SS 202B Signals & Systems (2016)	Final Exam Solution (Page 1/7)	Name: Student ID No.:

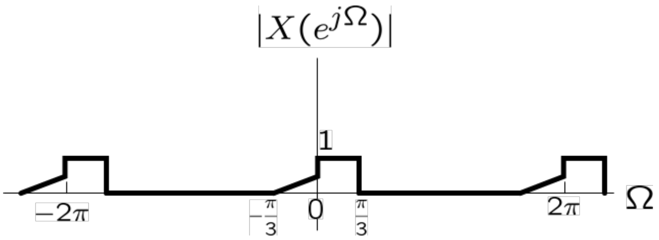
[Score table]

Prob.	1			2			3		4		5				Extra	Total
	5	5	10	5	5	5	20	15	5	5	5	5	5	5	P/F	100
Score																

Problems & Solutions

[Problem 1 – Fourier transform | 20 points]

Magnitude of Fourier transform $X(e^{j\Omega})$ of a discrete sequence $x[n]$ is shown below:

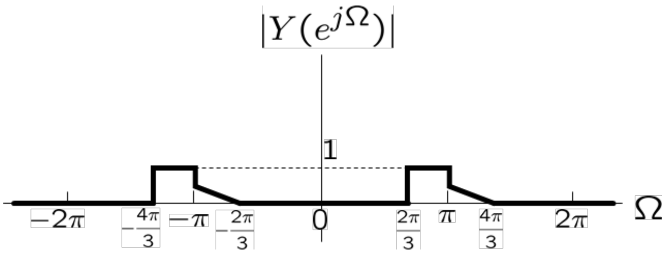


$(\measuredangle X(e^{j\Omega}) = 0)$

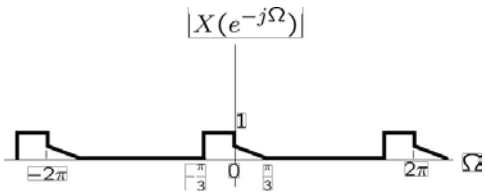
(a) **[5 pts]** Plot the (magnitude of) Fourier transform of $x[-n]$.

(b) **[5 pts]** Plot the Fourier transform of $\mathrm{Re}\{x[n]\}$.

(c) **[10 pts]** Express a sequence $y[n]$ in terms of $x[n]$, where its Fourier transform $Y(e^{j\Omega})$ is given by the following figure. $(\measuredangle Y(e^{j\Omega}) = 0)$



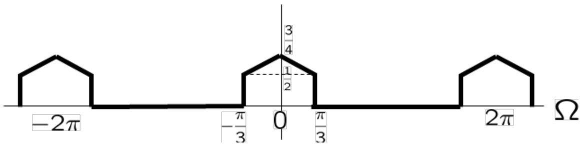
(a) Property of CTFT: $x[-n] \overset{CTFT}{\Leftrightarrow} X(e^{-j\Omega})$: Horizontal flip of $X(e^{j\Omega})$



(b) $\mathrm{Re}\{x[n]\} = (x[n] + x[n]^*)/2$, $x[n]^* \overset{DTFT}{\Leftrightarrow} X(e^{-j\Omega})^*$

$$\therefore \mathrm{Re}\{x[n]\} \overset{DTFT}{\Leftrightarrow} \frac{X(e^{j\Omega}) + X(e^{-j\Omega})^*}{2}$$

Because $\measuredangle X(e^{j\Omega}) = 0$, the magnitude spectrum is simply the addition of original and horizontally flipped spectrum.



(c) $Y(e^{j\Omega})$ is horizontal flip + frequency shift by π of $X(e^{j\Omega})$

$\therefore\ y[n] = x[-n]e^{j\pi n} = (-1)^n x[-n]$

[Problem 2 - Sampling | 15 points]

(a) **[5 pts]** Find the Fourier transform of signal $\mathrm{rect}_T(t)$ given by

$$\mathrm{rect}_T(t) = \begin{cases} 1 & -T \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

(a) **[5 pts]** Suppose that $X(j\omega)$ is the Fourier transform of $x(t) = \sum_{n=0}^{7} a_n \delta(t-2n)$, where a_n are 8 digits of <u>your student ID number</u>. For example, $a_n = (2,0,1,5,1,0,0,4)$ for student ID 20151004.

