[Score table] **Total Score : ________________**

Prob.	1			2			3					4					5
	a	b	c	a	b	c	a	b	c	d	e	a	b	c	d	e	
Total	5	10	10	5	5	10	5	5	5	5	5	5	5	5	5	5	5
Score																	

Your problems

[25 points] Chapter 4 Continuous-time Fourier transform

1. Figure 1 shows the magnitude and phase of the Fourier transform $X(j\omega)$ of a signal $x(t)$.

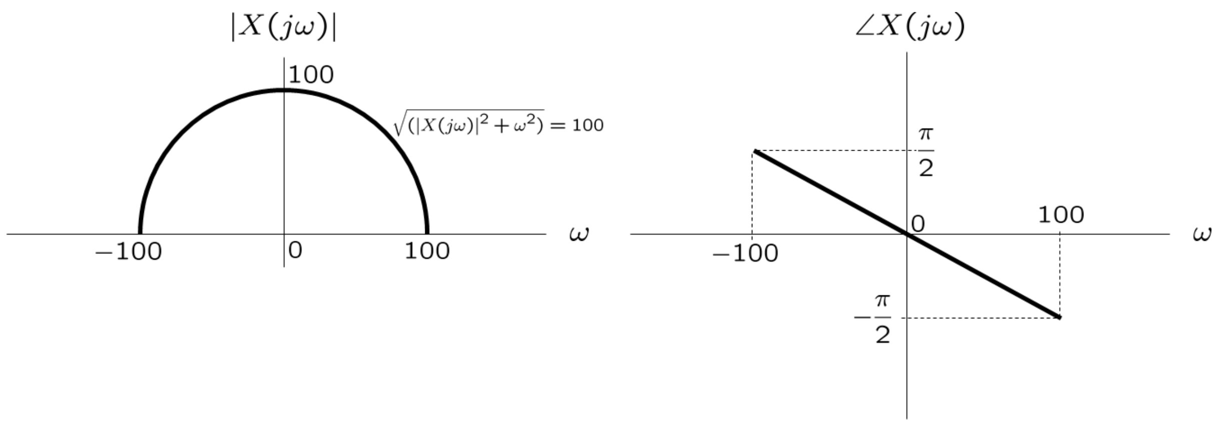
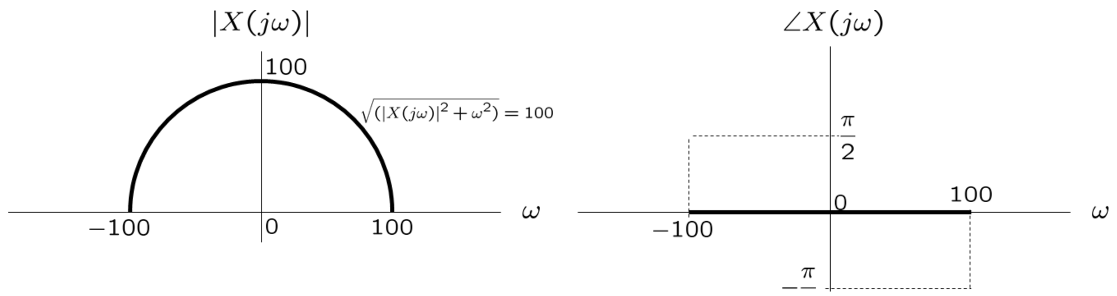


Figure 1

(a) [5 pts] Determine the value of this time signal $x(t)$ at $t = \pi/200$.

The signal is the time shift of the signal with zero phase. From the properties of Fourier transform, $x(t-t_0) \leftrightarrow X(j\omega)e^{-j\omega t_0}$, the time delay is $t_0 = \pi/200$. Therefore, $x(t_0)$ is equivalent to $x(0)$ with zero time shift(figure shown below). From the properties of Fourier transform, $x(0)$ is the area of $X(j\omega)$ divided by 2π. The half-circle area is $\pi(100)^2/2$, so $x(t_0) = (100)^2/4 = 2500$.



