Exeter Math Club Contest
January 26, 2013

Table of Contents

Organizing Information

- *Tournament Directors* Albert Chu, Ray Li
- *Tournament Supervisor* Zuming Feng
- *System Administrator and Webmaster* Albert Chu
- *Head Problem Writers* Ravi Jagadeesan, David Yang
- *Problem Committee* Leigh Marie Braswell, Vahid Fazel-Rezai, Chad Qian, Zhuo Qun (Alex) Song, Kuo-An (Andy) Wei
- *Solutions Editors* Leigh Marie Braswell, Vahid Fazel-Rezai, Chad Qian, Zhuo Qun (Alex) Song, Kuo-An (Andy) Wei
- *Problem Reviewers* Zuming Feng, Chris Jeuell, Palmer Mebane, Shijie (Joy) Zheng
- *Problem Contributors* Priyanka Boddu, Mickey Chao, Bofan Chen, Jiapei Chen, Xin Xuan (Vivian) Chen, Albert Chu, Trang Duong, Claudia Feng, Jiexiong (Chelsea) Ge, Mark Huang, Ravi Jagadeesan, Jay Lee, Ray Li, Calvin Luo, Zhuo Qun (Alex) Song, Elizabeth Wei, David Yang, Grace Yin
- *Treasurer* Jiexiong (Chelsea) Ge
- *Publicity* Claudia Feng
- *Primary Tournament Sponsor* We would like to thank Jane Street Capital for their generous support of this competition.
- *Tournament Sponsors* We would also like to thank the Phillips Exeter Academy Math Department and Art of Problem Solving for their support.

Contest day information

- *Head Proctor* Bofan Chen
- *Proctors* David (Anthony) Bau Jiapei Chen, Weihang (Frank) Fan, Jack Hirsch, Meena Jagadeesan, Min Sung (Matthew) Kim, Yuree Kim, Max Le, Francis Lee, Harry Lee, Jay Lee, Leo Lien, Kevin Ma, Eugene Park, Jeffrey Qiao, Rohit Rajiv, Arianna Serafini, Darius Shi, Angela Song, Sam Tan, Chris Vazan, Malachi Weaver, Alex Wei
- *Head Grader* Ravi Jagadeesan
- *Graders* Leigh Marie Braswell, Jiapei Chen, Albert Chu, Vahid Fazel-Rezai, Claudia Feng, Jiexiong (Chelsea) Ge, Chad Qian, Zhuo Qun (Alex) Song, Kuo-An (Andy) Wei, Dai Yang, David Yang
- *Judges* Zuming Feng, Greg Spanier
- *Head Runner* Mickey Chao
- *Runners* Janet Chen, Calvin Luo, Max Vachon Lizzie Wei

Chapter 1

EMC^2 2013 Problems

1.1 Individual Speed Test

Morning, January 26, 2013

There are 20 problems, worth 3 points each, to be solved in 20 minutes.

1. Determine how many digits the number 10^{10} has.

2. Let ABC be a triangle with $\angle ABC = 60^\circ$ and $\angle BCA = 70^\circ$. Compute $\angle CAB$ in degrees.

3. Given that $x : y = 2012 : 2$ and $y : z = 1 : 2013$, compute $x : z$. Express your answer as a common fraction.

4. Determine the smallest perfect square greater than 2400.

5. At 12:34 and 12:43, the time contains four consecutive digits. Find the next time after 12:43 that the time contains four consecutive digits on a 24-hour digital clock.

6. Given that $\sqrt{3^a \cdot 9^a \cdot 3^a} = 81^2$, compute a.

7. Find the number of positive integers less than 8888 that have a tens digit of 4 and a units digit of 2.

8. Find the sum of the distinct prime divisors of $1 + 2012 + 2013 + 2011 \cdot 2013$.

9. Albert wants to make 2×3 wallet sized prints for his grandmother. Find the maximum possible number of prints Albert can make using one 4×7 sheet of paper.

10. Let ABC be an equilateral triangle, and let D be a point inside ABC. Let E be a point such that ADE is an equilateral triangle and suppose that segments DE and AB intersect at point F. Given that $\angle CAD = 15^\circ$, compute $\angle DFB$ in degrees.

11. A *palindrome* is a number that reads the same forwards and backwards; for example, 1221 is a palindrome. An *almost-palindrome* is a number that is not a palindrome but whose first and last digits are equal; for example, 1231 and 1311 are an almost-palindromes, but 1221 is not. Compute the number of 4-digit almost-palindromes.

12. Determine the smallest positive integer n such that the sum of the digits of 11^n is not 2^n.

13. Determine the minimum number of breaks needed to divide an 8×4 bar of chocolate into 1×1 pieces. (When a bar is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

14. A particle starts moving on the number line at a time $t = 0$. Its position on the number line, as a function of time, is $x = (t-2012)^2 - 2012(t-2012) - 2013$. Find the number of positive integer values of t at which time the particle lies in the negative half of the number line (strictly to the left of 0).

15. Let A be a vertex of a unit cube and let B, C, and D be the vertices adjacent to A. The tetrahedron $ABCD$ is cut off the cube. Determine the surface area of the remaining solid.

16. In equilateral triangle ABC, points P and R lie on segment AB, points I and M lie on segment BC, and points E and S lie on segment CA such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PS = 2$, $RI = 3$, and $ME = 5$, compute the area of hexagon $PRIMES$.

17. Find the smallest odd positive integer with an odd number of positive integer factors, an odd number of distinct prime factors, and an odd number of perfect square factors.

18. Fresh Mann thinks that the expressions $2\sqrt{x^2-4}$ and $2(\sqrt{x^2}-\sqrt{4})$ are equivalent to each other, but the two expressions are not equal to each other for most real numbers x. Find all real numbers x such that $2\sqrt{x^2-4} = 2(\sqrt{x^2}-\sqrt{4})$.

19. Let m be the positive integer such that a 3×3 chessboard can be tiled by at most m pairwise incongruent rectangles with integer side lengths. If rotations and reflections of tilings are considered distinct, suppose that there are n ways to tile the chessboard with m pairwise incongruent rectangles with integer side lengths. Find the product mn.

20. Let ABC be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$. A triangle XYZ is said to be *friendly* if it intersects triangle ABC and it is a translation of triangle ABC. Let S be the set of points in the plane that are inside some friendly triangle. Compute the ratio of the area of S to the area of triangle ABC.

1.2 Individual Accuracy Test

Morning, January 26, 2013

There are 10 problems, worth 9 points each, to be solved in 30 minutes.

1. Find the largest possible number of consecutive 9's in which an integer between 10,000,000 and 13,371,337 can end. For example, 199 ends in two 9's, while 92,999 ends in three 9's.

2. Let $ABCD$ be a square of side length 2. Equilateral triangles ABP, BCQ, CDR, and DAS are constructed inside the square. Compute the area of quadrilateral $PQRS$.

3. Evaluate the expression $7 \cdot 11 \cdot 13 \cdot 1003 - 3 \cdot 17 \cdot 59 \cdot 331$.

4. Compute the number of positive integers c such that there is a non-degenerate obtuse triangle with side lengths $21, 29$, and c.

5. Consider a 5×5 board, colored like a chessboard, such that the four corners are black. Determine the number of ways to place 5 rooks on black squares such that no two of the rooks attack one another, given that the rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

6. Let $ABCD$ be a trapezoid of height 6 with bases AB and CD. Suppose that $AB = 2$ and $CD = 3$, and let F and G be the midpoints of segments AD and BC, respectively. If diagonals AC and BD intersect at point E, compute the area of triangle FGE.

7. A *regular octahedron* is a solid with eight faces that are congruent equilateral triangles. Suppose that an ant is at the center of one face of a regular octahedron of edge length 10. The ant wants to walk along the surface of the octahedron to reach the center of the opposite face. (Two faces of an octahedron are said to be opposite if they do not share a vertex.) Determine the minimum possible distance that the ant must walk.

8. Let $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$, and $D_1D_2D_3$ be triangles in the plane. All the sides of the four triangles are extended into lines. Determine the maximum number of pairs of these lines that can meet at 60° angles.

9. For an integer n, let $f_n(x)$ denote the function $f_n(x) = \sqrt{x^2 - 2012x + n} + 1006$. Determine all positive integers a such that $f_a(f_{2012}(x)) = x$ for all $x \geq 2012$.

10. Determine the number of ordered triples of integers (a, b, c) such that $(a + b)(b + c)(c + a) = 1800$.

1.3 Team Test

Morning, January 26, 2013

There are 10 problems, worth 30 points each, to be solved in 45 minutes.

1. Determine the number of ways to place 4 rooks on a 4×4 chessboard such that:

   (a) no two rooks attack one another, and

   (b) the main diagonal (the set of squares marked X below) does not contain any rooks.

X			
	X		
		X	
			X

   The rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

2. Seven students, numbered 1 to 7 in counter-clockwise order, are seated in a circle. Fresh Mann has 100 erasers, and he wants to distribute them to the students, albeit unfairly. Starting with person 1 and proceeding counter-clockwise, Fresh Mann gives i erasers to student i; for example, he gives 1 eraser to student 1, then 2 erasers to student 2, et cetera. He continues around the circle until he does not have enough erasers to give to the next person. At this point, determine the number of erasers that Fresh Mann has.

3. Let ABC be a triangle with $AB = AC = 17$ and $BC = 24$. Approximate $\angle ABC$ to the nearest multiple of 10 degrees.

4. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = \frac{3}{5}$ and for $n \geq 1$, $x_{n+1} = 2 - \dfrac{1}{x_n}$. Compute the product $x_1 x_2 x_3 \cdots x_{2013}$.

5. In equilateral triangle ABC, points P and R lie on segment AB, points I and M lie on segment BC, and points E and S lie on segment CA such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PR = 2$, $IM = 3$, and $ES = 5$, compute the area of hexagon $PRIMES$.

6. Let $f(a, b) = \frac{a^2}{a+b}$. Let A denote the sum of $f(i, j)$ over all pairs of integers (i, j) with $1 \leq i < j \leq 10$; that is,

$$A = (f(1,2) + f(1,3) + \cdots + f(1,10)) + (f(2,3) + f(2,4) + \cdots + f(2,10)) + \cdots + f(9,10).$$

   Similarly, let B denote the sum of $f(i, j)$ over all pairs of integers (i, j) with $1 \leq j < i \leq 10$; that is,

$$B = (f(2,1) + f(3,1) + \cdots + f(10,1)) + (f(3,2) + f(4,2) + \cdots + f(10,2)) + \cdots + f(10,9).$$

   Compute $B - A$.

7. Fresh Mann has a pile of seven rocks with weights 1, 1, 2, 4, 8, 16, and 32 pounds and some integer X between 1 and 64, inclusive. He would like to choose a set of the rocks whose total weight is exactly X pounds. Given that he can do so in more than one way, determine the sum of all possible values of X. (The two 1-pound rocks are indistinguishable.)

8. Let $ABCD$ be a convex quadrilateral with $AB = BC = CA$. Suppose that point P lies inside the quadrilateral with $AP = PD = DA$ and $\angle PCD = 30^\circ$. Given that $CP = 2$ and $CD = 3$, compute CA.

9. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = 2$, $x_2 = \frac{13}{2}$, and for $n \geq 1$,

$$x_{n+2} = 3 - \frac{3}{x_{n+1}} + \frac{1}{x_n x_{n+1}}.$$

Compute x_{100}.

10. Ten prisoners are standing in a line. A prison guard wants to place a hat on each prisoner. He has two colors of hats, red and blue, and he has 10 hats of each color. Determine the number of ways in which the prison guard can place hats such that among any set of consecutive prisoners, the number of prisoners with red hats and the number of prisoners with blue hats differ by at most 2.

1.4 Guts Test

Afternoon, January 26, 2013

There are 24 problems, with varying point values, to be solved in 75 minutes.

