EE202B Signals and Systems, Spring 2019 **Name:** ______________

Final Exam **Student ID No.:** ______________

June 12, 2019

Time Limit: 9:00 - 11:45 Lecturer: Jung-Woo Choi

Grade Table (for teacher use only)

Question:	1	2	3	4	5	6	Total
Points:	15	35	20	5	20	5	100
Score:							

1. (15 points) Determine whether each of the following statement is true or false.

   (a) (5 points) For the Laplace transform $H(s)$ of a continuous time (CT) LTI system, the system is causal if the region of convergence is on the right half plane.

   > Answer: false
   > When there is a negative time delay ($e^{s\tau}$), the system can be non-causal even if the ROC is on the right half plane.

   (b) (5 points) If a signal $x(t)$ is band-limited within $\omega \in [-\omega_M, \omega_M]$, the Nyquist rate to sample the signal $x^4(t)$ is given by $4\omega_M$.

   > Answer: false
   > When the same signals are multiplied in time, their spectra are convolved in frequency domain. This yields the doubled bandwidth. Therefore, $x^4(t)$ is band-limited in $\omega \in [-\omega_4 M, \omega_4 M]$, and the Nyquist rate to sample this signal is given by $\omega_s = 8\omega_M$.

   (c) (5 points) When the length of causal DT sequence is finite, its region of convergence is the entire z-plane.

   > Answer: false
   > Unlike the Laplace transform, the z-transform of $x[n]$, given by
   > $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$
   > can diverge as $|z|$ goes to zero for any nonzero value of $x[n]$.

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2. (35 points) Consider a LTI system with the condition of initial rest. The system satisfies the following differential equation.

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 2\frac{d^2x(t)}{dt^2} \tag{1}$$

   (a) (5 points) Determine the Laplace transform $H(s)$ and the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$.

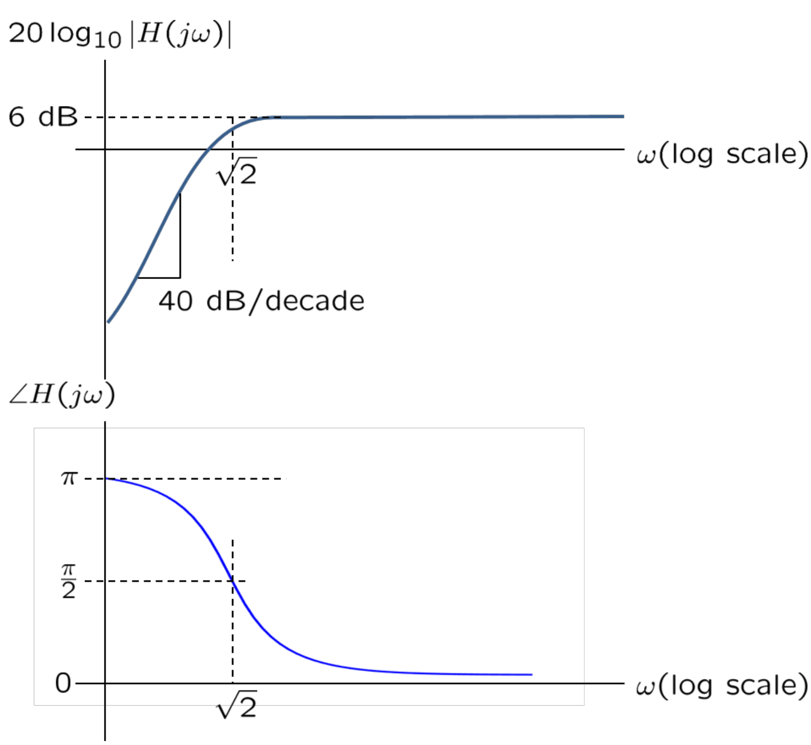
$$
\begin{aligned}
\left((j\omega)^2 + 2(j\omega) + 2\right) Y(j\omega) &= 2(j\omega)^2 X(j\omega) \\
H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} &= \frac{2(j\omega)^2}{(j\omega)^2 + 2(j\omega) + 2} \\
&= \frac{2(j\omega)^2}{(j\omega + 1 + j)(j\omega + 1 - j)}
\end{aligned}
$$

(b) (5 points) Determine whether this system is (a) underdamped (b) critically-damped (c) overdamped.

