EE202A Signals and Systems, Fall 2022
Oct 19, 2022
Time Limit: 13:00 - 15:45 Lecturer: Jung-Woo Choi

Score Table (for teacher use only)

Question:	1	2	3	4	5	Total
Points:	15	15	25	25	20	100
Score:						

This is a CLOSED-BOOK exam.
Please provide ALL DERIVATIONS and EXPLANATIONS with your answers.
Any communication with others during the exam will be regarded as a cheating case.

1. (15 points) (T/F) Determine whether the following statements are true/false. Justify your answers.

(a) (3 points) For an arbitrary real-valued signal $\tilde{x}_T[n]$, its Fourier series coefficients $a_k = b_k + jc_k$ can always be decomposed into even and odd signals.

(True) For arbitrary a_k, even and odd parts are given by $a_{even} = (a_k + a_{-k})/2$, $a_{odd} = (a_k - a_{-k})/2$. Since $a_k = a_{even} + a_{odd}$, this statement is always true.

(b) (3 points) [ST] The fundamental period of the signal $x(t) = \sin(4\pi t) + \cos(2\pi t)$ is 1.

(True) The fundamental period of $\sin(4\pi t)$ is 0.5 ($T_1 = 2\pi/\omega = 2\pi/(4\pi) = 0.5$). The second term has $T_2 = 1$. The least common multiple of T_1 and T_2 is 1.

(c) (3 points) The system represented by a running-integral operator ($y(t) = \int_{-\infty}^{t} x(\tau)d\tau$) is linear and time-invariant.

(True)

- Linearity

$$y(t) = \int_{-\infty}^{t} a * x_1(\tau) + b * x_2(\tau)d\tau = a\int_{-\infty}^{t} x_1(\tau)d\tau + b\int_{-\infty}^{t} x_2(\tau)d\tau$$

- Time invariance

$$\begin{aligned} y_1(t) &= y(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)d\tau \\ y_2(t) &= \int_{-\infty}^{t} x(\tau - t_0)d\tau \\ &= \int_{-\infty}^{t-t_0} x(\tau')d\tau' \\ y_1(t) &= y_2(t) \end{aligned}$$

(d) (3 points) For a causal LTI system with an impulse response $h(t)$, let $g(t) = h(t) * u(-t)$. Then another LTI system with the impulse response $g(t)$ is also causal.

(False)
$u(-t) = 0$ for $t > 0$. The convolution of two signals is given by

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} u(-\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{0} h(t-\tau)d\tau \\ &= \int_{t}^{\infty} h(\tau')d\tau' \quad (\tau' = t-\tau) \end{aligned}$$

To have a causal system, it should be guaranteed that $g(t) = 0$ for all $t < 0$. However, in the above integral, $h(\tau')$ is nonzero for positive τ'. Therefore, we cannot guarantee the causality of $g(t)$.

(e) (3 points) [ST] The Fourier series coefficients of the signal $x[n] = \sum_{m=-\infty}^{\infty}(-1)^m\delta[n-mN]$ is $a_k = (1-(-1)^k)/(2N)$.

(True)
The fundamental period of this signal is $2N$, and the fundamental frequency is $\Omega_0 = 2\pi/2N = \pi/N$. According to the analysis equation of FS expansion,

$$\begin{aligned} a_k &= \frac{1}{2N}\sum_{n=0}^{2N-1} x[n]e^{-jk\Omega_0 n} \\ &= \frac{1}{2N}\sum_{n=0}^{2N-1} (\delta[n]-\delta[n-N])e^{-jk\Omega_0 n} \\ &= (1-e^{-jk\Omega_0 N})/(2N) \qquad\qquad = (1-e^{-jk\pi})/(2N) \\ &= (1-(-1)^k)/(2N) \end{aligned}$$

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2. (15 points) Answer the following questions.

(a) (5 points) [ST] Find the signal obtained by the convolution of $x[n]$ and $h[n]$ for

$$x[n] = [1,\ 3,\ 2,\ 2], \quad h[n] = [0,\ 6,\ 4,\ 1]$$

$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$
$\therefore$ [0 6 22 25 23 10 2]

(b) (10 points) For the signal $h(t) = e^{-t}u(t)$, calculate the convolution $h_t(t) = h(t) * h(t) * h(t)$.

(solution) First, let's solve the convolution $h(t) * h(t)$ - for $t > 0$, it becomes

$$\begin{aligned} h(t) * h(t) &= \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau \\ &= e^{-t}\int_{0}^{t} 1 d\tau \\ &= te^{-t} \end{aligned}$$

- for $t \leq 0$, there is no overlap between $u(\tau)$ and $u(t-\tau)$ and the convolution becomes 0.

