

# Introduction to Computational Fluid Dynamics using OpenFOAM and Octave

Lakshman Anumolu  
Kumaresh Selvakumar  
(Session-3)

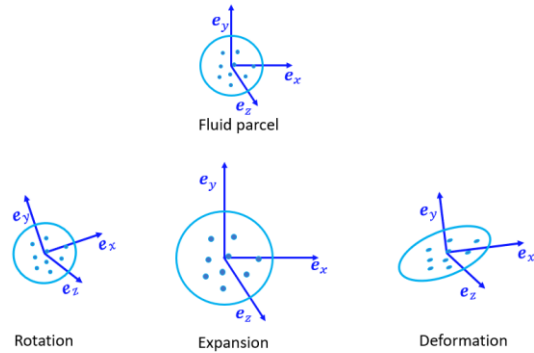
*Instructions: Mon, Wed, Thu (3:30PM-4:30PM IST)*

*Query session: Sundays 8:30AM-9:00AM IST*

# Quick Recap

# What Did We Discuss?

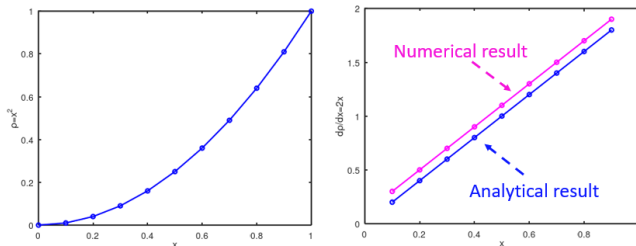
## Fluid Behavior



## Gradient

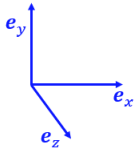
- Numerical Approximation

$$\begin{array}{ccccccc} \bullet & & \bullet & & \bullet & & \bullet \\ x_{i-1} & & x_i & & x_{i+1} & & x_{i+2} \end{array} \quad \left( \frac{\partial \rho}{\partial x} \right)_i = \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i} + O(\Delta x_i)$$



## Mathematical Operations

- Divergence

$$\nabla \cdot \mathbf{u} = \left( \frac{\partial}{\partial x} e_x + \frac{\partial}{\partial y} e_y + \frac{\partial}{\partial z} e_z \right) (u e_x + v e_y + w e_z) = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$


- Numerical approximation

- Same as earlier

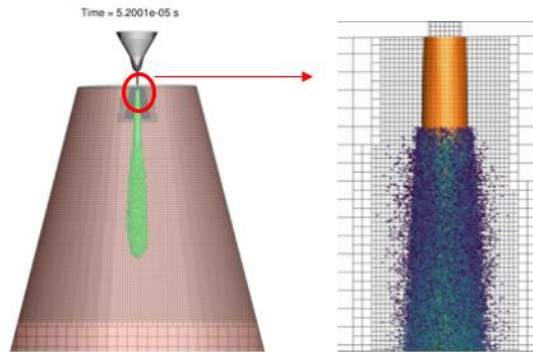
- Physical significance

- Positive value : Source or expansion of fluid volume
- Negative value: Sink
- Zero signifies incompressible nature or no change in volume

# What Did We Discuss?

$$\frac{D\phi(\mathbf{X}(\mathbf{p}_i, t), t)}{Dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} \frac{DX}{Dt} + \frac{\partial \phi}{\partial y} \frac{DY}{Dt} + \frac{\partial \phi}{\partial z} \frac{DZ}{Dt}$$

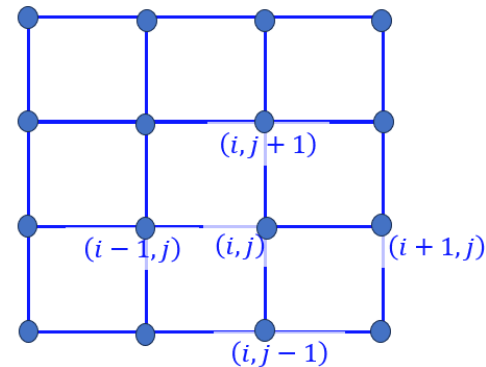
$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi$$



