HW9 solution

8.8 $f(y) = \frac{1}{\theta} e^{-\frac{y}{\theta}} \cdot (y>0)$, $E(Y) = \theta$, $var(Y) = \theta^2$

(a) $E(Y_1) = \theta$, $E\left(\frac{Y_1+Y_2}{2}\right) = \theta$, $E\left(\frac{Y_1+2Y_2}{3}\right) = \theta$, $E(\bar{Y}) = \frac{1}{n}\Sigma E(Y_i) = \theta$ $\Rightarrow$ U.E of θ

pdf of $Y_{(1)}$: $\frac{3}{\theta} \cdot e^{-\frac{y}{3}\theta}$ $\Leftrightarrow$ $E(Y_{(1)}) = \frac{\theta}{3}$ $\Rightarrow$ Bias estimator

(b) $var(Y_1) = \theta^2$, $var\left(\frac{Y_1+Y_2}{2}\right) = \frac{\theta^2}{2}$, $var\left(\frac{Y_1+2Y_2}{3}\right) = \frac{5}{9}\theta^2$, $\boxed{var(\bar{Y}) = \frac{\theta^2}{9}}$ $\Rightarrow$ smallest variance

8.11 $E((Y-3)^3) = E(Y^3) - 9E(Y^2) + 27 \cdot E(Y) - 27$

$\Rightarrow$ $\hat{\theta}_3 - 9\hat{\theta}_2 + 54$: U.E of $E((Y-3)^3)$

8.13 $Y \sim B(n,p)$, then $E(Y) = np$, $V(Y) = np(1-p)$

(a) $E\left(Y - \frac{Y^2}{n}\right) = E(Y) - \frac{1}{n}E(Y^2)$

$= np - \frac{1}{n}(n^2p^2 + np(1-p)) = (n-1)p(1-p) \neq np(1-p)$ $\Rightarrow$ biased estimator

(b) Using (a), easy to check $\hat{\theta} = \frac{n}{n-1} \cdot \left(Y - \frac{Y^2}{n}\right)$: U.E of $V(Y)$

8.16 (a), (b) using $Y = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$,

pdf of χ^2 with df. n-1 $= Gamma\left(\frac{n-1}{2}, 2\right)$

$E(S) = E\left(\frac{\sigma}{\sqrt{n-1}} \cdot \sqrt{Y}\right) = \frac{\sigma}{\sqrt{n-1}} \cdot E(\sqrt{Y}) = \frac{\sigma}{\sqrt{n-1}} \cdot \int_0^\infty y^{\frac{1}{2}} \cdot f(y)\, dy$

$= \frac{\sigma}{\sqrt{n-1}} \cdot \frac{\sqrt{2} \cdot \Gamma(n/2)}{\Gamma((n-1)/2)}$. U.E of σ : $\frac{\sqrt{n-1} \cdot \Gamma((n-1)/2)}{\sqrt{2} \cdot \Gamma(n/2)}$ $(\hat{\sigma})$

(c) $\bar{Y} - z_\alpha \cdot \hat{\sigma}$

8.18 Using pdf of order statistic (see lecture note),

pdf of $Y_{(1)}$ is given by $\frac{n}{\theta} \cdot \left(1 - \frac{y}{\theta}\right)^{n-1}$, $0 \le y \le \theta$

$E(Y_{(1)}) = \int_0^\theta \frac{ny}{\theta} \cdot \left(1 - \frac{y}{\theta}\right)^{n-1} dy = \frac{\theta}{n+1}$, $(n+1)Y_{(1)}$ is U.E of θ

8.23 (a) $\bar{x} \pm 2 \cdot \frac{s}{\sqrt{n}}$ $\Rightarrow$ $11.3 \pm 2 \times \frac{16.6}{\sqrt{467}}$

(b) $(\bar{Y}_c - \bar{Y}_R) \pm 2 \cdot \sqrt{\frac{S_c^2}{n_c} + \frac{S_R^2}{n_R}}$ $\Rightarrow$ $-1.3 \pm 2 \cdot \sqrt{\frac{104.04}{467} + \frac{91.04}{191}}$

(c) Similar as (b), $\hat{P}_c - \hat{P}_R \pm 2 \cdot \sqrt{\frac{\hat{P}_c(1-\hat{P}_c)}{n_c} + \frac{\hat{P}_R(1-\hat{P}_R)}{n_R}}$ $\Rightarrow$ $0.17 \pm 2 \cdot 0.08$

8.27 (a) $2\cdot\sqrt{\frac{0.601\cdot 0.399}{985}} = 0.031$

(b) by (a), 95% C.I is given as $(0.57, 0.63)$

$\Rightarrow$ The Republican candidate will be elected because he is expected to win more than half the votes.

(c) Sampling bias ... and more reasons

8.34 $\bar{Y}$ is U.E of λ. Hence we can use sample mean to estimate λ.

