[Score table]

Prob.	1-(1)				1-(2)		2-(1)	2-(2)			3-(1)			3-(2)		4	Ext1	Ext2	Ext3
	a	b	c	d	a	b		a	b	c	a	b	c	a	b				
Total	3	3	3	3	4	4	10	10	10	5	10	10	5	5	10	5	5	10	5

Problems & Solutions

[Chapter 1 | 20 points]

(1) Determine whether or not each of the following signals or systems satisfies the given property. Justify your answers. **[3 pts each, total 12 pts]**

(a) Periodic? $\mathcal{EV}\{\cos(\pi(|n|+1))\}$ ($\mathcal{EV}$: even function)

$f(n) = \cos(\pi|n|+\pi) = \mathrm{Re}\left[e^{j\pi|n|}e^{j\pi}\right] = -(-1)^{|n|}$

This is an even function ($f(-n) = f(n)$), so $\mathcal{EV}\{f(n)\} = f(n)$.

$f(0) = -1$

$f(1) = 1, \quad f(-1) = 1$ → periodic with period 2 **Answer: periodic**

$f(n) = f(n+2)$

(b) Linear? $y(t) = a|x(t)|$ (a : arbitrary complex constant)

$y_1(t) = a|bx(t)| \neq b(a|x(t)|) = by(t)$: not linear **Answer: nonlinear**

(c) Causal? $y(t) = \int_{-\infty}^{\infty} u(\tau+1)x(t-\tau)d\tau$

$y(t) = u(t+1) * x(t) = \int_{-\infty}^{\infty} u(\tau+1)x(t-\tau)\,\mathrm{d}\tau$ ($*$: convolution)

This convolution integral requires the future information of $x(t)$ (i.e., $x(t-\tau)$ for $\tau < 0$), because $u(\tau+1) \neq 0$ for $-1 \le \tau < 0$ **Answer: non-causal**

(d) Time invariant? $y[n] = \sum_{k=-\infty}^{\infty} (1+(-1)^k)x[n-k]$

$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k], \quad \text{with } h[k] = 1+(-1)^k$

➔ LTI because the output is described by the convolution sum of the input signal. **Answer: time-invariant**

(2) Let z denotes a complex variable ($z = x + jy = re^{j\theta}$). Then the complex conjugate is defined as $z^* = x - jy = re^{-j\theta}$.

(a) **[4 pts]** Derive the following relation: $\measuredangle\left(1/z\right) = \measuredangle z^*$ ($\measuredangle z = \theta$)

$$1/z = \frac{1}{x+jy} = \frac{x-jy}{(x+jy)(x-jy)} = \frac{x-jy}{|x^2+y^2|} = \frac{z^*}{|z|^2} \quad \rightarrow \quad \therefore \measuredangle(1/z) = \measuredangle z^*$$

(b) **[4 pts]** Express the magnitude $|z|$ and phase $\measuredangle z$ of $z = \dfrac{j\omega}{1+j\omega}$ in terms of ω.

$$|z| = \sqrt{zz^*} = \sqrt{\frac{j\omega}{1+j\omega}\frac{-j\omega}{1-j\omega}} = \sqrt{\frac{\omega^2}{1+\omega^2}} = \frac{\omega}{\sqrt{1+\omega^2}}$$

$$z = \frac{j\omega}{1+j\omega}\frac{1-j\omega}{1-j\omega} = \frac{\omega^2+j\omega}{1+\omega^2} \quad \rightarrow \quad \measuredangle z = \tan^{-1}(\frac{\omega}{\omega^2}) = \tan^{-1}(\frac{1}{\omega}) \text{ or } \frac{\pi}{2} - \tan^{-1}(\omega) \text{ or } \cot(\omega)$$

[Chapter 2 | 35 points]

(1) **[10 pts]** For the following pairs of waveforms, use the convolution integral to find the response $y(t)$ of the LTI system with impulse response $h(t)$ to the input $x(t)$. Sketch your results.