## Finite Difference – Finite Volume

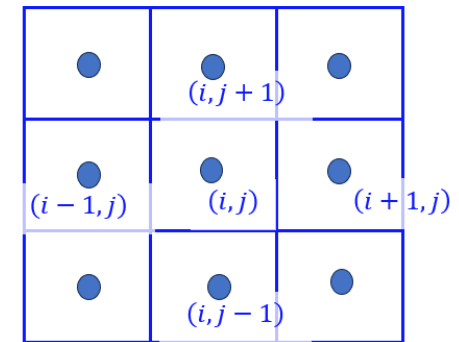
Differential form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$



Integral form

$$\frac{\partial}{\partial t} \int_V \rho dV + \oint_S \rho \mathbf{u} \cdot d\mathbf{S} = 0$$



# Current Session

# Overview

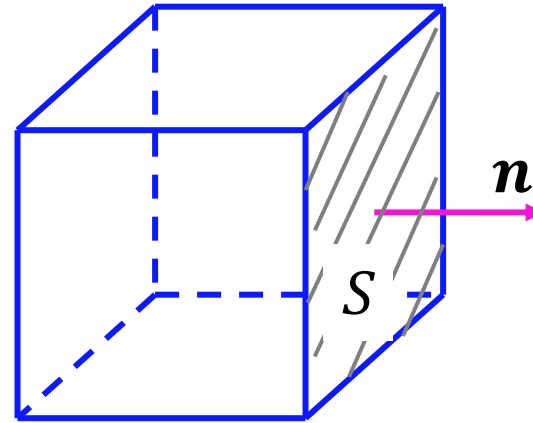
- Gauss Divergence Theorem
- Reynolds Transport Theorem

# Basic Quantities

- Surface Area Vector:  $\mathbf{S}$

$$\mathbf{S} = |\mathbf{S}|\mathbf{n}$$

$$\mathbf{S} = S\mathbf{n}$$



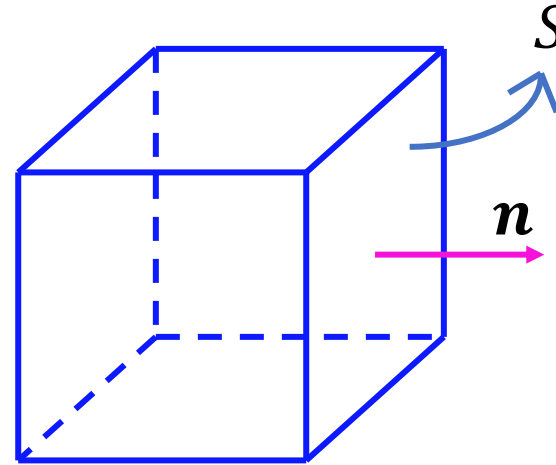
- Extensive property:  $\Phi$
- Intensive property:  $\phi$ 
  - Conserved property per unit mass
- Mass:  $\rho V = \int_V \rho dV$

# Gauss Divergence Theorem

- For a vector:  $\mathbf{F}$

$$\int (\nabla \cdot \mathbf{F}) dV \approx \sum \mathbf{F} \cdot \mathbf{S}$$

- Rate of change of a quantity over a control volume = Rate of flow through control surface.



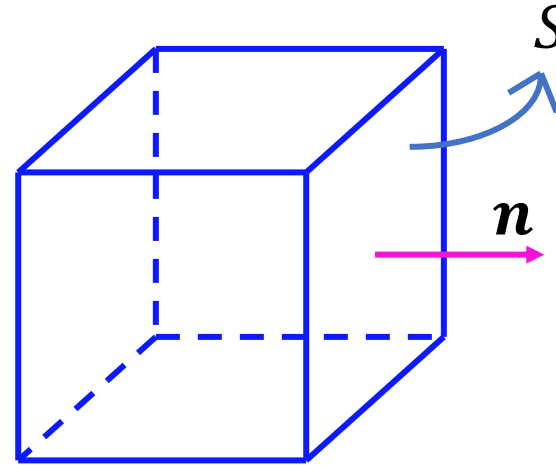


# Gauss Divergence Theorem

- For a vector:  $\mathbf{F}$

$$\int (\nabla \cdot \mathbf{F}) dV \approx \sum \mathbf{F} \cdot \mathbf{S}$$

Let:  $\mathbf{F} = x\mathbf{e}_x + y\mathbf{e}_y$



$$\int (\nabla \cdot \mathbf{F}) dV = \int \left( \frac{\partial}{\partial x} \mathbf{e}_x + \frac{\partial}{\partial y} \mathbf{e}_y \right) \cdot (x\mathbf{e}_x + y\mathbf{e}_y) dV$$

$$\nabla \cdot \mathbf{u} = \left( \frac{\partial}{\partial x} \mathbf{e}_x + \frac{\partial}{\partial y} \mathbf{e}_y + \frac{\partial}{\partial z} \mathbf{e}_z \right) (u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z) = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

