The HDG Code

version 1.0

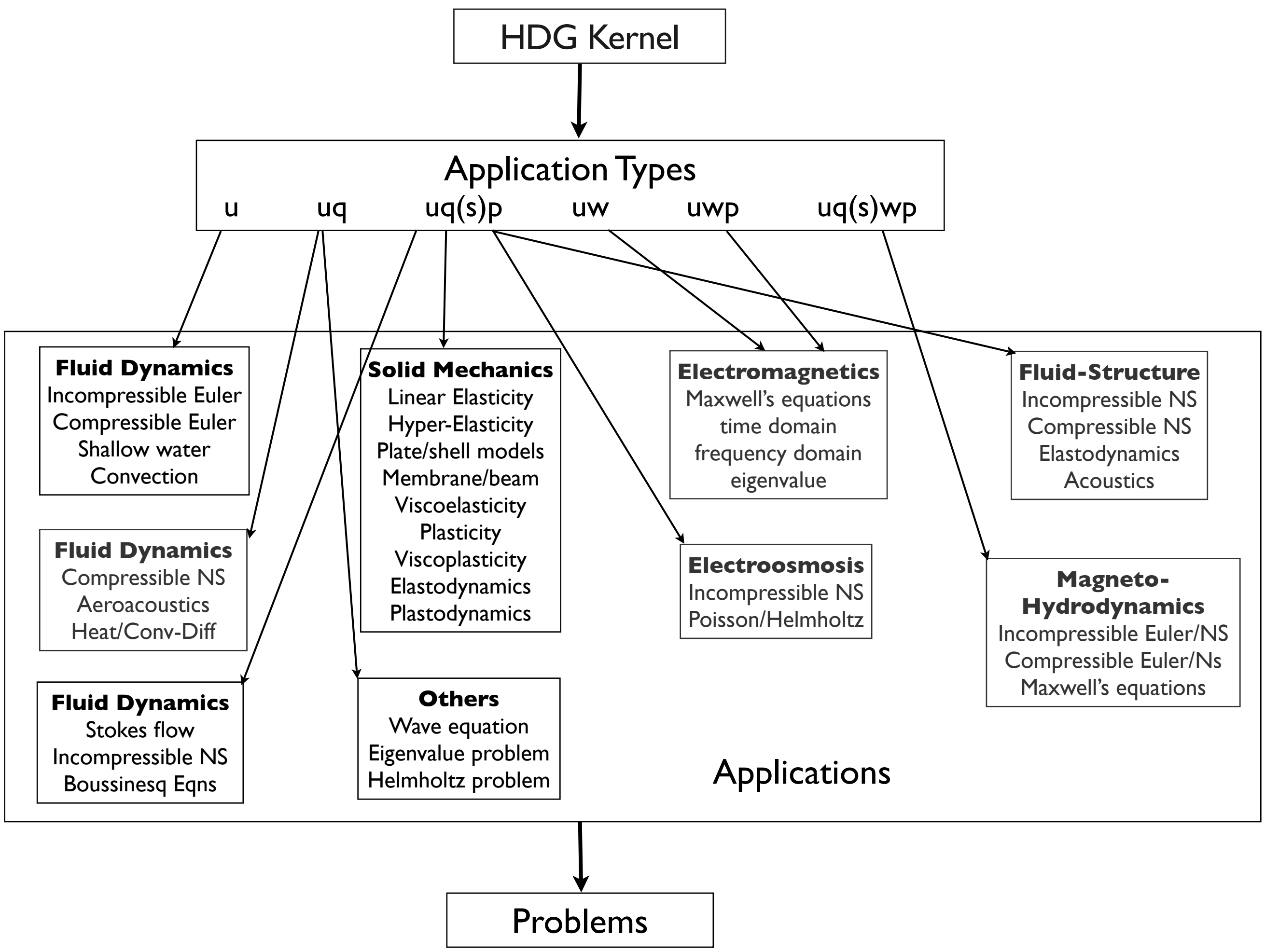

Figure 1: HDG Hierarchy.

1 HDG Kernel

HDG Kernel is responsible for forming and solving the linear system for all the Application Types. There are two main functions:

- `hdg_solve.m` assembles the Jacobian matrix and the Residual vector, and then solves the linear system. [1] This function calls `elemmat.m` and `localsolve.m` for the assembly of the linear system. The linear system stems from linearization of the nonlinear problem

$$\sum_{K\in\mathcal{T}_h} \left\langle \widehat{\boldsymbol{F}}(\boldsymbol{U}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{\mu} \right\rangle_{\partial K\backslash\partial\Omega} + \left\langle \widehat{\boldsymbol{F}}^b(\boldsymbol{U}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu} \right\rangle_{\partial\Omega} = 0, \quad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h. \tag{1}$$

Here $\widehat{\boldsymbol{F}}$ and $\widehat{\boldsymbol{F}}^b$ are the interior and boundary fluxes, respectively; and $\boldsymbol{g}$ is the boundary data. The definition of the boundary flux $\widehat{\boldsymbol{F}}^b$ depends on the applied boundary conditions and is different on different parts of the boundary $\partial\Omega$.

- `elemmat.m` computes the elemental Jacobian matrix and Residual vector by evaluating the following integrals, on the interior faces F,

$$\begin{aligned} \mathbb{J}_F &= \left\langle \Big(\frac{\partial \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)}{\partial \boldsymbol{U}_h} \delta \boldsymbol{U}_h^{\delta\boldsymbol{\lambda}} + \frac{\partial \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)}{\partial \widehat{\boldsymbol{u}}_h} \delta \widehat{\boldsymbol{u}}_h \Big) \cdot \boldsymbol{n}, \boldsymbol{\mu} \right\rangle_F, \\ \mathbb{R}_F &= -\left\langle \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \cdot \boldsymbol{n}, \boldsymbol{\mu} \right\rangle_F - \left\langle \Big(\frac{\partial \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)}{\partial \boldsymbol{U}_h} \delta \boldsymbol{U}_h^{\boldsymbol{r}} \Big) \cdot \boldsymbol{n}, \boldsymbol{\mu} \right\rangle_F; \end{aligned} \tag{2}$$

on the boundary faces $F \in \partial\Omega$,

$$\begin{aligned} \mathbb{J}_F &= \left\langle \Big(\frac{\partial \widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)}{\partial \boldsymbol{U}_h} \delta \boldsymbol{U}_h^{\delta\boldsymbol{\lambda}} + \frac{\partial \widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)}{\partial \widehat{\boldsymbol{u}}_h} \delta \widehat{\boldsymbol{u}}_h \Big) \cdot \boldsymbol{n}, \boldsymbol{\mu} \right\rangle_F, \\ \mathbb{R}_F &= -\left\langle \widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \cdot \boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu} \right\rangle_F - \left\langle \Big(\frac{\partial \widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)}{\partial \boldsymbol{U}_h} \delta \boldsymbol{U}_h^{\boldsymbol{r}} \Big) \cdot \boldsymbol{n}, \boldsymbol{\mu} \right\rangle_F. \end{aligned} \tag{3}$$

Here $\delta \boldsymbol{U}_h^{\delta\boldsymbol{\lambda}}$ and $\delta \boldsymbol{U}_h^{\boldsymbol{r}}$ are local solutions which are computed in `localsolve.m`.[2]

- Other functions, `hdg_solve_bdf.m` and `hdg_solve_dirk.m`, call `hdg_solve.m` to solve time-dependent problems using BDF and DIRK methods, respectively.

2 Application Types (PDE Forms)

Application Types include the local solver to several types of PDEs. The local solver is responsible for computing the local solutions $\delta \boldsymbol{U}_h^{\delta\boldsymbol{\lambda}}$ and $\delta \boldsymbol{U}_h^{\boldsymbol{r}}$. There are two main functions

- `localresid.m` computes the local matrices and vectors on each element.
- `localsolve.m` solve the local linear system for $\delta \boldsymbol{U}_h^{\delta\boldsymbol{\lambda}}$ and $\delta \boldsymbol{U}_h^{\boldsymbol{r}}$.

The form of the local linear system depends on the type of PDEs.

