**EE202B Signals and Systems, Spring 2019** **Name:** ______________

**Mid-term Exam** **Student ID No.:** ______________
**April 17, 2019**

**Time Limit: 9:00 - 11:45** Lecturer: Jung-Woo Choi

Grade Table (for teacher use only)

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Points: | 25 | 15 | 25 | 20 | 10 | 5 | 100 |
| Score: | | | | | | | |

1. (25 points) Determine whether each of the following statement is <u>true or false</u>. Justify your answers. ("Justify your answer" means you have to provide a reasoning.)

   (a) (5 points) The system described by $y(t) = x(t-2) + 1$ is linear.

   > Answer: **False**
   > For $y_1(t) = x_1(t-2) + 1$ and $y_2(t) = x_2(t-2) + 1$, $y_1(t) + y_2(t) \neq x_1(t-2) + x_2(t-2) + 1$. The constant addition (+1) makes the system nonlinear.

   (b) (5 points) If two signals $\tilde{x}[n]$ and $\tilde{y}[n]$ have fundamental periods $M$ and $N$, respectively, then the fundamental period of the signal $z[n] = x[n]y[n]$ is given by the least common multiple of $M$ and $N$. $(M,\ N > 0)$

   > Answer: **False**
   > For example, the multiplication of the same signal $x[n] = e^{j\omega_0 n}$ doubles the frequency: $x[n]x[n] = e^{j(2\omega_0)n}$ and halves the fundamental period. The statement is only true for the addition of two signals.

   (c) (5 points) When a system described by the LCCDE satisfies the condition of initial rest, the system's input-output relation can be written as $y(t) = \int_0^\infty h(\tau)x(t-\tau)d\tau$.

   > Answer: **True**
   > If a system described by LCCDE satisfies the condition of initial rest, the system is LTI and **causal**. The causal LTI system's impulse response $h(t)$ has non-zero values for $t > 0$. Therefore, the convolution integral can be written as $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_0^\infty h(\tau)x(t-\tau)d\tau$.

   (d) (5 points) The system described by $y(t) = \int_{-\infty}^{t} e^{(t-\tau)}x(\tau)d\tau$ is stable.

   > Answer: **False**
   > The integral is equivalent to the convolution $y(t) = (e^t u(t)) * x(t)$. Since the system's output is described by the convolution integral, the system is LTI. The total power of a LTI system's impulse response should be finite to be stable. However, $h(t) = e^t u(t)$ diverges as $t \to \infty$.

(e) (5 points) The system described by $y(t) = \int_{-\infty}^{t} e^{(t-\tau)}x(\tau)d\tau$ is causal.

Answer: **True**
The system's impulse response $h(t) = e^{t}u(t)$ is zero for $t < 0$.

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2. (15 points) Consider a LTI system described by the following LCCDE:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2x(t) \tag{1}$$

(a) (5 points) Determine the output $y(t)$ of the system for the input $x(t) = 2\cos\left(\frac{3}{2}t + 1\right)$.

The given input signal can be decomposed into two frequency components (Euler's relation):

$$\begin{aligned} x(t) &= 2\cos\left(\frac{3}{2}t + 1\right) \\ &= e^{j\left(\frac{3}{2}t+1\right)} + e^{-j\left(\frac{3}{2}t+1\right)} \end{aligned}$$

The complex exponential function is the eigenfunction of LCCDE. Therefore, the output is also described as the combination of two exponential functions: $y(t) = Ae^{j\frac{3}{2}t} + Be^{-j\frac{3}{2}t}$. Solving the LCCDE with this $x(t)$ and $y(t)$ for each frequency yields:

$$A\left(-\frac{9}{4} + 3j \times \frac{3}{2} + 2\right)e^{j\frac{3}{2}t} = 2e^{j} \cdot e^{j\frac{3}{2}t} \rightarrow A = -\frac{8e^{j}}{1 - 18j}$$
$$B\left(-\frac{9}{4} - 3j \times \frac{3}{2} + 2\right)e^{-j\frac{3}{2}t} = 2e^{-j} \cdot e^{-j\frac{3}{2}t} \rightarrow B = -\frac{8e^{-j}}{1 + 18j}$$

Accordingly, the output $y(t)$ is given by

$$y(t) = -\frac{8e^{j}}{1 - 18j}e^{j\frac{3}{2}t} - \frac{8e^{-j}}{1 + 18j}e^{-j\frac{3}{2}t}$$

(b) (10 points) With the condition of initial rest, calculate the impulse response $h(t)$ of the system.

