Chapter 1

The Problems

1.1 Individual Speed Test

Morning, January 30, 2010

There are 20 problems, worth 3 points each, and 20 minutes to solve as many problems as possible.

1. Evaluate $\dfrac{\sqrt{2} \cdot \sqrt{6}}{\sqrt{3}}$.

2. If 6% of a number is 1218, what is 18% of that number?

3. What is the median of $\{42, 9, 8, 4, 5, 1, 13666, 3\}$?

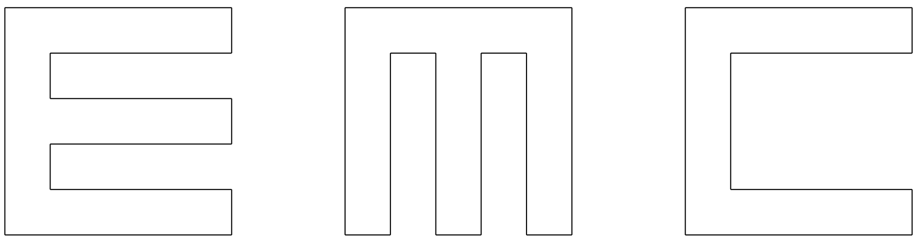
4. Define the operation $\heartsuit$ so that $i \heartsuit u = 5i - 2u$. What is $3 \heartsuit 4$?

5. How many 0.2-inch by 1-inch by 1-inch gold bars can fit in a 15-inch by 12-inch by 9-inch box?

6. A tetrahedron is a triangular pyramid. What is the sum of the number of edges, faces, and vertices of a tetrahedron?

7. Ron has three blue socks, four white socks, five green socks, and two black socks in a drawer. Ron takes socks out of his drawer blindly and at random. What is the least number of socks that Ron needs to take out to guarantee he will be able to make a pair of matching socks?

8. One segment with length 6 and some segments with lengths 10, 8, and 2 form the three letters in the diagram shown below. Compute the sum of the perimeters of the three figures.



9. How many integer solutions are there to the inequality $|x - 6| \leq 4$?

10. In a land for bad children, the flavors of ice cream are grass, dirt, earwax, hair, and dust-bunny. The cones are made out of granite, marble, or pumice, and can be topped by hot lava, chalk, or ink. How many ice cream cones can the evil confectioners in this ice-cream land make? (Every ice cream cone consists of one scoop of ice cream, one cone, and one topping.)

11. Compute the sum of the prime divisors of $245 + 452 + 524$.

12. In quadrilateral $SEAT$, $SE = 2$, $EA = 3$, $AT = 4$, $\angle EAT = \angle SET = 90°$. What is the area of the quadrilateral?

13. What is the angle, in degrees, formed by the hour and minute hands on a clock at 10:30 AM?

14. Three numbers are randomly chosen without replacement from the set $\{101, 102, 103, \ldots, 200\}$. What is the probability that these three numbers are the side lengths of a triangle?

15. John takes a 30-mile bike ride over hilly terrain, where the road always either goes uphill or downhill, and is never flat. If he bikes a total of 20 miles uphill, and he bikes at 6 mph when he goes uphill, and 24 mph when he goes downhill, what is his average speed, in mph, for the ride?

16. How many distinct six-letter words (not necessarily in any language known to man) can be formed by rearranging the letters in EXETER? (You should include the word EXETER in your count.)

17. A pie has been cut into eight slices of different sizes. Snow White steals a slice. Then, the seven dwarfs (Sneezy, Sleepy, Dopey, Doc, Happy, Bashful, Grumpy) take slices one by one according to the alphabetical order of their names, but each dwarf can only take a slice next to one that has already been taken. In how many ways can this pie be eaten by these eight persons?

18. Assume that n is a positive integer such that the remainder of n is 1 when divided by 3, is 2 when divided by 4, is 3 when divided by 5, $\ldots$, and is 8 when divided by 10. What is the smallest possible value of n?

19. Find the sum of all positive four-digit numbers that are perfect squares and that have remainder 1 when divided by 100.

20. A coin of radius 1 cm is tossed onto a plane surface that has been tiled by equilateral triangles with side length $20\sqrt{3}$ cm. What is the probability that the coin lands within one of the triangles?

1.2 Individual Accuracy Test

Morning, January 30, 2010

There are 10 problems, worth 9 points each, and 30 minutes to solve as many problems as possible.

