\sigma+2.5}\right) = -\tan^{-1}\left(\frac{\omega}{\sigma-2}\right) \pm 180^\circ(2k+1)$$

Taking tangents of both sides of last equation, we obtain

$$\frac{\dfrac{\omega+\sqrt{6.75}}{\sigma+2.5}+\dfrac{\omega-\sqrt{6.75}}{\sigma+2.5}}{1-\left(\dfrac{\omega+\sqrt{6.75}}{\sigma+2.5}\right)\left(\dfrac{\omega-\sqrt{6.75}}{\sigma+2.5}\right)}=-\frac{\omega}{\sigma-2}$$

or

$$\frac{2\omega(\sigma+2.5)}{\sigma^2+5\sigma+6.25-\omega^2+6.75}=-\frac{\omega}{\sigma-2}$$

which can be simplified to

$$2\omega(\sigma+2.5)(\sigma-2)=-\omega(\sigma^2+5\sigma+13-\omega^2)$$

or

$$\omega(3\sigma^2+6\sigma+3-\omega^2)=0$$

Further simplification of this last equation yields

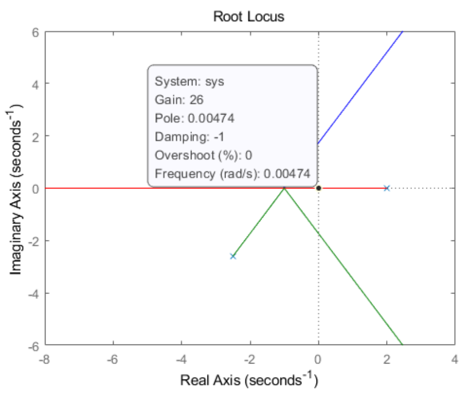
$$\omega\left(\sigma+1+\frac{1}{\sqrt{3}}\omega\right)\left(\sigma+1-\frac{1}{\sqrt{3}}\omega\right)=0$$

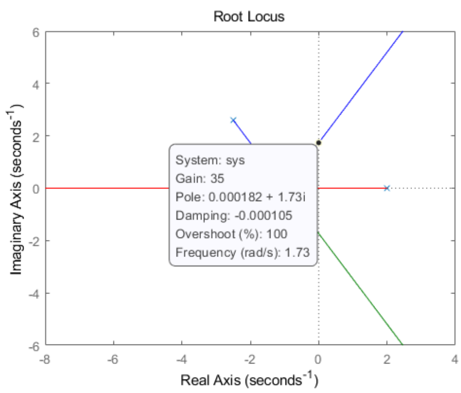
Which defines three lines:

$$\omega=0,\quad \sigma+1+\frac{1}{\sqrt{3}}\omega=0,\quad \sigma+1-\frac{1}{\sqrt{3}}\omega=0$$

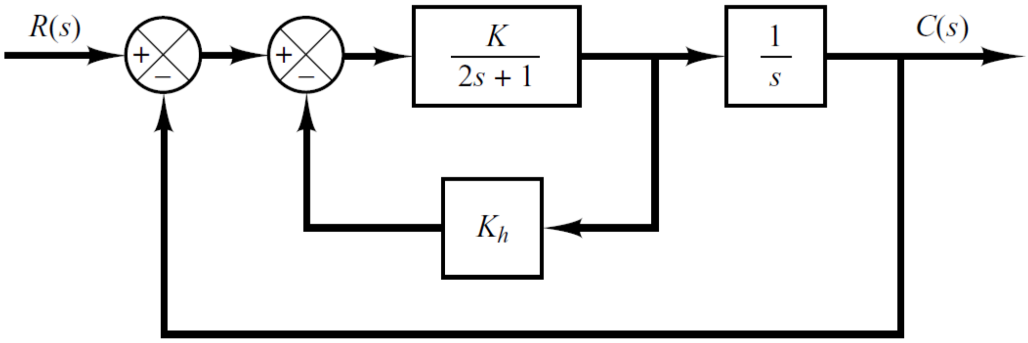
(c) From (a), the root-locus branches cross the imaginary axis at $\omega=\pm\sqrt{3}$ (where $K=35$) and $\omega=0$ (where $K=26$). Since the value of gain K at the origin is 26, the range of gain value K for stability is

$$26 < K < 35$$





2. (**10 points**) Consider the system shown in the following figure, which involves velocity feedback. Determine the values of the minimum gain K and the velocity feedback gain K_h so that the following specifications are satisfied:
   - Damping ratio of the closed-loop poles is 0.5
   - Settling time $\leq$ 1 sec
   - Static velocity error constant $K_v \geq 10 \text{ sec}^{-1}$
   - $0 < K_h < 1$



