

Introduction to Computational Fluid Dynamics using OpenFOAM and Octave

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(Session-4)

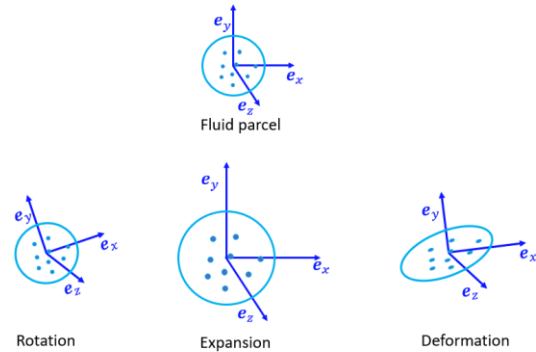
Instructions: Mon, Wed, Thu (5:30PM-6:30PM IST)

Query session: Sundays 8AM-8:30AM IST

Quick Recap

What Did We Discuss?

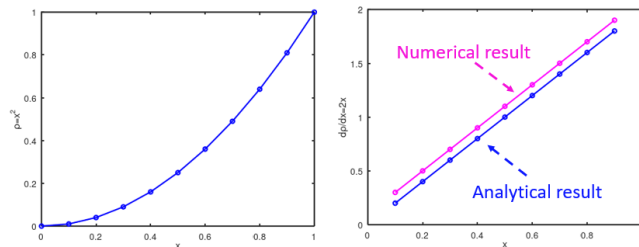
Fluid Behavior



Gradient

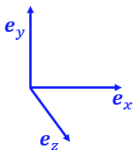
- Numerical Approximation

$$\begin{array}{ccccccc} \bullet & & \bullet & & \bullet & & \bullet \\ x_{i-1} & & x_i & & x_{i+1} & & x_{i+2} \end{array} \quad \left(\frac{\partial \rho}{\partial x} \right)_i = \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} + O(\Delta x_i)$$



Mathematical Operations

- Divergence

$$\nabla \cdot \mathbf{u} = \left(\frac{\partial}{\partial x} e_x + \frac{\partial}{\partial y} e_y + \frac{\partial}{\partial z} e_z \right) (u e_x + v e_y + w e_z) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$


- Numerical approximation

- Same as earlier

- Physical significance

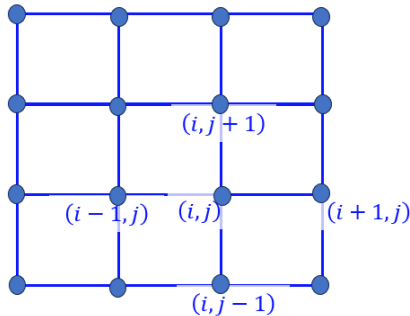
- Positive value : Source or expansion of fluid volume
- Negative value: Sink
- Zero signifies incompressible nature or no change in volume

What Did We Discuss?

Finite Difference – Finite Volume

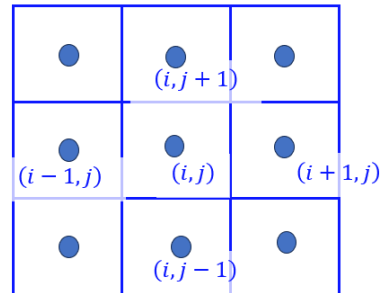
Differential form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$



Integral form

$$\frac{\partial}{\partial t} \int_V \rho dV + \oint_S \rho \mathbf{u} \cdot d\mathbf{S} = 0$$

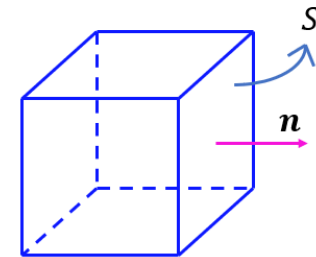


Gauss Divergence Theorem

- For a vector: \mathbf{F}

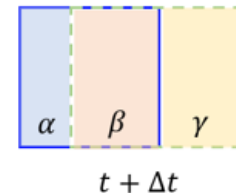
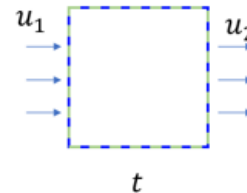
$$\int (\nabla \cdot \mathbf{F}) dV \approx \sum \mathbf{F}_f \cdot \mathbf{S}$$

- Rate of change of a quantity over a control volume = Rate of flow through control surface.



