IE241 final Practice solution (even number)

2. Let $V = Y_1 Y_2$, $U = -\ln(V)$

$$F_V(v) = P(Y_1Y_2 \le v) = \int_0^1 P(Y_1 \le \tfrac{v}{y_2} \mid Y_2 = y_2)\cdot f_{Y_2}(y_2)\,dy_2$$

$$P(Y_1 \le \tfrac{v}{y_2} \mid Y_2 = y_2) = \begin{cases} \frac{v}{y_2} & (y_2 \ge v) \\ 1 & (y_2 < v) \end{cases}$$

$$= \int_0^v \underbrace{f_{Y_2}(y_2)}_{=1}\,dy_2 + \int_v^1 \frac{v}{y_2}\cdot \underbrace{f_{Y_2}(y_2)}_{=1}\,dy_2$$

$$= \int_0^v 1\cdot dy_2 + \int_v^1 \frac{v}{y_2}\,dy_2 = v - v\ln(v)$$

$$f_V(v) = \frac{d}{dv}F_V(v) = -\ln(v)$$

$$\therefore f_U(u) = f_V(e^{-u})\cdot\left|\frac{d}{du}e^{-u}\right| = u\cdot e^{-u} \quad (u > 0)$$

4. (a) $\dfrac{(n-1)S_X^2}{\sigma^2} \sim \chi^2(n-1)$, $\dfrac{(m-1)S_Y^2}{\sigma^2} \sim \chi^2(m-1)$

and they are independent. $\therefore W \sim \chi^2_{n+m-2}$

(b) $\bar{X} - \bar{Y} \sim N(\mu_x - \mu_y, \sigma^2(\frac{1}{n} + \frac{1}{m}))$

$$\therefore \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim N(0,1)$$

(c) Remember $T = \dfrac{Z}{\sqrt{V/r}}$ ($Z \sim N(0,1)$, $V \sim \chi^2(r)$, Z & V are independent)

$\bar{X} \perp S_X^2$, $\bar{Y} \perp S_Y^2$, $\bar{X} - \bar{Y} \perp S_X^2$, $\bar{X} - \bar{Y} \perp S_Y^2$

$$\Rightarrow T = \frac{Z}{\sqrt{W/_{n+m-2}}} \sim t_{n+m-2}$$

6.

$$F_Y(y) = \begin{cases} 0 & (y \le 0) \\ \frac{1}{2}y^2 & (0 < y \le 1) \\ y - \frac{1}{2} & (1 < y \le 1.5) \\ 1 & (y > 1.5) \end{cases}, \quad F_U(u) = P(U \le u) = P\left(Y \le \frac{u+4}{10}\right)$$

$$F_U(u) = \begin{cases} 0 & (u \le -4) \\ \frac{1}{2}\left(\frac{u+4}{10}\right)^2 & (-4 < u \le 6) \\ \frac{u}{10} - \frac{1}{10} & (6 \le u \le 11) \\ 1 & (u > 11) \end{cases} \Rightarrow f(u) = \begin{cases} \frac{u+4}{100} & (-4 < u \le 6) \\ \frac{1}{10} & (6 < u \le 11) \\ 0 & (\text{elsewhere}) \end{cases}$$

8. (a) $F_{Y_{(1)}}(y) = P(Y_{(1)} \le y) = 1 - P(Y_{(1)} > y) = 1 - [P(Y_1 > y)]^n$

$= 1 - [1 - F(y)]^n$

$\therefore \quad f_{(1)}(y) = n[1 - F(y)]^{n-1} \cdot f(y)$

$F(y) = 1 - e^{-\frac{(y-2)}{4}}, \quad f_{(1)}(y) = n \cdot e^{-\frac{(y-2)(n-1)}{4}} \cdot \frac{1}{4} e^{-\frac{(y-2)}{4}}$

$\therefore f_{(1)}(y) = \frac{n}{4} \cdot e^{-\frac{n}{4}\cdot(y-2)} \quad (y \ge 2)$

(b) $\int_2^\infty \frac{ny}{4} \cdot e^{-\frac{n}{4}(y-2)}\, dy = \frac{1}{n} + 2$

10. $W = \frac{mV^2}{2}, \quad V = \sqrt{\frac{2W}{m}}, \quad \left|\frac{dv}{dw}\right| = \frac{1}{\sqrt{2mV}}$

$f_w(w) = f(v) \cdot \left|\frac{dv}{dw}\right| = a(2w/m) \cdot e^{-w/kT} \cdot \frac{1}{\sqrt{2mw}} \quad (w \ge 0)$

