Generalised linear models in R

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November 2018

Recap: Linear regression

Assumptions:

1. A **linear** mean function is relevant.
2. Variances are equal across all predicted values of the response (**homoscedatic**).
3. Errors are **normally** distributed.
4. Samples collected at **random**.
5. Errors are **independent**.

Generalised linear models (GLMs)

1. A linear mean (including any explanatory variables you want to) i.e $\beta_0 + \beta_1 X_1 + \ldots + \beta_p X_p$
2. A **link function** (like an "internal" transformation).
3. An **error structure**.
   So far we assumed normality $\epsilon \sim \mathcal{N}(0, \sigma^2)$

Link functions

Links your **mean** function to the *scale* of the **observed data** e.g.

$$E(Y) = g^{-1}\left(\beta_0 + \beta_1 X\right)$$

- $\mathbb{E}(Y)$ is the **expected value** (i.e. mean of Y).
- The function $g(\cdot)$ is known as the **link function**, and $g^{-1}(\cdot)$ denotes the **inverse** of $g(\cdot)$.

Simple linear regression is a special case of a GLM

1. A linear mean: $\beta_0 + \beta_1 X_1 + \ldots + \beta_p X_p$
2. An **error structure**: $\epsilon \sim \mathcal{N}(0, \sigma^2)$
3. **Link function**: **identity**
   $\mu = \beta_0 + \beta_1 X_1 + \ldots + \beta_p X_p$

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$
$$\mu = \beta_0 + \beta_1 X$$

GLMs in R

```
lm(height ~ weight, data=df)
```

Is equivalent to:

```
glm(height ~ weight, data=df, family=gaussian(link=identity))
```

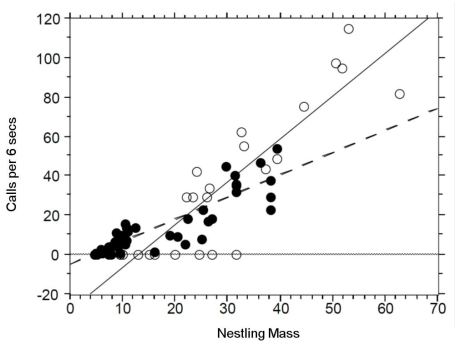
`family` specifies the error structure **and** link function

Default link functions

Family	Link
gaussian	identity
binomial	logit, probit or cloglog
poisson	log, identity or sqrt
Gamma	inverse, identity or log
inverse.gaussian	1/mu^2
quasi	user-defined
quasibinomial	logit
quasipoisson	log

Poisson regression

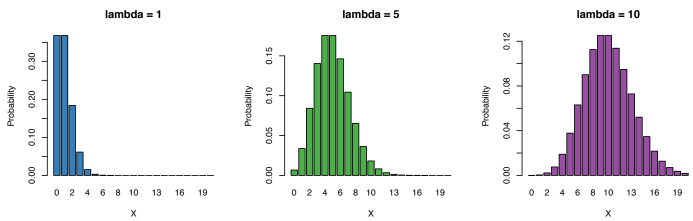
Count data is **discrete** and **non-negative**



$$Y \sim \mathcal{N}(\mu, \sigma^2) \qquad Y \sim \mathcal{P}ois(\mu)$$
$$\mu = \beta_0 + \beta_1 X \qquad \log \mu = \beta_0 + \beta_1 X$$

Poisson distribution

- **Discrete** variable, defined on the range $0, 1, \ldots, \infty$.
- A single **rate** parameter λ, where $\lambda > 0$.
- **Mean** $= \lambda$
- **Variance** $= \lambda$



Poisson regression

$$Y \sim \mathcal{P}ois(\lambda)$$
$$\log \lambda = \beta_0 + \beta_1 X$$

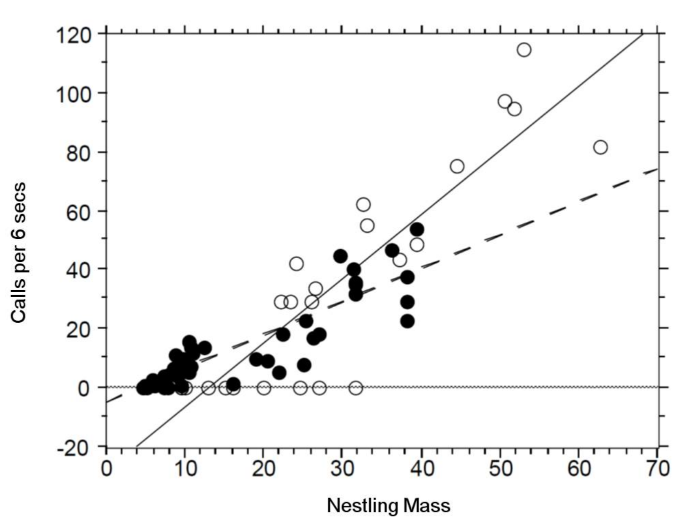
Using the rules of logarithm (i.e $\log \lambda = k$, then $\lambda = e^k$):

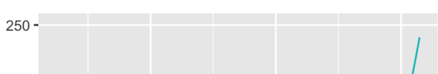
$$\log \lambda = \beta_0 + \beta_1 X$$
$$\lambda = e^{\beta_0 + \beta_1 X}$$