[1] Need to implement `globalsolve.m` to solve the linear system.
[2] Need to move the call to `fbou.m` and `fhat.m` from `elemmat.m` to `localsolve.m`

2.1 $\boldsymbol{u}$-Type

We solve the boundary value problem of the form.

$$
\begin{aligned}
-\nabla\cdot\boldsymbol{F}(\boldsymbol{u}) &= \boldsymbol{f}, && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g}_D, && \text{on } \Gamma_D.
\end{aligned}
\tag{4}
$$

The HDG method seeks $(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\in\boldsymbol{W}_h\times\boldsymbol{M}_h$ such that

$$
\begin{aligned}
(\boldsymbol{F}(\boldsymbol{u}_h),\nabla\boldsymbol{w})_{\mathcal{T}_h}-\langle\widehat{\boldsymbol{F}}(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= (\boldsymbol{f},\boldsymbol{w})_{\mathcal{T}_h}, && \forall\boldsymbol{w}\in\boldsymbol{W}_h,\\
\langle\widehat{\boldsymbol{F}}(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega}+\langle\widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}-\boldsymbol{g},\boldsymbol{\mu}\rangle_{\partial\Omega} &= 0, && \forall\boldsymbol{\mu}\in\boldsymbol{M}_h,
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
\widehat{\boldsymbol{F}}(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} &= \boldsymbol{F}(\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}-\mathbf{S}(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h),\\
\widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} &= \begin{cases}\widehat{\boldsymbol{u}}_h, & \text{on } \Gamma_D,\\ -\mathbf{S}(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h), & \text{on } \partial\Omega\backslash\Gamma_D,\end{cases}\\
\boldsymbol{g} &= \begin{cases}\boldsymbol{g}_D, & \text{on } \Gamma_D,\\ \mathbf{0}, & \text{on } \partial\Omega\backslash\Gamma_D.\end{cases}
\end{aligned}
\tag{6}
$$

Linearization of the first equation of (5) gives

$$
(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{u}}_h)\delta\boldsymbol{u}_h,\nabla\boldsymbol{w})_{\mathcal{T}_h}-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h+\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}=\boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h},\quad\forall\boldsymbol{w}\in\boldsymbol{W}_h,
\tag{7}
$$

where

$$
\boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}=(\boldsymbol{f},\boldsymbol{w})_{\mathcal{T}_h}-(\boldsymbol{F}(\overline{\boldsymbol{u}}_h),\nabla\boldsymbol{w})_{\mathcal{T}_h}+\langle\widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}.
\tag{8}
$$

As a result, $\delta\boldsymbol{u}_h^{\boldsymbol{r}}$ is the solution of

$$
(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{u}}_h)\delta\boldsymbol{u}_h,\nabla\boldsymbol{w})_{\mathcal{T}_h}-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h,\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}=\boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h},\quad\forall\boldsymbol{w}\in\boldsymbol{W}_h.
\tag{9}
$$

And $\delta\boldsymbol{u}_h^{\delta\boldsymbol{\lambda}}$ is the solution of

$$
(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{u}}_h)\delta\boldsymbol{u}_h,\nabla\boldsymbol{w})_{\mathcal{T}_h}-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h,\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}=\langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{\lambda},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h},\quad\forall\boldsymbol{w}\in\boldsymbol{W}_h.
\tag{10}
$$

The Eqns (9) and (10) define the local solver.

Next, we consider the time-dependent problem

$$
\begin{aligned}
\frac{\partial\boldsymbol{u}}{\partial t}-\nabla\cdot\boldsymbol{F}(\boldsymbol{u}) &= \boldsymbol{f}, && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g}_D, && \text{on } \Gamma_D.
\end{aligned}
\tag{11}
$$

The HDG method seeks $(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\in\boldsymbol{W}_h\times\boldsymbol{M}_h$ such that

$$
\begin{aligned}
\left(\frac{\boldsymbol{u}_h}{\Delta t},\boldsymbol{w}\right)_{\mathcal{T}_h}+(\boldsymbol{F}(\boldsymbol{u}_h),\nabla\boldsymbol{w})_{\mathcal{T}_h}-\langle\widehat{\boldsymbol{F}}(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= (\boldsymbol{f},\boldsymbol{w})_{\mathcal{T}_h}+\left(\frac{\boldsymbol{u}_h^{k-1}}{\Delta t},\boldsymbol{w}\right)_{\mathcal{T}_h}, && \forall\boldsymbol{w}\in\boldsymbol{W}_h,\\
\langle\widehat{\boldsymbol{F}}(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega}+\langle\widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}-\boldsymbol{g},\boldsymbol{\mu}\rangle_{\partial\Omega} &= 0, && \forall\boldsymbol{\mu}\in\boldsymbol{M}_h.
\end{aligned}
\tag{12}
$$

Linearization of the first equation of (12) gives

$$
\begin{aligned}
\left(\frac{\delta\boldsymbol{u}_h}{\Delta t},\boldsymbol{w}\right)_{\mathcal{T}_h}+(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{u}}_h)\delta\boldsymbol{u}_h,\nabla\boldsymbol{w})_{\mathcal{T}_h}-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h+\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}&\\
=\boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h},\quad\forall\boldsymbol{w}\in\boldsymbol{W}_h,&
\end{aligned}
\tag{13}
$$

where

$$\boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} + \left(\frac{\boldsymbol{u}_h^{k-1}}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - \left(\frac{\overline{\boldsymbol{u}}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - (\boldsymbol{F}(\overline{\boldsymbol{u}}_h), \nabla \boldsymbol{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \cdot \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h}. \quad (14)$$

As a result, $\delta \boldsymbol{u}_h^{\boldsymbol{r}}$ is the solution of

$$\left(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}} \boldsymbol{F}(\overline{\boldsymbol{u}}_h) \delta \boldsymbol{u}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} - \langle (\partial_{\boldsymbol{u}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \boldsymbol{u}_h, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} = \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, \quad \forall \boldsymbol{w} \in \boldsymbol{W}_h. \quad (15)$$

And $\delta \boldsymbol{u}_h^{\delta \boldsymbol{\lambda}}$ is the solution of

$$\left(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}} \boldsymbol{F}(\overline{\boldsymbol{u}}_h) \delta \boldsymbol{u}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} - \langle (\partial_{\boldsymbol{u}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \boldsymbol{u}_h, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} = \langle (\partial_{\widehat{\boldsymbol{u}}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{u}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \boldsymbol{\lambda}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h}, \quad \forall \boldsymbol{w} \in \boldsymbol{W}_h. \quad (16)$$

The Eqns (15) and (16) define the local solver.

2.2 $\boldsymbol{uq}$-Type

We solve the boundary value problem of the form.

$$\begin{array}{rcll} \boldsymbol{q} - \nabla \boldsymbol{u} & = & \boldsymbol{0}, & \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{q}) & = & \boldsymbol{f}, & \text{in } \Omega, \\ \boldsymbol{u} & = & \boldsymbol{g}_D, & \text{on } \Gamma_D, \\ \boldsymbol{B}^n(\boldsymbol{u}, \boldsymbol{q}) \cdot \boldsymbol{n} & = & \boldsymbol{g}_N, & \text{on } \Gamma_N. \end{array} \quad (17)$$

The HDG method seeks $(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times \boldsymbol{M}_h$ such that

$$\begin{array}{rcll} (\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} & = & \boldsymbol{0}, & \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\ (\boldsymbol{F}(\boldsymbol{u}_h, \boldsymbol{q}_h), \nabla \boldsymbol{w})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} & = & (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h}, & \forall \boldsymbol{w} \in \boldsymbol{W}_h, \\ \langle \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} + \langle \widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial \Omega} & = & 0, & \forall \boldsymbol{\mu} \in \boldsymbol{M}_h, \end{array} \quad (18)$$

where

$$\begin{array}{rcl} \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n} & = & \boldsymbol{F}(\widehat{\boldsymbol{u}}_h, \boldsymbol{q}_h) \cdot \boldsymbol{n} - \mathbf{S}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h)(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \\ \widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n} & = & \begin{cases} \widehat{\boldsymbol{u}}_h, & \text{on } \Gamma_D, \\ \boldsymbol{B}^n(\widehat{\boldsymbol{u}}_h, \boldsymbol{q}_h) \cdot \boldsymbol{n} - \mathbf{S}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h)(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), & \text{on } \Gamma_N, \end{cases} \\ \boldsymbol{g} & = & \begin{cases} \boldsymbol{g}_D, & \text{on } \Gamma_D, \\ \boldsymbol{g}_N, & \text{on } \Gamma_N. \end{cases} \end{array} \quad (19)$$