(Ans)

The closed-loop transfer function for the system is

$$\frac{C(s)}{R(s)} = \frac{K}{2s^2 + s + KK_h s + K} = \frac{\frac{K}{2}}{s^2 + \frac{1 + KK_h}{2}s + \frac{K}{2}}$$

From this equation, we obtain

$$\omega_n = \sqrt{K/2}, \qquad 2\zeta\omega_n = \frac{1 + KK_h}{2}$$

Since the damping ratio is specified as 0.5, we get

$$\omega_n = \frac{1 + KK_h}{2}$$

Therefore, we have

$$\frac{1 + KK_h}{2} = \sqrt{\frac{K}{2}}$$

The settling time is specified as

$$T_s = \frac{4}{\zeta\omega_n} = \frac{16}{(1 + KK_h)} \leq 1$$

Since the feedforward transfer function $G(s)$ is

$$G(s) = \frac{K}{2s + 1 + KK_h}\frac{1}{s}$$

the static velocity error constant K_v is

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s\frac{K}{2s + 1 + KK_h}\frac{1}{s} = \frac{K}{1 + KK_h}$$

This value must be equal to or greater than 10. Hence,

$$\frac{K}{1 + KK_h} \geq 10$$

Thus, the conditions to be satisfied can be summarized as follows:

$$\frac{1+KK_h}{2} = \sqrt{\frac{K}{2}} \tag{1}$$

$$\frac{16}{(1+KK_h)} \leq 1 \tag{2}$$

$$\frac{K}{1+KK_h} \geq 10 \tag{3}$$

From equation (1) and (2), we get

$$16 \leq 1+KK_h = \sqrt{2K}, \quad \text{or} \quad 128 < K$$

From equation (3), we obtain

$$\frac{K}{10} \geq 1+KK_h = \sqrt{2K}, \quad \text{or} \quad K \geq 200$$

If we choose $K = 200$, then we get

$$1+KK_h = \sqrt{2K} = 20$$

or

$$K_h = 19/200 = 0.095$$

Thus, we determined a set of values of K and K_h as follows:

$$K = 200, \qquad K_h = 0.095$$

With these values, all specifications are satisfied.

3. (**10 points**) (a) Draw a Nyquist plot for the unity-feedback control system with the open-loop transfer function. Using the Nyquist stability criterion, determine the range of K for stability of the closed-loop system. (**5 points**)

$$G(s) = \frac{K(1-s)}{1+s}$$

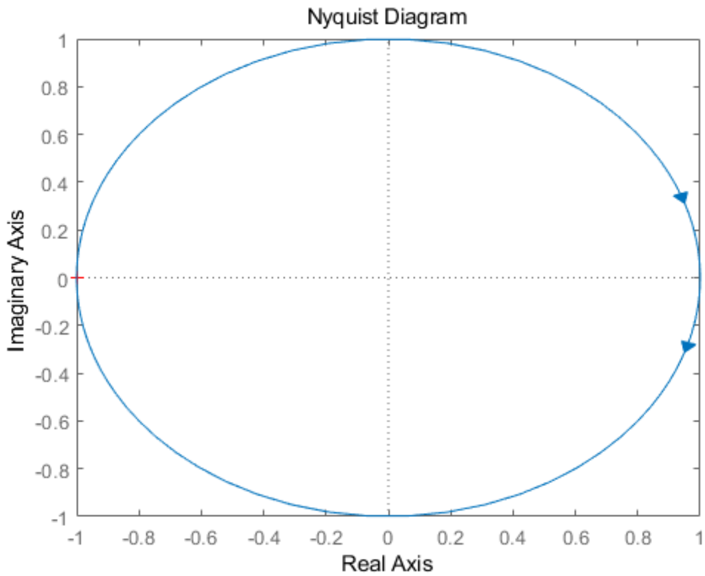
(b) Consider the unity-feedback system whose open-loop transfer function is

$$G(s) = \frac{Ke^{-2s}}{s}.$$

Find the maximum value of K for which the system is stable. (**5 points**)