(b) [10 pts] Express the signal $x_2(t)$ of Figure 2 in terms of $x(t)$, when the Fourier transform $X_2(j\omega)$ is given as follows:

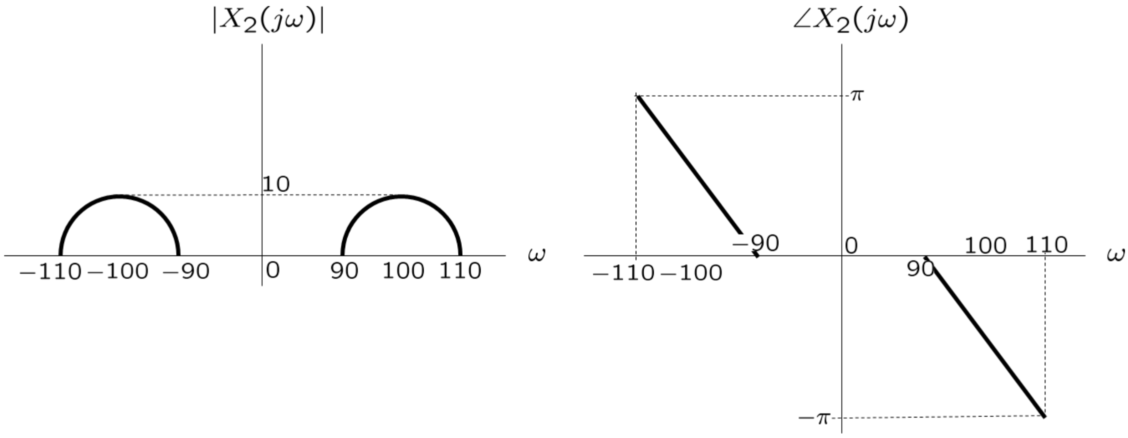
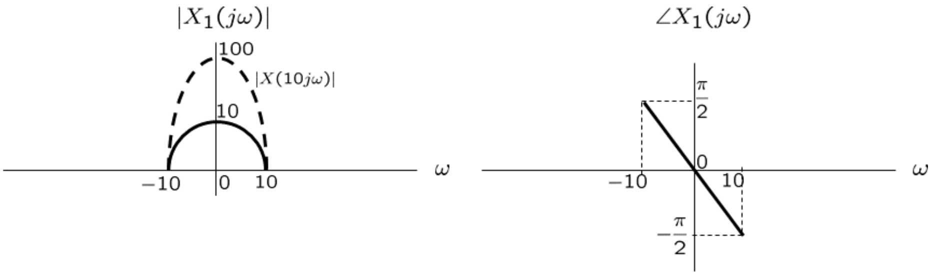


Figure 2

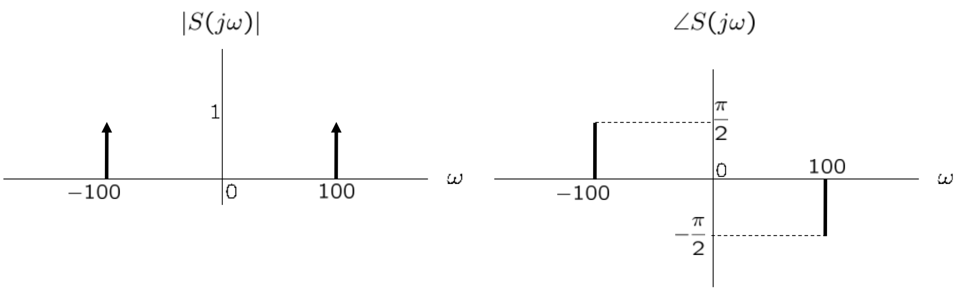
The given signal is $1/10$ scaling of the original signal, modulated by $100\,\text{rad/s}$ to both sides.

The scaling: $X_1(j\omega) = \dfrac{X(10j\omega)}{10}$ → $x_1(t) = \dfrac{1}{10}\left(\dfrac{1}{10}x\left(\dfrac{t}{10}\right)\right)$ --- (1)



Modulation: the phase has opposite signs for positive and negative frequencies -> sinusoidal amplitude modulation

$$X_2(j\omega) = \left[X_1(j\omega) * S(j\omega)\right] = \frac{1}{2\pi}\left[X_1(j\omega) * 2\pi \cdot \left(e^{-j\frac{\pi}{2}}\delta(\omega-\omega_0) + e^{j\frac{\pi}{2}}\delta(\omega+\omega_0)\right)\right], \quad \omega_0 = 100$$



(continues on the next page)

From the multiplication-convolution property,

$$x_2(t) = x_1(t) \times (-je^{j\omega_0 t} + je^{-j\omega_0 t}) = x_1(t) \times 2\sin(\omega_0 t) \quad \text{----(2)}$$

From (1) and (2), $x_2(t) = \dfrac{1}{50}\sin(100t)\,x\left(\dfrac{t}{10}\right)$.

