Exeter Math Club Contest
January 29, 2011

Contents

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- *Tournament Supervisor* Zuming Feng
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- *Problem Committee* Ravi Bajaj, Chong Gu, Ravi Jagadeesan, Gwangseung (Eric) Kim, Yong Wook (Spencer) Kwon, Ray Li, Abraham Shin, Dai Yang, David Yang
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- *Publicity* Claudia Feng, Jiexiong (Chelsea) Ge, Chieh-Ming (Jamin) Liu
- *Participating Organizations* Acton, Bigelow Middle School, Chesapeake Science Point Charter School, Century Chinese Language School, Debrown, Eaglebrook School, Fudan School, Homeschool, Honey Creek School, IDEAMath, Jonas Clarke Middle School, Lincoln Middle School, Math Girls, R.J. Grey Jr. High School, Roxbury Latin Middle School, The Sage School, Sycamore School, William Diamond Middle School
- *Primary Tournament Sponsor* We would like to thank Jane Street Capital for their generous support of this competition.

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- *Tournament Sponors* We would also like to thank the Phillips Exeter Academy Math Department and Art of Problem Solving for their support.

Contest day information

- *Proctors*

Proctor	Team 1 (code)	Team 2 (code)	Location
Ted Lee *(Head Proctor)*	Clarke A (7)	Sycamore-Honey Creek (175)	ACD 008 (Swift)
Will Zhang *(Head Proctor)*	Clarke B (11)	Dolphin (19)	ACD 007 (Seidenberg)
Gwangseung (Eric) Kim	Panda (23)	Bigelow (31)	ACD 109 (Marshall)
Payut (Paul) Pantawongdecha	Penguin (29)	Lincoln (41)	ACD 004 (Feng)
Theo Motzkin	Eaglebrook 1 (43)	debrown (101)	ACD 102 (Girard)
Lucy Yu	Eaglebrook 2 (47)	Fudan 1 (166)	ACD 105 (Kaminski)
Harlin Lee	Infinity 1 (132)	Sycamore (134)	ACD 108 (Keeble)
Harini Kannan	Infinity 2 (133)	Clarke C (136)	ACD 009 (Coogan)
Jiapei Chen	Clarke D (137)	Fudan 2 (167)	ACD 104 (Spanier)
Adisa Kruayatidee	Infinity 3 (141)	Fudan 3 (168)	ACD 106 (Chen)
Shi-Fan Chen	Infinity 4 (142)	Fudan 4 (169)	ACD 021 (Stahr)
Theo Gao	Honey Creek (143)	Liu-hui (152)	ACD 103 (Wolfson)
Ki Hong Ahn	Zu-Chong-zhi (153)	WMD (156)	ACD 006 (Parris)
Eugenia Kang	R J Grey (160)	Math Girls (162)	ACD 206 (Bergofsky)
Kaitlin Kimberling	Epic Ninja Cows (164)	AHA (172)	ACD 207 (Holden)
Claudia Feng	Sage 2 (165)	Diamond (174)	ACD 107 (Mallinson)
Elizabeth Gong	Infinity Indiv. (176)		ACD 126 (Golay)

- *Head Graders* Albert Chu, Yong Wook (Spencer) Kwon,
- *Graders* Ravi Charan, Andrew Holzman, Ravi Jagadeesan, Daniel Li, Ray Li, Abraham Shin, David Xiao, Dai Yang, David Yang
- *Judges* Zuming Feng, Greg Spanier
- *Runners* Michelle (Hannah) Jung *(Head Runner)*, Jiexiong (Chelsea) Ge, Lealia Xiong
- *Breakfast and Lunch Setup* Jiexiong (Chelsea) Ge, Yvonne Guu, Chanthana Kruayatidee
- *Lunch Vouchers and Coaches/Parents Room* Ravi Bajaj, Chong Gu

Chapter 1

EMC^2 2011 Problems

1.1 Individual Speed Test

Morning, January 29, 2011

There are 20 problems, worth 3 points each, and 20 minutes to solve as many problems as possible.

1. Euclid eats $\frac{1}{7}$ of a pie in 7 seconds. Euler eats $\frac{1}{5}$ of an identical pie in 10 seconds. Who eats faster?

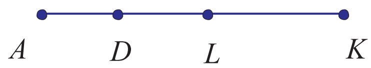
2. Given that $\pi = 3.1415926\ldots$, compute the circumference of a circle of radius 1. Express your answer as a decimal rounded to the nearest hundred thousandth (i.e. 1.234562 and 1.234567 would be rounded to 1.23456 and 1.23457, respectively).