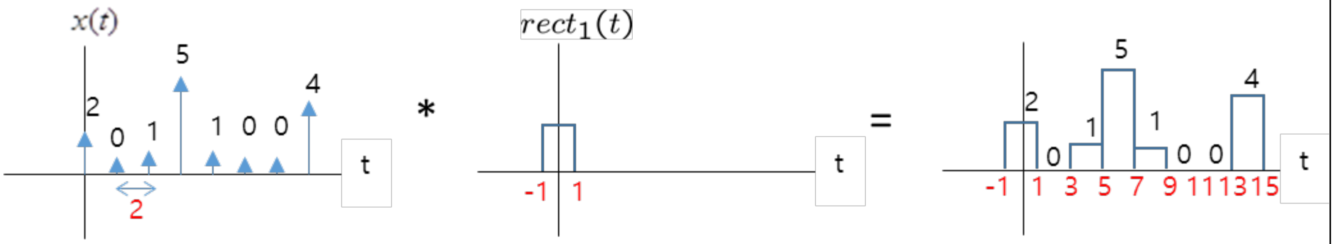
Derive the inverse Fourier transform $y(t)$ of $Y(j\omega) = X(j\omega)\dfrac{2\sin(\omega)}{\omega}$ and plot your result.

(b) **[5 pts]** For $z(t) = x(t) * \dfrac{\sin(t)}{\pi t}$ ($*$: convolution), <u>find Nyquist rate</u> ω_s in (rad/sec) for sampling $z(t)$ without aliasing.

(a) $\mathrm{Rect}_T(j\omega) = \int_{-T}^{T} 1 \cdot e^{-j\omega t} dt = \dfrac{e^{j\omega T} - e^{-j\omega T}}{j\omega} = 2\dfrac{\sin \omega T}{\omega}$ or $2T\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)$

(b) $2\dfrac{\sin\omega}{\omega} \overset{DTFT}{\Leftrightarrow} \mathrm{rect}_1(t)$ --- (from (a), T=1)

$Y(j\omega) = X(j\omega)\dfrac{2\sin(\omega)}{\omega} \overset{DTFT}{\Leftrightarrow} x(t) * \mathrm{rect}_1(t)$



(c) $Z(j\omega) = X(j\omega)\mathcal{F}\left\{\dfrac{\sin(t)}{\pi t}\right\}$ (convolution-multiplication property).

Since the signal $x(t)$ is sampled by $T_s = 2$, its Fourier transform is periodic and not band-limited. So the Nyquist rate is determined by $\mathcal{F}\left\{\dfrac{\sin(t)}{\pi t}\right\} = \begin{cases} 1 & for\ \omega < 1 \\ 0 & otherwise \end{cases} \rightarrow \omega_M = 1$

$\therefore\ \omega_s = 2\omega_M = 2 \text{ rad/sec}$

[Problem 3 – Characterization | 35 points]

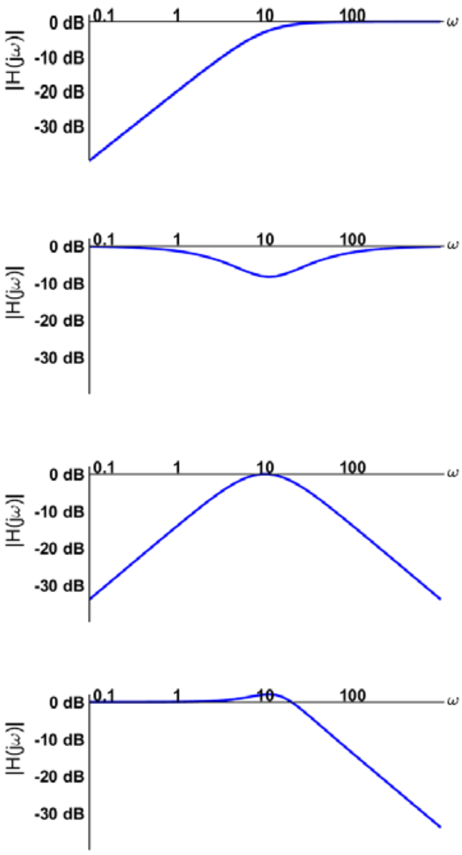
(a) **[20 pts; 5 pts per each correct connection]** For each system described in the left side of the following problem, find and connect to its corresponding bode plot on the right pane. Assume that all systems are LTI and causal, and satisfy the condition of initial rest. ($y(t)$: output, $x(t)$: input, $h(t)$: impulse response, $H(j\omega)$: frequency response)

$$\frac{d^2y(t)}{dt^2} + 20\frac{dy(t)}{dt} + 100y(t) = 20\frac{dx(t)}{dt}$$

$$h(t) = 20e^{-10t}\cos(10t)u(t)$$

$$H(j\omega) = \left(\frac{j\omega-(10+5j)}{j\omega+(10+5j)}\right)\cdot\left(\frac{j\omega+(10-5j)}{j\omega-(10-5j)}\right)$$

$$\frac{dy^2(t)}{dt^2} - 20\frac{dy(t)}{dt} + 100y(t) = \frac{d^2x(t)}{dt^2} + 10\frac{dx(t)}{dt}$$



(b) **[15 pts]** For four systems of problem (a), answer to the following questions. (Hint: derive Laplace transform and use pole-zero map)

(1) Find the highest natural frequency ω_n. Ans.) $\omega_n =$ ________

(2) Is there any unstable system(s) among these four systems? If so, write down the system number [A]~[D].

