Modified on Wednesday 23rd September, 2020, 10:50:5110:50

Sep 23, 2020

- Eigenvalues and eigenvectors. [4.4, 8.8]
- Review
  - Permutation, determinant, minor, cofactor, Cramer's rule
  - $\det(AB) = \det(BA)$, $\det(A) = \det(A^T)$
  - What can we say about $\det(PAP^{-1})$?
  - What can we say about $\det(P)$ for an orthogonal matrix P?
- Let A be an $n \times n$ matrix with real entries. Regard A as a linear operator on $\mathbb{R}^n$, i.e., a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. For all $\mathbf{v} \in \mathbb{R}^n$, we have $A\mathbf{v} \in \mathbb{R}^n$.
- A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is said to be an **eigenvector** of A with **eigenvalue** λ, if there is a scalar λ such that $A\mathbf{v} = \lambda\mathbf{v}$. In other words, a scalar λ is called an eigenvalue of A, if there exists a nonzero vector $\mathbf{v}$, called an eigenvector of A corresponding to λ, such that $A\mathbf{v} = \lambda\mathbf{v}$.
- Since $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $A\mathbf{v} - \lambda\mathbf{v} = (A - \lambda I_n)\mathbf{v} = \mathbf{0}$, nonzero vector $\mathbf{v}$ is an eigenvector of A with eigenvalue λ if and only if $\mathbf{v}$ is a nontrivial solution to homogeneous linear system $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$.
- Let A be an $n \times n$ matrix with real entries. The following are equivalent.
  - λ is an eigenvalue of A.
  - $A - \lambda I_n$ is singular.
  - $\det(\lambda I_n - A) = 0$.

  It follows that A is singular if an only if 0 is an eigenvalue of A.
- For an $n \times n$ matrix A, $\det(tI_n - A)$ in which t is a variable is called the **characteristic polynomial** of A. Notice that $\det(tI_n - A)$ is a polynomial in t of degree n with real coefficients. The polynomial equation $\det(tI_n - A) = 0$ is called the **characteristic equation** of A.
- For an $n \times n$ real matrix A, the characteristic polynomial $\det(tI_n - A)$ is a monic polynomial of degree n with real coefficients. Let $p_A(t) = \det(tI_n - A)$.
- λ is an eigenvalue of A if and only if $t = \lambda$ is a solution of $\det(tI_n - A) = 0$.
- Let λ be an eigenvalue of A. The null space of $\lambda I_n - A$, $\{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \lambda\mathbf{v}\} = \{\mathbf{v} : (\lambda I_n - A)\mathbf{v} = \mathbf{0}\}$, is called the **eigenspace** of A corresponding to eigenvalue λ. What can we say about the dimension of the eigenspace of A?
- Examples.
  - Find the eigenvalues and eigenspaces for A.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$p_A(t) = \begin{vmatrix} t-1 & -2 \\ -4 & t-3 \end{vmatrix} = (t-1)(t-3) - 8 = t^2 - 4t - 5 = (t+1)(t-5)$$

Eigenvalues: -1 and 5

Eigenvectors for eigenvalue -1: $A\mathbf{v} = (-1)\mathbf{v}$

$$\begin{bmatrix} (-1)-1 & -2 \\ -4 & (-1)-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v} = c\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad c \neq 0$$

Eigenspace for eigenvalue -1:

$$\left\{ c\begin{bmatrix} 1 \\ -1 \end{bmatrix} : c \in \mathbb{R} \right\}$$

Eigenvectors for eigenvalue 5: $A\mathbf{v} = 5\mathbf{v}$

$$\begin{bmatrix} 5-1 & -2 \\ -4 & 5-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v} = c\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad c \neq 0$$

Eigenspace for eigenvalue 5:

$$\left\{ c\begin{bmatrix} 1 \\ 2 \end{bmatrix} : c \in \mathbb{R} \right\}$$

- Let $p(t) = t^n + c_1t^{n-1} + \cdots + c_n$ be a monic polynomial of degree n with complex coefficients. By the fundamental theorem of algebra, $p(t)$ factors over $\mathbb{C}$ into linear factors, $p(t) = (t-a_1)^{d_1}\cdots(t-a_m)^{d_m}$ with distinct a_i's.

