1.1 Individual Speed Test

Morning, January 26, 2013

There are 20 problems, worth 3 points each, to be solved in 20 minutes.

1. Determine how many digits the number 10^{10} has.

2. Let ABC be a triangle with $\angle ABC = 60^\circ$ and $\angle BCA = 70^\circ$. Compute $\angle CAB$ in degrees.

3. Given that $x : y = 2012 : 2$ and $y : z = 1 : 2013$, compute $x : z$. Express your answer as a common fraction.

4. Determine the smallest perfect square greater than 2400.

5. At 12:34 and 12:43, the time contains four consecutive digits. Find the next time after 12:43 that the time contains four consecutive digits on a 24-hour digital clock.

6. Given that $\sqrt{3^a \cdot 9^a \cdot 3^a} = 81^2$, compute a.

7. Find the number of positive integers less than 8888 that have a tens digit of 4 and a units digit of 2.

8. Find the sum of the distinct prime divisors of $1 + 2012 + 2013 + 2011 \cdot 2013$.

9. Albert wants to make 2×3 wallet sized prints for his grandmother. Find the maximum possible number of prints Albert can make using one 4×7 sheet of paper.

10. Let ABC be an equilateral triangle, and let D be a point inside ABC. Let E be a point such that ADE is an equilateral triangle and suppose that segments DE and AB intersect at point F. Given that $\angle CAD = 15^\circ$, compute $\angle DFB$ in degrees.

11. A *palindrome* is a number that reads the same forwards and backwards; for example, 1221 is a palindrome. An *almost-palindrome* is a number that is not a palindrome but whose first and last digits are equal; for example, 1231 and 1311 are an almost-palindromes, but 1221 is not. Compute the number of 4-digit almost-palindromes.

12. Determine the smallest positive integer n such that the sum of the digits of 11^n is not 2^n.

13. Determine the minimum number of breaks needed to divide an 8×4 bar of chocolate into 1×1 pieces. (When a bar is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

14. A particle starts moving on the number line at a time $t = 0$. Its position on the number line, as a function of time, is $x = (t-2012)^2 - 2012(t-2012) - 2013$. Find the number of positive integer values of t at which time the particle lies in the negative half of the number line (strictly to the left of 0).

15. Let A be a vertex of a unit cube and let B, C, and D be the vertices adjacent to A. The tetrahedron $ABCD$ is cut off the cube. Determine the surface area of the remaining solid.

16. In equilateral triangle ABC, points P and R lie on segment AB, points I and M lie on segment BC, and points E and S lie on segment CA such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PS = 2$, $RI = 3$, and $ME = 5$, compute the area of hexagon $PRIMES$.

17. Find the smallest odd positive integer with an odd number of positive integer factors, an odd number of distinct prime factors, and an odd number of perfect square factors.

18. Fresh Mann thinks that the expressions $2\sqrt{x^2-4}$ and $2(\sqrt{x^2}-\sqrt{4})$ are equivalent to each other, but the two expressions are not equal to each other for most real numbers x. Find all real numbers x such that $2\sqrt{x^2-4} = 2(\sqrt{x^2}-\sqrt{4})$.

19. Let m be the positive integer such that a 3×3 chessboard can be tiled by at most m pairwise incongruent rectangles with integer side lengths. If rotations and reflections of tilings are considered distinct, suppose that there are n ways to tile the chessboard with m pairwise incongruent rectangles with integer side lengths. Find the product mn.

20. Let ABC be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$. A triangle XYZ is said to be *friendly* if it intersects triangle ABC and it is a translation of triangle ABC. Let S be the set of points in the plane that are inside some friendly triangle. Compute the ratio of the area of S to the area of triangle ABC.