(c) [10 pts] The signal $x(t)$ is fed into a system $h(t)$ and produces the output $y(t)$. When the Fourier transform of the output signal $y(t)$ is given as follows, derive the system's impulse response $h(t)$.

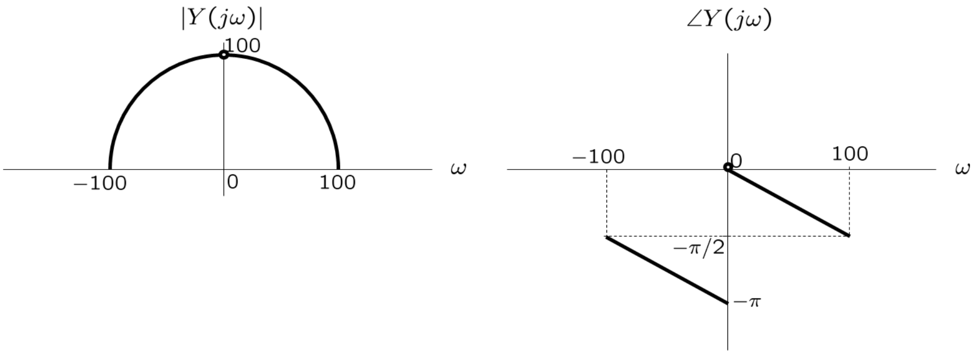
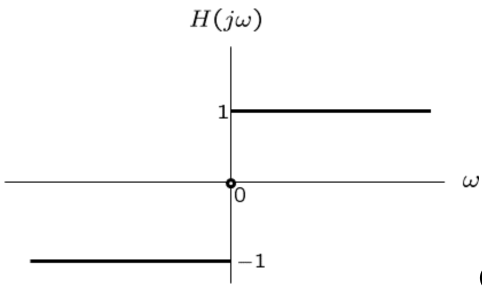


Figure 3

The phase for $\omega < 0$ is subtracted by $-\pi$, which means that $Y(j\omega) = H(j\omega)X(j\omega)$ with $H(j\omega)$ being the Signum function in the frequency domain defined by

$$H(j\omega) = 2U(\omega) - 1$$

where $U(\omega)$ is the unit step function in the frequency domain.



(note: $-1 = e^{j\pi} = e^{-j\pi}$)

The inverse Fourier transform of the Signum function is given by

$$\frac{dH(j\omega)}{d\omega} = 2\frac{dU(\omega)}{d\omega} = 2\delta(\omega)$$

$$-jth(t) = \frac{2}{2\pi} = \frac{1}{\pi} \qquad (H(j\omega) \text{ has no DC component, } \mathcal{F}^{-1}\{\delta(\omega)\} = \frac{1}{2\pi})$$

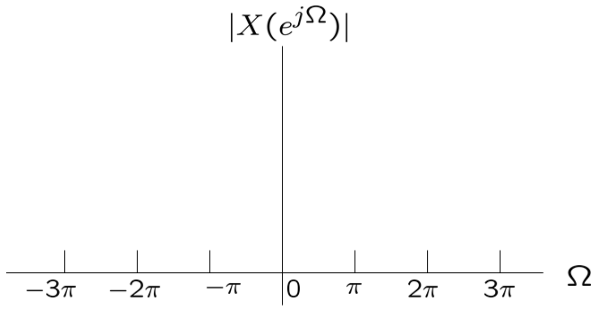
$$\therefore h(t) = \frac{j}{\pi t}$$

[20 points] Chapter 5 & 7 Discrete Fourier transform and Sampling

2. (a) [5 pts] Now the signal $x(t)$ of Figure 1 is sampled at a rate of ω_s. Determine the Nyquist rate required to avoid aliasing.

