EE202C Signals and Systems, Final Exam, 2021 Fall
Lecturer: Jung Woo Choi 9:00 - 11:15 am

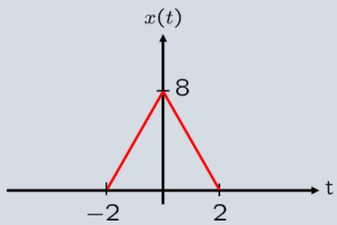
[Instructions]

1. You need to put your **both hands** on the desk.
2. You **cannot leave the Zoom session** without finishing the exam. If you have to leave, submit your pdf answer sheets to the KLMS. (The link for submission will be notified during the exam).
3. Write down your **name and student number** on the **top-right corner** of your answer sheet (only on the first page).
4. Please write down the **page number** on the bottom of your answer sheet.
5. Don't forget to write down the **problem number**.
6. Describe the whole derivation. You cannot earn the full score if you simply write down only the final answer. Especially, "justify your answer" means you need to prove or validate the answer.
7. The exam will end at 11:15 am. If you have any problem with the KLMS submission, then submit via email to jwoo@kaist.ac.kr. Late submission after the end of the exam session cannot be accepted.
8. The **solution** will be uploaded to KLMS at 12:00 pm. However, there can be mistakes in the initial solution. If you have any claim on the solution, please post to Classum during this week. Any wrong answers will be fixed and reuploaded to KLMS.

Problem 1 (5)	Problem 2 (20)			Problem 3 (25)				Problem 4 (15)		Problem 5 (35)				
a	a	b	c	a	b	c	d	a	b	a	b	c	d	e
5	10	5	5	5	10	5	5	5	10	5	10	10	5	5

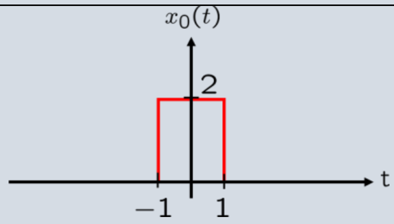
Problem 1 (5 pts) Fourier transform (Inspired by the question from Hoyun Jeong)

Consider a triangular signal $x(t)$ shown below:



a) [5 pts] Find the Fourier transform $Y(j\omega)$ of $y(t) = x(t) * x(t)$.

First, consider a rectangular signal $x_0(t) = 2\text{rect}(t/2)$.



Its Fourier transform is given by $X_0(j\omega) = 4\text{sinc}(\omega/\pi)$.
Then the signal $x(t)$ can be rewritten as

$$x(t) = x_0(t) * x_0(t)$$

Therefore, the convolution can be expressed as

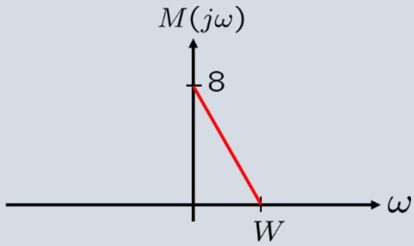
$$y(t) = x(t) * x(t) = x_0(t) * x_0(t) * x_0(t) * x_0(t)$$

From the convolution-multiplication property of Fourier transform, we have

$$\begin{aligned} Y(j\omega) &= \left(X_0(j\omega)\right)^4 \\ &= 256\,\text{sinc}^4(\omega/\pi) \\ &= \left(4\frac{\sin\omega}{\omega}\right)^4 \end{aligned}$$

Problem 2. (20 pts) Fourier transform & Sampling theorem

Let $m(t)$ be a continuous-time signal whose Fourier transform $M(j\omega)$ is shown below:

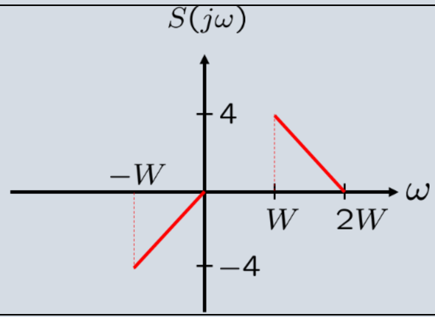


a) **[10 pts]** Consider a new signal $s(t) = jm(t)\sin(Wt)$. Express its Fourier transform $S(j\omega)$ in terms of $M(j\omega)$.

