1.1 Individual Speed Test

Morning, January 28, 2012

There are 20 problems, worth 3 points each, to be solved in 20 minutes.

1. Evaluate
$$\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}.$$

2. A regular hexagon and a regular n-sided polygon have the same perimeter. If the ratio of the side length of the hexagon to the side length of the n-sided polygon is $2:1$, what is n?

3. How many nonzero digits are there in the decimal representation of $2\cdot 10\cdot 500\cdot 2500$?

4. When the numerator of a certain fraction is increased by 2012, the value of the fraction increases by 2. What is the denominator of the fraction?

5. Sam did the computation $1-10\cdot a+22$, where a is some real number, except he messed up his order of operations and computed the multiplication last; that is, he found the value of $(1-10)\cdot(a+22)$ instead. Luckily, he still ended up with the right answer. What is a?

6. Let $n! = n\cdot(n-1)\cdots 2\cdot 1$. For how many integers n between 1 and 100 inclusive is $n!$ divisible by 36?

7. Simplify the expression $\sqrt{\dfrac{3\cdot 27^3}{27\cdot 3^3}}$.

8. Four points A, B, C, D lie on a line in that order such that $\dfrac{AB}{CB}=\dfrac{AD}{CD}$. Let M be the midpoint of segment AC. If $AB = 6, BC = 2$, compute $MB\cdot MD$.

9. Allan has a deck with 8 cards, numbered $1, 1, 2, 2, 3, 3, 4, 4$. He pulls out cards without replacement, until he pulls out an even numbered card, and then he stops. What is the probability that he pulls out exactly 2 cards?

10. Starting from the sequence $(3, 4, 5, 6, 7, 8, \dots)$, one applies the following operation repeatedly. In each operation, we change the sequence
$$(a_1, a_2, a_3, \dots, a_{a_1-1}, a_{a_1}, a_{a_1+1}, \dots)$$
to the sequence
$$(a_2, a_3, \dots, a_{a_1}, a_1, a_{a_1+1}, \dots).$$
(In other words, for a sequence starting with x, we shift each of the next $x-1$ term to the left by one, and put x immediately to the right of these numbers, and keep the rest of the terms unchanged. For example, after one operation, the sequence is $(4, 5, 3, 6, 7, 8, \dots)$, and after two operations, the sequence becomes $(5, 3, 6, 4, 7, 8, \dots)$. How many operations will it take to obtain a sequence of the form $(7, \dots)$ (that is, a sequence starting with 7)?

11. How many ways are there to place 4 balls into a 4×6 grid such that no column or row has more than one ball in it? (Rotations and reflections are considered distinct.)

12. Point P lies inside triangle ABC such that $\angle PBC = 30^\circ$ and $\angle PAC = 20^\circ$. If $\angle APB$ is a right angle, find the measure of $\angle BCA$ in degrees.

13. What is the largest prime factor of $9^3 - 4^3$?

14. Joey writes down the numbers 1 through 10 and crosses one number out. He then adds the remaining numbers. What is the probability that the sum is less than or equal to 47?

15. In the coordinate plane, a *lattice point* is a point whose coordinates are *integers*. There is a pile of grass at every lattice point in the coordinate plane. A certain cow can only eat piles of grass that are at most 3 units away from the origin. How many piles of grass can she eat?

16. A book has 1000 pages numbered $1, 2, \ldots, 1000$. The pages are numbered so that pages 1 and 2 are back to back on a single sheet, pages 3 and 4 are back to back on the next sheet, and so on, with pages 999 and 1000 being back to back on the last sheet. How many pairs of pages that are back to back (on a single sheet) share no digits in the same position? (For example, pages 9 and 10, and pages 89 and 90.)

17. Find a pair of integers (a, b) for which $\dfrac{10^a}{a!} = \dfrac{10^b}{b!}$ and $a < b$.

18. Find all ordered pairs (x, y) of real numbers satisfying

$$\begin{cases} -x^2 + 3y^2 - 5x + 7y + 4 = 0 \\ 2x^2 - 2y^2 - x + y + 21 = 0 \end{cases}$$

19. There are six blank fish drawn in a line on a piece of paper. Lucy wants to color them either red or blue, but will not color two adjacent fish red. In how many ways can Lucy color the fish?

20. There are sixteen 100-gram balls and sixteen 99-gram balls on a table (the balls are visibly indistinguishable). You are given a balance scale with two sides that reports which side is heavier or that the two sides have equal weights. A weighing is defined as reading the result of the balance scale: For example, if you place three balls on each side, look at the result, then add two more balls to each side, and look at the result again, then two weighings have been performed. You wish to pick out two different sets of balls (from the 32 balls) with equal numbers of balls in them but different total weights. What is the minimal number of weighings needed to ensure this?