3.1 Individual Speed Test Solutions

1. Evaluate $\dfrac{\sqrt{2} \cdot \sqrt{6}}{\sqrt{3}}$.

   Solution. The answer is $(\sqrt{2})^2 = \boxed{2}$.

2. If 6% of a number is 1218, what is 18% of that number?

   Solution. The answer is $1218 \cdot 3 = \boxed{3654}$.

3. What is the median of $\{42, 9, 8, 4, 5, 1, 13666, 3\}$?

   Solution. The answer is $\dfrac{5+8}{2} = \boxed{\dfrac{13}{2}}$. We can arrange these elements in increasing order: 1, 3, 4, 5, 8, 9, 42, 13666. Since there are an even number of values, the median is the average of the two middle numbers 5 and 8.

4. Define the operation $\heartsuit$ so that $i \heartsuit u = 5i - 2u$. What is $3 \heartsuit 4$?

   Solution. The answer is $5 \cdot 3 - 2 \cdot 4 = \boxed{7}$.

5. How many 0.2-inch by 1-inch by 1-inch gold bars can fit in a 15-inch by 12-inch by 9-inch box?

   Solution. The answer is $15 \cdot 5 \cdot 12 \cdot 9 = 3^4 \cdot 5^2 \cdot 2^2 = \boxed{8100}$.

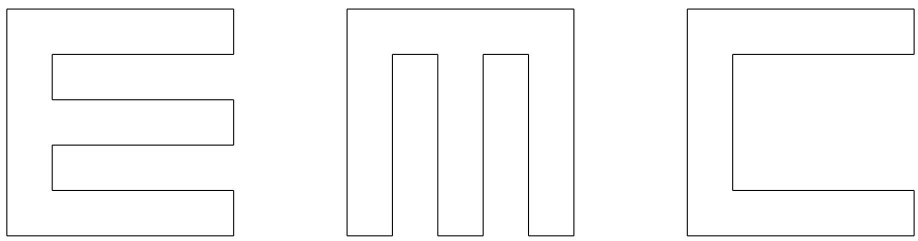
6. A tetrahedron is a triangular pyramid. What is the sum of the number of edges, faces, and vertices of a tetrahedron?

   Solution. The answer is $6 + 4 + 4 = \boxed{14}$.

7. Ron has three blue socks, four white socks, five green socks, and two black socks in a drawer. Ron takes socks out of his drawer blindly and at random. What is the least number of socks that Ron needs to take out to guarantee he will be able to make a pair of matching socks?

   Solution. The answer is $\boxed{5}$. There are only four different colors. Among any five socks, at least two of them are of the same color.

8. One segment with length 6 and some segments with lengths 10, 8, and 2 form the three letters in the diagram shown below. Compute the sum of the perimeters of the three figures.

   Solution. The answer is $30 \cdot 4 + 10 \cdot 8 = \boxed{200}$. For each letter, we can rearrange segments with lengths less than 8 to form a square of side 10. The sum of the perimeters of the three squares is 120. There are 10 additional segments of length 8.

9. How many integer solutions are there to the inequality $|x - 6| \leq 4$?

   Solution. The answer is $\boxed{9}$. The possible values of $x - 6$ are $0, \pm 1, \pm 2, \pm 3, \pm 4$.

10. In a land for bad children, the flavors of ice cream are grass, dirt, earwax, hair, and dust-bunny. The cones are made out of granite, marble, or pumice, and can be topped by hot lava, chalk, or ink. How many ice cream cones can the evil confectioners in this ice-cream land make? (Every ice cream cone consists of one scoop of ice cream, one cone, and one topping.)

    Solution. The answer is $5 \cdot 3 \cdot 3 = \boxed{45}$.

11. Compute the sum of the prime divisors of $245 + 452 + 524$.

    Solution. The answer is $11 + 3 + 37 = \boxed{51}$. Note that $245 + 452 + 524 = 222 + 444 + 555 = (2 + 4 + 5) \cdot 111 = 11 \cdot 3 \cdot 37$.

12. In quadrilateral $SEAT$, $SE = 2$, $EA = 3$, $AT = 4$, $\angle EAT = \angle SET = 90°$. What is the area of the quadrilateral?

    Solution. The answer is $5 + 6 = \boxed{11}$. The quadrilateral is the union of two right triangles EAT (with area 6 and $ET = 5$) and SET (with area 5).

13. What is the angle, in degrees, formed by the hour and minute hands on a clock at 10:30 AM?

    Solution. The answer is $\boxed{135}$.

14. Three numbers are randomly chosen without replacement from the set $\{101, 102, 103, \ldots, 200\}$. What is the probability that these three numbers are the side lengths of a triangle?

