[Score table]

Prob.	Chapter 1					Chapter 2					Chapter 3				Matlab	Total
	3	3	3	3	3	10	5	5	10	5	10	10	10	10	10	100
Score																

Problems & Solutions

[Chapter 1 | 17 points]

Determine whether or not each of the following statement is true or false. Justify your answers.

(1-1) **[3 pts] (True/False)** $\left(\sqrt{j}\right)^n$ is periodic

(True) $\left(\sqrt{j}\right)^n = \left(\sqrt{e^{j\frac{\pi}{2}}}\right)^n = e^{j\frac{\pi}{4}n} \Rightarrow e^{j\frac{\pi}{4}(n+8m)} = e^{j\frac{\pi}{4}n}e^{j2\pi m} = e^{j\frac{\pi}{4}n}$ (periodic with fundamental period 8)

(1-2) **[3 pts] (True/False)** $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau)d\tau$ is invertible

(True) Assume that $y_1(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x_1(\tau)d\tau$ and $y_2(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x_2(\tau)d\tau$. Then from $y_2(t) = y_1(t) \Rightarrow$ $x_2(t) = x_1(t)$, the system is invertible.

(1-3) **[3 pts] (True/False)** The energy of a signal $x(t)$ is equal to the sum of energies of its even and odd functions, i.e.,

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt$$

(True) $\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} \left\{x_e{}^2(t) + 2x_e(t)x_o(t) + x_o{}^2(t)\right\} dt$. Because $x_e(t)x_o(t)$ is an odd function, its integration will be zero. Therefore, $\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt$.

(1-4) **[3 pts] (True/False)** The system represented by the following input/output relation is linear.

$$y(t) = \int_{t-1}^{t} x(\tau)d\tau + 1$$

(False) If we give $a \cdot x(t)$ as the input signal, $y(t) = a \cdot \int_{t-1}^{t} x(\tau)d\tau + 1 \neq a \cdot y(t)$.

(1-5) **[5 pts]** A complex signal $x(t) = Ae^{j\phi}e^{j\omega t}$ (A is real and $A > 0$). Express the average power of its real part in terms of A. The average power of a signal $f(t)$ is given by

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |f(t)|^2\,dt$$

Answer) $A^2/2$

$\mathrm{Re}\{x(t)\} = A\cdot\cos(\omega t+\phi)$, the average power is $\frac{1}{T}\int_0^T |A\cos(\omega t+\phi)|^2\,dt = \frac{A^2}{2}$.

[Chapter 2 | 35 points]

Consider a causal linear time-invariant (LTI) system described by the difference equation

$$y[n] + e^{-2\beta}y[n-2] = x[n] - e^{-4\beta}x[n-4] \qquad (\beta > 0) \qquad \text{--- (1)}$$

(2-1) **[10 pts]** Find impulse response $h[n]$ of this system.

Answer) $h[n] = \delta[n] - e^{-2\beta}\delta[n-2]$

- Impulse response:

$y[n]$ for $x[n] = \delta[n]$ $\rightarrow$ $y[n] + e^{-2\beta}y[n-2] = \delta[n] - e^{-4\beta}\delta[n-4]$

- Causal LTI system → Condition of initial rest:

$\delta[n] = 0$ for $n < 0$ $\rightarrow$ $y[n] = 0$ for $n < 0$

- Substitute impulse input:

$y[0] = \delta[0] = 1$

$y[1] = \delta[1] = 0$

$y[2] = \delta[2] - e^{-2\beta}y[0] = -e^{-2\beta}y[0] = -e^{-2\beta}$

$y[3] = 0$

$y[4] = -e^{-4\beta}\delta[0] - e^{-2\beta}y[2] = 0$

and $y[n] = 0$ for $n > 4$

Therefore $y[n] = \delta[n] - e^{-2\beta}\delta[n-2]$ and $h[n] = \delta[n] - e^{-2\beta}\delta[n-2]$

(2-2) **[5 pts]** Determine frequency response $H(e^{j\omega})$ of this system.