$$j\sin(Wt) = \frac{1}{2}\left(e^{jWt} - e^{-jWt}\right) \xrightleftharpoons{\mathcal{F}} \frac{1}{2}\left(2\pi\delta(\omega-W) - 2\pi\delta(\omega+W)\right)$$

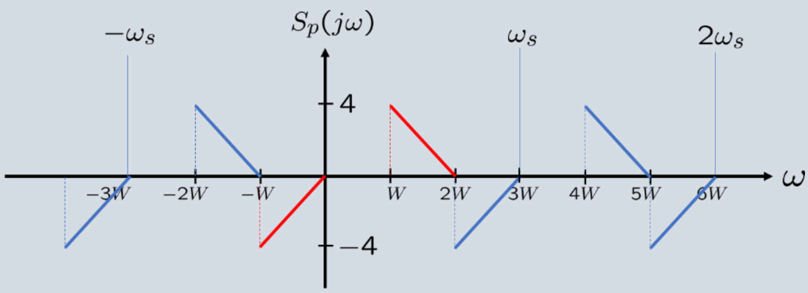
From the multiplication-convolution theorem,

$$\begin{aligned} S(j\omega) &= \frac{1}{2\pi}M(j\omega) * \frac{1}{2}\left(2\pi\delta(\omega-W) - 2\pi\delta(\omega+W)\right) \\ &= \frac{1}{2}\left(M(j(\omega-W)) - M(j(\omega+W))\right) \end{aligned}$$



b) **[5 pts]** Determine the lowest sampling frequency ω_s for sampling the signal $s(t)$ without aliasing.

From the result of (a), $S(j\omega)$ consists of $M(j\omega)$ shifted by $\pm W$, and its bandwidth is given by $2W-(-W)=3W$. From the Nyquist sampling theorem, $\omega_s \ge 3W$ should be satisfied to avoid aliasing. Therefore, the lowest sampling rate is $3W$.

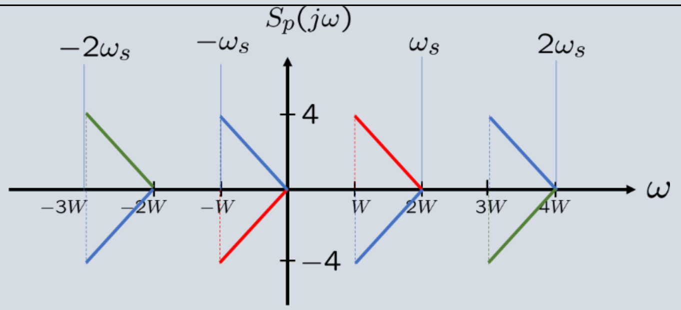


c) **[5 pts]** Suppose that the signal $s(t)$ is sampled at a sampling rate of $2W$. Plot the magnitude of $S(e^{j\omega})$ obtained from the DT Fourier transform of the sampled discrete-time sequence $s[n]$.

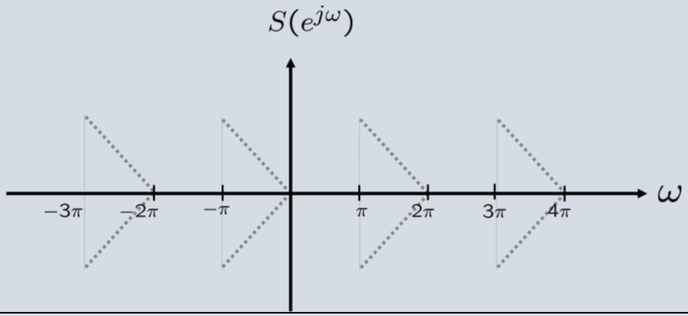
With the sampling rate $\omega_s = 2W$, the positive and negative values of aliased components are cancelled out by each other. Therefore, the sampled function $S_p(j\omega)=0$.

This can be also confirmed by the sampling of $s(t)=jm(t)\sin(Wt)$ with $t=nT=n\dfrac{2\pi}{\omega_s}=n\dfrac{\pi}{W}$.