1.4.1 Round 1

1. [6pts] Five girls and three boys are sitting in a room. Suppose that four of the children live in California. Determine the maximum possible number of girls that could live somewhere outside California.

2. [6pts] A 4-meter long stick is rotated 60° about a point on the stick 1 meter away from one of its ends. Compute the positive difference between the distances traveled by the two endpoints of the stick, in meters.

3. [6pts] Let
$$f(x) = 2x(x-1)^2 + x^3(x-2)^2 + 10(x-1)^3(x-2).$$
Compute $f(0) + f(1) + f(2)$.

1.4.2 Round 2

4. [8pts] Twenty boxes with weights $10, 20, 30, \ldots, 200$ pounds are given. One hand is needed to lift a box for every 10 pounds it weighs. For example, a 40 pound box needs four hands to be lifted. Determine the number of people needed to lift all the boxes simultaneously, given that no person can help lift more than one box at a time.

5. [8pts] Let ABC be a right triangle with a right angle at A, and let D be the foot of the perpendicular from vertex A to side BC. If $AB = 5$ and $BC = 7$, compute the length of segment AD.

6. [8pts] There are two circular ant holes in the coordinate plane. One has center $(0, 0)$ and radius 3, and the other has center $(20, 21)$ and radius 5. Albert wants to cover both of them completely with a circular bowl. Determine the minimum possible radius of the circular bowl.

1.4.3 Round 3

7. [10pts] A line of slope -4 forms a right triangle with the positive x and y axes. If the area of the triangle is 2013, find the square of the length of the hypotenuse of the triangle.

8. [10pts] Let ABC be a right triangle with a right angle at B, $AB = 9$, and $BC = 7$. Suppose that point P lies on segment AB with $AP = 3$ and that point Q lies on ray BC with $BQ = 11$. Let segments AC and PQ intersect at point X. Compute the positive difference between the areas of triangles APX and CQX.

9. [10pts] Fresh Mann and Sophy Moore are racing each other in a river. Fresh Mann swims downstream, while Sophy Moore swims $\frac{1}{2}$ mile upstream and then travels downstream in a boat. They start at the same time, and they reach the finish line 1 mile downstream of the starting point simultaneously. If Fresh Mann and Sophy Moore both swim at 1 mile per hour in still water and the boat travels at 10 miles per hour in still water, find the speed of the current.

1.4.4 Round 4

10. [12pts] The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. The first few terms of the Fibonacci sequence are $0, 1, 1, 2, 3, 5, 8, 13$. Every positive integer can be expressed as the sum of nonconsecutive, distinct, positive Fibonacci numbers; for example, $7 = 5 + 2$. Express 121 as the sum of nonconsecutive, distinct, positive Fibonacci numbers. (It is not permitted to use both a 2 and a 1 in the expression.)

11. [12pts] There is a rectangular box of surface area 44 whose space diagonals have length 10. Find the sum of the lengths of all the edges of the box.

12. [12pts] Let ABC be an acute triangle, and let D and E be the feet of the altitudes to BC and CA, respectively. Suppose that segments AD and BE intersect at point H with $AH = 20$ and $HD = 13$. Compute $BD \cdot CD$.

1.4.5 Round 5

13. [14pts] In coordinate space, a *lattice point* is a point all of whose coordinates are integers. The lattice points (x, y, z) in three-dimensional space satisfying $0 \leq x, y, z \leq 5$ are colored in n colors such that any two points that are $\sqrt{3}$ units apart have different colors. Determine the minimum possible value of n.

14. [14pts] Determine the number of ways to express 121 as a sum of strictly increasing positive Fibonacci numbers.

15. [14pts] Let $ABCD$ be a rectangle with $AB = 7$ and $BC = 15$. Equilateral triangles ABP, BCQ, CDR, and DAS are constructed outside the rectangle. Compute the area of quadrilateral $PQRS$.

1.4.6 Round 6

Each of the three problems in this round depends on the answer to one of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.

16. [16pts] Let C be the answer to problem 18. Suppose that x and y are real numbers with $y > 0$ and

$$\begin{cases} x + y = C \\ x + \frac{1}{y} = -2. \end{cases}$$

Compute $y + \frac{1}{y}$.

17. [16pts] Let A be the answer to problem 16. Let PQR be a triangle with $\angle PQR = 90^\circ$, and let X be the foot of the perpendicular from point Q to segment PR. Given that $QX = A$, determine the minimum possible area of triangle PQR.

18. [16pts] Let B be the answer to problem 17 and let $K = 36B$. Alice, Betty, and Charlize are identical triplets, only distinguishable by their hats. Every day, two of them decide to exchange hats. Given that they each have their own hat today, compute the probability that Alice will have her own hat in K days.

1.4.7 Round 7

19. [16pts] Find the number of positive integers a such that all roots of $x^2 + ax + 100$ are real and the sum of their squares is at most 2013.

20. [16pts] Determine all values of k such that the system of equations

$$\begin{cases} y = x^2 - kx + 1 \\ x = y^2 - ky + 1 \end{cases}$$

has a real solution.

21. [16pts] Determine the minimum number of cuts needed to divide an $11 \times 5 \times 3$ block of chocolate into $1 \times 1 \times 1$ pieces. (When a block is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

1.4.8 Round 8

22. [18pts] A sequence that contains the numbers $1, 2, 3, \ldots, n$ exactly once each is said to be a *permutation* of length n. A permutation $w_1 w_2 w_3 \cdots w_n$ is said to be *sad* if there are indices $i < j < k$ such that $w_j > w_k$ and $w_j > w_i$. For example, the permutation 3142**7**5**6** is sad because $7 > 6$ and $7 > 1$. Compute the number of permutations of length 11 that are not sad.

23. [18pts] Let ABC be a triangle with $AB = 39, BC = 56$, and $CA = 35$. Compute $\angle CAB - \angle ABC$ in degrees.

24. [18pts] On a strange planet, there are n cities. Between any pair of cities, there can either be a one-way road, two one-way roads in different directions, or no road at all. Every city has a name, and at the source of every one-way road, there is a signpost with the name of the destination city. In addition, the one-way roads only intersect at cities, but there can be bridges to prevent intersections at non-cities. Fresh Mann has been abducted by one of the aliens, but Sophy Moore knows that he is in Rome, a city that has no roads leading out of it. Also, there is a direct one-way road leading from each other city to Rome. However, Rome is the secret police's name for the so-described city; its official name, the name appearing on the labels of the one-way roads, is unknown to Sophy Moore. Sophy Moore is currently in Athens and she wants to head to Rome in order to rescue Fresh Mann, but she does not know the value of n. Assuming that she tries to minimize the number of roads on which she needs to travel, determine the maximum possible number of roads that she could be forced to travel in order to find Rome. Express your answer as a function of n.

Chapter 2

EMC^2 2013 Solutions

2.1 Individual Speed Test Solutions

1. Determine how many digits the number 10^{10} has.

   Solution. The answer is $\boxed{11}$.

   The number is written as a 1 followed by ten 0's, for a total of 11 digits.

2. Let ABC be a triangle with $\angle ABC = 60^\circ$ and $\angle BCA = 70^\circ$. Compute $\angle CAB$ in degrees.

   Solution. The answer is $\boxed{50^\circ}$.

   The sum of the three angles in a triangle is 180°, and thus $\angle CAB = 180^\circ - 60^\circ - 70^\circ = 50^\circ$.

3. Given that $x : y = 2012 : 2$ and $y : z = 1 : 2013$, compute $x : z$. Express your answer as a common fraction.

   Solution. The answer is $\boxed{\dfrac{1006}{2013}}$.

   We have
   $$\frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} = \frac{2012}{2} \cdot \frac{1}{2013} = \frac{1006}{2013}.$$

4. Determine the smallest perfect square greater than 2400.

   Solution. The answer is $\boxed{2401}$.

   Note that $2401 = 49^2$.

5. At 12:34 and 12:43, the time contains four consecutive digits. Find the next time after 12:43 that the time contains four consecutive digits on a 24-hour digital clock.

   Solution. The answer is $\boxed{13:02 \text{ or } 1:02}$.

   12:03, 12:30, 12:34, and 12:43 are the only times between 12:00 and 12:59, inclusive, that satisfy the condition, and 13:02 is the first time after 12:59 that satisfies the condition.

6. Given that $\sqrt{3^a \cdot 9^a \cdot 3^a} = 81^2$, compute a.

   Solution. The answer is $\boxed{4}$.

   We have $9^4 = 81^2 = \sqrt{3^a \cdot 9^a \cdot 3^a} = \sqrt{81^a} = 9^a$, which implies that $a = 4$.

7. Find the number of positive integers less than 8888 that have a tens digit of 4 and a units digit of 2.

   Solution. The answer is $\boxed{89}$.

   We want to find the number of positive integers less than 8888 that end in 42. The minimum number that satisfies this conditions is 42, which we can write as 0042, and the maximum is 8842. The two digits in front of the 42 can range from 00 to 88, so there are a total of $88 + 1 = 89$ numbers.

8. Find the sum of the distinct prime divisors of $1 + 2012 + 2013 + 2011 \cdot 2013$.

Solution. The answer is $\boxed{75}$.

Grouping terms together appropriately, we have

$$1 + 2012 + 2013 + 2011 \cdot 2013 = 2013 + 2013 + 2011 \cdot 2013 = 2013 \cdot 2013 = 2013^2.$$

Because $2013 = 3 \cdot 11 \cdot 61$, the prime divisors of 2013^2 are 3, 11, and 61, which sum to 75.

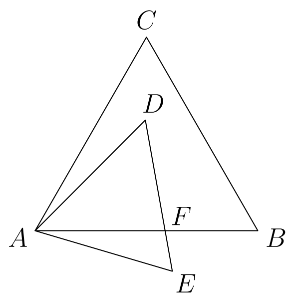
9. Albert wants to make 2×3 wallet sized prints for his grandmother. Find the maximum possible number of prints Albert can make using one 4×7 sheet of paper.

Solution. The answer is $\boxed{4}$.

The area of the sheet of paper is $4 \cdot 7 = 28$, while the area of each wallet sized print is $2 \cdot 3 = 6$. We can cut out at most $\frac{28}{6} = 4\frac{2}{3}$ prints, which implies that we can make at most 4 prints. We can place four 2×3 prints in a 4×6 sheet of paper, which in turn fits inside the 4×7 sheet of paper.

10. Let ABC be an equilateral triangle, and let D be a point inside ABC. Let E be a point such that ADE is an equilateral triangle and suppose that segments DE and AB intersect at point F. Given that $\angle CAD = 15°$, compute $\angle DFB$ in degrees.

Solution. The answer is $\boxed{105°}$.



Because $\angle CAD = 15°$, we have $\angle DAB = 45°$. Considering triangle ADF, the fact that the angles of a triangle sum to $180°$ implies that $\angle DFA = 180° - 45° - 60° = 75°$, which implies that $\angle DFB = 105°$.

11. A *palindrome* is a number that reads the same forwards and backwards; for example, 1221 is a palindrome. An *almost-palindrome* is a number that is not a palindrome but whose first and last digits are equal; for example, 1231 and 1311 are an almost-palindromes, but 1221 is not. Compute the number of 4-digit almost-palindromes.

Solution. The answer is $\boxed{810}$.

Because the first digit cannot be 0, there are 9 choices for the first digit, which is the same as the last digit. The second digit can be any number from 0 to 9, so there are 10 choices for the second digit. Because the middle digits must be different, there are only 9 choices for the third digit. Therefore, there are a total of $9 \times 10 \times 9 = 810$ almost-palindromes.

12. Determine the smallest positive integer n such that the sum of the digits of 11^n is not 2^n.

Solution. The answer is $\boxed{5}$.

For $n = 1, 2, 3, 4$, we have that $11^1 = 11$, $11^2 = 121$, $11^3 = 1331$, $11^4 = 14641$, and in all these cases, the sum of the digits of 11^n is 2^n. For $n = 5$, we have $11^5 = 161051$, whose digits sum to $14 \neq 2^4$.