Answer) For $x[n] = e^{j\omega n}$, $y[n] = H(e^{j\omega})e^{j\omega n}$ → $H(e^{j\omega}) = 1 - e^{-2\beta}e^{-2j\omega}$

(2-3) **[5 pts]** Describe the characteristic of the frequency response $H(e^{j\omega})$ obtained from Prob. (2-2): low-pass, high-pass, band-pass?

Answer) Band-pass

Solution)

$$|H(e^{j\omega})| = \sqrt{(1-e^{-2\beta}\cos(2\omega))^2 + e^{-4\beta}\sin^2(2\omega)} = \sqrt{1+e^{-4\beta} - 2e^{-2\beta}\cos(2\omega)}$$

$$= \sqrt{(1+e^{-2\beta})^2 + 2(1-e^{-2\beta}\cos(2\omega))}$$

: has a maximum when $\omega = \pm\frac{\pi}{2}$.

$|H(e^{j\omega})|$ increases at $\omega \in [0, \frac{1}{2}\pi]$

$|H(e^{j\omega})|$ decreases at $\omega \in [\frac{1}{2}\pi, \pi]$

(2-4) **[10 pts]** Find impulse response $h_{inv}[n]$ of the causal LTI inverse system of (a) such that $h[n] * h_{inv}[n] = \delta[n]$. (Hint: switch $x[n]$ and $y[n]$ of the given differential equation (1) and find its impulse response.)

Answer) $h_{inv}[n] = \begin{cases} e^{-\beta n}u[n] & \text{for even } n, \\ 0 & \text{for odd } n \end{cases}$

Solution) Let switch $x[n] \,\&\, y[n] \rightarrow x[n] + e^{-2\beta}x[n-2] = y[n] - e^{-4\beta}y[n-4]$

- Impulse response:

$y[n]$ for $x[n] = \delta[n] \rightarrow y[n] - e^{-4\beta}y[n-4] = \delta[n] + e^{-2\beta}\delta[n-2]$

- Condition of initial rest:

$\delta[n] = 0$ for $n < 0 \rightarrow y[n] = 0$ for $n < 0$

- Substitute impulse input:

same method as (2-1)

Therefore $h_{inv}[n] = \begin{cases} e^{-\beta n}u[n] & \text{for even } n, \\ 0 & \text{for odd } n \end{cases}$

(2-5) **[5 pts]** Determine whether the derived inverse system is stable or not.

Answer) Stable

$\sum_{k=-\infty}^{\infty} |h_{inv}[k]| = \sum_{k=0}^{\infty} e^{-2\beta k} = \frac{1}{1-e^{-2\beta}}$: converges

[Chapter 3 | 40 pts]

(3-1) **[10 pts]** Suppose that the Fourier series coefficients of a periodic signal $\tilde{x}_T(t)$ with fundamental period $T = 4$ (Figure 3-1) are given by a_k .

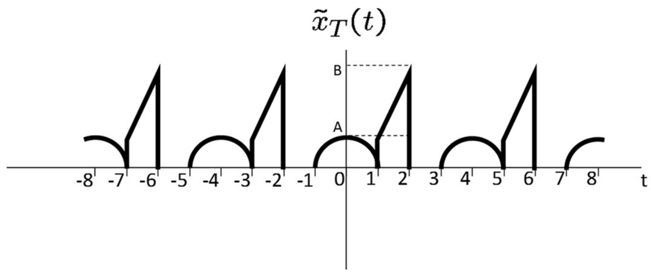


Figure 3-1

Express real and imaginary parts of the Fourier series coefficients b_k of the following signal in terms of a_k .

