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- Matrix algebra, linear systems. [2.1, 2.2, 3.1, 3.2]
- For an $m \times n$ matrix $A = (a_{ij})$ and a column vector $\mathbf{x} = (x_i)$, the matrix product $A\mathbf{x}$ is

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

  Let A_i denote the ith column of A. Then we have

$$A\mathbf{x} = x_1A_1 + x_2A_2 + \cdots + x_nA_n.$$

  This tells us that $A\mathbf{x}$ is a linear combination of the column vectors of A with coefficients $x_1, x_2, \ldots, x_n$.
- Question: Given a vector $\mathbf{b} = (b_i) \in \mathbb{R}^n$, determine whether $\mathbf{b}$ can be expressed as a linear combination of the columns of A. This is equivalent to finding coefficients $x_1, x_2, \ldots, x_n$ satisfying

$$A\mathbf{x} = x_1A_1 + x_2A_2 + \cdots + x_nA_n = \mathbf{b}.$$

- **System of linear equations.** A linear equation in variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

  where $a_1, a_2, \ldots, a_n$ and b are numbers. If we have several linear equations in the same variables, then we have a system of linear equations. For example,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

  is a linear system with n variables and m equations, which can be written as

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

  If we let $A = (a_{ij})$ be the $m \times n$ matrix with (i, j)-entry a_{ij}, then the above is equivalent to

$$A\mathbf{x} = \mathbf{b}.$$

  The matrix A above is called the coefficients matrix of the linear system. A solution to the above linear system is a vector $\mathbf{x} \in \mathbb{R}^n$ satisfying each equation in the system.
- The **augmented matrix** associated to $A\mathbf{x} = \mathbf{b}$, denoted by $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$.

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

- A vector $\mathbf{v} \in \mathbb{R}^n$ is called a **solution** to a linear system $A\mathbf{x} = \mathbf{b}$, if $A\mathbf{v} = \mathbf{b}$.
- Given a linear system, $A\mathbf{x} = \mathbf{b}$, we can ask some questions:
  - Does it have a solution?
  - If it has a solution, how many?
  - Is there a systematic, or algorithmic method to find all solutions or to conclude that there is no solution?
- **Elementary row operations**
  - Multiply a row by a nonzero constant.
  - Add a multiple of a row to another row.
  - Interchange two rows.
- **Gaussian elimination** of $A\mathbf{x} = \mathbf{b}$: Begin with the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$, apply a sequence of elementary row operations to obtain a row echelon form of A.

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 4 & 6 \\ 3 & 3 & 6 & 8 & 11 \\ 0 & 2 & 2 & 2 & 8 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 3 \\ 3 & 3 & 6 & 8 & 11 \\ 0 & 2 & 2 & 2 & 8 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 & 8 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 3 \\ 0 & 2 & 2 & 2 & 8 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} \mathbf{1} & 1 & 2 & 2 & 3 \\ 0 & \mathbf{1} & 1 & 1 & 4 \\ 0 & 0 & 0 & \mathbf{1} & 1 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 &= 3 \\ x_2 + x_3 + x_4 &= 4 \\ x_4 &= 1 \end{aligned}$$

This system can be solved by **back substitution**, determining values of $x_n, x_{n-1}, \ldots, x_1$ in this order.

$$\begin{aligned} x_4 &= 1 \\ x_3 &= s \quad (s \text{ is a variable}) \\ x_2 &= 4 - x_3 - x_4 = 4 - s - 1 = -s + 3 \\ x_1 &= 3 - x_2 - 2x_3 - 2x_4 = 3 - (-s + 3) - 2s - 2 \cdot 1 = -s - 2 \end{aligned}$$

$$\{(-s-2, -s+3, s, 1) : s \in \mathbb{R}\}$$

$$(x_1, x_2, x_3, x_4) = s(-1, -1, 1, 0) + (-2, 3, 0, 1), \quad s \in \mathbb{R}$$

  - **leading** 1: Every leading 1, total three, is indicated by bold face in the last augmented matrix in the above.
  - **pivot position**: The position of a leading 1 is a pivot position. In the above matrix, pivot positions are $(1, 1), (2, 2), (3, 4)$.
  - **pivot column**: The column of a pivot position is a pivot column. In this case, the first, the second and the fourth columns are pivot columns.
  - **leading variable**: The variable corresponding to the column of a leading 1. In this case, x_1, x_2, x_4 are leading variables.
  - **free variable**: Any variable which is not a leading variable is called a free variable. So free variables correspond to columns which are not pivot columns. In our example, x_3 is a free variable. Free variables are assigned new variables during the back substitution.