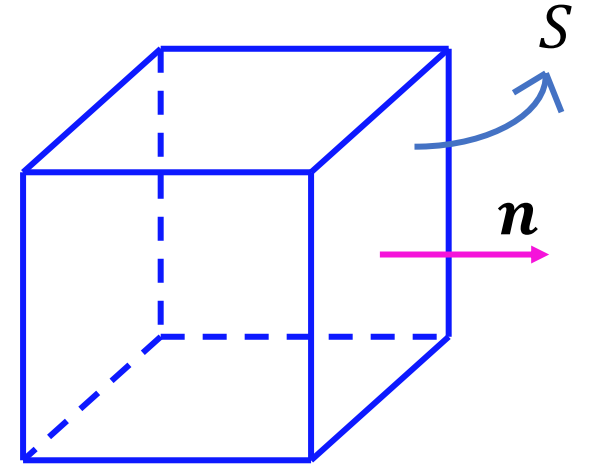
# Gauss Divergence Theorem

$$\int (\nabla \cdot \mathbf{F}) dV \approx \sum \mathbf{F} \cdot \mathbf{S}$$

Let:  $\mathbf{F} = x\mathbf{e}_x + y\mathbf{e}_y$

$$\begin{aligned} \int (\nabla \cdot \mathbf{F}) dV &= \int \left( \frac{\partial}{\partial x} \mathbf{e}_x + \frac{\partial}{\partial y} \mathbf{e}_y \right) \cdot (x\mathbf{e}_x + y\mathbf{e}_y) dV \\ &= \int \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) dV = 2V \end{aligned}$$

$$\sum \mathbf{F} \cdot \mathbf{S} = \sum (x\mathbf{e}_x + y\mathbf{e}_y) \cdot \mathbf{S}$$

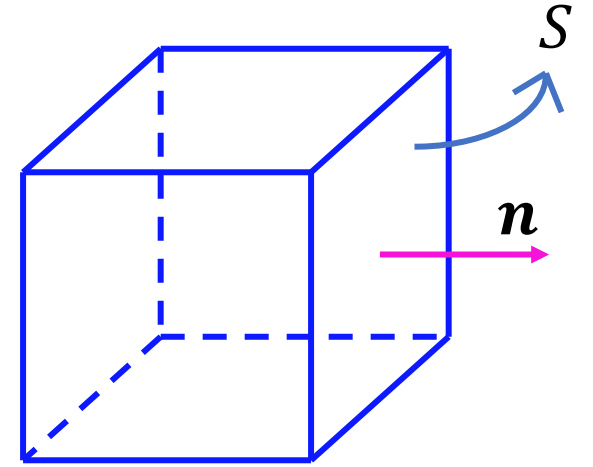


$$S = \Delta x \Delta y$$

$$V = \Delta x \Delta y \Delta z$$

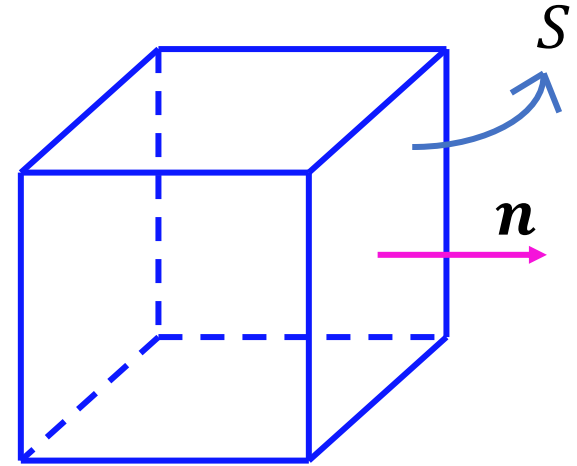
# Gauss Divergence Theorem

$$\begin{aligned}\sum \mathbf{F} \cdot \mathbf{S} &= \sum (x\mathbf{e}_x + y\mathbf{e}_y) \cdot \mathbf{S} \\&= \left(x + \frac{\Delta x}{2}\right)S - \left(x - \frac{\Delta x}{2}\right)S - \left(y - \frac{\Delta y}{2}\right)S + \left(y + \frac{\Delta y}{2}\right)S \\&= \Delta x S + \Delta y S \\&= 2V\end{aligned}$$