The signal is bandlimited within $|\omega| \le 100 = \omega_M$, and hence, the Nyquist rate is $\omega_s = 2\omega_M = 200$

(b) [5 pts] Suppose that the sampling rate is $\omega_s = 1000$ rad/sec, and the signal is converted to a discrete sequence $x[n]$ after the sampling. Plot the magnitude of the discrete Fourier transform $X(e^{j\Omega})$. Also indicate the zero-crossing point and peak amplitude of $|X(e^{j\Omega})|$.

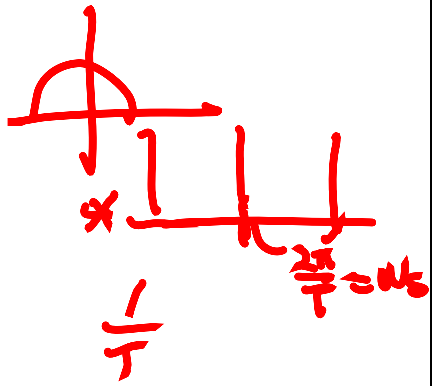


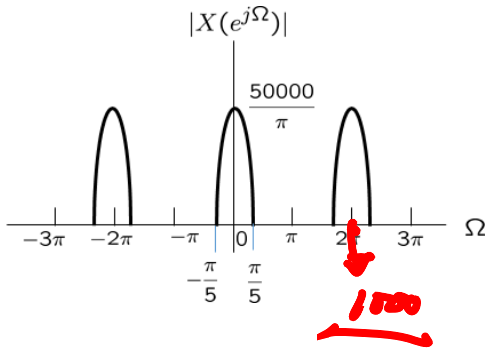
From the sampling $\omega T = \Omega$,

$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\Omega - 2\pi k)/T) = \frac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} X(j(\Omega - 2\pi k)\tfrac{\omega_s}{2\pi})$$

Frequency scaling $\omega_s = 1000 \to \Omega = 2\pi$: $\omega = 100 \to \dfrac{2\pi}{10} = \dfrac{\pi}{5}$

Amplitude scaling by $\dfrac{\omega_s}{2\pi}$: $\dfrac{1000}{2\pi}100 = \dfrac{50000}{\pi}$





(c) [10 pts] For a signal $y[n]$ satisfying the following relation, plot magnitude of discrete Fourier transform $|Y(e^{j\omega})|$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[5k]\left(\frac{\sin(\frac{\pi}{5}(n-5k))}{\frac{\pi}{5}(n-5k)}\right) \quad (1)$$

Consider a signal $x_5[n] = \begin{cases} x[n] & \text{for } n = 5k \\ 0 & \text{otherwise} \end{cases}$

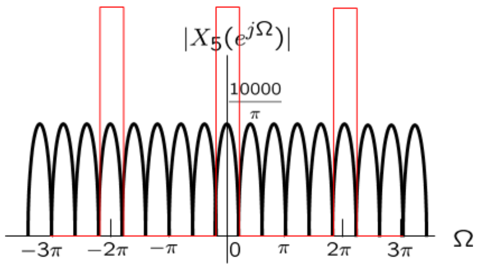
Equation (1) is then the convolution of $x_5[n]$ and $\dfrac{\sin(\frac{\pi}{5}n)}{\frac{\pi}{5}n}$

$$y[n] = \sum_{m=-\infty}^{\infty} x_5[m]\left(\frac{\sin(\frac{\pi}{5}(n-m))}{\frac{\pi}{5}(n-m)}\right) = \sum_{k=-\infty}^{\infty} x[5k]\left(\frac{\sin(\frac{\pi}{5}(n-5k))}{\frac{\pi}{5}(n-5k)}\right) \quad \text{--(1)}$$

The Fourier transform of $x_N[n]$ is given by,

$$X_N(e^{j\Omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j\left(\Omega - k\frac{2\pi}{N}\right)}) \quad \text{(textbook Eq. 7.42)}$$

Magnitude scaling: $1/N$, repetition with period $2\pi/N$ (solid black curve shown below).