3. Alice bikes to Wonderland, which is 6 miles from her house. Her bicycle has two wheels, and she also keeps a spare tire with her. If each of the three tires must be used for the same number of miles, for how many miles will each tire be used?

4. Simplify $\frac{2010 \cdot 2010}{2011}$ to a mixed number. (For example, $2\frac{1}{2}$ is a mixed number while $\frac{5}{2}$ and 2.5 are not.)

5. There are currently 175 problems submitted for EMC^2. Chris has submitted 51 of them. If nobody else submits any more problems, how many more problems must Chris submit so that he has submitted $\frac{1}{3}$ of the problems?

6. As shown in the diagram below, points D and L are located on segment AK, with D between A and L, such that $\frac{AD}{DK} = \frac{1}{3}$ and $\frac{DL}{LK} = \frac{5}{9}$. What is $\frac{DL}{AK}$?



7. Find the number of possible ways to order the letters G, G, e, e, e such that two neighboring letters are never G and e in that order.

8. Find the number of odd composite integers between 0 and 50.

9. Bob tries to remember his 2-digit extension number. He knows that the number is divisible by 5 and that the first digit is odd. How many possibilities are there for this number?

10. Al walks 1 mile due north, then 2 miles due east, then 3 miles due south, and then 4 miles due west. How far, in miles, is he from his starting position? (Assume that the Earth is flat.)

11. When n is a positive integer, $n!$ denotes the product of the first n positive integers; that is, $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. Given that $7! = 5040$, compute $8! + 9! + 10!$.

12. Sam's phone company charges him a per-minute charge as well as a connection fee (which is the same for every call) every time he makes a phone call. If Sam was charged \$4.88 for an 11-minute call and \$6.00 for a 19-minute call, how much would he be charged for a 15-minute call?

13. For a positive integer n, let s_n be the sum of the n smallest primes. Find the least n such that s_n is a perfect square (the square of an integer).

14. Find the remainder when 2011^{2011} is divided by 7.

15. Let a, b, c, and d be 4 positive integers, each of which is less than 10, and let e be their least common multiple. Find the maximum possible value of e.

16. Evaluate $100 - 1 + 99 - 2 + 98 - 3 + \cdots + 52 - 49 + 51 - 50$.

17. There are 30 basketball teams in the Phillips Exeter Dorm Basketball League. In how ways can 4 teams be chosen for a tournament if the two teams Soule Internationals and Abbot United cannot be chosen at the same time?

18. The numbers $1, 2, 3, 4, 5, 6$ are randomly written around a circle. What is the probability that there are four neighboring numbers such that the sum of the middle two numbers is less than the sum of the other two?

19. What is the largest positive 2-digit factor of $3^{2^{2011}} - 2^{2^{2011}}$?

20. Rhombus $ABCD$ has vertices $A = (-12, -4)$, $B = (6, b)$, $C = (c, -4)$ and $D = (d, -28)$, where b, c, and d are integers. Find a constant m such that the line $y = mx$ divides the rhombus into two regions of equal area.

1.2 Individual Accuracy Test

Morning, January 29, 2011

There are 10 problems, worth 9 points each, and 30 minutes to solve as many problems as possible.

1. What is the maximum number of points of intersection between a square and a triangle, assuming that no side of the triangle is parallel to any side of the square?

2. Two angles of an isosceles triangle measure 80° and x°. What is the sum of all the possible values of x?

3. Let p and q be prime numbers such that $p+q$ and $p+7q$ are both perfect squares. Find the value of pq.

4. Anna, Betty, Carly, and Danielle are four pit bulls, each of which is either wearing or not wearing lipstick. The following three facts are true:

   (1) Anna is wearing lipstick if Betty is wearing lipstick.
   (2) Betty is wearing lipstick only if Carly is also wearing lipstick.
   (3) Carly is wearing lipstick if and only if Danielle is wearing lipstick

   The following five statements are each assigned a certain number of points:

   (a) Danielle is wearing lipstick if and only if Carly is wearing lipstick. (This statement is assigned 1 point.)
   (b) If Anna is wearing lipstick, then Betty is wearing lipstick. (This statement is assigned 6 points.)
   (c) If Betty is wearing lipstick, then both Anna and Danielle must be wearing lipstick. (This statement is assigned 10 points.)
   (d) If Danielle is wearing lipstick, then Anna is wearing lipstick. (This statement is assigned 12 points.)
   (e) If Betty is wearing lipstick, then Danielle is wearing lipstick. (This statement is assigned 14 points.)