Remark: In fact, for $n \geq 5$, the sum of the digits of 11^n is strictly less than 2^n. Indeed, by the Binomial Theorem, we have

$$2^n = (1+1)^n = \binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \cdots + \binom{n}{0},$$

and by the Binomial Theorem again we have

$$11^n = (10+1)^n = \binom{n}{n}10^n + \binom{n}{n-1}10^{n-1} + \binom{n}{n-2}10^{n-2} + \cdots + \binom{n}{0}10^0.$$

The above expression for 11^n is the sum of 2^n powers of 10. When we add the powers of 10 and write the result in base 10, some powers of 10 may be grouped together and therefore contribute only 1 to the sum of the digits of 2^n. If this occurs, the sum of the digits of 11^n will be strictly less than 2^n. Because $\binom{n}{2} \geq 10$ for $n \geq 5$ and $\binom{n}{k} < 10$ for all $0 \leq k \leq n \leq 4$, this grouping occurs for $n \geq 5$.

13. Determine the minimum number of breaks needed to divide an 8×4 bar of chocolate into 1×1 pieces. (When a bar is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

    Solution. The answer is $\boxed{5}$.

    Each break multiplies the number of pieces by a factor of at most 2. The final configuration is composed of $32 = 2^5$ pieces, which implies that at least 5 breaks are required. To complete the process in five breaks, we can break the 8×4 block into two 4×4 blocks of chocolate, and then stack the two pieces. Repeating this process gives thirty-two 1×1 pieces after 5 breaks.

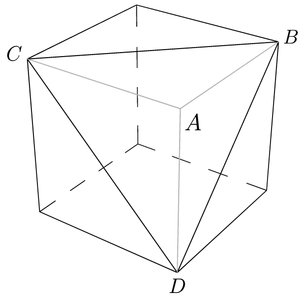
14. A particle starts moving on the number line at a time $t = 0$. Its position on the number line, as a function of time, is $x = (t-2012)^2 - 2012(t-2012) - 2013$. Find the number of positive integer values of t at which time the particle lies in the negative half of the number line (strictly to the left of 0).

    Solution. The answer is $\boxed{2013}$.

    Let $s = t - 2012$. Substituting s into the original equation, we have $x = s^2 - 2012s - 2013 = (s+1)(s-2013)$. The value of x is negative for $-1 < s < 2013$, and thus s can range from 0 to 2012, inclusive. Therefore, thetr are $4024 - 2012 + 1 = 2013$ values of s which correspond to 2013 values of t.

15. Let A be a vertex of a unit cube and let B, C, and D be the vertices adjacent to A. The tetrahedron $ABCD$ is cut off the cube. Determine the surface area of the remaining solid.

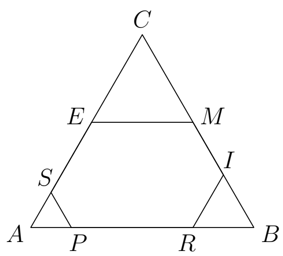
    Solution. The answer is $\boxed{\dfrac{9+\sqrt{3}}{2}}$.



The surface area of a unit cube is $6 \cdot (1 \cdot 1) = 6$. When the tetrahedron $ABCD$ is cut away, the triangles ABC, ABD, and ACD are removed from the surface area, but the equilateral triangle BCD is added to the surface area. The total area of the triangles ABC, ABD, and ACD is $3 \cdot (\frac{1 \cdot 1}{2}) = \frac{3}{2}$. By the Pythagorean Theorem, we have $BC = CD = DB = \sqrt{2}$, which implies that triangle BCD has area $\frac{(\sqrt{2})^2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$. Thus, the surface area of the solid is $6 - \frac{3}{2} + \frac{\sqrt{3}}{2} = \frac{9+\sqrt{3}}{2}$.

16. In equilateral triangle ABC, points P and R lie on segment AB, points I and M lie on segment BC, and points E and S lie on segment CA such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PS = 2$, $RI = 3$, and $ME = 5$, compute the area of hexagon $PRIMES$.

Solution. The answer is $\boxed{\dfrac{83\sqrt{3}}{4}}$.



Because $PRIMES$ is equiangular, the opposite sides of $PRIMES$ must be pairwise parallel; namely, SP and IM are parallel, PR and ME are parallel, and RI and ES are parallel. The first statement implies that SP is parallel to BC, which implies that triangles ABC and APS are similar. Analogously, the triangle ABC is also similar to the triangles RBI and EMC, which implies that all three triangles are equilateral. Given that $PS = 2$, we can compute the area of triangle APS: It is $\frac{PS^2\sqrt{3}}{4} = \sqrt{3}$. Similarly, the area of triangle RBI is $\frac{RI^2\sqrt{3}}{4} = \frac{9\sqrt{3}}{4}$, and the area of triangle EMC is $\frac{ME^2\sqrt{3}}{4} = \frac{25\sqrt{3}}{4}$. We can find the area of the hexagon $PRIMES$ by subtracting the area of the three triangles APS, BIR, CME from the area of ABC. Therefore, $PRIMES$ has area $\frac{121\sqrt{3}}{4} - \sqrt{3} - \frac{9\sqrt{3}}{4} - \frac{25\sqrt{3}}{4} = \frac{83\sqrt{3}}{4}$.

17. Find the smallest odd positive integer with an odd number of positive integer factors, an odd number of distinct prime factors, and an odd number of perfect square factors.

Solution. The answer is $\boxed{81}$.

If a positive integer n has an odd number of positive integer factors, it is a perfect square. Indeed, if we pair a divisor a of n with $\frac{n}{a}$, the only factor that could be paired with itself is $\sqrt{n}$ (the converse also holds). Therefore, a positive integer n satisfying the problem's conditions must be a square. Furthermore, a square b^2 divides n if and only if $\sqrt{b}$ divides $\sqrt{n}$, and thus $\sqrt{n}$ must have an odd number of factors, which implies that $\sqrt{n}$ is itself a perfect square. It follows that n must be a perfect fourth power. The conditions that n is odd and n has an odd number of prime factors yield that $3^4 = 81$, which has one prime factor, is the smallest possible value of n.

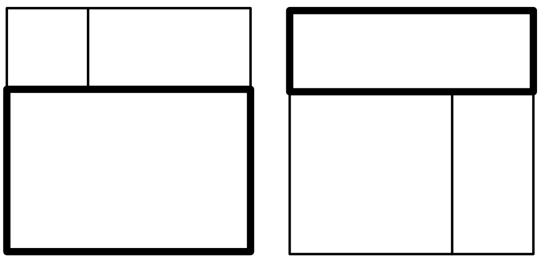
18. Fresh Mann thinks that the expressions $2\sqrt{x^2-4}$ and $2(\sqrt{x^2}-\sqrt{4})$ are equivalent to each other, but the two expressions are not equal to each other for most real numbers x. Find all real numbers x such that $2\sqrt{x^2-4} = 2(\sqrt{x^2}-\sqrt{4})$.

    Solution. The answer is $\boxed{-2 \text{ and } 2}$.

    Notice that if the condition holds for a value x, it also holds for $-x$ because $x^2 = (-x)^2$. Thus, we may restrict our consideration to nonnegative values of x, where the condition reduces to $2\sqrt{x^2-4} = 2x-4$. Dividing both sides by 2 and then squaring, we get: $x^2-4 = (x-2)^2 = x^2-4x+4$, which is equivalent to $4x = 8$, or $x = 2$. Substituting into the original equation, $x = 2$ indeed satisfies the condition. Thus, we have shown that the only nonnegative value of x satisfying the condition is 2, and thus, the only values of x over the real numbers are -2 and 2.

19. Let m be the positive integer such that a 3×3 chessboard can be tiled by at most m pairwise incongruent rectangles with integer side lengths. If rotations and reflections of tilings are considered distinct, suppose that there are n ways to tile the chessboard with m pairwise incongruent rectangles with integer side lengths. Find the product mn.

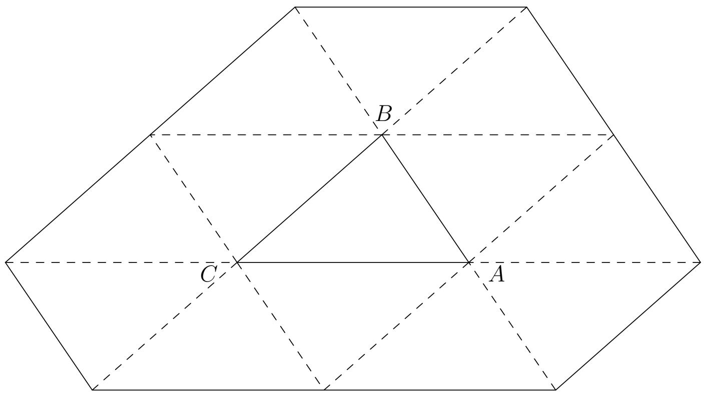
    Solution. The answer is $\boxed{48}$.



First of all, we establish that the rectangles must have sides parallel to those of the chessboard because they must combine to tile the entire chessboard. Next, the vertices of the rectangle must be intersections of lines of the chessboard, because the rectangles have integer side lengths. Finally, we note that we cannot have four rectangles because for each value of n between 1 and 3, inclusive, there is only one rectangle up to congruence with an area n, which implies that four potential pairwise incongruent rectangles have an area of at least $1+2+3+4 = 10$, larger than the area of the chessboard. With this information in mind, we attempt to construct all possible tilings with 3 rectangles. First note that if three vertices of the chessboard are vertices of the same rectangle, then that rectangle must be the entire chessboard. Thus, when tilling the chessboard with 3 rectangles, there must be one rectangle for which two vertices are also vertices of the chessboard. This rectangle can be either a 1×3 or a 2×3 rectangle, and can share any of the four sides of the chessboard. If the rectangle is a 1×3, then we are left to split the remaining 2×3 into 2 rectangles, for which there are only two ways. Similarly, if the rectangle is a 2×3, then we are left to split the remaining 1×3 into 2 rectangles, for which there are also only two ways. Thus, $m = 3$, and $n = 4 \cdot (2+2) = 16$, implying that $mn = 48$.

20. Let ABC be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$. A triangle XYZ is said to be *friendly* if it intersects triangle ABC and it is a translation of triangle ABC. Let S be the set of points in the plane that are inside some friendly triangle. Compute the ratio of the area of S to the area of triangle ABC.

    Solution. The answer is $\boxed{13}$.



See the figure above for a graph of the region S. The region can be tiled by 13 copies of triangle ABC.

2.2 Individual Accuracy Test Solutions

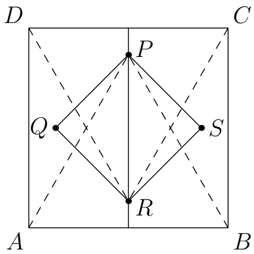
1. Find the largest possible number of consecutive 9's in which an integer between 10,000,000 and 13,371,337 can end. For example, 199 ends in two 9's, while 92,999 ends in three 9's.

   Solution. The answer is $\boxed{6}$.

   The two smallest positive integers that end in at least seven consecutive nines are 9,999,999 and 19,999,999. Because $9,999,999 < 10,000,000 < 13,371,337 < 19,999,999$, an integer between 10,000,000 and 13,371,337 cannot end in seven or more consecutive 9's. There are three integers in these bounds that end in six consecutive 9's: 10,999,999; 11,999,999; and 12,999,999.

2. Let $ABCD$ be a square of side length 2. Equilateral triangles ABP, BCQ, CDR, and DAS are constructed inside the square. Compute the area of quadrilateral $PQRS$.

   Solution. The answer is $\boxed{8 - 4\sqrt{3}}$.