$x(t) = at+b$, $h(t) = \frac{4}{3}(u(t)-u(t-1)) - \frac{1}{3}\delta(t-2)$ (a, b : positive real scalar)

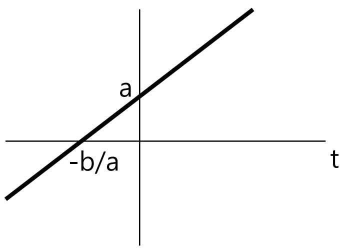
- Distributive property:

$$x(t)*(h_1(t)+h_2(t)) = x(t)*h_1(t) + x(t)*h_2(t)$$

$$(\ h_1(t) = \tfrac{4}{3}(u(t)-u(t-1)),\ h_2(t) = -\tfrac{1}{3}\delta(t-2)\ )$$

- Convolution of h_1 & h_2

$$\begin{aligned} x(t)*h_1(t) &= \tfrac{4}{3}\int_{-\infty}^{\infty} x(t-\tau)(u(\tau)-u(\tau-1))\,\mathrm{d}\tau \\ &= \tfrac{4}{3}\int_0^1 x(t-\tau)\,\mathrm{d}\tau = \tfrac{4}{3}\int_0^1 [a(t-\tau)+b]\,\mathrm{d}\tau \\ &= \frac{4}{3}\left(-\frac{a}{2}\tau^2 + (at+b)\tau \Big|_{\tau=0}^{\tau=1} \right) \\ &= \frac{4}{3}\left(at - \frac{a}{2} + b \right) \end{aligned}$$



$$\begin{aligned} x(t)*h_2(t) &= x(t) * -\tfrac{1}{3}\delta(t-2) \\ &= -\frac{1}{3}x(t-2) \\ &= -\frac{1}{3}\left(a(t-2)+b\right) \end{aligned}$$

$$\therefore x(t)*h_1(t) + x(t)*h_2(t) = at+b$$

(2) Answer to the following questions.

(2-1) **[10 pts]** Find the impulse response $h[n]$ of a system described by the following difference equation with the condition of initial rest($y[n]=0$ for $n<n_0$ *if* $x[n]=0$ for $n<n_0$).

$$y[n]-2y[n-1]=x[n]$$

- Impulse response: $y[n]$ for $x[n]=\delta[n]$ → $y[n]-2y[n-1]=\delta[n]$
- Condition of initial rest: $\delta[n]=0$ for $n<0$ → $y[n]=0$ for $n<0$

$$y[0]-\underbrace{2y[-1]}_{=0\ \text{(initial rest)}}=\delta[0]=1 \quad\rightarrow\quad y[0]=1$$

$$y[1]-2\underbrace{y[0]}_{=1}=\delta[1]=0 \quad\rightarrow\quad y[1]=2$$

$$y[2]-2\underbrace{y[1]}_{=12}=\delta[2]=0 \quad\rightarrow\quad y[2]=4$$

$$\begin{array}{c} y[0]=1 \\ y[n]=2y[n-1] \quad \text{for } n\geq 1 \end{array} \quad\leftrightarrow\quad h[n]=2^n u[n]$$

(2-2) **[10 pts]** Determine the frequency response $H(e^{j\omega})$ of the system described in (2-1). [*Hint*: use the eigenfunction $x[n]=e^{j\omega n}$ to find out the response $y[n]$, and determine the frequency response using this result.]

$$\left.\begin{array}{r} x[n]=e^{j\omega n} \\ y[n]=H(e^{j\omega})e^{j\omega n} \end{array}\right\} \rightarrow H(e^{j\omega})(1-2e^{-j\omega})e^{j\omega n}=e^{j\omega n}$$

$$\therefore H(e^{j\omega})=\frac{1}{1-2e^{-j\omega}}$$

(2-3) **[5 pts]** Discuss whether this system is stable or not.
Determine whether the system's response is FIR or IIR. Justify your answer.

For $h[n]=2^n u[n]$, $\sum_{n=-\infty}^{\infty}|h(n)|$ diverges. → $\therefore$ Not stable
System has an impulse response of infinite length → IIR

[Chapter 3 | 40 pts]

(1) **[25 pts]** Suppose that the Fourier series coefficients of the signal $\tilde{x}_T(t)$ shown in Figure 3-1 is given by a_k.

