Introduction to Computational Fluid Dynamics using OpenFOAM and Octave

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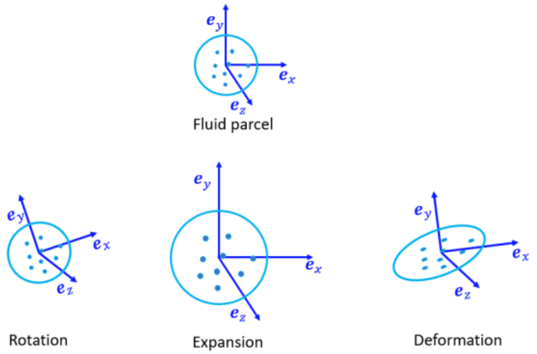
Kumaresh Selvakumar
(Session-4)

Instructions: Mon, Wed, Thu (3:30PM-4:30PM IST)
Query session: Sundays 8:30AM-9:00AM IST

Quick Recap

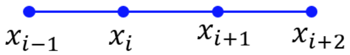
What Did We Discuss?

Fluid Behavior

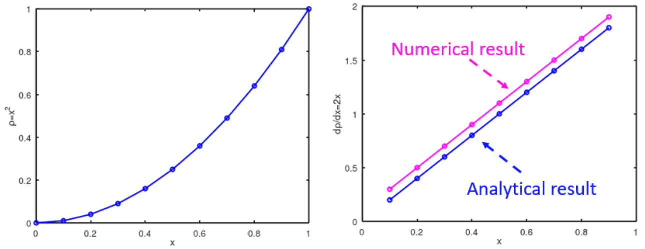


Gradient

- Numerical Approximation



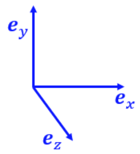
$$\left(\frac{\partial\rho}{\partial x}\right)_i = \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} + O(\Delta x_i)$$



Mathematical Operations

- Divergence

$$\nabla \cdot \boldsymbol{u} = \left(\frac{\partial}{\partial x}\boldsymbol{e}_x + \frac{\partial}{\partial y}\boldsymbol{e}_y + \frac{\partial}{\partial z}\boldsymbol{e}_z\right)(u\boldsymbol{e}_x + v\boldsymbol{e}_y + w\boldsymbol{e}_z) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$



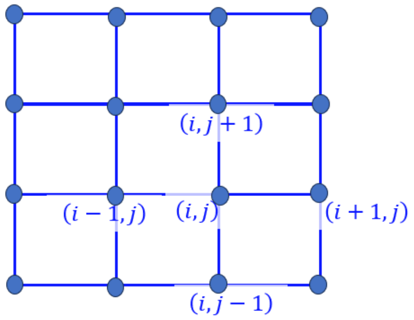
- Numerical approximation
  - Same as earlier
- Physical significance
  - Positive value : Source or expansion of fluid volume
  - Negative value: Sink
  - Zero signifies incompressible nature or no change in volume

What Did We Discuss?

Finite Difference – Finite Volume

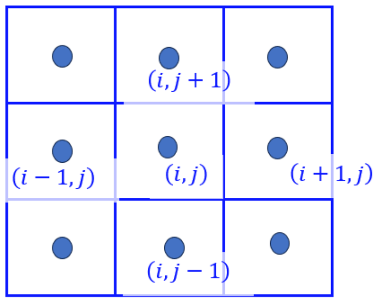
Differential form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$



Integral form

$$\frac{\partial}{\partial t}\int_V \rho dV + \oint_S \rho \boldsymbol{u} \cdot d\boldsymbol{S} = 0$$

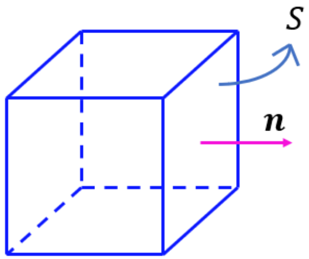


Gauss Divergence Theorem

- For a vector: $\boldsymbol{F}$

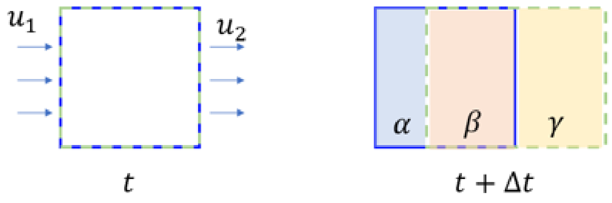
$$\int (\boldsymbol{\nabla} \cdot \boldsymbol{F}) \boldsymbol{dV} \approx \sum \boldsymbol{F_f} \cdot \boldsymbol{S}$$

- Rate of change of a quantity over a control volume = Rate of flow through control surface.



