# Homework 3

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## Problem 3.50

A missile protection system consists of n radar sets operating independently, each with a probability of .9 of detecting a missile entering a zone that is covered by all of the units.

*sol a)*

missile rader 5개가 각각의 확률이 0.9로 감지한다고 한다. 먼저 rader 4개가 감지할 확률을 구하면 다음과 같다.

$P(Y = 4) = 5 \times 0.9^4 \times 0.1 = 0.32805$

다음으로 1개 이상의 rader가 감지할 확률을 구하면 다음과 같다.

$P(Y \geq 1) = 1 - P(Y = 0) = 1 - 0.1^5 = 0.99999$

*sol b)*

문제에서 구해야하는 것은 $n$개의 rader가 있을 때 $P(Y \geq 1) \geq 0.999$를 만족하는 최소의 $n$을 구하는 것이다. $n$개의 rader가 있을 때 1개 이상이 감지할 확률을 구하면 다음과 같다.

$P(Y \geq 1) = 1 - \binom{n}{0} \times 0.1^n \geq 0.999$

따라서, 계산을 하면 최소의 $n$은 3임을 알 수 있다.

## Problem 3.55

Suppose that Y is a binomial random variable with n ¿ 2 trials and success probability p. Use the technique presented in Theorem 3.7 and the fact that $E\{Y(Y-1)(Y-2)\} = E(Y^3) - 3E(Y^2) + 2E(Y)$ to derive $E(Y^3)$.

*sol)*

$Y$의 경우에는 $binomial$한 상황이다. $n \geq 2$에 대해 식을 쓰면 다음과 같다.

$$
\begin{aligned}
E(Y(Y-1)(Y-2)) &= \sum_{y=0}^{n} y(y-1)(y-2)\binom{n}{y}p^y(1-p)^{n-y} \\
&= \sum_{y=0}^{n} \frac{y(y-1)(y-2)n!}{y!(n-y)!}p^y(1-p)^{n-y} \\
&= \sum_{y=3}^{n} \frac{n(n-1)(n-2)(n-3)!}{(y-3)!(n-3-(y-3))!}p^y(1-p)^{n-y} \\
&= n(n-1)(n-2)\sum_{y=3}^{n} \frac{(n-3)!}{(y-3)!(n-3-(y-3))!}p^y(1-p)^{n-y} \\
&= n(n-1)(n-2)p^3\sum_{x=0}^{n-3} \frac{(n-3)!}{(x)!(n-3-x)!}p^x(1-p)^{n-3-x} \\
&= n(n-1)(n-2)p^3 = E(Y^3) - 3E(Y^2) + 2E(Y)
\end{aligned}
$$

따라서, 이를 바탕으로 $E(Y^3)$을 구하면 다음과 같다.

$E(Y^3) = 3n(n-1)p^2 + n(n-1)(n-2)p^3 - 2np$

## Problem 3.62

Goranson and Hall (1980) explain that the probability of detecting a crack in an airplane wing is the product of p1, the probability of inspecting a plane with a wing crack; p2, the probability of inspecting the detail in which the crack is located; and p3, the probability of detecting the damage.

*sol a)*

확률을 곱으로 표현할 수 있는 이유는 세 가지 events의 indepnedence를 가정했기 때문입니다.

*sol b)*

$p_1 = 0.9$, $p_2 = 0.8$, $p_3 = 0.5$이므로 $p_{detect} = p_1 \times p_2 \times p_3 = 0.36$이다.

세 가지에 대해 binomial 하므로,

$\Rightarrow\ P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{3}{0} \times 0.64^3 = 0.737856$이다.

## Problem 3.66

Suppose that Y is a random variable with a geometric distribution. Show that

a) $\sum_y p(y) = \sum_{y=1}^{\infty} q^{y-1}p = 1$

b) $\frac{p(y)}{p(y-1)} = q$ for y=2,3,... This ratio is less than 1, implying that the geometric probabilities are monotonically decreasing as a function of y. If Y has a geometric distribution, what value of Y is the most likely (has the highest probability)?

*sol a)*

정의에 의해,

$\sum_y p(y) = \sum_{y=1}^{\infty} = q^{y-1}p$이다.

$$\sum_{y=1}^{\infty} q^{y-1}p = p\sum_{y=1}^{\infty} q^{y-1} = p\sum_{x=0}^{\infty} q^x = \frac{p}{1-q} = 1$$

*sol b)*

$\frac{p(y)}{p(y-1)} = \frac{q^{y-1}p}{q^{y-2}p} = q$로 정리할 수 있다.

또한 monotonically decreasing as a function of y인 event에서 가장 크게 할려면, $Y = 1$이여야한다.

## Problem 3.70

An oil prospector will drill a succession of holes in a given area to find a productive well. The probability that he is successful on a given trial is .2.

a) What is the probability that the third hole drilled is the first to yield a productive well?

b) If the prospector can afford to drill at most ten wells, what is the probability that he will fail to find a productive well?