(3) Is there any system(s) with complex valued impulse response(s)? Ans.) Yes / No

Answers:

(a) **[A]-(3), [B]-(4), [C]-(2), [D]-(1)**

[A] $\frac{dy^2(t)}{dt^2}+20\frac{dy(t)}{dt}+100y(t)=20\frac{dx(t)}{dt}$ → $H(j\omega)=\frac{20(j\omega)}{(j\omega)^2+20(j\omega)+100}=20\frac{j\omega}{(j\omega+10)^2}$

For small ω, $H(j\omega)\sim\frac{1}{5}j\omega$, For high ω, $H(j\omega)\sim 20\frac{1}{j\omega}$ → **(3)**

[B]

$$h(t)=20e^{-10t}\cos(10t)u(t)\overset{CTFT}{\Leftrightarrow}10\left\{\frac{1}{(j\omega+10-j10)}+\frac{1}{(j\omega+10+j10)}\right\}$$
$$=20\left\{\frac{j\omega+10}{(j\omega+10-j10)(j\omega+10+j10)}\right\}$$

For small ω, $H(j\omega)\sim 1$, For high ω, $H(j\omega)\sim 20\frac{1}{j\omega}$ → **(4)**

[C]

$$H(j\omega)=\left(\frac{j\omega-(10+5j)}{j\omega+(10+5j)}\right)\cdot\left(\frac{j\omega+(10-5j)}{j\omega-(10-5j)}\right)$$

For small ω, $H(j\omega)\sim 1$, For high ω, $H(j\omega)\sim 1$ → **(2)**

[D] $\frac{dy(t)}{dt}-20\frac{dy(t)}{dt}+100y(t)=\frac{d^2x(t)}{dt^2}+10\frac{dx(t)}{dt}$ → $H(j\omega)=\frac{j\omega(j\omega+10)}{(j\omega)^2-20(j\omega)+100}=\frac{j\omega(j\omega+10)}{(j\omega-10)^2}$

For small ω, $H(j\omega)\sim j\omega/10$, For high ω, $H(j\omega)\sim 1$ → **(1)**

(b)

(1) Consider pole locations of 4 cases. Natural frequency ω_n corresponds to the distance of pole location from the origin.

[A] $p_1,p_2=-10 \to \omega_n=10$

[B] $p_1,p_2=-10\pm 10j \to \omega_n=10\sqrt{2}$

[C] $p_1,p_2=\pm 10+5j \to \omega_n=\sqrt{125}$

[D] $p_1,p_2=10 \to \omega_n=10$

$\therefore$ **highest natural frequency is** $10\sqrt{2}$

(2) Because the given systems are causal, ROCs are on the right-hand side from the rightmost pole.

[A] $\mathrm{Re}\{s\}>-10$, [B] $\mathrm{Re}\{s\}>-10$, [C] $\mathrm{Re}\{s\}>10$, [D] $\mathrm{Re}\{s\}>10$

A system is unstable ROC includes jw axis. $\therefore$ **System [C]&[D] are unstable Ans.: Yes**

(3) if h(t) is real, the pole-zero locations are symmetric for positive and negative frequencies.

$\therefore$ **Poles of system [C] are not conjugate even.**

[Problem 4 – Laplace transform | 10 points]

A causal LTI system with system response $H(s)$ is given as follows:

$$H(s) = \frac{s+1}{s-10},$$

(1) **[5 pts]** Design an additional system $G(s)$ that makes the total system($H(s)G(s)$) be stable. The total system's magnitude response should be the same as $|H(j\omega)|$.

(2) **[5 pts]** Discuss the type of frequency response $G(j\omega)$ and find an impulse response $g(t) = \mathcal{F}^{-1}\{G(j\omega)\}$.

(1) For a causal system of which system function is rational, poles should be on LHP(so that jw-axis is within ROC). The system $H(s)$ is unstable because of a pole at $s = 10$. In order to cancel the pole without changing $|H(s)|$, an all-pass system of $|G(s)|$=1 is required. Such $G(s)$ is given by

$$G(s) = \frac{s-10}{s+10}.$$

When multiplied by $H(s)$, the pole on RHP is cancelled and a new pole is created at $s = -10$. That is,

$$H(s)G(s) = \left(\frac{s+1}{s-10}\right)\left(\frac{s-10}{s+10}\right) = \frac{s+1}{s+10}$$

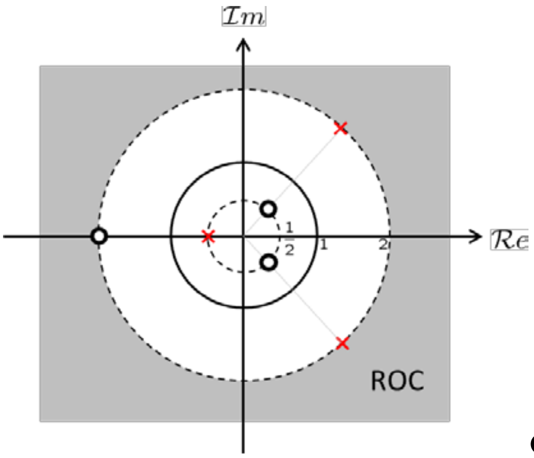
Therefore, the total system is stable.