Reynolds Transport Theorem

$$\frac{d}{dt}\int_{V_S} \rho\phi dV = \frac{d}{dt}\int_{V_{CV}} \rho\phi dV + \int_S \rho\phi \boldsymbol{u} \cdot \boldsymbol{n} dS$$

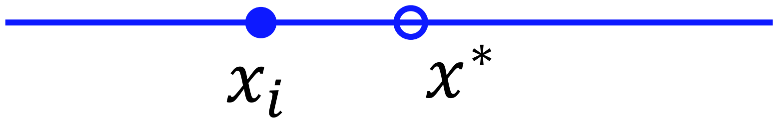


Current Session

Overview

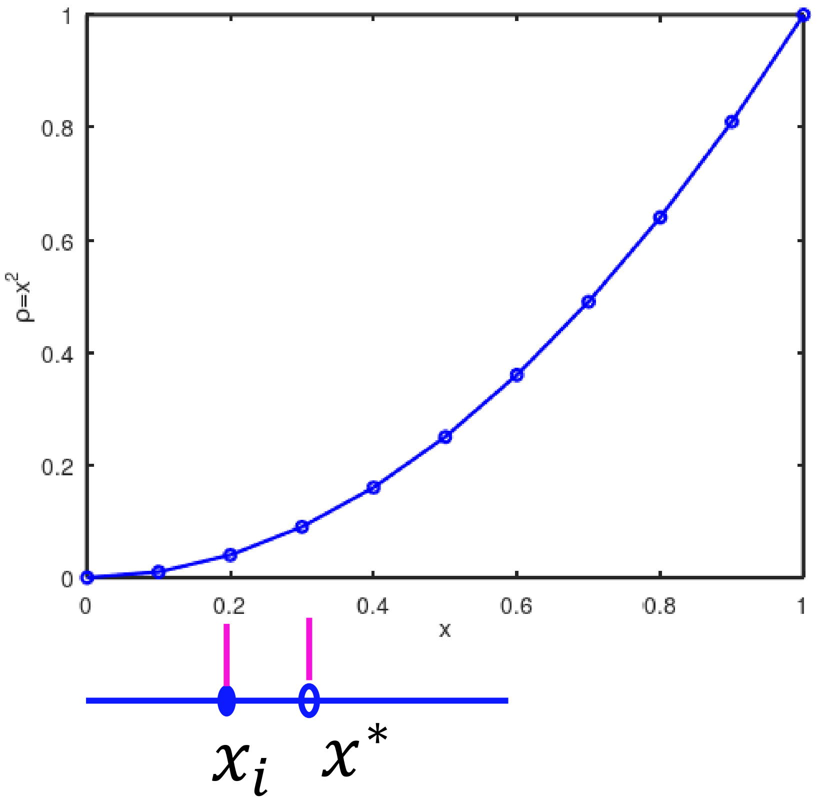
- Taylor series analysis
- Numerical discretization

Taylor Series



$$\rho(x^*) = \rho(x_i) + \frac{(x^* - x_i)}{1!}\left(\frac{d\rho}{dx}\right)_i + \frac{(x^* - x_i)^2}{2!}\left(\frac{d^2\rho}{dx^2}\right)_i + \frac{(x^* - x_i)^3}{3!}\left(\frac{d^3\rho}{dx^3}\right)_i + \cdots$$

Taylor Series: Discrete Operations



$$\rho(x^*) = \rho(x_i) + \frac{(x^* - x_i)}{1!}\left(\frac{d\rho}{dx}\right)_i + \frac{(x^* - x_i)^2}{2!}\left(\frac{d^2\rho}{dx^2}\right)_i + \frac{(x^* - x_i)^3}{3!}\left(\frac{d^3\rho}{dx^3}\right)_i + \cdots$$