(Ans)

(a) For $K = 1$, the Nyquist plot is as follows:



The stability requirement of the unity feedback control system with

$$G(j\omega) = \frac{K(1-j\omega)}{(1+j\omega)}$$

is that $-K$ be greater than -1, or

$$K < 1$$

Since we assume that $K > 0$, the condition for stability is

$$0 < K < 1.$$

(b)

$$G(j\omega) = \frac{Ke^{-2j\omega}}{j\omega}$$

$$\angle G(j\omega) = \angle(\cos 2\omega - j\sin 2\omega) - \frac{\pi}{2}$$

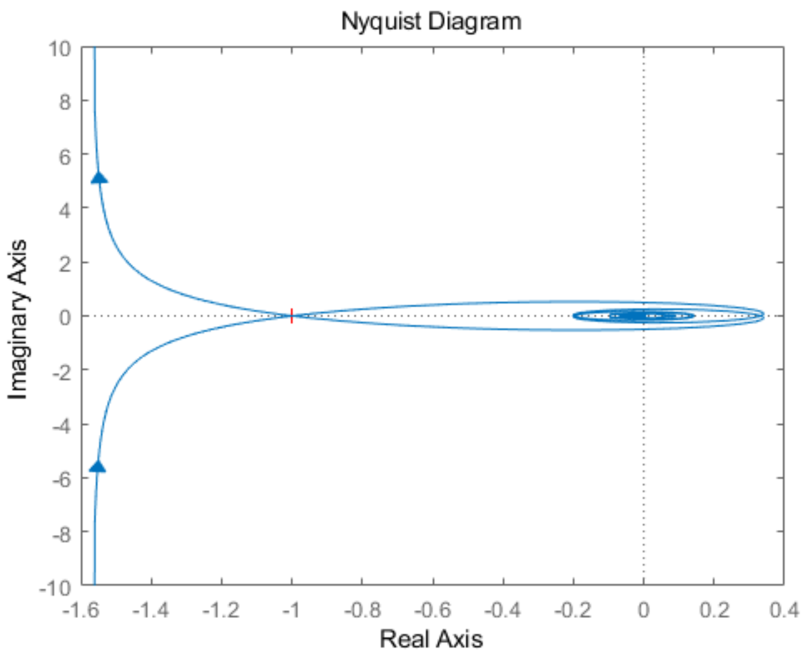
$$= -2\omega - \frac{\pi}{2}$$

The phase angle became equal to $-180°$ at $2\omega = \pi/2$ rad/sec. For stability, the magnitude $|G(j\omega)|$ at $\omega = \pi/4$ must be less than unity. Hence,

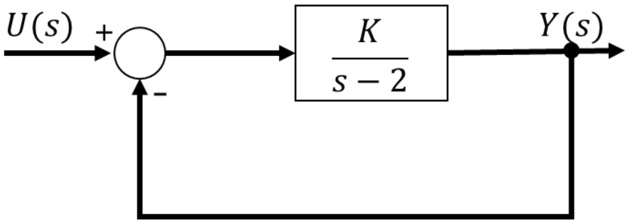
$$|G(j\omega)| = \frac{K}{\omega}$$

We require that $K < \pi/4$ for stability.

The Nyquist plot for $K = \pi/4$ is shown as follows:



4. (**10 points**) Consider the closed-loop system shown below. Determine the critical value of K for stability by using the Nyquist stability criterion.