(2) $G(j\omega)$ is all pass system with impulse response

$$\mathcal{F}^{-1}\left\{\frac{j\omega-10}{j\omega+10}\right\} = \mathcal{F}^{-1}\left\{1 - \frac{20}{j\omega+10}\right\} = \delta(t) - 20e^{-10t}u(t)$$

[Problem 5 – z-transform | 20 points]

A pole-zero map of a discrete-time system $h[n]$ is shown below:



$\mathbf{o}$: zeros , $\times$: poles

(1) **[5 pts]** Describe the type of frequency response of this system

(A) All-pass (B) High-pass (C) Low-pass

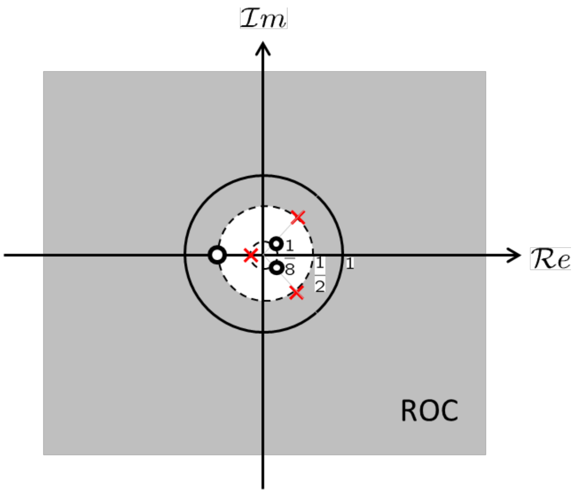
(2) **[5 pts]** Draw the pole-zero map of $g[n] = \left(\frac{1}{4}\right)^n h[n]$ (hint: $G(z) = \mathcal{F}\left\{g[n]r^{-n}\right\}$ for $z = re^{j\omega}$).

(3) **[5 pts]** Discuss whether $g[n]$ is stable or not. Justify your answer.

(4) **[5 pts]** Draw the pole-zero map of $f[n] = e^{j\pi n}h[n]$ (hint: $\mathcal{F}\left\{f[n]r^{-n}\right\} = \sum_{n=-\infty}^{\infty} f[n]r^{-n}e^{-j\omega n}$)

(1) All zeros are reciprocal of poles: **(A) all-pass system**

(2) $G(z) = \mathcal{F}\left\{g[n]r^{-n}\right\} = \mathcal{F}\left\{h[n]4^{-n}r^{-n}\right\} = \mathcal{F}\left\{h[n](4r)^{-n}\right\} = H(4z)$ → compression in radius by 4 times



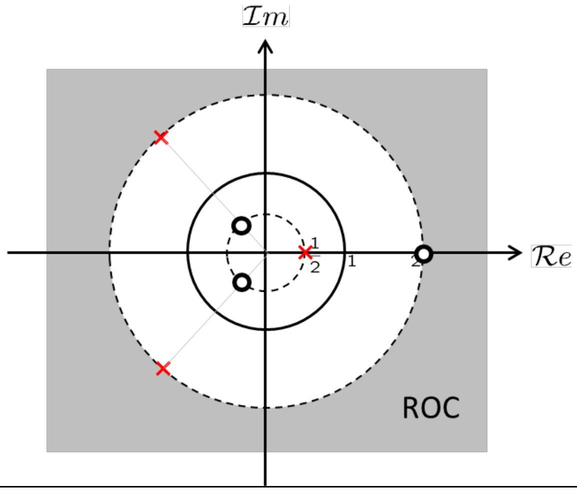
(3) Now all poles are within the unit circle → **system is stable**

(4) $F(z) = \mathcal{F}\left\{f[n]r^{-n}\right\} = \sum_{n=-\infty}^{\infty} f[n]r^{-n}e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} e^{j\pi n} h[n] r^{-n} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n] r^{-n} e^{-j(\omega-\pi)n}$$

$$= H(re^{j(\omega-\pi)})$$

➔ rotation in angle by π.



[Extra Problem | Only for students absent more than 8 times]

Prove the initial value theorem $h[0] = \lim_{z\to\infty} H(z)$, where $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$ and $h[n]$ are a causal system's responses.

➔ refer to the text book