HW#2 Solution

E3.2 A robot-arm drive system for one joint can be represented by the differential equation [8]

$$\frac{dv(t)}{dt} = -k_1 v(t) - k_2 y(t) + k_3 i(t),$$

where $v(t)$ = velocity, $y(t)$ = position, and $i(t)$ is the control-motor current. Put the equations in state variable form and set up the matrix form for $k_1 = k_2 = 1$.

(Ans)

By the relation between velocity and position, we can get the below equation

$$\frac{dy}{dt} = v,$$

and the given statement

$$\frac{dv}{dt} = -k_1 v(t) - k_2 y(t) + k_3 i(t)$$

can be written in matrix form as

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix}\begin{pmatrix} y \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ k_3 \end{bmatrix} i.$$

If we let $u = i, k_1 = k_2 = 1$, for the state variable form,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

we can get the coefficients as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ k_3 \end{bmatrix}, \quad \text{and } \mathbf{x} = \begin{pmatrix} y \\ v \end{pmatrix}.$$

E3.4 Obtain a state variable matrix for a system with a differential equation

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 20u(t).$$

(Ans)

If we define the state vector **x** as

$$\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T, \text{where } x_1 = y(t), x_2 = y(t)', x_3 = y(t)'',$$

then we can get the differential equation as follows:

$$x_3' = -8x_1 - 6x_2 - 4x_3 + 20u(t).$$

Now, you can obtain the state variable matrix as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -6 & -4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0] .$$

E3.5 A system is represented by a block diagram as shown in Figure E3.5. Write the state equations in the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

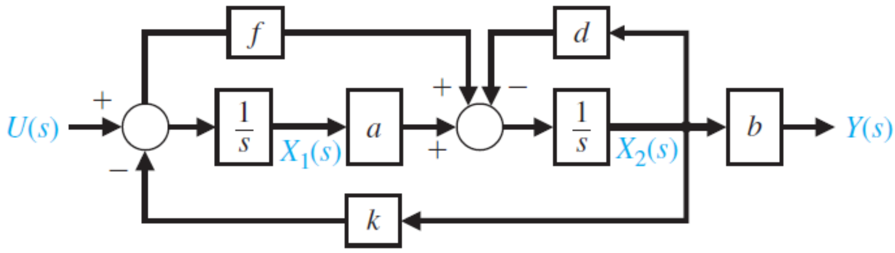


FIGURE E3.5 Block diagram.

(Ans)

From the block diagram we determine that the state equations are

$$\dot{x_2} = -(fk + d)x_2 + ax_1 + fu$$
$$\dot{x_1} = -kx_2 + u$$

and the output equation is

$$y = bx_2.$$

Define a state vector **x** like

$$\mathbf{x} = [x_1 \quad x_2]^T$$

then you can derive the state equations in a matrix from as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -k \\ a & -(fk + d) \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ f \end{bmatrix}, \qquad \mathbf{C} = [0 \quad b], \qquad \mathbf{D} = [0].$$

E3.9 A multi-loop block diagram is shown in Figure E3.9. The state variables are denoted by $x_1(t)$ and $x_2(t)$. (a) Determine a state variable representation of the closed-loop system where the output is denoted by $y(t)$ and the input is $r(t)$. (b) Determine the characteristic equation.

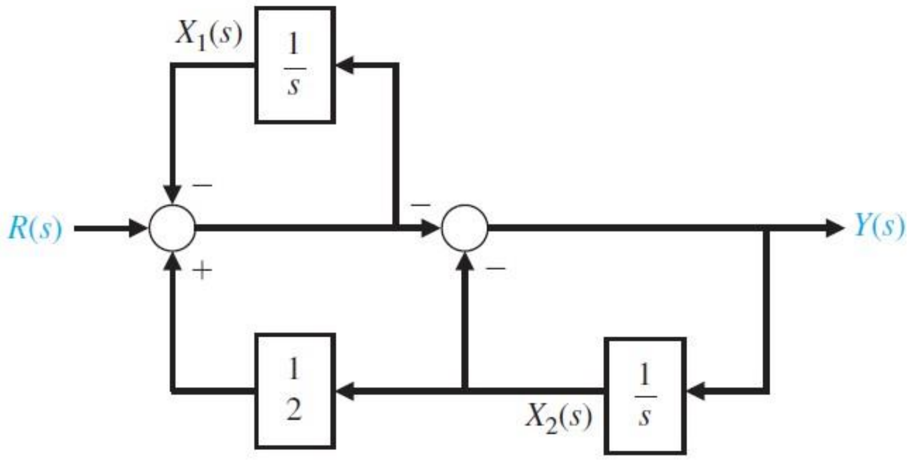


FIGURE E3.9 Multi-loop feedback control system.