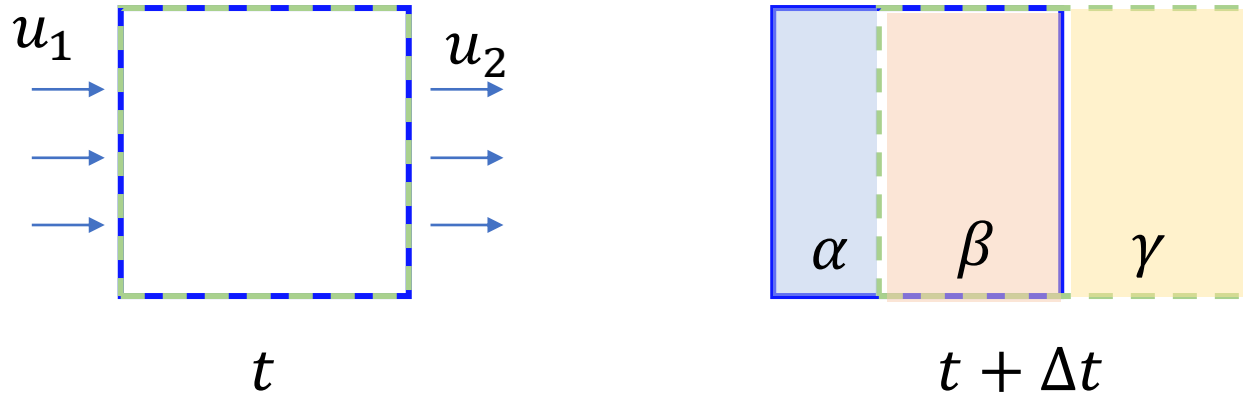
# Reynolds Transport Theorem

$$\frac{d}{dt} \int_{V_S} \rho \phi dV = \frac{d}{dt} \int_{V_{CV}} \rho \phi dV + \int_S \rho \phi \mathbf{u} \cdot \mathbf{n} dS$$



# Reynolds Transport Theorem

$$\frac{d}{dt} \int_{V_S} \rho \phi dV = \frac{d}{dt} \int_{V_{CV}} \rho \phi dV + \int_S \rho \phi \mathbf{u} \cdot \mathbf{n} dS$$



$$(1) \quad \Phi_S(t) = \Phi_{CV}(t)$$

$$(2) \quad \Phi_S(t + \Delta t) = \Phi_{CV}(t + \Delta t) - \Phi_\alpha + \Phi_\gamma$$

# Reynolds Transport Theorem

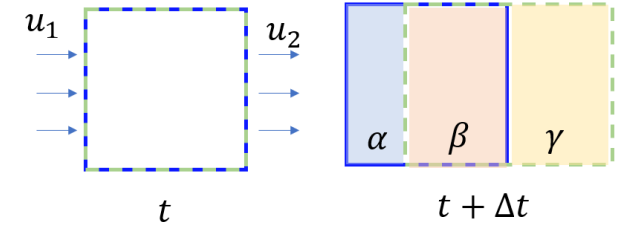
$$(1) \quad \Phi_S(t) = \Phi_{CV}(t)$$

$$(2) \quad \Phi_S(t + \Delta t) = \Phi_{CV}(t + \Delta t) - \Phi_\alpha + \Phi_\gamma$$

Subtract (1) from (2)

$$\left( \frac{\Phi(t + \Delta t) - \Phi(t)}{\Delta t} \right)_{System} = \left( \frac{\Phi(t + \Delta t) - \Phi(t)}{\Delta t} \right)_{CV} - \dot{\Phi}_\alpha + \dot{\Phi}_\gamma$$

$$\frac{d\Phi}{dt}_{System} = \frac{d\Phi}{dt}_{CV} - \dot{\Phi}_\alpha + \dot{\Phi}_\gamma$$



