General physics 2 Homework #07 Solution

The mirrors in Fig.1 make a 40° angle. A light ray enters parallel to the symmetry axis, as shown.

1. What is the reflection angle?

(a) 10° (b) 70° (c) 30° (d) 50°

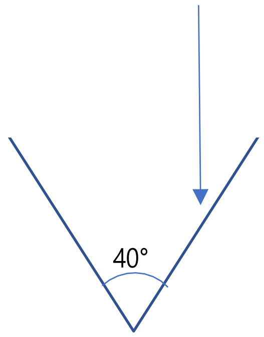


Figure 1

Sol)

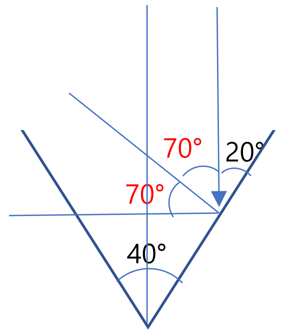


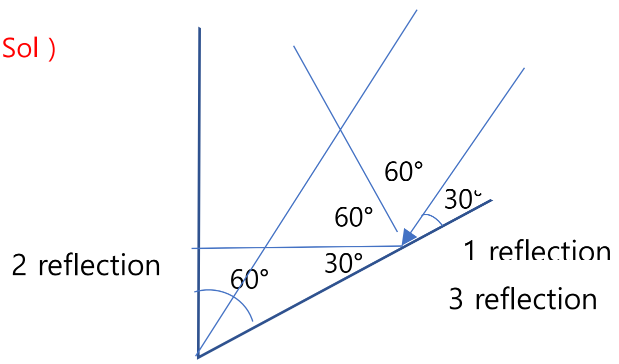
Figure 2

(b)

2. Suppose the 40° angle in Fig.1 is changed to 60°. A ray enters the mirror system parallel to the axis. How many reflections does it make?

(a) 7 (b) 5 (c) 3 (d) 1

Sol)



(c)

3. Light is incident on an air-glass interface, and the refracted light in the glass makes a 19° angle with the normal to the interface. The air's refractive index is 1 and The glass has refractive index 1.5. And $\sin(19°) = 1/3$. Find the incidence angle.

(a) 9.5°　　　(b) 55°　　　(c) 80°　　　(d) 30°

Sol) 1 * $\sin(\theta) = \frac{3}{2} sin19° = \frac{1}{2}$

Θ = 30°

(d)

4. What is the critical angle for light propagating in glass with n = 1.5 when the glass is immersed in water with n = 1.3? choose the answer that is closest to your calculations.

(a) 60°　　　(b) 15°　　　(c) 90°　　　(d) 40°

Sol) $\sin\theta = \frac{1.3}{1.5}$

$$\theta = \sin^{-1}\left(\frac{1.3}{1.5}\right) \cong 60°$$

(a)

5. In two dimensional Euclidean space, there is a mirror A: {(x, y)| y = 4 and x is any real number}. A ray that started at the position B(0, 2) was reflected in mirror A and reached position C(8, 0). Find the length of the path of the ray from B to C.

<Solution>

Using symmetry, we can realize that the length of real path can be expressed by the path from D(0, 6) to C(8, 0). Because angle of reflection should be equal to angle of incidence, the path of ray from D to C should be straight line and its length is 10. Therefore, the length of the path of the ray from C to D is 10.

Answer: 10.

<Scoring Criteria> (Total score:15 point)

Case1. Correct answer. (15 point)

Case2. Wrong answer. (0 point) (no partial point)

6. In two dimensional Euclidean space, the reflective index is $2n$ $(n > 0)$ in the region R: $\{(x, y)| -2 < x < 2, -3 < y < 1\}$, and the reflective index is n outside the region R. A ray that started at origin (0 ,0) propagate with direction (a, b) ($a, b > 0$, it means that gradient of the ray is $\frac{b}{a}$ in x-y plane as it started). What is the condition of $\frac{a}{b}$ for the situation that there will be total internal reflections at A :$\{(x, y)|\ 0 < x < 2,\ y = 1\}$ and B :$\{(x, y)|\ x = 2,\ -3 < y < 1\}$.

<Solution>

The necessary condition that the ray will reach A and B is $\frac{2}{5} < \frac{a}{b} < 2$.

For total internal reflection, angles of incident at A and B should be greater than critical angle (for total internal reflection).

It implies that $\frac{1}{2} < \sin\theta$ or $\frac{1}{\sqrt{3}} < \tan\theta$, (where θ is incident angle at A or B), and $\tan\theta = \frac{a}{b}, \frac{b}{a}$ at A and B. Therefore, the condition for that there will be total internal reflections at A and B is $\frac{1}{\sqrt{3}} < \frac{a}{b} < \sqrt{3}$. The answer is $\frac{1}{\sqrt{3}} < \frac{a}{b} < \sqrt{3}$.

Answer: $\frac{1}{\sqrt{3}} < \frac{a}{b} < \sqrt{3}$.