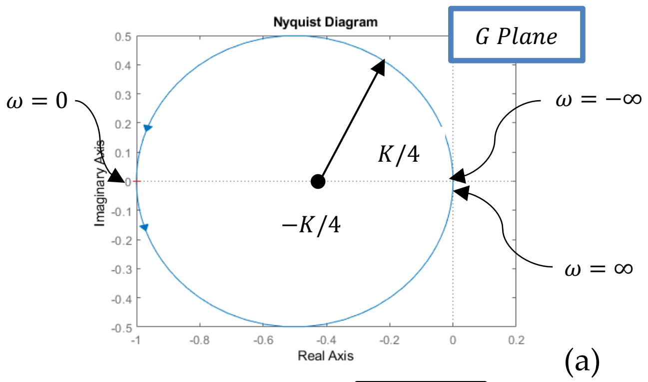


(Ans)

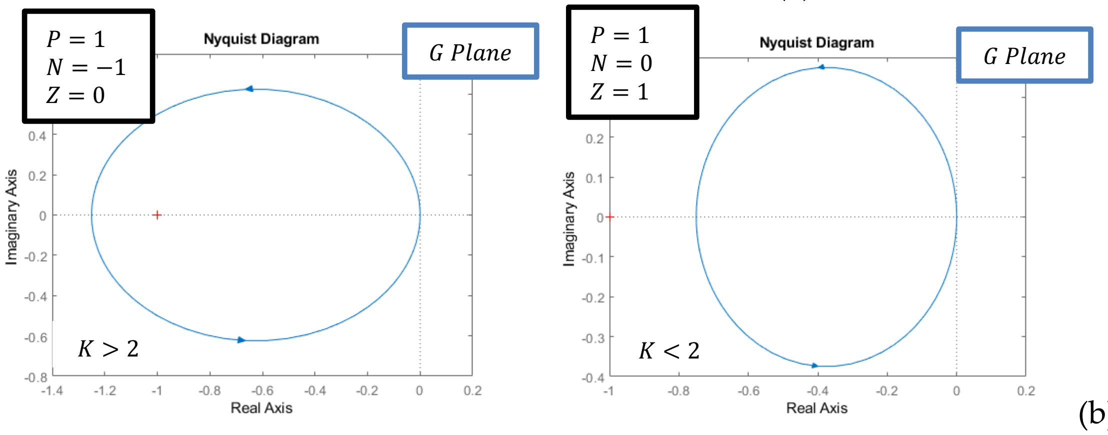
The polar plot of

$$G(j\omega) = \frac{K}{j\omega - 2}$$

is a circle with center at $-K/4$ on the negative real axis and radius $K/4$, as shown in Figure (a). As ω is increased from $-\infty$ to ∞, the $G(j\omega)$ locus makes a counterclockwise rotation. In this system, $P = 1$ because there is one pole of $G(s)$ in the right-half s plane. For the closed-loop system to be stable, Z must be equal to zero. Therefore, $N = Z - P$ must be equal to -1, or there must be one counterclockwise encirclement of the $-1 + j0$ point for stability. (If there is no encirclement of the $-1 + j0$ point, the system is unstable.) Thus, for stability, K must be greater than 2, and $\boldsymbol{K = 2}$ gives the stability limit. Figure (b) shows both stable and unstable cases of $G(j\omega)$ plots.



(a)

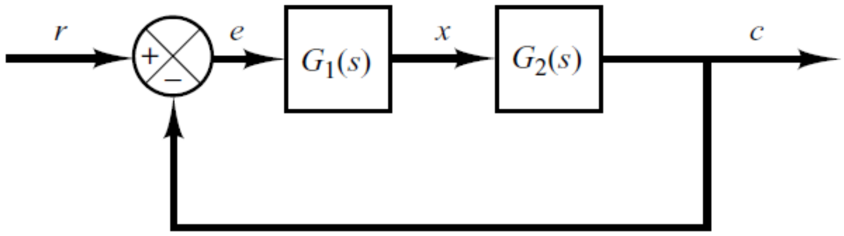


(b)

5. (**10 points**) A closed-loop control system may include an unstable element within the loop. When the Nyquist stability criterion is to be applied to such a system, the frequency-response curves for the unstable element should be obtained. How can we obtain the frequency-response curves experimentally for such an unstable element? Suggest a possible approach to the experimental determination of the frequency response of an unstable linear element.

(Ans)

One possible approach is to measure the frequency-response characteristics of the unstable element by using it as a part of a stable system.



Consider the system shown above. Suppose that the element $G_1(s)$ is unstable. The complete system may be made stable by choosing a suitable linear element $G_2(s)$. We apply a sinusoidal signal at the input. At steady state, all signals in the loop will be sinusoidal. We measure the signals $e(t)$, the input to the unstable element, and $x(t)$, the output of the unstable element. By changing the frequency [and possibly the amplitude for the convenience of measuring $e(t)$ and $x(t)$] of the input sinusoid and repeating this process, it is possible to obtain the frequency response of the unstable linear element.

