Introduction to Computational Fluid Dynamics using OpenFOAM and Octave

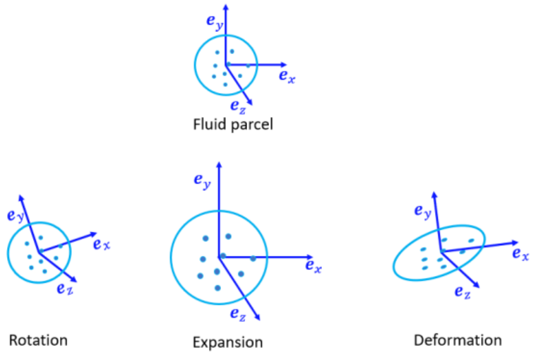
Lakshman Anumolu
Kumaresh Selvakumar
(Session-3)

Instructions: Mon, Wed, Thu (5:30PM-6:30PM IST)
Query session: Sundays 8AM-8:30AM IST

Quick Recap

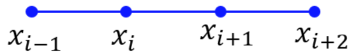
What Did We Discuss?

Fluid Behavior

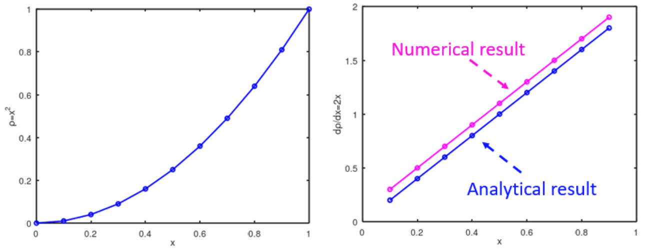


Gradient

- Numerical Approximation



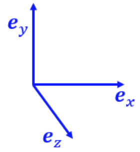
$$\left(\frac{\partial \rho}{\partial x}\right)_i = \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} + O(\Delta x_i)$$



Mathematical Operations

- Divergence

$$\nabla \cdot \boldsymbol{u} = \left(\frac{\partial}{\partial x}\boldsymbol{e}_x + \frac{\partial}{\partial y}\boldsymbol{e}_y + \frac{\partial}{\partial z}\boldsymbol{e}_z\right)(u\boldsymbol{e}_x + v\boldsymbol{e}_y + w\boldsymbol{e}_z) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

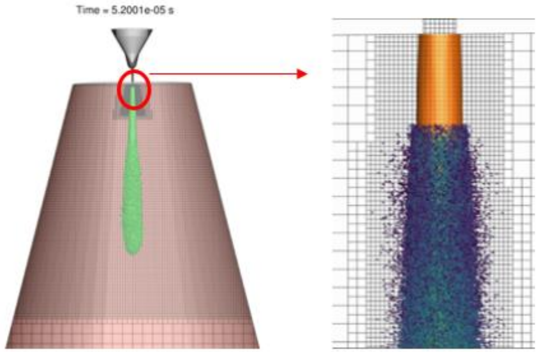


- Numerical approximation
  - Same as earlier
- Physical significance
  - Positive value : Source or expansion of fluid volume
  - Negative value: Sink
  - Zero signifies incompressible nature or no change in volume

What Did We Discuss?

$$\frac{D\phi(\boldsymbol{X}(\boldsymbol{p}_i,t),t)}{Dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{DX}{Dt} + \frac{\partial\phi}{\partial y}\frac{DY}{Dt} + \frac{\partial\phi}{\partial z}\frac{DZ}{Dt}$$

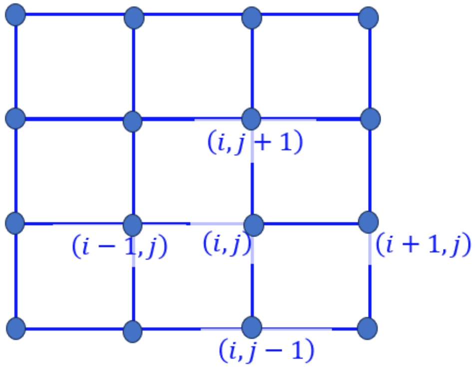
$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \boldsymbol{u}\cdot\nabla\phi$$



