Name:

Student ID No.:

[Score table]

Prob.	Problem 1 (20)				Problem 2 (35)				Problem 3 (35)				Matlab	Total
	5	5	5	5	10	10	5	10	10	10	10	5	10	100
Score														

Problems & Solutions

[Problem 1 | 20 points]

Determine whether or not each of the following statement is true or false. Justify your answers.

(1-1) **[5 pts] (True/False)** If the fundamental period of a periodic sequence $\tilde{x}_N[n]$ is an odd number N, the fundamental period of a sequence $(-1)^n \tilde{x}_N[n]$ is $2N$.

(1-2) **[5 pts] (True/False)** If A_x and A_y are average values of periodic signals $\tilde{x}(t)$ and $\tilde{y}(t)$, respectively, then the average value of $\tilde{x}(t) * \tilde{y}(t) = \int_0^T \tilde{x}(\tau)\tilde{y}(t-\tau)d\tau$ is $A_x A_y$. The periods of two signals are the same.

(1-3) **[5 pts] (True/False)** For the input-output pair $x(t) = 2e^{j\omega t}$ and $y(t) = 4e^{2j\omega t}$, there is a LTI system satisfying this input-output relation.

(1-4) **[5 pts] (True/False)** If a continuous-time system is linear and time invariant, its input-output relation can be always described in terms of a linear constant coefficient differential equation: $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$

(1-1)

(True) $(-1)^{n+2N} \tilde{x}_N[n+2N] = (-1)^n \tilde{x}_N[n]$

(1-2)

(False) The LTI system's input-output relation is described by the convolution. If LTI system exists, $e^{j\omega t}$ is an eigenfunction of the convolution operator. Since the output signal has a different exponential form, there cannot be an LTI system satisfying this input-output relation.

(1-3)

(False)

$$\frac{1}{T}\int_0^T \left(\tilde{x}(t) * \tilde{y}(t)\right)dt = \frac{1}{T}\int_0^T \left(\int_0^T \tilde{x}(\tau)\tilde{y}(t-\tau)d\tau\right)dt$$
$$= \int_0^T \tilde{x}(\tau)\left\{\frac{1}{T}\int_0^T \tilde{y}(t-\tau)dt\right\}d\tau$$
$$= A_y\int_0^T \tilde{x}(\tau)d\tau = TA_xA_y$$

(1-4)

(False) For example, a delay system $y(t) = x(t-\tau)$ is LTI, but the output cannot be expressed in form of the LCCDE.

[Problem 2 | 35 points]

(2-1) [**10 pts**] Consider a system satisfying the condition of initial rest and described by the differential equation:

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 6y(t) = \frac{dx(t)}{dt} + x(t).$$

Find impulse response $h(t)$ of this system.

(Hint: one of your HW problem)

Answer) $h(t) = \left(\frac{4}{5}e^{3t} + \frac{1}{5}e^{-2t}\right)u(t)$

The system described by LCCDE satisfies the condition of initial rest → Causal, LTI

- Impulse response: let $w(t) = \frac{dx(t)}{dt} + x(t)$ (FIR system), $\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 6y(t) = w(t)$ (IIR system)

The impulse responses of these two systems are given by

$h_1(t) = \frac{d\delta(t)}{dt} + \delta(t)$, and by solving the equivalent problem with homogeneous DE and auxiliary conditions:

$$\frac{d^2h_2(t)}{dt^2} - \frac{dh_2(t)}{dt} - 6h_2(t) = 0, \quad \left.\frac{dh_2(t)}{dt}\right|_{t=0^+} = 1, \ h_2(0^+) = 0$$