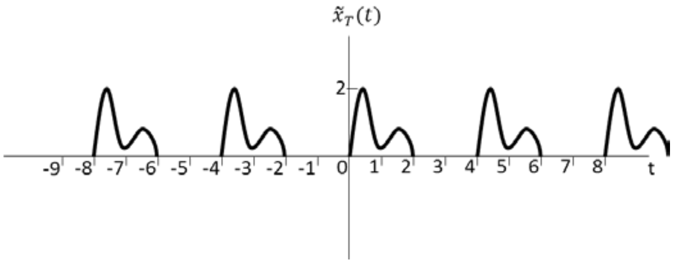


Figure 3-1

(a) [**10 pts**] Express the Fourier series coefficients b_k of the following signal in terms of a_k.

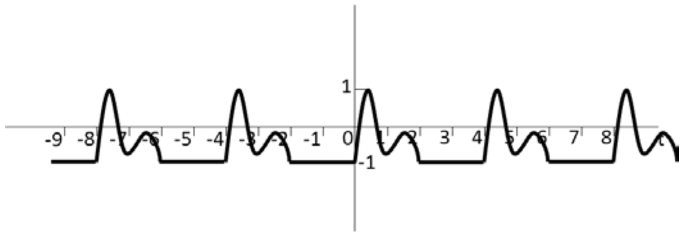


Figure 3-2

$x_1(t) = x(t) - 1$, $T = 4$ (fundamental period)

From the linear property of Fourier series expansion: $\underbrace{\mathcal{FS}[x_1(t)]}_{b_k} = \underbrace{\mathcal{FS}[x(t)]}_{a_k} - \underbrace{\mathcal{FS}[1]}_{c_k}$

$$\mathcal{FS}[1] = c_k = \begin{cases} \frac{1}{4}\int_0^4 1e^{j0\omega_0 t}dt = 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

$$b_k = a_k - c_k = \begin{cases} a_0 - 1 & \text{for } k = 0 \\ a_k & \text{for } k \neq 0 \end{cases}$$

(b) [**10 pts**] Express the Fourier series coefficients b_k of the following signal in terms of a_k.

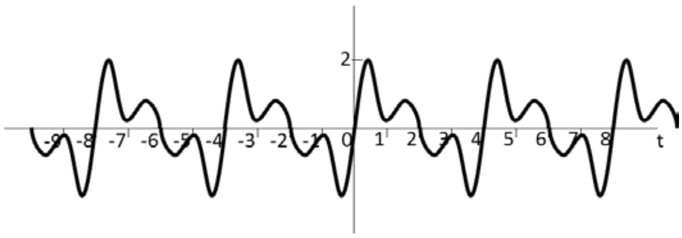


Figure 3-3

$x_1(t) = x(t) - x(-t)$ → $x(t) \overset{\mathcal{FS}}{\leftrightarrow} a_k$, $x(-t) \overset{\mathcal{FS}}{\leftrightarrow} a_{-k}$ (flipping) $\quad \therefore b_k = a_k - a_{-k}$

Another solution: because $x(t)$ is real, $a_{-k} = a_k^*$ → $b_k = a_k - a_k^* = 2\,\mathrm{Im}\{a_k\}$

(c) [**5 pts**] Let $A = \int_0^4 |x(t)|^2\,dt$ for the curve shown in Fig. 3.1. Express $\sum_{k=-\infty}^{\infty} |a_k|^2$ in terms of A.

From Parseval's relation, $\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \quad \rightarrow \quad \sum_{k=-\infty}^{\infty} |a_k|^2 = \frac{A}{4}$

(2) **[10 pts]** Solve the following two problems

(a) **[5 pts]** For the discrete-time periodic signal $x[n] = 2 + \cos\left(\frac{2\pi}{3}n\right) + 4\sin\left(\frac{5\pi}{3}n\right)$, determine the fundamental period N of this signal and the Fourier series coefficients a_k such that

$$x[n]=\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \quad .$$

$$x[n]=\underbrace{2}_{periodic\,with 1}+\underbrace{\cos\left(\frac{2\pi}{3}n\right)}_{periodic\,with\,3}+\underbrace{4\sin\left(\frac{5\pi}{3}n\right)}_{periodic\,\,with\,6}$$