Thus we are effectively modelling the observed counts using an exponential distribution

```
glm(outcome ~ explanatory, data=df, family=poisson(link=log))
```


Cuckoo data



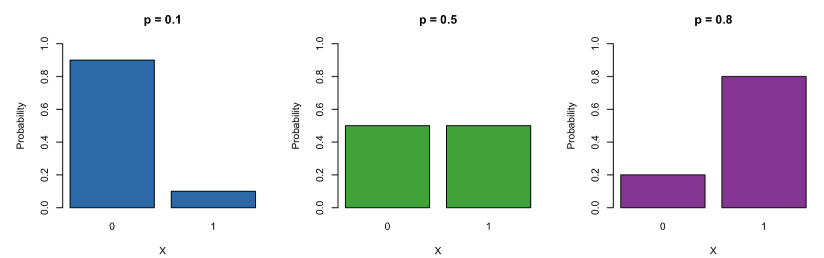


Logistic regression

Consider a **categorical** response variable with two levels (e.g pass/fail). These type of **binary** data are assumed to be **Bernoulli** distributed:

$$Y \sim \mathcal{B}ern(p)$$

- A **probability** parameter p, where $0 < p < 1$.
- **Mean** $= p$
- **Variance** $= p(1-p)$



Logistic regression

$$Y \sim \mathcal{N}(\mu, \sigma^2) \qquad Y \sim \mathcal{Pois}(\lambda) \qquad Y \sim \mathcal{Bern}(p)$$

$$\mu = \beta_0 + \beta_1 X \qquad \log \lambda = \beta_0 + \beta_1 X \qquad ?? = \beta_0 + \beta_1 X$$

$$Y \sim \mathcal{Bern}(p)$$

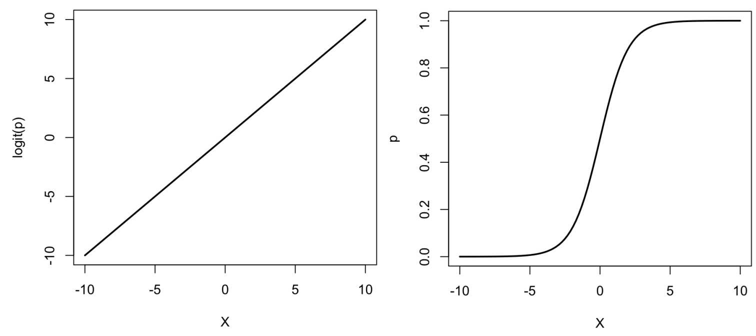
$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 X$$

$$\text{logit}(p) = \beta_0 + \beta_1 X$$

Logistic regression

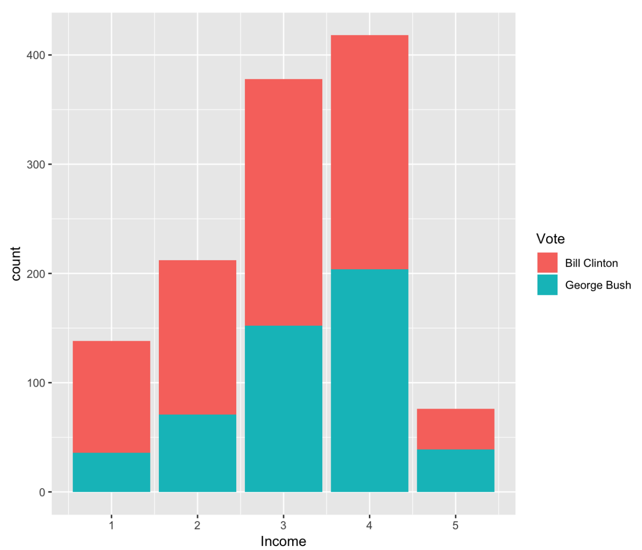
$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 X$$

$$p = \frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}}$$




```
glm(response ~ explanatory, data=df, family=binomial(link=logit))
```


1992 US election survey



1992 US election survey

```
fit <- glm(Vote ~ Income, data=USA, family=binomial(link=logit))
summary(fit)
```

```
##
## Call:
## glm(formula = Vote ~ Income, family = binomial(link = logit),
##     data = USA)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -1.2699  -1.0162  -0.8998   1.2152   1.6199
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  -1.3017     0.1828  -7.122 1.06e-12 ***
## Income        0.3033     0.0551   5.505 3.69e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 1655.0  on 1221  degrees of freedom
## Residual deviance: 1623.5  on 1220  degrees of freedom
## AIC: 1627.5
##
## Number of Fisher Scoring iterations: 4
```

1992 US election survey

$$Y \sim \mathcal{B}ern(p)$$
$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 X$$

- `(Intercept)` $= \beta_0 =$ -1.3
- `Income` $= \beta_1 =$ 0.303

It is common to interpret variables according to some **central tendency** e.g at the central income category (i.e $X = 3$)

$$\begin{aligned} P(\text{Republican vote at } X = 3) &= \text{logit}^{-1}\left(-1.3 + 0.3 \times 3\right) \\ &= \frac{e^{-1.3+0.3\times 3}}{1 + e^{-1.3+0.3\times 3}} \\ &= 0.48. \end{aligned}$$