Spring 2024 IE 241 – Final Practice

1. Let a random variable Y have a uniform distribution on $[0, 1]$.

   (a) Calculate the moment generating function of $X = -2\ln(Y)$. (Hint: $Y = e^{\ln(Y)}$)

   (b) Show that X has a Gamma distribution. Identify α and β of the Gamma distribution.

2. Let Y_1 and Y_2 be independent and uniformly distributed over the interval $(0, 1)$. Find the probability distribution of $U = -\ln(Y_1 Y_2)$.

3. Let $Y_1, Y_2, \ldots, Y_n$ be independent continuous random variables with distribution function $F(y)$ and density function $f(y)$. Denote the ordered random variables Y_i by $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$, where $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$.

   (a) Derive the density function of $Y_{(n)}$.

   (b) Let $Y_1, Y_2, \ldots, Y_n$ be independent continuous random variables, each with a beta distribution, with $\alpha = \beta = 2$. Find the density function of $Y_{(n)}$.

4. Suppose we have two independent samples from two different normal distributions with the same variance but possibly different expected values:

$$(X_1, X_2, \cdots, X_n) \overset{iid}{\sim} N(\mu_X, \sigma^2), \quad (Y_1, Y_2, \cdots, Y_m) \overset{iid}{\sim} N(\mu_Y, \sigma^2).$$

   Denote the sample means by $\bar{X}$ and $\bar{Y}$, and the sample variances by

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \quad S_Y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y})^2.$$

   (a) Show that

$$W = \frac{(n-1)S_X^2}{\sigma^2} + \frac{(m-1)S_Y^2}{\sigma^2} \sim \chi^2_{n+m-2}.$$

   (b) Show that

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim N(0, 1).$$

   (c) Using (a) and (b), show that

$$T = \frac{Z}{\sqrt{W/(n+m-2)}} \sim t_{n+m-2}.$$

5. Let a random variable Y have a uniform distribution on $[0, 1]$.

(a) Calculate the moment generating function of $X = -2\ln(Y)$. (Hint: $Y = e^{\ln(Y)}$)

(b) Show that X has a Gamma distribution. Identify α and β of the Gamma distribution.

6. A supplier of kerosene has a weekly demand Y possesing a probability density function given by

$$f(y) = \begin{cases} y, & 0 \leq y \leq 1 \\ 1, & 1 < y \leq 1.5 \\ 0, & \text{elsewhere} \end{cases}$$

with measurements in hundreds of gallons. The supplier's profit is given by $U = 10Y - 4$. Find the density function of U.

7. Suppose that $Y_1, Y_2, \ldots, Y_n$ is a random sample from a normal distribution with mean μ and variance σ^2.

(a) Find $E(S^2)$ and $V(S^2)$ where

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2.$$

(b) Show that

$$U = \frac{(Y_1 - Y_2)^2}{2\sigma^2}$$

follows a χ^2 distribution with 1 df.

(c) Show that

$$V = \frac{(Y_3 - Y_4)}{\sqrt{(Y_1 - Y_2)^2}}$$

follows a t distribution with 1 df and

$$W = \frac{(Y_3 - Y_4)^2}{(Y_1 - Y_2)^2}$$

follows a F distribution with (1,1) df.

8. Let $Y_1, Y_2, \ldots, Y_n$ be independent continuous random variables with distribution function $F(y)$ and density function $f(y)$. Denote the ordered random variables Y_i by $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$, where $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$.

(a) Derive the density function of $Y_{(1)}$.

(b) In (a), suppose that

$$f(y) = \frac{1}{4}e^{-(y-2)/4}, \quad y \geq 2,$$

and 0 for $y < 2$. Find $E(Y_{(1)})$.

9. Suppose that a unit of mineral ore contains a proportion Y_1 of metal A and a proportion Y_2 of metal B. Experience has shown that the joint probability density function of Y_1 and Y_2 is given by

$$f(y_1, y_2) = 2, \quad 0 \le y_1 \le 1,\ 0 \le y_2 \le 1;\ 0 \le y_1 + y_2 \le 1.$$

Let $U = Y_1 + Y_2$, the proportion of either meta A or B per unit. Find the probability density function of U.

10. The speed of a molecule in a uniform gas at equilibrium is a random variable V whose density function is given by

$$f(v) = av^2 e^{-mv^2/(2kT)}, \quad v > 0$$

where k, T, and m denote Boltzmann's constant, the absolute temperature, and the mass of the molecule, respectively. Derive the distribution of $W = mV^2/2$, the kinetic energy of the molecule.

11. Let $W_1, W_2, \ldots, W_r$ be independent Geometric random variables with the probability of success given by p.

    (a) Find the probability function of $Y = \sum_{i=1}^{r} W_i$.

    (b) Find the probability function of W_1 given $\sum_{i=1}^{r} W_i = m$.

12. Assume that when a number is rounded off to an integer, the round-off error is represented as a uniform random variable on $[-\frac{1}{2}, \frac{1}{2}]$. Suppose that 48 numbers are rounded off and then added. Find the probability that the accumulated round-off error exceeds 4 in its absolute value .

13. Suppose that Y_1 is a random variable from $N(0, 1^2)$, Y_2 is a random variable from $N(0, 2^2)$, and Y_3 is a random variable from $N(0, 3^2)$. Note that there is no unique answer for the following questions.

    (a) Suggest a statistic that is a function of all three Y_i, and that has chi-square distribution with 1 degrees of freedom.