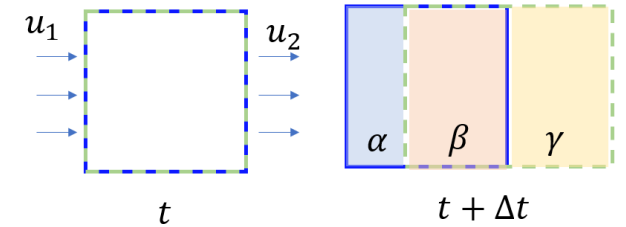
# Reynolds Transport Theorem

$$\frac{d\Phi}{dt}_{system} = \frac{d\Phi}{dt}_{cv} - \dot{\Phi}_{\alpha} + \dot{\Phi}_{\gamma}$$

$$\Phi_{\alpha} = \phi_{\alpha} m_{\alpha} = \phi_{\alpha} \rho_{\alpha} V_{\alpha} = \phi_{\alpha} \rho_{\alpha} (u_1 \Delta t) S$$

Using  $\Phi = \int \rho \phi dV$  and net flux as  $\int \rho \phi \mathbf{u} \cdot \mathbf{n} dS$

$$\frac{d}{dt} \int_{V_S} \rho \phi dV = \frac{d}{dt} \int_{V_{cv}} \rho \phi dV + \int_S \rho \phi \mathbf{u} \cdot \mathbf{n} dS$$



# Conservation Laws

- Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0; \quad \nabla \cdot (\rho \mathbf{u}) = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}$$

- Conservation of momentum

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g};$$

- Scalar conservation law

$$\frac{\partial \rho \phi}{\partial t} + \nabla \cdot (\rho \mathbf{u} \phi) = \nabla \cdot (\Gamma \nabla \phi) + S_\phi$$

$$\frac{d}{dt} \int_{V_S} \rho \phi dV = \frac{d}{dt} \int_{V_{CV}} \rho \phi dV + \int_S \rho \phi \mathbf{u} \cdot \mathbf{n} dS$$

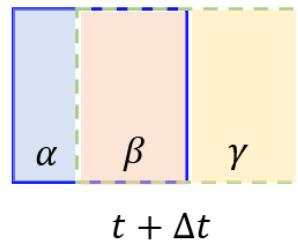
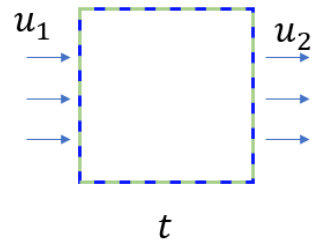


# Quick Summary

# What Did We Discuss?

$$\int (\nabla \cdot \mathbf{F}) dV \approx \sum \mathbf{F} \cdot \mathbf{S}$$

## Reynolds Transport Theorem

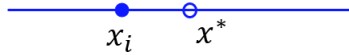


$$\frac{d}{dt} \int_{V_S} \rho \phi dV = \frac{d}{dt} \int_{V_{CV}} \rho \phi dV + \int_S \rho \phi \mathbf{u} \cdot \mathbf{n} dS$$

# Next Session

- Taylor series analysis
- Numerical discretization

## Taylor Series: Discrete Operations

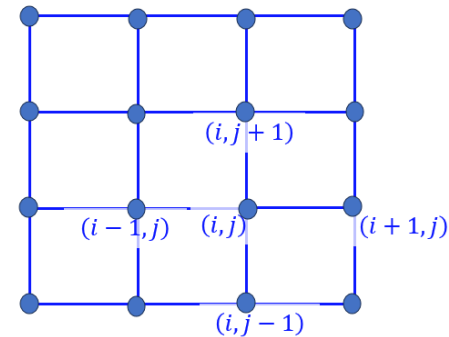


$$\rho(x^*) = \rho(x_i) + (x^* - x_i) \left( \frac{\partial \rho}{\partial x} \right)_i + (x^* - x_i)^2 \left( \frac{\partial^2 \rho}{\partial x^2} \right)_i + (x^* - x_i)^3 \left( \frac{\partial^3 \rho}{\partial x^3} \right)_i + \dots$$

## Finite Difference – Finite Volume

Differential form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$



Integral form

$$\frac{\partial}{\partial t} \int_V \rho dV + \oint_S \rho \mathbf{u} \cdot d\mathbf{S} = 0$$