Reynolds Transport Theorem

$$\frac{d}{dt} \int_{V_S} \rho \phi dV = \frac{d}{dt} \int_{V_{CV}} \rho \phi dV + \int_S \rho \phi \mathbf{u} \cdot \mathbf{n} dS$$

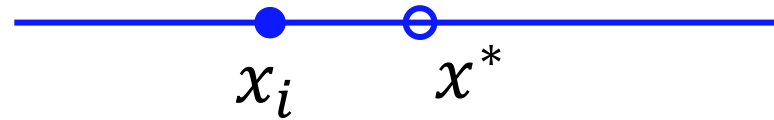


Current Session

Overview

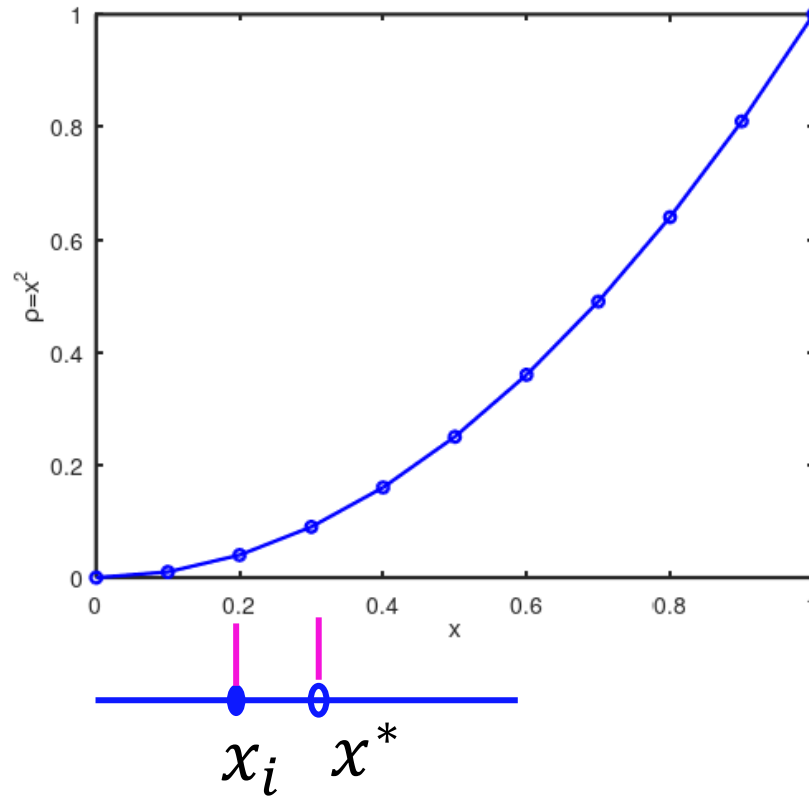
- Taylor series analysis
- Numerical discretization

Taylor Series



$$\rho(x^*) = \rho(x_i) + \frac{(x^* - x_i)}{1!} \left(\frac{d\rho}{dx} \right)_i + \frac{(x^* - x_i)^2}{2!} \left(\frac{d^2\rho}{dx^2} \right)_i + \frac{(x^* - x_i)^3}{3!} \left(\frac{d^3\rho}{dx^3} \right)_i + \dots$$

Taylor Series: Discrete Operations



$$\rho(x^*) = \rho(x_i) + \frac{(x^* - x_i)}{1!} \left(\frac{d\rho}{dx} \right)_i + \frac{(x^* - x_i)^2}{2!} \left(\frac{d^2\rho}{dx^2} \right)_i + \frac{(x^* - x_i)^3}{3!} \left(\frac{d^3\rho}{dx^3} \right)_i + \dots$$