<Scoring Criteria> (Total score:15 point)

Case1. Correct answer. (15 point)

Case2. Wrong answer. (0 point) (no partial point)

Problem 7. **(20pt)**

Solution)
If $n_z = 0$, the equation of path is $(n_xt + x_0)\hat{x} + (n_yt + y_0)\hat{y} + (n_zt + z_0)\hat{z}$ for all t. (n_z term can be ignored because it is zero.) **(3pt)**

In case of $n_z \neq 0$, the equation of path is $(n_xt + x_0)\hat{x} + (n_yt + y_0)\hat{y} + (n_zt + z_0)\hat{z}$ in $z \geq 0$ region. $(t \geq -\frac{z_0}{n_z})$ **(3pt)**

In $z \leq 0$ region, due to the snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \qquad \textbf{(2pt)} \tag{1}$$

where $\sin\theta_1 = \sqrt{\frac{n_x^2+n_y^2}{n_x^2+n_y^2+n_z^2}}$, $\sin\theta_2 = \sqrt{\frac{n_x'^2+n_y'^2}{n_x'^2+n_y'^2+n_z'^2}}$.
We know that, $n_x' = n_x, n_y' = n_y$
By calculation,

$$n_z' = \sqrt{(n_x^2 + n_y^2)\frac{(n_2^2 - n_1^2)}{n_1^2} + \frac{n_2^2}{n_1^2}n_z^2} \qquad \textbf{(9pt)} \tag{2}$$

The intersection point of ray of light and interface is $(x_0 - \frac{n_xz_0}{n_z}, y_0 - \frac{n_yz_0}{n_z}, 0)$.
Therefore, the equation of path is

$$(n_xt + x_0 - \frac{n_xz_0}{n_z})\hat{x} + (n_yt + y_0 - \frac{n_yz_0}{n_z})\hat{y} + n_z't\hat{z} \quad \textbf{(3pt)} \tag{3}$$

where n_z' is same as above.

Problem 8. **(10pt)**

Solution)
For the case of $n_1 \geq n_2$, if incident angle is larger than critical angle, refraction does not occur. **(5pt)**. The condition is that $\sin\theta_1 \geq \frac{n_2}{n_1}$. Then it gives

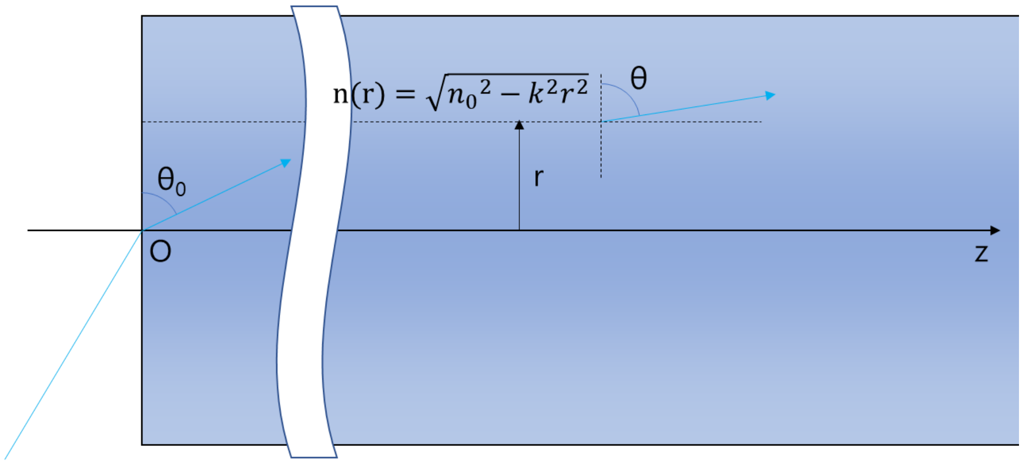
$$1 + \frac{n_z^2}{n_x^2 + n_y^2} \leq \frac{n_1^2}{n_2^2} \tag{4}$$

When we solve the equation (4)

$$0 < n_z \leq \sqrt{\frac{(n_1^2 - n_2^2)}{n_2^2} \cdot (n_x^2 + n_y^2)} \qquad \textbf{(5pt)} \tag{5}$$

(Advanced Problem) (20pts)

Materials can be made to have a refractive index gradient, or a refractive index as a function of position. Consider a ray of light incident on one end of a glass cylinder, as show in the diagram. The refracted ray is at an angle θ_0 with the glass surface. If the refractive index gradient of the glass is $n(r) = \sqrt{{n_0}^2 - k^2 r^2}$ for radius r from the axis, how does the ray of light propagate inside the glass cylinder? If possible, write the radial trajectory as r(z), a function of z. Assume that the size of the cylinder is not a limiting condition for the ray of light.



Solution)

As Hint 1 implies, the term n·sinθ is constant along the path of light inside the glass cylinder. Using the initial condition of the refracted ray, we can write that for any arbitrary point along the ray

$$n_0 \sin\theta_0 = n(r) \sin\theta$$

(**+5pts** for correctly utilizing Hint 1 to deduce the above formula)

To find the path of the ray as a function r(z), we must first substitute θ as a function of r and z. Since the slope of r(z) would be cot θ, we can write $\cot\theta = \frac{dr}{dz}$ and thus

$$n_0 \sin\theta_0 = \frac{n(r)}{\sqrt{1 + (\frac{dr}{dz})^2}}$$

Using the given refractive index gradient and solving for $\frac{dr}{dz}$ we find

$$\frac{dr}{dz} = \frac{\sqrt{n_0^2 \cos^2\theta_0 - k^2 r^2}}{n_0 \sin\theta_0}$$

(**+10pts** for correctly using the definition of slope to deduce the above formula)

Integrating over both sides, using Hint 2, and solving for the initial condition z(r=0)=0, we get

$$z = \frac{n_0 \sin\theta_0}{k} \tan^{-1} \frac{kr}{\sqrt{n_0^2 \cos^2\theta_0 - k^2 r^2}}$$

This is z as a function of r, so we can easily calculate r(z) to find

$$r = \frac{1}{k} \sin \frac{kz}{n_0 \sin\theta_0}$$

Thus, the ray of light inside a cylinder with the given refractive index gradient follows a sinusoidal curve.

(**+5pts** for correctly calculating the answer)

Partial score may be given for...

- Only qualitative analysis of the path of light (**10pts**)

- Errors in calculation (-**1pts** per criteria)