6. (**10 points**) Consider a lead-lag compensator $G_c(s)$ defined by

$$G_c(s) = \frac{K_c\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\beta}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)}.$$

Calculate the frequency ω_1 where the phase angle of $G_c(j\omega)$ becomes zero. This compensator acts as a lag compensator for $0 < \omega < \omega_1$ and acts as a lead compensator for $\omega_1 < \omega < \infty$.

(Hint: You can use the relationship $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}$.)

(Ans)

The angle of $G_c(j\omega)$ is given by

$$\angle G_c(j\omega) = \angle(j\omega + \frac{1}{T_1}) + \angle(j\omega + \frac{1}{T_2}) - \angle(j\omega + \frac{\beta}{T_1}) - \angle(j\omega + \frac{1}{\beta T_2})$$
$$= \tan^{-1}\omega T_1 + \tan^{-1}\omega T_2 - \tan^{-1}\frac{\omega T_1}{\beta} - \tan^{-1}\omega T_2\beta$$

With the relationship $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}$, we can get

$$\tan^{-1}a + \tan^{-1}b = \tan^{-1}\frac{a+b}{1-ab}$$

Thus, the angle of $G_c(j\omega)$ becomes

$$\angle G_c(j\omega) = \tan^{-1}\frac{\omega T_1 + \omega T_2}{1 - \omega^2 T_1 T_2} - \tan^{-1}\frac{\frac{\omega T_1}{\beta} + \omega T_2 \beta}{1 - \omega^2 T_1 T_2}$$

We can calculate the frequency ω_1 by setting $\angle G_c(j\omega_1) = 0$.

$$\angle G_c(j\omega_1) = \tan^{-1}\frac{\omega_1 T_1 + \omega_1 T_2}{1 - {\omega_1}^2 T_1 T_2} - \tan^{-1}\frac{\frac{\omega_1 T_1}{\beta} + \omega_1 T_2 \beta}{1 - {\omega_1}^2 T_1 T_2} = 0$$