→ $h_2(t) = A_1e^{-2t} + A_2e^{3t}$, $-2A_1 + 3A_2 = 1, \ A_1 + A_2 = 0$

→ $A_1 = -\frac{1}{5}, \ A_2 = \frac{1}{5}$

Convolution of two systems' impulse responses yields

$$
\begin{aligned}
h_1(t)*h_2(t) &= \frac{1}{5}\left(\frac{d\delta(t)}{dt}+\delta(t)\right)*\left\{\left(e^{3t}-e^{-2t}\right)u(t)\right\} \\
&= \frac{1}{5}\left(e^{3t}-e^{-2t}\right)u(t)+\frac{1}{5}\frac{d}{dt}\left\{\left(e^{3t}-e^{-2t}\right)u(t)\right\} \\
&= \frac{1}{5}\left(e^{3t}-e^{-2t}\right)u(t)+\frac{1}{5}\left(e^{3t}-e^{-2t}\right)\delta(t)+\frac{1}{5}\left(3e^{3t}+2e^{-2t}\right)u(t) \\
&= \left(\frac{4}{5}e^{3t}+\frac{1}{5}e^{-2t}\right)u(t)+\frac{1}{5}\left(e^{3t}-e^{-2t}\right)\delta(t) \\
&= \left(\frac{4}{5}e^{3t}+\frac{1}{5}e^{-2t}\right)u(t) \qquad (\because x(t)\delta(t)=x(0)\delta(t))
\end{aligned}
$$

(2-2) **[10 pts]** Determine the output of the system given by (2-1) for the input signal $x(t)=3\cos(t)$. Express the output $y(t)$ only in terms of $\sin(t)$ and $\cos(t)$. (Hint: eigenvalue and eigenfunction)

Answer) For a steady signal $x(t)=3\cos(t)=3\dfrac{e^{jt}+e^{-jt}}{2}$, only the particular solution exist. The output from the FIR system $w(t)=\dfrac{dx(t)}{dt}+x(t)$ is given by

$$w(t)=3(\cos t-\sin t)=3\left(\frac{e^{jt}+e^{-jt}}{2}-\frac{e^{jt}-e^{-jt}}{2j}\right)=3\frac{1+j}{2}e^{jt}+3\frac{1-j}{2}e^{-jt}$$

For a single frequency input $w(t)=e^{j\omega t}$, the IIR system's output will be in form of $y(t)=H(j\omega)e^{j\omega t}$

$$\rightarrow\; H(j\omega)\left((j\omega)^2-(j\omega)-6\right)e^{j\omega t}=e^{j\omega t} \;\rightarrow\; H(j\omega)=\frac{1}{(j\omega)^2-(j\omega)-6}$$

For $\omega=1,\quad H(j)=\dfrac{1}{-7-j},\; H(-j)=\dfrac{1}{-7+j}$

Therefore, the output signal is given by

$$
\begin{aligned}
&3\left(\frac{1+j}{2}\frac{1}{-7-j}e^{jt}+\frac{1-j}{2}\frac{1}{-7+j}e^{-jt}\right)=\frac{3}{100}\left((1+j)(-7+j)e^{jt}+(1-j)(-7-j)e^{-jt}\right) \\
&=\frac{3}{100}\left((-8-6j)e^{jt}+(-8+6j)e^{-jt}\right)=\frac{3}{25}\left(-4\frac{e^{jt}+e^{-jt}}{2}+3\frac{e^{jt}-e^{-jt}}{2j}\right) \\
&=\frac{\left(-12\cos t+9\sin t\right)}{25}
\end{aligned}
$$

(2-3) **[5 pts]** Determine the BIBO stability of this system. If stable, show that the system has a bounded output $|y(t)|<\infty$ for any bounded input $|x(t)|<B$. If not, justify your answer.

Answer) Unstable

Solution) The system is unstable because the impulse response $h(t)=\left(\frac{4}{5}e^{3t}+\frac{1}{5}e^{-2t}\right)u(t)$ grows to infinity as $t\to\infty$.

(2-4) **[10 pts]** Consider a causal LTI system that produces output $y[n]=(-\frac{1}{2})^n u[n]+2u[n]$ for input $x[n]=u[n]$. Find impulse response $h[n]$ of this LTI system.
(Hint: use the relation between impulse and step responses)

Answer) $3(-\frac{1}{2})^n u[n]$

Solution) $y[n]$ is a step response, so its impulse response can be obtained by $h[n]=y[n]-y[n-1]$.

$$
\begin{aligned}
h[n] &= y[n]-y[n-1] \\
&= (-\tfrac{1}{2})^n u[n]+2u[n]-(-\tfrac{1}{2})^{n-1}u[n-1]-2u[n-1] \\
&= (-\tfrac{1}{2})^n u[n]+\underbrace{2\delta[n]+2(-\tfrac{1}{2})^n u[n-1]}_{2\left(-\frac{1}{2}\right)^n u[n]} = 3(-\tfrac{1}{2})^n u[n]
\end{aligned}
$$

[Chapter 3 | 35 pts]

(3-1) **[10 pts]** Suppose that the Fourier series coefficients of a periodic sequence $\tilde{x}_N[n]$ with fundamental period $N=6$ (Figure 3-1) are given by a_k .