Answer: (a) underdamped
The system has two complex roots and damping is less than 1 ($\zeta = 1/\sqrt{2} < 1$).

(c) (10 points) Draw the bode plot of the frequency response $H(j\omega)$ derived in the problem (a). Show both magnitude and phase responses.
(Specify (1) the break frequency, (2) slopes of the magnitude (in dB/decade) curve, (3) magnitude (in dB) at the pass-band, and (4) phases at $\omega = 0,\ \infty$, break frequency.)

The denominator of the frequency response has the standard form of the second-order system. $((j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1)$ Therefore, the break frequency ω_n is given by $\sqrt(2)$.
The numerator has the second order zeros with increasing slope. Therefore, the system has a typical 2nd order high-pass response of 20dB/decade slope.
At the pass-band($\omega >> \sqrt(2)$), $|H(j\omega)|$ 2. Accordingly, the pass-band magnitude is 6dB.
For the phase, the numerator's phase begins from π ($-1 = e^{j\pi}$). As the frequency increases, the phase is compensated by the denominator's phase and approaches to 0.



(d) (10 points) For the impulse response $h(t)$ of this system, find the $g(t) = \int_{-\infty}^{t} h(\tau)d\tau$.

(1) From the CFT property on running integral,

$$\int_{-\infty}^{t} h(\tau)d\tau \Leftrightarrow \frac{1}{j\omega}H(j\omega) + \pi H(0)\delta(\omega)$$

Since $H(0) = 0$, only the first term is nonzero.

$$G(j\omega) = \frac{1}{j\omega}H(j\omega) = \frac{2(j\omega)}{(j\omega+1+j)(j\omega+1-j)}$$

Applying the partial fraction expansion yields

$$G(j\omega) = \frac{1}{j\omega}H(j\omega) = 2(j\omega)\cdot\frac{1}{2j}\left(\frac{1}{j\omega+1-j} - \frac{1}{j\omega+1+j}\right)$$

Using the CFT property on the derivative ($dh(t)/dt \Longleftrightarrow (j\omega)H(jw)$) and the relation $e^{-at}u(t) \Longleftrightarrow 1/(j\omega+a)$, we have

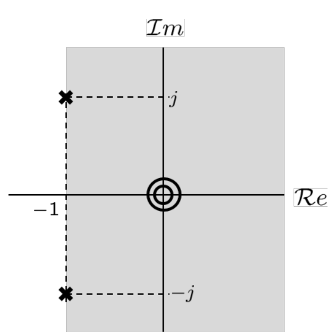
$$\begin{aligned} g(t) &= 2\frac{d}{dt}\left[e^{-t}\frac{1}{2j}\left(e^{jt} - e^{-jt}\right)u(t)\right] \\ &= 2\frac{d}{dt}[e^{-t}\sin(t)u(t)] \\ &= 2\left[(-e^{-t}\sin(t) + e^{-t}\cos(t)]u(t)) + e^{-t}\sin(t)\delta(t)\right] \end{aligned}$$

From the sifting property ($x(t)\delta(t) = x(0)\delta(t)$), the last term is zero. Consequently, we have

$$g(t) = 2e^{-t}[\cos(t) - \sin(t)]u(t)$$

(e) (5 points) Draw the pole-zero plot of $H(j\omega)$ in the s-plane with ROC. Is this system stable?

Answer: stable
When a LTI system satisfies the condition of initial rest, the system is causal. Therefore, the ROC is given by the right half plane from the rightmost pole location. The poles of the given system are at $s = -1 \pm j$, so ROC is given by $Re[s] > -1$. In addition there are two zeros at $s = 0$.
Since the ROC includes the $j\omega$ axis, the system is stable.