12. $Y_1, Y_2 \cdots \sim U(-\frac{1}{2}, \frac{1}{2})$, Let $C = \Sigma Y_i \sim N(0, \frac{n}{12})$ 48 $= N(0, 4)$

$P(|C| > 4) = 2 \cdot P(C > 4) = 2 \cdot P(Z > 2) \simeq 0.05$

14. (a) $f(y_1, \cdots y_n \mid \theta) = \frac{1}{\beta^{2n}} \cdot \prod_{i=1}^{n} y_i \cdot e^{-y_i/\beta} = \frac{1}{\beta^{2n}} \cdot e^{-\frac{1}{\beta}\sum y_i} \cdot \left(\prod_{i=1}^{n} y_i\right)$

by factorization thm, $\sum y_i$ is SS for β

As we know, $\sum y_i \sim \text{gamma}(2n, \beta)$, $E(\sum y_i) = 2n\beta$.

(b) by (a), MVUE of β : $\frac{\sum y_i}{2n}$, $Var\left(\frac{\sum y_i}{2n}\right) = \frac{\beta^2}{2n}$

(c) done in class

(d) let $\tilde{\beta} = \frac{Y_1}{2}$, $E(\tilde{\beta}) = \beta$, but does not converge if $n \to \infty$

So $\tilde{\beta}$ not consistent estimator.

16. (a) $f(y \mid \theta) = \frac{3y^2}{\theta^3} \quad (0 \le y \le \theta) = \frac{3y^2}{\theta^3} \cdot I_{(0,\theta)}(y)$, $I_{(0,\theta)}(y) = \begin{cases} 1 & (0 < y < \theta) \\ 0 & (\text{o.w}) \end{cases}$

$L(\theta) = \frac{3^n \cdot \prod_{i=1}^{n} y_i^2}{\theta^3} \cdot \prod_{i=1}^{n} I_{(0,\theta)}(y_i) = \frac{3^n \cdot \prod_{i=1}^{n} y_i^2}{\theta^3} \cdot I_{(0,\theta)}(Y_{(n)})$.

By factorization thm, $Y_{(n)}$ is SS for θ.

Cheak by yourself. (using pdf / CDF or $Y_{(n)}$)

(b) $E(Y_{(n)}) = \frac{3n}{3n+1}\theta$, since $Y_{(n)}$ is SS for θ,

$\frac{3n+1}{3n} Y_{(n)}$ is MVUE for θ. (by Rao-Blackwell thm)

18. $P(\bar{Y} - 1 < \mu < \bar{Y} + 1) = P(-\bar{Y} - 1 < -\mu < -\bar{Y} + 1)$

$= P\left(\frac{-1}{\frac{3}{\sqrt{n}}} < \frac{\bar{Y} - \mu}{\frac{3}{\sqrt{n}}} = Z < \frac{1}{\frac{3}{\sqrt{n}}}\right) = P\left(|Z| < \frac{\sqrt{n}}{3}\right) = 0.95$

$\frac{\sqrt{n}}{3} \simeq 1.96 \Rightarrow n \simeq 35$

20. See HW9 8.93

22. (a), (b) $f(y_1, \cdots y_n \mid \theta) = (\theta+1)^n \left(\prod_{i=1}^{n} y_i\right)^\theta$

$$\log f(y_1, \cdots y_n \mid \theta) = n \log (\theta+1) + \theta \sum_{i=1}^{n} \log y_i$$

By factorization thm, $W_i = \sum \log y_i$ is SS for θ

(c) Let $X = -\log Y$, $f_X(x) = (\theta+1) e^{-\theta x} \cdot \left|-e^{-x}\right| = (\theta+1) e^{-(\theta+1)x}$ $(x>0)$

then $X \sim \exp(\theta+1)$

$E(X) = \frac{1}{\theta+1} = \gamma$, $E(\bar{X}) = E\left(\frac{1}{n} \sum -\log y_i\right) = \frac{1}{\theta+1}$

Since $\sum \log y_i$ is M.S.S, $-\frac{1}{n} \sum \log y_i$ is MVUE of $\frac{1}{\theta+1}$

(d) $Var\left(-\frac{1}{n} \sum \log y_i\right) = \frac{1}{n(\theta+1)^2} \xrightarrow{n\to\infty} 0$: consistent.

(e) by (d), I suggest $\hat{\theta} = \frac{1}{-\frac{1}{n} \sum \log y_i} - 1$