Finite Difference – Finite Volume

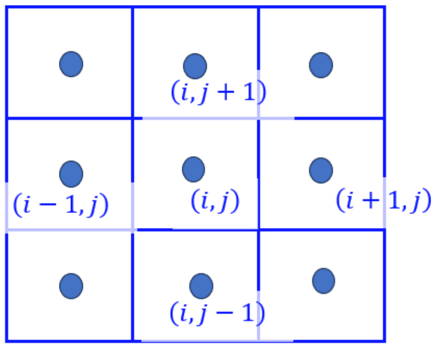
Differential form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0$$



Integral form

$$\frac{\partial}{\partial t}\int_V \rho dV + \oint_S \rho\boldsymbol{u}\cdot d\boldsymbol{S} = 0$$



Current Session

Overview

- Gauss Divergence Theorem

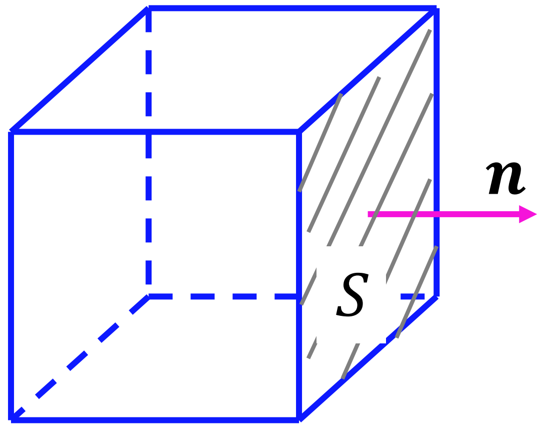
- Reynolds Transport Theorem

Basic Quantities

- Surface Area Vector: $\boldsymbol{S}$

$$\boldsymbol{S} = |\boldsymbol{S}|\boldsymbol{n}$$

$$\boldsymbol{S} = S\boldsymbol{n}$$

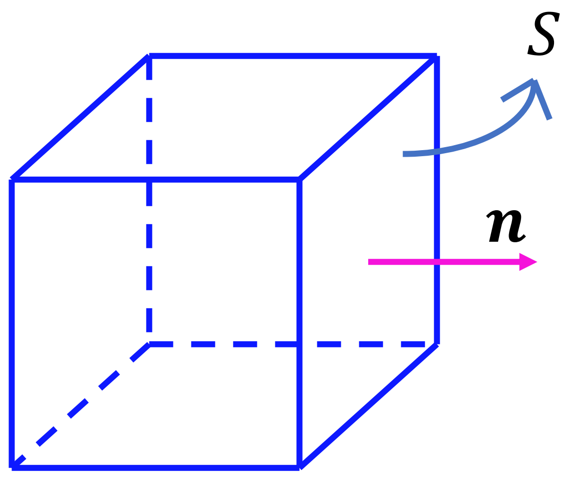


- Extensive property: $\boldsymbol{\Phi}$
- Intensive property: ϕ
  - Conserved property per unit mass
- Mass: $\rho V = \int_V \rho dV$

Gauss Divergence Theorem

- For a vector: $\boldsymbol{F}$

$$\int (\boldsymbol{\nabla} \cdot \boldsymbol{F}) \boldsymbol{dV} \approx \sum \boldsymbol{F} \cdot \boldsymbol{S}$$

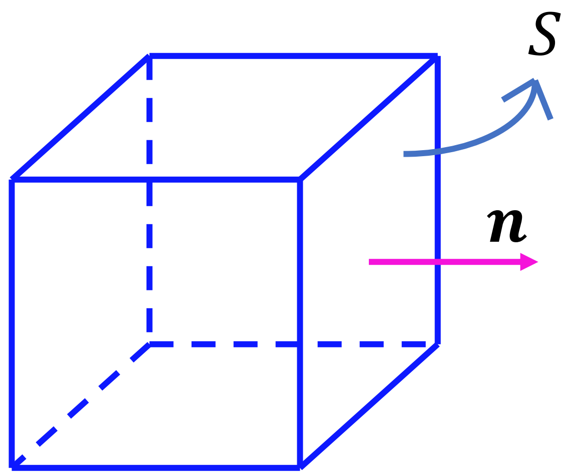


  - Rate of change of a quantity over a control volume = Rate of flow through control surface.