1. Calculate $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)^2$.

2. Find the 2010^{th} digit after the decimal point in the expansion of $\frac{1}{7}$.

3. If you add 1 liter of water to a solution consisting of acid and water, the new solutions will contain of 30% water. If you add another 5 liters of water to the new solution, it will contain $36\frac{4}{11}\%$ water. Find the number of liters of acid in the original solution.

4. John places 5 indistinguishable blue marbles and 5 indistinguishable red marbles into two distinguishable buckets such that each bucket has at least one blue marble and one red marble. How many distinguishable marble distributions are possible after the process is completed?

5. In quadrilateral $PEAR$, $PE = 21$, $EA = 20$, $AR = 15$, $RE = 25$, and $AP = 29$. Find the area of the quadrilateral.

6. Four congruent semicircles are drawn within the boundary of a square with side length 1. The center of each semicircle is the midpoint of a side of the square. Each semicircle is tangent to two other semicircles. Region $\mathcal{R}$ consists of points lying inside the square but outside of the semicircles. The area of $\mathcal{R}$ can be written in the form $a - b\pi$, where a and b are positive rational numbers. Compute $a + b$.

7. Let x and y be two numbers satisfying the relations $x \geq 0$, $y \geq 0$, and $3x + 5y = 7$. What is the maximum possible value of $9x^2 + 25y^2$?

8. In the Senate office in Exie-land, there are 6 distinguishable senators and 6 distinguishable interns. Some senators and an equal number of interns will attend a convention. If at least one senator must attend, how many combinations of senators and interns can attend the convention?

9. Evaluate $(1^2 - 3^2 + 5^2 - 7^2 + 9^2 - \cdots + 2009^2) - (2^2 - 4^2 + 6^2 - 8^2 + 10^2 - \cdots + 2010^2)$.

10. Segment EA has length 1. Region $\mathcal{R}$ consists of points P in the plane such that $\angle PEA \geq 120^\circ$ and $PE < \sqrt{3}$. If point X is picked randomly from the region $\mathcal{R}$, the probability that $AX < \sqrt{3}$ can be written in the form $a - \frac{\sqrt{b}}{c\pi}$, where a is a rational number, b and c are positive integers, and b is not divisible by the square of a prime. Find the ordered triple (a, b, c).

1.3 Team Round

Morning, January 30, 2010

There are 15 problems, worth 20 points each, and 30 minutes to solve as many problems as possible.

1. A very large lucky number N consists of eighty-eight 8s in a row. Find the remainder when this number N is divided by 6.

2. If 3 chickens can lay 9 eggs in 4 days, how many chickens does it take to lay 180 eggs in 8 days?

3. Find the ordered pair (x, y) of real numbers satisfying the conditions $x > y$, $x+y = 10$, and $xy = -119$.

4. There is pair of similar triangles. One triangle has side lengths 4, 6, and 9. The other triangle has side lengths 8, 12 and x. Find the sum of two possible values of x.

5. If $x^2 + \frac{1}{x^2} = 3$, there are two possible values of $x + \frac{1}{x}$. What is the smaller of the two values?

6. Three flavors (chocolate strawberry, vanilla) of ice cream are sold at Brian's ice cream shop. Brian's friend Zerg gets a coupon for 10 free scoops of ice cream. If the coupon requires Zerg to choose an even number of scoops of each flavor of ice cream, how many ways can he choose his ice cream scoops? (For example, he could have 6 scoops of vanilla and 4 scoops of chocolate. The order in which Zerg eats the scoops does not matter.)

7. David decides he wants to join the West African Drumming Ensemble, and thus he goes to the store and buys three large cylindrical drums. In order to ensure none of the drums drop on the way home, he ties a rope around all of the drums at their mid sections so that each drum is next to the other two. Suppose that each drum has a diameter of 3.5 feet. David needs m feet of rope. Given that $m = a\pi + b$, where a and b are rational numbers, find sum $a + b$.

