DIY $f(y_1, y_2) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(y_1-\mu_1)^2}{2\sigma_1^2}\right) \times \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{(y_2-\mu_2)^2}{2\sigma_2^2}\right)$

$= f(y_1) \times f(y_2)$

- Marginal

$$\begin{bmatrix} Y_1 \sim N(\mu_1, \sigma_1^2) \\ Y_2 \sim N(\mu_2, \sigma_2^2) \end{bmatrix}$$

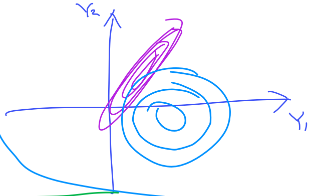
Show

$$\int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(y_1-\mu_1)^2}{2\sigma_1^2}\right)$$

- $\underline{Y} \sim N(\underline{\mu}, \Sigma)$
- $\underline{a}'\underline{Y} \sim N(\underline{a}'\underline{\mu}, \underline{a}'\Sigma\underline{a})$ $\forall a$

$$Y_1 = (1, 0)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \underline{a}'\underline{Y}$$



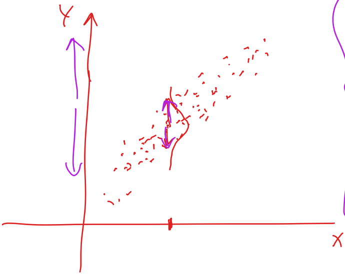
☆☆ Conditional of $Y_1 \mid Y_2 = y_2$

$$f(y_1 \mid y_2) = \frac{f(y_1, y_2)}{f(y_2)} = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}Q\right)}{\frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{(y_2-\mu_2)^2}{2\sigma_2^2}\right)}$$

$=$

: density of

$$N\Big(\overbrace{\mu_1+\rho\cdot\frac{\sigma_1}{\sigma_2}(y_2-\mu_2)}^{E(Y_1|Y_2=y_2)},\ \overbrace{\sigma_1^2(1-\rho^2)}^{V(Y_1|Y_2=y_2)}\Big)$$



① Conditional mean is a linear ft of y_2.

② Conditional variance does not depend on y_2

③ Conditional variance $\leq$ Variance of Y_1

$\Downarrow$

Why we do regression.

Conditional Expectations

mean

Conditional expectation of Y_1, given that $Y_2 = y_2$

$$E[Y_1|Y_2 = y_2] = \int_{-\infty}^{\infty} y_1 f(y_1|y_2) dy_1$$

; mean of conditional distribution

$E(Y_1|Y_2)$ is a function of Y_2, thus a random variable. thus, it has a distribution itself.

a ft of Y_2 → w.r.t. conditional dist. of $Y_1|Y_2$.

$$E(Y_1) = E[E(Y_1|Y_2)] \quad ①$$

→ w.r.t. marginal of Y_2

$$\boxed{V(Y_1) = \overbrace{E[V(Y_1|Y_2)]}^{a} + \overbrace{V[E(Y_1|Y_2)]}^{b}} \quad ②$$

Total variance theorm.

① $\text{RHS} = \int E(Y_1|Y_2)\, f_2(y_2)\, dy_2 = \int \left[\int y_1 \cdot f(y_1|y_2)\, dy_1\right] f_2(y_2)\, dy_2$

$$= \int\left[\int y_1 \cdot \frac{f(y_1,y_2)}{f_2(y_2)}\, dy_1\right] f_2(y_2)\, dy_2$$

$$= \iint y_1 f(y_1,y_2)\, dy_1\, dy_2 = E(Y_1)$$

---

② $a = E_{Y_2}\left(E_{Y_1|Y_2}(Y_1^2|Y_2) - E_{Y_1|Y_2}(Y_1|Y_2)^2 \right)$

$= E\left(E(Y_1^2|Y_2)\right) - E\left(E(Y_1|Y_2)^2\right)$

$= E(Y_1^2) - E\left(\cancel{E(Y_1|Y_2)^2}\right)$

$b = V\left(E(Y_1|Y_2)\right) = E\left(\cancel{E(Y_1|Y_2)^2}\right) - \left[E\left[E(Y_1|Y_2)\right)\right]^2$

$= \quad // \quad - E(Y_1)^2$

$a + b = E(Y_1^2) - E(Y_1)^2 = V(Y_1)$

□

$$Y_1 \mid Y_1 = y_1 \sim \begin{cases} y_1 & \text{w.p. } 1 \\ 0 & \text{o.w.} \end{cases}$$

$$V(Y_1 \mid Y_1 = y_1) = 0$$

$$E(Y_1 \mid Y_1 = y_1) = y_1$$

Eg 32)

Y : # defectives out of 10 $\sim B(10, p)$

p : prob. defect. $\sim U(0, \frac{1}{4})$

- $Y \mid p \sim B(10, p)$

- $$E(Y) = E(E(Y \mid P)) = E(10p) = 10 \cdot \frac{1}{8}$$

- $$V(Y) = V(E(Y \mid P)) + E(V(Y \mid P))$$

$$= V(10p) + E(10p(1-p)) = 100 \cdot \frac{(\frac{1}{4})^2}{12} + 10E(p) - 10E(p^2)$$

.....

1139

N : # jobs per week $\sim$ Poisson(λ)

Y : no. hrs completing each job $\sim$ Gamma(α, β)

Total hours completing all jobs in a week

$$T = \sum_{i=1}^{N} Y_i \quad , \quad Y_i \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$$

$$N \sim \text{Poisson}(\lambda)$$

$$E(T) = E(E(T|N)) = E(N\alpha\beta)$$

$$= \alpha\beta\lambda$$

$T|N \sim \text{Gamma}(N\alpha, \beta)$

$$V(T)$$

$$= V(E(T|N)) + E(V(T|N))$$

$$= V(N\alpha\beta) + E(N\alpha\beta^2) = \alpha^2\beta^2\lambda + \alpha\beta^2\lambda$$