$\sqrt{V(\bar{Y})} = \sqrt{\lambda/n}$ is U.E of standard error.

8.38 (a). $E(\hat{\theta}) = \frac{1-p}{p^2}$, $\hat{\theta} = \frac{Y^2-Y}{2} \Rightarrow$ U.E of $V(Y)$

(b) $\sqrt{\widehat{V(Y)}} = \frac{\sqrt{1-\hat{p}}}{\hat{p}}$, $\hat{p} = \frac{1}{Y} \Leftrightarrow \sqrt{\widehat{V(Y)}} = \sqrt{Y^2-Y}$

Hence $2.se = 2\sqrt{Y^2-Y}$ is bound of the error estimation.

8.44 (a) easy to check

(b) Let $U = Y/\theta$, $F(U) = P(U \le u) = P(Y \le u\theta) = F_Y(u\theta)$

$\therefore 2u(1-u) \Rightarrow$ does not depend on unknown θ

(c) $P(U \le \alpha) = 0.9 \Leftrightarrow 2\alpha(1-\alpha) = 0.9$ then $\alpha \simeq 0.6838$

Hence 90% lower bound of $\theta = \frac{Y}{0.6838}$

8.60 (a) $\bar{x} \pm z_{0.005}\cdot\frac{6}{\sqrt{n}} \Leftrightarrow 98.25 \pm 2.576\cdot\frac{0.73}{\sqrt{130}}$

(b) not contain. It is possible that the standard for normal is not valid.

8.63 (a) $\hat{p} \pm z_{0.025}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \Leftrightarrow 0.78 \pm 1.96\cdot\sqrt{\frac{0.78\cdot 0.22}{1000}}$

(b) $1.96\cdot\sqrt{\frac{0.78\cdot 0.22}{1000}} = 2.6\% \neq 3.1\%$.

If $p = 0.5$, $1.96\cdot\sqrt{\frac{0.5\times 0.5}{1000}} \simeq 3.1\%$

8.68 (a) $var(Y_1 - Y_2) = var(Y_1) + var(Y_2) - 2\,cov(Y_1, Y_2)$
$= np_1(1-p_1) + np_2(1-p_2) + 2np_1p_2$

(b) $\hat{p}_1 = 0.06$, $\hat{p}_2 = 0.16$ under 95% confidence coefficient,

$\hat{p}_1 - \hat{p}_2 \pm z_{0.025} \cdot \sqrt{var(\hat{p}_1 - \hat{p}_2)} \iff 0.06 - 0.16 \pm \sqrt{\frac{0.06\times0.94 + 0.16\times0.84 + 2\times0.06\times0.16}{500}}$

8.74. 95% CI $\rightarrow$ $\mu \pm 0.1 \iff \bar{Y} \pm z_{0.025} \cdot \frac{\sigma}{\sqrt{n}}$

Then $\sqrt{n} \geq 1.96 \times \frac{0.5}{0.1} = 9.8$

$\therefore n = 97$

Because of bias, it is not valid.

8.88 $t_{11,0.05} = 1.796$, 90% CI of μ is $9 \pm 1.796 \cdot \frac{6.4}{\sqrt{12}}$ ($\bar{x} = 9$, $s = 6.4$ $\Rightarrow$ easy to compute)

8.93 (a) $2\bar{X} + \bar{Y} \sim N\left(2\mu_1 + \mu_2, \frac{4\sigma^2}{n} + \frac{3\sigma^2}{m}\right)$. σ^2 is known,

95% C.I is $2\bar{X} + \bar{Y} \pm z_{0.025} \cdot \sqrt{\frac{4\sigma^2}{n} + \frac{3\sigma^2}{m}}$

(b) $\frac{(n-1)S_X^2}{\sigma^2} + \frac{(m-1)S_Y^2}{\sigma^2/3} \sim \chi^2(n+m-2)$, $S_X = \frac{1}{n-1}\Sigma(X_i - \bar{X})^2$, $S_Y = \frac{1}{m-1}\Sigma(Y_i - \bar{Y})^2$

by using $T = \frac{2\bar{X} + \bar{Y} - (2\mu_1 + \mu_2)}{\hat{\sigma} \cdot \sqrt{\frac{4}{n} + \frac{3}{m}}}$, $\hat{\sigma}^2 = \frac{(n-1)S_X^2 + 3(m-1)S_Y^2}{n+m-2}$

(cf) $T = \frac{Z \sim N(0,1)}{\sqrt{V/r}}$, $V \sim \chi^2(r)$

95% C.I is $2\bar{X} + \bar{Y} \pm t_{0.025} \cdot \hat{\sigma} \cdot \sqrt{\frac{4}{n} + \frac{3}{m}}$

8.96 $n = 10$, $s^2 = 63.5$. Then $\chi^2_{0.95} = 3.3251$ $\chi^2_{0.05} = 16.9190$

The 90% C.I for σ^2 is $(33.79, 171.90)$