Linear Algebra

WEDNESDAY, AUGUST, 9, 2017

Agenda

- Recall
- Last Week's Exercise
- Linear Transformations, Null Spaces, and Ranges

Recall

Basis

A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

Dimension

A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vector in each basis for V is called the dimension of V and is denoted by $dim(V)$. A vector space that is not finite-dimensional is called infinite-dimensional.

Last Week's Exercise

Section 1.6: 1, 9, 11, 14, 23, 29(a)

Linear Transformations, Null Spaces, and Ranges

Definition

Let V and W be vector spaces (over F). We call a function $T : V \to W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

1. $T(x + y) = T(x) + T(y)$ and
2. $T(cx) = cT(x)$

Remark

We often simply call T linear.

If T is linear, and for all $x, y \in V$, for all $c \in F$, then

1. $T(0) = 0$
2. $T(cx + y) = cT(x) + T(y)$
3. $T(x - y) = T(x) - T(y)$.

Last, if T is linear if and only if, $x_1, \cdots, x_n \in V$ and $a_1, \cdots, a_n \in F$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i).$$

Example 1

$T(x) = 3x$ is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.

Example 2

$T(x) = x^2$ is not a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.

Theorem (1/2)

Let V and W be vector spaces. If $\{v_1, \cdots, v_n\}$ is a basis for V, then given any $w_1, \cdots, w_n$ in W, there exists a linear transformation $T : V \to W$ which sends v_i to w_i, where $i = 1, 2, \cdots, n$.

Theorem (2/2)

Proof

For all v in V can be written as an linear combination
$v = a_1v_1 + \cdots + a_nv_n$ in a unique way.
If $T : V \to W$ is a linear transformation, then

$$T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n),$$

so the image of every v in V is uniquely determined once the images of $v_1, \cdots, v_n$ are chosen.

Definitions

Let V and W be vector spaces, and let $T : V \to W$ be linear. We define the null space (or kernel) $N(T)$ of T to be the set of all vectors x in V such that $T(x) = 0$; that is,
$N(T) = \{x \in V : T(x) = 0\}$.

We define the range (or image) $R(T)$ of T to be the subset of W consisting of all images (under T) of vectors in V; that is,
$R(T) = \{T(x) : x \in V\}$.

Example 3 (1/2)

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$

Verify that

$$N(T) = \{(a, a, 0) : a \in \mathbb{R}\}$$

and

$$R(T) = \mathbb{R}^2.$$

Example 3 (2/2)

SOLUTION

It is obvious that $N(T) = \{(a, a, 0) : a \in \mathbb{R}\}$, then it remains to verify that $R(T) = \mathbb{R}^2$.

Since $\{e_1 = (1, 0), e_2 = (0, 1)\}$ is the standard basis for $\mathbb{R}^2$, we are able to verify it if we can find $x, y \in \mathbb{R}^3$, such that $T(x) = e_1$ and $T(y) = e_2, .$

By elementary algebra,

$$x = (b, b + 1, 0), y = (c, c, 0.5)$$

where $b, c \in \mathbb{R}$.

Theorem 2.1 (1/3)

Let V and W be vector spaces and $T : V \to W$ be linear. Then $N(T)$ and $R(T)$ are subspaces of V and W, respectively.

Theorem 2.1 (2/3)

PROOF

To clarity the notation, we use the symbols 0_V and 0_W to denote the zero vectors of V and W, respectively.

Since $T(0_V) = 0_W$, we have that $0_V \in N(T)$. Let $x, y \in N(T)$ and $c \in F$. Then $T(x+y) = T(x) + T(y) = 0_W + 0_W = 0_W$, and $T(cx) = cT(x) = c0_W = 0_W$. Hence $x + y \in N(T)$ and $cx \in N(T)$, so that $N(T)$ is a subspace of V.

Theorem 2.1 (3/3)

Because $T(0_V) = 0_W$, we have that $0_W \in R(T)$. Now let $x, y \in R(T)$ and $c \in F$. Then there exists v and w in V such that $T(v) = x$ and $T(w) = y$. So $T(v + w) = T(v) + T(w) = x + y$, and $T(cv) = cT(v) = cx$. Thus $x + y \in R(T)$ and $cx \in R(T)$, so $R(T)$ is a subspace of W.