- number of free variables, number of pivot positions, number of leading 1's.
  What are the relations among these three numbers?
- **row echelon form of** A: The last matrix of the above example has special properties which make the system easy to solve.

$$\begin{bmatrix} 2 & 2 & 4 & 4 \\ 3 & 3 & 6 & 8 \\ 0 & 2 & 2 & 2 \end{bmatrix} \to \quad \cdots \quad \to \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  A matrix is said to be in a **row echelon form**, if
  - zero rows are at the bottom of the matrix,
  - in each row the first nonzero entry is 1, called a leading 1, and
  - the positions of leading 1's move to the right when rows are scanned from top to bottom.

  A matrix is said to be in a **reduced row echelon form**, if it satisfies the above three conditions and in addition, if
  - each leading 1 is the unique nonzero entry in its column.
- In solving the above linear system, we may continue applying elementary row operations to obtain a reduced row echelon form of A.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

  From the last matrix, we can write down solutions very easily. This method is called the **Gauss-Jordan elimination**.
- A linear system is said to be **consistent**, if it has at least one solution; **inconsistent**, otherwise. The following linear system, which is a slight modification of the previous example, indicated by red), is inconsistent, as can be seen from the last matrix.

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 4 & 4 \\ 3 & 3 & 6 & 6 & 8 \\ 0 & 2 & 2 & 2 & 8 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 2 \\ 3 & 3 & 6 & 6 & 8 \\ 0 & 2 & 2 & 2 & 8 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 2 & 2 & 8 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

- A linear system of the form $A\mathbf{x} = \mathbf{0}$, i.e., $\mathbf{b}$ is the zero vector in $A\mathbf{x} = \mathbf{b}$, is said to be **homogeneous**. If a linear system is not homogeneous, then it is **nonhomogeneous**.
- A homogeneous system $A\mathbf{x} = \mathbf{0}$ has a solution consisting of all zero entries, called the **trivial solution**, and may have other solutions, called **nontrivial solutions**.
- An $n \times n$ matrix is called a square matrix. If A, B are square matrices of size $n \times n$, then both AB and BA are defined and of size $n \times n$.
- Properties of the matrix product. Let A, B, C be $n \times n$ matrices.
  - $A(BC) = (AB)C$: matrix product is associative.
  - In general, $AB \neq BA$: matrix product is not commutative.
- I_n, the identity matrix of order n, the $n \times n$ identity matrix.

$$I_n = (a_{ij}), \quad a_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

  For any $m \times n$ matrix A, $I_m A = A I_n = A$.

- $\mathbf{0}_n$ or $\mathbf{0}_{n\times n}$, the zero matrix of order n. For any $m \times n$ matrix A, $\mathbf{0}_m A = A\mathbf{0}_n = \mathbf{0}_{m\times n}$.
- Note that for any $n \times n$ matrix A, $AI_n = I_n A = A$. I_n is a multiplicative unity. So we wonder if there is a B such that $AB = BA = I_n$. If it exists, B is said to be **invertible** (or **nonsingular**) and is called the **inverse** of A, denoted by A^{-1}.
- Questions:
  - Does every nonzero square matrix have the inverse?
  - Is the inverse, if it exists, unique?
  - Is there a simple method to find the inverse?
- **Power** of a $n \times n$ square matrix A: $A^0 = I_n$, $A^n = AA\cdots A$. If A is invertible, $A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots A^{-1}$.
- **Transpose** of an $m \times n$ matrix A, denoted by A^T, is defined to be the $n \times m$ matrix that is obtained by making the rows of A into columns. $(AB)^T = B^T A^T$.
- Question: If A is invertible, then is A^T also invertible?
- **Trace** of a square matrix A, denoted by $\mathrm{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$