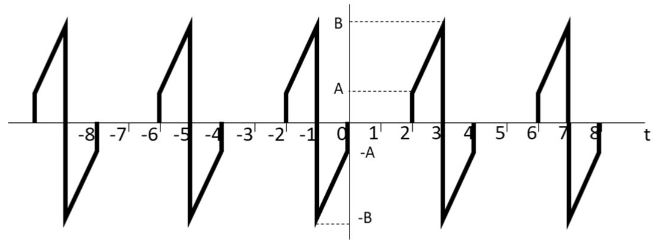


Figure 3-2

Answer)

Real part: $2\sin(k\omega_0)\cdot \mathrm{Im}\{a_k\}$

Imag part: $2\cos(k\omega_0)\cdot \mathrm{Im}\{a_k\}$.

Solution) Figure 3-2 is the 1 sec delay of the odd part of Fig. 3-1 ($\tilde{x}_T(t-1) - \tilde{x}(-(t-1))$).

From the properties of Fourier series, $F.S.\{\tilde{x}_T(t-1)\} = e^{-jk\omega_0}\cdot a_k$ and $F.S.\{\tilde{x}_T(-(t-1))\} = e^{-jk\omega_0}\cdot a_k^*$.

Therefore,

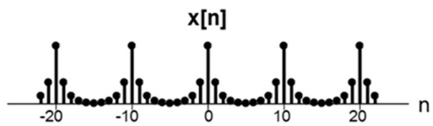
$$F.S.\{\tilde{x}_T(t-1) - \tilde{x}_T(-(t-1))\} = e^{-jk\omega_0}\left(a_k - a_k^*\right)$$
$$= e^{-jk\omega_0}\cdot 2j\cdot \mathrm{Im}\{a_k\}.$$

$\rightarrow$ Real part: $2\sin(k\omega_0)\cdot \mathrm{Im}\{a_k\}$

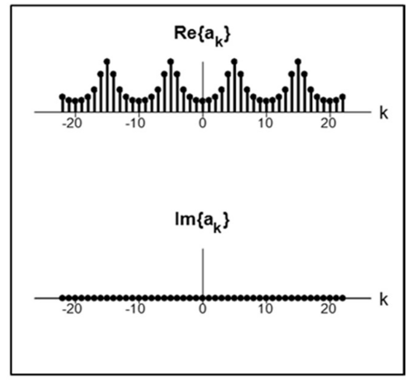
$\rightarrow$ Imag part: $2\cos(k\omega_0)\cdot \mathrm{Im}\{a_k\}$.

(3-2) [**10 pts**] Exponential sequences of period $N = 10$ and their Fourier series coefficients (both real & imaginary parts) are shown below. Find matching pairs & justify your answers.

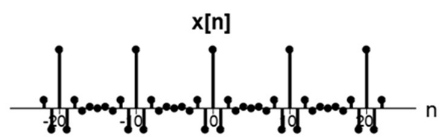
(A)



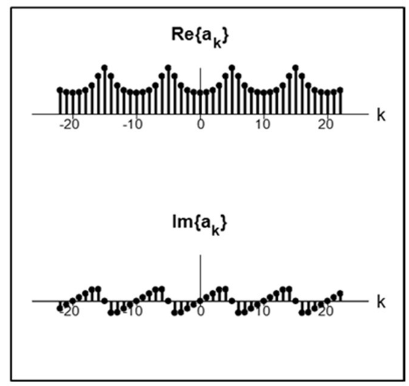
(1)



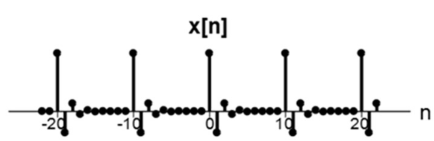
(B)



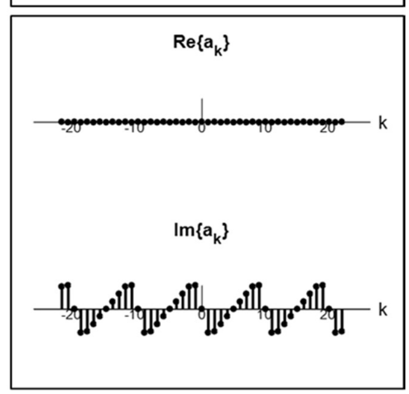
(2)