Taylor Series: Discrete Operations

$$\rho(x_{i+1}) = \rho(x_i) + \frac{(x_{i+1} - x_i)}{1!}\left(\frac{d\rho}{dx}\right)_i + \frac{(x_{i+1} - x_i)^2}{2!}\left(\frac{d^2\rho}{dx^2}\right)_i + \frac{(x_{i+1} - x_i)^3}{3!}\left(\frac{d^3\rho}{dx^3}\right)_i + \cdots$$

$$\rho(x_{i+1}) = \rho(x_i) + (x_{i+1} - x_i)\left(\frac{d\rho}{dx}\right)_i + O\left(\Delta x_i^2\right); \qquad \Delta x_i = (x_{i+1} - x_i)$$

$$\rho(x_{i+1}) = \rho(x_i) + \Delta x_i\left(\frac{d\rho}{dx}\right)_i + O\left(\Delta x_i^2\right)$$

Taylor Series: Discrete Operations (1st order)

$$\rho(x_{i+1}) = \rho(x_i) + \Delta x_i \left(\frac{d\rho}{dx}\right)_i + O(\Delta x_i^2)$$

$$\left(\frac{d\rho}{dx}\right)_i = \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} + \frac{1}{\Delta x_i} O(\Delta x_i^2)$$

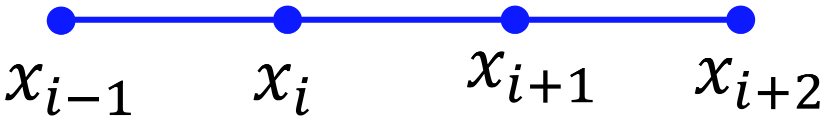
$$\left(\frac{d\rho}{dx}\right)_i = \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} + O(\Delta x_i)$$

$$\left(\frac{d\rho}{dx}\right)_i \approx \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i}$$