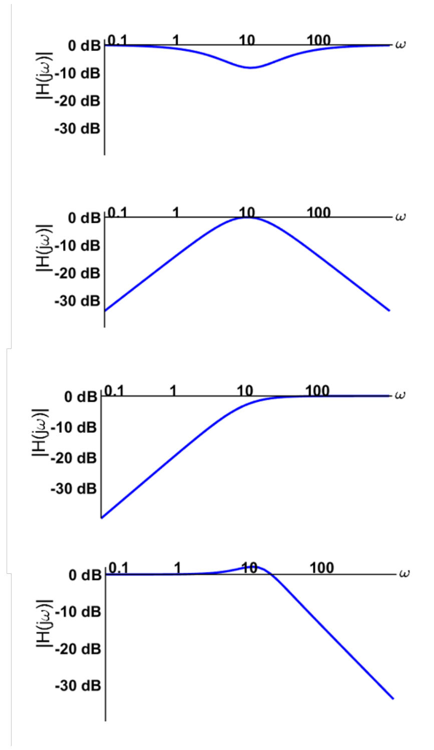
3. (20 points) [5 pts per each correct connection] For each system described in the left pane, find and connect to the corresponding bode plot on the right pane. Assume that all systems are LTI and causal, and satisfy the condition of initial rest.
(y(t):output, x(t):input, h(t):impulse response, H(jω): frequency response)

$$H(j\omega) = \frac{20j\omega}{(j\omega+10)^2} \quad (1) \qquad (A)$$

$$h(t) = 20e^{-10t}\cos(10t)u(t) \quad (2) \qquad (B)$$

$$H(j\omega) = \frac{j\omega-(10+5j)}{j\omega+(10+5j)} \cdot \frac{j\omega+(10-5j)}{j\omega-(10-5j)} \quad (3) \qquad (C)$$

$$H(j\omega) = \frac{j\omega(j\omega+10)}{(j\omega-10)^2} \quad (4) \qquad (D)$$



Answer: (1)- (B), (2) - (D), (3) - (A), (4) - (C)
(Refer to the solution of 2016 Final exam.)

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4. (5 points) A signal $x(t)$ is band-limited within $\omega \in [-\omega_M, \omega_M]$. Determine the Nyquist rate to sample the following signal $y(t)$ without aliasing.

$$y(t) = x(t/2)\sin(2\omega_M t + 3) \tag{2}$$

Answer: $5\omega_M$
From the CFT property $x(at) \Longleftrightarrow 1/|a|X(j\omega/a)$, the first term $x(t/2)$ results in the compression of spectrum and decreases the bandwidth by the factor of two.
Next, the multiplication with $\sin(2\omega_M t + 3)$ yields the modulation of spectrum by $2\omega_M$. Accordingly, the spectrum lies between $\omega_M \in [-\omega_M/2, \omega_M/2] \pm 2\omega_M$. The highest frequency is $2.5\omega_M$, so the Nyquist rate is $2 \times 2.5\omega_M = 5\omega_M$.

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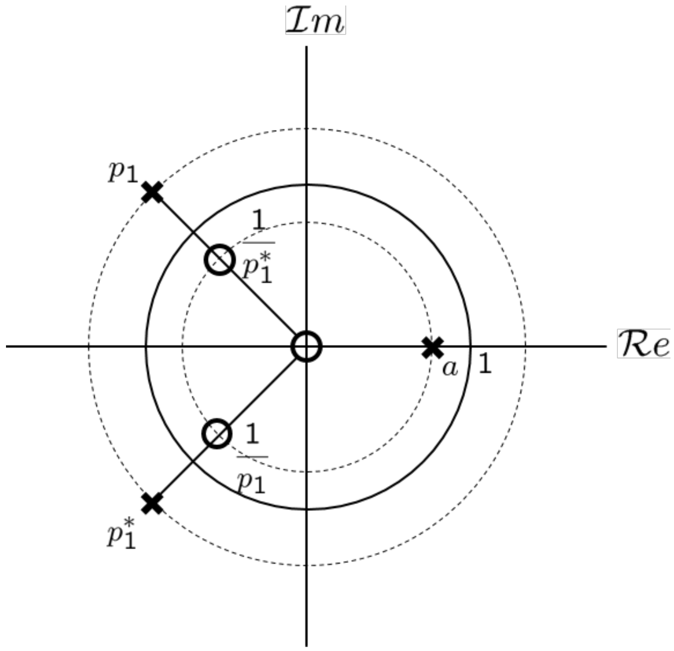
5. (20 points) A pole-zero map of a causal LTI discrete-time system $h[n]$ is shown below:



