1 Tensor Basics

1.1 Definitions

Transformation rules for tensors:

Covariant tensor of order 1 $\iff \bar{T}_i = T_r \frac{\partial x^r}{\partial \bar{x}^i}$

Contravariant tensor of order 1 $\iff \bar{T}^i = T^r \frac{\partial \bar{x}^i}{\partial x^r}$

Mixed tensor: a combination of the above types.

(n, m) - tensor: tensor with contravariant order of n, covariant order of m

1.2 transform rules for differential form

$\partial f = \frac{\partial f}{\partial c^i} \partial c^i = \frac{\partial f}{\partial p^i} \partial p^i$

basis covector transformation:

$\partial p^i = \frac{\partial p^i}{\partial c^j} \partial c^j$, coefficients: $\frac{\partial p^i}{\partial c^j} \iff$ contravariant

covector components transformation:

$\frac{\partial f}{\partial p^j} = \frac{\partial c^i}{\partial p^j} \frac{\partial f}{\partial c^i}$, coefficients: $\frac{\partial c^i}{\partial p^j} \iff$ covariant

1.3 transform rules for vector

$\frac{\partial}{\partial \lambda} = \frac{\partial c^i}{\partial \lambda} \frac{\partial}{\partial c^i} = \frac{\partial p^i}{\partial \lambda} \frac{\partial}{\partial p^i}$

basis vector transformation:

$\frac{\partial}{\partial p^j} = \frac{\partial c^i}{\partial p^j} \frac{\partial}{\partial c^i}$, coefficients: $\frac{\partial c^i}{\partial p^j} \iff$ covariant

vector components transformation:

$\frac{\partial p^i}{\partial \lambda} = \frac{\partial p^i}{\partial c^j} \frac{\partial c^j}{\partial \lambda}$, coefficients: $\frac{\partial p^i}{\partial c^j} \iff$ contravariant

1.4 Tensor Manipulations and Operations

let ST be the inner product of tensors S and T where 2 indices of opposite types are equated and take the sum of the products for the equated index. Resulting tensor order decreases by 2.

let $[ST]$ be the outer product of tensors S and T where product is taken over all permutation of indices. Resulting tensor order adds up from the input tensors.

Tensor order of 0 is an invariant.

Self tensor contraction: set 2 indices of opposite types for the current tensor and perform sum over the equated index. Resulting tensor order decreases by 2.

equality of composite operations:
contract $\circ$ outer_product $\iff$ inner_product

2 Metric Tensor

$g_{ij} = (\frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j})$, dot product of basis vectors

$\tilde{g}_{ij} = \frac{\partial}{\partial \tilde{c}^i} \cdot \frac{\partial}{\partial \tilde{c}^j}$

$\tilde{g}_{ij} = \frac{\partial}{\partial c^i} \frac{\partial c^i}{\partial \tilde{c}^i} \cdot \frac{\partial}{\partial c^j} \frac{\partial c^j}{\partial \tilde{c}^j}$

$\tilde{g}_{ij} = \frac{\partial}{\partial c^i} \cdot \frac{\partial}{\partial c^j} \frac{\partial c^i}{\partial \tilde{c}^i} \frac{\partial c^j}{\partial \tilde{c}^j}$

$\tilde{g}_{ij} = g_{ij} \frac{\partial c^i}{\partial \tilde{c}^i} \frac{\partial c^j}{\partial \tilde{c}^j} \iff$ (0,2)-tensor

Arc length using metric tensor:

$\| \frac{\partial \vec{R}}{\partial \lambda} \|^2 = \frac{\partial \vec{R}}{\partial u^i} \frac{\partial u^i}{\partial \lambda} \cdot \frac{\partial \vec{R}}{\partial u^j} \frac{\partial u^j}{\partial \lambda}$

$\| \frac{\partial \vec{R}}{\partial \lambda} \|^2 = \frac{\partial u^i}{\partial \lambda} \frac{\partial u^j}{\partial \lambda} (\frac{\partial \vec{R}}{\partial u^i} \cdot \frac{\partial \vec{R}}{\partial u^j})$

$\| \frac{\partial \vec{R}}{\partial \lambda} \|^2 = \frac{\partial u^i}{\partial \lambda} \frac{\partial u^j}{\partial \lambda} g_{ij}$

Generalized inner product:

convert tensor to opposite type using g before doing an inner product

3 Tensor Derivatives

let T be a contravariant tensor, then:

$\bar{T}^i = T^r \frac{\partial \bar{x}^i}{\partial x^r}$, where $T = T^i(x(t))$

Take derivative wrt. t:

$\frac{\partial \bar{T}^i}{\partial t} = \frac{\partial T^r}{\partial t} \frac{\partial \bar{x}^i}{\partial x^r} + T^r \frac{\partial^2 \bar{x}^i}{\partial x^s \partial x^r} \frac{\partial x^s}{\partial t}$ (not a general tensor)

$\bar{x}^i$ being a linear functions of $x^r \implies \frac{\partial \bar{T}^i}{\partial t} = \frac{\partial T^r}{\partial t} \frac{\partial \bar{x}^i}{\partial x^r}$ (it's a general tensor)

Goal: Would like the curvature of a curve to be an intrinsic concept, independent of coordinate systems. Use Christoffel symbols to solve it.

ref: Schaum Chapter 6

3.1 Christoffel Symbol of the 1st Kind

$\Gamma_{ijk} = \frac{1}{2}\left(-\frac{\partial(g_{ij})}{\partial x^k} + \frac{\partial(g_{jk})}{\partial x^i} + \frac{\partial(g_{ki})}{\partial x^j}\right)$

