

HW 6 solution

5.2

a. The sample space for the toss of three balanced coins with probabilities are below:

Outcome	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
(y_1, y_2)	(3, 1)	(3, 1)	(2, 1)	(1, 1)	(2, 2)	(1, 2)	(1, 3)	(0, -1)
probability	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

b. $F(2, 1) = p(0, -1) + p(1, 1) + p(2, 1) = 1/2$.

5.9

a. Since the density must integrate to 1, evaluate $\int_0^1 \int_0^{y_2} k(1 - y_2) dy_1 dy_2 = k/6 = 1$, so $k = 6$.

b. Note that since $Y_1 \leq Y_2$, the probability must be found in two parts:

$$P(Y_1 \leq 3/4, Y_2 \geq 1/2) = \int_{1/2}^1 \int_{1/2}^1 6(1 - y_2) dy_1 dy_2 + \int_{1/2}^{3/4} \int_{y_1}^1 6(1 - y_2) dy_2 dy_1 = 24/64 + 7/64 = 31/64$$

5.15

a. $P(Y_1 < 2, Y_2 > 1) = \int_1^2 \int_1^{y_1} e^{-y_1} dy_2 dy_1 = \int_1^2 \int_{y_2}^2 e^{-y_1} dy_1 dy_2 = e^{-1} - 2e^{-2}$.

b. $P(Y_1 \geq 2Y_2) = \int_0^\infty \int_{2y_2}^\infty e^{-y_2} dy_1 dy_2 = 1/2$.

c. $P(Y_1 - Y_2 \geq 1) = P(Y_1 \geq Y_2 + 1) = \int_0^\infty \int_{y_2+1}^\infty e^{-y_1} dy_1 dy_2 = e^{-1}$.

5.20

a. The marginal probability function is given in the table below.

y_2	-1	1	2	3
$p_2(y_2)$	1/8	4/8	2/8	1/8

b. $P(Y_1 = 3 \mid Y_2 = 1) = \frac{P(Y_1=3, Y_2=1)}{P(Y_2=1)} = \frac{1/8}{4/8} = 1/4$.

5.21

a. The marginal distribution of Y_1 is hypergeometric with $N = 9$, $n = 3$, and $r = 4$.

b. Similar to part a, the marginal distribution of Y_2 is hypergeometric with $N = 9$, $n = 3$, and $r = 3$. Thus,

$$P(Y_1 = 1 \mid Y_2 = 2) = \frac{P(Y_1 = 1, Y_2 = 2)}{P(Y_2 = 2)} = \frac{\binom{4}{1} \binom{3}{2} \binom{2}{0}}{\binom{9}{3}} / \frac{\binom{3}{2} \binom{6}{1}}{\binom{9}{3}} = 2/3.$$

c. Similar to part b,

$$P(Y_3 = 1 | Y_2 = 1) = P(Y_1 = 1 | Y_2 = 1) = \frac{P(Y_1 = 1, Y_2 = 1)}{P(Y_2 = 1)} = \frac{\binom{3}{1}\binom{2}{1}\binom{4}{1}}{\binom{9}{3}} / \frac{\binom{3}{1}\binom{6}{2}}{\binom{9}{3}} = 8/15.$$

d. Same

5.27

a.

$$f_1(y_1) = \int_{y_1}^1 6(1 - y_2) dy_2 = 3(1 - y_1)^2, 0 \leq y_1 \leq 1$$

$$f_2(y_2) = \int_0^{y_2} 6(1 - y_2) dy_1 = 6y_2(1 - y_2), 0 \leq y_2 \leq 1$$

b. $P(Y_2 \leq 1/2 | Y_1 \leq 3/4) = \frac{\int_0^{1/2} \int_0^{y_2} 6(1 - y_2) dy_1 dy_2}{\int_0^{3/4} 3(1 - y_1)^2 dy_1} = 32/63.$

c. $f(y_1 | y_2) = 1/y_2, 0 \leq y_1 \leq y_2 \leq 1.$

d. $f(y_2 | y_1) = 2(1 - y_2) / (1 - y_1)^2, 0 \leq y_1 \leq y_2 \leq 1.$

e. From part d, $f(y_2 | 1/2) = 8(1 - y_2), 1/2 \leq y_2 \leq 1.$ Thus, $P(Y_2 \geq 3/4 | Y_1 = 1/2) = 1/4.$

5.32

a. $f_1(y_1) = \int_{y_1}^{2-y_1} 6y_1^2 y_2 dy_2 = 12y_1^2(1 - y_1), 0 \leq y_1 \leq 1.$

b. This marginal density must be constructed in two parts:

$$f_2(y_2) = \begin{cases} \int_0^{y_2} 6y_1^2 y_2 dy_1 = 2y_2^4 & 0 \leq y_2 \leq 1 \\ \int_0^{2-y_2} 6y_1^2 y_2 dy_1 = 2y_2(2 - y_2)^3 & 1 \leq y_2 \leq 2 \end{cases}.$$

c. $f(y_2 | y_1) = \frac{1}{2}y_2 / (1 - y_1), y_1 \leq y_2 \leq 2 - y_1.$

d. Using the density found in part c, $P(Y_2 < 1.1 | Y_1 = .6) = \frac{1}{2} \int_{.6}^{1.1} y_2 / .4 dy_2 = .53$