   What is the sum of the points assigned to the statements that *must* be true? (For example, if only statements (a) and (d) are true, then the answer would be $1+12=13$.)

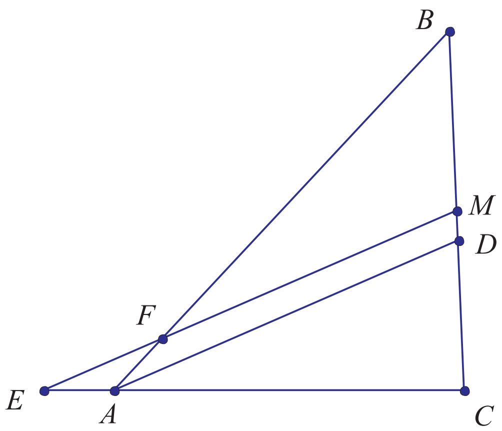
5. Let $f(x)$ and $g(x)$ be functions such that $f(x)=4x+3$ and $g(x)=\frac{x+1}{4}$. Evaluate $g(f(g(f(42))))$.

6. Let A, B, C, and D be consecutive vertices of a regular polygon. If $\angle ACD = 120^\circ$, how many sides does the polygon have?

7. Fred and George have a fair 8-sided die with the numbers $0, 1, 2, 9, 2, 0, 1, 1$ written on the sides. If Fred and George each roll the die once, what is the probability that Fred rolls a larger number than George?

8. Find the smallest *positive* integer t such that $(23t)^3-(20t)^3-(3t)^3$ is a perfect square.

9. In triangle ABC, $AC = 8$ and $AC < AB$. Point D lies on side BC with $\angle BAD = \angle CAD$. Let M be the midpoint of BC. The line passing through M parallel to AD intersects lines AB and AC at F and E, respectively. If $EF = \sqrt{2}$ and $AF = 1$, what is the length of segment BC? (See the following diagram.)



10. There are 2011 evenly spaced points marked on a circular table. Three segments are randomly drawn between pairs of these points such that no two segments share an endpoint on the circle. What is the probability that each of these segments intersects the other two? (See the following diagram.)

1.3 Team Test

Morning, January 29, 2011

There are 15 problems, worth 20 points each, and 30 minutes to solve as many problems as possible.

1. Velociraptor A is located at $x = 10$ on the number line and runs at 4 units per second. Velociraptor B is located at $x = -10$ on the number line and runs at 3 units per second. If the velociraptors run towards each other, at what point do they meet?

2. Let n be a positive integer. There are n non-overlapping circles in a plane with radii $1, 2, \ldots, n$. The total area that they enclose is at least 100. Find the minimum possible value of n.

3. How many integers between 1 and 50, inclusive, are divisible by 4 but not 6?

4. Let $a \star b = 1 + \frac{b}{a}$. Evaluate $((((((1 \star 1) \star 1) \star 1) \star 1) \star 1) \star 1) \star 1$.

5. In acute triangle ABC, D and E are points inside triangle ABC such that $DE \parallel BC$, B is closer to D than it is to E, $\angle AED = 80^\circ$, $\angle ABD = 10^\circ$, and $\angle CBD = 40^\circ$. Find the measure of $\angle BAE$, in degrees.

6. Al is at $(0, 0)$. He wants to get to $(4, 4)$, but there is a building in the shape of a square with vertices at $(1, 1)$, $(1, 2)$, $(2, 2)$, and $(2, 1)$. Al cannot walk inside the building. If Al is not restricted to staying on grid lines, what is the shortest distance he can walk to get to his destination?

7. Point $A = (1, 211)$ and point $B = (b, 2011)$ for some integer b. For how many values of b is the slope of AB an integer?

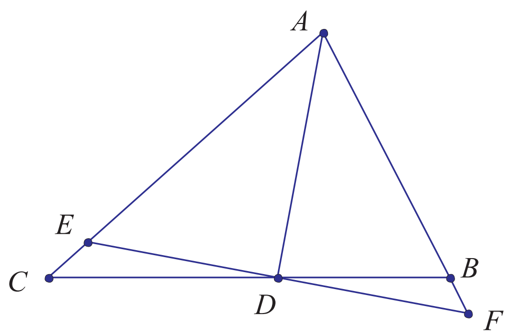
8. A palindrome is a number that reads the same forwards and backwards. For example, $1, 11$ and 141 are all palindromes. How many palindromes between 1 and 1000 are divisible by 11?