Linearization of the first two equations of (18) gives

$$\begin{array}{rcll} (\delta \boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \delta \widehat{\boldsymbol{u}}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} & = & \boldsymbol{r}(\boldsymbol{v}), & \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\ (\partial_{\boldsymbol{u}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h) \delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h) \delta \boldsymbol{q}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} & & & \\ -\langle (\partial_{\boldsymbol{u}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \boldsymbol{q}_h) \cdot \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} & & & \\ -\langle (\partial_{\widehat{\boldsymbol{u}}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} & = & \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, & \forall \boldsymbol{w} \in \boldsymbol{W}_h, \end{array} \quad (20)$$

where

$$\begin{aligned} \boldsymbol{r}(\boldsymbol{v}) &= -(\overline{\boldsymbol{q}}_h, \boldsymbol{v})_{\mathcal{T}_h} - (\overline{\boldsymbol{u}}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle \overline{\widehat{\boldsymbol{u}}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, \\ \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} - (\boldsymbol{F}(\overline{\boldsymbol{U}}_h), \nabla\boldsymbol{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}. \end{aligned} \tag{21}$$

As a result, $(\delta\boldsymbol{u}_h^{\boldsymbol{r}}, \delta\boldsymbol{q}_h^{\boldsymbol{r}})$ is the solution of

$$\begin{aligned} (\delta\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v}\in\boldsymbol{V}_h, \\ (\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & && \\ -\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w}\in\boldsymbol{W}_h, \end{aligned} \tag{22}$$

And $(\delta\boldsymbol{u}_h^{\delta\boldsymbol{\lambda}}, \delta\boldsymbol{q}_h^{\delta\boldsymbol{\lambda}})$ is the solution of

$$\begin{aligned} (\delta\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} &= \langle\delta\boldsymbol{\lambda}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, && \forall \boldsymbol{v}\in\boldsymbol{V}_h, \\ (\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & && \\ -\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{\lambda})\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}, && \forall \boldsymbol{w}\in\boldsymbol{W}_h, \end{aligned} \tag{23}$$

The Eqns (22) and (23) define the local solver.

We consider the time-dependent problem

$$\begin{aligned} \boldsymbol{q} - \nabla\boldsymbol{u} &= \boldsymbol{0}, && \text{in } \Omega, \\ \frac{\partial\boldsymbol{u}}{\partial t} - \nabla\cdot\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{q}) &= \boldsymbol{f}, && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{g}_D, && \text{on } \Gamma_D, \\ \boldsymbol{B}^n(\boldsymbol{u}, \boldsymbol{q})\cdot\boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N. \end{aligned} \tag{24}$$

The HDG method seeks $(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{W}_h\times\boldsymbol{V}_h\times\boldsymbol{M}_h$ such that

$$\begin{aligned} (\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle\widehat{\boldsymbol{u}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{0}, && \forall \boldsymbol{v}\in\boldsymbol{V}_h, \\ \left(\frac{\boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\boldsymbol{F}(\boldsymbol{u}_h, \boldsymbol{q}_h), \nabla\boldsymbol{w})_{\mathcal{T}_h} - \langle\widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} + \left(\frac{\boldsymbol{u}_h^{k-1}}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h}, && \forall \boldsymbol{w}\in\boldsymbol{W}_h, \\ \langle\widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} + \langle\widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega} &= 0, && \forall \boldsymbol{\mu}\in\boldsymbol{M}_h, \end{aligned} \tag{25}$$

Linearization of the first two equations of (25) gives

$$\begin{aligned} (\delta\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle\delta\widehat{\boldsymbol{u}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v}\in\boldsymbol{V}_h, \\ \left(\frac{\delta\boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & && \\ -\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} & && \\ -\langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w}\in\boldsymbol{W}_h, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \boldsymbol{r}(\boldsymbol{v}) &= -(\overline{\boldsymbol{q}}_h, \boldsymbol{v})_{\mathcal{T}_h} - (\overline{\boldsymbol{u}}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle\overline{\widehat{\boldsymbol{u}}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, \\ \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} + \left(\frac{\boldsymbol{u}_h^{k-1}}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - \left(\frac{\overline{\boldsymbol{u}}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - (\boldsymbol{F}(\overline{\boldsymbol{U}}_h), \nabla\boldsymbol{w})_{\mathcal{T}_h} + \langle\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}. \end{aligned} \tag{27}$$

As a result, $(\delta \boldsymbol{u}_h^{\boldsymbol{r}}, \delta \boldsymbol{q}_h^{\boldsymbol{r}})$ is the solution of

$$
\begin{aligned}
(\delta \boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
\left(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta \boldsymbol{q}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle (\partial_{\boldsymbol{u}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{q}_h) \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial \mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h,
\end{aligned}
\tag{28}
$$

And $(\delta \boldsymbol{u}_h^{\delta \boldsymbol{\lambda}}, \delta \boldsymbol{q}_h^{\delta \boldsymbol{\lambda}})$ is the solution of

$$
\begin{aligned}
(\delta \boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} &= \langle \delta \boldsymbol{\lambda}, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial \mathcal{T}_h}, && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
\left(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta \boldsymbol{q}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle (\partial_{\boldsymbol{u}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{q}_h) \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial \mathcal{T}_h} &= \langle (\partial_{\widehat{\boldsymbol{u}}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{\lambda}) \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial \mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h,
\end{aligned}
\tag{29}
$$

The Eqns (28) and (29) define the local solver.

We consider the wave propagation problem

$$
\begin{aligned}
\frac{\partial \boldsymbol{q}}{\partial t} - \nabla \boldsymbol{u} &= \boldsymbol{0}, && \text{in } \Omega, \\
\frac{\partial \boldsymbol{u}}{\partial t} - \nabla \cdot \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{q}) &= \boldsymbol{f}, && \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{g}_D, && \text{on } \Gamma_D, \\
\boldsymbol{B}^n(\boldsymbol{u}, \boldsymbol{q}) \cdot \boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N.
\end{aligned}
\tag{30}
$$

The HDG method seeks $(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times \boldsymbol{M}_h$ such that

$$
\begin{aligned}
\left(\frac{\boldsymbol{q}_h}{\Delta t}, \boldsymbol{v}\right)_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial \mathcal{T}_h} &= \left(\frac{\boldsymbol{q}_h^{k-1}}{\Delta t}, \boldsymbol{v}\right)_{\mathcal{T}_h}, && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
\left(\frac{\boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\boldsymbol{F}(\boldsymbol{u}_h, \boldsymbol{q}_h), \nabla \boldsymbol{w})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial \mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} + \left(\frac{\boldsymbol{u}_h^{k-1}}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h, \\
\langle \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \langle \widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial \Omega} &= 0, && \forall \boldsymbol{\mu} \in \boldsymbol{M}_h,
\end{aligned}
\tag{31}
$$

Linearization of the first two equations of (31) gives

$$
\begin{aligned}
\left(\frac{\delta \boldsymbol{q}_h}{\Delta t}, \boldsymbol{v}\right)_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \delta \widehat{\boldsymbol{u}}_h, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial \mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
\left(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta \boldsymbol{q}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle (\partial_{\boldsymbol{u}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{q}_h) \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial \mathcal{T}_h} & \\
-\langle (\partial_{\widehat{\boldsymbol{u}}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial \mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h,
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
\boldsymbol{r}(\boldsymbol{v}) &= \left(\frac{\boldsymbol{q}_h^{k-1}}{\Delta t}, \boldsymbol{v}\right)_{\mathcal{T}_h} - \left(\frac{\overline{\boldsymbol{q}}_h}{\Delta t}, \boldsymbol{v}\right)_{\mathcal{T}_h} - (\overline{\boldsymbol{u}}_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle \overline{\widehat{\boldsymbol{u}}}_h, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial \mathcal{T}_h}, \\
\boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} + \left(\frac{\boldsymbol{u}_h^{k-1}}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - \left(\frac{\overline{\boldsymbol{u}}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - (\boldsymbol{F}(\overline{\boldsymbol{U}}_h), \nabla \boldsymbol{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial \mathcal{T}_h}.
\end{aligned}
\tag{33}
$$

As a result, $(\delta \boldsymbol{u}_h^{\boldsymbol{r}}, \delta \boldsymbol{q}_h^{\boldsymbol{r}})$ is the solution of

$$
\begin{aligned}
\left(\frac{\delta \boldsymbol{q}_h}{\Delta t}, \boldsymbol{v}\right)_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
\left(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta \boldsymbol{q}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} & && \\
-\langle (\partial_{\boldsymbol{u}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{q}_h)\cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h,
\end{aligned}
\tag{34}
$$

And $(\delta \boldsymbol{u}_h^{\delta\boldsymbol{\lambda}}, \delta \boldsymbol{q}_h^{\delta\boldsymbol{\lambda}})$ is the solution of

$$
\begin{aligned}
\left(\frac{\delta \boldsymbol{q}_h}{\Delta t}, \boldsymbol{v}\right)_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} &= \langle \delta\boldsymbol{\lambda}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
\left(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta \boldsymbol{q}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} & && \\
-\langle (\partial_{\boldsymbol{u}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta \boldsymbol{q}_h)\cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \langle (\partial_{\widehat{\boldsymbol{u}}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{\lambda})\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h,
\end{aligned}
\tag{35}
$$

The Eqns (34) and (35) define the local solver.

