[Instructions]

1. You need to put your **both hands** on the desk.
2. You **cannot leave the Zoom session** without finishing the exam. If you have to leave, submit your pdf answer sheets to the KLMS. (The link for submission will be notified during the exam).
3. Write down your **name and student number** on the **top-right corner** of your answer sheet (only on the first page).
4. Please write down the **page number** on the bottom of your answer sheet.
5. Don't forget to write down the **problem number**.
6. Describe the whole derivation. You cannot earn the full score if you simply write down only the final answer. Especially, "justify your answer" means you need to prove or validate the answer.
7. The exam will end at 11:00 am. If you have any problem with the KLMS submission, then submit via email to jwoo@kaist.ac.kr. Late submission after the end of the exam session cannot be accepted.
8. The **solution** will be uploaded to KLMS at 12:00 pm. However, there can be mistakes in the initial solution. If you have any claim on the solution, please post to Classum during this week. Any wrong answers will be fixed and reuploaded to KLMS.

Problem 1		Problem 2		Problem 3				Problem 4				Problem 5	
a	b	a	b	c	d	a	b	a	b	c	d	a	B
5	5	5	10	10	5	10	10	5	5	5	10	5	10

Problem 1 (10 pts)

Given the following system $y[n]=a^n x[n],\ (a>0)$, answer to the following questions:

a) [5 pts] Determine whether the system is 1) causal, 2) linear, 3) time invariant.

1) Causality
The output $y[n]$ only depends on the present input $x[n]$. Therefore, the system is causal.

2) Linearity
$$ax_1[n]+bx_2[n]=ay_1[n]+by_2[n]$$. The system is linear.

3) Time invariance
$$x_1=[n-n_0]\rightarrow y_1[n]=a^n x[n-n_0]$$
$$y_1[n]\neq y[n-n_0]=a^{n-n_0}x[n-n_0]$$
Therefore, for $a\neq 1$, the system is not time invariant.
Only for a=1, $y[n]=x[n]$ and the system is time invariant. (+1)

b) [5 pts] What is the condition of a to have the BIBO stable system?

For a bounded input $|x[n]| \le B$, the output is $|y[n]| = |a^n||x[n]| = B|a^n|$.

To have the bounded output for all n, $|a| = 1$ ($a = 1$)

Problem 2 (15 pts)

Consider a CT signal $x(t) = \sum_{n=-\infty}^{\infty} e^{-\alpha(t-nT)}(u(t-nT) - u(t-(n+1)T))$.

a) [5 pts] Determine whether $x(t)$ is periodic or not. Justify your answer.

$$x(t+T) = \sum_{n=-\infty}^{\infty} e^{-\alpha(t-(n-1)T)}(u(t-(n-1)T) - u(t-nT))$$

If we make a substitution $n' = n-1$,

$$x(t+T) = \sum_{n'=-\infty}^{\infty} e^{-\alpha(t-n'T)}(u(t-n'T) - u(t-(n'+1)T)) = x(t)$$

Therefore, the signal $x(t)$ is periodic with a fundamental period T.

b) [10 pts] For $\alpha = \frac{1}{T}$ $(T > 0)$, find the Fourier series expansion coefficients a_k of $x(t)$. (note: if necessary, don't forget to apply the relation $e^{j2\pi n} = 1$)

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_0^T e^{-\alpha t}e^{-jk\omega_0 t}dt$$

$$= -\frac{1}{T}\cdot\frac{1}{(\alpha + jk\omega_0)}e^{-(\alpha+jk\omega_0)t}\Big|_0^T = \frac{1}{T}\cdot\frac{1}{\alpha + jk\omega_0}\left(1 - e^{-(\alpha+jk\omega_0)T}\right)$$

Using $\omega_0 = \frac{2\pi}{T}$, $\alpha = \frac{1}{T}$,

$$a_k = \frac{(1 - e^{-(1-2jk\pi)})}{(1+2jk\pi)} = \frac{1-e^{-1}}{1+2jk\pi}$$

Problem 3 (35 pts)

For a signal $h(t) = e^{-at}u(t)$, answer to the following questions

a) [10 pts] Find the convolution of $g(t) = e^{-bt}u(t)$ and $h(t)$. (note: consider both cases $a = b$ and $a \ne b$. Also be careful in treating the time interval where the result of the convolution is nonzero.)

$$h(t)*g(t)=\int_{-\infty}^{\infty}h(t-\tau)g(\tau)d\tau$$
$$=\int_{-\infty}^{\infty}e^{-a(t-\tau)}e^{-b\tau}u(t-\tau)u(\tau)d\tau \quad ---(1)$$
$$=\int_{0}^{t}e^{-a(t-\tau)}e^{-b\tau}d\tau\cdot u(t)$$
$$=e^{-at}u(t)\int_{0}^{t}e^{(a-b)\tau}d\tau$$
$$=\begin{cases}\dfrac{e^{-bt}-e^{-at}}{a-b}u(t) & \text{for } a\neq b\\ te^{-at}u(t) & \text{for } a=b\end{cases}$$