9. Suppose x, y, z are real numbers that satisfy:

$$\begin{aligned} x + y - z &= 5 \\ y + z - x &= 7 \\ z + x - y &= 9 \end{aligned}$$

   Find $x^2 + y^2 + z^2$.

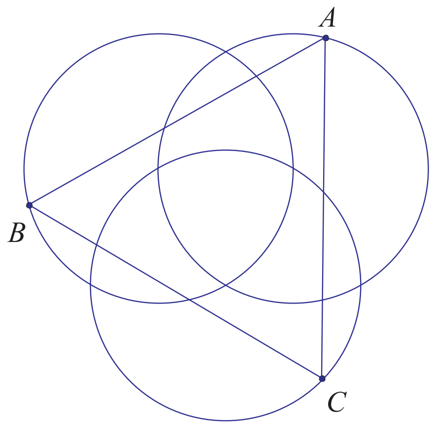
10. In triangle ABC, $AB = 3$ and $AC = 4$. The bisector of angle A meets BC at D. The line through D perpendicular to AD intersects lines AB and AC at F and E, respectively. Compute $EC - FB$. (See the following diagram.)



11. Bob has a six-sided die with a number written on each face such that the sums of the numbers written on each pair of opposite faces are equal to each other. Suppose that the numbers 109, 131, and 135 are written on three faces which share a corner. Determine the maximum possible sum of the numbers on the three remaining faces, given that all three are positive primes less than 200.

12. Let d be a number chosen at random from the set $\{142, 143, \ldots, 198\}$. What is the probability that the area of a rectangle with perimeter 400 and diagonal length d is an integer?

13. There are 3 congruent circles such that each circle passes through the centers of the other two. Suppose that A, B, and C are points on the circles such that each circle has exactly one of A, B, or C on it and triangle ABC is equilateral. Find the ratio of the maximum possible area of ABC to the minimum possible area of ABC. (See the following diagram.)



14. Let k and m be constants such that for all triples (a, b, c) of positive real numbers,

$$\sqrt{\frac{4}{a^2} + \frac{36}{b^2} + \frac{9}{c^2} + \frac{k}{ab}} = \left|\frac{2}{a} + \frac{6}{b} + \frac{3}{c}\right| \quad \text{if and only if} \quad am^2 + bm + c = 0.$$

Find k.

15. A bored student named Abraham is writing n numbers $a_1, a_2, \ldots, a_n$. The value of each number is either 1, 2, or 3; that is, a_i is 1, 2 or 3 for $1 \leq i \leq n$. Abraham notices that the ordered triples

$$(a_1, a_2, a_3), \quad (a_2, a_3, a_4), \quad \ldots, \quad (a_{n-2}, a_{n-1}, a_n), \quad (a_{n-1}, a_n, a_1), \quad (a_n, a_1, a_2)$$

are distinct from each other. What is the maximum possible value of n? Give the answer n, along with an example of such a sequence. Write your answer as an ordered pair. (For example, if the answer were 5, you might write $(5, 12311)$.)

1.4 Guts Test

Afternoon, January 29, 2011

1.4.1 Round 1

1. [3pts] In order to make good salad dressing, Bob needs a 0.9% salt solution. If soy sauce is 15% salt, how much water, in mL, does Bob need to add to 3 mL of pure soy sauce in order to have a good salad dressing?

2. [3pts] Alex the Geologist is buying a canteen before he ventures into the desert. The original cost of a canteen is \$20, but Alex has two coupons. One coupon is \$3 off and the other is 10% off the entire remaining cost. Alex can use the coupons in any order. What is the least amount of money he could pay for the canteen?

3. [3pts] Steve and Yooni have six distinct teddy bears to split between them, including exactly 1 blue teddy bear and 1 green teddy bear. How many ways are there for the two to divide the teddy bears, if Steve gets the blue teddy bear and Yooni gets the green teddy bear? (The two do not necessarily have to get the same number of teddy bears, but each teddy bear must go to a person.)

1.4.2 Round 2

4. [5pts] In the currency of Mathamania, 5 wampas are equal to 3 kabobs and 10 kabobs are equal to 2 jambas. How many jambas are equal to twenty-five wampas?