   Because triangle ABP is equilateral, the distance from point P to line AB is $\frac{\sqrt{3}}{2} \cdot AB = \sqrt{3}$. Similarly, the distance from point R to line CD is $\sqrt{3}$, and the distance between lines AB and CD is 2, which implies that $PR = \sqrt{3}+\sqrt{3}-2 = 2\sqrt{3}-2$. Similarly, we have $QS = 2\sqrt{3}-2$. By symmetry, quadrilateral $PQRS$ is a rhombus. Therefore, the area of rhombus $PQRS$ is $\frac{PR \cdot QS}{2} = \frac{(2\sqrt{3}-2)\cdot(2\sqrt{3}-2)}{2} = \frac{16-8\sqrt{3}}{2} = 8 - 4\sqrt{3}$.

3. Evaluate the expression $7 \cdot 11 \cdot 13 \cdot 1003 - 3 \cdot 17 \cdot 59 \cdot 331$.

   Solution. The answer is $\boxed{8024}$.

   Notice that $1003 = 17 \cdot 59$. The expression can be factored as

   $$1003(7 \cdot 11 \cdot 13 - 3 \cdot 331) = 1003 \cdot 8 = 8024.$$

4. Compute the number of positive integers c such that there is a non-degenerate obtuse triangle with side lengths $21, 29$, and c.

   Solution. The answer is $\boxed{25}$.

   Consider a triangle with sides $x \le y \le z$. By the triangle inequality, $x + y > z$. The triangle is obtuse if and only if $x^2 + y^2 < z^2$. We do casework to determine all possible values of c.

   - *Case 1:* $c \le 29$. The following two inequalities must be satisfied:

     $$21 + c > 29$$
     $$21^2 + c^2 < 29^2.$$

     From the first we find that $c > 8$. The second yields $c < 20$. There are 11 possible values of c between 9 and 19 inclusive.

- *Case 2:* $c > 29$. The following two inequalities must be satisfied:

$$21 + 29 > c$$
$$21^2 + 29^2 < c^2.$$

From the first we find that $c < 50$. The second yields $c > \sqrt{1282} > 35$. There are 14 possible values of c between 36 and 49 inclusive.

Combining the two cases, there are $11 + 14 = 25$ possible integers c.

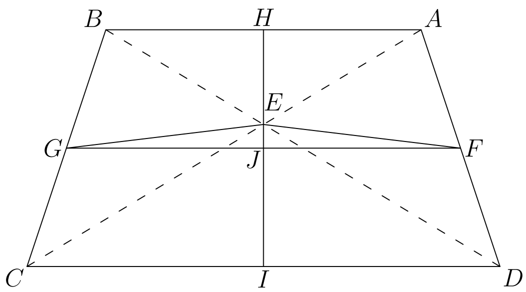
5. Consider a 5×5 board, colored like a chessboard, such that the four corners are black. Determine the number of ways to place 5 rooks on black squares such that no two of the rooks attack one another, given that the rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

Solution. The answer is $\boxed{12}$.

We start by placing the first rook in the leftmost column of the board. There are three choices of a black square. There are two choices of a placement for the in the second column. Neither of these black squares is in a row that could contain the first rook, so they are both valid possibilities. The third column has three black squares. However, there are only two options for the placement of the rook in the third column, since the first rook already occupies the row of one of the three black squares. Similarly, there is $2 - 1 = 1$ option for the rook in the fourth column, as the second rook is already in the row of one of the two black squares in the fourth column. With four rows and four columns already occupied, the fifth rook's position is necessarily on a black square and is determined. Thus, there are $3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 12$ ways to place the rooks.

6. Let $ABCD$ be a trapezoid of height 6 with bases AB and CD. Suppose that $AB = 2$ and $CD = 3$, and let F and G be the midpoints of segments AD and BC, respectively. If diagonals AC and BD intersect at point E, compute the area of triangle FGE.

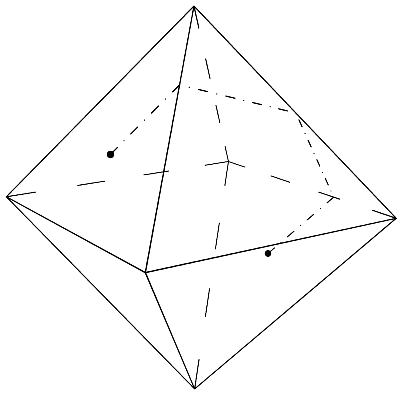
Solution. The answer is $\boxed{\frac{3}{4}}$.

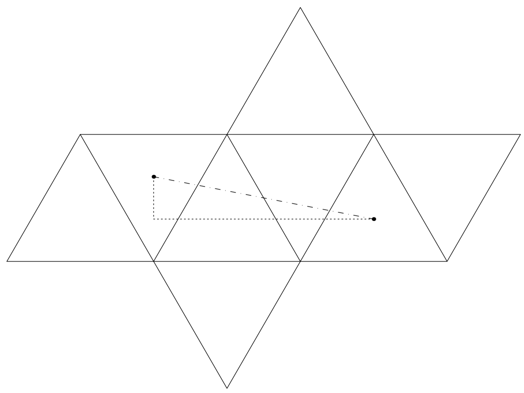


The diagram is not drawn to scale. Because FG is a midline of the trapezoid, it is parallel to the bases AB and CD. Let the line perpendicular to the bases passing through E intersect line AB at H, line FG at J, and line DC at I. Because HI is a height of the trapezoid, we have $HI = 6$. Because FG is a midline of the trapezoid, $HJ = JI = 3$. From the similarity $ABE \sim CDE$, it follows that $\frac{EH}{EI} = \frac{AB}{CD} = \frac{2}{3}$. Thus we have $3 \cdot EH = 2 \cdot EI$, which implies that $6 = EH + EI = \frac{5}{3}EI$, or $EI = \frac{18}{5}$, and therefore we have $EJ = EI - JI = \frac{18}{5} - 3 = \frac{3}{5}$. It follows that the area of triangle FGE is $\frac{EJ \cdot FG}{2} = \frac{\frac{3}{5} \cdot \frac{5}{2}}{2} = \frac{3}{4}$.

7. A *regular octahedron* is a solid with eight faces that are congruent equilateral triangles. Suppose that an ant is at the center of one face of a regular octahedron of edge length 10. The ant wants to walk along the surface of the octahedron to reach the center of the opposite face. (Two faces of an octahedron are said to be opposite if they do not share a vertex.) Determine the minimum possible distance that the ant must walk.

Solution. The answer is $\boxed{\dfrac{10\sqrt{21}}{3}}$.



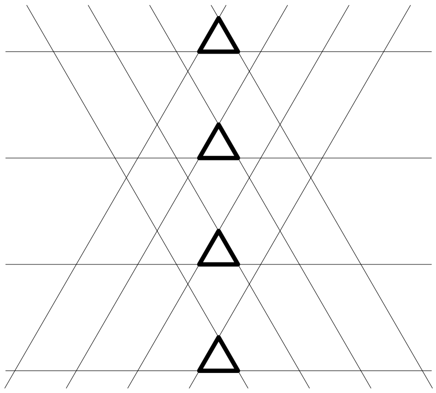


First, we cut the octahedron out into a net, as shown in the right figure above. Considering two opposite faces, we see that they are the outer triangles of a parallelogram made out of four equilateral triangles. The starting and ending points of the ant are the centers of the triangles, and thus we can form a right triangle whose hypotenuse is the line connecting the two points. The base has length $10 + \frac{10}{2} = 10 + 5 = 15$, and the height has length $5\sqrt{3} - 2 \cdot \frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}}$. Therefore, the hypotenuse has length $\sqrt{15^2 + \frac{5}{\sqrt{3}}^2} = \sqrt{225 + \frac{25}{3}} = \sqrt{\frac{700}{3}} = \sqrt{\frac{2100}{9}} = \frac{10\sqrt{21}}{3}$.

8. Let $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$, and $D_1D_2D_3$ be triangles in the plane. All the sides of the four triangles are extended into lines. Determine the maximum number of pairs of these lines that can meet at 60° angles.

Solution. The answer is $\boxed{48}$.

There are two possible slopes that will intersect any given line at a 60° angle, either at 60° or 120° counterclockwise from the line. Therefore, given any line ℓ in the place and any triangle PQR, at most two of the extensions of the sides of triangle PQR can intersect ℓ at a 60° angle. Letting ℓ range over all extensions of the sides of the four triangles and letting PQR range over the four triangles yields at most $2 \cdot 12 \cdot 4$ intersections at 60° angles. However, every intersection will be counted twice; for example, if A_1A_2 and B_1B_2 intersect at a 60° angle, then this will be counted for both $\ell = A_1A_2$ and $\ell = B_1B_2$. Therefore, there are at most $12 \cdot 4 = 48$ such intersections. When the four triangles are congruent equilateral triangles that are oriented in the same way, the diagram below verifies that 48 intersections at 60° angles are possible.



9. For an integer n, let $f_n(x)$ denote the function $f_n(x) = \sqrt{x^2 - 2012x + n} + 1006$. Determine all positive integers a such that $f_a(f_{2012}(x)) = x$ for all $x \geq 2012$.

   Solution. The answer is $\boxed{2,022,060}$.

   By completing the square, we have

   $$f_a(x) = \sqrt{(x - 1006)^2 + n - 1006^2} + 1006.$$

   Simplifying $f_a(f_{2012}(x))$, we have

   $$\begin{aligned} f_a(f_{2012}(x)) &= \sqrt{(f_{2012}(x) - 1006)^2 + a - 1006^2} + 1006 \\ &= \sqrt{\left(\sqrt{(x-1006)^2 + 2012 - 1006^2}\right)^2 + a - 1006^2} + 1006 \\ &= \sqrt{(x - 1006)^2 + a + 2012 - 2 \cdot 1006^2} + 1006. \end{aligned}$$

   In order for $f_a(f_{2012}(x)) = x$, the expression inside the square root must equal $(x - 1006)^2$, which implies that $a + 2012 - 2 \cdot 1006^2 = 0$ or $a = 2 \cdot 1006^2 - 2012 = 2,022,060$.

10. Determine the number of ordered triples of integers (a, b, c) such that $(a + b)(b + c)(c + a) = 1800$.

    Solution. The answer is $\boxed{576}$.

    Let $a + b = x, b + c = y$, and $a + c = z$. Consider the system of equations

    $$\begin{cases} a + b &= x \\ b + c &= y \\ a + c &= z. \end{cases}$$

    By adding the three equations and dividing by 2, we find that

    $$a + b + c = \frac{x + y + z}{2}.$$

    Combined with the first equation, we have $c = \frac{y+z-x}{2}$, and similarly $a = \frac{x+z-y}{2}$ and $b = \frac{x+y-z}{2}$. Therefore, for each possible ordered triple (x, y, z), there exists a unique ordered triple (a, b, c), and $a, b,$and c are integers when $x + y + z$ is even. Thus, $x, y,$ and z must all be even, or exactly two of them must be odd.

- *Case 1:* x, y, and z are all even. Because $1800 = 2^3 \cdot 3^2 \cdot 5^2$, and each of x, y, and z must have a factor of 2, the only factors that remain to be divided among x, y, and z are those of $3^2 \cdot 5^2$. The possible unordered triples, dividing x, y,and z by 2 and ignoring possible negative signs, are (225, 1, 1), (75, 3, 1), (45, 5, 1), (25, 9, 1), (25, 3, 3), (15, 3, 5), (15, 15, 1), and (9, 5, 5). In each unordered triple of three distinct numbers, there are $(6)(4) = 24$ ordered triples, since there are 6 ways to assign the numbers to x, y, and z and $\binom{3}{2} + 1 = 4$ ways to distribute either two or zero negative signs among x, y, and$z$. In each unordered triple of two distinct numbers, there are $(3)(4) = 12$ ordered triples, since there are 3 ways to assign the numbers to x, y, z and $\binom{3}{2} + 1 = 4$ ways to distribute either two or zero negative signs among x, y, and$z$. There are four unordered triples of three distinct numbers and four unordered triples of two distinct numbers for a total of $(4)(24) + (4)(12) = 144$ ordered triples in this case.
- *Case 2:* Exactly two of x, y, and z are odd. Then one of x, y, and z must contain all three factors of 2. Once again, the only factors that remain to be divided among x, y, and$z$ are those of $3^2 \cdot 5^2$. The possible unordered triples, disregarding the factor of 8 in one of x, y, and$z$, are the same as in *Case 1*, as are the number of ways to assign the numbers to x, y, and$z$. There are 3 choices for which of x, y, and$z$ contains the factor of 8, so there are $3 \cdot 144 = 432$ ordered triples in this case.