(Ans)

Analyzing the block diagram yields

$$\dot{x_1} = -x_1 + \frac{1}{2}x_2 + r$$

$$\dot{x_2} = x_1 - \frac{3}{2}x_2 - r$$

$$y = x_1 - \frac{3}{2}x_2 - r.$$

Define a state vector $\mathbf{x}$ like

$$\mathbf{x} = [x_1 \quad x_2]^T.$$

In a state-variable form, we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & \frac{1}{2} \\ 1 & -\frac{3}{2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} r, \ \ y = \begin{bmatrix} 1 & -\frac{3}{2} \end{bmatrix}\mathbf{x} - r.$$

The characteristic equation is obtained from $\det[s\mathbf{I} - \mathbf{A}]$ as follows:

$$s^2 + \frac{5}{2}s + 1 = (s+2)\left(s + \frac{1}{2}\right) = 0.$$

E3.11 Determine a state variable representation for the system described by the transfer function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{4(s+3)}{(s+2)(s+6)}.$$

(Ans)

Case 1: Phase variable canonical form

The transfer function is

$$T(s) = \frac{4s+12}{s^2+8s+12}.$$

A state variable representation is

$$\mathbf{x} = \mathbf{A}\dot{\mathbf{x}} + \mathbf{B}r$$
$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -12 & -8 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 12 & 4 \end{bmatrix}.$$

(Another answer)

Case 2: Input feedforward canonical form

$$\mathbf{A} = \begin{bmatrix} -8 & 1 \\ -12 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(Another answer)

Case 3: Diagonal canonical form

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 3 \end{bmatrix}.$$

E3.16 Two carts with negligible rolling friction are connected as shown in Figure E3.16. An input force is $u(t)$. The output is the position of cart 2, that is, $y(t) = q(t)$. Determine a state space representation of the system.

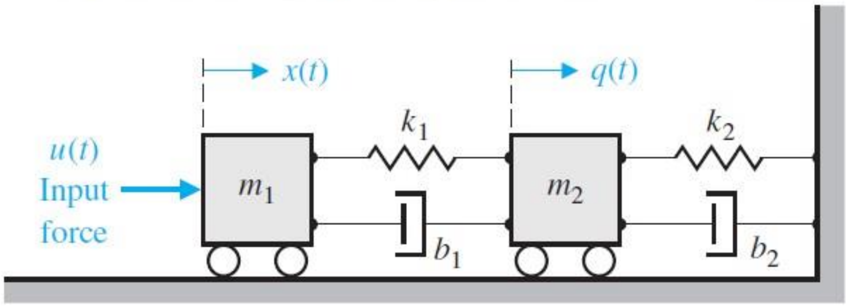


FIGURE E3.16 Two carts with negligible rolling friction.

(Ans)

The governing equations of motion are

$$m_1\ddot{x} + k_1(x-q) + b_1(\dot{x}-\dot{q}) = u(t)$$
$$m_2\ddot{q} + k_2 q + b_2\dot{q} + b_1(\dot{q}-\dot{x}) + k_1(q-x) = 0.$$

Let $x_1 = x$, $x_2 = \dot{x}$, $x_3 = q$ and $x_4 = \dot{q}$. Then,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_1}{m_1} & -\dfrac{b_1}{m_1} & \dfrac{k_1}{m_1} & \dfrac{b_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \dfrac{k_1}{m_2} & \dfrac{b_1}{m_2} & -\dfrac{k_1+k_2}{m_2} & -\dfrac{b_1+b_2}{m_2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \dfrac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u(t).$$

Since the output is $y(t) = q(t)$, then

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x}.$$

E3.20 For the simple pendulum shown in Figure E3.20, the nonlinear equations of motion are given by

$$\ddot{\theta}(t) + \frac{g}{L}\sin\theta(t) + \frac{k}{m}\dot{\theta}(t) = 0,$$

where g is gravity, L is the length of the pendulum, m is the mass attached at the end of the pendulum (assume the rod is massless), and k is the coefficient of friction at the pivot point.