Taylor Series: Discrete Operations



$$\rho(x_{i+1}) = \rho(x_i) + \frac{(x_{i+1} - x_i)}{1!} \left(\frac{d\rho}{dx} \right)_i + \frac{(x_{i+1} - x_i)^2}{2!} \left(\frac{d^2\rho}{dx^2} \right)_i + \frac{(x_{i+1} - x_i)^3}{3!} \left(\frac{d^3\rho}{dx^3} \right)_i + \dots$$

$$\rho(x_{i+1}) = \rho(x_i) + (x_{i+1} - x_i) \left(\frac{d\rho}{dx} \right)_i + O(\Delta x_i^2); \quad \Delta x_i = (x_{i+1} - x_i)$$

$$\rho(x_{i+1}) = \rho(x_i) + \Delta x_i \left(\frac{d\rho}{dx} \right)_i + O(\Delta x_i^2)$$

Taylor Series: Discrete Operations (1st order)

$$\rho(x_{i+1}) = \rho(x_i) + \Delta x_i \left(\frac{d\rho}{dx} \right)_i + O(\Delta x_i^2)$$

$$\left(\frac{d\rho}{dx} \right)_i = \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} + \frac{1}{\Delta x_i} O(\Delta x_i^2)$$

$$\left(\frac{d\rho}{dx} \right)_i = \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} + O(\Delta x_i)$$

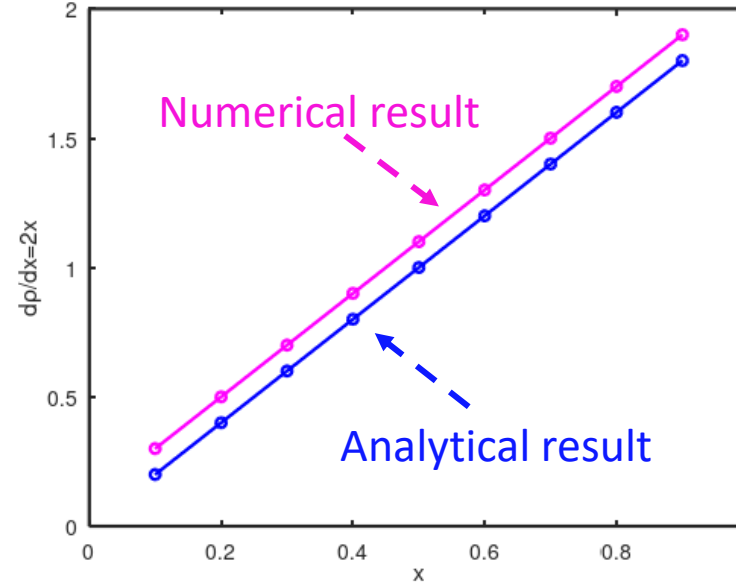
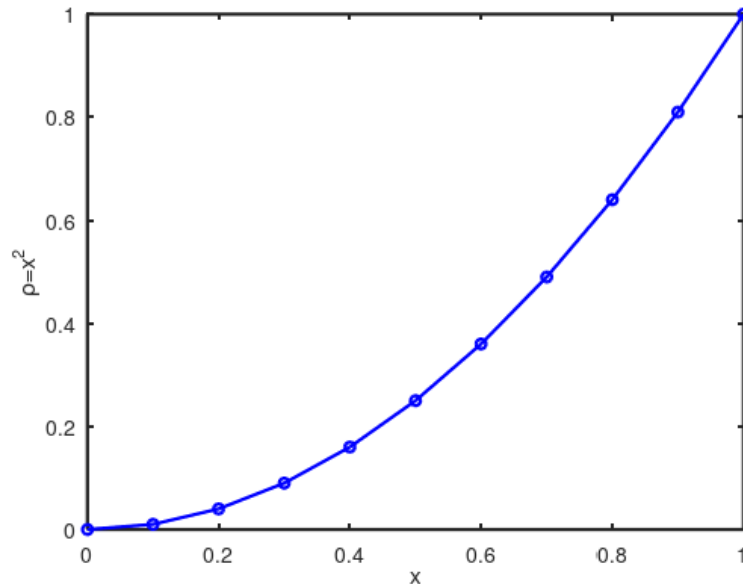
$$\left(\frac{d\rho}{dx} \right)_i \approx \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} \quad \leftarrow \text{First order upwind scheme}$$