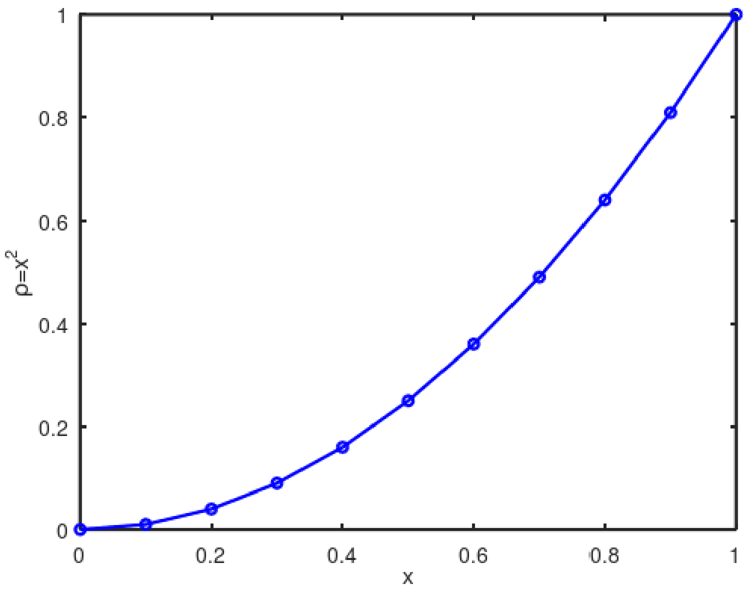
First order upwind scheme

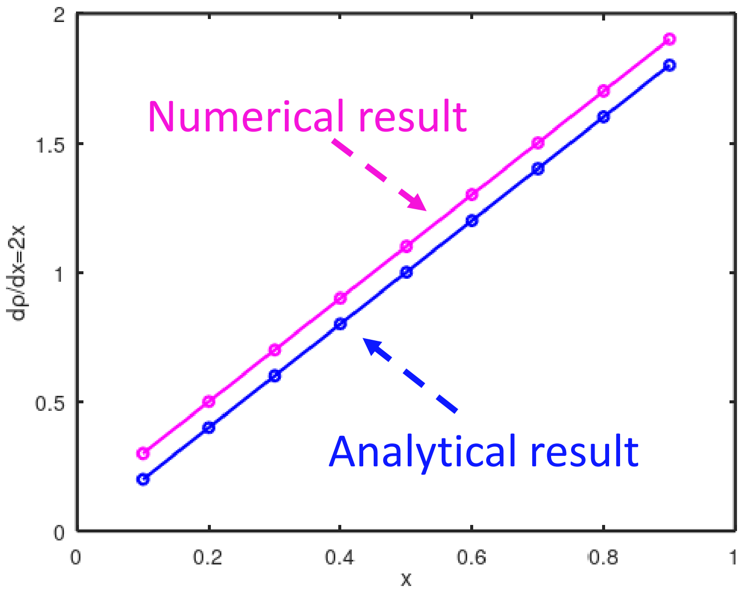
First order upwind scheme

- Numerical Approximation



$$\left(\frac{d\rho}{dx}\right)_i = \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} + O(\Delta x_i)$$





Central Difference Scheme (2nd order)

$$\rho(x_{i+1}) = \rho(x_i) + \frac{(x_{i+1} - x_i)}{1!}\left(\frac{d\rho}{dx}\right)_i + \frac{(x_{i+1} - x_i)^2}{2!}\left(\frac{d^2\rho}{dx^2}\right)_i + \frac{(x_{i+1} - x_i)^3}{3!}\left(\frac{d^3\rho}{dx^3}\right)_i + \cdots$$

$$(1)\quad \rho(x_{i+1}) = \rho(x_i) + \Delta x_i\left(\frac{d\rho}{dx}\right)_i + \frac{\Delta x_i^2}{2}\left(\frac{d^2\rho}{dx^2}\right)_i + O\left(\Delta x_i^3\right)$$

$$(2)\quad \rho(x_{i-1}) = \rho(x_i) - \Delta x_i\left(\frac{d\rho}{dx}\right)_i + \frac{\Delta x_i^2}{2}\left(\frac{d^2\rho}{dx^2}\right)_i + O\left(\Delta x_i^3\right)$$

Central Difference Scheme (2nd order)

$$(1)\quad \rho(x_{i+1}) = \rho(x_i) + \Delta x_i \left(\frac{d\rho}{dx}\right)_i + \frac{\Delta x_i^2}{2}\left(\frac{d^2\rho}{dx^2}\right)_i + O(\Delta x_i^3)$$

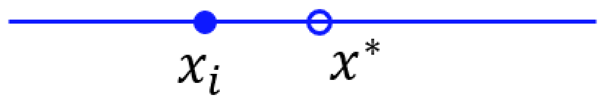
$$(2)\quad \rho(x_{i-1}) = \rho(x_i) - \Delta x_i \left(\frac{d\rho}{dx}\right)_i + \frac{\Delta x_i^2}{2}\left(\frac{d^2\rho}{dx^2}\right)_i + O(\Delta x_i^3)$$

Subtract (2) from (1)

$$\rho(x_{i+1}) - \rho(x_{i-1}) = 2\Delta x_i \left(\frac{d\rho}{dx}\right)_i + O(\Delta x_i^3)$$

$$\boxed{\left(\frac{d\rho}{dx}\right)_i = \frac{\rho(x_{i+1}) - \rho(x_{i-1})}{2\Delta x_i} + O(\Delta x_i^2)}$$

Summary



$$\rho(x^*) = \rho(x_i) + \frac{(x^* - x_i)}{1!}\left(\frac{d\rho}{dx}\right)_i + \frac{(x^* - x_i)^2}{2!}\left(\frac{d^2\rho}{dx^2}\right)_i + \frac{(x^* - x_i)^3}{3!}\left(\frac{d^3\rho}{dx^3}\right)_i + \cdots$$

$$\left(\frac{d\rho}{dx}\right)_i \approx \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i}$$