It becomes

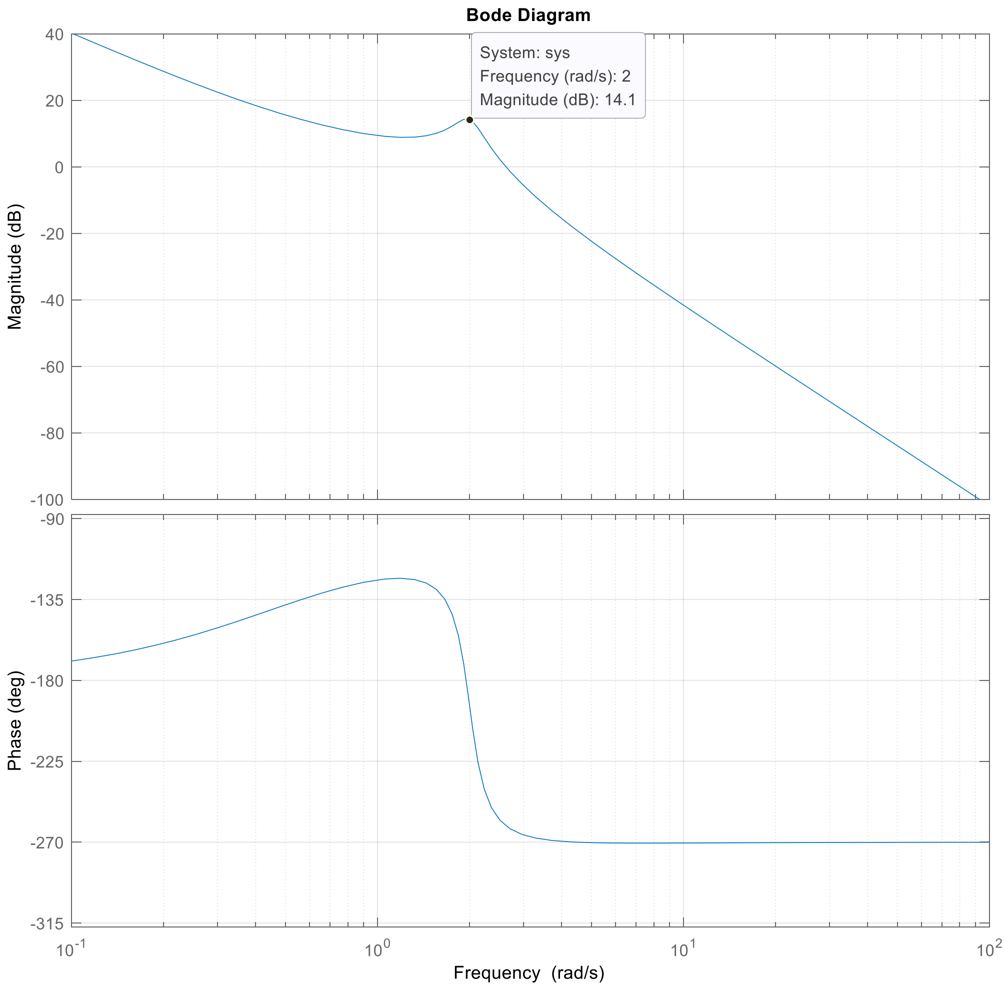
$$\tan^{-1}\frac{\omega_1 T_1 + \omega_1 T_2}{1 - {\omega_1}^2 T_1 T_2} = \tan^{-1}\frac{\frac{\omega_1 T_1}{\beta} + \omega_1 T_2 \beta}{1 - {\omega_1}^2 T_1 T_2}$$
$$\frac{\omega_1 T_1 + \omega_1 T_2}{1 - {\omega_1}^2 T_1 T_2} = \frac{\frac{\omega_1 T_1}{\beta} + \omega_1 T_2 \beta}{1 - {\omega_1}^2 T_1 T_2}$$
$$\frac{T_1 + T_2}{1 - {\omega_1}^2 T_1 T_2} = \frac{\frac{T_1}{\beta} + T_2 \beta}{1 - {\omega_1}^2 T_1 T_2}$$

If $1 - {\omega_1}^2 T_1 T_2 \neq 0$, there is no solution for ω_1.

If $1 - {\omega_1}^2 T_1 T_2 = 0$, $\tan^{-1}\frac{\omega_1 T_1 + \omega_1 T_2}{1 - {\omega_1}^2 T_1 T_2} = \tan^{-1}\frac{\frac{\omega_1 T_1}{\beta} + \omega_1 T_2 \beta}{1 - {\omega_1}^2 T_1 T_2} = \tan^{-1}(\infty) = 90°$

Therefore, if we set ω_1 as $\frac{1}{\sqrt{T_1 T_2}}$, the angle of $G_c(j\omega_1)$ becomes 0.

7. (**10 points**) A Bode plot of the open-loop transfer function $G(s)$ of a unity-feedback control system is shown in the figure below. It is known that the open-loop transfer function is minimum phase. From the plot, it can be seen that there is a pair of complex-conjugate poles at $\omega = 2$ rad/sec.
   (a) Determine the damping ratio of the quadratic term involving these complex-conjugate poles. (**3 points**)
   (b) Determine the open-loop transfer function $G(s)$. (**7 points**)



(Ans)

(a) The maximum value $M_{p\omega}$ of the frequency response is $(2\zeta\sqrt{1-\zeta^2})^{-1}$. The maximum value point has 14.1 of magnitude,

$$14.1 = 20\log_{10}(2\zeta\sqrt{1-\zeta^2})^{-1}$$

Thus, we can get

$$\zeta = 0.1$$

Or this can be obtained from Fig. 8.10.