Because $s(nT)=jm(nT)\sin\left(Wn\dfrac{\pi}{W}\right)=jm(nT)\sin(n\pi)=0$, no signal is detected by the sampling.



Accordingly, you do not need to draw anything except a zero line.



Problem 3 (25 pts) Laplace transform and ROC

a) **[5 pts]** For $X(s)=\dfrac{1}{a^2-s^2}$ with the region of convergence (ROC) $\mathrm{Re}\{s\}>a>0$, find its inverse Laplace transform $x(t)$.

$$X(s)=\frac{1}{a^2-s^2}=\frac{1}{a-s}\cdot\frac{1}{a+s}$$
$$=\frac{1}{2a}\left(\frac{1}{s+a}-\frac{1}{s-a}\right)$$
$$x(t)=\frac{1}{2a}\left(e^{-at}u(t)-e^{at}u(t)\right)$$

b) **[10 pts] (Inspired by the Question from Sangmyoung Lee)**

For $X(s)=\dfrac{1}{a^2-s^2}$ with its ROC given by $-a<\mathrm{Re}\{s\}<a$, find the inverse Laplace transform $x(t)$. ($a>0$)

The Laplace transform is the same, but its ROC is bounded.

$$X(s)=-\frac{1}{2a}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)$$

To have a bounded ROC, the signal corresponding $1/(s-a)$ should be a left-sided signal, and the one for $1/(s+a)$ should be the right-sided signal, such that the intersection of ROC exists.

$$x(t)=\frac{1}{2a}\left(e^{-at}u(t)+e^{at}u(-t)\right) \quad \text{or} \quad \frac{1}{2a}e^{-a|t|}$$

c) **[5 pts] (Question from Huiil Cha, Euijin Shin)**

For $X_n(s)=\dfrac{1}{(s+a)^n}$ with ROC: $\mathrm{Re}\{s\}>-a$, show that its inverse Laplace transform is given by $x(t)=\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t)$ (n : integer)

$X_n(s)$ and $X_{n-1}(s)$ are related by the differentiation operator. That is,

$$\begin{aligned}\frac{d}{ds}\left(X_{n-1}(s)\right)&=\frac{d}{ds}\left(\frac{1}{(s+a)^{n-1}}\right)\\&=\frac{-(n-1)}{(s+a)^n}\\&=-(n-1)X_n(s)\end{aligned}$$

Therefore, $X_n(s)$ can be expressed in terms of $X_1(s)$ as

$$\begin{aligned}X_n(s)&=-\frac{1}{n-1}\frac{dX_{n-1}(s)}{ds}\\&=\left(-\frac{1}{n-1}\right)\times\left(-\frac{1}{n-2}\right)\times\cdots\times\left(-\frac{1}{1}\right)\times\left(\frac{d^{n-1}}{ds^{n-1}}X_1(s)\right)\\&=\frac{(-1)^{n-1}}{(n-1)!}\left(\frac{d^{n-1}}{ds^{n-1}}X_1(s)\right) \quad \cdots(1)\end{aligned}$$

From the properties of Laplace transform, we have

$$X_1(s)=\frac{1}{s+a} \xrightleftharpoons{\mathcal{L}} x_1(t)=e^{-at}u(t)$$

$$\frac{d^n}{ds^n}X(s) \xrightleftharpoons{\mathcal{L}} (-t)^n x(t)$$

From these properties and (1), the time domain signal can be derived as

$$\therefore x(t)=\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$$

d) [5 pts] (Inspired by the question from Chaeyun Jeong)

Using the relation of (c), calculate $\int_0^\infty \frac{t^3}{6}e^{-3t}dt$

From the relation of (c), we have $\dfrac{t^3e^{-3t}}{6}u(t) \xrightleftharpoons{\mathcal{L}} \dfrac{1}{(s+3)^4}$.