Gauss Divergence Theorem

- For a vector: $\boldsymbol{F}$

$$\int (\boldsymbol{\nabla} \cdot \boldsymbol{F}) \boldsymbol{dV} \approx \sum \boldsymbol{F} \cdot \boldsymbol{S}$$



Let: $\boldsymbol{F} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$

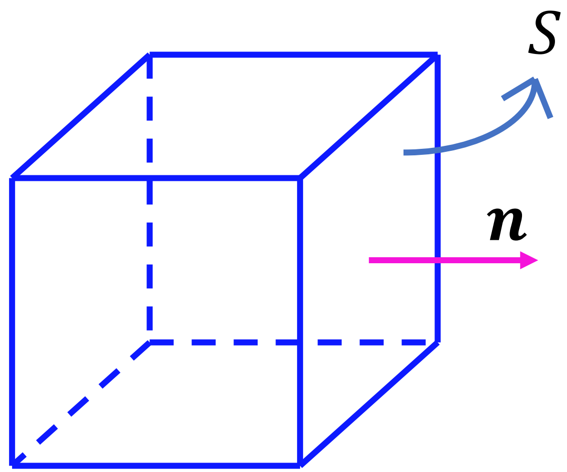
$$\int (\boldsymbol{\nabla} \cdot \boldsymbol{F}) \boldsymbol{dV} = \int \left(\frac{\partial}{\partial x} \boldsymbol{e}_x + \frac{\partial}{\partial y} \boldsymbol{e}_y \right) \cdot (x\boldsymbol{e}_x + y\boldsymbol{e}_y) \boldsymbol{dV}$$

$$\nabla \cdot \boldsymbol{u} = \left(\frac{\partial}{\partial x} \boldsymbol{e}_x + \frac{\partial}{\partial y} \boldsymbol{e}_y + \frac{\partial}{\partial z} \boldsymbol{e}_z \right) (u\boldsymbol{e}_x + v\boldsymbol{e}_y + w\boldsymbol{e}_z) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

Gauss Divergence Theorem

$$\boxed{\int (\nabla \cdot \boldsymbol{F}) d\boldsymbol{V} \approx \sum \boldsymbol{F} \cdot \boldsymbol{S}}$$

Let: $\boldsymbol{F} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$



$$\int (\nabla \cdot \boldsymbol{F}) d\boldsymbol{V} = \int \left(\frac{\partial}{\partial x} \boldsymbol{e}_x + \frac{\partial}{\partial y} \boldsymbol{e}_y \right) \cdot (x\boldsymbol{e}_x + y\boldsymbol{e}_y) d\boldsymbol{V}$$

$$= \int \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) d\boldsymbol{V} = 2V$$

$$S = \Delta x \Delta y$$

$$V = \Delta x \Delta y \Delta z$$

$$\sum \boldsymbol{F} \cdot \boldsymbol{S} = \sum (x\boldsymbol{e}_x + y\boldsymbol{e}_y) \cdot \boldsymbol{S}$$

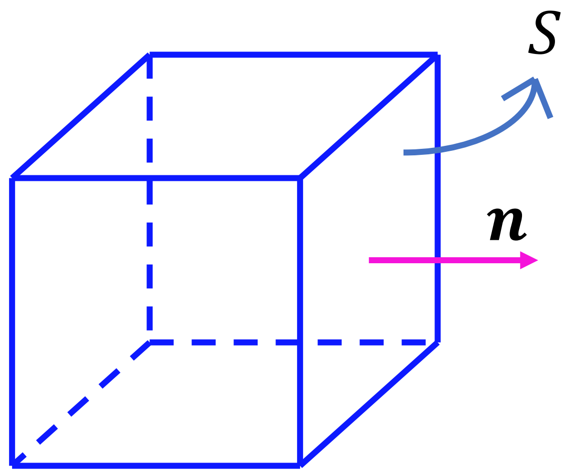
Gauss Divergence Theorem

$$\sum \boldsymbol{F} \cdot \boldsymbol{S} = \sum \left(x\boldsymbol{e}_x + y\boldsymbol{e}_y\right) \cdot \boldsymbol{S}$$

$$= \left(x + \frac{\Delta x}{2}\right) S - \left(x - \frac{\Delta x}{2}\right) S - \left(y - \frac{\Delta y}{2}\right) S + \left(y + \frac{\Delta y}{2}\right) S$$