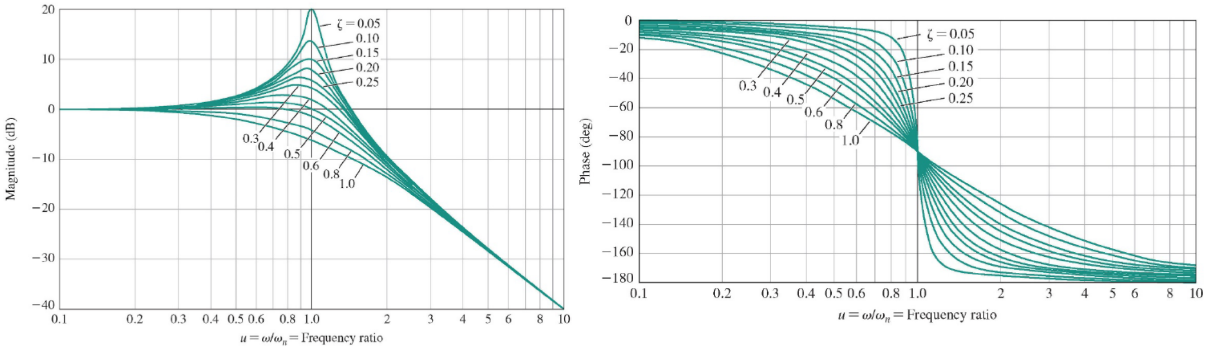


FIGURE 8.10 Bode diagram for $G(j\omega) = [1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]^{-1}$.

(b) We can find that there is a corner frequency at $\omega = 0.5$ rad/sec and the slope of the magnitude curve in the low-frequency region is -40 dB/decade.
Thus, $G(j\omega)$ can be determined as follows:

$$G(j\omega) = \frac{K(\frac{j\omega}{0.5} + 1)}{(j\omega)^2[1 + 0.1(j\omega) + (\frac{j\omega}{2})^2]}$$

Also, we can find $|G(j(0.1))| = 40\text{dB}$, the gain K can be determined to be 1.
Hence, the transfer function $G(s)$ can be determined to be

$$G(s) = \frac{\frac{s}{0.5} + 1}{s^2[1 + 0.1(s) + (\frac{s}{2})^2]} = \frac{4(2s + 1)}{s^2(4 + 0.4s + s^2)}$$

8. (**10 points**) A state-space representation of a system in the controllable canonical form is given by

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -0.5 & -1.5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u \tag{8.1}$$

$$y = \begin{bmatrix}0.5 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{8.2}$$

The same system may be represented by the following state-space equation, which is in the observable canonical form:

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & -0.5\\ 1 & -1.5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0.5\\ 1\end{bmatrix}u \tag{8.3}$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{8.4}$$

(a) Show that the state-space representation given by Equations (8.1) and (8.2) gives a system that is state controllable, but not observable. Show, on the other hand, that the state-space representation defined by Equations (8.3) and (8.4) gives a system that is not completely state controllable, but is observable. (**6 points**)

(b) Explain what causes the apparent difference in the controllability and observability of the same system. (**4 points**)

(Ans)

(a) Consider the system defined by Equation (8.1) and (8.2). The rank of the controllability matrix

$$[\,\mathbf{B} \mid \mathbf{AB}\,] = \begin{bmatrix}0 & 1\\ 1 & -1.5\end{bmatrix}$$

is 2. Hence, the system is completely state controllable. The rank of the observability matrix

$$[\,\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*\,] = \begin{bmatrix}0.5 & 1\\ -0.5 & -1\end{bmatrix}$$

is 1. Hence the system is not observable.

Next consider the system defined by Equation (8.3) and (8.4). The rank of the controllability matrix

$$[\,\mathbf{B} \mid \mathbf{AB}\,] = \begin{bmatrix}0.5 & -0.5\\ 1 & -1\end{bmatrix}$$

is 1. Hence, the system is not completely state controllable. The rank of the observability matrix

$$[\,\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*\,] = \begin{bmatrix}0 & 1\\ 1 & -1.5\end{bmatrix}$$

is 2. Hence, the system is observable.

(b) The apparent difference in the controllability and observability of the same system is caused by the fact that the original system has a pole-zero cancellation in the transfer function. The transfer function can be obtained as follows:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

If we use Equations (8.1) and (8.2), then

$$\begin{aligned} G(s) &= \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0.5 & s+1.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{s+0.5}{s^2+1.5s+0.5} \\ &= \frac{s+0.5}{(s+1)(s+0.5)} \end{aligned}$$

If we use Equations (8.3) and (8.4), then

$$\begin{aligned} G(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0.5 \\ -1 & s+1.5 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \\ &= \frac{s+0.5}{s^2+1.5s+0.5} \\ &= \frac{s+0.5}{(s+1)(s+0.5)} \end{aligned}$$

The same transfer functions can be obtained in both cases. Clearly, cancellation occurs in this transfer function.