Thus, there are a total of $144 + 432 = 576$ ordered triples of integers (a, b, c).

2.3 Team Test Solutions

1. Determine the number of ways to place 4 rooks on a 4×4 chessboard such that:

   (a) no two rooks attack one another, and

   (b) the main diagonal (the set of squares marked X below) does not contain any rooks.

X			
	X		
		X	
			X

   The rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

   Solution. The answer is $\boxed{9}$.

   To form a valid board, the rook in the leftmost column can occupy one of 3 places. The rook in the column that has an X in the row of the leftmost rook also has 3 possible places. From there, the placements of the remaining two rooks are determined. The number of boards is thus $3 \cdot 3 = 9$.

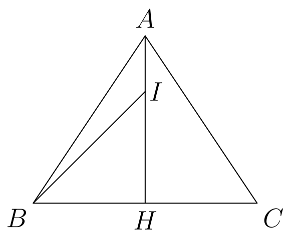
2. Seven students, numbered 1 to 7 in counter-clockwise order, are seated in a circle. Fresh Mann has 100 erasers, and he wants to distribute them to the students, albeit unfairly. Starting with person 1 and proceeding counter-clockwise, Fresh Mann gives i erasers to student i; for example, he gives 1 eraser to student 1, then 2 erasers to student 2, et cetera. He continues around the circle until he does not have enough erasers to give to the next person. At this point, determine the number of erasers that Fresh Mann has.

   Solution. The answer is $\boxed{1}$.

   Fresh Mann distributes $1 + 2 + 3 + \cdots + 7 = \frac{7 \cdot 8}{2} = 28$ erasers each time he walks around the circle. Therefore, after walking around the circle three times, he will have $100 - 3 \cdot 28 = 16$ erasers left. Then, he distributes $1 + 2 + 3 + 4 + 5 = \frac{5 \cdot 6}{2} = 15$ erasers to the first five students. At this point, he has 1 eraser left, but he needs 6 erasers to continue. Thus, he stops with 1 eraser left.

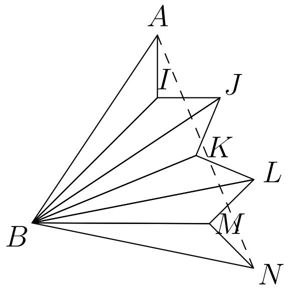
3. Let ABC be a triangle with $AB = AC = 17$ and $BC = 24$. Approximate $\angle ABC$ to the nearest multiple of 10 degrees.

   Solution. The answer is $\boxed{50^\circ}$.



   The diagram is not drawn to scale. Let H be the foot of the altitude from point A to segment BC. Because $\triangle ABC$ is isosceles, $BH = CH = 12$ holds. The Pythagorean theorem then implies that $AH = \sqrt{AB^2 - BH^2} = \sqrt{145}$. Note that

Now construct a point I such that I is on segment AH, and $HI = 12$. It is clear that $\triangle HIB$ is a $45^\circ - 45^\circ - 90^\circ$ right triangle, and thus $\angle IBH = 45^\circ$. We have shown that $\angle ABC > 45^\circ$, and it is intuitively clear that $\angle ABC < 55^\circ$, which implies that the answer is 50°. The remainder of the solution is devoted to a careful proof of the fact that $\angle ABI < 10^\circ$, which implies that $\angle ABC < 55^\circ$.



Let J be the reflection of A across line BI, let K be the reflection of I across line BJ, let L be the reflection of J across line BK, let M be the reflection of K across line BL, and finally, let N be the reflection of L across line BM. Due to the reflections, we have the following congruences:

$$\triangle ABI \cong \triangle JBI \cong \triangle JBK \cong \triangle LBK \cong \triangle LBM \cong \triangle NBM.$$

Hence, we have $\angle ABN = 6 \cdot \angle ABI$.

Furthermore, because

$$AN < AI + IJ + JK + KL + LM + MN = 6\sqrt{145} - 72,$$

we have that $AB > AN$. One may easily verify that $17 > 6\sqrt{145} - 72$. (Because $89^2 = 7921 > 5220 = 36 \cdot 145$, we have that $89 > 6\sqrt{145}$.) Since triangle ABN is isosceles, we have $\angle ABN < 60^\circ$, and thus $\angle ABI < 10^\circ$, as claimed.

Remark: The ratio $\frac{17}{12}$ is extremely close to $\sqrt{2}$. In fact, one can verify that among all rational numbers of denominator at most 12, $\frac{17}{12}$ is closest to $\sqrt{2}$.

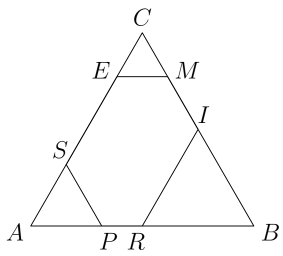
4. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = \frac{3}{5}$ and for $n \geq 1$, $x_{n+1} = 2 - \dfrac{1}{x_n}$. Compute the product $x_1 x_2 x_3 \cdots x_{2013}$.

   Solution. The answer is $\boxed{-\dfrac{4021}{5}}$.

   Consider the new sequence $a_n = x_1 x_2 \cdots x_n$. Then, let $a_0 = 1$. We have that $a_1 = \frac{3}{5}$. Multiplying the recursive relation for $\{x_n\}$ by $x_1 x_2 \cdots x_n$, we have $a_{n+1} = 2a_n - a_{n-1}$, or $a_{n+1} - a_n = a_n - a_{n-1}$. Thus, $\{a_n\}$ is an arithmetic sequence with initial term $a_0 = 1$ and common difference $\frac{3}{5} - 1 = -\frac{2}{5}$, which implies that $x_1 x_2 x_3 \cdots x_{2013} = a_{2013} = 1 - 2013 \cdot \frac{2}{5} = -\frac{4021}{5}$.

5. In equilateral triangle ABC, points P and R lie on segment AB, points I and M lie on segment BC, and points E and S lie on segment CA such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PR = 2$, $IM = 3$, and $ES = 5$, compute the area of hexagon $PRIMES$.

Solution. The answer is $\boxed{\dfrac{289\sqrt{3}}{16}}$.



Note: *For this solution and all succeeding solutions, the area of a polygon $\mathcal{P}$ will be denoted by $[\mathcal{P}]$.*

Because hexagon $PRIMES$ is equiangular, all of its internal angles are $120°$, which implies that all the internal angles of triangles APS, BIR, CEM are $60°$. It follows that triangles APS, BIR, and CEM are equilateral. Let $AP = AS = x, BI = BR = y$, and $CE = CM = z$. Then, we have $11 = AB = AP + PR + RB = x + y + 2$, which implies that $x + y = 9$. Similarly, we have $y + z = 8$ and $z + x = 6$. Adding the three equations together and dividing by 2 yields that $x + y + z = \frac{23}{2}$. It follows that $x = (x + y + z) - (y + z) = \frac{23}{2} - 8 = \frac{7}{2}$, and similarly we have $y = \frac{23}{2} - 6 = \frac{11}{2}$ and $z = \frac{23}{2} - 9 = \frac{5}{2}$. Therefore, the area of hexagon $PRIMES$ is

$$\begin{aligned}[ABC] - ([APS] + [BIR] + [CEM]) &= \frac{\sqrt{3}}{4}\left(AB^2 - (x^2 + y^2 + z^2)\right) = \frac{\sqrt{3}}{4}\left(121 - \frac{195}{4}\right) \\ &= \frac{\sqrt{3}}{4} \cdot \frac{289}{4} = \frac{289\sqrt{3}}{16}.\end{aligned}$$

6. Let $f(a, b) = \frac{a^2}{a+b}$. Let A denote the sum of $f(i, j)$ over all pairs of integers (i, j) with $1 \le i < j \le 10$; that is,

$$A = (f(1, 2) + f(1, 3) + \cdots + f(1, 10)) + (f(2, 3) + f(2, 4) + \cdots + f(2, 10)) + \cdots + f(9, 10).$$

Similarly, let B denote the sum of $f(i, j)$ over all pairs of integers (i, j) with $1 \le j < i \le 10$; that is,

$$B = (f(2, 1) + f(3, 1) + \cdots + f(10, 1)) + (f(3, 2) + f(4, 2) + \cdots + f(10, 2)) + \cdots + f(10, 9).$$

Compute $B - A$.

Solution. The answer is $\boxed{165}$.

Define $g(a, b) = f(b, a) - f(a, b) = \frac{b^2}{a+b} - \frac{a^2}{a+b} = b - a$. The value $B - A$ is then the sum of $g(i, j)$ over all pairs of integers (i, j) with $1 \le i < j \le 10$. For k an integer between 1 and 9, inclusive, k appears in the sum as $g(i, j)$ for $10 - k$ pairs (i, j), namely $(1, k + 1), (2, k + 2), \ldots, (10 - k, 10)$. Therefore, we have $B - A = 1 \cdot 9 + 2 \cdot 8 + \cdots + 9 \cdot 1 = 165$.

7. Fresh Mann has a pile of seven rocks with weights 1, 1, 2, 4, 8, 16, and 32 pounds and some integer X between 1 and 64, inclusive. He would like to choose a set of the rocks whose total weight is exactly X pounds. Given that he can do so in more than one way, determine the sum of all possible values of X. (The two 1-pound rocks are indistinguishable.)

Solution. The answer is $\boxed{992}$.

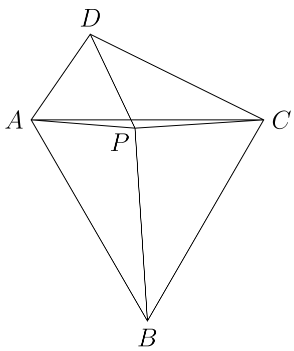
Clearly, 64 can only be achieved in one way: pick all the rocks.

With the seven rocks form a "big pile" consisting of rocks with weights 1, 2, 4, 8, 16, and 32 pounds and a "little pile" consisting of just the other 1-pound rock. The numbers 1 to 63 have a unique 7-digit representation in base two, and thus there is exactly one way to pick from only the big pile to achieve every total weight between 1 and 63. It follows that any other choice of rocks must include the 1-pound rock from the little pile. All odd total weights from 1 to 63 must be achieved using exactly one 1-pound rock, and using the 1-pound rock from the little pile is equivalent to using the 1-pound rock from the big pile, which yields the same set of rocks. Thus, each odd weight can be achieved in only one way.

However, for any even total weight of Y pounds, there is a unique way to achieve $Y - 1$ pounds using a set of rocks solely from the big pile, including the 1-pound rock (since $Y - 1$ is odd). Using these rocks in addition to the 1-pound rock from the little pile, we can form a new set of rocks with total weight Y (which uses both 1-pound rocks). Therefore, the numbers that can be represented in more than one way are the even numbers that are between 2 and 62, inclusive, which sum to $2 + 4 + \cdots + 62 = 2(1 + 2 + \cdots + 31) = 31 \cdot 32 = 992$.

8. Let $ABCD$ be a convex quadrilateral with $AB = BC = CA$. Suppose that point P lies inside the quadrilateral with $AP = PD = DA$ and $\angle PCD = 30°$. Given that $CP = 2$ and $CD = 3$, compute CA.

Solution. The answer is $\boxed{\sqrt{13}}$.