The Laplace transform can be rewritten in an integral form as

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \frac{t^3 e^{-3t}}{6} u(t) e^{-st} dt = \frac{1}{(s+3)^4}$$

The integral of the problem is the special case of $s = 0$. Therefore, we have

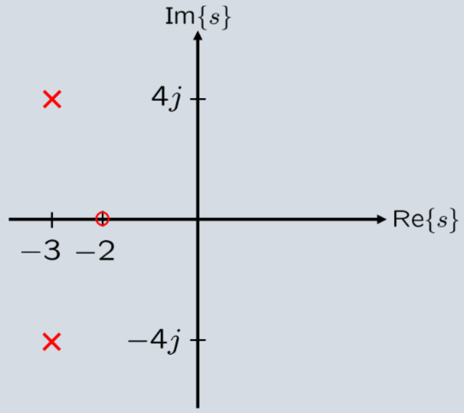
$$\int_{-\infty}^{\infty} \frac{t^3}{6} e^{-3t} u(t) e^{-0t} dt = \frac{1}{(0+3)^4} = \frac{1}{81}$$

Problem 4 (15 pts) Characterization of Systems

Consider the LTI system satisfying the following LCCDE:

$$b\frac{dy^2(t)}{dt^2} + a\frac{dy(t)}{dt} + y(t) = f\frac{d^2x(t)}{dt^2} + g\frac{dx(t)}{dt} + x(t).$$

The pole-zero map of this system is given as follows (note: single zero at s=-2):



a) **[5 pts] (Inspired by the question from Yuil Kim)**
Determine constants a, b, f, g and the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$.

The s-domain representation from the given LCCDE can be written as

$$H(s) = \frac{fs^2 + gs + 1}{bs^2 + as + 1} \quad \text{---(1)}$$

From the pole-zero plot, the system response can also be written as

$$H(s) = A\frac{s+2}{\big(s+(3-4j)\big)\big(s+(3-4j)\big)} = A\frac{s+2}{s^2+6s+25}$$

$$= \frac{2A}{25} \cdot \frac{\frac{1}{2}s+1}{\frac{1}{25}s^2 + \frac{6}{25}s + 1} \quad \text{---(2)}$$

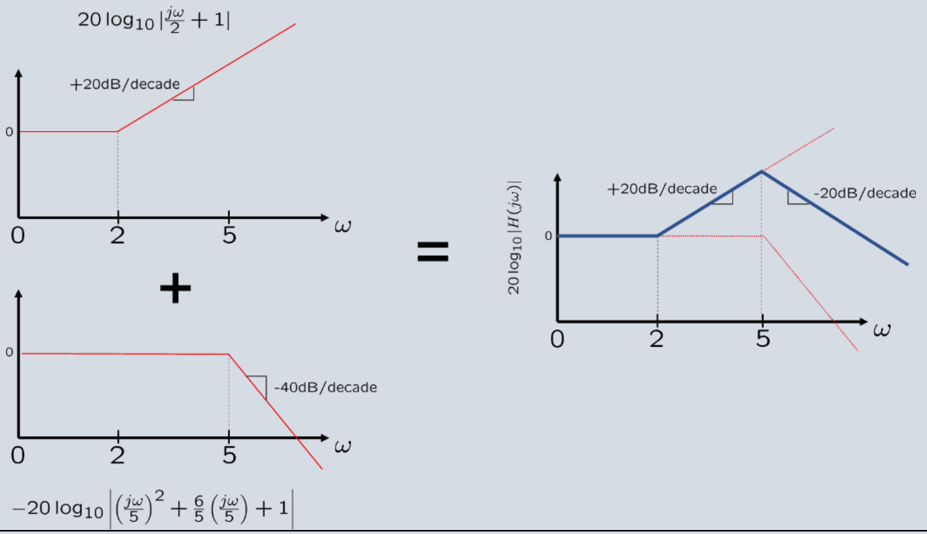
By comparing (1) & (2), we get $a=\frac{6}{25},\ b=\frac{1}{25},\ f=0,\ g=\frac{1}{2}.\quad (A=\frac{25}{2})$

The frequency response is given by $H(j\omega)=\frac{25}{2}\cdot\frac{j\omega+2}{(j\omega)^2+6j\omega+25}$

b) **[10 pts]** Draw the Bode plot of $20\log_{10}|H(j\omega)|$. Indicate the magnitude at zero frequency in dB scale and slopes of asymptotic lines.