First order upwind

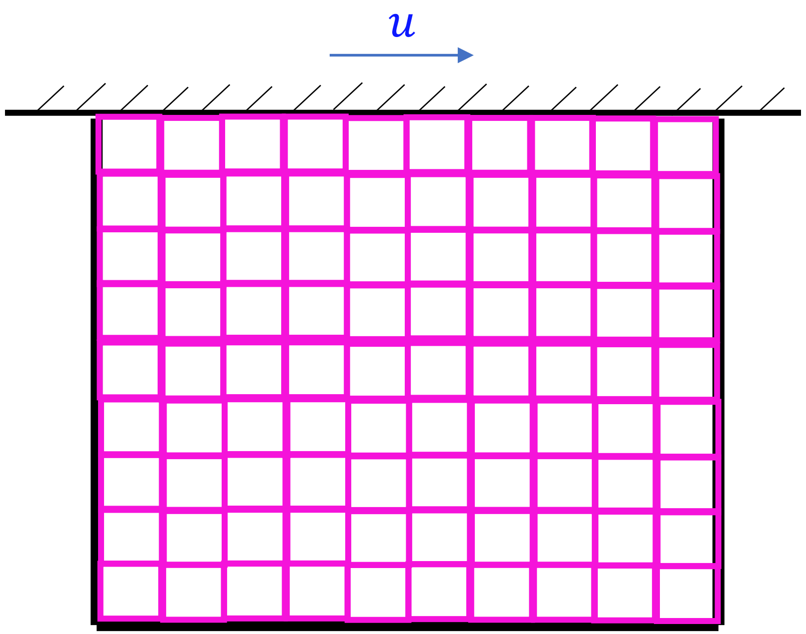
$$\left(\frac{d\rho}{dx}\right)_i \approx \frac{\rho(x_{i+1}) - \rho(x_{i-1})}{2\Delta x_i}$$

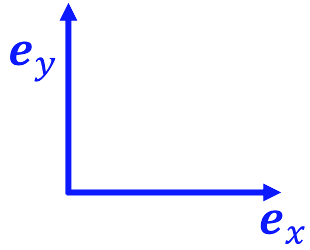
Second order central difference

Exercise

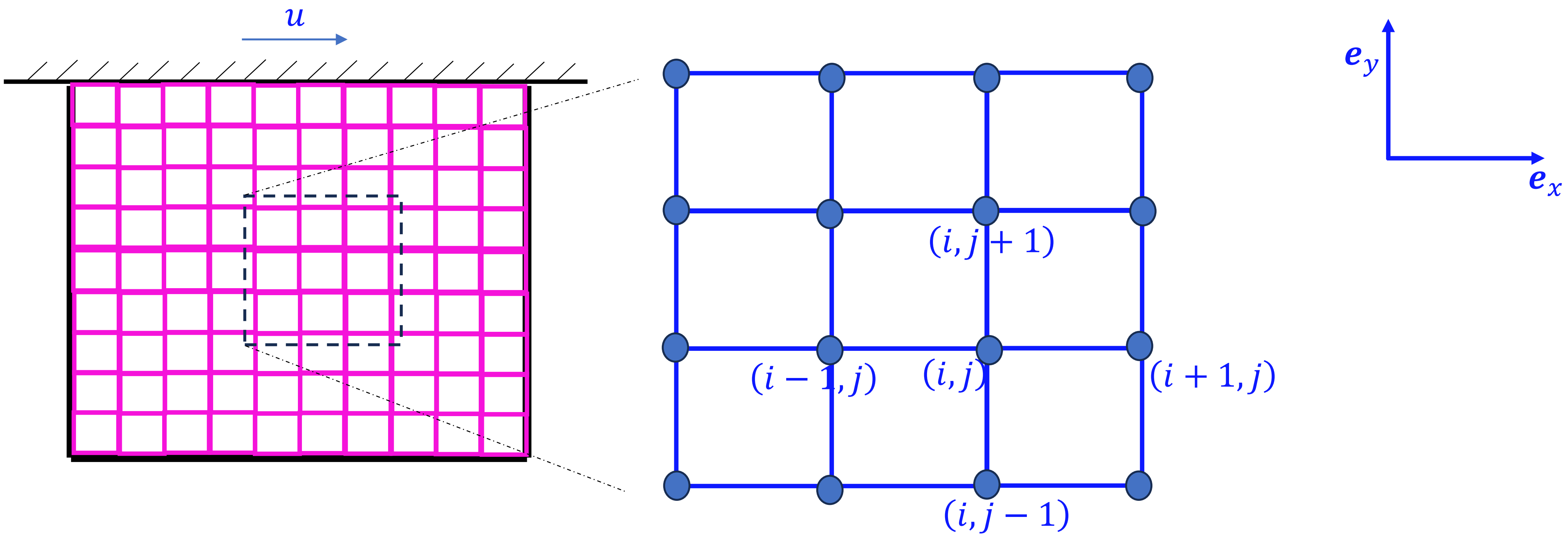
- Derive expression for $\left(\frac{d^2\rho}{dx^2}\right)_i$
- What is the accuracy of the resultant expression?

