1.4 Guts Test

Afternoon, January 28, 2012

There are 24 problems, with varying point values, to be solved in 75 minutes.

1.4.1 Round 1

1. [6pts] Ravi has a bag with 100 slips of paper in it. Each slip has one of the numbers 3, 5, or 7 written on it. Given that half of the slips have the number 3 written on them, and the average of the values on all the slips is 4.4, how many slips have 7 written on them?

2. [6pts] In triangle ABC, point D lies on side AB such that $AB \perp CD$. It is given that $\frac{CD}{BD} = \frac{1}{2}$, $AC = 29$, and $AD = 20$. Find the area of triangle BCD.

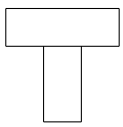
3. [6pts] Compute $(123 + 4)(123 + 5) - 123 \cdot 132$.

1.4.2 Round 2

4. [8pts] David is evaluating the terms in the sequence $a_n = (n + 1)^3 - n^3$ for $n = 1, 2, 3, \ldots$ (that is, $a_1 = 2^3 - 1^3, a_2 = 3^3 - 2^3, a_3 = 4^3 - 3^3$, and so on). Find the first composite number in the sequence. (An positive integer is *composite* if it has a divisor other than 1 and itself.)

5. [8pts] Find the sum of all positive integers strictly less than 100 that are not divisible by 3.

6. [8pts] In how many ways can Alex draw the diagram below without lifting his pencil or retracing a line? (Two drawings are different if the order in which he draws the edges is different, or the direction in which he draws an edge is different).



1.4.3 Round 3

7. [10pts] Fresh Mann is a 9th grader at Euclid High School. Fresh Mann thinks that the word *vertices* is the plural of the word *vertice*. Indeed, vertices is the plural of the word *vertex*. Using all the letters in the word *vertice*, he can make m 7-letter sequences. Using all the letters in the word *vertex*, he can make n 6-letter sequences. Find $m - n$.

8. [10pts] Fresh Mann is given the following expression in his Algebra 1 class:

$$101 - 102 = 1.$$

Fresh Mann is allowed to move some of the digits in this (incorrect) equation to make it into a correct equation. What is the minimal number of digits Fresh Mann needs to move?

9. [10pts] Fresh Mann said, "The function $f(x) = ax^2 + bx + c$ passes through 6 points. Their x-coordinates are consecutive positive integers, and their y-coordinates are $34, 55, 84, 119, 160$, and 207, respectively." Sophy Moore replied, "You've made an error in your list," and replaced one of Fresh Mann's numbers with the correct y-coordinate. Find the corrected value.

1.4.4 Round 4

10. [12pts] An assassin is trying to find his target's hotel room number, which is a three-digit positive integer. He knows the following clues about the number:

    (a) The sum of any two digits of the number is divisible by the remaining digit.

    (b) The number is divisible by 3, but if the first digit is removed, the remaining two-digit number is not.

    (c) The middle digit is the only digit that is a perfect square.

    Given these clues, what is a possible value for the room number?

11. [12pts] Find a positive real number r that satisfies

$$\frac{4 + r^3}{9 + r^6} = \frac{1}{5 - r^3} - \frac{1}{9 + r^6}.$$

12. [12pts] Find the largest integer n such that there exist integers x and y between 1 and 20 inclusive with

$$\left| \frac{21}{19} - \frac{x}{y} \right| < \frac{1}{n}.$$

1.4.5 Round 5

13. [14pts] A unit square is rotated 30° counterclockwise about one of its vertices. Determine the area of the intersection of the original square with the rotated one.

14. [14pts] Suppose points A and B lie on a circle of radius 4 with center O, such that $\angle AOB = 90^\circ$. The perpendicular bisectors of segments OA and OB divide the interior of the circle into four regions. Find the area of the smallest region.

15. [14pts] Let $ABCD$ be a quadrilateral such that $AB = 4$, $BC = 6$, $CD = 5$, $DA = 3$, and $\angle DAB = 90^\circ$. There is a point I inside the quadrilateral that is equidistant from all the sides. Find AI.

1.4.6 Round 6

The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.

16. [16pts] Let C be the answer to problem 18. Compute

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{C^2}\right).$$

17. [16pts] Let A be the answer to problem 16. Let $PQRS$ be a square, and let point M lie on segment PQ such that $MQ = 7PM$ and point N lie on segment PS such that $NS = 7PN$. Segments MS and NQ meet at point X. Given that the area of quadrilateral $PMXN$ is $A - \frac{1}{2}$, find the side length of the square.

18. [16pts] Let B be the answer to problem 17 and let $N = 6B$. Find the number of ordered triples (a, b, c) of integers between 0 and $N - 1$, inclusive, such that $a + b + c$ is divisible by N.

1.4.7 Round 7

19. [16pts] Let k be the units digit of $\underbrace{7^{7^{7^{7^{7^{7^{7}}}}}}}_{\text{Seven 7s}}$. What is the largest prime factor of the number consisting of k 7's written in a row?

20. [16pts] Suppose that $E = 7^7, M = 7$, and $C = 7 \cdot 7 \cdot 7$. The characters E, M, C, C are arranged randomly in the following blanks.

$$\underline{\hspace{2em}} \times \underline{\hspace{2em}} \times \underline{\hspace{2em}} \times \underline{\hspace{2em}}$$

Then one of the multiplication signs is chosen at random and changed to an equals sign. What is the probability that the resulting equation is true?

21. [16pts] During a recent math contest, Sophy Moore made the mistake of thinking that 133 is a prime number. Fresh Mann replied, "To test whether a number is divisible by 3, we just need to check whether the sum of the digits is divisible by 3. By the same reasoning, to test whether a number is divisible by 7, we just need to check that the sum of the digits is a multiple of 7, so 133 is clearly divisible by 7." Although his general principle is false, 133 is indeed divisible by 7. How many three-digit numbers are divisible by 7 and have the sum of their digits divisible by 7?

1.4.8 Round 8

22. [18pts] A *look-and-say sequence* is defined as follows: starting from an initial term a_1, each subsequent term a_k is found by reading the digits of a_{k-1} from left to right and specifying the number of times each digit appears consecutively. For example, 4 would be succeeded by 14 ("One four."), and 31337 would be followed by 13112317 ("One three, one one, two three, one seven.")

If a_1 is a random two-digit positive integer, find the probability that a_4 is at least six digits long.

23. [18pts] In triangle ABC, $\angle C = 90°$. Point P lies on segment BC and is not B or C. Point I lies on segment AP, and $\angle BIP = \angle PBI = \angle CAB$. If $\dfrac{AP}{BC} = k$, express $\dfrac{IP}{CP}$ in terms of k.

24. [18pts] A subset of $\{1, 2, 3, \ldots, 30\}$ is called *delicious* if it does not contain an element that is 3 times another element. A subset is called *super delicious* if it is delicious and no delicious set has more elements than it has. Determine the number of super delicious subsets.