2.3 $\boldsymbol{u}\boldsymbol{q}p$-Type

We solve the boundary value problem of the form

$$
\begin{aligned}
\boldsymbol{q} - \nabla \boldsymbol{u} &= \boldsymbol{0}, && \text{in } \Omega, \\
-\nabla\cdot \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{q}, p) &= \boldsymbol{f}, && \text{in } \Omega, \\
\epsilon\, p + \nabla\cdot \boldsymbol{u} &= 0, && \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{g}_D, && \text{on } \Gamma_D, \\
\boldsymbol{B}^n(\boldsymbol{u}, \boldsymbol{q}, p)\cdot \boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N.
\end{aligned}
\tag{36}
$$

The HDG method seeks $(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times \boldsymbol{M}_h$ such that

$$
\begin{aligned}
(\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{0}, && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
(\boldsymbol{F}(\boldsymbol{u}_h, \boldsymbol{q}_h, p_h), \nabla \boldsymbol{w})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{q}_h, p_h, \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h, \\
(\epsilon p_h, s)_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, s\rangle_{\partial\mathcal{T}_h} &= 0, && \forall s \in P_h, \\
\langle \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, p_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} + \langle \widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h, \boldsymbol{q}_h, p_h, \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega} &= 0, && \forall \boldsymbol{\mu} \in \boldsymbol{M}_h,
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
\widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{q}_h, p_h, \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} &= \boldsymbol{F}(\widehat{\boldsymbol{u}}_h, \boldsymbol{q}_h, p_h)\cdot\boldsymbol{n} - \mathbf{S}(\boldsymbol{u}_h, \boldsymbol{q}_h, p_h, \widehat{\boldsymbol{u}}_h)(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \\
\widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h, \boldsymbol{q}_h, p_h, \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} &= \begin{cases} \widehat{\boldsymbol{u}}_h, & \text{on } \Gamma_D, \\ \boldsymbol{B}^n(\widehat{\boldsymbol{u}}_h, \boldsymbol{q}_h, p_h)\cdot\boldsymbol{n} - \mathbf{S}(\boldsymbol{u}_h, \boldsymbol{q}_h, p_h, \widehat{\boldsymbol{u}}_h)(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), & \text{on } \Gamma_N, \end{cases} \\
\boldsymbol{g} &= \begin{cases} \boldsymbol{g}_D, & \text{on } \Gamma_D, \\ \boldsymbol{g}_N, & \text{on } \Gamma_N. \end{cases}
\end{aligned}
\tag{38}
$$

Linearizing (37) gives

$$\begin{aligned}
(\delta \boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \delta\widehat{\boldsymbol{u}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v}\in \boldsymbol{V}_h,\\
(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} & \\
-\langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w}\in \boldsymbol{W}_h,\\
(\epsilon\delta p_h, s)_{\mathcal{T}_h} - (\delta\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} + \langle\delta\widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, s\rangle_{\partial\mathcal{T}_h} &= r(s), && \forall s\in P_h,\\
\langle(\partial_{\boldsymbol{U}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{U}_h + \partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} & \\
+\langle(\partial_{\boldsymbol{U}}\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{U}_h + \partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega} &= r(\boldsymbol{\mu}), && \forall \boldsymbol{\mu}\in \boldsymbol{M}_h,
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
\boldsymbol{r}(\boldsymbol{v}) &= -(\overline{\boldsymbol{q}}_h, \boldsymbol{v})_{\mathcal{T}_h} - (\overline{\boldsymbol{u}}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle\overline{\widehat{\boldsymbol{u}}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h},\\
\boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} - (\boldsymbol{F}(\overline{\boldsymbol{U}}_h), \nabla\boldsymbol{w})_{\mathcal{T}_h} + \langle\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h},\\
r(s) &= -(\epsilon\overline{p}_h, s)_{\mathcal{T}_h} + (\overline{\boldsymbol{u}}_h, \nabla s)_{\mathcal{T}_h} - \langle\overline{\widehat{\boldsymbol{u}}}_h\cdot\boldsymbol{n}, s\rangle_{\partial\mathcal{T}_h},\\
\boldsymbol{r}(\mu) &= -\langle\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\cdot\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} - \langle\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\cdot\boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega}.
\end{aligned} \tag{40}$$

Applying the Augmented Lagrangian approach to solve (39) gives

$$\begin{aligned}
(\delta \boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \delta\widehat{\boldsymbol{u}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v}\in \boldsymbol{V}_h,\\
(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} & \\
-\langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w}\in \boldsymbol{W}_h,\\
((\epsilon+\beta)\delta p_h, s)_{\mathcal{T}_h} - (\delta\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} + \langle\delta\widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, s\rangle_{\partial\mathcal{T}_h} &= r(s) + (\beta\delta p_h^{m-1}, s)_{\mathcal{T}_h}, && \forall s\in P_h,\\
\langle(\partial_{\boldsymbol{U}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{U}_h + \partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} & \\
+\langle(\partial_{\boldsymbol{U}}\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{U}_h + \partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega} &= r(\boldsymbol{\mu}), && \forall \boldsymbol{\mu}\in \boldsymbol{M}_h,
\end{aligned} \tag{41}$$

where $\beta > 0$ is the AL coefficient and δp_h^{m-1} is the pressure increment at the previous pseudo timestep. The AL approach is a trick to avoid introducing the mean of the pressure in the local solver. As a result, $\delta\boldsymbol{U}_h^{\boldsymbol{r}}$ is the solution of

$$\begin{aligned}
(\delta \boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v}\in \boldsymbol{V}_h,\\
(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w}\in \boldsymbol{W}_h,\\
((\epsilon+\beta)\delta p_h, s)_{\mathcal{T}_h} - (\delta\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} &= r(s) + (\beta\delta p_h^{m-1}, s)_{\mathcal{T}_h}, && \forall s\in P_h.
\end{aligned} \tag{42}$$

And $\delta\boldsymbol{U}_h^{\delta\boldsymbol{\lambda}}$ is the solution of

$$
\begin{aligned}
(\delta\boldsymbol{q}_h,\boldsymbol{v})_{\mathcal{T}_h}+(\delta\boldsymbol{u}_h,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} &= \langle\delta\boldsymbol{\delta}_h,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, && \forall\boldsymbol{v}\in\boldsymbol{V}_h,\\
(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h+\partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h,\nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h+\partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{\lambda}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}, && \forall\boldsymbol{w}\in\boldsymbol{W}_h,\\
((\epsilon+\beta)\delta p_h,s)_{\mathcal{T}_h}-(\delta\boldsymbol{u}_h,\nabla s)_{\mathcal{T}_h} &= -\langle\delta\boldsymbol{\lambda}_h\cdot\boldsymbol{n},s\rangle_{\partial\mathcal{T}_h}, && \forall s\in P_h.
\end{aligned}
\tag{43}
$$

The Eqns (42) and (43) define the local solver.

We solve the time-dependent problem

$$
\begin{aligned}
\boldsymbol{q}-\nabla\boldsymbol{u} &= \boldsymbol{0}, && \text{in } \Omega,\\
\frac{\partial\boldsymbol{u}}{\partial t}-\nabla\cdot\boldsymbol{F}(\boldsymbol{u},\boldsymbol{q},p) &= \boldsymbol{f}, && \text{in } \Omega,\\
\epsilon\, p+\nabla\cdot\boldsymbol{u} &= 0, && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g}_D, && \text{on } \Gamma_D,\\
\boldsymbol{B}^n(\boldsymbol{u},\boldsymbol{q},p)\cdot\boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N.
\end{aligned}
\tag{44}
$$

