HW#5 Solution

E6.10 Consider a feedback system with closed-loop transfer function

$$T(s) = \frac{4}{s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4}.$$

Is the system stable?

(Ans)

The characteristic equation is

$$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0 \ .$$

And the Routh array is given by

$$\begin{array}{c|ccc l}
s^5 & 1 & 8 & 7 & \\
s^4 & 4 & 8 & 4 & \\
s^3 & 6 & 6 & 0 & \\
s^2 & 4 & 4 & & \Leftrightarrow A(s) = 4s^2 + 4 \\
s^1 & 0 & 0 & & \Leftarrow \text{Row of 0's} \\
s^o & & & &
\end{array}$$

The auxiliary equation is

$$A(s) = 4s^2 + 4 \ .$$

Solving $A(s) = 0$ yields two poles on the imaginary axis at $s = \pm j$.
Thus, it is marginally stable.

E6.24 Consider the system represented in state variable form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t),$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k & -k & -k \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 0 \quad 0], \mathbf{D} = [0].$$

(a) What is the system transfer function? (b) For what values of k is the system stable?

(Ans)

(a) The transfer function is

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$
$$= [1 \quad 0 \quad 0] \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k & k & s+k \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$= [1 \quad 0 \quad 0] \begin{bmatrix} s^2+ks+k & s+k & 1 \\ -k & s^2+ks & s \\ -ks & -ks-k & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\Delta(s)}$$
$$= \frac{1}{\Delta(s)}$$

where $\Delta(s) = s^3 + ks^2 + ks + k$. Thus, the transfer function is

$$G(s) = \frac{1}{s^3 + ks^2 + ks + k} \, .$$

(b) You can derive the Routh array as follows:

$$\begin{array}{c|cc} s^3 & 1 & k \\ s^2 & k & k \\ s^1 & k-1 & \\ s^o & k & \end{array}$$

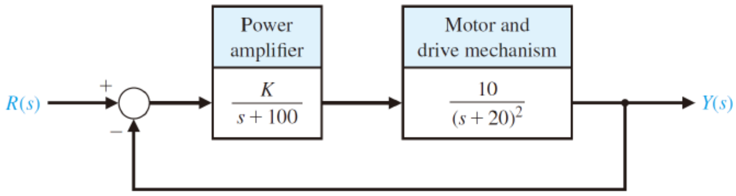
For stability, all the numbers in the first column are positive.

Thus, we require

$$k > 1 \ .$$

P6.9 A cassette tape storage device has been designed for mass-storage [1]. It is necessary to control the velocity of the tape accurately. The speed control of the tape drive is represented by the system shown in Figure P6.9.

(a) Determine the limiting gain for a stable system.
(b) Determine a suitable gain so that the percent overshoot to a step command is $P.O. = 5\%$.



(Ans)

(a) The closed-loop characteristic equation is

$$1 + GH(s) = 1 + \frac{10K}{(s+100)(s+20)^2}$$

or

$$s^3 + 140s^2 + 4400s + 40000 + 10K = 0 \ .$$

You can derive the Routh array as follows:

$$\begin{array}{c|cc} s^3 & 1 & 4400 \\ s^2 & 140 & 40000 + 10K \\ s^1 & b & \\ s^o & 40000 + 10K & \end{array}$$

where

$$b = \frac{140(4400) - (40000 + 10K)}{140} \ .$$

Examining the first column and requiring all the terms to be positive, we determine that the system is stable if

$$-4000 < K < 57600 \ .$$

(b) The desired characteristic equation polynomial is

$$\begin{aligned}&(s+b)(s^2+1.38\omega_n+\omega_n^2)\\&=s^3+(1.38\omega_n+b)s^2+(\omega_n^2+1.38\omega_n b)s+b\omega_n^2\end{aligned}$$

where we have used the fact that $\xi = 0.69$ to achieve a 5% overshoot, and ω_n and b are to be determined. The actual characteristic polynomial is

$$s^3+140s^2+4400s+40000+10K=0\ .$$

Equating the coefficients of the actual and desired characteristic polynomials, and solving for K, b, and ω_n, yields

$$b=104.2,\qquad \omega_n=25.9,\qquad \text{and}\qquad K=3003\ .$$

So, a suitable gain is $K = 3003$.

P6.12 A system has the third-order characteristic equation

$$s^3 + as^2 + bs + c = 0,$$

where a, b, and c are constant parameters. Determine the necessary and sufficient conditions for the system to be stable. Is it possible to determine stability of the system by just inspecting the coefficients of the characteristic equation?