Draw segment BP. Because triangles ABC and APD are equilateral, we have that $\angle PAB = 60° - \angle CAP = \angle DAC$. Therefore, $AB = AC$ and $AP = AD$ imply that triangles ABP and ACD are congruent by SAS. It follows that $BP = 3$, and $\angle APB = \angle ADC$. Because the angles of a convex quadrilateral sum to 360°, we have

$$\angle BPC = 360° - \angle APB - \angle CPA = 360° - \angle ADC - \angle CPA = \angle DAP + \angle PCD = 90°.$$

By the Pythagorean Theorem, we have $CA = CB = \sqrt{2^2 + 3^2} = \sqrt{13}$.

9. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = 2$, $x_2 = \frac{13}{2}$, and for $n \geq 1$,

$$x_{n+2} = 3 - \frac{3}{x_{n+1}} + \frac{1}{x_n x_{n+1}}.$$

Compute x_{100}.

Solution. The answer is $\boxed{\dfrac{49601}{48610}}$.

This problem is similar to problem 3 of the Team Round. Once again, let $a_n = x_1x_2x_3\cdots x_n$, and let $a_0 = 1$. We have that $a_1 = 2$ and $a_2 = 13$. Multiplying the recursion by $x_1x_2x_3\cdots x_{n+1}$ and rearranging the terms, we have $a_{n+2} - 3a_{n+1} + 3a_n - a_{n-1} = 0$. We claim that there are real numbers a, b, c such that $a_n = an^2 + bn + c$ for all n. Assuming that this holds, the initial values of a_n yield that $(a, b, c) = (5, -4, 1)$, which would imply that

$$x_{2013} = \frac{a_{2013}}{a_{2012}} = \frac{5 \cdot 100^2 - 4 \cdot 100 + 1}{5 \cdot 99^2 - 4 \cdot 99 + 1} = \frac{49601}{48610}.$$

Consider the sequence $a'_n = 5n^2 - 4n + 1$, and we will prove that $a_n = a'_n$ for all n. This clearly holds for $n = 0, 1$, and 2. Then, it suffices to prove that $a'_{m+2} - 3a'_{m+1} + 3a'_m - a'_{m-1} = 0$ for all $m \geq 1$, which would imply that for all $n \geq 3$, $a_n = a'_n$. We compute the above expression progressively; we have $a'_m - a'_{m-1} = 5(m^2 - (m-1)^2) - 4 = 10m - 9$. This implies that $a'_{m+1} - 2a'_m + a'_{m-1} = (a'_{m+1} - a'_m) + (a'_m - a'_{m-1}) = 10$, which yields that $a'_{m+2} - 3a'_{m+1} + 3a'_m - a'_{m-1} = (a'_{m+2} - 2a'_{m+1} + a'_m) - (a'_{m+1} - 2a'_m + a'_{m-1}) = 0$, as claimed.

Remark: For a sequence $\{a_n\}$ define the sequence $\Delta\{a_n\} = \{a_1 - a_0, a_2 - a_1, \ldots\}$. The operator Δ is called the *finite difference operator*. We showed that the sequences satisfying $\Delta^3\{a_n\} = \{0, 0, \ldots\}$ are exactly those that satisfy $a_n = f(n)$ for some quadratic function n. In fact, $\Delta^{k+1}\{a_n\} = \{0, 0, \ldots\}$ holds if and only if $a_n = f(n)$ for some polynomial f of degree at most k.

10. Ten prisoners are standing in a line. A prison guard wants to place a hat on each prisoner. He has two colors of hats, red and blue, and he has 10 hats of each color. Determine the number of ways in which the prison guard can place hats such that among any set of consecutive prisoners, the number of prisoners with red hats and the number of prisoners with blue hats differ by at most 2.

Solution. The answer is $\boxed{94}$.

For $0 \leq n \leq 10$, let $f(n)$ be the number of prisoners among the first n prisoners who are wearing red hats minus the number wearing blue hats. For the rest of this problem, assume that $a < b$. Notice that $f(b) - f(a)$ is the number of prisoners wearing red hats minus the number of prisoners wearing blue hats, among the prisoners $a+1, a+2, a+3, \ldots, b$. It follows from the definition of f that $|f(n) - f(n-1)| = 1$ for all n. We want to find the number of hat configurations such that f takes on at most three consecutive values: indeed, if $|f(b) - f(a)| > 2$, then among the prisoners $a+1, a+2, a+3, \ldots, b$, the number of prisoners with red hats and the number of prisoners with blue hats differ by more than 2.

We claim that there are 32 choices of f such that $-2 \leq f(n) \leq 0$. Indeed, if $f(n) = 0$ or $f(n) = -2$, then we must have $f(n+1) = -1$, and if $f(n) = -1$, we have two choices for $f(n+1)$. It follows that we make five choices in determining $f(1), f(2), f(3), \ldots, f(10)$, which yields $2^5 = 32$ functions f. Similarly, there are 32 functions f with $-1 \leq f(n) \leq 1$ and 32 functions f with $0 \leq f(n) \leq 2$. However, we double-count two functions: the function

$$g(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -1 & \text{otherwise} \end{cases}$$

is counted both in the case $-2 \leq f(n) \leq 0$ and the case $-1 \leq f(n) \leq 1$, while $-g(n)$ is counted both in the case $-1 \leq f(n) \leq 1$ and the case $0 \leq f(n) \leq 2$, and these are the only functions that are counted more than once. Thus, there are $3 \cdot 32 - 2 = 94$ functions f, which yields 94 hat configurations.

2.4 Guts Test Solutions

2.4.1 Round 1

1. [6pts] Five girls and three boys are sitting in a room. Suppose that four of the children live in California. Determine the maximum possible number of girls that could live somewhere outside California.

   Solution. The answer is $\boxed{4}$.

   Because there are eight students in total, four of whom live in California, exactly four must live outside California. Hence at most four girls live outside California. It is indeed possible for four girls to live outside California when one girl and three boys live in California, and thus the answer is 4.

2. [6pts] A 4-meter long stick is rotated $60°$ about a point on the stick 1 meter away from one of its ends. Compute the positive difference between the distances traveled by the two endpoints of the stick, in meters.

   Solution. The answer is $\boxed{\dfrac{2\pi}{3}}$.

   The endpoint of the stick that is closer to the center of rotation traces out a $60°$ arc of a circle of radius 1 meter, and thus the point travels $\frac{1}{6} \cdot 2 \cdot \pi$ meters. The farther end traces out a $60°$ arc of radius of 3 meters, and therefore the point travels $\frac{1}{6} \cdot 2 \cdot 3 \cdot \pi$ meters. The difference is $\frac{6\pi}{6} - \frac{2\pi}{6} = \frac{2\pi}{3}$.

3. [6pts] Let
   $$f(x) = 2x(x-1)^2 + x^3(x-2)^2 + 10(x-1)^3(x-2).$$
   Compute $f(0) + f(1) + f(2)$.

   Solution. The answer is $\boxed{25}$.

   Conveniently, when we input the values into the function, exactly two of the three terms always become 0. Therefore, we have $f(0) = (10)(-1)^3(-2) = 20$, $f(1) = (1)^3(-1)^2 = 1$, and $f(2) = (2)(2)(1) = 4$, which implies that $f(0) + f(1) + f(2) = 20 + 1 + 4 = 25$.

2.4.2 Round 2

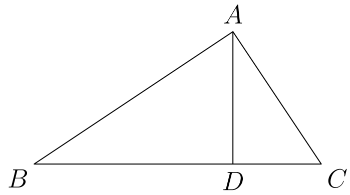
4. [8pts] Twenty boxes with weights $10, 20, 30, \ldots, 200$ pounds are given. One hand is needed to lift a box for every 10 pounds it weighs. For example, a 40 pound box needs four hands to be lifted. Determine the number of people needed to lift all the boxes simultaneously, given that no person can help lift more than one box at a time.

   Solution. The answer is $\boxed{110}$.

   The 10 pound box needs one hand to lift, which requires one person, while the 20 pound box needs two hands, which also requires one person. The 30 and 40 pound boxes need 3 and 4 hands respectively, so each needs 2 people to lift. In general, the $10(2k-1)$ pound and $10(2k)$ pound boxes each require k people to lift. Therefore, the answer is is $1 + 1 + 2 + 2 + 3 + 3 + \cdots + 10 + 10 = 2\left(\frac{10 \cdot 11}{2}\right) = 110$.

5. [8pts] Let ABC be a right triangle with a right angle at A, and let D be the foot of the perpendicular from vertex A to side BC. If $AB = 5$ and $BC = 7$, compute the length of segment AD.

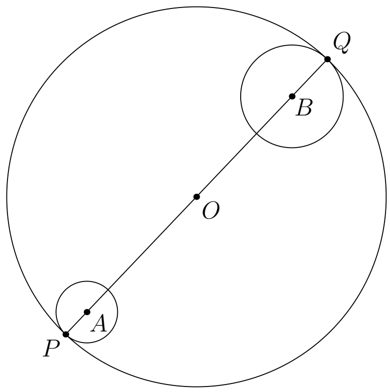
Solution. The answer is $\boxed{\dfrac{10\sqrt{6}}{7}}$.



By the Pythagorean Theorem, we have $AC = \sqrt{7^2 - 5^2} = \sqrt{24} = 2\sqrt{6}$. We compute the area of triangle ABC in two ways; it equals $\frac{1}{2}AB \cdot AC = 5\sqrt{6}$, and it equals $\frac{1}{2}AD \cdot BC = \frac{7}{2}AD$. Setting the two expressions equal yields that $AD = \frac{10\sqrt{6}}{7}$.

6. [8pts] There are two circular ant holes in the coordinate plane. One has center $(0, 0)$ and radius 3, and the other has center $(20, 21)$ and radius 5. Albert wants to cover both of them completely with a circular bowl. Determine the minimum possible radius of the circular bowl.

Solution. The answer is $\boxed{\dfrac{37}{2}}$.



Let the centers $A = (0, 0)$ and $B = (20, 21)$. Let line AB intersect circle A at P and circle B at Q so that both centers lie between P and Q. Note that $PQ = PA + AB + BQ = 3 + \sqrt{20^2 + 21^2} + 5 = 37$, so the radius of the bowl must be at least $\frac{37}{2}$. Conversely, a circle centered at the midpoint of segment PQ of radius $\frac{37}{2}$ covers both holes.

2.4.3 Round 3

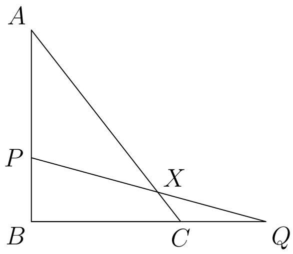
7. [10pts] A line of slope -4 forms a right triangle with the positive x and y axes. If the area of the triangle is 2013, find the square of the length of the hypotenuse of the triangle.

Solution. The answer is $\boxed{\dfrac{34221}{2}}$.

Let the line intersect the x axis at $(k, 0)$. Because it has slope -4, the line must intersect the y axis at $(0, 4k)$. The right triangle then has legs of lengths k and $4k$. Because the area of the triangle is 2013, we have $\frac{k \cdot 4k}{2} = 2013$, or $k^2 = \frac{2013}{2}$. By the Pythagorean Theorem, the square of the length of the hypotenuse is the sum of the squares of its legs, which is $k^2 + (4k)^2 = 17k^2 = \frac{34221}{2}$.

8. [10pts] Let ABC be a right triangle with a right angle at B, $AB = 9$, and $BC = 7$. Suppose that point P lies on segment AB with $AP = 3$ and that point Q lies on ray BC with $BQ = 11$. Let segments AC and PQ intersect at point X. Compute the positive difference between the areas of triangles APX and CQX.

Solution. The answer is $\boxed{\frac{3}{2}}$.



