# 1 Tensor Basics

## 1.1 Definitions

TODO

## 1.2 Tensor Manipulations and Operations

TODO

## 2 Tensor Derivatives

let $T$ be a contravariant tensor, then:

$\bar{T}^i = T^r \frac{\partial \bar{x}^i}{\partial x^r}$, where $T = T^i(x(t))$

Take derivative wrt. $t$:

$\frac{\partial \bar{T}^i}{\partial t} = \frac{\partial T^r}{\partial t} \frac{\partial \bar{x}^i}{\partial x^r} + T^r \frac{\partial^2 \bar{x}^i}{\partial x^s \partial x^r} \frac{\partial x^s}{\partial t}$ (not a general tensor)

$\bar{x}^i$ being a linear functions of $x^r \implies \frac{\partial \bar{T}^i}{\partial t} = \frac{\partial T^r}{\partial t} \frac{\partial \bar{x}^i}{\partial x^r}$ (it's a general tensor)

Goal: Would like the curvature of a curve to be an intrinsic concept, independent of coordinate systems. Use Christoffel symbols to solve it.

ref: Schaum Chapter 6

### 2.1 Christoffel Symbol of the 1st Kind

$\Gamma_{ijk} = \frac{1}{2} \left( -\frac{\partial(g_{ij})}{\partial x^k} + \frac{\partial(g_{jk})}{\partial x^i} + \frac{\partial(g_{ki})}{\partial x^j} \right)$

Let $g_{ijk} = \frac{\partial(g_{ij})}{\partial(x^k)}$

$\Gamma_{ijk} = \frac{1}{2}(-g_{ijk} + g_{jki} + g_{kij})$

Cyclic permute and add:

$\Gamma_{ijk} + \Gamma_{jki} = g_{kij}$

Driving transformation law for $\Gamma_{ijk}$:
By using $g_{ij}$ derivative and its cyclic permutatations,
$\bar{\Gamma}_{ijk} = \frac{1}{2}(-\bar{g}_{ijk} + \bar{g}_{jki} + \bar{g}_{kij})$

where:

$\bar{g}_{ijk} = \frac{\partial}{\partial \bar{x}^k} \left( g_{rs} \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} \right)$ (expand this)

Rearrange and simplify to:

$\bar{\Gamma}_{ijk} = \Gamma_{rst} \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial x^t}{\partial \bar{x}^k} + g_{rs} \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^k}$ (not a general tensor)

ref: Schaum Chapter 6

### 2.2 Christoffel Symbol of the 2nd Kind

$\Gamma_{ij}^k g^{kr} \Gamma_{ijk}$ (raising an index of Christoffel Symbol of the 1st kind.

Similar properties as $\Gamma_{ijk}$: symmetry in lower indices, $\Gamma_{ij}^k$ vanish if $g_{ij}$ are all constant.

Transformation rule for $\Gamma_{ij}^k$; use $\Gamma_{ijk}$ in derivation:

$\bar{\Gamma}_{ij}^k = \bar{g}^{kr} \bar{\Gamma}_{ijr}$

$\bar{\Gamma}_{ij}^k = \left( g^{sr} \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial \bar{x}^r}{\partial x^t} \right) \bar{\Gamma}_{ijr}$

$\bar{\Gamma}_{ij}^k = \left( g^{sr} \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial \bar{x}^r}{\partial x^t} \right) \left( \Gamma_{uvw} \frac{\partial x^u}{\bar{x}^i} \frac{\partial x^v}{\bar{x}^j} \frac{\partial x^w}{\partial \bar{x}^r} + g_{uv} \frac{\partial^2 x^u}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial x^v}{\partial \bar{x}^r} \right)$

$\bar{\Gamma}_{ij}^k = g^{st} \Gamma_{uvt} \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial x^v}{\partial \bar{x}^j} + g^{st} g_{ut} \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial^2 x^u}{\partial \bar{x}^i \partial \bar{x}^j}$

$\bar{\Gamma}_{ij}^k = \Gamma_{uv}^s \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial x^v}{\bar{x}^j} + \frac{\partial \bar{x}^k}{\partial x^u} \frac{\partial^2 x^u}{\partial \bar{x}^i \partial \bar{x}^j}$

$\bar{\Gamma}_{ij}^k = \Gamma_{uv}^w \frac{\partial \bar{x}^k}{\partial x^w} \frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial x^v}{\partial \bar{x}^j} + \frac{\partial \bar{x}^k}{\partial x^w} \frac{\partial^2 x^w}{\partial \bar{x}^i \partial \bar{x}^j}$ (not a general tensor)

linear coordinate transform $\implies \Gamma_{ij}^k$ becomes a tensor

$\frac{\partial^2 x^u}{\partial \bar{x}^i \partial \bar{x}^j} = \bar{\Gamma}_{ij}^k \frac{\partial x^w}{\partial \bar{x}^k} - \Gamma_{uv}^w \frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial x^v}{\partial \bar{x}^j}$

ref: Schaum Chapter 6

### 2.3 Covariant Derivative of Covariant Vector

let $T$ be a covariant tensor, then:

$\bar{T}_i = T_r \frac{\partial x^r}{\partial \bar{x}^i}$

$\frac{\bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_r}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^i} + T_r \frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^i}$

use $\frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}$ from earlier:

$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_r}{\partial x^s} \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^i} + T_r \left( \bar{\Gamma}_{ik}^s \frac{\partial x^r}{\partial \bar{x}^s} - \Gamma_{st}^r \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^t}{\partial \bar{x}^k} \right)$

$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_r}{\partial x^s} \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^i} + \bar{\Gamma}_{ik}^s \bar{T}_s - \Gamma_{st}^r T_r \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^t}{\partial \bar{x}^k}$

rename indices:

$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_r}{\partial x^s} \frac{\partial x^s}{\partial \bar{x}^k} \frac{x^r}{\partial \bar{x}^i} + \bar{\Gamma}_{ik}^t \bar{T}_t - \Gamma_{rs}^t T_t \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^k}$

$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \bar{\Gamma}_{ik}^t \bar{T}_t + \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^i} \left( \frac{\partial T_r}{\partial x^s} - \Gamma_{rs}^t T_t \right)$

$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} - \bar{\Gamma}_{ik}^t \bar{T}_t = \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^i} \left( \frac{\partial T_r}{\partial x^s} - \Gamma_{rs}^t T_t \right)$ is a (0,2)-tensor

components of covariant derivative wrt. $x^k$ of a covariant vector $T = (T_i)$:

$T,_k = (T_{i,k}) = \frac{\partial T_i}{\partial x^k} - \Gamma_{ik}^t T_t$

$g_{ij}$ are constants $\implies$ covariant derivatives and partial derivatives coincide

ref: Schaum Chapter 6

### 2.4 Covariant Derivative of Contravariant Vector

Similar to previous section, covariant derivative wrt. $x^k$ of a contravariant vector $T = (T^i)$ is:

$T,_k = (T_{,k}^i) = \frac{\partial T^i}{\partial x^k} + \Gamma_{tk}^i T^t$

Covariant derivative of any rensor is a tensor that has 1 additional covariant order more than the original tensor.

ref: Schaum Chapter 6

### 2.5 Absolute Differentiation Along Curve

TODO