5.38

This is the identical setup as in Ex. 5.34.

a. $f(y_1, y_2) = f(y_2 | y_1) f_1(y_1) = 1/y_1, 0 \leq y_2 \leq y_1 \leq 1.$

b. Note that $f(y_2 | 1/2) = 1/2, 0 \leq y_2 \leq 1/2.$ Thus, $P(Y_2 < 1/4 | Y_1 = 1/2) = 1/2.$

c. The probability of interest is $P(Y_1 > 1/2 | Y_2 = 1/4).$ So, the necessary conditional density is $f(y_1 | y_2) = f(y_1, y_2) / f_2(y_2) = \frac{1}{y_1(-\ln y_2)}, 0 \leq y_2 \leq y_1 \leq 1.$ Thus,

$$P(Y_1 > 1/2 | Y_2 = 1/4) = \int_{1/2}^1 \frac{1}{y_1 \ln 4} dy_1 = 1/2.$$

5.42

Let $Y = \#$ of defects per yard. Then,

$$p(y) = \int_0^\infty P(Y = y, \lambda) d\lambda = \int_0^\infty P(Y = y | \lambda) f(\lambda) d\lambda = \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!} e^{-\lambda} d\lambda = \left(\frac{1}{2}\right)^{y+1}, y = 0, 1, 2, \dots$$

5.65

a. The marginal density for Y_1 is $f_1(y_1) = \int_0^\infty [(1 - \alpha(1 - 2e^{-y_1}))(1 - 2e^{-y_2})] e^{-y_1 - y_2} dy_2$

$$\begin{aligned} &= e^{-y_1} \left[\int_0^\infty e^{-y_2} dy_2 - \alpha(1 - 2e^{-y_1}) \int_0^\infty (e^{-y_2} - 2e^{-2y_2}) dy_2 \right] \\ &= e^{-y_1} \left[\int_0^\infty e^{-y_2} dy_2 - \alpha(1 - 2e^{-y_1})(1 - 1) \right] = e^{-y_1}, \end{aligned}$$

which is the exponential density with a mean of 1.

b. By symmetry, the marginal density for Y_2 is also exponential with $\beta = 1$.

c. When $\alpha = 0$, then $f(y_1, y_2) = e^{-y_1 - y_2} = f_1(y_1) f_2(y_2)$ and so Y_1 and Y_2 are independent.

Now, suppose Y_1 and Y_2 are independent. Then, $E(Y_1 Y_2) = E(Y_1) E(Y_2) = 1$. So,

$$\begin{aligned} E(Y_1 Y_2) &= \int_0^\infty \int_0^\infty y_1 y_2 [(1 - \alpha(1 - 2e^{-y_1}))(1 - 2e^{-y_2})] e^{-y_1 - y_2} dy_1 dy_2 \\ &= \int_0^\infty \int_0^\infty y_1 y_2 e^{-y_1 - y_2} dy_1 dy_2 - \alpha \left[\int_0^\infty y_1 (1 - 2e^{-y_1}) e^{-y_1} dy_1 \right] \times \left[\int_0^\infty y_2 (1 - 2e^{-y_2}) e^{-y_2} dy_2 \right] \\ &= 1 - \alpha \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{2}\right) = 1 - \alpha/4. \text{ This equals 1 only if } \alpha = 0. \end{aligned}$$

5.81

Since Y_1 and Y_2 are independent, $E(Y_2/Y_1) = E(Y_2) E(1/Y_1)$.

$$E(Y_2/Y_1) = E(Y_2) E(1/Y_1) = \frac{1}{2} \int_0^\infty y_2 e^{-y_2/2} dy_2 \left[\frac{1}{4} \int_0^\infty e^{-y_1/2} dy_1 \right] = 2 \left(\frac{1}{2} \right) = 1.$$

5.84

All answers use results proven for the geometric distribution and independence:

a. $E(Y_1) = E(Y_2) = 1/p$, $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 0$.

b. $E(Y_1^2) = E(Y_2^2) = (1 - p)/p^2 + (1/p)^2 = (2 - p)/p^2$. $E(Y_1 Y_2) = E(Y_1) E(Y_2) = 1/p^2$.

c. $E[(Y_1 - Y_2)^2] = E(Y_1^2) - 2E(Y_1 Y_2) + E(Y_2^2) = 2(1 - p)/p^2$. $V(Y_1 - Y_2) = V(Y_1) + V(Y_2) = 2(1 - p)/p^2$.

d. Use Tchebysheff's theorem with $k = 3$.

5.100

- a. $E(Y_1) = E(Z) = 0, E(Y_2) = E(Z^2) = 1$.
 - b. $E(Y_1 Y_2) = E(Z^3) = 0$ (odd moments are 0).
 - c. $\text{Cov}(Y_1, Y_1) = E(Z^3) - E(Z)E(Z^2) = 0$.
 - d. $P(Y_2 > 1 \mid Y_1 > 1) = P(Z^2 > 1 \mid Z > 1) = 1 \neq P(Z^2 > 1)$.
- Thus, Y_1 and Y_2 are dependent.