Note that the area of triangle PQB is $\frac{1}{2}BP \cdot BQ = 33$ and the area of triangle ABC is $\frac{1}{2}AB \cdot BC = \frac{63}{2}$. It follows that

$$\begin{aligned}
[CQX] - [APX] &= ([PQB] - [BPXC]) + ([ABC] - [BPXC]) \\
&= [PQB] - [ABC] \\
&= 33 - \frac{63}{2} = \frac{3}{2}.
\end{aligned}$$

9. [10pts] Fresh Mann and Sophy Moore are racing each other in a river. Fresh Mann swims downstream, while Sophy Moore swims $\frac{1}{2}$ mile upstream and then travels downstream in a boat. They start at the same time, and they reach the finish line 1 mile downstream of the starting point simultaneously. If Fresh Mann and Sophy Moore both swim at 1 mile per hour in still water and the boat travels at 10 miles per hour in still water, find the speed of the current.

Solution. The answer is $\boxed{\frac{7}{29}}$.

Let the speed of the current be c miles per hour. Then, it takes $\frac{1}{1+c}$ hours for Fresh Mann to swim 1 mile downstream at a rate of $1 + c$ miles per hour, so it must also take Sophy Moore the same amount of time to complete her journey. Her first leg swimming upstream 0.5 miles at $1 - c$ miles per hour takes $\frac{0.5}{1-c}$ hours and her second leg boating downstream 1.5 miles takes $\frac{1.5}{10+c}$, so we must have

$$\frac{1}{1+c} = \frac{0.5}{1-c} + \frac{1.5}{10+c}.$$

We solve this equation to find c:

$$\begin{aligned}
\frac{2}{1+c} &= \frac{1}{1-c} + \frac{3}{10+c} \\
2(1-c)(10+c) &= (1+c)(10+c) + 3(1+c)(1-c) \\
20 - 18c - 2c^2 &= (10 + 11c + c^2) + (3 - 3c^2) \\
29c &= 7 \\
c &= \frac{7}{29}.
\end{aligned}$$

2.4.4 Round 4

10. [12pts] The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. The first few terms of the Fibonacci sequence are $0, 1, 1, 2, 3, 5, 8, 13$. Every positive integer can be expressed as the sum of nonconsecutive, distinct, positive Fibonacci numbers; for example, $7 = 5 + 2$. Express 121 as the sum of nonconsecutive, distinct, positive Fibonacci numbers. (It is not permitted to use both a 2 and a 1 in the expression.)

Solution. The answer is $\boxed{89 + 21 + 8 + 3}$.

The Fibonacci numbers less than 121 are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, and 89 in order. We consider whether we can include or exclude certain numbers, starting from 89 and working downward.

If we don't use 89, the largest sum we can get is $55 + 21 + 8 + 3 + 1 = 88 < 121$, and thus we must use 89. The remaining Fibonacci numbers must sum to 32. We cannot include 34 or 55 because they are both greater than 32. If we don't use 21, then the total sum is at most $89 + 13 + 5 + 2 + 1 = 110 < 121$, and thus we have to use 21. The remaining of the summands must sum $32 - 21 = 11$.

We cannot include 13 because it is too large. If we don't use 8, then the total sum is at most $89 + 21 + 5 + 3 + 1 = 119 < 121$, and thus use 8. This leaves $11 - 8 = 3$ for the numbers 1, 1, 2, and 3. Since 2 and 1 are consecutive, we must use the 3. This yields the unique expression $121 = 89+21+8+3$.

Remark: Zeckendorff's Theorem states that every positive integer can be expressed as a sum of nonconsecutive, distinct, positive Fibonacci numbers, and that such an expression is unique. The solution presented above follows the theorem's classical proof.

11. [12pts] There is a rectangular box of surface area 44 whose space diagonals have length 10. Find the sum of the lengths of all the edges of the box.

Solution. The answer is $\boxed{48}$.

Let the rectangular box have edge lengths a, b, and c. Then, the surface area is

$$2ab + 2bc + 2ca = 44$$

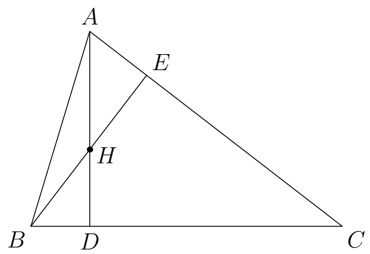
and the space diagonal is $\sqrt{a^2 + b^2 + c^2} = 10$, so

$$a^2 + b^2 + c^2 = 100.$$

Adding the two equations yields $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a + b + c)^2 = 144$, which implies that $a + b + c = \sqrt{144} = 12$. The sum of the lengths of the edges is $4(a + b + c) = 48$.

12. [12pts] Let ABC be an acute triangle, and let D and E be the feet of the altitudes to BC and CA, respectively. Suppose that segments AD and BE intersect at point H with $AH = 20$ and $HD = 13$. Compute $BD \cdot CD$.

    Solution. The answer is $\boxed{429}$.



We have $AD = AH + HD = 33$. Triangles BDH and ADC are similar by AA similarity, from which it follows that $\frac{CD}{AD} = \frac{HD}{BD}$. Cross multiplying, we have

$$BD \cdot CD = AD \cdot HD = 33 \cdot 13 = 429.$$

2.4.5 Round 5

13. [14pts] In coordinate space, a *lattice point* is a point all of whose coordinates are integers. The lattice points (x, y, z) in three-dimensional space satisfying $0 \le x, y, z \le 5$ are colored in n colors such that any two points that are $\sqrt{3}$ units apart have different colors. Determine the minimum possible value of n.

    Solution. The answer is $\boxed{2}$.

    The points $(0, 0, 0)$ and $(1, 1, 1)$ are $\sqrt{3}$ units apart, and thus they must have different colors, which implies that $n \ge 2$. To construct a coloring for $n = 2$, color the points with $z = 0, 2, 4$ black and color the points with $z = 1, 3, 5$ white. The distance between any two lattice points is of the form $\sqrt{a^2 + b^2 + c^2}$ for some integers a, b, c, and $\sqrt{3}$ can only be achieved when $a = b = c = 1$. However, if the z-coordinates of two points differ by 1, then the points are different colors in the given coloring.

    Another possible construction is to color a point (x, y, z) black if $x+y+z$ is odd and white if $x+y+z$ is even.

14. [14pts] Determine the number of ways to express 121 as a sum of strictly increasing positive Fibonacci numbers.

    Solution. The answer is $\boxed{8}$.

    Let $(f_1, f_2, \dots, f_{10}) = (1, 2, 3, 5, 8, 13, 21, 34, 55, 89)$ be the first ten Fibonacci numbers (excluding the first 1 because it is repeated). We have some observations:

    (a) If 89 is one of the summands, neither 34 nor 55 can be chosen (otherwise the sum will be over 121). If 89 is not one of the summands, 34 and 55 are both required to be chosen (otherwise the sum will be less than 121). Because $34 + 55 = 89$, we know that the numbers chosen from $f_1, f_2, \dots, f_7$ must add up to $121 - 89 = 32$.

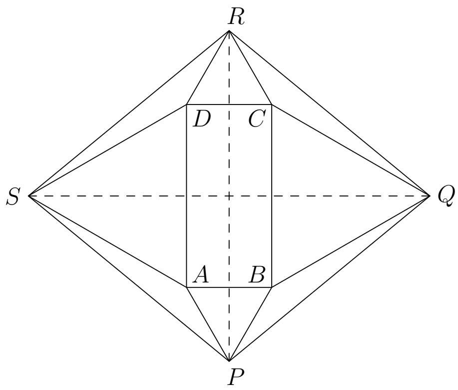
(b) If $f_7 = 21$ is not chosen, then because $f_1 + f_2 + f_3 + f_4 + f_5 + f_6 = 32$, and thus we must use all of $f_1, \dots, f_6$.

(c) If $f_7 = 21$ is chosen, then $f_6 = 13$ can not be one of the summands because $21 + 13 = 34 > 32$. We can list out the remaining ways to express 32: $(1, 2, 3, 5, 21)$, $(1, 2, 8, 21)$, and $(3, 8, 21)$.

From part (a), we have two ways to choose summands form the first 89 of the sum. For either case, we can form the remaining 32 without $f_7 = 21$ in one way and with $f_7 = 21$ in three ways. This gives a total of $2 \cdot (3 + 1) = 8$ solutions.

15. [14pts] Let $ABCD$ be a rectangle with $AB = 7$ and $BC = 15$. Equilateral triangles ABP, BCQ, CDR, and DAS are constructed outside the rectangle. Compute the area of quadrilateral $PQRS$.

Solution. The answer is $\boxed{210 + 137\sqrt{3}}$.



By symmetry, quadrilateral $PQRS$ is a rhombus. The heights of equilateral triangles ABP and CDR are both $h_1 = \frac{7\sqrt{3}}{2}$, and the heights of equilateral triangles BCQ and DAS are both $h_2 = \frac{15\sqrt{3}}{2}$. Therefore, we have $PR = 15 + 2h_1 = 15 + 7\sqrt{3}$ and $QS = 7 + 2h_2 = 7 + 15\sqrt{3}$. The area of $PQRS$ is $\frac{1}{2}PR \cdot QS = \frac{(7+15\sqrt{3})(15+7\sqrt{3})}{2} = 210 + 137\sqrt{3}$.

2.4.6 Round 6

Each of the three problems in this round depends on the answer to one of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.

16. [16pts] Let C be the answer to problem 18. Suppose that x and y are real numbers with $y > 0$ and

$$\begin{cases} x + y = C \\ x + \frac{1}{y} = -2. \end{cases}$$

Compute $y + \frac{1}{y}$.

Solution. The answer is $\boxed{\dfrac{\sqrt{85}}{3}}$.

Subtracting the two equations, we have $y-\frac{1}{y}=C+2$. Squaring then yields that $y^2-2+\frac{1}{y^2}=(C+2)^2$, from which it follows that $y^2+2+\frac{1}{y^2}=(C+2)^2+4$. Taking square roots of both sides yields that $y+\frac{1}{y}=\sqrt{(C+2)^2+4}$ because $y>0$. Substituting $C=\frac{1}{3}$ from problem 18 yields an answer of $\frac{\sqrt{85}}{3}$.

17. [16pts] Let A be the answer to problem 16. Let PQR be a triangle with $\angle PQR=90^\circ$, and let X be the foot of the perpendicular from point Q to segment PR. Given that $QX=A$, determine the minimum possible area of triangle PQR.

    Solution. The answer is $\boxed{\dfrac{85}{9}}$.

    Because $\angle PQX=\angle QRX$ and $\angle QPX=\angle RQX$, triangles PQX and QRX are similar, which implies that $\frac{PX}{XQ}=\frac{XQ}{XR}$. It follows that $PX\cdot XR=XQ^2=A^2$. Because the square of any real number is nonnegative, we have

$$\begin{aligned}\left(\sqrt{PX}-\sqrt{XR}\right)^2&\geq 0\\ PX-2\sqrt{PX\cdot XR}+XR&\geq 0\\ PR=PX+XR&\geq 2\sqrt{A^2}\end{aligned}$$

    Therefore, the area of $\triangle PQR$ is $\frac{1}{2}PR\cdot QX\leq\frac{1}{2}2A\cdot A=A^2$. Equality is achieved at $PX=XR$, and therefore the answer is A^2. Substituting $A=\frac{1}{3}$ from problem 16 yields an answer of $\frac{85}{9}$.

18. [16pts] Let B be the answer to problem 17 and let $K=36B$. Alice, Betty, and Charlize are identical triplets, only distinguishable by their hats. Every day, two of them decide to exchange hats. Given that they each have their own hat today, compute the probability that Alice will have her own hat in K days.

    Solution. The answer is $\boxed{\dfrac{1}{3}}$.