From the properties of Fourier transform, $\dfrac{\sin(Wn)}{\pi n} \Leftrightarrow \text{Rect}(W)$, where $\text{Rect}(W)$ is a rectangular function of $2W$ width($\pm W$). Therefore,

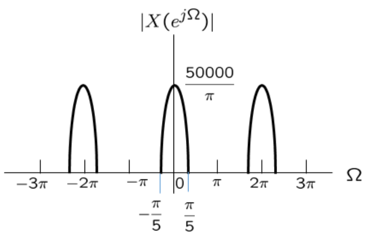
$$\mathcal{F}\left\{\frac{\sin(\frac{\pi}{5}n)}{\frac{\pi}{5}n}\right\} = 5\text{Rect}(\tfrac{\pi}{5}), \quad \text{(red curve)}$$

From (1) and convolution-multiplication property, it can be shown that the result is the same.

$$\mathcal{F}\{y[n]\} = \mathcal{F}\{x_5[n]\}\mathcal{F}\left\{\frac{\sin(\frac{\pi}{5}n)}{\frac{\pi}{5}n}\right\} = X_5(e^{j\Omega}) \times 5\text{Rect}(\tfrac{\pi}{5})$$

$$= \frac{1}{5}\sum_{k=0}^{N-1} X(e^{j\left(\Omega - k\frac{2\pi}{N}\right)}) \times 5\text{Rect}(\tfrac{\pi}{5}) = X(e^{j\Omega})$$

Therefore, the answer is the same as that of (b).



[25 points] Chapter 6 Time and Frequency Characterization

3. Consider a LTI system with the condition of initial rest. The system satisfies the following differential equation.

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+y(t)=10x(t)-\frac{dx(t)}{dt} \qquad (2)$$

(a) **[5 pts]** Determine the <u>Laplace transform</u> $H(s)$ and the <u>frequency response</u> $H(j\omega)=Y(j\omega)/X(j\omega)$ of the given system.

$$(s^2+2s+1)Y(s)=(10-s)X(s)$$

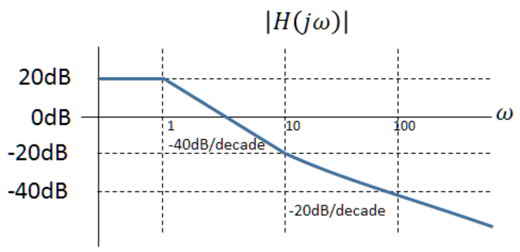
$$H(s)=\frac{Y(s)}{X(s)}=\frac{10-s}{s^2+2s+1}=\frac{10-s}{(s+1)^2}$$

$$\therefore H(j\omega)=\frac{10-j\omega}{(j\omega+1)^2}$$

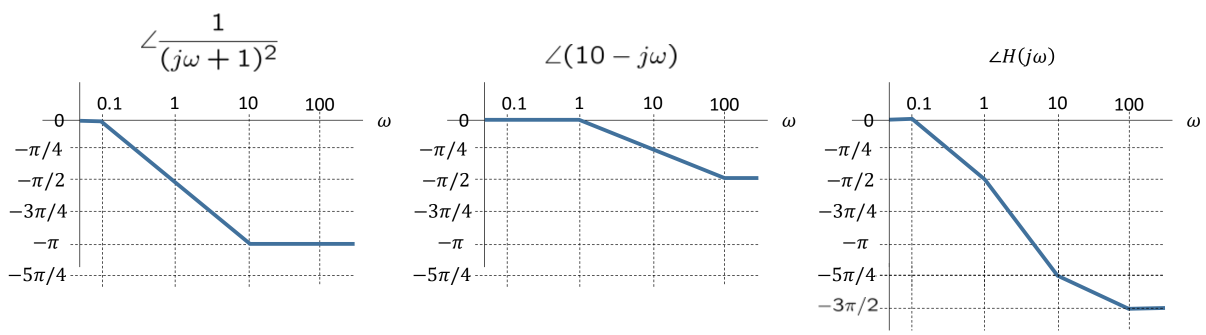
(b) **[5 pts]** Draw the bode plot of $H(j\omega)$ with respect to ω. Show both magnitude and phase responses.