$N=$ the smallest common multiple = 6

$$x[n]=2e^{j0\left(\frac{2\pi}{6}\right)n}+\frac{1}{2}\left(e^{j2\left(\frac{2\pi}{6}\right)n}+e^{-j2\left(\frac{2\pi}{6}\right)n}\right)+\frac{4}{2j}\left(e^{j5\left(\frac{2\pi}{6}\right)n}-e^{-j5\left(\frac{2\pi}{6}\right)n}\right)$$

$a_{0+6m}=2$, $a_{2+6m}=a_{-2+6m}=0.5$, $a_{5+6m}=-2j$, $a_{-5+6m}=2j$, $a_{k+6m}=0$ otherwise (for arbitrary integer m)

<Note that the Fourier series coefficients of DT signal are periodic with the period N >

(b) **[10 pts]** Suppose that the frequency response of a system is given by $H(e^{j\omega})=\dfrac{1}{1-2e^{-6j\omega}}$. Determine the output $y[n]$ of this system for the input signal $x[n]$ given by (2-(a)).

$e^{j\omega_k n}=e^{jk\omega_0 n}=e^{jk\left(\frac{2\pi}{N}\right)n}$ is the eigenvector of the LTI system,

so the output becomes $y[n]=\sum_k H(e^{j\omega_k})(a_k e^{j\omega_k n})$ for $x[n]=\sum_k a_k e^{j\omega_k n}$

$$a_0=2,\ a_2=0.5,\ a_{-2}=0.5,\ a_1=2j,\ a_{-1}=-2j,\ a_3=a_{-3}=0$$

$$\omega_0=0,\omega_1=\frac{\pi}{3},\omega_2=\frac{2\pi}{3},\omega_3=\pi$$

$$y[n]=(-1)2e^{j0\left(\frac{2\pi}{6}\right)n}+(-1)\frac{1}{2}\left(e^{j2\left(\frac{2\pi}{6}\right)n}+e^{-j2\left(\frac{2\pi}{6}\right)n}\right)+(-1)\frac{4}{2j}\left(e^{j5\left(\frac{2\pi}{6}\right)n}-e^{-j5\left(\frac{2\pi}{6}\right)n}\right)=-x[n]$$

[Chapter 4 | 5 pts]

Use the Fourier transform analysis equation $X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$ to calculate the Fourier transform of $e^{-2|t-1|}$.

Problem of HW #6

$$\begin{aligned}X(j\omega)&=\int_{-\infty}^{\infty}e^{-2|t-1|}e^{-j\omega t}dt\\&=\int_{1}^{\infty}e^{-2(t-1)}e^{-j\omega t}dt+\int_{-\infty}^{1}e^{-2(1-t)}e^{-j\omega t}dt\\&=e^{2}\left.\frac{e^{(-2-j\omega)t}}{-2-j\omega}\right|_{1}^{\infty}+e^{-2}\left.\frac{e^{(2-j\omega)t}}{2-j\omega}\right|_{-\infty}^{1}\\&=e^{2}\frac{e^{-2-j\omega}}{2+j\omega}+e^{-2}\frac{e^{2-j\omega}}{2-j\omega}=e^{-j\omega}\left(\frac{1}{2+j\omega}+\frac{1}{2-j\omega}\right)=\frac{4}{4+\omega^{2}}e^{-j\omega}\end{aligned}$$

[Extra Problems] Choose and solve the following problems as you wish.
You can use extra pages to write down your answers.

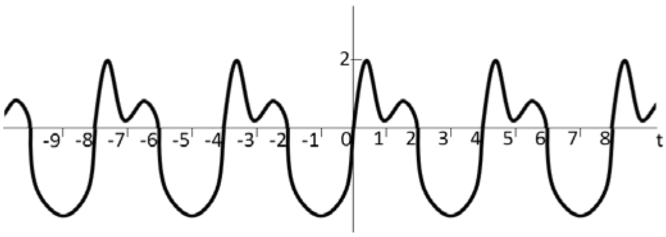
1. **[5 pts]** The Fourier series coefficients a_k of a periodic signal $\tilde{x}_T(t)$ of period T can be obtained by the following analysis equation:

$$a_k = \frac{1}{T}\int_{t_0}^{t_0+T} \tilde{x}_T(t)e^{-jk\omega_0 t}dt.$$

Show that the result of above integral is independent of t_0 for the periodic signal $\tilde{x}_T(t)$.