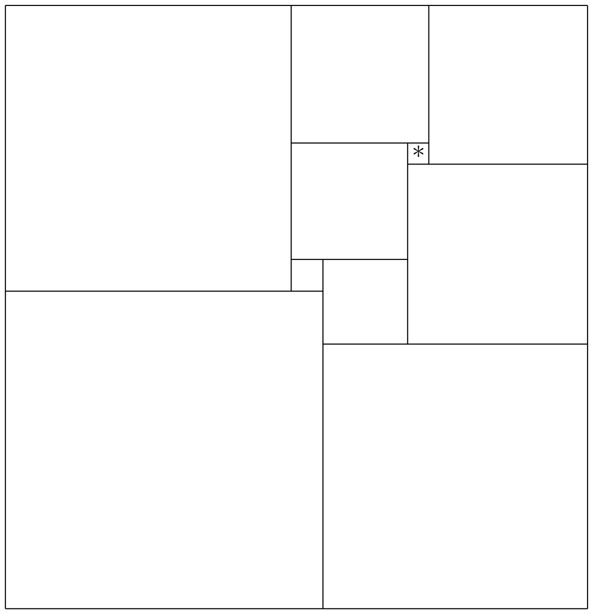
8. Segment AB is the diameter of a semicircle of radius 24. A beam of light is shot from a point $12\sqrt{3}$ from the center of the semicircle, and perpendicular to AB. How many times does it reflect off the semicircle before hitting AB again?

9. A cube is inscribed in a sphere of radius 8. A smaller sphere is inscribed in the same sphere such that it is externally tangent to one face of the cube and internally tangent to the larger sphere. The maximum value of the ratio of the volume of the smaller sphere to the volume of the larger sphere can be written in the form $\frac{a-\sqrt{b}}{36}$, where a and b are positive integers. Find the product ab.

10. How many ordered pairs (x, y) of integers are there such that $2xy + x + y = 52$?

11. Three musketeers looted a caravan and walked off with a chest full of coins. During the night, the first musketeer divided the coins into three equal piles, with one coin left over. He threw it into the ocean and took one of the piles for himself, then went back to sleep. The second musketeer woke up an hour later. He divided the remaining coins into three equal piles, and threw out the one coin that was left over. He took one of the piles and went back to sleep. The third musketeer woke up and divided the remaining coins into three equal piles, threw out the extra coin, and took one pile for himself. The next morning, the three musketeers gathered around to divide the coins into three equal piles. Strangely enough, they had one coin left over this time as well. What is the minimum number of coins that were originally in the chest?

12. The diagram shows a rectangle that has been divided into ten squares of different sizes. The smallest square is 2×2 (marked with $*$). What is the area of the rectangle (which looks rather like a square itself)?



13. Let $A = (3, 2)$, $B = (0, 1)$, and P be on the line $x + y = 0$. What is the minimum possible value of $AP + BP$?

14. Mr. Mustafa the number man got a $6 \times x$ rectangular chess board for his birthday. Because he was bored, he wrote the numbers 1 to $6x$ starting in the upper left corner and moving across row by row (so the number $x + 1$ is in the 2^{nd} row, 1^{st} column). Then, he wrote the same numbers starting in the upper left corner and moving down each column (so the number 7 appears in the 1^{st} row, 2^{nd} column). He then added up the two numbers in each of the cells and found that some of the sums were repeated. Given that x is less than or equal to 100, how many possibilities are there for x?

15. Six congruent equilateral triangles are arranged in the plane so that every triangle shares at least one whole edge with some other triangle. Find the number of distinct arrangements. (Two arrangements are considered the same if one can be rotated and/or reflected onto another.)

EXETER MATH CLUB COMPETITION

1.4 Guts Round

Afternoon, January 30, 2010

1.4.1 Round 1

1. [5pts] Define the operation $\clubsuit$ so that $a \clubsuit b = a^b + b^a$. Then, if $2 \clubsuit b = 32$, what is b?

2. [5pts] A square is changed into a rectangle by increasing two of its sides by $p\%$ and decreasing the two other sides by $p\%$. The area is then reduced by 1%. What is the value of p?

3. [5pts] What is the sum, in degrees, of the internal angles of a heptagon?

4. [5pts] How many integers in between $\sqrt{47}$ and $\sqrt{8283}$ are divisible by 7?

1.4.2 Round 2

5. [8pts] Some mutant green turkeys and pink elephants are grazing in a field. Mutant green turkeys have six legs and three heads. Pink elephants have 4 legs and 1 head. There are 100 legs and 37 heads in the field. How many animals are grazing?

6. [8pts] Let $A = (0, 0)$, $B = (6, 8)$, $C = (20, 8)$, $D = (14, 0)$, $E = (21, -10)$, and $F = (7, -10)$. Find the area of the hexagon $ABCDEF$.

7. [8pts] In Moscow, three men, Oleg, Igor, and Dima, are questioned on suspicion of stealing Vladimir Putin's blankie. It is known that each man either always tells the truth or always lies. They make the following statements:

   (a) Oleg: I am innocent!

   (b) Igor: Dima stole the blankie!

   (c) Dima: I am innocent!

   (d) Igor: I am guilty!