  If coefficients c_i are all real, then $p(t)$ factors over $\mathbb{R}$ into linear and quadratic factors.

- The characteristic polynomial $p_A(t)$. Let A be an $n \times n$ real matrix.
  - If $A = (a_{ij})$ is upper triangular, then $p_A(t) = \prod_{i=1}^n (t - a_{ii})$. Every a_{ii} is an eigenvalue.
  - If $p_A(t) = t^n - c_1t^{n-1} + c_2t^{n-2} - \cdots + (-1)^n c_n$, then $c_n = \det(A)$ and $c_1 = \mathrm{tr}(A)$.
  - If $p_A(\lambda) = 0$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then λ is a complex($\mathbb{C}$) eigenvalue but not a real($\mathbb{R}$) eigenvalue.
  - If $p_A(t)$ factors into linear factors over $\mathbb{C}$, i.e., $p_A(t) = \prod_{i=1}^n (t - \lambda_i)$ with $\lambda_i \in \mathbb{C}$, then $\det(A) = \prod_{i=1}^n \lambda_i$ and $\mathrm{tr}(A) = \sum_{i=1}^n \lambda_i$. In general, $p_A(t)$ doesn't factor into linear factors over $\mathbb{R}$.
  - Which matrices A allow a factorization of $p_A(t)$ into linear factors over $\mathbb{R}$?
  - $p_{P^{-1}AP}(t) = p_A(t)$ for any invertible P.

- Find the eigenvalues and eigenspaces for A.

$$A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$p_A(t) = \begin{vmatrix} t-3 & 4 \\ -4 & t-3 \end{vmatrix} = (t-3)^2 + 16 = t^2 - 6t + 25 = (t-3-4i)(t-3+4i)$$

Complex eigenvalues: $3+4i$ and $3-4i$

Complex eigenvectors for complex eigenvalue $3+4i$: $A\mathbf{v} = (3+4i)\mathbf{v}$

$$\begin{bmatrix} (3+4i)-3 & 4 \\ -4 & (3+4i)-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v} = c\begin{bmatrix} i \\ 1 \end{bmatrix}, \quad c \in \mathbb{C} \setminus \{0\}$$

Complex eigenspace for eigenvalue $3+4i$:

$$\left\{ c\begin{bmatrix} i \\ 1 \end{bmatrix} : c \in \mathbb{C} \right\}$$

Do the same steps with complex eigenvalue $3-4i$.

Complex eigenspace for complex eigenvalue $3-4i$:

$$\left\{ c \begin{bmatrix} -i \\ 1 \end{bmatrix} : c \in \mathbb{C} \right\}$$

- Recall: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. $\mathcal{S}$: the standard basis. $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$: an ordered basis for $\mathbb{R}^n$.

$$P = [\mathrm{id}]_{\mathcal{S},\mathcal{B}} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$
$$[T]_{\mathcal{S}} = P[T]_{\mathcal{B}}P^{-1}$$
$$[T]_{\mathcal{B}} = P^{-1}[T]_{\mathcal{S}}P$$

- We can define eigenvectors and eigenvalues for a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$. A real number λ is an eigenvalue of T if there is a nonzero vector $\mathbf{v}$, called an eigenvector, such that $T(\mathbf{v}) = \lambda\mathbf{v}$. In fact, the eigenvalues of T coincide with eigenvalues of $[T]_{\mathcal{B}}$ for any ordered basis $\mathcal{B}$ of $\mathbb{R}^n$.
- Let $\mathbf{v}$ be an eigenvector of A corresponding to eigenvalue λ. If $|\lambda| < 1$, then $\|A^k\mathbf{v}\| = |\lambda^k|\|\mathbf{v}\|$ approaches 0 as $k \to \infty$.
- If $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ, then it is an eigenvector of $p(A)$ for any polynomial $p(x)$ in x corresponding to the eigenvalue $p(\lambda)$.

  Let $p(x) = 3x^4 + x^2 + 1$. $p(A) = 3A^4 + A^2 + I$. $p(A)\mathbf{v} = p(\lambda)\mathbf{v} = (3\lambda^4 + \lambda^2 + 1)\mathbf{v}$.