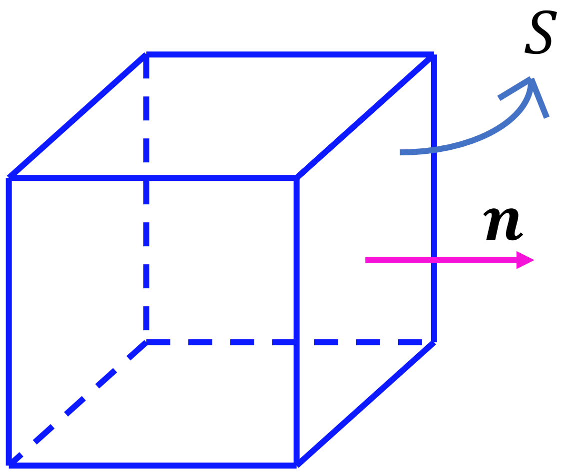
$$= \Delta x S + \Delta y S$$

$$= 2\mathrm{V}$$



Reynolds Transport Theorem

$$\boxed{\frac{d}{dt}\int_{V_S} \rho\phi dV = \frac{d}{dt}\int_{V_{CV}} \rho\phi dV + \int_{S} \rho\phi \boldsymbol{u}\cdot\boldsymbol{n} dS}$$



Reynolds Transport Theorem

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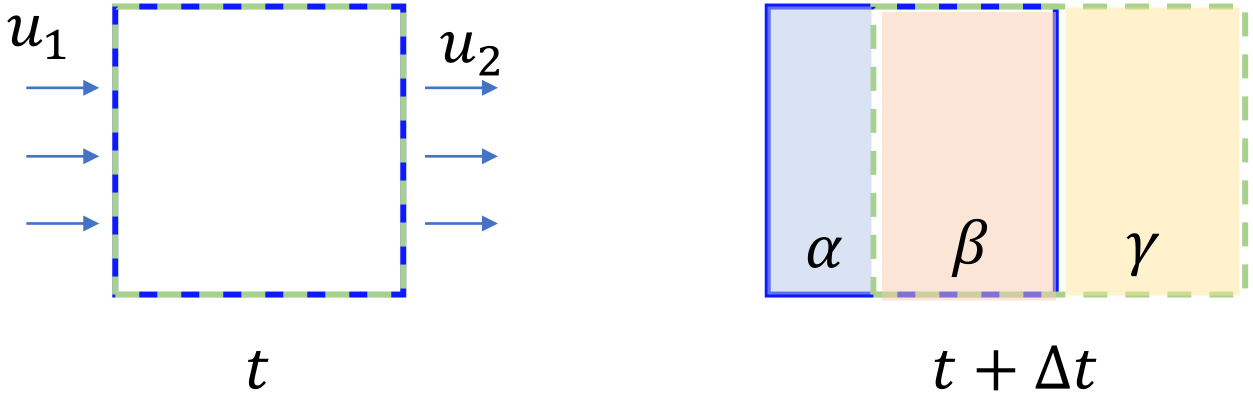
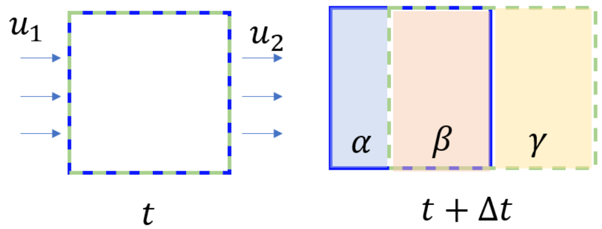


(1) $\Phi_S(t) = \Phi_{CV}(t)$

(2) $\Phi_S(t+\Delta t) = \Phi_{CV}(t+\Delta t) - \Phi_\alpha + \Phi_\gamma$

Reynolds Transport Theorem



$$(1) \quad \Phi_S(t) = \Phi_{CV}(t)$$

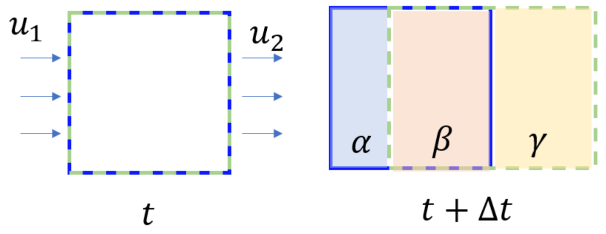
$$(2) \quad \Phi_S(t+\Delta t) = \Phi_{CV}(t+\Delta t) - \Phi_\alpha + \Phi_\gamma$$