9. (**10 points**) Consider the control system described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Assuming the state feedback control law $u(t) = -\mathbf{K}\mathbf{x}(t)$ where $\mathbf{K} = [k_1 \quad k_2]$, determine the constants k_1 and k_2 so that the following performance index is minimized:

$$J = \int_0^\infty \mathbf{x}^T(t)\mathbf{x}(t)dt.$$

Consider only the case where the initial condition is

$$x(0) = \begin{bmatrix} c \\ c \end{bmatrix}$$

where c is a constant. Choose the natural frequency to be 3 rad/s.

(Ans)
Substituting $u(t) = -\mathbf{K}\mathbf{x}(t)$ into $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$, we obtain

$$\dot{\mathrm{x}} = \mathbf{A}\mathrm{x} - \mathbf{BK}\mathrm{x}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [-k_1 x_1 \quad -k_2 x_2]$$
$$= \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus,

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

Elimination of x_2 from upper equation yields

$$\ddot{x}_1 + k_2\dot{x}_1 + k_1 x_1 = 0$$

Since the undamped natural frequency is specified as 3 rad/s, we obtain

$$k_1 = \omega_n^2 = 9$$

Therefore,

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -9 & -k_2 \end{bmatrix}$$

$\mathbf{A}-\mathbf{BK}$ is a stable matrix if $k_2 > 0$. Our problem now is to determine the value of k_2 so that the performance index

$$J = \int_0^\infty \mathbf{x}^T\mathbf{x}dt = \mathbf{x}^T(0)\mathbf{P}(0)\mathbf{x}(0)$$

is minimized, where the matrix $\mathbf{P}$ is determined from following equation

$$(\mathbf{A}-\mathbf{BK})^*\mathbf{P} + \mathbf{P}(\mathbf{A}-\mathbf{BK}) = -(\mathbf{Q} + \mathbf{K}^*\mathbf{RK})$$

Since in this system $\mathbf{Q} = \mathbf{I}$ and $\mathbf{R} = \mathbf{0}$, this last equation can be simplified to

$$(\mathbf{A}-\mathbf{BK})^*\mathbf{P} + \mathbf{P}(\mathbf{A}-\mathbf{BK}) = -\mathbf{I}$$

Since the system involves only real vectors and real matrices, $\mathbf{P}$ becomes a real symmetric matrix. Then

$$(\mathbf{A}-\mathbf{BK})^*\mathbf{P} + \mathbf{P}(\mathbf{A}-\mathbf{BK}) = -\mathbf{I}$$

can be written as

$$\begin{bmatrix} 0 & -9 \\ 1 & -k_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -9 & -k_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving for the matrix $\mathbf{P}$, we obtain

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{k_2^2 + 90}{18k_2} & \dfrac{1}{18} \\ \dfrac{1}{18} & \dfrac{5}{9k_2} \end{bmatrix}$$

The performance index is then

$$\begin{aligned} J &= \mathbf{x}^T(0)\mathbf{P}(0)\mathbf{x}(0) \\ &= \begin{bmatrix} c & c \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} c \\ c \end{bmatrix} = (p_{11} + p_{12})c^2 + (p_{12} + p_{22})c^2 \\ &= \frac{c^2(k_2^2 + 2k_2 + 100)}{18k_2} \end{aligned}$$

To minimize J, we differentiate J with respect to k_2 and set $\partial J/\partial k_2$ equal to zero as follows:

$$\frac{\partial J}{\partial k_2} = \left(\frac{1}{9} - \frac{5}{9k_2^2} - \frac{k_2^2 + 90}{18k_2^2}\right)c^2 = 0$$

Hence,

$$k_2 = 10$$

With this value of k_2, we have $\partial^2 J/\partial k_2^2 > 0$. Thus, the minimum value of J is obtained by substituting $k_2 = 10$ into J, or

$$J_{\min} = \frac{11c^2}{9}$$

The designed system has the control law

$$u = -9x_1 - 10x_2$$

The designed system is optimal in that it results in a minimum value for the performance index J under the assumed initial condition.