Refer to the solution on KLMS

2. **[10 pts]** Express the Fourier series coefficients a_k of Figure 3-1 using the Fourier series coefficients b_k of the following signal. ($a_k = f(b_k)$)



By denoting the new signal as $\tilde{y}_2(t)$, its Fourier series representation is given by

$$\tilde{y}_2(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\left(\frac{\pi}{2}\right)t}$$

The original signal can be expanded as

$$\tilde{x}_4(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{4}\right)t} \quad \text{(periodic with } T = 4\text{)}$$

The signal $\tilde{x}_4$ of Figure 3-1 is the multiplication of $\tilde{y}_2(t)$ and periodic square signal shifted by 1

$$\tilde{x}_2(t) = \tilde{y}_2(t)\tilde{r}_4(t-1)$$

where the rectangle function is defined as

$$\tilde{r}_T(t) = \begin{cases} 1 & \text{for } -T_0 \le t < T_0 \\ 0 & \text{for } -\dfrac{T}{2} < t < -T_0 \text{ or } T_0 < t < \dfrac{T}{2} \end{cases} \quad (T = 4,\ T_0 = 1 \text{ in this case}).$$

FS of the rectangular function: $\tilde{r}_T \overset{\mathcal{FS}}{\leftrightarrow} d_k = \dfrac{2T_0}{T}\text{sinc}\left(k\dfrac{2T_0}{T}\right) \rightarrow \frac{1}{2}\text{sinc}(\frac{k}{2}) = \dfrac{\sin(k\pi/2)}{k\pi}$

Time shifting property of FS: $\tilde{r}_T(t-1) \overset{\mathcal{FS}}{\leftrightarrow} d_k e^{-jk\omega_0} = \frac{1}{2}\text{sinc}(\frac{k}{2})e^{-jk\frac{\pi}{2}} = \frac{1}{2}\text{sinc}(\frac{k}{2})(-j)^k \equiv d_k'$

Multiplication – convolution property of FS: $\tilde{y}_T(t)\tilde{r}_T(t-1) \overset{\mathcal{FS}}{\leftrightarrow} \sum_{\ell=-\infty}^{\infty} b_{k-\ell} d'_\ell$

$$a_k = \sum_{\ell=-\infty}^{\infty} b_{k-\ell} d'_\ell = \tfrac{1}{2} \sum_{\ell=-\infty}^{\infty} b_{k-\ell} \operatorname{sinc}(\tfrac{\ell}{2})(-j)^\ell$$

$$= \tfrac{1}{2} b_k + \sum_{m=-\infty}^{\infty} \frac{b_{k-2m-1}}{j(2m+1)\pi}$$

(→ $\ell = 2m+1$ or 0)

3. **[5 pts]** Prove the following relation for a real-valued periodic signal $\tilde{x}_T(t)$ and its Fourier series coefficients a_k.

$$2\sum_{k=1}^{\infty} |a_k|^2 = \frac{1}{T}\int_0^T |\tilde{x}_T(t)|^2\, dt - \left|\frac{1}{T}\int_0^T \tilde{x}_T(t)dt\right|^2$$

From Parseval's relation, $\frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$ -- (1)

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = |a_0|^2 + \sum_{k=1}^{\infty} |a_k|^2 + \sum_{k=-\infty}^{-1} |a_k|^2$$

$$= |a_0|^2 + \sum_{k=1}^{\infty} |a_k|^2 + \sum_{k=1}^{\infty} |a_{-k}|^2$$

For a real-valued signal, $|a_{-k}| = |a_k^*| = |a_k|$ (conjugate symmetry)

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = |a_0|^2 + 2\sum_{k=1}^{\infty} |a_k|^2 \quad \text{-- (2)}$$

However, $a_0 = \frac{1}{T}\int_T \tilde{x}_T(t)dt$ (from analysis equation with $k=0$) --(3)

From (1), (2), (3), $\left|\frac{1}{T}\int_0^T \tilde{x}_T(t)dt\right|^2 + 2\sum_{k=1}^{\infty} |a_k|^2 = \frac{1}{T}\int_0^T |\tilde{x}_T(t)|^2\, dt$