*sol a)*

$p_{success} = 0.2$

따라서, $0.8^2 \times 0.2 = 0.128$이다.

*sol b)*

이 경우에는 다 실패했다고 생각하면, $0.8^{10} = 0.10737$이다.

## Problem 3.79

If Y has a geometric distribution with success probability .3, what is the largest value, y0, such that $P(Y > y0) \geq .1$?

*sol)*

n번째에서 첫 번째 앞면을 얻는 확률을 생각해보자.

$P(Y = n) = \frac{1}{2}^{n-1}\frac{1}{2} = \frac{1}{2}^n$

$E(Y) = \frac{1}{p} = 2$가 됨을 알 수 있다.

## Problem 3.80

Two people took turns tossing a fair die until one of them tossed a 6. Person A tossed first, B second, A third, and so on. Given that person B threw the first 6, what is the probability that B obtained the first 6 on her second toss (that is, on the fourth toss overall)?

*sol)*

주사위에서 6의 눈을 넘겨줄 때 까지 진행되는 게임으로 먼저 B가 6을 넘길 확률을 구해보면 된다.
즉, 2,4,6,8... 번째의 게임에서 6을 넘겨줄 확률을 생각하면 된다.
problem 3.77에 의해, $P(Y = oddint) = \frac{p}{1-q^2}$임을 이용할 수 있다.
$P(Bpass6) = 1 - P(Y = 1, 3, 5, ...) = 1 - p\frac{1}{1-q^2} = \frac{5}{11}$이다.
이 중, Y=4인 것을 고려한 조건부 확률을 계산하면 다음과 같다.
$P(Y = 4|pass\,6) = \frac{\frac{25}{36} \times \frac{1}{6}}{\frac{5}{11}} = 0.25463$

## Problem 3.85

Find $E(Y(Y - 1))$ for a geometric random variable Y by finding $\sum_{y=1}^{\infty} \frac{d^2}{dq^2}(q^y)$ Use this result to find the variance of Y .

*sol)*

$E(Y(Y - 1))$을 찾아보자. 이때, Y는 geometric random variable이다.
또한, $\frac{d^2}{dq^2}(q^y) = y(y - 1)q^{y-2}$이다.
즉, $\frac{d^2}{dq^2}(\sum q^y) = \sum_{y=2}^{\infty} y(y - 1)q^{y-2}$

$$
\begin{aligned}
E(Y(Y-1)) &= \sum_{y=1}^{\infty} y(y-1)q^{y-1}p \\
&= pq\sum_{y=1}^{\infty} y(y-1)q^{y-2} \\
&= pq\sum_{y=2}^{\infty} y(y-1)q^{y-2} \\
&= pq(\frac{d^2}{dq^2}(\sum_{y=2}^{\infty} q^y)) \\
&= pq(\frac{d^2}{dq^2}(\frac{1}{1-q} - (1-q))) \\
&= \frac{2pq}{(1-q)^3} = \frac{2q}{p^2}
\end{aligned}
$$

이를 이용해서 $V(Y) = E(Y(Y - 1)) + E(Y) - E(Y)^2$이니 정리하면,
$V(Y) = \frac{q}{p^2}$가 된다.

## Problem 3.88

If Y is a geometric random variable, define $Y^* = Y - 1$. If Y is interpreted as the number of the trial on which the first success occurs, then $Y^*$ can be interpreted as the number of failures before the first success. If $Y^* = Y - 1, P(Y^* = y) = P(Y - 1 = y) = P(Y = y + 1)$ for y = 0, 1, 2,.... Show that $P(Y^* = y) = q^y p$

*sol)*

$P(Y^* = y) = P(Y - 1 = y) = P(Y = y + 1)$이므로,

$P(Y^* = y) = P(Y - 1 = y) = P(Y = y + 1) = q^{y+1-1}p = q^y p$로 정리된다.

따라서, $P(Y^* = y) = q^y p$이다.

## Problem 3.96

The telephone lines serving an airline reservation office are all busy about 60% of the time.

a) If you are calling this office, what is the probability that you will complete your call on the first try? The second try? The third try?

b) If you and a friend must both complete calls to this office, what is the probability that a total of four tries will be necessary for both of you to get through?

*sol a)*

첫 번째 시도에 성공할 확률 : 0.6

두 번째 시도에 성공할 확률 : $0.4 \times 0.6 = 0.24$

세 번째 시도에 성공할 확률 : $0.4^2 \times 0.6 = 0.144$

*sol b)*

둘이 합쳐서 시도가 4번 걸리는 경우를 묻고 있으므로 이는 negative binomial distribution을 활용하면 된다. r=2, p=0.4인 경우로 계산하면 다음과 같다.