To find an impulse response, we need to solve the DE with $x(t) = \delta(t)$

$$\begin{aligned} \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) &= 2x(t) \\ \frac{1}{2}\frac{d^2h(t)}{dt^2} + \frac{3}{2}\frac{dh(t)}{dt} + h(t) &= \delta(t) \end{aligned}$$

This problem is equivalent to

$$\frac{1}{2}\frac{d^2h(t)}{dt^2} + \frac{3}{2}\frac{dh(t)}{dt} + h(t) = 0$$

with auxiliary conditions

$$\frac{1}{2}\frac{dh(t)}{dt}\bigg|_{t=0^+} = 1, \quad h(0^+) = 0$$

The solution is therefore given by

$$h(t) = -2e^{-2t}u(t) + 2e^{-t}u(t)$$

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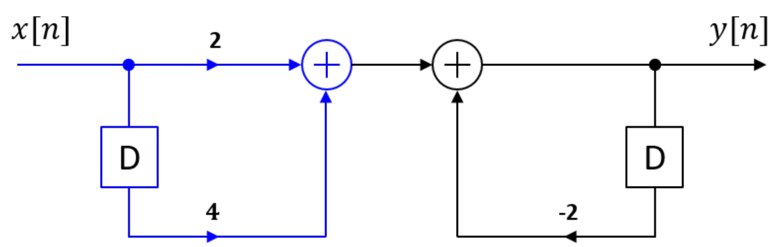
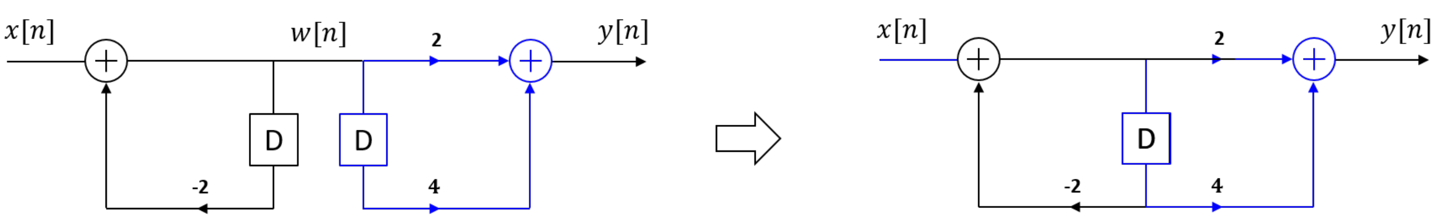
3. (25 points) A discrete-time LTI system is described by the following block diagram




(a) (5 points) Derive the LCCDE of the given system.

$$y[n] = 2x[n] + 4x[n-1] - 2y[n-1]$$

(b) (5 points) Draw the block diagram for the Direct form II realization.




IIR FIR

Switching the order of FIR & IIR systems

Direct form II realization

(c) (5 points) When the system satisfies the condition of initial rest, determine the impulse response $h[n]$ of this system.

First, derive the impulse response of the IIR part: $w[n] = x[n] - 2w[n-1]$.
For $x[n] = \delta[n]$,

$$\begin{aligned} n = 0 &\rightarrow w[0] = 1 - 0 = 1 \\ n = 1 &\rightarrow w[1] = 0 - 2 = -2 \\ n = 2 &\rightarrow w[2] = 0 - 2 \times -2 = 4 \\ \ldots &\quad w[n] = (-2)^n u[n] \end{aligned}$$

Next, the FIR part:

$$\begin{aligned} y[n] &= 2w[n] + 4w[n-1] \\ &= 2(-2)^n u[n] + 4(-2)^{n-1} u[n-1] \\ &= (-2)^n (2u[n] - 2u[n-1]) \\ &= 2 \cdot (-2)^n \delta[n] \\ &= 2\delta[n] \text{ (sifting property: } \delta[n]x[n] = \delta[n]x[0]) \end{aligned}$$

(d) (5 points) Determine whether the system is stable or not. Justify your answer.

Answer: **Stable**

As derived in (c), squared sum of system's impulse response has finite value. Therefore, the system is stable.

(e) (5 points) Derive the frequency response $H(e^{j\omega})$ of the system.