Note that the integral (1) is zero for t<0, so $u(t)$ should be described in the answer.

b) [5 pts] Express the derivative $\dfrac{dh(t)}{dt}$ in terms of $h(t)$ and unit impulse $\delta(t)$.

$$\frac{dh(t)}{dt}=-ae^{-at}u(t)+e^{-at}\delta(t)=-ah(t)+\delta(t)$$
$$\rightarrow \frac{dh(t)}{dt}+ah(t)=\delta(t)$$

(note: we used $h(t)=e^{-at}u(t)$ and the sifting property $x(t)\delta(t)=x(0)\delta(t)$)

c) [10 pts] When the $h(t)$ of a) is the impulse response of an LTI system, determine the LCCDE (Linear Constant Coefficients Differential Equation) of its **LTI inverse**. (You can utilize the result of b) to express the LTI inverse.)

From the result of b), we can consider the second system that takes $h(t)$ as the input and satisfies the following LCCDE

$$y(t)=\frac{dx(t)}{dt}+ax(t) \quad ---(1)$$

If we feed the impulse response $h(t)$ of the first system as the input to the second system, then its output becomes the unit impulse (from the result of (b)).

$$\delta(t)=\frac{dh(t)}{dt}+ah(t)$$

Therefore, the system satisfying (1) is the LTI inverse that reverts $h(t)$ to $\delta(t)$.

d) [10 pts] Using the result of a), find the impulse response of the system satisfying the following LCCDE and condition of initial rest:

$$\frac{d^2y(t)}{dt^2}+2a\frac{dy(t)}{dt}+a^2y(t)=x(t)$$
$$\Rightarrow\left(\frac{d}{dt}+a\right)\left[\left(\frac{d}{dt}+a\right)y(t)\right]=x(t)$$

(Hint: consider $\left(\frac{d}{dt}+a\right)y(t)=\frac{dy(t)}{dt}+ay(t)=w(t)$).

The given LCCDE can be considered as the sequential interconnection of two identical systems satisfying $\frac{dy(t)}{dt}+ay(t)=w(t)$, $\frac{dw(t)}{dt}+aw(t)=x(t)$.

The impulse response of each system can be found by solving the homogeneous DE and inhomogeneous auxiliary condition.

$$\frac{dh(t)}{dt}+ah(t)=0, \quad \text{where } h(0^+)=1$$

Therefore, $h(t)=e^{-at}u(t)$.

For the serial interconnection of two systems, its impulse response is given by the convolution of two impulse responses $h_2(t)=h(t)*h(t)$. Therefore, the impulse response of the total system $h_2(t)$ is identical to the $b=a$ case of problem a).

$$\therefore h_2(t)=h(t)*h(t)=te^{-at}u(t)$$

Problem 4. (25 pts)

The Fourier series coefficients of a periodic signal $\tilde{x}_T(t)$ with a fundamental period T are given by a_k at the frequencies $\omega_k=k\frac{2\pi}{T}$. Express the Fourier series coefficients of the following signals in terms of a_k.

a) [5 pts] $\tilde{x}_T\left(\frac{1}{2}t+1\right)$

Consider two steps of mappings: $x_1(t)=x(t+1)$, $x_2(t)=x_1\left(\frac{1}{2}t\right)$

From the first mapping, $b_k=a_ke^{jk\omega_0}=a_ke^{jk\frac{2\pi}{T}}$.

From the second mapping, $c_k=b_k$, but with $\omega_0'=\omega_0/2$.

Therefore, $c_k=a_ke^{jk\frac{2\pi}{T}}$.

b) [5 pts] $\tilde{x}_T^*(t-1)$

First mapping $x_1(t) = x(t-1)$: $b_k = a_k e^{-jk\frac{2\pi}{T}}$

Second mapping $x_2(t) = x_1(t)^*$: $c_k = b_{-k}^* = a_{-k}^* \left(e^{-j(-k)\frac{2\pi}{T}} \right)^* = a_{-k}^* e^{-jk\frac{2\pi}{T}}$

c) [5 pts] $\tilde{x}_T(t-T) + \tilde{x}_T(-t+T)$

$\tilde{x}_T(t-T)$: $b_k = a_k e^{-jk\frac{2\pi}{T}T} = a_k e^{-j2\pi k} = a_k$
$\tilde{x}_T(-t+T)$: $c_k = a_{-k}$
$\therefore b_k + c_k = a_k + a_{-k}$

d) [10 pts] $\tilde{x}_T(t)\widetilde{\mathrm{rect}}_T\left(\frac{2}{T}t\right)$

note: the periodic signal $\widetilde{\mathrm{rect}}_T\left(\frac{t}{L}\right) = \begin{cases} 1 & \text{for } |t| < \frac{L}{2} \\ 0 & \text{for } \frac{L}{2} < |t| < \frac{T}{2} \end{cases}$ for the first period.