First order upwind scheme

- Numerical Approximation



$$\left(\frac{d\rho}{dx}\right)_i = \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} + O(\Delta x_i)$$



Central Difference Scheme (2nd order)



$$\rho(x_{i+1}) = \rho(x_i) + \frac{(x_{i+1} - x_i)}{1!} \left(\frac{d\rho}{dx} \right)_i + \frac{(x_{i+1} - x_i)^2}{2!} \left(\frac{d^2\rho}{dx^2} \right)_i + \frac{(x_{i+1} - x_i)^3}{3!} \left(\frac{d^3\rho}{dx^3} \right)_i + \dots$$

$$(1) \quad \rho(x_{i+1}) = \rho(x_i) + \Delta x_i \left(\frac{d\rho}{dx} \right)_i + \frac{\Delta x_i^2}{2} \left(\frac{d^2\rho}{dx^2} \right)_i + O(\Delta x_i^3)$$

$$(2) \quad \rho(x_{i-1}) = \rho(x_i) - \Delta x_i \left(\frac{d\rho}{dx} \right)_i + \frac{\Delta x_i^2}{2} \left(\frac{d^2\rho}{dx^2} \right)_i + O(\Delta x_i^3)$$

Central Difference Scheme (2nd order)

$$(1) \quad \rho(x_{i+1}) = \rho(x_i) + \Delta x_i \left(\frac{d\rho}{dx} \right)_i + \frac{\Delta x_i^2}{2} \left(\frac{d^2\rho}{dx^2} \right)_i + O(\Delta x_i^3)$$

$$(2) \quad \rho(x_{i-1}) = \rho(x_i) - \Delta x_i \left(\frac{d\rho}{dx} \right)_i + \frac{\Delta x_i^2}{2} \left(\frac{d^2\rho}{dx^2} \right)_i + O(\Delta x_i^3)$$

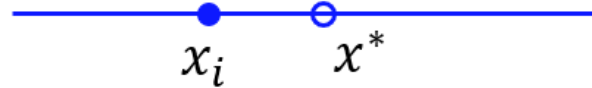


Subtract (2) from (1)

$$\rho(x_{i+1}) - \rho(x_{i-1}) = 2\Delta x_i \left(\frac{d\rho}{dx} \right)_i + O(\Delta x_i^3)$$

$$\left(\frac{d\rho}{dx} \right)_i = \frac{\rho(x_{i+1}) - \rho(x_{i-1}))}{2\Delta x_i} + O(\Delta x_i^2)$$

Summary



$$\rho(x^*) = \rho(x_i) + \frac{(x^* - x_i)}{1!} \left(\frac{d\rho}{dx} \right)_i + \frac{(x^* - x_i)^2}{2!} \left(\frac{d^2\rho}{dx^2} \right)_i + \frac{(x^* - x_i)^3}{3!} \left(\frac{d^3\rho}{dx^3} \right)_i + \dots$$



$$\left(\frac{d\rho}{dx} \right)_i \approx \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i}$$

First order upwind

$$\left(\frac{d\rho}{dx} \right)_i \approx \frac{\rho(x_{i+1}) - \rho(x_{i-1}))}{2\Delta x_i}$$