(a) Linearize the equations of motion about the equilibrium condition $\theta_0 = 0°$.

(b) Obtain a state variable representation of the system. The system output is the angle $\theta(t)$.

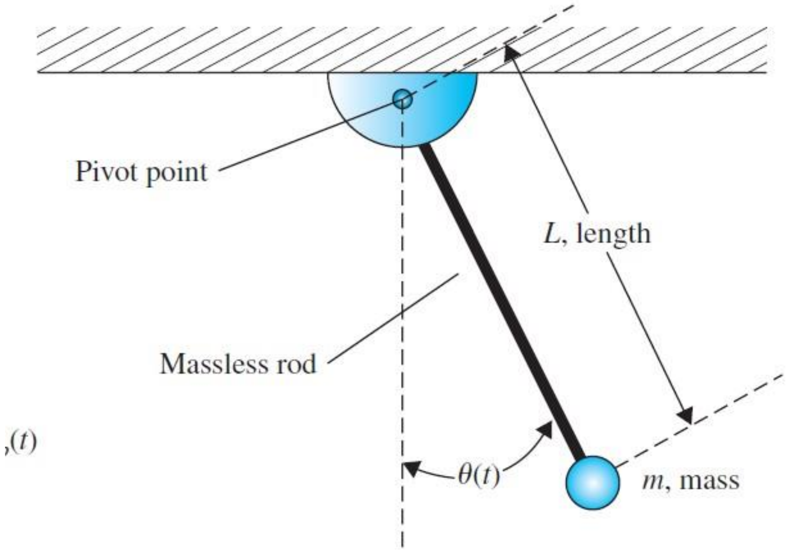


FIGURE E3.20 Simple pendulum.

(Ans)

(a) The linearized equation can be derived from the observation that $\sin\theta \approx \theta$ when $\theta \approx 0$. In this case, the linearized equation is

$$\ddot{\theta}(t) + \frac{k}{m}\dot{\theta}(t) + \frac{g}{L}\theta(t) = 0.$$

(b) Let the state variable $\mathbf{x} = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$ and system output $y = \theta$. Then, the system in state variable form can be represented as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$
$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{k}{m} \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0], \text{ and } \mathbf{x}(0) = \begin{bmatrix}\theta(0) \\ \dot{\theta}(0)\end{bmatrix}.$$

E3.23 Consider a system modeled via the third-order differential equation

$$\dddot{x}(t) + 3\ddot{x}(t) + 3\dot{x}(t) + x(t) = \dddot{u}(t) + 2\ddot{u}(t) + 4\dot{u}(t) + u(t).$$

Develop a state variable representation and obtain a block diagram of the system assuming the output is $x(t)$ and the input is $u(t)$.

(Ans)

By applying Laplace Transform, the given differential equation can be expressed as follows:

$$(s^3 + 3s^2 + 3s + 1)X(s) = (s^3 + 2s^2 + 4s + 1)U(s)$$
$$\rightarrow \frac{X(s)}{U(s)} = \frac{s^3 + 2s^2 + 4s + 1}{s^3 + 3s^2 + 3s + 1}.$$

To obtain a block diagram of the system, we can develop the equation as follows:

$$\frac{X(s)}{U(s)} = \frac{s^3 + 2s^2 + 4s + 1}{s^3 + 3s^2 + 3s + 1} \cdot \frac{Z_1(s)}{Z_1(s)}$$

where

$z_1(t)$: a new state variable, $Z_1(s) = \mathcal{L}\{z_1(t)\}$.