Numerical Discretization (Grid layout)



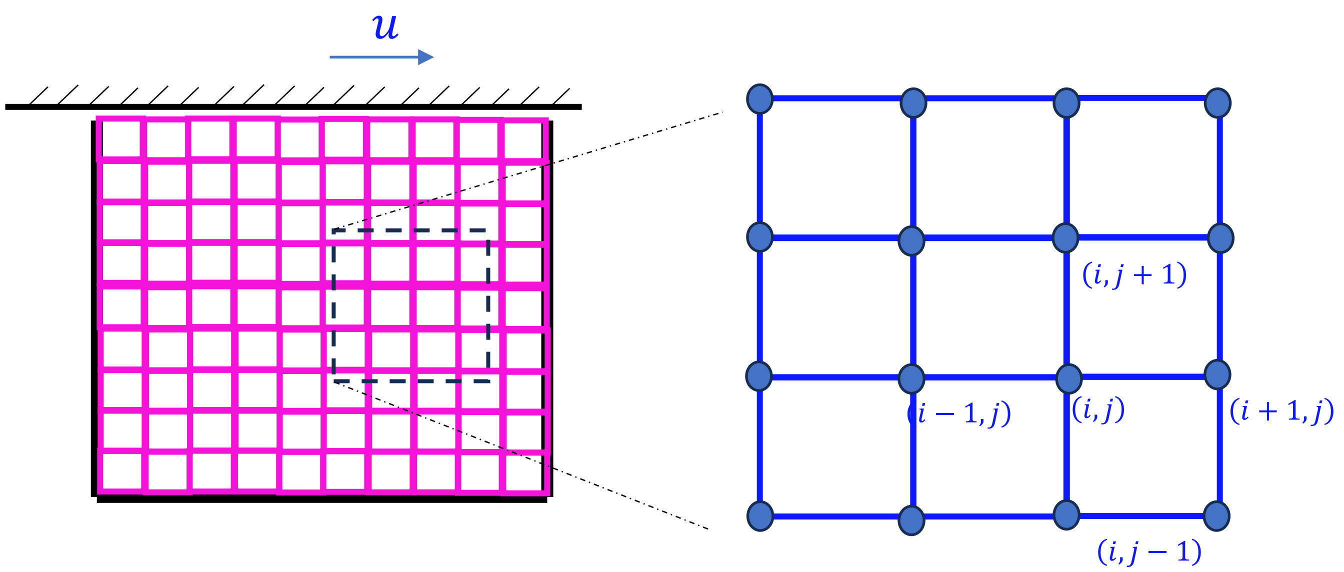


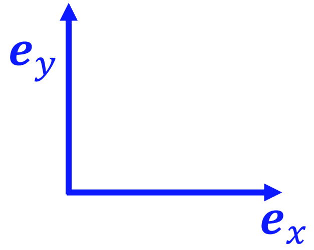
Numerical Discretization (Grid layout)

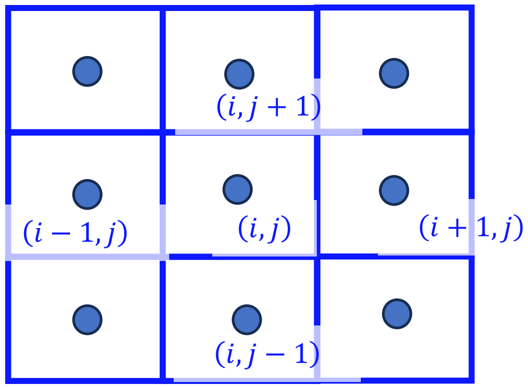


- Flow properties are assigned at each grid locations.