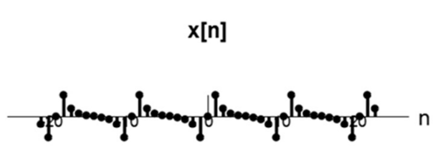
(C)



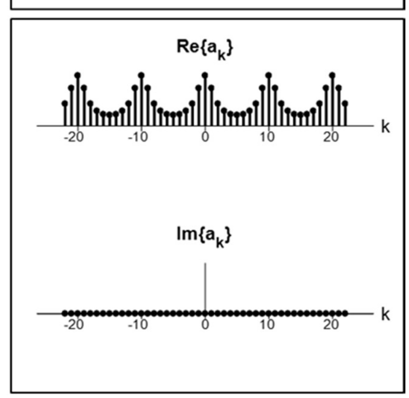
(3)



(D)



(4)



Answer) **(A)- ( ), (B)-( ), (C)-( ), (D)-( )**

Answer) **(A)- (4), (B)-(1), (C)-(2), (D)-(3)**

Solution)

By property of Fourier series, if the $x[n]$ is even function, then a_k is a purely real for all k. Therefore, (A) and (B) must be matched with (1) and (4).

In Fourier series, when $k = 0$, $a_0 = \sum_{n=-\infty}^{\infty} x[n]$. The infinite sum of (A) is larger than (B), so it should be matched with (4) which has larger a_k. Hence, (B) should be matched with (1).

(D) is an odd function, therefore the Fourier series coefficients of (D) are a purely imaginary. As a result, (D) corresponds to (3), and because (C) is neither even nor odd function, (C) should be matched with (2).

(3-3) **[10 pts]** Three signals of the same period and area S are given in Figure 3-3. The Fourier series coefficients of these signals are denoted by a_k. Which one of these signals has the smallest $\sum_{k=1}^{\infty}|a_k|^2$? Justify your answer.

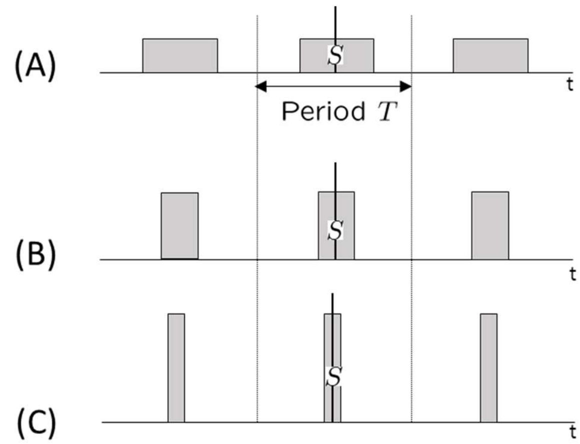


Figure 3-3

Answer) (A)

Solution)

For a real signal, $\sum_{k=1}^{\infty}|a_k|^2 = \frac{1}{2}\left[\left(\sum_{k=-\infty}^{\infty}|a_k|^2\right) - |a_0|^2\right]$

By Parseval's relation, $\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty}|a_k|^2$.

From the property of Fourier series expansion (temporal mean), $a_0 = \frac{1}{T}\int_T x(t)dt$

$$\sum_{k=1}^{\infty}|a_k|^2 = \frac{1}{2}\left[\frac{1}{T}\int_T |x(t)|^2\,dt - \left(\frac{1}{T}\int_T x(t)dt\right)^2\right] \quad \text{--- (1)}$$

For a rectangular signal of duration L and area S, the level of signal is S/L. Therefore, $\frac{1}{T}\int_T |x(t)|^2\,dt = \frac{1}{T}\cdot\left(\frac{S}{L}\right)^2\cdot L = \frac{S^2}{LT}$. Likewise, the temporal mean is given by $a_0 = \frac{S}{T}$.