    Each day, one of the following three exchanges happens: Alice and Betty switch hats, Betty and Charlize switch hats, or Charlize and Alice switch hats. If Alice has her own hat on day $K-1$, there is a $\frac{1}{3}$ probability of Alice keeping her own hat on day K, namely when Betty and Charlize switch hats that day. If Alice doesn't have her own hat on day $K-1$, there is also a $\frac{1}{3}$ probability that she gets her hat on day K, namely when Alice switches with the girl who had her hat on day $K-1$. Thus, Alice has a $\frac{1}{3}$ chance of having her own hat on day K. This value is independent of B, provided that $K>0$. Indeed, B is the area of a non-degenerate triangle, which must be positive.

2.4.7 Round 7

19. [16pts] Find the number of positive integers a such that all roots of $x^2+ax+100$ are real and the sum of their squares is at most 2013.

    Solution. The answer is $\boxed{28}$.

    Let $x^2+ax+100=(x-m)(x-n)$ where m and n are the real roots of the original quadratic equation. After expanding the right-hand side, we equate the linear and constant terms, which yields $m+n=-a$

and $mn = 100$. Thus, we have $a^2 - 200 = (m+n)^2 - 2mn = m^2 + n^2 \geq 2013$, according to the problem's condition. Therefore, we have $a^2 \geq 2213$. Because m and n are real, the discriminant is nonnegative, which implies that $a^2 - 400 \geq 0$, or $a^2 \geq 400$. As a result, we have $400 \leq a^2 \geq 2213$, or $20 \leq a \leq 47$, which yields 28 possible values for a. Conversely, if $20 \leq a \leq 47$, then the discriminant is positive and $m^2 + n^2 \leq 2013$.

20. [16pts] Determine all values of k such that the system of equations

$$\begin{cases} y = x^2 - kx + 1 \\ x = y^2 - ky + 1 \end{cases}$$

has a real solution.

Solution. The answer is $\boxed{k \leq -3 \text{ or } k \geq 1}$.

The graphs of the two equations are parabolas that are reflections of one another over the line $y = x$. If the parabola $y = x^2 - kx + 1$ does not intersect the line $y = x$, then the parabola lies completely above the line, which implies that the graph of $x = y^2 - ky + 1$ lies below the line. Therefore, the two parabolas cannot intersect. Conversely, if the parabola $y = x^2 - kx + 1$ intersects the line $y = x$ at a point, then that point lies on the other parabola as well. It follows that the two parabolas intersect if and only if the quadratic equation $x = x^2 - kx + 1$ has a real solution. This is equivalent to $x^2 - (k+1)x + 1 = 0$ having a real solution, which occurs if and only if the discriminant $\Delta = (k+1)^2 - 4$ is nonnegative. Therefore, the two parabolas intersect if and only if $|k+1| \geq 2$, which occurs when $k \geq 1$ and when $k \leq -3$.

21. [16pts] Determine the minimum number of cuts needed to divide an $11 \times 5 \times 3$ block of chocolate into $1 \times 1 \times 1$ pieces. (When a block is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

Solution. The answer is $\boxed{9}$.

We make the following more general claim: Given an $a \times b \times c$ block, at least $p + q + r$ cuts are needed to break it into unit cubes, where p, q, and r are integers such that $2^{p-1} < a \leq 2^p$, $2^{q-1} < b \leq 2^q$, and $2^{r-1} < c \leq 2^r$.

We prove the claim by induction on $a + b + c$.. For the base case of a $1 \times 1 \times 1$ block, we have $a = b = c = 1$, which implies that $p + q + r = 0$; indeed, zero cuts are required to turn this block into a unit cube.

Now, we assume that our claim is true for all blocks with sum of dimensions less than or equal to $n - 1$, and consider an $a \times b \times c$ block with $a + b + c = n$. Note that the first cut will produce two blocks, both having two dimensions the same as the original and the other dimension split. Without loss of generality, suppose that the edge of length a is split. Then, one of the two blocks has dimensions $d \times b \times c$ with $d \geq \frac{a}{2}$. Let s be the integer such that $2^{s-1} < d \leq 2^s$. Because $d \geq \frac{a}{2}$, we have $d > 2^{p-2}$, which implies that $s > p - 2$. By the inductive hypothesis, at least $s + q + r \geq p + q + r - 1$ cuts are required to divide the $d \times b \times c$ block into unit cubes. In addition to the initial cut, this implies at least $p + q + r$ cuts are required to divide the $a \times b \times c$ block into unit cubes, which completes the proof of the inductive step.

Applying the above claim to our problem, we see that it takes at least $4 + 3 + 2 = 9$ cuts to fully divide an $11 \times 5 \times 3$ block. We can also show that this minimum is attainable, by the following process:

(a) Cut the $11 \times 5 \times 3$ into three blocks of $11 \times 5 \times 1$. This uses two cuts.

(b) Stack the $11 \times 5 \times 1$ blocks together, and now cut each of them into two $11 \times 2 \times 1$ blocks and one $11 \times 1 \times 1$ block. This uses two cuts.

(c) Stack the six resulting $11 \times 2 \times 1$ blocks together, and cut them to form twelve $11 \times 1 \times 1$ blocks , for a grand total of fifteen $11 \times 1 \times 1$ blocks. This uses one cut.

(d) Stack all fifteen blocks together, and cut each of them into two $4 \times 1 \times 1$ blocks and one $3 \times 1 \times 1$ block. This uses two cuts.

(e) Using only one cut, we splice the thirty $4 \times 1 \times 1$ blocks into sixty $2 \times 1 \times 1$ blocks, and the fifteen $3 \times 1 \times 1$ blocks into fifteen $2 \times 1 \times 1$ blocks and fifteen $1 \times 1 \times 1$ blocks.

(f) Finally, cut the seventy-five $2 \times 1 \times 1$ blocks into one hundred and fifty $1 \times 1 \times 1$ blocks.

This process uses a total of $2 + 2 + 1 + 2 + 1 + 1 = 9$ cuts, which implies that 9 is indeed the minimum possible number of cuts.

2.4.8 Round 8

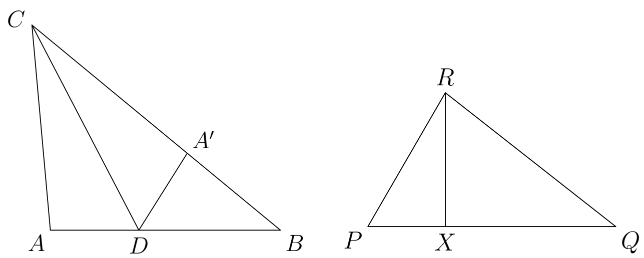
22. [18pts] A sequence that contains the numbers $1, 2, 3, \ldots, n$ exactly once each is said to be a *permutation* of length n. A permutation $w_1w_2w_3 \cdots w_n$ is said to be *sad* if there are indices $i < j < k$ such that $w_j > w_k$ and $w_j > w_i$. For example, the permutation $\mathbf{3142756}$ is sad because $7 > 6$ and $7 > 1$. Compute the number of permutations of length 11 that are not sad.

Solution. The answer is $\boxed{1024}$.

Call a permutation *happy* if it is not sad. We can construct a happy sequence of n numbers by adding the number n to a happy sequence of $n - 1$ numbers. Since n is greater than any number in the permutation of length $n - 1$, n must be inserted at either the start or end; otherwise the two numbers adjacent to n will both be smaller than n, which would yield a sad permutation. Thus, for each permutation of length $n - 1$, there are two permutations of length n. Since there is 1 happy permutation of length 1, there are 2 happy permutations of length 2, 4 of length 3, and so on until $2^{10} = 1024$ happy permutations of length 11.

23. [18pts] Let ABC be a triangle with $AB = 39, BC = 56$, and $CA = 35$. Compute $\angle CAB - \angle ABC$ in degrees.

Solution. The answer is $\boxed{60^\circ}$.



Let D be the foot of the angle bisector from C to AB, and let A' be the reflection of A over line CD. Triangles CAD and $CA'D$ are congruent, which implies that $\angle CAB = \angle DA'C$. Furthermore, by exterior angles, we have $\angle DA'C = \angle A'DB + \angle ABC$, which implies that $\angle CAB - \angle ABC = \angle AD'B$. By the Angle-Bisector Theorem, we have $\frac{AD}{DB} = \frac{AC}{CB} = \frac{5}{8}$, which implies that $A'D = AD = 15$ and

$DB = 24$. In addition, we have $BA' = BC - A'C = 56 - 35 = 21$. Therefore, triangle $A'BC$ is similar to a triangle with side lengths 5, 7, and 8. Let PQR be a triangle with $PQ = 8$, $QR = 7$, and $RP = 5$. Then, we have $\angle AD'B = \angle RPQ$. Let X be the foot of the perpendicular from point R to side PQ; the area of triangle ABC is $\frac{8XR}{2} = 4XR$. However, by Heron's Formula, the area of triangle PQR is $\sqrt{10 \cdot 5 \cdot 3 \cdot 2} = 10\sqrt{3}$, which implies that $XR = \frac{5\sqrt{3}}{2}$. Therefore, RPX is a $30 - 60 - 90$ right triangle, which implies that $\angle RPQ = 60°$. It follows that $\angle CAB = \angle ABC = 60°$.

24. [18pts] On a strange planet, there are n cities. Between any pair of cities, there can either be a one-way road, two one-way roads in different directions, or no road at all. Every city has a name, and at the source of every one-way road, there is a signpost with the name of the destination city. In addition, the one-way roads only intersect at cities, but there can be bridges to prevent intersections at non-cities. Fresh Mann has been abducted by one of the aliens, but Sophy Moore knows that he is in Rome, a city that has no roads leading out of it. Also, there is a direct one-way road leading from each other city to Rome. However, Rome is the secret police's name for the so-described city; its official name, the name appearing on the labels of the one-way roads, is unknown to Sophy Moore. Sophy Moore is currently in Athens and she wants to head to Rome in order to rescue Fresh Mann, but she does not know the value of n. Assuming that she tries to minimize the number of roads on which she needs to travel, determine the maximum possible number of roads that she could be forced to travel in order to find Rome. Express your answer as a function of n.

Solution. The answer is $\boxed{n-1}$.

We will first show that Sophy Moore can guarantee to have arrived at Rome by traveling in $n - 1$ roads, and then giving an example of a road map where Sophy Moore is required to use at least $n - 1$ moves to arrive at Rome.

In order to arrive at Rome in at most $n - 1$ moves, Sophy Moore can employ the following strategy: At every city, she looks at the labels of all the one-way roads on which she could travel. If there are no one-way roads, then she has arrived at Rome, because otherwise there would be a one-way road to Rome. If there are some one-way roads, then she then chooses one of the destination cities to which she has not yet been; she can always do this so long as she has not yet arrived at Rome, because until her arrival at Rome, there will always be a one-way road to Rome in the set of destination cities to which she has not yet been. Because this guarantees that she visits each city at most once, she will eventually visit Rome. In addition, because there are only. This guarantees that she visits each city at most once, and thus she will arrive at Rome after visiting at most n cities, or traveling on at most $n - 1$ roads.

A possible map configuration that forces Sophy Moore to use $n - 1$ roads to visit Rome is as follows: Name the cities $1, 2, 3, ..., n$; where 1 is Athens and n is Rome. Between every two cities, there is a one-way road leading from the city with smaller number to the city with larger number. Notice that if Sophy Moore is, at any point in time, at the city named m, $(1 \leq m \leq n - 1)$, then the cities $m + 1, m + 2, \ldots, n$ will all look identical to her, because they appear identical from the perspective of all cities named $1, 2, \ldots, m$, and if Sophy Moore is currently at the city named m, then she has only been to cities in the set $\{1, 2, \ldots, m\}$. Thus we can assume, under worst-case scenario, that she travels to the city named $m + 1$. This means that she will need to travel $m - 1$ roads under this road configuration and worst-case scenario in order to arrive at Rome.

Because Sophy Moore has a strategy which allows her to arrive at Rome within $n - 1$ road travels, and there can be a road configuration for which she could potentially be required to use $n - 1$ road travels, the answer is $n - 1$.