(a) (5 points) Describe the characteristic of this system:
(A) All-pass (B) High-pass (C) Low-pass (D) Low-shelving (E) High-shelving

Answer: (C) Low-pass The poles and zeros on the left half plane form the all-pass filter, and hence, do not contribute to the magnitude response. The pole on the right half plane and zero at the origin has a form $1/(1 - az^{-1})$ with $a < 1$, so the total system exhibits the low-pass response.

$$\begin{aligned} H(z) &= A\frac{z^{-1} - p_1^*}{1 - p_1 z^{-1}} \cdot \frac{z^{-1} - p_1}{1 - p_1^* z^{-1}} \cdot \frac{1}{1 - az^{-1}} \\ |H(z)| &= \left| \frac{A}{1 - az^{-1}} \right| \end{aligned}$$

where A is the unknown gain of the system.

(b) (5 points) Suppose that the frequency and impulse responses of the system are $H(e^{j\omega})$ and $h[n]$, respectively. When $h[0]$=1, express the zero-frequency magnitude response of the system ($|H(e^{j\omega})|_{\omega=0}$) in terms of a and p_1.

To find the gain of the system at $\omega = 0$, we first need to identify the system gain A.
From the initial value theorem, $h[0] = \lim_{z\to\infty} H(z)$. Substituting $z = \infty$ to the system response $H(z)$ gives

$$A|p_1|^2 = 1 \quad \rightarrow \quad A = 1/|p_1|^2$$

The zero-frequency response therefore can be written as

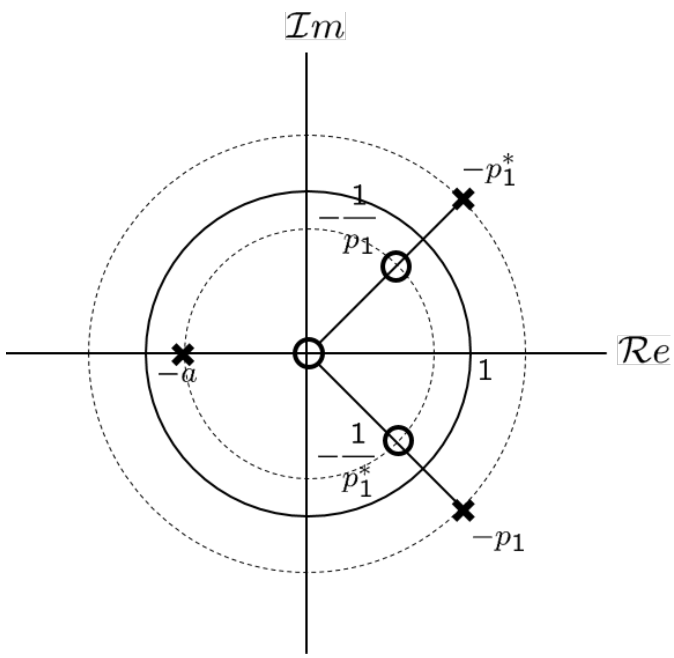
$$|H(1)| = \left|\frac{A}{1-a}\right| = \frac{1}{|p_1|^2(1-a)}$$

(c) (5 points) Draw the pole-zero map of $g[n] = (-1)^n h[n]$.

From the definition of z-transform,

$$\begin{aligned} G(z) &= \sum_{n=-\infty}^{\infty} g[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n h[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} h[n](-z)^{-n} = H(-z) \end{aligned}$$

Therefore, we have $H(-z) = G(z)$, and $-z$ represents the rotation of 180 degree in z-domain.



(d) (5 points) Determine whether $g[n]$ is stable or not.

Answer: unstable
Since the original system is causal, the ROC is outside of the outermost pole p_1. The ROC does not change due to the rotation, so the g[n] has the same ROC. Since the pole p_1 lies outside from the unit circle, the ROC does not include the unit circle. Therefore, the system is unstable.

6. (5 points) [Matlab Problem] A sampled signal vector x of length 100 is transformed using fft() function as shown below:

   X = fft(x(:));

   When the sampling rate for the signal x was f_s=24000 Hz, find the frequency resolution Δf (difference in frequency between two adjacent frequency components).

   Answer: $\Delta f = f_s/N = 240\text{Hz}$