Definitions

Let V and W be vector spaces, and let $T : V \to W$ be linear. If $N(T)$ and $R(T)$ are finite-dimensional, then we define the nullity of T, denoted $nullity(T)$, and the rank of T, denoted $rank(T)$, to be the dimensions of $N(T)$ and $R(T)$, respectively.

Theorem 2.3 (Dimension Theorem) (1/4)

Let V and W be vector spaces, and let $T : V \to W$ be linear, If V is finite-dimensional, then

$$nullity(T) + rank(T) = dim(V).$$

Theorem 2.3 (Dimension Theorem) (2/4)

PROOF

Suppose that $dim(V) = n$, $dim(N(T)) = k$, and $\{v_1, \cdots, v_k\}$ is a basis for $N(T)$. We may extend $\{v_1, \cdots, v_k\}$ to a basis $\beta = \{v_1, \cdots, v_n\}$ for V. We claim that $S = \{T(v_{k+1}), \cdots, T(v_n)\}$ is a basis for $R(T)$.

First we prove that S generates $R(T)$, we have

$$\begin{aligned} R(T) &= span(\{T(v_1), \cdots, T(v_n)\}) \\ &= span(\{T(v_{k+1}), \cdots, T(v_n)\}) = span(S) \end{aligned}$$

Theorem 2.3 (Dimension Theorem) (3/4)

Now we prove that S is linearly independent. Suppose that

$$\sum_{i=k+1}^{n} b_i T(v_i) = 0 \quad \forall\, b_i \in F.$$

Using the fact that T is linear, we have

$$T\left(\sum_{i=k+1}^{n} b_i v_i\right) = 0.$$

Theorem 2.3 (Dimension Theorem) (4/4)

So

$$\sum_{i=k+1}^{n} b_i v_i \in N(T).$$

Hence that exist $c_1, \cdots, c_k \in F$ such that

$$\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i.$$

Since β is a basis for V, we have $b_i = 0$ for all i. Hence S is linearly independent. Notice that this argument also shows that $T(v_{k+1}), \cdots, T(v_n)$ are distinct; therefore $rank(T) = n - k$.

Example 4

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x, y, z) \to (x, y)$$

$$dim(N(T)) = 1, dim(R(T)) = 2$$

$$S : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x, y, z) \to (x, 0)$$

$$dim(N(S)) = 2, dim(R(S)) = 1$$

Example 5

$$T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$$
$$f(x) \to \frac{d}{dx} f(x)$$

$$N(T) = \{a_0 | a_0 \in \mathbb{R}\}$$
$$R(T) = \{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} | a_i \in \mathbb{R}\}$$

$$dim(N(T)) = 1, dim(R(T)) = n$$
$$dim(N(T)) + dim(R(T)) = 1 + n = dim(P_n(\mathbb{R}))$$

Theorem 2.4 (1/2)

Let V and W be vector spaces, and let $T : V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$.

Theorem 2.4 (2/2)

PROOF ($\Longrightarrow$)

Suppose that T is one-to-one and $x \in N(T)$. Then $T(x) = 0 = T(0)$. Since T is one-to-one, we have $x = 0$. Hence $N(T) = \{0\}$.

PROOF ($\Longleftarrow$)

Now assume that $N(T) = \{0\}$, and suppose that $T(x) = T(y)$. Then $0 = T(x) - T(y) = T(x - y)$. Therefore $x - y \in N(T) = \{0\}$. So $x - y = 0$, or $x = y$. This means that T is one-to-one.

Theorem 2.5 (1/2)

Let V and W be vector spaces of equal (finite) dimension, and let $T : V \to W$ be linear. Then the following are equivalent.

1. T is one-to-one.
2. T is onto.
3. $rank(T) = dim(V)$.

Theorem 2.5 (2/2)

PROOF

From the dimension theorem, we have

$$nullity(T) + rank(T) = dim(V).$$

Now, we have that T is one-to-one if and only if $N(T) = \{0\}$, if and only if $nullity(T) = 0$, if and only if $rank(T) = dim(V)$, if and only if $rank(T) = dim(W)$, and if and only if $dim(R(T)) = dim(W)$. This equality is equivalent to $R(T) = W$ (Theorem 1.11), the definition of T being onto.

Exercise

Section 2.1: 1, 5, 6, 7, 9(bd), 10, 11, 12, 13, 14, 15, 16, 20