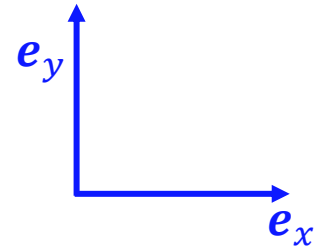
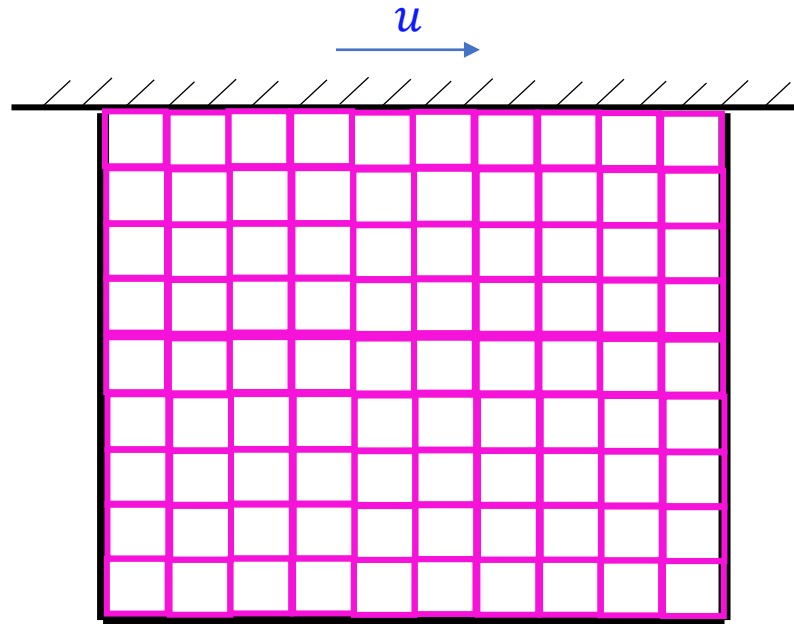
Second order central difference

Exercise

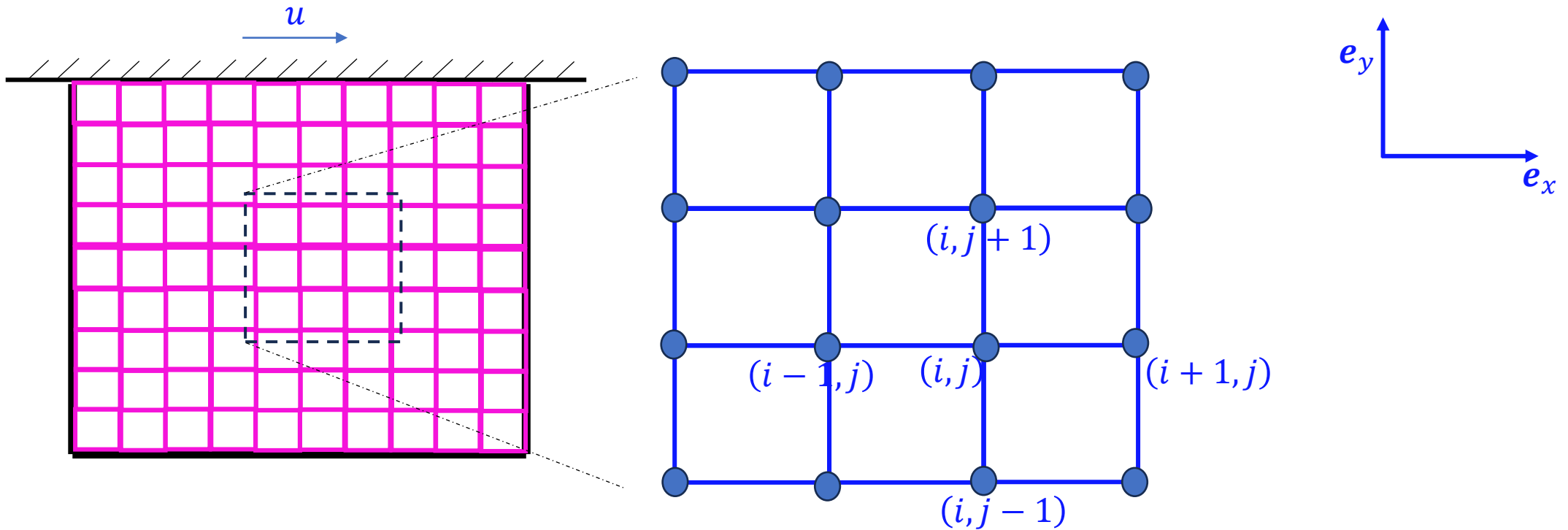


- Derive expression for $\left(\frac{d^2\rho}{dx^2}\right)_i$
- What is the accuracy of the resultant expression?