    Solution. The answer is $\boxed{1}$. Note that the sum of any two numbers in the set is greater than any number in the set.

15. John takes a 30-mile bike ride over hilly terrain, where the road always either goes uphill or downhill, and is never flat. If he bikes a total of 20 miles uphill, and he bikes at 6 mph when he goes uphill, and 24 mph when he goes downhill, what is his average speed, in mph, for the ride?

    Solution. The answer is $\dfrac{30}{\frac{20}{6} + \frac{10}{24}} = \boxed{8}$.

16. How many distinct six-letter words (not necessarily in any language known to man) can be formed by rearranging the letters in EXETER? (You should include the word EXETER in your count.)

    Solution. The answer is $\dfrac{6!}{3!} = \boxed{120}$.

17. A pie has been cut into eight slices of different sizes. Snow White steals a slice. Then, the seven dwarfs (Sneezy, Sleepy, Dopey, Doc, Happy, Bashful, Grumpy) take slices one by one according to the alphabetical order of their names, but each dwarf can only take a slice next to one that has already been taken. In how many ways can this pie be eaten by these eight persons?

    Solution. The answer is $8 \cdot 2^6 = 2^9 = \boxed{512}$. Snow White has 8 choices. Each of Bashful, Doc, Dopey, Grumpy, Happy, and Sleepy has two choices. Then, there is only one slice left for Sneezy to eat.

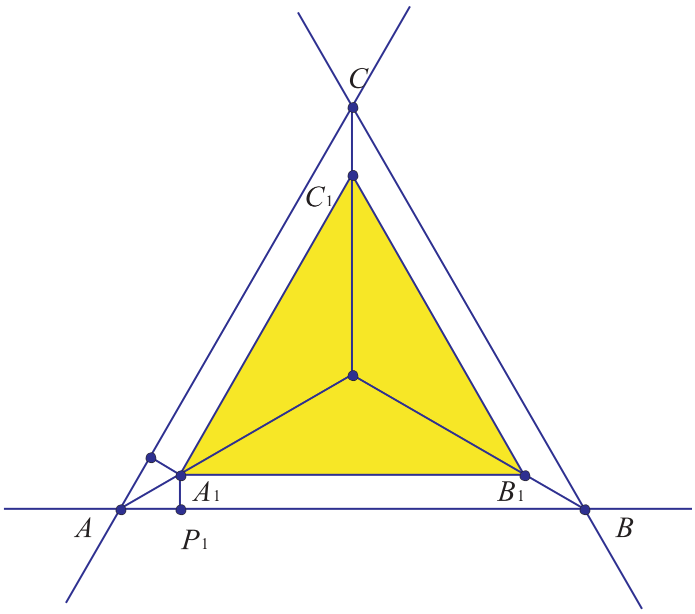
18. Assume that n is a positive integer such that the remainder of n is 1 when divided by 3, is 2 when divided by 4, is 3 when divided by 5, ..., and is 8 when divided by 10. What is the smallest possible value of n?

Solution. The answer is $\operatorname{lcm}(3, 4 \ldots, 10) - 2 = 5 \cdot 7 \cdot 8 \cdot 9 - 2 = \boxed{2518}$. Adding 2 to the desired number yields a number that is divisible by each of $3, 4, \ldots, 10$.

19. Find the sum of all positive four-digit numbers that are perfect squares and that have remainder 1 when divided by 100.

Solution. The answer is $49^2 + 51^2 + 99^2 = 2401 + 2601 + 9801 = \boxed{14803}$. Let $\overline{abcd}$ be such a number. Then $\overline{abcd} = (10x \pm 1)^2 = 100x^2 \pm 20x + 1$, implying that $x = 5$ or $x = 10$.

20. A coin of radius 1 cm is tossed onto a plane surface that has been tiled by equilateral triangles with side length $20\sqrt{3}$ cm. What is the probability that the coin lands within one of the triangles?



Solution. The answer is $\frac{(18\sqrt{3})^2}{(20\sqrt{3})^2} = \boxed{\frac{81}{100}}$. Assume that the center of the coin lies within the boundary of triangle ABC. In order for the coin to lie completely within the triangle, the center of the circle must lie in triangle $A_1B_1C_1$, where the distances between pairs of parallel lines A_1B_1 and AB, B_1C_1 and BC, and C_1A_1 and CA are all equal to 1. Let P_1 be the foot of the perpendicular from A_1 to line AB. It is not difficult to see that $A_1P_1 = 1$ and $AP_1 = \sqrt{3}$, from which it follows that $A_1B_1 = 18\sqrt{3}$ and the ratio between the areas of triangles $A_1B_1C_1$ and ABC is equal to $\frac{(18\sqrt{3})^2}{(20\sqrt{3})^2}$.