From (1), $\sum_{k=1}^{\infty}|a_k|^2 = \frac{1}{2}\left[\frac{S^2}{LT} - \frac{S^2}{T^2}\right] = \frac{S^2}{2T}\left[\frac{1}{L} - \frac{1}{T}\right]$.

The largest L gives the smallest $\sum_{k=1}^{\infty}|a_k|^2$. Therefore, (A) has the smallest $\sum_{k=1}^{\infty}|a_k|^2$.

(3-4) **[10 pts]** Suppose that a periodic function $\tilde{f}_T(t)$ has Fourier series coefficients of a_k. Prove that the maximum <u>real</u> value of the Fourier series coefficients b_k of the signal $\tilde{g}_T(t) = \left|\tilde{f}_T(t)\right|^2$ is equal to $\sum_{k=-\infty}^{\infty} |a_k|^2$.

$\tilde{f}_T(t)^* \Leftrightarrow a_{-k}^*$ (conjugation property)

$\tilde{g}_T(t) = \tilde{f}_T(t)^* \tilde{f}_T(t) \quad \Leftrightarrow a_{-k}^* * a_k = \sum_{m=-\infty}^{\infty} a_{-m}^* a_{k-m} = \sum_{m=-\infty}^{\infty} a_m^* a_{k+m}$ (multiplication $\Leftrightarrow$ convolution property)

Therefore, the problem is asking the maximum real value of $b_k = \sum_{m=-\infty}^{\infty} a_m^* a_{k+m}$.

Consider an inequality $\sum_{m=-\infty}^{\infty} | a_m - a_{k+m} |^2 \geq 0$.

$$\Rightarrow \quad \sum_{m=-\infty}^{\infty} | a_m - a_{k+m} |^2 = \sum_{m=-\infty}^{\infty} \left(|a_m|^2 + |a_{k+m}|^2 - a_m^* a_{k+m} - a_m a_{k+m}^* \right) \geq 0$$

Let $I = \sum_{m=-\infty}^{\infty} |a_m|^2 = \sum_{m=-\infty}^{\infty} |a_{k+m}|^2$, and $\sum_{m=-\infty}^{\infty} (a_m^* a_{k+m} + a_m a_{k+m}^*) = \sum_{m=-\infty}^{\infty} 2\,\mathrm{Re}[a_m^* a_{k+m}]$.

The inequality can now be rewritten as

$$\sum_{m=-\infty}^{\infty} |a_m|^2 \geq \mathrm{Re}\left\{ \sum_{m=-\infty}^{\infty} a_m^* a_{k+m} \right\} = \mathrm{Re}\{b_k\}$$

The equality (=) holds when $k = 0 \rightarrow \quad \mathrm{Re}\left\{ \sum_{m=-\infty}^{\infty} a_m^* a_m \right\} = \sum_{m=-\infty}^{\infty} |a_m|^2$. Therefore, the maximum of $\mathrm{Re}\{b_k\}$ is given by $\sum_{k=-\infty}^{\infty} |a_k|^2$

[Matlab problem]

(10 pts) Suppose that a matrix A is given as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Express the output Y of the following operation in Matlab.

```
b=1j *(0:4);

c= zeros(1,8);

c(2) = 1;

Y= c*A*b'
```

Answer) **-** j

Sol)

$$c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 & j & 2j & 3j & 4j \end{bmatrix} \rightarrow b' = \begin{bmatrix} 0 \\ -j \\ -2j \\ -3j \\ -4j \end{bmatrix}$$: conjugate transpose

$$c * A = [-1 \;\; 1 \;\; 0 \;\; 0 \;\; 0]$$ (2nd row)

$$cAb' = [-1\, 1\, 0\, 0\, 0] \begin{bmatrix} 0 \\ -j \\ -2j \\ -3j \\ -4j \end{bmatrix} = -j$$