The HDG method seeks $(\boldsymbol{u}_h,\boldsymbol{q}_h,\widehat{\boldsymbol{u}}_h)\in\boldsymbol{W}_h\times\boldsymbol{V}_h\times\boldsymbol{M}_h$ such that

$$
\begin{aligned}
(\boldsymbol{q}_h,\boldsymbol{v})_{\mathcal{T}_h}+(\boldsymbol{u}_h,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h}-\langle\widehat{\boldsymbol{u}}_h,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{0}, && \forall\boldsymbol{v}\in\boldsymbol{V}_h,\\
\left(\frac{\boldsymbol{u}_h}{\Delta t},\boldsymbol{w}\right)_{\mathcal{T}_h}+(\boldsymbol{F}(\boldsymbol{U}_h),\nabla\boldsymbol{w})_{\mathcal{T}_h}-\langle\widehat{\boldsymbol{F}}(\boldsymbol{U}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= (\boldsymbol{f},\boldsymbol{w})_{\mathcal{T}_h}+\left(\frac{\boldsymbol{u}_h^{k-1}}{\Delta t},\boldsymbol{w}\right)_{\mathcal{T}_h}, && \forall\boldsymbol{w}\in\boldsymbol{W}_h,\\
(\epsilon p_h,s)_{\mathcal{T}_h}-(\boldsymbol{u}_h,\nabla s)_{\mathcal{T}_h}+\langle\widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n},s\rangle_{\partial\mathcal{T}_h} &= 0, && \forall s\in P_h,\\
\langle\widehat{\boldsymbol{F}}(\boldsymbol{U}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega}+\langle\widehat{\boldsymbol{F}}^b(\boldsymbol{U}_h,\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}-\boldsymbol{g},\boldsymbol{\mu}\rangle_{\partial\Omega} &= 0, && \forall\boldsymbol{\mu}\in\boldsymbol{M}_h,
\end{aligned}
\tag{45}
$$

Linearizing (45) gives

$$
\begin{aligned}
(\delta\boldsymbol{q}_h,\boldsymbol{v})_{\mathcal{T}_h}+(\delta\boldsymbol{u}_h,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h}-\langle\delta\widehat{\boldsymbol{u}}_h,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall\boldsymbol{v}\in\boldsymbol{V}_h,\\
\left(\frac{\delta\boldsymbol{u}_h}{\Delta t},\boldsymbol{w}\right)_{\mathcal{T}_h}+(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h+\partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h,\nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h+\partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h} & \\
-\langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall\boldsymbol{w}\in\boldsymbol{W}_h,\\
(\epsilon\delta p_h,s)_{\mathcal{T}_h}-(\delta\boldsymbol{u}_h,\nabla s)_{\mathcal{T}_h}+\langle\delta\widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n},s\rangle_{\partial\mathcal{T}_h} &= r(s), && \forall s\in P_h,\\
\langle(\partial_{\boldsymbol{U}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta U_h+\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} & \\
+\langle(\partial_{\boldsymbol{U}}\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{U}_h+\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}-\boldsymbol{g},\boldsymbol{\mu}\rangle_{\partial\Omega} &= r(\boldsymbol{\mu}), && \forall\boldsymbol{\mu}\in\boldsymbol{M}_h,
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
\boldsymbol{r}(\boldsymbol{v}) &= -(\overline{\boldsymbol{q}}_h, \boldsymbol{v})_{\mathcal{T}_h} - (\overline{\boldsymbol{u}}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle \overline{\widehat{\boldsymbol{u}}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, \\
\boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} + \left(\frac{\boldsymbol{u}_h^{k-1}}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - \left(\frac{\boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - (\boldsymbol{F}(\overline{\boldsymbol{U}}_h), \nabla\boldsymbol{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}, \\
r(s) &= -(\epsilon\overline{p}_h, s)_{\mathcal{T}_h} + (\overline{\boldsymbol{u}}_h, \nabla s)_{\mathcal{T}_h} - \langle \overline{\widehat{\boldsymbol{u}}}_h\cdot\boldsymbol{n}, s\rangle_{\partial\mathcal{T}_h}, \\
\boldsymbol{r}(\mu) &= -\langle \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\cdot\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} - \langle \widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\cdot\boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega}.
\end{aligned}
\tag{47}
$$

Applying the Augmented Lagrangian approach to solve (46) yields

$$
\begin{aligned}
(\delta\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \delta\widehat{\boldsymbol{u}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall\boldsymbol{v}\in\boldsymbol{V}_h, \\
\left(\frac{\delta\boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} & \\
-\langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall\boldsymbol{w}\in\boldsymbol{W}_h, \\
((\epsilon+\beta)\delta p_h, s)_{\mathcal{T}_h} - (\delta\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} + \langle \delta\widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, s\rangle_{\partial\mathcal{T}_h} &= r(s) + (\beta\delta p_h^{m-1}, s)_{\mathcal{T}_h}, && \forall s\in P_h, \\
\langle(\partial_{\boldsymbol{U}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{U}_h + \partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} & \\
+\langle(\partial_{\boldsymbol{U}}\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{U}_h + \partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega} &= r(\boldsymbol{\mu}), && \forall\boldsymbol{\mu}\in\boldsymbol{M}_h,
\end{aligned}
\tag{48}
$$

As a result, $\delta\boldsymbol{U}_h^{\boldsymbol{r}}$ is the solution of

$$
\begin{aligned}
(\delta\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall\boldsymbol{v}\in\boldsymbol{V}_h, \\
\left(\frac{\delta\boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall\boldsymbol{w}\in\boldsymbol{W}_h, \\
((\epsilon+\beta)\delta p_h, s)_{\mathcal{T}_h} - (\delta\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} &= r(s) + (\beta\delta p_h^{m-1}, s)_{\mathcal{T}_h}, && \forall s\in P_h.
\end{aligned}
\tag{49}
$$

And $\delta\boldsymbol{U}_h^{\delta\boldsymbol{\lambda}}$ is the solution of

$$
\begin{aligned}
(\delta\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} &= \langle \delta\boldsymbol{\delta}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, && \forall\boldsymbol{v}\in\boldsymbol{V}_h, \\
\left(\frac{\delta\boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{\lambda}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}, && \forall\boldsymbol{w}\in\boldsymbol{W}_h, \\
((\epsilon+\beta)\delta p_h, s)_{\mathcal{T}_h} - (\delta\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} &= -\langle \delta\boldsymbol{\lambda}_h\cdot\boldsymbol{n}, s\rangle_{\partial\mathcal{T}_h}, && \forall s\in P_h.
\end{aligned}
\tag{50}
$$

The Eqns (49) and (50) define the local solver.

We solve the wave propagation problem

$$
\begin{aligned}
\frac{\partial \boldsymbol{q}}{\partial t} - \nabla \boldsymbol{u} &= \boldsymbol{0}, && \text{in } \Omega,\\
\frac{\partial \boldsymbol{u}}{\partial t} - \nabla \cdot \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{q}, p) &= \boldsymbol{f}, && \text{in } \Omega,\\
\epsilon \frac{\partial p}{\partial t} + \nabla \cdot \boldsymbol{u} &= 0, && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g}_D, && \text{on } \Gamma_D,\\
\boldsymbol{B}^n(\boldsymbol{u}, \boldsymbol{q}, p) \cdot \boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N.
\end{aligned} \tag{51}
$$

The HDG method seeks $(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times \boldsymbol{M}_h$ such that

$$
\begin{aligned}
\left(\frac{\boldsymbol{q}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= \left(\frac{\boldsymbol{q}_h^{k-1}}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h}, && \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
\left(\frac{\boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\boldsymbol{F}(\boldsymbol{U}_h), \nabla \boldsymbol{w})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{F}}(\boldsymbol{U}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} + \left(\frac{\boldsymbol{u}_h^{k-1}}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h,\\
\left(\epsilon \frac{p_h}{\Delta t}, s\right)_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, s \rangle_{\partial \mathcal{T}_h} &= \left(\epsilon \frac{p_h^{k-1}}{\Delta t}, s\right)_{\mathcal{T}_h}, && \forall s \in P_h,\\
\langle \widehat{\boldsymbol{F}}(\boldsymbol{U}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \langle \widehat{\boldsymbol{F}}^b(\boldsymbol{U}_h, \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial \Omega} &= 0, && \forall \boldsymbol{\mu} \in \boldsymbol{M}_h,
\end{aligned} \tag{52}
$$

