EE202A Signals and Systems, Fall 2022
December 14, 2022
Time Limit: 13:00 - 15:45 Lecturer: Jung-Woo Choi

Score Table (for teacher use only)

Question:	1	2	3	4	5	Total
Points:	20	20	20	15	25	100
Score:						

This is a CLOSED-BOOK exam.
Please provide ALL DERIVATIONS and EXPLANATIONS with your answers.
Any communication with others during the exam will be regarded as a cheating case.

1. (20 points) Fourier Transform

   (a) (10 points) Derive the inverse Fourier transform $x(t)$ of the spectrum

$$X(j\omega) = \int_{-\infty}^{\omega} Y(j\eta)d\eta$$

in terms of $y(t)$. (Do not answer simply using duality. Provide full derivations)
(Hint) Utilize properties of inverse Fourier transform:

$$\delta(\omega) \overset{ICTFT}{\Longleftrightarrow} \frac{1}{2\pi}, \qquad \frac{d}{d\omega}A(j\omega) \overset{ICTFT}{\Longleftrightarrow} (-jt)a(t), \qquad X(j\omega) * Y(j\omega) \overset{ICTFT}{\Longleftrightarrow} 2\pi x(t)y(t)$$

(Answer)

$$x(t) = -\frac{1}{jt}y(t) + \pi y(0)\delta(t)$$

(Solution) The running integral is the convolution with unit step function. Therefore,

$$X(j\omega) = Y(j\omega) * H(j\omega), \quad x(t) = 2\pi y(t)h(t) \tag{1}$$

where $H(j\omega)$ is the unit step function defined in the frequency domain, and $h(t)$ is its inverse Fourier transform. Let's define the sign function as

$$H(j\omega) = \frac{1}{2}(1 + \mathrm{SGN(j\omega)}) \tag{2}$$

The above equation satisfies

$$\frac{d}{d\omega}H(j\omega) = \delta(\omega) = \frac{1}{2}\frac{d}{d\omega}\mathrm{SGN(j\omega)}$$

Since the sign function has no dc value, its inverse Fourier transform can be derived as

$$\mathrm{SGN}(j\omega) \overset{ICTFT}{\Longleftrightarrow} \mathrm{sgn}(t) = \frac{1}{-j\pi t}$$

(Using properties of CTFT)

$$\delta(\omega) \overset{ICTFT}{\Longleftrightarrow} \frac{1}{2\pi}$$

$$\frac{d}{d\omega}A(j\omega) \overset{ICTFT}{\Longleftrightarrow} (-jt)a(t)$$

From this result and (1)-(2), the inverse Fourier transform of $X(j\omega)$ is given by

$$x(t) = -\frac{1}{jt}y(t) + \pi y(t)\delta(t) = -\frac{1}{jt}y(t) + \pi y(0)\delta(t) \quad \text{(sifting property)}$$

(b) (10 points) Find the time signal $x(t)$ whose Fourier transform is given by

$$X(j\omega) = \begin{cases} \frac{\omega+1}{2} & \text{for } |\omega| < 1 \\ 0 & \text{for } |\omega| > 1 \end{cases}$$

(Answer)

$$x(t) = -\frac{\sin t}{j2\pi t^2} + \frac{e^{jt}}{2\pi jt}$$

(Solution) Consider $Y(j\omega)$ defined as

$$Y(j\omega) = \frac{d}{d\omega}X(j\omega) = \begin{cases} \frac{1}{2} & \text{for } |\omega| < 1 \\ 0 & \text{for } |\omega| > 1 \end{cases} \overset{ICTFT}{\Longleftrightarrow} y(t) = \frac{\sin t}{2\pi t}$$

Then, $X(j\omega) = \int_{-\infty}^{\omega} Y(j\eta)d\eta - U(j(\omega - 1))$. From the result of prob. 1a,

$$\begin{aligned} x(t) &= -\frac{1}{jt}y(t) + \pi y(0)\delta(t) - \frac{1}{2\pi}\left(\frac{-1}{jt} + \pi\delta(t)\right)e^{jt} \\ &= -\frac{\sin t}{j2\pi t^2} + \frac{\delta(t)}{2} + \frac{1}{2\pi}\frac{1}{jt}e^{jt} - \frac{\delta(t)}{2} \quad (y(0) = 1/2\pi) \\ &= -\frac{\sin t}{j2\pi t^2} + \frac{e^{jt}}{2\pi jt} \end{aligned}$$