   (e) Oleg: Yes, Igor is indeed guilty!

   If exactly one of Oleg, Igor, and Dima is guilty of the theft, who is the thief??

8. [8pts] How many 11-letter sequences of E's and M's have at least as many E's as M's?

1.4.3 Round 3

9. [11pts] John is entering the following summation $31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39$ in his calculator. However, he accidently leaves out a plus sign and the answer becomes 3582. What is the number that comes before the missing plus sign?

10. [11pts] Two circles of radius 6 intersect such that they share a common chord of length 6. The total area covered may be expressed as $a\pi + \sqrt{b}$, where a and b are integers. What is $a + b$?

11. [11pts] Alice has a rectangular room with 6 outlets lined up on one wall and 6 lamps lined up on the opposite wall. She has 6 distinct power cords (red, blue, green, purple, black, yellow). If the red and green power cords cannot cross, how many ways can she plug in all six lamps?

12. [11pts] Tracy wants to jump through a line of 12 tiles on the floor by either jumping onto the next block, or jumping onto the block two steps ahead. An example of a path through the 12 tiles may be: 1 step, 2 steps, 2 steps, 2 steps, 1 step, 2 steps, 2 steps. In how many ways can Tracy jump through these 12 tiles?

1.4.4 Round 4

13. [14pts] What is the units digit of the number $(2^1 + 1)(2^2 - 1)(2^3 + 1)(2^4 - 1)\ldots(2^{2010} - 1)$?

14. [14pts] Mr. Fat noted that on January 2, 2010, the display of the day is 01/02/2010, and the sequence 01022010 is a palindrome (a number that reads the same forwards and backwards). How many days does Mr. Fat need to wait between this palindrome day and the last palindrome day of this decade?

15. [14pts] Farmer Tim has a 30-meter by 30-meter by $30\sqrt{2}$-meter triangular barn. He ties his goat to the corner where the two shorter sides meet with a 60-meter rope. What is the area, in square meters, of the land where the goat can graze, given that it cannot get inside the barn?

16. [14pts] In triangle ABC, $AB = 3$, $BC = 4$, and $CA = 5$. Point P lies inside the triangle and the distances from P to two of the sides of the triangle are 1 and 2. What is the maximum distance from P to the third side of the triangle?

1.4.5 Round 5

17. [17pts] Let Z be the answer to the third question on this guts quadruplet. If $x^2 - 2x = Z - 1$, find the positive value of x.

18. [17pts] Let X be the answer to the first question on this guts quadruplet. To make a FATRON2012, a cubical steel body as large as possible is cut out from a solid sphere of diameter X. A TAFTRON2013 is created by cutting a FATRON2012 into 27 identical cubes, with no material wasted. What is the length of one edge of a TAFTRON2013?

19. [17pts] Let Y be the smallest integer greater than the answer to the second question on this guts quadruplet. Fred posts two distinguishable sheets on the wall. Then, Y people walk into the room. Each of the Y people signs up on 0, 1, or 2 of the sheets. Given that there are at least two people in the room other than Fred, how many possible pairs of lists can Fred have?

20. [17pts] Let A, B, C, be the respective answers to the first, second, and third questions on this guts quadruplet. At the Robot Design Convention and Showcase, a series of robots are programmed such that each robot shakes hands exactly once with every other robot of the same height. If the heights of the 16 robots are 4, 4, 4, 5, 5, 7, 17, 17, 17, 34, 34, 42, 100, A, B, and C feet, how many handshakes will take place?

1.4.6 Round 6

21. [20pts] Determine the number of ordered triples (p, q, r) of primes with $1 < p < q < r < 100$ such that $q - p = r - q$.

22. [20pts] For numbers a, b, c, d such that $0 \le a, b, c, d \le 10$, find the minimum value of $ab + bc + cd + da - 5a - 5b - 5c - 5d$.

23. [20pts] Daniel has a task to measure 1 gram, 2 grams, 3 grams, 4 grams , . . . , all the way up to n grams. He goes into a store and buys a scale and six weights of his choosing (so that he knows the value for each weight that he buys). If he can place the weights on either side of the scale, what is the maximum value of n?

24. [20pts] Given a Rubik's cube, what is the probability that at least one face will remain unchanged after a random sequence of three moves? (A Rubik's cube is a 3 by 3 by 3 cube with each face starting as a different color. The faces (3 by 3) can be freely turned. A move is defined in this problem as a 90 degree rotation of one *face either clockwise or counter-clockwise. The center square on each face–six in total–is fixed.)*