(1) The Fourier series coefficients of $\widetilde{\mathrm{rect}}_T\left(\frac{t}{L}\right)$ are given by $b_k = \frac{L}{T}\mathrm{sinc}\left(k\frac{L}{T}\right)$.

In this case, $L = T/2$. So, $b_k = \frac{1}{2}\mathrm{sinc}\left(\frac{1}{2}k\right)$.

(2) The multiplication of two signals in time = convolution of FS coefficients with respect to k.

$$c_k = \sum_{m=-\infty}^{\infty} a_{k-m} b_m = \frac{1}{2}\sum_{m=-\infty}^{\infty} a_{k-m}\mathrm{sinc}\left(\frac{m}{2}\right)$$
$$= \sum_{m=-\infty}^{\infty} a_m b_{m-k} = \frac{1}{2}\sum_{m=-\infty}^{\infty} a_m \mathrm{sinc}\left(\frac{m-k}{2}\right)$$

where $\mathrm{sinc}\left(\frac{m}{2}\right) = \dfrac{\sin\frac{\pi m}{2}}{\frac{\pi m}{2}} = \frac{2}{\pi m}\sin\left(\frac{\pi m}{2}\right)$

Problem 5. (15 pts) Consider the following system described by a LCCDE:

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt} + x(t)$$

(a) [5 pts] Derive the output $y(t)$ of the system for the input signal $x(t) = e^{j\omega t}$.

For $x(t) = e^{j\omega t}$, let $y(t) = H(j\omega)e^{j\omega t}$,

$$j\omega H(j\omega)e^{j\omega t} = (j\omega + 1)e^{j\omega t}$$

$$\therefore H(j\omega) = \frac{j\omega + 1}{j\omega}$$

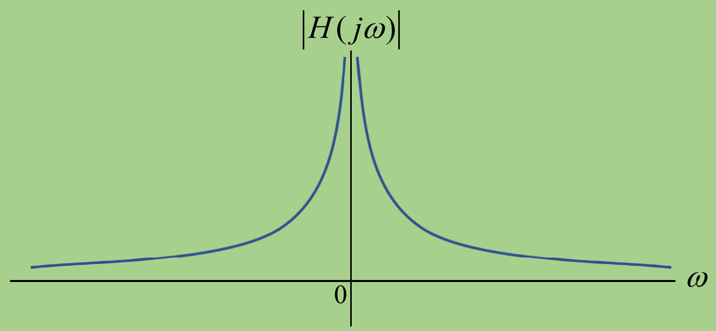
$$y(t) = \frac{j\omega + 1}{j\omega} e^{j\omega t}$$

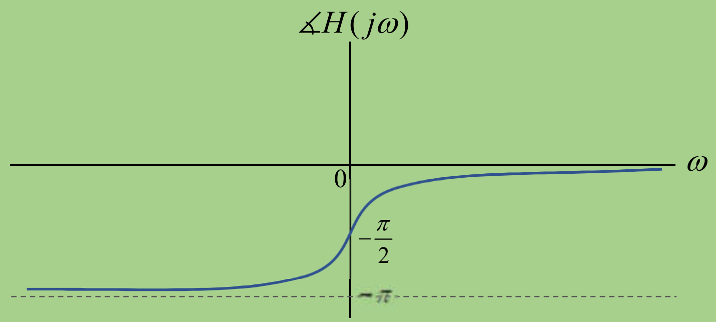
(b) [10 pts] From the result of (a), determine the frequency response $H(j\omega)$ of the system. Also, **Derive** and **plot** its **magnitude** response $|H(j\omega)|$ and **phase** response $\measuredangle H(j\omega)$ with respect to ω ($\omega \geq 0$) in two separate graphs.

(note: mark the asymptotic values of $\measuredangle H(j\omega)$ for $\omega \to 0, \infty$)

$$|H(j\omega)| = \sqrt{1 + \frac{1}{\omega^2}},$$

$$\begin{aligned}\measuredangle H(j\omega) &= \measuredangle \text{numerator of } H(j\omega) - \measuredangle \text{denominator of } H(j\omega) \\ &= \tan^{-1}(\omega) - \measuredangle j\omega \\ &= \tan^{-1}(\omega) - \frac{\pi}{2} \quad (\measuredangle j = \measuredangle e^{j\frac{\pi}{2}} = \frac{\pi}{2})\end{aligned}$$





Plotting for the region $\omega > 0$ is considered as a correct answer.