$$\therefore h(t) * h(t) = te^{-t}u(t)$$

- The convolution $h_t(t)$ can be calculated as

$$\begin{aligned}
&\text{case 1: for } t > 0, \\
&\qquad h_t(t) = \left(te^{-t}u(t)\right) * e^{-t}u(t) \\
&\qquad\qquad = \int_{-\infty}^{\infty} \tau e^{-\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau \\
&\qquad\qquad = e^{-t}\int_0^t \tau d\tau \\
&\qquad\qquad = \frac{t^2}{2}e^{-t} \\
&\text{case 2: for } t \leq 0, \\
&\qquad h_t(t) = 0
\end{aligned}$$

$$\therefore\ h_t(t) = \frac{t^2}{2}e^{-t}u(t)$$

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3. (25 points) The LCCDE of a DT system satisfying the condition of initial rest is given as follows:

$$y[n] - y[n+1] = -x[n+1]$$

(a) (5 points) Derive the impulse response of this system

(Solution) Substituting $n' = n+1$, we can obtain the LCCDE given by

$$y[n'-1] - y[n'] = -x[n'] \qquad \rightarrow \qquad y[n'] - y[n'-1] = x[n']$$

Solving this LCCDE is equivalent to solving the homogeneous equation with an inhomogeneous auxiliary condition. That is,

$$y[n] - y[n-1] = 0 \ \text{ with } \ y[0] = 1$$

Let $y[n] = Az^n$ to find the homogeneous solution. Then the above equation becomes

$$Az^n(1 - z^{-1}) = 0, \quad \rightarrow \quad z = 1, \quad A = 1$$

$$\therefore h[n] = 1 \cdot 1^n u[n] = u[n]$$

(b) (5 points) Determine whether this system is causal or not.

(Solution) The system described by LCCDE and satisfying the condition of initial rest is causal.
This can be double-checked from the right-sided impulse response $h[n] = u[n]$.

(c) (5 points) Determine whether this system is BIBO stable or not.

(Solution) The system is unstable because its absolute sum is not finite.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |u[n]| \to \infty$$

(d) (5 points) Derive the LCCDE of the LTI inverse.

(Solution) The impulse response $h_{inv}[n]$ of LTI inverse should satisfy $h[n] * h_{inv}[n] = \delta[n]$. The function that transforms $h[n] = u[n]$ to unit impulse is the first-order difference. Therefore, the LCCDE of the inverse system is

$$y[n] = x[n] - x[n-1]$$

(e) (5 points) Using the solution of Prob. 3(a), derive the impulse response of the following system

$$y[n] - y[n+1] = -x[n+1] - x[n]$$

(Solution) Substituting $n' = n+1$, we can obtain the LCCDE given by

$$y[n'-1] - y[n'] = -x[n'] - x[n'-1] \qquad \rightarrow \qquad y[n'] - y[n'-1] = x[n'] + x[n'-1]$$

This system can be regarded as the serial interconnection of two systems satisfying

$$\begin{aligned} w[n] &= x[n] + x[n-1] \\ y[n] - y[n-1] &= w[n] \end{aligned}$$

The impulse response of the first system is simply $h_1[n] = \delta[n] + \delta[n-1]$, and that of the second system is $h_2[n] = u[n]$ from the result of Prob. 3(a). Therefore, the convolution of two impulse responses gives

$$h[n] = h_1[n] * h_2[n] = u[n] + u[n-1]$$

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4. (25 points) [ST] A periodic signal $\tilde{x}_T(t)$ with fundamental period T is given by

$$\tilde{x}_T(t) = \begin{cases} 1 & \text{for } 0 \le t < L \\ 0 & \text{for } L < t \le T \end{cases}$$

(a) (5 points) Find the Fourier series expansion a_k

(Solution) This signal is a rectangular function shifted by $\tau = L/2$: $\text{rect}_T\left(\frac{t-L/2}{L}\right)$.
First, consider the FS expansion of a rectangular function

$$\text{rect}_T\left(\frac{t}{L}\right) \quad \overset{FS}{\longleftrightarrow} \quad c_k = \frac{L}{T}\text{sinc}\left(\frac{L}{T}k\right)$$

From the property of FS ($x(t-\tau) \overset{FS}{\longleftrightarrow} c_k e^{-jk\omega_0\tau}$), the FS coefficients of the delayed signal are given by

$$\text{rect}_T\left(\frac{t-L/2}{L}\right) \overset{FS}{\longleftrightarrow} \quad a_k = \frac{L}{T}\text{sinc}\left(\frac{L}{T}k\right) e^{-jk\pi(L/T)}$$

(b) (10 points) When $L = T/2$, express a_0, a_{2m} and a_{2m+1} without using trigonometric functions (sin, cos, $e^{(\cdot)}$). (m: integer)

(Solution) For $L = T/2$, a_k can be simplified to

$$\begin{aligned}
a_k &= \frac{L}{T}\text{sinc}\left(\frac{L}{T}k\right) e^{-jk\omega_0 L/2} = \frac{1}{2}\text{sinc}\left(\frac{1}{2}k\right) e^{-jk\frac{2\pi}{T}\frac{T}{4}} \\
&= (-j)^k \frac{1}{2}\text{sinc}\left(\frac{1}{2}k\right) \\
&= (-j)^k \frac{1}{2}\frac{\sin(k\pi/2)}{k\pi/2} \\
&= \begin{cases} \frac{1}{2} & \text{for } \ k=0 \\ (-j)^{2m}\frac{1}{2}\frac{\sin(m\pi)}{m\pi} & \text{for } \ k=2m \\ (-j)^{2m+1}\frac{1}{2}\frac{\sin((2m+1)\pi/2)}{(2m+1)\pi/2} & \text{for } \ k=2m+1 \end{cases} \\
&= \begin{cases} \frac{1}{2} & \text{for } \ k=0 \\ 0 & \text{for } \ k=2m \\ \frac{1}{j(2m+1)\pi} & \text{for } \ k=2m+1 \end{cases}
\end{aligned}$$