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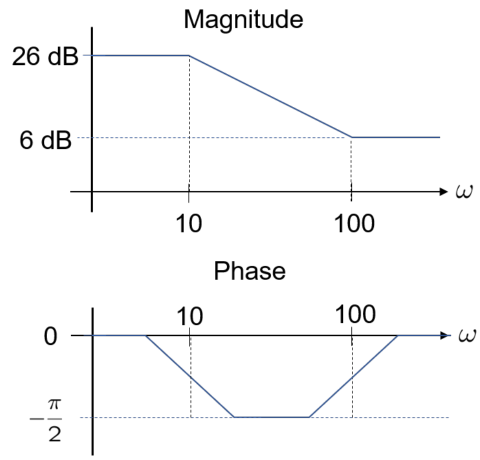
2. (20 points) For LTI systems satisfying the condition of initial rest, answer to the following questions.

   (a) (10 points) The bode plot of the system is shown below. Derive its (A) frequency response $H(j\omega)$ (B) impulse response $h(t)$ and (C) linear constant coefficient differential equation.



(Answer)

$$H(j\omega) = 2\frac{j\omega + 100}{j\omega + 10}, \qquad h(t) = 2\delta(t) + 180e^{-10t}u(t), \qquad \frac{dy(t)}{dt} + 10y(t) = 2\frac{dx(t)}{dt} + 200x(t)$$

(Solution) Gain at zero frequency is 26 dB $= 20\log_{10} 20$.

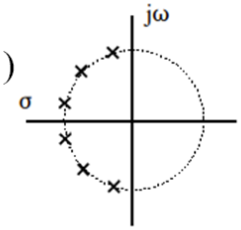
$$\begin{aligned} H(j\omega) &= 20 \cdot \frac{10}{j\omega + 10} \cdot \frac{j\omega + 100}{100} \\ &= 2\frac{j\omega + 100}{j\omega + 10} \\ h(t) &= F^{-1}\{2\frac{j\omega + 100}{j\omega + 10}\} \\ &= 2F^{-1}\{1 + \frac{90}{j\omega + 10}\} \\ &= 2(\delta(t) + 90e^{-10t}u(t)) \end{aligned}$$

From the frequency response, the LCCDE can be written as

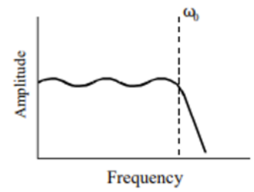
$$\frac{dy(t)}{dt} + 10y(t) = 2\frac{dx(t)}{dt} + 200x(t)$$

(b) (10 points) Three pole-zero plots in the s-domain are shown in the left column of the figure. Find matching frequency responses from the right column.
(Answer in the following form: (1)–(A), (2)–(B), (3)–(C))

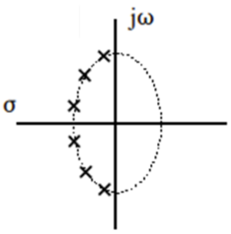
(1)



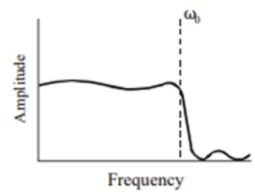
(A)



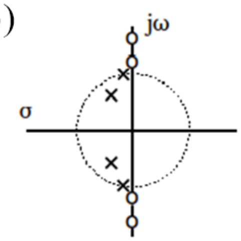
(2)



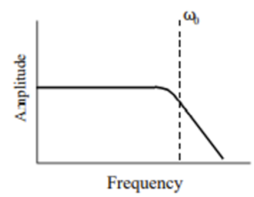
(B)



(3)



(C)



(Answer) (1) - (C), (2) - (A), (3) - (B)
(Solution)
In the first pole-zero plot of (1), poles are on a circle of the same radius. Therefore, the break frequencies of 2nd-order systems are all the same, and only damping ratios are different. This corresponds to (c), which has no fluctuations in the pass-band and stop-band.
Meanwhile, the poles of (2) have different radii, which yield the three pass-band peaks due to the combination of three 2nd-order responses of different break frequencies.
The last one, (3), consists of two 2nd-order poles and zeros, which give two pass-band peaks and two stop-band notches.