Let $y[n] = Y(e^{j\omega})e^{j\omega n}$, $x[n] = X(e^{j\omega})e^{j\omega n}$.
Then, from the difference equation,

$$\begin{aligned} Y(e^{j\omega})(1 + 2e^{-j\omega}) &= X(e^{j\omega})(2 + 4e^{-j\omega}) \\ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} &= \frac{2 + 4e^{-j\omega}}{1 + 2e^{-j\omega}} = 2 \end{aligned}$$

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4. (20 points) Let $\tilde{x}[n]$ be an arbitrary real-valued signal with the fundamental period $N$. Its even and odd parts are denoted by

$$\tilde{x}_e[n] = \mathcal{E}v\{\tilde{x}[n]\}$$
$$\tilde{x}_o[n] = \mathcal{O}d\{\tilde{x}[n]\}.$$

(a) (5 points) Show that

$$\sum_{n=<N>} \tilde{x}^2[n] = \sum_{n=<N>} \tilde{x}_e^2[n] + \sum_{n=<N>} \tilde{x}_o^2[n] \tag{2}$$

By definition, $\tilde{x}_e[n] = \frac{\tilde{x}[n]+\tilde{x}[-n]}{2}$ and $\tilde{x}_o[n] = \frac{\tilde{x}[n]-\tilde{x}[-n]}{2}$.

$$\begin{aligned} \tilde{x}_e^2[n] + \tilde{x}_o^2[n] &= \frac{\tilde{x}^2[n] + \tilde{x}^2[-n]}{2} \\ \longrightarrow \sum_{n=<N>} \tilde{x}_e^2[n] + \tilde{x}_o^2[n] &= \sum_{n=<N>} \tilde{x}^2[n] + \sum_{n=<N>} \tilde{x}^2[-n] \\ \text{However, } \sum_{n=<N>} \tilde{x}^2[-n] &= \sum_{n=<N>} \tilde{x}^2[n]. \text{ (sum of a periodic signal over a single period)} \\ \therefore \sum_{n=<N>} \tilde{x}^2[n] &= \sum_{n=<N>} \tilde{x}_e^2[n] + \sum_{n=<N>} \tilde{x}_o^2[n] \end{aligned}$$

(b) (10 points) Let $b_k$ and $c_k$ be the real and imaginary parts of the Fourier series coefficients $a_k$ of $\tilde{x}[n]$, respectively.

$$\tilde{x}[n] \overset{\mathcal{FS}}{\Longleftrightarrow} a_k = b_k + jc_k \tag{3}$$

For the periodic convolution of two coefficients $d_k = b_k * c_k$, find the value of $d_0$.

Because $\tilde{x}[n]$ is a real-valued function, its even part $\tilde{x}_e[n]$ has Fourier series coefficients $b_k$, and $\tilde{x}_o[n]$ has $jc_k$.

$$\tilde{x}_e[n] \overset{\mathcal{FS}}{\Longleftrightarrow} b_k, \quad \tilde{x}_o[n] \overset{\mathcal{FS}}{\Longleftrightarrow} jc_k$$

The convolution of two Fourier series coefficiens is the multiplication in time domain. Therefore, we have

$$\tilde{x}_e[n]x_o[n] \overset{\mathcal{FS}}{\Longleftrightarrow} b_k * jc_k = d_k$$

The value of $d_k$ at $k = 0$ corresponds to the mean value of its time signal:

$$d_0 = \frac{1}{N} \sum_{n=<N>} \tilde{x}_e[n]\tilde{x}_o[n]$$

However, from the result of Eq.(2), $\sum_{n=<N>} \tilde{x}_e[n]\tilde{x}_o[n]$ is equal to zero because:

$$\begin{aligned} \sum_{n=<N>} \tilde{x}^2[n] &= \sum_{n=<N>} (\tilde{x}_e[n] + \tilde{x}_o[n])^2 \\ &= \sum_{n=<N>} \tilde{x}_e^2[n] + \sum_{n=<N>} \tilde{x}_o^2[n] \end{aligned}$$

Therefore, $d_0 = 0$.

(c) (5 points) Calculate the mean value of $c_k$.

The mean value of of $c_k$ corresponds to the zero-time value in time domain. Since $jc_k$ is the Fourier series coefficients of the odd function $x_o[n]$, the zero-time value of $x_o[0] = 0$.