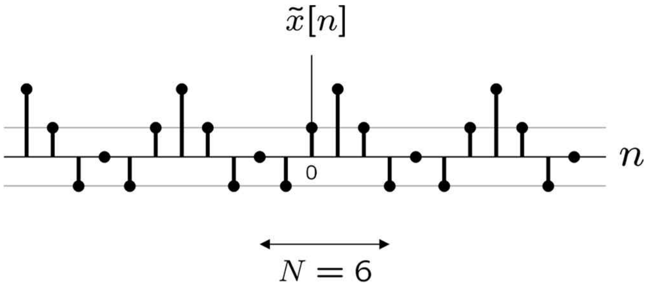
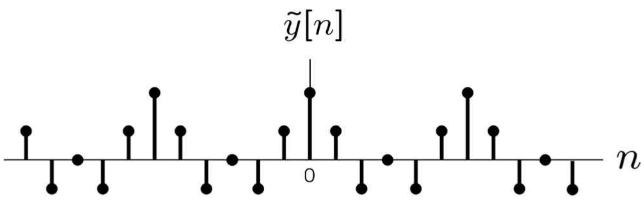


Figure 3-1

When the real and imaginary parts of a_k are given by b_k and c_k ($a_k=b_k+ic_k$), find the ratio $d_k=\dfrac{c_k}{b_k}$. (Hint: the shape of the signal)

Answer) $-\tan\left(\frac{\pi}{3}k\right)$

Consider a sequence $\tilde{y}_N[n]$ centered at the origin ($\tilde{x}_N[n]=\tilde{y}_N[n-1]$).



Since $\tilde{y}_N[n]$ is real and even ($\tilde{y}_N[n]=\tilde{y}_N[-n]$), its Fourier series coefficients are purely real. Let the real coefficients be g_k.

From the relation $\tilde{x}_N[n]=\tilde{y}_N[n-1]$, the Fourier series coefficients of $\tilde{x}_N[n]$ are given by

$$a_k = g_k e^{-jk\Omega_0} = g_k e^{-j\frac{2\pi}{N}k} = g_k e^{-j\frac{2\pi}{6}k}$$
$$=\left(g_k\cos\left(\frac{\pi}{3}k\right)-jg_k\sin\left(\frac{\pi}{3}k\right)\right)$$
$$d_k = -\frac{\sin\left(\frac{\pi}{3}k\right)}{\cos\left(\frac{\pi}{3}k\right)} = -\tan\left(\frac{\pi}{3}k\right)$$

(3-2) [10 pts] Calculate the periodic convolution of two signals $\tilde{x}[n]$ and $\tilde{y}[n]$, given by

$$\tilde{x}[n]=\cos\left(\frac{\pi}{16}n\right)\ ,\quad \tilde{y}[n]=\sin\left(\frac{\pi}{8}n\right)$$

(Hint: the periodic convolution is defined as $\tilde{x}[n]*\tilde{y}[n]=\sum_{k=<N>}\tilde{x}[k]\tilde{y}[n-k]$ for the common period N of two periodic signals. What is the Fourier Series property for the cyclic convolution?)

Answer) 0

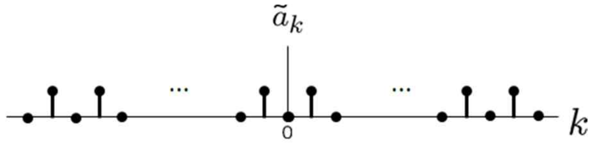
- Utilize the convolution-multiplication property of discrete Fourier series. ($\tilde{x}_N[n]*\tilde{y}_N[n]\overset{DFS}{\Leftrightarrow}Na_k b_k$)
- Consider the discrete Fourier series expansion of $\tilde{x}[n]$ and $\tilde{y}[n]$. Since both signals are periodic with period $N=32$, the fundamental frequency is given by $\Omega_0=\frac{2\pi}{N}=\frac{\pi}{16}$.