Linearizing (52) gives

$$
\begin{aligned}
\left(\frac{\delta \boldsymbol{q}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \delta \widehat{\boldsymbol{u}}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
\left(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h) \delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \boldsymbol{F}(\overline{\boldsymbol{U}}_h) \delta \boldsymbol{q}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} & \\
- \langle (\partial_{\boldsymbol{u}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \boldsymbol{u}_h + \partial_{\boldsymbol{q}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \boldsymbol{q}_h) \cdot \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} & \\
- \langle (\partial_{\widehat{\boldsymbol{u}}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h,\\
\left(\epsilon \frac{\delta p_h}{\Delta t}, s\right)_{\mathcal{T}_h} - (\delta \boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} + \langle \delta \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, s \rangle_{\partial \mathcal{T}_h} &= r(s), && \forall s \in P_h,\\
\langle (\partial_{\boldsymbol{U}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \boldsymbol{U}_h + \partial_{\widehat{\boldsymbol{u}}} \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} & \\
+ \langle (\partial_{\boldsymbol{U}} \widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \boldsymbol{U}_h + \partial_{\widehat{\boldsymbol{u}}} \widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \delta \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial \Omega} &= r(\boldsymbol{\mu}), && \forall \boldsymbol{\mu} \in \boldsymbol{M}_h,
\end{aligned} \tag{53}
$$

where

$$
\begin{aligned}
\boldsymbol{r}(\boldsymbol{v}) &= \left(\frac{\boldsymbol{q}_h^{k-1}}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} + \left(\frac{\overline{\boldsymbol{q}}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - (\overline{\boldsymbol{u}}_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle \overline{\widehat{\boldsymbol{u}}}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h},\\
\boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} + \left(\frac{\boldsymbol{u}_h^{k-1}}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - \left(\frac{\overline{\boldsymbol{u}}_h}{\Delta t}, \boldsymbol{w}\right)_{\mathcal{T}_h} - (\boldsymbol{F}(\overline{\boldsymbol{U}}_h), \nabla \boldsymbol{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \cdot \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h},\\
r(s) &= \left(\epsilon \frac{p_h^{k-1}}{\Delta t}, s\right)_{\mathcal{T}_h} - \left(\epsilon \frac{\overline{p}_h}{\Delta t}, s\right)_{\mathcal{T}_h} + (\overline{\boldsymbol{u}}_h, \nabla s)_{\mathcal{T}_h} - \langle \overline{\widehat{\boldsymbol{u}}}_h \cdot \boldsymbol{n}, s \rangle_{\partial \mathcal{T}_h},\\
\boldsymbol{r}(\mu) &= -\langle \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \cdot \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} - \langle \widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \cdot \boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial \Omega}.
\end{aligned} \tag{54}
$$

Applying the Augmented Lagrangian approach to solve (53) yields

$$
\begin{aligned}
\Big(\frac{\delta \boldsymbol{q}_h}{\Delta t}, \boldsymbol{v}\Big)_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \delta\widehat{\boldsymbol{u}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v}\in \boldsymbol{V}_h,\\
\Big(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\Big)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle (\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} & \\
-\langle (\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w}\in \boldsymbol{W}_h,\\
\Big((\frac{\epsilon}{\delta t}+\beta)\delta p_h, s\Big)_{\mathcal{T}_h} - (\delta\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} + \langle \delta\widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, s\rangle_{\partial\mathcal{T}_h} &= r(s) + (\beta\delta p_h^{m-1}, s)_{\mathcal{T}_h}, && \forall s\in P_h,\\
\langle (\partial_{\boldsymbol{U}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{U}_h + \partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} & \\
+\langle (\partial_{\boldsymbol{U}}\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{U}_h + \partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}^b(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega} &= r(\boldsymbol{\mu}), && \forall \boldsymbol{\mu}\in \boldsymbol{M}_h,
\end{aligned}
\tag{55}
$$

As a result, $\delta\boldsymbol{U}_h^{\boldsymbol{r}}$ is the solution of

$$
\begin{aligned}
\Big(\frac{\delta \boldsymbol{q}_h}{\Delta t}, \boldsymbol{v}\Big)_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v}\in \boldsymbol{V}_h,\\
\Big(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\Big)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle (\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w}\in \boldsymbol{W}_h,\\
\Big((\frac{\epsilon}{\delta t}+\beta)\delta p_h, s\Big)_{\mathcal{T}_h} - (\delta\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} &= r(s) + (\beta\delta p_h^{m-1}, s)_{\mathcal{T}_h}, && \forall s\in P_h.
\end{aligned}
\tag{56}
$$

And $\delta\boldsymbol{U}_h^{\delta\boldsymbol{\lambda}}$ is the solution of

$$
\begin{aligned}
\Big(\frac{\delta \boldsymbol{q}_h}{\Delta t}, \boldsymbol{v}\Big)_{\mathcal{T}_h} + (\delta \boldsymbol{u}_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} &= \langle \delta\boldsymbol{\delta}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, && \forall \boldsymbol{v}\in \boldsymbol{V}_h,\\
\Big(\frac{\delta \boldsymbol{u}_h}{\Delta t}, \boldsymbol{w}\Big)_{\mathcal{T}_h} + (\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{q}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle (\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{q}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{q}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \langle (\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h,\overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{\lambda}_h)\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}, && \forall \boldsymbol{w}\in \boldsymbol{W}_h,\\
\Big((\frac{\epsilon}{\delta t}+\beta)\delta p_h, s\Big)_{\mathcal{T}_h} - (\delta\boldsymbol{u}_h, \nabla s)_{\mathcal{T}_h} &= -\langle \delta\boldsymbol{\lambda}_h\cdot\boldsymbol{n}, s\rangle_{\partial\mathcal{T}_h}, && \forall s\in P_h.
\end{aligned}
\tag{57}
$$

The Eqns (56) and (57) define the local solver.

2.4 $\boldsymbol{uw}$-Type

We solve the boundary value problem of the form.

$$
\begin{aligned}
\boldsymbol{w} + \nabla\times\boldsymbol{u} &= \boldsymbol{0}, && \text{in } \Omega,\\
-\nabla\times\boldsymbol{F}(\boldsymbol{u},\boldsymbol{w}) - k^2\boldsymbol{u} &= \boldsymbol{f}, && \text{in } \Omega,\\
\boldsymbol{u}\times\boldsymbol{n} &= \boldsymbol{g}_D, && \text{on } \Gamma_D,\\
\boldsymbol{B}^n(\boldsymbol{u},\boldsymbol{w})\times\boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N.
\end{aligned}
\tag{58}
$$

The HDG method seeks $(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times \boldsymbol{M}_h$ such that

$$
\begin{aligned}
(\boldsymbol{w}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \times \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h \times \boldsymbol{n}, \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{0}, && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
-(\boldsymbol{F}(\boldsymbol{u}_h, \boldsymbol{w}_h), \nabla \times \boldsymbol{p})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h), \boldsymbol{p} \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - (k^2 \boldsymbol{u}_h, \boldsymbol{p})_{\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{p})_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h, \\
\langle \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h), \boldsymbol{\mu} \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} + \langle \widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h) - \boldsymbol{g}, \boldsymbol{\mu} \times \boldsymbol{n}\rangle_{\partial\Omega} &= 0, && \forall \boldsymbol{\mu} \in \boldsymbol{M}_h,
\end{aligned}
\tag{59}
$$

where

$$
\begin{aligned}
\widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h) &= \boldsymbol{F}(\widehat{\boldsymbol{u}}_h, \boldsymbol{w}_h) - \mathbf{S}(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h)(\boldsymbol{u}_h \times \boldsymbol{n} - \widehat{\boldsymbol{u}}_h \times \boldsymbol{n}), \\
\widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h) &= \begin{cases} \widehat{\boldsymbol{u}}_h \times \boldsymbol{n}, & \text{on } \Gamma_D, \\ \boldsymbol{B}^n(\widehat{\boldsymbol{u}}_h, \boldsymbol{w}_h) - \mathbf{S}(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h)(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \times \boldsymbol{n}, & \text{on } \Gamma_N, \end{cases} \\
\boldsymbol{g} &= \begin{cases} \boldsymbol{g}_D, & \text{on } \Gamma_D, \\ \boldsymbol{g}_N, & \text{on } \Gamma_N. \end{cases}
\end{aligned}
\tag{60}
$$

In the two dimensional case, we have

$$
\begin{aligned}
(w_h, v)_{\mathcal{T}_h} + (u_x, dv/dy)_{\mathcal{T}_h} - (u_y, dv/dx)_{\mathcal{T}_h} - \langle u_x n_y - u_y n_x, v\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{0}, && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
-(\boldsymbol{F}(\boldsymbol{u}_h, \boldsymbol{w}_h), \nabla \times \boldsymbol{p})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h), \boldsymbol{p} \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - (k^2 \boldsymbol{u}_h, \boldsymbol{p})_{\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{p})_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h, \\
\langle \widehat{\boldsymbol{F}}(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h), \boldsymbol{\mu} \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} + \langle \widehat{\boldsymbol{F}}^b(\boldsymbol{u}_h, \boldsymbol{w}_h, \widehat{\boldsymbol{u}}_h) - \boldsymbol{g}, \boldsymbol{\mu} \times \boldsymbol{n}\rangle_{\partial\Omega} &= 0, && \forall \boldsymbol{\mu} \in \boldsymbol{M}_h,
\end{aligned}
\tag{61}
$$