(c) (5 points) Find the FS coefficients b_k of the signal $\tilde{y}_T(t)$ defined as

$$\tilde{y}_T(t) = \begin{cases} 0 & \text{for } \ 0 \le t < L \\ 1 & \text{for } \ L < t \le T \end{cases}$$

(Solution) Utilizing the relation $\tilde{y}_T(t) = 1 - \tilde{x}_T(t)$ and the result of Prob. (4-2), we have

$$b_k = \delta_{k0} - a_k = \delta_{k0} - \frac{L}{T}\text{sinc}\left(\frac{L}{T}k\right) e^{-jk\pi(L/T)}$$

(d) (5 points) Derive the FS coefficients of the signal $\tilde{z}_T(t) = \tilde{x}_T(t) * \tilde{y}_T(-t)$

(Solution) From the FS properties $\tilde{x}_T(t) * \tilde{y}_T(t) \overset{FS}{\longleftrightarrow} Ta_k b_k$ and $\tilde{y}_T(-t) \overset{FS}{\longleftrightarrow} b_{-k}$,

$$\begin{aligned}
\tilde{z}_T(t) = \tilde{x}_T(t) * \tilde{y}_T(-t) &\overset{FS}{\longleftrightarrow} \quad d_k = Ta_k b_{-k} \\
d_k &= T\frac{L}{T}\text{sinc}\left(\frac{L}{T}k\right) e^{-jk\omega_0 L/2}\left(\delta_{k0} - \frac{L}{T}\text{sinc}\left(\frac{L}{T}k\right) e^{jk\omega_0 L/2}\right) \\
&= T\left(\frac{L}{T}\delta_{k0} - \frac{L^2}{T^2}\text{sinc}^2\left(\frac{L}{T}k\right)\right) \\
&= L\delta_{k0} - \frac{L^2}{T}\text{sinc}^2\left(\frac{L}{T}k\right)
\end{aligned}$$

5. (20 points) The LTI system satisfying the condition of initial rest is described by the following LCCDE:

$$\frac{d^3y(t)}{dt^3} + 3\frac{dy^2(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = x(t) \tag{1}$$

(a) (5 points) Find the frequency response $H(j\omega)$ of the system at frequency ω. (Hint) $f^3 + 3f^2 + 3f + 1 = (1+f)^3$.

(Solution) Substituting $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)e^{j\omega t}$ yields

$$H(j\omega) = \frac{1}{(j\omega)^3 + 3(j\omega)^2 + 3(j\omega) + 1} = \frac{1}{(j\omega + 1)^3}$$

(b) (5 points) Describe the filter characteristic of this system (for example, low-pass, high-pass, band-pass). Justify your answer.

(Solution) $|H(j\omega)| \to 1$ for $\omega \to 0$, and $|H(j\omega)| \to 0$ for $\omega \to \infty$. Therefore, the system has a low-pass characteristic.

(c) (5 points) The system of Eq. (1) can be decomposed into a serial interconnection of three first-order systems. For the following three LCCDEs of the first-order systems, determine a, b, and c.

$$\frac{dy(t)}{dt} + ay(t) = z(t) \tag{2}$$

$$\frac{dz(t)}{dt} + bz(t) = w(t) \tag{3}$$

$$\frac{dw(t)}{dt} + cw(t) = x(t) \tag{4}$$

(Solution) Substituting Eq. (2) to (3) and also substituting the result to Eq. (4) gives

$$\frac{d^3y(t)}{dt^3} + (a+b+1)\frac{dy^2(t)}{dt^2} + (ab+a+b)\frac{dy(t)}{dt} + c = x(t)$$

Comparing this to Eq. (1) yields

$$\begin{aligned} a+b+1 = 3, \quad ab+a+b = 3, \quad c = 1 \\ \longrightarrow \quad a+b = 2, \quad ab = 1, \quad c = 1 \\ \longrightarrow \quad a = 1, \ b = 1, \ c = 1 \end{aligned}$$

(d) (5 points) Using the result of Prob. 5(c), derive the impulse response $h(t)$ of the total system.

(Solution) The three systems are identical, and their impulse response is given by $h_1(t) = e^{-t}u(t)$. The serial interconnection of systems is equivalent to the sequential convolution of their impulse responses, so $h(t) = h_1(t) * h_1(t) * h_1(t)$. From the result of Prob. 2(b), the total impulse response is given by

$$h(t) = \frac{t^2}{2} e^{-t} u(t)$$

[End of Problem]