    (b) Suggest a statistic that has F distribution with 1 and 2 degrees of freedom.

    (c) Suggest a statistic that has t distribution with 2 degrees of freedom.

14. Let $Y_1, \ldots, Y_n$ be i.i.d. random variables from Gamma$(2, \beta)$, with probability density function

$$f(y) = \frac{1}{\beta^2} y e^{-y/\beta}, \quad y > 0.$$

(a) Find a sufficient statistic for β. Find the expectation of the sufficient statistic that you found.

(b) Find a MVUE of β. Find the variance of your MVUE.

(c) Propose a consistent estimator of $\text{Var}(Y) = 2\beta^2$. Give a reason why your estimator is consistent.

(d) Can you find an unbiased estimator of β that is not consistent? Give a reason why your estimator is not consistent.

15. Suppose that $Y_1, \ldots, Y_n$ is a random sample from a probability density function

$$f(y|\alpha, \theta) = \alpha y^{\alpha-1}/\theta^{\alpha}, \quad 0 \leq y \leq \theta$$

Assume that θ is known.

(a) Find an estimator for α by using the method of moments.

(b) Find the MLE of α.

(c) Find the MLE of $E(Y)$.

16. Let $Y_1, \ldots, Y_n$ be a random sample from a population with density function

$$f(y|\theta) = \frac{3y^3}{\theta^3}, \quad 0 \leq y \leq \theta$$

(a) Show that $Y_{(n)}$ is sufficient for θ.

(b) Find a MVUE of θ.

17. Suppose that $Y_1, \ldots, Y_n$ constitute a random sample from a population with probability density function

$$f(y) = \frac{1}{\theta+1} e^{-y/(\theta+1)}, \ y > 0, \theta > -1.$$

(a) Show that $\bar{Y}$ is biased.

(b) Find MSE of $\bar{Y}$.

(c) Suggest an unbiased estimator for θ, by modifying $\bar{Y}$.

(d) Find the variance of the estimator in (c).

18. Let $\bar{Y}$ be the mean of a random sample of size n from $N(\mu, 9)$. Find n such that $P(\bar{Y} - 1 < \mu < \bar{Y} + 1) = .95$.

19. Let Y be a random sample from a distribution with the pdf

$$f(y|\theta) = e^{-(y-\theta)}, \quad y > \theta.$$

(a) Show that $Y - \theta$ is a pivotal quantity.

(b) Construct a 95% confidence interval for θ.

20. A factory operates with two machines of type I and one machine of type II. The weekly repair costs, X, for type I machines are normally distributed with mean μ_1 and variance σ^2. The weekly repair costs, Y, for type II machines are also normally distributed, but with mean μ_2 and variance $3\sigma^2$. Therefor the expected repair cost per week for the factory is $2\mu_1 + \mu_2$. Suppose you are given a random sample $X_1, X_2, \ldots, X_n$ on the repair costs of type I machines, and an independent random sample $Y_1, Y_2, \ldots, Y_n$ on the repair costs of type II machines. Show how you would construct a 90% confidence interval for $2\mu_1 + \mu_2$

    (a) if σ^2 is known.

    (b) if σ^2 is not known.

21. In a poll taken among college students, 300 of 500 fraternity men favored a certain proposition whereas 64 of 100 nonfraternity men favored it. Estimate the difference in proportions favoring the proposition and place a 2-standard deviation bound on the error of estimation. How many fraternity and nonfraternity men must be included in a poll if we wish to obtain an estimate, correct to within .05, for the difference in the proportions favoring the proposition? Assume that the groups will be of equal size and that $p = .6$ will suffice as an approximation of both proportions.

22. Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with the density function

$$f(y|\theta) = (\theta + 1)y^\theta, \quad 0 \le y \le 1$$

    (a) Find a sufficient statistic for θ.

    (b) Show that $W_n = \sum_{i=1}^n \log Y_i$ is also a sufficient statistic for θ.

    (c) Find a MVUE of $(\theta + 1)^{-1}$.

    (d) Show that the MVUE in (c) is consistent.

    (e) Suggest a consistent estimator of θ.

23. Let $Y_1, \ldots, Y_n$ be i.i.d. random variables from normal distribution with mean 0 and variance σ^2.

    (a) Find a (minimal) sufficient statistic for σ^2.

    (b) Find a MVUE of σ^2.

    (c) Suggest a consistent estimator of $\sigma = \sqrt{\sigma^2}$.

24. Let $Y_1, \ldots, Y_7$ be i.i.d. random variables from Gamma$(2, \beta)$.

   (a) Find a (minimal) sufficient statistic for β.

   (b) Find an exact 95% confidence interval for β. (Hint: Use a pivotal quantity)

   (c) Use (incorrectly) the central limit theorem to calculate an approximate 95% confidence interval for β.
   (Hint: $\frac{\bar{Y}-E(\bar{Y})}{\sqrt{V(\bar{Y})}} \approx N(0,1)$)

   (d) Discuss the difference between the two intervals. Which has wider length?

25. A certain type of electronic component has a life time Y (in hours) with probability density function given by
$$f(y|\theta) = \frac{1}{\theta^2} y e^{-y/\theta}, \quad y > 0.$$

   (a) Find the MLE θ of θ based on a sample of size n from this Gamma distribution.

   (b) Find $E(\hat{\theta})$ and $V(\hat{\theta})$.

   (c) What is the MLE for the variance of Y?