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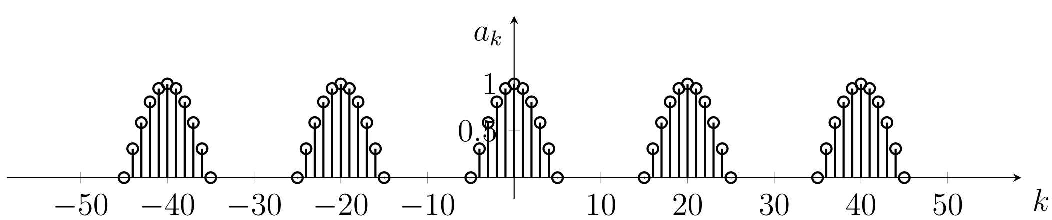
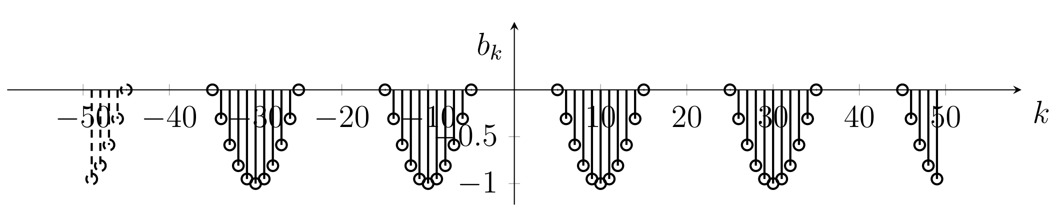
5. (10 points) The real-valued Fourier series coefficients $a_k$ of a periodic discrete-time sequence $\tilde{x}[n]$ is given as follows:




(a) (5 points) The Fourier series coefficients $b_k$ of a signal $\tilde{y}[n]$ is also shown below. Using the relation of two graphs, express $\tilde{y}[n]$ in terms of $\tilde{x}[n]$.

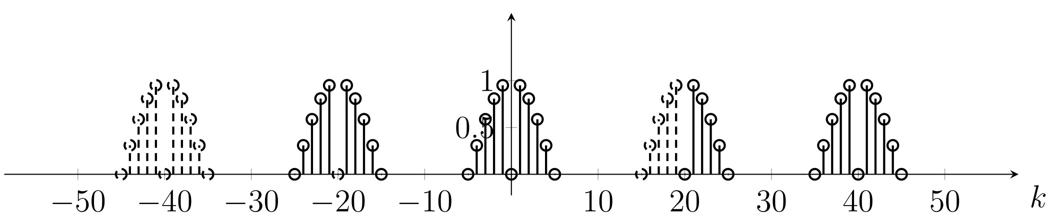


The fundamental period of $a_k$ is $N = 20$.
The signal $b_k$ is obtained by shifting $a_k$ by half of its period ($N/2 = 10$) and inverting its sign. Therefore,

$$\begin{aligned} b_k &= -a_{k-10} \\ \tilde{y}[n] &= -\tilde{x}[n]e^{j10\omega_0 n} \\ &= -\tilde{x}[n]e^{j10\frac{2\pi}{N}n} = -\tilde{x}[n]e^{j\pi n} \\ &= -(-1)^n\tilde{x}[n] \end{aligned}$$

(b) (5 points) Plot the Fourier series coefficients of a signal $\tilde{x}[n] - 1$ (Also mark the maximum magnitude and fundamental period).

Fourier Series coefficients of a constant are given by a series of impulses $\delta[k - nN]$ for the period $N$ and arbitrary integer $n$. Therefore, the subtraction (-1) reduces the zero frequency component and its periodic replicas by one.

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6. (5 points) [Matlab Problem] Complete a code for calculating the convolution of signals $x[n] = [7,\ 5,\ 3,\ 1]$ and $h[n] = [2,\ 4]$. Choose correct expressions among the given examples to complete the code. Then calculate the result of convolution $y[n]$.

Answer: (A)- ( 4 ), (B) - ( 3 ), (C) - ([ 14 38 26 14 4 ] )

A. (1) x=7:-2:end (2) x=fliplr(1:7) (3) x=flipud(1:2:7) (4) x=7:-2:1

B. (1) h=2ˆ(1:2) (2) h=2:4 (3) h=2.ˆ(1:2) (4) h=2:1:4

C. Calculate the result of y = conv(h,x);