Numerical Discretization (Grid layout)

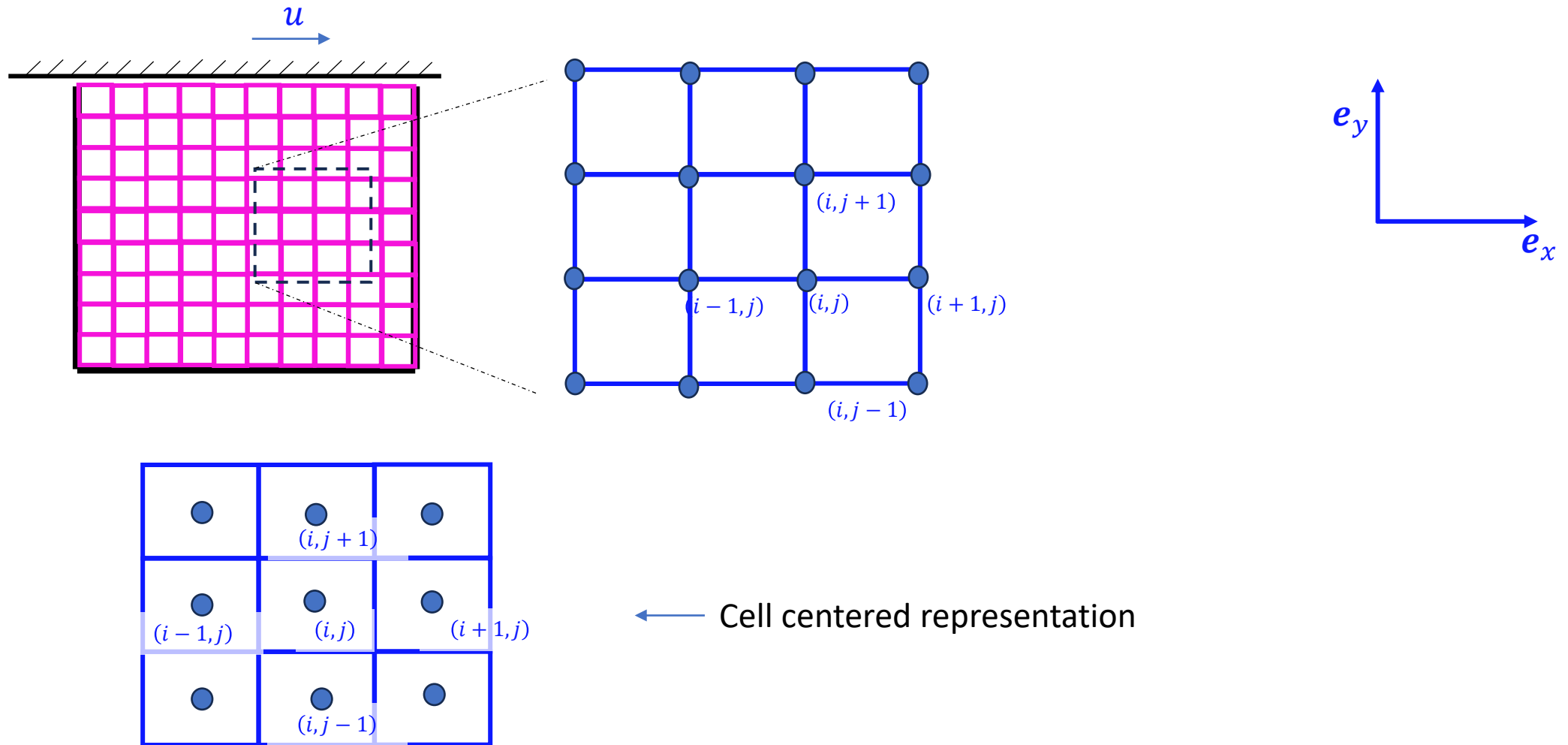


Numerical Discretization (Grid layout)