Numerical Discretization



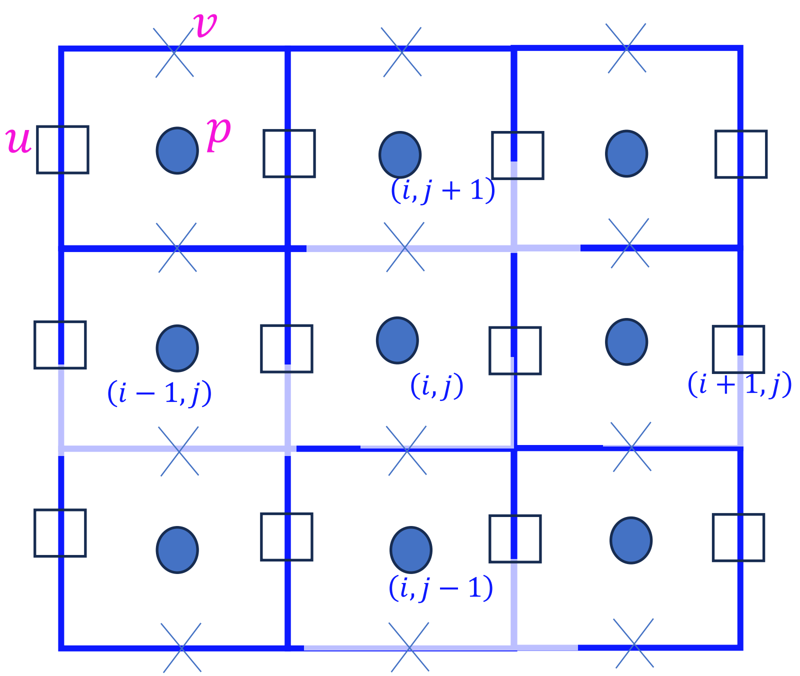




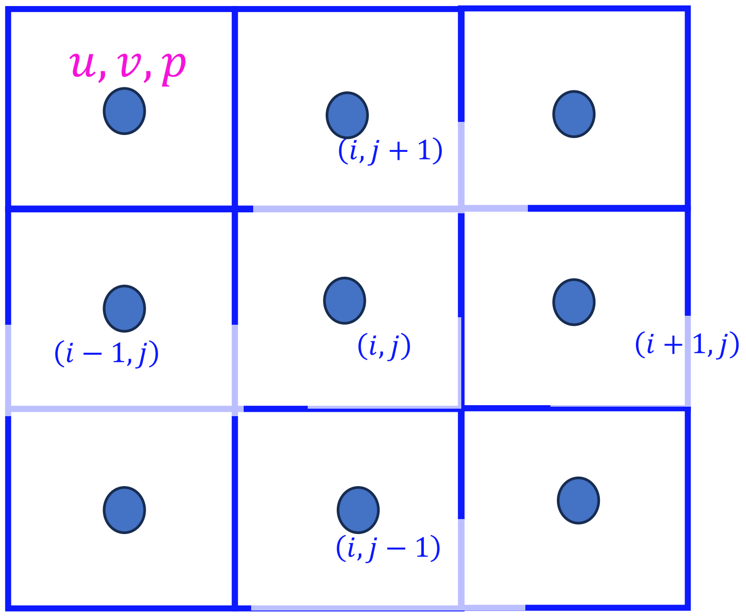
Cell centered representation

Numerical Discretization

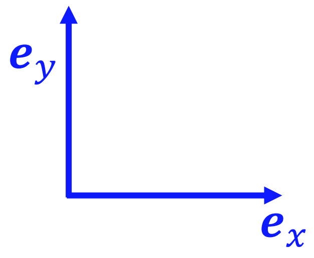
- Cell centered representation



Staggered grid



Collocated grid



Exercise

- Derive expression for $\left(\frac{d^2\rho}{dx^2}\right)_i$
- What is the accuracy of the resultant expression?