$P(Y = 4) = \binom{3}{1} \times 0.4^2 \times 0.6^2 = 0.1728$

## Problem 3.118

Five cards are dealt at random and without replacement from a standard deck of 52 cards. What is the probability that the hand contains all 4 aces if it is known that it contains at least 3 aces?

*sol)*

52장의 카드에서 5장을 비복원추출하는 상황으로 간주할 수 있다. 먼저, 3 aces를 뽑을 확률을 구하면

다음과 같다.
$P(Y=3) = \frac{\binom{4}{3}\times\binom{48}{2}}{\binom{52}{5}}$
다음으로. 4장 다 aces를 뽑을 확률을 구하면 다음과 같다.
$P(Y=4) = \frac{\binom{4}{4}\times\binom{48}{1}}{\binom{52}{5}}$
따라서 조건부 확률을 구하면, $P(Y=4\,|\,Y\geq 3) = 0.010527$이다.

## Problem 3.120

The sizes of animal populations are often estimated by using a capture–tag–recapture method. In this method k animals are captured, tagged, and then released into the population. Some time later n animals are captured, and Y , the number of tagged animals among the n, is noted. The probabilities associated with Y are a function of N, the number of animals in the population, so the observed value of Y contains information on this unknown N. Suppose that k = 4 animals are tagged and then released. A sample of n = 3 animals is then selected at random from the same population. Find P(Y = 1) as a function of N. What value of N will maximize P(Y = 1)?

*sol*)

population은 N개 이고, sample은 k개, unsampled는 N-k개가 존재한다.
전체 뽑는 경우의 수는 $\binom{N}{3}$이다. $P(Y=1)$을 계산하면 다음과 같다.
$P(Y=1) = \frac{\binom{4}{1}\times\binom{N-4}{2}}{\binom{N}{3}} = \frac{12(N-4)(N-5)}{N(N-1)(N-2)}$
위와 같은 식에서 최대가 되도록 하는 N 값을 찾으면 11, 12임을 파악할 수 있다.

## Problem 3.124

Approximately 4% of silicon wafers produced by a manufacturer have fewer than two large flaws. If Y , the number of flaws per wafer, has a Poisson distribution, what proportion of the wafers have more than five large flaws? [Hint: Use Table 3, Appendix 3.]

*sol*)

Table 3, Appendix 3.의 Poisson distribution을 참조하게 되면, 문제의 상황은 $P(Y\leq 2) = 0.04, \lambda = 6.6$인 상황이다.
또한, Table에 의하면, $P(Y>5) = 1 - P(Y\leq 5) = 1 - 0.355 = 0.645$로 계산된다.

## Problem 3.130

A parking lot has two entrances. Cars arrive at entrance I according to a Poisson distribution at an average of three per hour and at entrance II according to a Poisson distribution at an average of four per hour. What is the probability that a total of three cars will arrive at the parking lot in a given hour? (Assume that the numbers of cars arriving at the two entrances are independent.)

*sol*)

유형 I의 경우에는 $\lambda = 3$인 case이고, 유형 II의 경우에는 $\lambda = 4$인 case로 생각하고 접근하면 된다.
$P(Y = 3) = P(Y_I = 0, Y_{II} = 3) + P(Y_I = 1, Y_{II} = 2) + ... + P(Y_I = 3, Y_{II} = 0)$인데, independent 하므로, $P(Y = 3) = P(Y_I = 0)P(Y_{II} = 3) + P(Y_I = 1)P(Y_{II} = 2) + ... + P(Y_I = 3)P(Y_{II} = 0)$ 이다. Poisson distribution을 계산하게 되면 그 값은 0.0521이 된다.

## Problem 3.136

Increased research and discussion have focused on the number of illnesses involving the organism Escherichia coli (10257:H7), which causes a breakdown of red blood cells and intestinal hemorrhages in its victims (http://www.hsus.org/ace/11831, March 24, 2004). Sporadic outbreaks of E.coli have occurred in Colorado at a rate of approximately 2.4 per 100,000 for a period of two years.

a) If this rate has not changed and if 100,000 cases from Colorado are reviewed for this year, what is the probability that at least 5 cases of E.coli will be observed?

b) If 100,000 cases from Colorado are reviewed for this year and the number of E.coli cases exceeded 5, would you suspect that the state's mean E.coli rate has changed? Explain.

*sol*)

$\lambda = 2.4$인 Poisson distribution case이다. Table 3, Appendix 3.의 Poisson distribution을 참조하게 되면 다음을 얻을 수 있다.

a) $P(Y \geq 5) = 1 - P(\leq 4) = 1 - 0.904 = 0.096$이다.

b) $P(Y > 5) = 1 - P(\leq 5) = 1 - 0.964 = 0.036$이다.

이와 같은 경우에서는 이 사건과 관련된 확률이 작기 때문에 아마도 비율 역시 바뀌었을 것이라는 추정을 할 수 있다.