$$20\log_{10}|H(j\omega)| = 20\log_{10}\left|\frac{\frac{j\omega}{2}+1}{\left(\frac{j\omega}{5}\right)^2+\frac{6}{5}\left(\frac{j\omega}{5}\right)+1}\right|$$

$$= 20\log_{10}\left|\frac{j\omega}{2}+1\right| - 20\log_{10}\left|\left(\frac{j\omega}{5}\right)^2+\frac{6}{5}\left(\frac{j\omega}{5}\right)+1\right|$$



Problem 5 (35 pts) Laplace and z-transform (Questions from TA Seongrae Kim)

A CT LTI system satisfying the condition of initial rest is described in terms of the differential equation

$$\frac{d^2y(t)}{dt^2}+\sqrt{2}\frac{dy(t)}{dt}+y(t)=x(t)$$

(a) [5 pts] Find the system response $H(s)$ of this system.

$$H(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

(b) [10 pts] Now we want to design a DT system $H(z)$ that preserve the frequency characteristics of the CT system $H(s)$. For this purpose, we first consider a sampled signal

$h_p(t)$ of the ideal impulse response $h(t)$ defined as

$$h_p(t) = \sum_{n=-\infty}^{\infty} h(nT)\delta(t-nT)$$

where T is the sampling time. **Show that the Laplace transform $H_p(s)$ of $h_p(t)$ is equal to the z-transform $H(z)$ of $h[n]=h(nT)$ when $z=e^{sT}$.**

$$\begin{aligned}
\mathcal{L}\{h_p(t)\} &= \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} h(nT)\delta(t-nT)\right)e^{-st}dt \\
&= \sum_{n=-\infty}^{\infty} h[n]\int_{-\infty}^{\infty}\delta(t-nT)e^{-st}dt \\
&= \sum_{n=-\infty}^{\infty} h[n]e^{-snT} \\
&= \sum_{n=-\infty}^{\infty} h[n]\left(e^{sT}\right)^{-n} \\
&= \sum_{n=-\infty}^{\infty} h[n]z^{-n}, \quad z=e^{sT}
\end{aligned}$$

c) [10 pts] The relation $z=e^{sT}$ of (b) can be approximated by the series expansion as

$$z=e^{sT} \rightarrow s=\frac{1}{T}\ln z$$

$$s=\frac{1}{T}\ln z=\frac{2}{T}\left[\frac{z-1}{z+1}+\frac{1}{3}\left(\frac{z-1}{z+1}\right)^3+\cdots\right]\simeq\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

Using this approximation, find <u>pole locations</u> of the z-transform $H(z)$ approximating the system response derived in (a). Assume that $T=2\sqrt{2}$.

$$\begin{aligned}
H(z) &= \frac{1}{\frac{1}{2}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1} \\
&= \frac{2\left(1+z^{-1}\right)^2}{\left(1-z^{-1}\right)^2+2\left(1-z^{-2}\right)+2(1+z^{-1})^2} \\
&= \frac{2(1+z^{-1})^2}{z^{-2}+2z^{-1}+5} \\
&= \frac{2(1+z^{-1})^2}{\left(z^{-1}-(-1+2j)\right)\left(z^{-1}-(-1-2j)\right)}
\end{aligned}$$

$$z^{-1}=-1\pm 2j \rightarrow z=\frac{-1\pm 2j}{5}.$$

Therefore, two poles are at $\frac{-1\pm 2j}{5}$

d) [5 pts] Determine whether the $H(z)$ is stable or not. Justify your answer.

- Since the system is causal (satisfying the condition of initial rest), the impulse response is right-sided. Accordingly, the ROC is the exterior region from the outermost pole locations.
- From the result of (c), the poles of this system are inside of the unit circle.

$$|z| = \frac{\sqrt{5}}{5} < 1$$

- Therefore, the ROC includes the unit circle and the system is stable.

e) [5 pts] Find the initial value of the impulse response $h[n]$.

From the initial value theorem, $h[0] = \lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{2(1+z^{-1})^2}{z^{-2}+2z^{-1}+5} = \frac{2}{5}$