Let $g_{ijk} = \frac{\partial(g_{ij})}{\partial(x^k)}$

$\Gamma_{ijk} = \frac{1}{2}(-g_{ijk} + g_{jki} + g_{kij})$

Cyclic permute and add:

$\Gamma_{ijk} + \Gamma_{jki} = g_{kij}$

Driving transformation law for Γ_{ijk}:
By using g_{ij} derivative and its cyclic permutatations,
$\bar{\Gamma}_{ijk} = \frac{1}{2}(-\bar{g}_{ijk} + \bar{g}_{jki} + \bar{g}_{kij})$

where:

$\bar{g}_{ijk} = \frac{\partial}{\partial \bar{x}^k}\left(g_{rs} \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j}\right)$ (expand this)

Rearrange and simplify to:

$\bar{\Gamma}_{ijk} = \Gamma_{rst} \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial x^t}{\partial \bar{x}^k} + g_{rs} \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^k}$ (not a general tensor)

ref: Schaum Chapter 6

3.2 Christoffel Symbol of the 2nd Kind

$\Gamma^k_{ij} = g^{kr}\Gamma_{ijr}$ (raising an index of Christoffel Symbol of the 1st kind.

Similar properties as Γ_{ijk}: symmetry in lower indices, Γ^k_{ij} vanish if g_{ij} are all constant.

Transformation rule for Γ^k_{ij}; use Γ_{ijk} in derivation:

$\bar{\Gamma}^k_{ij} = \bar{g}^{kr}\bar{\Gamma}_{ijr}$

$\bar{\Gamma}^k_{ij} = \left(g^{sr} \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial \bar{x}^r}{\partial x^t}\right) \bar{\Gamma}_{ijr}$

$\bar{\Gamma}^k_{ij} = \left(g^{sr} \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial \bar{x}^r}{\partial x^t}\right) \left(\Gamma_{uvw} \frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial x^v}{\partial \bar{x}^j} \frac{\partial x^w}{\partial \bar{x}^r} + g_{uv} \frac{\partial^2 x^u}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial x^v}{\partial \bar{x}^r}\right)$

$\bar{\Gamma}^k_{ij} = g^{st}\Gamma_{uvt} \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial x^v}{\partial \bar{x}^j} + g^{st} g_{ut} \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial^2 x^u}{\partial \bar{x}^i \partial \bar{x}^j}$

$\bar{\Gamma}^k_{ij} = \Gamma^s_{uv} \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial x^v}{\partial \bar{x}^j} + \frac{\partial \bar{x}^k}{\partial x^u} \frac{\partial^2 x^u}{\partial \bar{x}^i \partial \bar{x}^j}$

$\bar{\Gamma}^k_{ij} = \Gamma^w_{uv} \frac{\partial \bar{x}^k}{\partial x^w} \frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial x^v}{\partial \bar{x}^j} + \frac{\partial \bar{x}^k}{\partial x^w} \frac{\partial^2 x^w}{\partial \bar{x}^i \partial \bar{x}^j}$ (not a general tensor)

linear coordinate transform $\implies \Gamma^k_{ij}$ becomes a tensor

$\frac{\partial^2 x^u}{\partial \bar{x}^i \partial \bar{x}^j} = \bar{\Gamma}^k_{ij} \frac{\partial x^w}{\partial \bar{x}^k} - \Gamma^w_{uv} \frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial x^v}{\partial \bar{x}^j}$

ref: Schaum Chapter 6

3.3 Covariant Derivative of Covariant Vector

let T be a covariant tensor, then:

$\bar{T}_i = T_r \frac{\partial x^r}{\partial \bar{x}^i}$

$\frac{\bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_r}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^i} + T_r \frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^i}$

use $\frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}$ from earlier:

$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_r}{\partial x^s} \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^i} + T_r \left(\bar{\Gamma}^s_{ik} \frac{\partial x^r}{\partial \bar{x}^s} - \Gamma^r_{st} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^t}{\partial \bar{x}^k}\right)$

$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_r}{\partial x^s} \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^i} + \bar{\Gamma}^s_{ik} \bar{T}_s - \Gamma^r_{st} T_r \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^t}{\partial \bar{x}^k}$

rename indices:

$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_r}{\partial x^s} \frac{\partial x^s}{\partial \bar{x}^k} \frac{x^r}{\partial \bar{x}^i} + \bar{\Gamma}^t_{ik} \bar{T}_t - \Gamma^t_{rs} T_t \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^k}$

$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \bar{\Gamma}^t_{ik} \bar{T}_t + \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^i} \left(\frac{\partial T_r}{\partial x^s} - \Gamma^t_{rs} T_t\right)$

$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} - \bar{\Gamma}^t_{ik} \bar{T}_t = \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^i} \left(\frac{\partial T_r}{\partial x^s} - \Gamma^t_{rs} T_t\right)$ is a (0,2)-tensor

components of covariant derivative wrt. x^k of a covariant vector $T = (T_i)$:

$T_{,k} = (T_{i,k}) = \frac{\partial T_i}{\partial x^k} - \Gamma^t_{ik} T_t$

g_{ij} are constants $\implies$ covariant derivatives and partial derivatives coincide

ref: Schaum Chapter 6

3.4 Covariant Derivative of Contravariant Vector

Similar to previous section, covariant derivative wrt. x^k of a contravariant vector $T = (T^i)$ is:

$T_{,k} = (T^i_{,k}) = \frac{\partial T^i}{\partial x^k} + \Gamma^i_{tk} T^t$

Covariant derivative of any rensor is a tensor that has 1 additional covariant order more than the original tensor.

ref: Schaum Chapter 6

3.5 Absolute Differentiation Along Curve

TODO

4 Geodesic

TODO