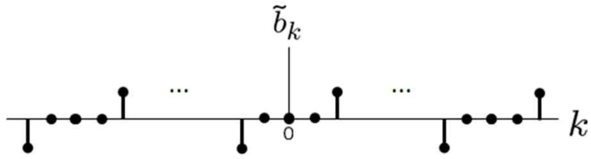
$$a_k=\frac{1}{32}\sum_{n=0}^{31}x[n]e^{jk\Omega_0 n}=\delta[k-1\pm 32\ell]+\delta[k+1\pm 32\ell]$$
$$\left(\because \cos(k_0\Omega_0 n)=\frac{e^{jk_0\Omega_0 n}+e^{-jk_0\Omega_0 n}}{2},\ e^{jk_0\Omega_0 n}\overset{DFS}{\Leftrightarrow}\delta[k-k_0\pm\ell N]\right)$$

$$b_k = \frac{1}{32}\sum_{n=0}^{31} y[n]e^{jk\Omega_0 n}$$

$$= \frac{1}{2j}\left(\delta[k-2\pm 32\ell] - \delta[k+2\pm 32\ell]\right)$$

$$\left(\because \sin(2\Omega_0 n) = \frac{e^{j2\Omega_0 n} - e^{-j2\Omega_0 n}}{2j},\ e^{j2\Omega_0 n} \overset{DFS}{\Leftrightarrow} \delta[k-2\pm \ell N]\right)$$





The multiplication of two Fourier series coefficients are zero. $c_k = Na_k b_k = 0$. Therefore, the cyclic convolution also becomes zero.

(3-3) **[10 pts]** Find the squared sum $\sum_{n=-\infty}^{\infty} |h[n]|^2$ of a sequence $h[n] = \frac{1}{n\pi}\sin\left(n\frac{\pi}{4}\right)$.

(Hint: I didn't say that n is the time index.)

Answer) $\sum_{n=-\infty}^{\infty} |h[n]|^2 = \frac{1}{4}$

Solution)

$$h[n] = \frac{1}{n\pi}\sin\left(n\frac{\pi}{4}\right) = \frac{1}{4}\mathrm{sinc}\left(\frac{n}{4}\right)$$

Let's rewrite this sequence as $a_k = \frac{1}{4}\mathrm{sinc}\left(\frac{k}{4}\right)$. Then there would be a periodic signal $\tilde{x}_T(t)$ with the Fourier series coefficients a_k. From the Fourier Series expansion of the rectangular function, $\tilde{x}_T(t)$ is given by

$$\tilde{x}_T(t) = \widetilde{\mathrm{rect}}_{T=4L}\left(\frac{t}{L}\right)$$

From the Parseval's relation,

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = \frac{1}{T}\int_T |\tilde{x}_T(t)|^2\, dt$$

$$= \frac{1}{4L}\int_{-L/2}^{L/2} 1^2\, dt$$

$$= \frac{1}{4}$$

(3-4) **[5 pts]** Find the continuous-time Fourier transform $X(j\omega)$ of $x(t) = e^{-a|t|} + 1 \quad (a > 0)$.

(Hint: $e^{j\omega_0 t} \overset{\mathcal{CTFT}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$)

$$e^{-a|t|} \overset{CTFT}{\Leftrightarrow} \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2} \quad , \quad 1 = e^{j0t} \overset{CTFT}{\Leftrightarrow} 2\pi\delta(0)$$

$$X(j\omega) = \int_{-\infty}^{\infty} [x_1(t) + x_2(t)] e^{-j\omega t} dt = \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$
$$= X_1(j\omega) + X_2(j\omega)$$

Therefore, $X(j\omega) = \dfrac{2a}{a^2+\omega^2} + 2\pi\delta(\omega)$.

[Matlab problem]

(10 pts) Suppose that a matrix A is given as follows:

$$A = \begin{bmatrix} 1 & 2 & -1i & -2i \\ 3 & 4 & -3i & -4i \\ 5 & 6 & -5i & -6i \\ 7 & 8 & -6i & -8i \end{bmatrix}$$

Express the output Y of the following Matlab command.

```
Y =  A(2:3,end-3)' * A(2,3:end)';
```

(Hint: mind the trap that was in the exam 2017)

Answer) `29i`
Sol)

A(2:3, end-3) : 2^{nd} and 3^{rd} rows, 4-3=1^{st} column → $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

X': complex conjugate transpose of X
A(2:3, end-3)' → [3, 5]
Likewise,
A(2,3:end) : 2^{nd} row, 3^{rd} and 4^{th} columns of A → [-3i, -4i]
A(2,3:end)' = $\begin{bmatrix} 3i \\ 4i \end{bmatrix}$

Therefore, A(2:3,end-3)' * A(2,3:end)' = $\begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} 3i \\ 4i \end{bmatrix} = 29i$