(Ans)

The characteristic equation is

$$s^3 + as^2 + bs + c = 0.$$

So, the Routh array is

$$\begin{array}{c|cc} s^3 & 1 & b \\ s^2 & a & c \\ s^1 & \frac{ab-c}{a} & \\ s^0 & c & \end{array}$$

For the system to be stable, we require that $a > 0$, $ab - c > 0$ and $c > 0$. When $a > 0$ and $c > 0$, we know that $b > 0$. So, a necessary condition for stability is that all coefficients a, b, and c be positive. The necessary and sufficient conditions for stability also require that $b > c/a$, in addition to $a > 0$ and $c > 0$.

P6.21 Consider the system described in state variable form by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{C} = [1 \quad -1],$$

and where $k_1 \neq k_2$ and both k_1 and k_2 are real numbers.

(a) Compute the state transition matrix $\Phi(t, 0)$.

(b) Compute the eigenvalues of the system matrix $\mathbf{A}$.

(c) Compute the roots of the characteristic polynomial. (d) Discuss the results of parts (a)–(c) in terms of stability of the system.

(Ans)

(a)

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}$$
$$= \frac{1}{s^2 + k_2 s + k_1}\begin{bmatrix} s + k_2 & 1 \\ -k_1 & s \end{bmatrix}$$
$$= \frac{1}{(s + p_1)(s + p_2)}\begin{bmatrix} s + k_2 & 1 \\ -k_1 & s \end{bmatrix}$$

where

$$p_1 p_2 = k_1 \text{ and } p_1 + p_2 = k_2 \text{ .}$$

The state transition matrix is

$$\Phi(t, 0) = L^{-1}\{\Phi(s)\}$$

$$= L^{-1}\left\{\begin{bmatrix} \dfrac{\frac{-p_1 + k_2}{-p_1 + p_2}}{s + p_1} + \dfrac{\frac{p_2 - k_2}{-p_1 + p_2}}{s + p_2} & \dfrac{\frac{1}{-p_1 + p_2}}{s + p_1} + \dfrac{\frac{-1}{-p_1 + p_2}}{s + p_2} \\ \dfrac{\frac{-k_1}{-p_1 + p_2}}{s + p_1} + \dfrac{\frac{k_1}{-p_1 + p_2}}{s + p_2} & \dfrac{\frac{-p_1}{-p_1 + p_2}}{s + p_1} + \dfrac{\frac{p_2}{-p_1 + p_2}}{s + p_2} \end{bmatrix}\right\}$$

$$= L^{-1}\left\{\frac{1}{-p_1+p_2}\begin{bmatrix} \dfrac{-p_1+k_2}{s+p_1}+\dfrac{p_2-k_2}{s+p_2} & \dfrac{1}{s+p_1}+\dfrac{-1}{s+p_2} \\ \dfrac{-k_1}{s+p_1}+\dfrac{k_1}{s+p_2} & \dfrac{-p_1}{s+p_1}+\dfrac{p_2}{s+p_2} \end{bmatrix}\right\}$$

So the answer is

$$\Phi(t,0) = \frac{1}{p_2-p_1}\begin{bmatrix} (k_2-p_1)e^{-p_1 t}-(k_2-p_2)e^{-p_2 t} & e^{-p_1 t}-e^{-p_2 t} \\ -k_1 e^{-p_1 t}+k_1 e^{-p_2 t} & -p_1 e^{-p_1 t}+p_2 e^{-p_2 t} \end{bmatrix}.$$

(b) The eigenvalues are given by the solution to $\det|\lambda\mathbf{I}-\mathbf{A}| = \lambda^2 + k_2\lambda + k_1 = 0$. Therefore the eigenvalues are

$$\lambda_{1,2} = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2}.$$

(c) The transfer function is given by

$$G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = -\frac{s-1}{s^2+k_2 s+k_1}.$$

Therefore the characteristic equation is $s^2 + k_2 s + k_1 = 0$.

The poles are given by the solution to the characteristic equation,

$$s_{1,2} = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2}.$$

(d) For (a), p_1 and p_2 are positive for stability. It means k_1 and k_2 are also positive. For (b), the eigenvalues are in the left hand s-plane and for (c), the poles are in the left hand s-plane for stability. In those cases, k_1 and k_2 are also positive. We can find that the values for $\lambda_{1,2}$ and $s_{1,2}$ are the same. Also, the eigenvalues are the same as the values of $-p_1$ and $-p_2$. So, if the eigenvalues are negative, then the elements of the state transition matrix will decay exponentially.