Then, we can derive the following equations:

$$X(s) = (s^3 + 2s^2 + 4s + 1)Z_1(s)$$
$$\rightarrow x(t) = \dddot{z}_1(t) + 2\ddot{z}_1(t) + 4\dot{z}_1(t) + z_1(t) \quad (8.1)$$
$$U(s) = (s^3 + 3s^2 + 3s + 1)Z_1(s)$$
$$\rightarrow u(t) = \dddot{z}_1(t) + 3\ddot{z}_1(t) + 3\dot{z}_1(t) + z_1(t). \quad (8.2)$$

Putting (8.2) into (8.1), we get

$$\dddot{z}_1(t) = -3\ddot{z}_1(t) - 3\dot{z}_1(t) - z_1(t) + u(t)$$
$$x(t) = -\ddot{z}_1(t) + \dot{z}_1(t) + u(t)$$

And define a state variable **z**

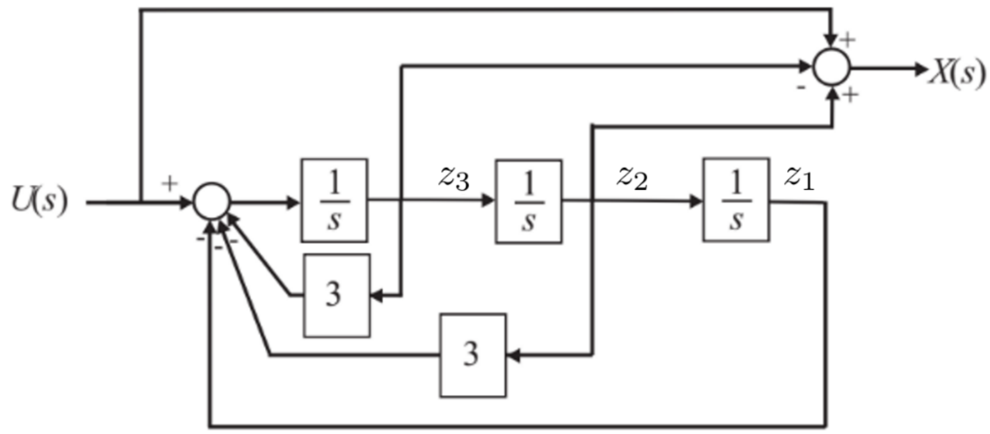
$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \text{ and } z_2(t) = \dot{z}_1(t), z_3(t) = \dot{z}_2(t) = \ddot{z}_1(t)$$

Then, the system in state variable form and block diagram can be represented as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$
$$x = \mathbf{C}\mathbf{z} + \mathbf{D}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix},\ \mathbf{D} = [1].$$



P3.7 An automatic depth-control system for a robot submarine is shown in Figure P3.7. The depth is measured

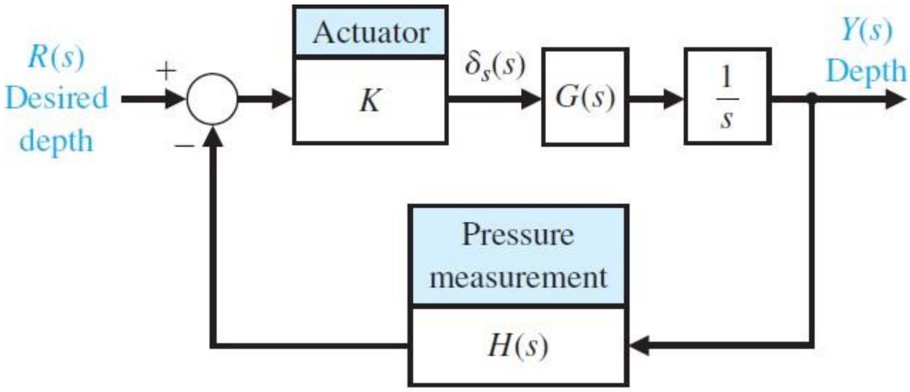


FIGURE P3.7 Submarine depth control.

by a pressure transducer. The gain of the stern plane actuator is $K = 1$ when the vertical velocity is 25 m/s. The submarine has the transfer function

$$G(s) = \frac{(s+1)^2}{s^2+1},$$

and the feedback transducer is $H(s) = 2s + 1$. Determine a state variable representation for the system.