Magnitude response

- Starting point: $H(0)=10$ → +20dB



Phase response



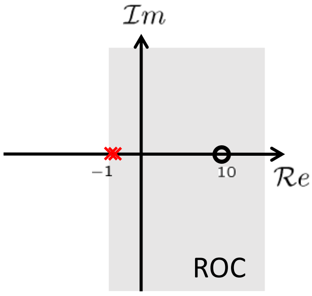
(c) **[5 pts]** Determine the <u>impulse response</u> $h(t)$ of this system.

LTI, Condition of initial rest → causal system → right-sided

$$H(j\omega)=\frac{10-j\omega}{(j\omega+1)^2}=\frac{(-1-j\omega)+11}{(j\omega+1)^2}=-\frac{1}{j\omega+1}+\frac{11}{(j\omega+1)^2}$$

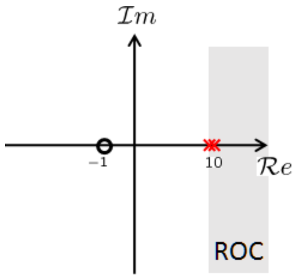
$$\therefore h(t)=-e^{-t}u(t)+11\cdot te^{-t}u(t)$$

(d) **[5 pts]** Draw the pole-zero plot in the s-plane. Is this system stable?



Rational $H(s)$, Causal system → ROC is on the right-side of the rightmost pole.
ROC includes imaginary axis → Stable

(e) **[5 pts]** An inverse system $H_{inv}(s)$ of a system $H(s)$ is defined as the system of which transfer function is $1/H(s)$. Determine whether the inverse of the given system($H(s)^{-1}$) can be causal and stable at the same time. Justify your answer.



Inverse of $H(s)$ changes the location of poles and zeros. The causal inverse system should have ROC on the right-side from s=10, which does not include the imaginary axis → cannot be stable & causal

[25 points] Chapter 10 Z-transform

4. The pole-zero plot of a LTI system with impulse response $h[n]$ is shown in Figure 4.

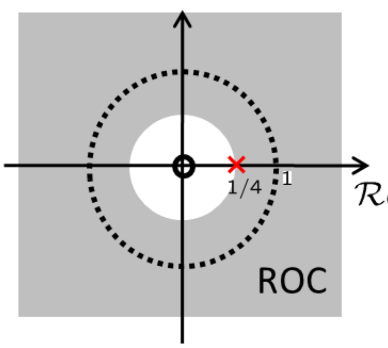


Figure 4

(a) [10 pts] When the zero-frequency response is $H(e^{j\omega})|_{\omega=0} = 2$, identify the <u>system response</u> $H(z)$ From $H(z)$, describe the <u>system's characteristic</u>(ex. Low-pass, high-pass, band-pass, all-pass, low-shelf, high-shelf …).

$$H(z) = G\frac{z}{(z-\frac{1}{4})} = G\frac{1}{1-\frac{1}{4}z^{-1}}, \qquad H(e^0 = 1) = G\frac{1}{3/4} = 2 \;\rightarrow\; G = \frac{3}{2}$$

$$\therefore H(z) = \frac{3}{2}\cdot\frac{1}{1-\frac{1}{4}z^{-1}}$$ (Low-pass: see textbook p. 468)

(b) Find the impulse response $h[n]$ of the system. Verify your result at $n=0$ using initial value theorem.

ROC is outside from the pole → Causal system

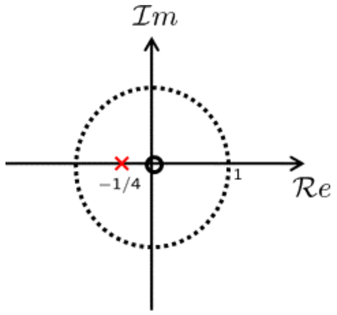
$$h[n] = \frac{3}{2}\left(\frac{1}{4}\right)^n u[n]$$

$$\lim_{z\to\infty} H(z) = \frac{3}{2} \text{ at } n = 0$$

(c) [5 pts] Draw the pole-zero plot of $(-1)^n h[n]$ in z-domain. Is this system stable?

$$H_{new}(z) = \sum_{n=-\infty}^{\infty} (-1)^n h[n] z^{-n} = \sum_{n=-\infty}^{\infty} h[n](-z)^{-n} = H(-z)$$