Subtract (1) from (2)

$$\left(\frac{\Phi(t+\Delta t) - \Phi(t)}{\Delta t}\right)_{System} = \left(\frac{\Phi(t+\Delta t) - \Phi(t)}{\Delta t}\right)_{CV} - \dot{\Phi}_\alpha + \dot{\Phi}_\gamma$$

$$\frac{d\Phi}{dt}_{System} = \frac{d\Phi}{dt}_{CV} - \dot{\Phi}_\alpha + \dot{\Phi}_\gamma$$

Reynolds Transport Theorem



$$\left.\frac{d\Phi}{dt}\right|_{System} = \left.\frac{d\Phi}{dt}\right|_{CV} - \dot{\Phi}_\alpha + \dot{\Phi}_\gamma$$

$$\Phi_\alpha = \phi_\alpha m_\alpha = \phi_\alpha \rho_\alpha V_\alpha = \phi_\alpha \rho_\alpha (u_1 \Delta t) S$$

Using $\Phi = \int \rho\phi dV$ and net flux as $\int \rho\phi \boldsymbol{u}\cdot\boldsymbol{n} dS$

$$\frac{d}{dt}\int_{V_S} \rho\phi dV = \frac{d}{dt}\int_{V_{CV}} \rho\phi dV + \int_{S} \rho\phi \boldsymbol{u}\cdot\boldsymbol{n} dS$$

Conservation Laws

$$\frac{d}{dt}\int_{V_S} \rho\phi dV = \frac{d}{dt}\int_{V_{CV}} \rho\phi dV + \int_S \rho\phi \boldsymbol{u}\cdot\boldsymbol{n} dS$$

- Conservation of mass

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0; \quad \nabla\cdot(\rho\boldsymbol{u}) = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}$$

- Conservation of momentum

$$\frac{\partial\rho\boldsymbol{u}}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}\boldsymbol{u}) = -\nabla p + \nabla\cdot\boldsymbol{\sigma} + \rho\boldsymbol{g};$$

- Scalar conservation law

$$\frac{\partial\rho\phi}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}\phi) = \nabla\cdot(\Gamma\nabla\phi) + S_\phi$$

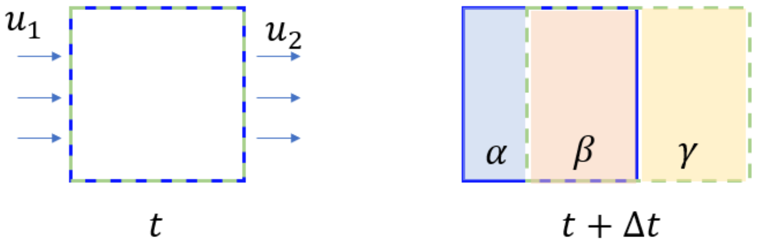
Quick Summary

What Did We Discuss?

$$\int (\boldsymbol{\nabla} \cdot \boldsymbol{F}) d\boldsymbol{V} \approx \sum \boldsymbol{F} \cdot \boldsymbol{S}$$

Reynolds Transport Theorem

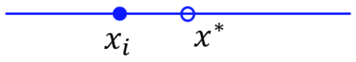
$$\frac{d}{dt} \int_{V_S} \rho \phi dV = \frac{d}{dt} \int_{V_{CV}} \rho \phi dV + \int_S \rho \phi \boldsymbol{u} \cdot \boldsymbol{n} dS$$



Next Session

- Taylor series analysis
- Numerical discretization

Taylor Series: Discrete Operations

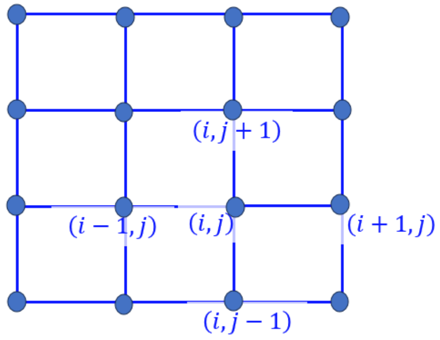


$$\rho(x^*) = \rho(x_i) + (x^* - x_i)\left(\frac{\partial \rho}{\partial x}\right)_i + (x^* - x_i)^2\left(\frac{\partial^2 \rho}{\partial x^2}\right)_i + (x^* - x_i)^3\left(\frac{\partial^3 \rho}{\partial x^3}\right)_i + \cdots$$

Finite Difference – Finite Volume

Differential form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$



Integral form

$$\frac{\partial}{\partial t}\int_V \rho dV + \oint_S \rho \boldsymbol{u} \cdot d\boldsymbol{S} = 0$$