Linearization of the first two equations of (18) gives

$$
\begin{aligned}
(\delta\boldsymbol{w}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta\boldsymbol{u}_h, \nabla \times \boldsymbol{v})_{\mathcal{T}_h} - \langle \delta\widehat{\boldsymbol{u}}_h, \boldsymbol{v} \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{w}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{w}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{w}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{w}_h) \times \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} & \\
-\langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\widehat{\boldsymbol{u}}_h) \times \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h,
\end{aligned}
\tag{62}
$$

where

$$
\begin{aligned}
\boldsymbol{r}(\boldsymbol{v}) &= -(\overline{\boldsymbol{w}}_h, \boldsymbol{v})_{\mathcal{T}_h} - (\overline{\boldsymbol{u}}_h, \nabla \times \boldsymbol{v})_{\mathcal{T}_h} + \langle \overline{\widehat{\boldsymbol{u}}}_h, \boldsymbol{v} \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, \\
\boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} - (\boldsymbol{F}(\overline{\boldsymbol{U}}_h), \nabla\boldsymbol{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h) \times \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
\tag{63}
$$

As a result, $(\delta\boldsymbol{u}_h^{\boldsymbol{r}}, \delta\boldsymbol{w}_h^{\boldsymbol{r}})$ is the solution of

$$
\begin{aligned}
(\delta\boldsymbol{w}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta\boldsymbol{u}_h, \nabla \times \boldsymbol{v})_{\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{v}), && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{w}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{w}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{w}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{w}_h) \times \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \boldsymbol{r}(\boldsymbol{w})_{\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h,
\end{aligned}
\tag{64}
$$

And $(\delta\boldsymbol{u}_h^{\delta\boldsymbol{\lambda}}, \delta\boldsymbol{w}_h^{\delta\boldsymbol{\lambda}})$ is the solution of

$$
\begin{aligned}
(\delta\boldsymbol{w}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\delta\boldsymbol{u}_h, \nabla \times \boldsymbol{v})_{\mathcal{T}_h} &= \langle \delta\boldsymbol{\lambda}, \boldsymbol{v} \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, && \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
(\partial_{\boldsymbol{u}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{w}}\boldsymbol{F}(\overline{\boldsymbol{U}}_h)\delta\boldsymbol{w}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} & \\
-\langle(\partial_{\boldsymbol{u}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{u}_h + \partial_{\boldsymbol{w}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{w}_h) \times \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= \langle(\partial_{\widehat{\boldsymbol{u}}}\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}_h, \overline{\widehat{\boldsymbol{u}}}_h)\delta\boldsymbol{\lambda}) \times \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}, && \forall \boldsymbol{w} \in \boldsymbol{W}_h,
\end{aligned}
\tag{65}
$$

The Eqns (22) and (23) define the local solver.

3 Conservation Laws on Moving Domain

Let $\Omega_{\boldsymbol{x}}(t) \in \mathbb{R}^{d=3}$ be a time-dependent domain whose spatial point is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$. In this domain we consider a system of conservation laws written in the Eulerian framework as

$$\left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{F}(\boldsymbol{u}, \nabla_{\boldsymbol{x}} \boldsymbol{u}) = 0, \quad \text{in } \Omega_{\boldsymbol{x}}(t) \times (0, T], \tag{66}$$

where $\boldsymbol{u} \in \mathbb{R}^m$ is a vector of m conserved variables and $\boldsymbol{F} \in \mathbb{R}^{m \times d}$ contains d flux vectors of dimension m. Examples of conservation laws include the Navier-Stokes equations, the equations of solid motion, and the Maxwell's equations.

4 Transformed Conservation Laws on Reference Domain

4.1 Mapping

Let us denote by $\Omega_{\boldsymbol{X}}$ a fixed (time-independent) reference domain with spatial coordinate $\boldsymbol{X} = (X_1, X_2, X_3)$. Our goal is to transform the system of conservation laws on the moving domain $\Omega_{\boldsymbol{x}}(t)$ to an equivalent system on the reference domain $\Omega_{\boldsymbol{X}}$. This will permit the use of standard discretization techniques to discretize the resulting system on the reference domain. Toward this end we assume that there exists a one-to-one mapping $\boldsymbol{\phi}(\boldsymbol{X}, t)$ that maps every point $\boldsymbol{X} \in \Omega_{\boldsymbol{X}}$ to a point $\boldsymbol{x} \in \Omega_{\boldsymbol{x}}(t)$:

$$\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X}, t). \tag{67}$$

The deformation gradient and velocity are then given by

$$\boldsymbol{G} = \nabla_{\boldsymbol{X}} \boldsymbol{\phi}, \qquad \boldsymbol{v_g} = \left.\frac{\partial \boldsymbol{\phi}}{\partial t}\right|_{\boldsymbol{X}}. \tag{68}$$

Let $g = \det(\boldsymbol{G})$ be the determinant of the deformation gradient $\boldsymbol{G}$. We have that

$$d\Omega_{\boldsymbol{x}} = g d\Omega_{\boldsymbol{X}}, \tag{69}$$

where $d\Omega_{\boldsymbol{x}} = dx_1 dx_2 dx_3$, $d\Omega_{\boldsymbol{X}} = dX_1 dX_2 dX_3$ denote volume elements in $\Omega_{\boldsymbol{x}}(t)$ and $\Omega_{\boldsymbol{X}}$, respectively. From this relation, we can obtain the Nanson's formula

$$\boldsymbol{n_x} d\mathcal{A}_{\boldsymbol{x}} = g \boldsymbol{G}^{-T} \boldsymbol{n_X} d\mathcal{A}_{\boldsymbol{X}}, \tag{70}$$

where $d\mathcal{A}_{\boldsymbol{x}}$ and $d\mathcal{A}_{\boldsymbol{X}}$ denote area elements in $\Omega_{\boldsymbol{x}}(t)$ and $\Omega_{\boldsymbol{X}}$, respectively. Note here that $\boldsymbol{n_x}$ and $\boldsymbol{n_X}$ are the unit vectors which are outward-pointing and normal to $\mathcal{A}_{\boldsymbol{x}}$ and $\mathcal{A}_{\boldsymbol{X}}$, respectively.

Note that the differentiation of the matrix determinant satisfies the following identity

$$\partial(\det \boldsymbol{A}) = \det \boldsymbol{A} \, \mathrm{tr}(\partial(\boldsymbol{A}) \ \boldsymbol{A}^{-1}), \tag{71}$$

where $\mathrm{tr}(\boldsymbol{A}) = A_{ii}$ is the trace of the matrix $\boldsymbol{A}$.

4.2 Piola Transformation

The above mapping has the following property

$$\nabla_{\boldsymbol{X}} \cdot (g \boldsymbol{G}^{-T}) = \boldsymbol{0}, \tag{72}$$

which is known as the Piola identity. The proof of the Piola identity is quite simple

$$\int_{\Omega_{\boldsymbol{X}}} \nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{G}^{-T}) d\Omega_{\boldsymbol{X}} = \int_{\partial\Omega_{\boldsymbol{X}}} g\boldsymbol{G}^{-T}\boldsymbol{n}_{\boldsymbol{X}} d\mathcal{A}_{\boldsymbol{X}} = \int_{\partial\Omega_{\boldsymbol{x}}} \boldsymbol{n}_{\boldsymbol{x}} d\mathcal{A}_{\boldsymbol{x}} = \int_{\Omega_{\boldsymbol{x}}} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{I} d\Omega_{\boldsymbol{x}} = \boldsymbol{0}, \tag{73}$$

by the Gauss divergence theorem and the Nanson's formula. For arbitrary tensors $\boldsymbol{W}$ and $\boldsymbol{V}$ we recall the Piola transformation

$$\boldsymbol{W} = g\boldsymbol{V}\boldsymbol{G}^{-T}, \tag{74}$$

and the inverse Piola transformation

$$\boldsymbol{V} = g^{-1}\boldsymbol{W}\boldsymbol{G}^{T}. \tag{75}$$

By the properties of the gradient and divergence operators and using the Piola identity, we obtain

$$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{W} = \nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{V}\boldsymbol{G}^{-T}) = \nabla_{\boldsymbol{X}}\boldsymbol{V} : g\boldsymbol{G}^{-T} + \boldsymbol{V}\nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{G}^{-T}) = g(\nabla_{\boldsymbol{X}} \cdot \boldsymbol{V})\boldsymbol{G}^{-T} = g\nabla_{\boldsymbol{x}} \cdot \boldsymbol{V}\ . \tag{76}$$