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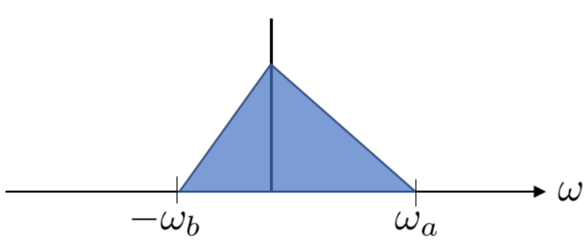
3. (20 points) [ST] For a CT signal $x(t)$ given by

$$x(t) = \frac{\sin(3\pi t)}{\pi t} e^{j2\pi t},$$

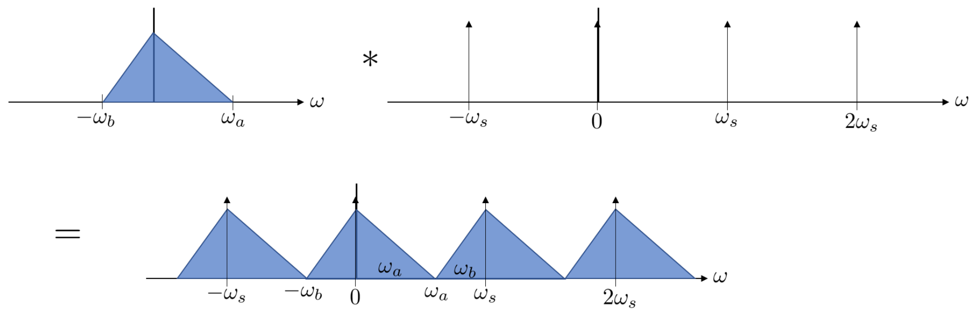
determine the Nyquist sampling rate by answering the following questions.

(a) (5 points) Consider an asymmetric signal $Y(j\omega)$ defined in the frequency domain as shown below:



Determine the Nyquist sampling rate ω_s to sample the given signal without aliasing artifact.

(Answer) $\omega_s = \omega_a + \omega_b$
(Solution) The aliasing artifact occurs due to the convolution with periodic impulses of period ω_s. To avoid the overlap between repeated spectra, the period of impulses should be greater than $\omega_s = \omega_a + \omega_b$ (see figure below).



(b) (10 points) Determine the Fourier transform $X(j\omega)$ of the signal $x(t) = \frac{\sin(3\pi t)}{\pi t} e^{j2\pi t}$.

(Answer) $X(j\omega) = \begin{cases} 1 & \text{for} \quad -\pi < \omega < 5\pi \\ 0 & \text{otherwise} \end{cases}$
(Solution) From the properties of Fourier transform

$$\frac{\sin(Wt)}{\pi t} \overset{CTFT}{\Longleftrightarrow} \text{Rect}_{2W}(\omega) = \begin{cases} 1 & \text{for} \quad -W < \omega < W \\ 0 & \text{otherwise} \end{cases}$$

$$e^{j\omega_0 t} \overset{CTFT}{\Longleftrightarrow} 2\pi\delta(\omega - \omega_0)$$

$$y(t)z(t) \overset{CTFT}{\Longleftrightarrow} \frac{1}{2\pi} Y(j\omega) * Z(j\omega),$$

CTFT of $x(t)$ is given by

$$\begin{aligned} X(j\omega) &= \delta(\omega - 2\pi) * \text{Rect}_{6\pi}(\omega) \\ &= \text{Rect}_{6\pi}(\omega - 2\pi) \\ &= \begin{cases} 1 & \text{for} \quad -\pi < \omega < 5\pi \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3}$$

(c) (5 points) Determine the Nyquist rate ω_s and the corresponding sampling time T to sample the signal $X(j\omega)$ without aliasing. ($\omega_s = ?$, $T = ?$)

(Solution) Utilizing the results of (3) and problem 4-(a), the Nyquist rate is given by

$$\omega_s = 5\pi + \pi = 6\pi, \quad T = \frac{2\pi}{\omega_s} = \frac{1}{3}$$

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4. (15 points) For a CT signal $x(t) = e^{-t}u(t)$, answer the following questions.

(a) (5 points) Determine Laplace transform $X(s)$ of $x(t)$ and region of convergence (ROC).

(Answer) $X(s) = \frac{1}{s+1}$, ROC: $\text{Re}\{s\} > -1$.
(Solution)

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(s+1)t}dt = -\frac{1}{s+1}e^{-(s+1)t}\Big|_{t=0}^{\infty} \\ &= \frac{1}{s+1} \quad \text{for} \quad \text{Re}\{s\} > -1 \end{aligned}$$

(b) (10 points) [ST] The signal $x(t)$ is sampled with the sampling period T. For $x_c(t)$ given by

$$x_c(t) = \sum_{n=0}^{\infty} e^{-nT}\delta(t - nT),$$

derive its Laplace transform $X_c(s)$ and determine the ROC. (Hint: use the property of Laplace transform $\delta(t-\tau) \overset{\mathcal{L}}{\Longleftrightarrow} e^{-s\tau}$ (ROC: entire s-plane))