5.108

Y_1 has a gamma distribution with $\alpha = 2$ and $\beta = 1$, and Y_2 has an exponential distribution with $\beta = 1$. Thus, $E(Y_1 - Y_2) = 2(1) - 1 = 1$. Also, since

$$E(Y_1 Y_2) = \int_0^\infty \int_0^{y_1} y_1 y_2 e^{-y_1} dy_2 dy_1 = 3, \text{Cov}(Y_1, Y_1) = 3 - 2(1) = 1,$$

$$V(Y_1 - Y_2) = 2(1)^2 + 1^2 - 2(1) = 1.$$

Since a value of 4 minutes is four three standard deviations above the mean of 1 minute, this is not likely.

5.126

Using formulas for means, variances, and covariances for the multinomial:

$$\begin{aligned} E(Y_1) &= 10(.1) = 1 & V(Y_1) &= 10(.1)(.9) = .9 \\ E(Y_2) &= 10(.05) = .5 & V(Y_2) &= 10(.05)(.95) = .475 \\ \text{Cov}(Y_1, Y_2) &= -10(.1)(.05) = -.05 \end{aligned}$$

So,

$$\begin{aligned} E(Y_1 + 3Y_2) &= 1 + 3(.5) = 2.5 \\ V(Y_1 + 3Y_2) &= .9 + 9(.475) + 6(-.05) = 4.875 \end{aligned}$$

5.128

The marginal distribution for Y_1 is found by

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2.$$

Making the change of variables $u = (y_1 - \mu_1) / \sigma_1$ and $v = (y_2 - \mu_2) / \sigma_2$ yields

$$f_1(y_1) = \frac{1}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp \left[-\frac{1}{2(1-\rho^2)} (u^2 + v^2 - 2\rho uv) \right] dv.$$

To evaluate this, note that $u^2 + v^2 - 2\rho uv = (v - \rho u)^2 + u^2(1 - \rho^2)$ so that

$$f_1(y_1) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-u^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp \left[-\frac{1}{2(1-\rho^2)} (v - \rho u)^2 \right] dv,$$

So, the integral is that of a normal density with mean ρu and variance $1 - \rho^2$. Therefore,

$$f_1(y_1) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(y_1 - \mu_1)^2 / 2\sigma_1^2}, -\infty < y_1 < \infty,$$

which is a normal density with mean μ_1 and standard deviation σ_1 . A similar procedure will show that the marginal distribution of Y_2 is normal with mean μ_2 and standard deviation σ_2 .

5.129

$$\frac{f(y_1, y_2)}{f(y_2)} = \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}} \exp \left[\frac{\sigma_1^2(u-v\rho)^2}{-2\sigma_1^2(1-\rho^2)} \right] \text{ for } u, v \text{ from 5.128.}$$

Since $\sigma_1(u - v\rho) = y_1 - u_1 - \rho \frac{\sigma_1}{\sigma_2}(y_2 - u_2)$,

this is pdf of normal distribution with mean $u_1 + \rho \frac{\sigma_1}{\sigma_2}(y_2 - u_2)$ and variance $\sigma_1^2(1 - \rho^2)$.

5.131

a. The joint distribution of Y_1 and Y_2 is simply the product of the marginals $f_1(y_1)$ and $f_2(y_2)$ since they are independent. It is trivial to show that this product of density has the form of the bivariate normal density with $\rho = 0$.

b. Following the result of Ex. 5.130, let $a_1 = a_2 = b_1 = 1$ and $b_2 = -1$.

Thus, U_1 and U_2 have a bivariate normal distribution

Also, $\sum_{i=1}^n a_i b_j = 0$ so U_1 and U_2 are independent.

5.141

$$E(Y_2) = E(E(Y_2 | Y_1)) = E(Y_1/2) = \frac{\lambda}{2}$$

$$V(Y_2) = E[V(Y_2 | Y_1)] + VE(Y_2 | Y_1) = E[Y_1^2/12] + V[Y_1/2] = (2\lambda^2)/12 + (\lambda^2)/2 = \frac{2\lambda^2}{3}.$$

5.142

$$\text{a. } E(Y) = E[E(Y|p)] = E(np) = nE(p) = \frac{n\alpha}{\alpha+\beta}.$$

$$\text{b. } V(Y) = E[V(Y | p)] + VE(Y | p) = E[np(1-p)] + V(np) = nE(p - p^2) + n^2V(p).$$

Now:

$$nE(p - p^2) = \frac{n\alpha}{\alpha + \beta} - \frac{n\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)}$$

$$n^2V(p) = \frac{n^2\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

$$\text{So, } V(Y) = \frac{n\alpha}{\alpha + \beta} - \frac{n\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)} + \frac{n^2\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} = \frac{n\alpha\beta(\alpha + \beta + n)}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$