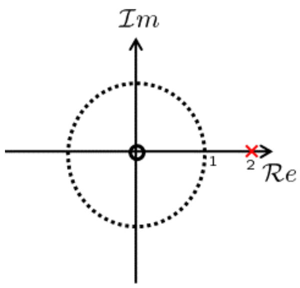
Now the pole is at z=-1/4 : ROC still includes unit circle → stable



(d) [5 pts] Draw the pole-zero plot of the $8^n h[n]$ in z -domain. Is this system stable?

$$H_{new}(z) = \sum_{n=-\infty}^{\infty} (8)^n h[n] z^{-n} = \sum_{n=-\infty}^{\infty} h[n](z/8)^{-n} = H(z/8)$$

Now the pole is at z=1/4*8=2: ROC excludes the unit circle → unstable



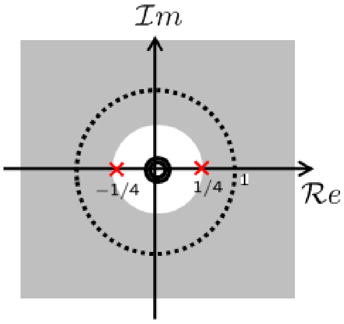
(e) [5 pts] Consider a discrete-time sequence $h_2[n]$ defined as

$$h_2[n] = \begin{cases} h[n], & n = 2m \\ 0 & \text{otherwise} \end{cases}$$

(1) Draw the pole locations of $h_2[n]$ and discuss the ROC of this sequence. (2) Derive the difference equation that can describe this system.

$$h_2[n]=\frac{1}{2}h[n](1+(-1)^n) \rightarrow H_2(z)=\frac{1}{2}\big(H(z)+H(-z)\big)$$

$$\therefore H_2(z)=\frac{3}{4}\cdot\left(\frac{1}{1-\frac{1}{4}z^{-1}}+\frac{1}{1+\frac{1}{4}z^{-1}}\right)=\frac{3}{2}\cdot\frac{1}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{4}z^{-1})}=\frac{3}{2}\cdot\frac{z^2}{(z-\frac{1}{4})(z+\frac{1}{4})}$$



$\therefore$ ROC is the same

$$H_2(z)=\frac{3}{2}\cdot\frac{1}{(1-\frac{1}{16}z^{-2})}=\frac{Y(z)}{X(z)}$$

$$2\left(y[n]-\tfrac{1}{16}y[n-2]\right)=3x[n]$$

$$\therefore 2y[n]-\tfrac{1}{8}y[n-2]=3x[n]$$

[5 points] Chapter 4 Continuous-time Fourier transform

5. [5 pts] *Correlation* of signals $x(t)$ and $y(t)$ is defined as follows:

$$c_{xx}(t)=\int_{-\infty}^{\infty}x(\tau)x(t+\tau)d\tau\,,\quad c_{xy}(t)=\int_{-\infty}^{\infty}x(t+\tau)y(\tau)d\tau \qquad (3)$$

Suppose that an unknown LTI system with an impulse response $h(t)$ was driven by a input signal $x(t)$, and then its output $y(t)$ was measured. For a special input signal $x(t)$, the impulse response $h(t)$ can be directly obtained from the correlation of the output and input signal($c_{xy}(t)$). That is,

$$c_{xy}(t)=h(t) \qquad (4)$$

What is $|C_{xx}(\omega)|=\left|\mathcal{F}\{c_{xx}(t)\}\right|$? (1) First, show that $c_{xy}(t)=x(-t)*y(t)$. (2) From this result, find out the characteristic of $c_{xx}(t)$ to satisfy Eq. (4) and derive the Fourier transform of such $c_{xx}(t)$.

$$
\begin{aligned}
c_{xy}(t) &= \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(t-(-\tau))y(\tau)d\tau \\
&= x(-t) * y(t)
\end{aligned}
$$

Since $y(t) = h(t) * x(t)$,

$$
\begin{aligned}
c_{xy}(t) &= x(-t) * y(t) \\
&= x(-t) * h(t) * x(t) \\
&= \underbrace{x(-t) * x(t)}_{c_{xx}(t)} * h(t) = h(t)
\end{aligned}
$$

Therefore, $c_{xx}(t) = \delta(t)$ and $\mathcal{F}\{c_{xx}(t)\} = 1$