(Answer)

$$X(s) = \frac{1}{1 - e^{-(s+1)T}}, \quad \text{ROC: Re}\{s\} > -1$$

(Solution) From the property of Laplace transform,

$$\begin{aligned} \delta(t-\tau) &\overset{\mathcal{L}}{\Longleftrightarrow} e^{-s\tau} \quad \text{(ROC: entire s-plane)} \\ \sum_{n=0}^{\infty} e^{-nT}\delta(t-nT) &\overset{\mathcal{L}}{\Longleftrightarrow} X(s) = \sum_{n=0}^{\infty} e^{-nT}e^{-nsT} \\ X(s) &= \sum_{n=0}^{\infty}(e^{-(s+1)T})^n = \frac{1}{1-e^{-(s+1)T}} \end{aligned} \tag{4}$$

The above series converges when $\left|e^{-(s+1)T}\right| < 1$. Therefore, the exponential function $e^{-(\sigma+1)T}$ should be less than 1 for $s = \sigma + j\omega$. This condition gives the ROC specified as $\text{Re}\{s\} > -1$.

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5. (25 points) Z-transform problems

(a) (10 points) Let $x[m]$ be a signal whose rational z-transform $X(z)$ contains a pole at $z = 1/2$. Given that

$$x_1[n] = \left(\frac{1}{4}\right)^n x[n]$$

is absolutely summable and

$$x_2[n] = \left(\frac{1}{8}\right)^n x[n]$$

is not absolutely summable, determine whether $x[n]$ is left-sided, right-sided, or two sided. Justify your answer.

(Answer) two-sided signal
(Solution)
Consider the z-transform of $x[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

If $x_1[n]$ is absolutely summable, then the above z-transform has ROC at $z = 4$ because the following sum converges:

$$X(4) = \sum_{n=-\infty}^{\infty} x[n]4^{-n}$$

With the same token, $z = 8$ does not belong ROC.
Since the original pole location is at $z = 1/2$, the ROC cannot be the interior region including $z = 4$. At the same time, the ROC should exclude $z = 8$. Therefore, the ROC is the ring between $z = 1/2, 8$, which implies that the original signal is the two-sided sequence.

(b) (10 points) Consider the following two system functions for stable LTI systems. Determine whether the corresponding systems are causal.

$$\text{(A)}\quad H(z) = \frac{1 - \frac{4}{3}z^{-1} + \frac{1}{2}z^{-2}}{z^{-1}(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}, \qquad \text{(B)}\quad H(z) = \frac{z - \frac{1}{2}}{z^2 + \frac{1}{2}z - \frac{3}{16}}$$

(A) causal / noncausal, (B) causal / noncausal (answer to your answer sheet, not here)

(Answer) (A) Noncausal, (B) causal
(Solution)
(A) To be a causal system, the rational system function should include $z = \infty$ as ROC. However, this system diverges as $z \to \infty$. Therefore, this system is not causal.
(B) Two poles of this system are at $z = 1/4, \ -3/4$ ($|z| = 1/4, \ 3/4$). Because the system is stable, ROC includes the unit circle ($|z| = 1$). The ROC cannot include poles, so ROC is given by the exterior region from the outermost pole ($|z| = 3/4$). The ROC also includes $z = \infty$ since $H(z) \to 0$ as $z \to \infty$. Consequently, the system is causal.

(c) (5 points) For the system response

$$h[n] = 6\left(\frac{1}{2}\right)^n u[n] + 10\left(\frac{4}{3}\right)^n u[-n-1]$$

Calculate the z-transform of $h[n]$ and determine the stability of system

(Answer)

$$H(z) = \frac{-4(1 + \frac{3}{4}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \frac{4}{3}z^{-1})}, \quad \text{ROC:}|z| > |1/2| \cap |z| < |4/3|, \quad \text{stable}$$

(Solution)

$$
\begin{aligned}
a^n u[n] &\overset{z}{\Longleftrightarrow} \frac{1}{1-az^{-1}},\ \text{ROC :}|z| > |a| \\
-a^n u[-n-1] &\overset{z}{\Longleftrightarrow} \frac{1}{1-az^{-1}},\ \text{ROC :}|z| < |a| \\
H(z) &= \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{10}{1-\frac{4}{3}z^{-1}} \\
&= \frac{-4(1+\frac{3}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{4}{3}z^{-1})} : \text{ROC:}|z| > |1/2| \cap |z| < |4/3|
\end{aligned} \tag{5}
$$

ROC includes the unit circle = system is stable.

[End of Problem]