It thus follows that

$$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{W} = g\nabla_{\boldsymbol{x}} \cdot (g^{-1}\boldsymbol{W}\boldsymbol{G}^{T}), \qquad \nabla_{\boldsymbol{x}} \cdot \boldsymbol{V} = g^{-1}\nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{V}\boldsymbol{G}^{-T})\ . \tag{77}$$

We further derive the Piola relationships for arbitrary vectors $\boldsymbol{w}$ and $\boldsymbol{v}$ as follows. Given the Piola tranformation

$$\boldsymbol{w} = g\boldsymbol{G}^{-1}\boldsymbol{v}, \qquad \boldsymbol{v} = g^{-1}\boldsymbol{G}\boldsymbol{w}. \tag{78}$$

we similarly obtain

$$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{w} = \nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{G}^{-1}\boldsymbol{v}) = \nabla_{\boldsymbol{X}}\boldsymbol{v} : g\boldsymbol{G}^{-T} + \boldsymbol{v}^{T}\nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{G}^{-T}) = g(\nabla_{\boldsymbol{X}} \cdot \boldsymbol{v})\boldsymbol{G}^{-T} = g\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}\ . \tag{79}$$

by using the properties of the gradient and divergence operators as well as the Piola identity. It thus follows that

$$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{w} = g\nabla_{\boldsymbol{x}} \cdot (g^{-1}\boldsymbol{G}\boldsymbol{v}), \qquad \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = g^{-1}\nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{G}^{-1}\boldsymbol{v})\ . \tag{80}$$

4.3 Kinematic Relations

We derive here some useful kinematic relations for later use. First, we use the chain rule to obtain the material time derivative

$$\begin{aligned} \left.\frac{\partial \alpha(\boldsymbol{x},t)}{\partial t}\right|_{\boldsymbol{X}} &= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}}\alpha \cdot \frac{\partial \boldsymbol{x}}{\partial t} \\ &= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v}_g \cdot \nabla_{\boldsymbol{x}}\alpha, \end{aligned} \tag{81}$$

for any scalar quantity α. For the Jacobian $g = \det(\boldsymbol{G})$, its material time derivative is given by

$$\begin{aligned} \left.\frac{\partial g(\boldsymbol{x},t)}{\partial t}\right|_{\boldsymbol{X}} &= g\ \mathrm{tr}\left(\left.\frac{\partial \boldsymbol{G}}{\partial t}\right|_{\boldsymbol{X}}\boldsymbol{G}^{-1}\right) \\ &= g\ \mathrm{tr}\left(\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{X}}\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}}\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}\right) \\ &= g\ \mathrm{tr}\left(\frac{\partial \boldsymbol{v_g}}{\partial \boldsymbol{x}}\right) \\ &= g\ \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v_g} \\ &= \nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{G}^{-1}\boldsymbol{v}_g)\ . \end{aligned} \tag{82}$$

It thus follows that the Jacobian g satisfies

$$\frac{\partial g}{\partial t}\Big|_{\boldsymbol{X}} - \nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{G}^{-1}\boldsymbol{v}_g) = 0 \ . \tag{83}$$

4.4 Transformed Conservation Laws

It follows from the above results to obtain

$$\begin{aligned}
\mathbf{0} &= \frac{\partial \boldsymbol{u}}{\partial t}\Big|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{F}(\boldsymbol{u}, \nabla_{\boldsymbol{x}}\boldsymbol{u}) \\
&= \frac{\partial \boldsymbol{u}}{\partial t}\Big|_{\boldsymbol{X}} - \boldsymbol{v}_g \cdot \nabla_{\boldsymbol{x}}\boldsymbol{u} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{F} \\
&= \frac{\partial \boldsymbol{u}}{\partial t}\Big|_{\boldsymbol{X}} - \boldsymbol{v}_g \cdot \nabla_{\boldsymbol{x}}\boldsymbol{u} + g^{-1}\nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{F}\boldsymbol{G}^{-T}) \\
&= g^{-1}\frac{\partial g\boldsymbol{u}}{\partial t}\Big|_{\boldsymbol{X}} - g^{-1}\boldsymbol{u}\frac{\partial g}{\partial t}\Big|_{\boldsymbol{X}} - \boldsymbol{v}_g \cdot \nabla_{\boldsymbol{x}}\boldsymbol{u} + g^{-1}\nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{F}\boldsymbol{G}^{-T}) \\
&= g^{-1}\frac{\partial g\boldsymbol{u}}{\partial t}\Big|_{\boldsymbol{X}} - \boldsymbol{u}(\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}_{\boldsymbol{g}}) - \boldsymbol{v}_g \cdot \nabla_{\boldsymbol{x}}\boldsymbol{u} + g^{-1}\nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{F}\boldsymbol{G}^{-T}) \\
&= g^{-1}\frac{\partial g\boldsymbol{u}}{\partial t}\Big|_{\boldsymbol{X}} - \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{u} \otimes \boldsymbol{v}_{\boldsymbol{g}}) + g^{-1}\nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{F}\boldsymbol{G}^{-T}) \\
&= g^{-1}\frac{\partial g\boldsymbol{u}}{\partial t}\Big|_{\boldsymbol{X}} - g^{-1}\nabla_{\boldsymbol{X}} \cdot (g(\boldsymbol{u} \otimes \boldsymbol{v}_{\boldsymbol{g}})\boldsymbol{G}^{-T}) + g^{-1}\nabla_{\boldsymbol{X}} \cdot (g\boldsymbol{F}\boldsymbol{G}^{-T}) \\
&= g^{-1}\left(\frac{\partial g\boldsymbol{u}}{\partial t}\Big|_{\boldsymbol{X}} + \nabla_{\boldsymbol{X}} \cdot (g(\boldsymbol{F} - \boldsymbol{u} \otimes \boldsymbol{v}_{\boldsymbol{g}})\boldsymbol{G}^{-T})\right)
\end{aligned} \tag{84}$$

Therefore, we arrive at the transformed system of conservation laws in the reference domain as

$$\frac{\partial (g\boldsymbol{u})}{\partial t}\Big|_{\boldsymbol{X}} + \nabla_{\boldsymbol{X}} \cdot (g(\boldsymbol{F}(\boldsymbol{u}, \nabla_{\boldsymbol{x}}\boldsymbol{u}) - \boldsymbol{u} \otimes \boldsymbol{v}_{\boldsymbol{g}})\boldsymbol{G}^{-T}) = 0 \ . \tag{85}$$

We now introduce $\boldsymbol{U} = g\boldsymbol{u}$ and thus have

$$\nabla_{\boldsymbol{x}}\boldsymbol{u} = (\nabla_{\boldsymbol{X}}\boldsymbol{u})\boldsymbol{G}^{-1} = (\nabla_{\boldsymbol{X}}(\boldsymbol{U}/g))\boldsymbol{G}^{-1} = (g^{-1}\nabla_{\boldsymbol{X}}\boldsymbol{U} + \boldsymbol{U} \otimes \nabla_{\boldsymbol{X}} g^{-1})\boldsymbol{G}^{-1}. \tag{86}$$

We finally have

$$\frac{\partial \boldsymbol{U}}{\partial t}\Big|_{\boldsymbol{X}} + \nabla_{\boldsymbol{X}} \cdot \boldsymbol{F}_{\boldsymbol{X}}(\boldsymbol{U}, \nabla_{\boldsymbol{X}}\boldsymbol{U}) = 0, \quad \Omega_{\boldsymbol{X}} \times [0, T] \ . \tag{87}$$

where

$$\boldsymbol{F}_{\boldsymbol{X}}(\boldsymbol{U}, \nabla_{\boldsymbol{X}}\boldsymbol{U}) = g(\boldsymbol{F}(g^{-1}\boldsymbol{U}, (g^{-1}\nabla_{\boldsymbol{X}}\boldsymbol{U} + \boldsymbol{U} \otimes \nabla_{\boldsymbol{X}} g^{-1})\boldsymbol{G}^{-1}) - g^{-1}\boldsymbol{U} \otimes \boldsymbol{v}_{\boldsymbol{g}})\boldsymbol{G}^{-T}. \tag{88}$$

5 Shells

$$\begin{aligned}
\mathbf{F} - \nabla\boldsymbol{\varphi} &= 0, && \text{in } \Omega, \\
-\nabla \cdot (\mathbf{D}(\mathbf{F}) + pJ\mathbf{F}^{-T}) &= \boldsymbol{b}, && \text{in } \Omega, \\
\varepsilon\, p - (J-1) &= 0, && \text{in } \Omega, \\
\boldsymbol{\varphi} &= \boldsymbol{g}_D, && \text{on } \partial\Omega_D, \\
\mathbf{P}\boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \partial\Omega_N.
\end{aligned} \tag{89}$$