(Ans)

Given $K = 1$, we have

$$KG(s) \cdot \frac{1}{s} = \frac{(s+1)^2}{s(s^2+1)}.$$

And from the given block diagram, the transfer function can be represented as follows:

$$\frac{Y(s)}{R(s)} = T(s) = \frac{kG(s)}{s + H(s) \cdot kG(s)}.$$

We then compute the closed-loop transfer function as

$$T(s) = \frac{s^2+2s+1}{3s^3+5s^2+5s+1} = \frac{\frac{1}{3}s^2+\frac{2}{3}s+\frac{1}{3}}{s^3+\frac{5}{3}s^2+\frac{5}{3}s+\frac{1}{3}}.$$

By defining a state vector $\mathbf{x}$, the state variable model is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/3 & -5/3 & -5/3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \mathbf{x}\,,$$

where $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T$ and $x_1 = x(t), x_2 = x(t)', x_3 = x(t)''$.

P3.10 Many control systems must operate in two dimensions, for example, the x- and the y-axes. A two-axis control system is shown in Figure P3.10, where a set of state variables is identified. The gain of each axis is K_1 and K_2, respectively. (a) Obtain the state differential equation. (b) Find the characteristic equation from the **A** matrix. (c) Determine the state transition matrix for $K_1 = 1$ and $K_2 = 2$.

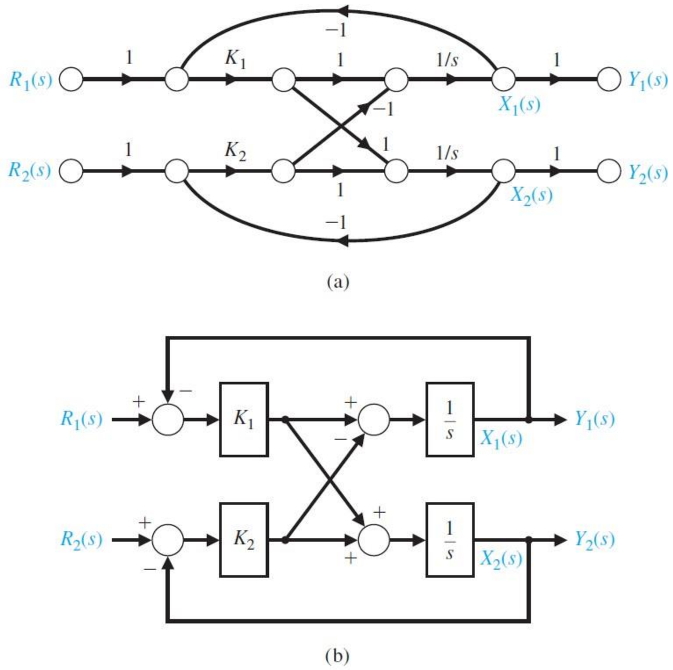


FIGURE P3.10 Two-axis system. (a) Signal-flow graph. (b) Block diagram model.

(Ans)

(a) From the signal flow diagram, we determine that a state-space model is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -K_1 & K_2 \\ -K_1 & -K_2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} K_1 & -K_2 \\ K_1 & K_2 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x}.$$

(b) The characteristic equation is

$$\det[s\mathbf{I} - \mathbf{A}] = s^2 + (K_2 + K_1)s + 2K_1K_2 = 0\,,$$

$$\text{where } \mathbf{A} = \begin{bmatrix} -K_1 & K_2 \\ -K_1 & -K_2 \end{bmatrix}.$$

(c) When $K_1 = 1$ and $K_2 = 2$,

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -1 & -2 \end{bmatrix}.$$

The state transition matrix associated with $\mathbf{A}$ in s-domain is

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \begin{bmatrix} s+2 & 1 \\ -2 & s+1 \end{bmatrix}.$$

Then, the state transition matrix is finally derived as follows:

$$\phi(t) = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}$$

$$= e^{-\frac{3}{2}t} \begin{bmatrix} \cos(\alpha t) + \frac{1}{2\alpha}\sin(\alpha t) & \frac{2}{\alpha}\sin(\alpha t) \\ -\frac{1}{\alpha}\sin(\alpha t) & \cos(\alpha t) - \frac{1}{2\alpha}\sin(\alpha t) \end{bmatrix}, \text{where } \alpha = \frac{\sqrt{7}}{2}.$$