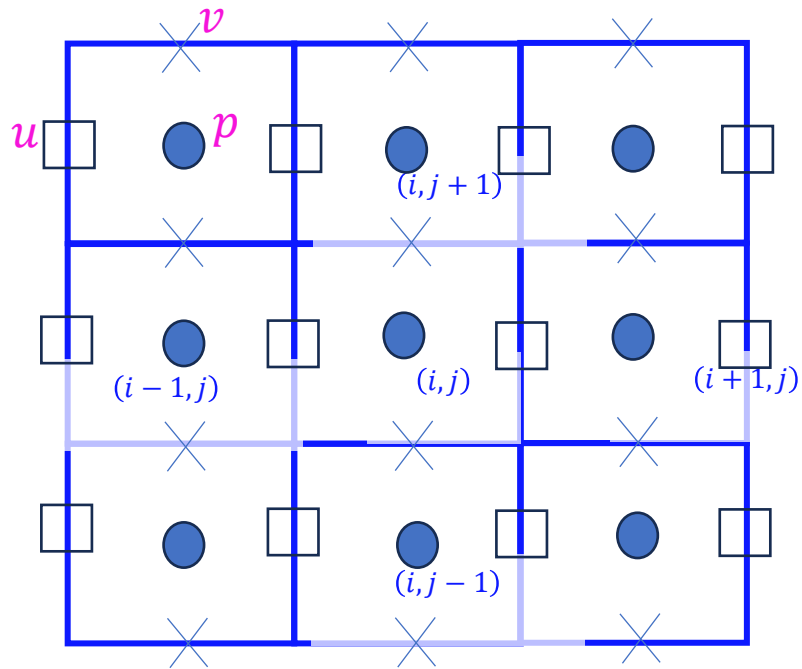
- Flow properties are assigned at each grid locations.

Numerical Discretization

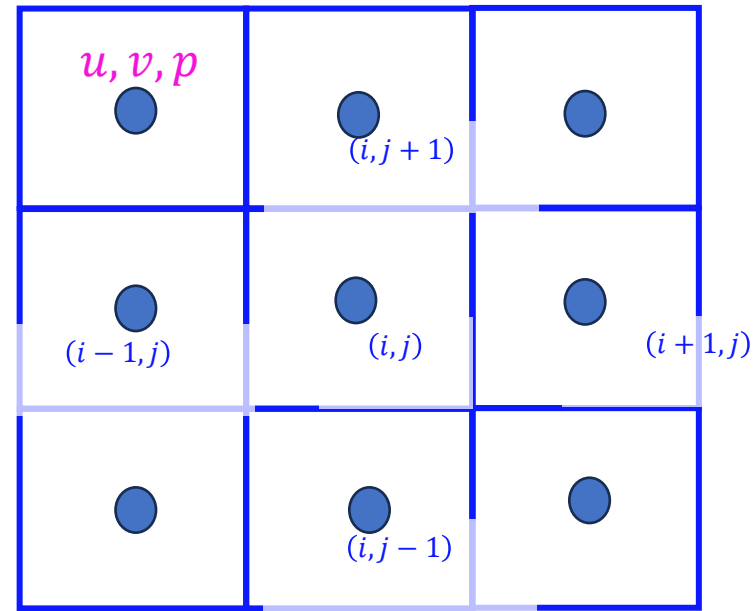


Numerical Discretization

- Cell centered representation



Staggered grid



Collocated grid

Exercise



- Derive expression for $\left(\frac{d^2\rho}{dx^2}\right)_i$
- What is the accuracy of the resultant expression?