5. [5pts] A sphere has a volume of 81π. A new sphere with the same center is constructed with a radius that is $\frac{1}{3}$ the radius of the original sphere. Find the volume, in terms of π, of the region between the two spheres.

6. [5pts] A frog is located at the origin. It makes four hops, each of which moves it either 1 unit to the right or 1 unit to the left. If it also ends at the origin, how many 4-hop paths can it take?

1.4.3 Round 3

7. [6pts] Nick multiplies two consecutive positive integers to get $4^5 - 2^5$. What is the smaller of the two numbers?

8. [6 pts] In rectangle $ABCD$, E is a point on segment CD such that $\angle EBC = 30°$ and $\angle AEB = 80°$. Find $\angle EAB$, in degrees.

9. [6pts] Mary's secret garden contains clones of Homer Simpson and WALL-E. A WALL-E clone has 4 legs. Meanwhile, Homer Simpson clones are human and therefore have 2 legs each. A Homer Simpson clone always has 5 donuts, while a WALL-E clone has 2. In Mary's secret garden, there are 184 donuts and 128 legs. How many WALL-E clones are there?

1.4.4 Round 4

10. [7pts] Including Richie, there are 6 students in a math club. Each day, Richie hangs out with a different group of club mates, each of whom gives him a dollar when he hangs out with them. How many dollars will Richie have by the time he has hung out with every possible group of club mates?

11. [7pts] There are seven boxes in a line: three empty, three holding \$10 each, and one holding the jackpot of $\$1,000,000$. From the left to the right, the boxes are numbered 1, 2, 3, 4, 5, 6 and 7, in that order. You are told the following:

    - No two adjacent boxes hold the same contents.
    - Box 4 is empty.
    - There is one more \$10 prize to the right of the jackpot than there is to the left.

    Which box holds the jackpot?

12. [7pts] Let a and b be real numbers such that $a + b = 8$. Let c be the minimum possible value of $x^2 + ax + b$ over all real numbers x. Find the maximum possible value of c over all such a and b.

1.4.5 Round 5

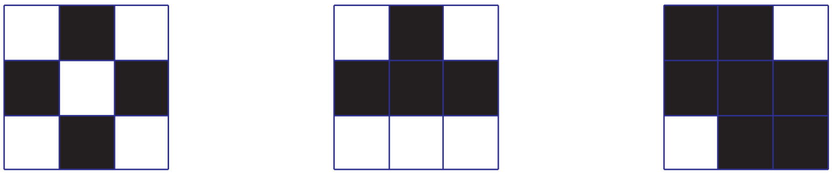
13. [9pts] Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 12$. Let M be the midpoint of CD, and P be a point on BM such that $BP = BC$. Find the area of $ABPD$.

14. [9pts] The number 19 has the following properties:

    - It is a 2-digit positive integer.
    - It is the two leading digits of a 4-digit perfect square, because $1936 = 44^2$.

    How many numbers, including 19, satisfy these two conditions?

15. [9pts] In a 3×3 grid, each unit square is colored either black or white. A coloring is considered "nice" if there is at most one white square in each row or column. What is the total number of nice colorings? Rotations and reflections of a coloring are considered distinct. (For example, in the three squares shown below, only the rightmost one has a nice coloring.)



1.4.6 Round 6

16. [11pts] Let $a_1, a_2, \ldots, a_{2011}$ be a sequence of numbers such that $a_1 = 2011$ and $a_1 + a_2 + \cdots + a_n = n^2 \cdot a_n$ for $n = 1, 2, \ldots 2011$. (That is, $a_1 = 1^2 \cdot a_1$, $a_1 + a_2 = 2^2 \cdot a_2, \ldots$) Compute a_{2011}.

17. [11pts] Three rectangles, with dimensions 3×5, 4×2, and 6×4, are each divided into unit squares which are alternately colored black and white like a checkerboard. Each rectangle is cut along one of its diagonals into two triangles. For each triangle, let m be the total black area and n the total white area. Find the maximum value of $|m - n|$ for the 6 triangles.

18. [11pts] In triangle ABC, $\angle BAC = 90°$, and the length of segment AB is 2011. Let M be the midpoint of BC and D the midpoint of AM. Let E be the point on segment AB such that $EM \parallel CD$. What is the length of segment BE?

1.4.7 Round 7

19. [12pts] How many integers from 1 to 100, inclusive, can be expressed as the difference of two perfect squares? (For example, $3 = 2^2 - 1^2$).

20. [12pts] In triangle ABC, $\angle ABC = 45°$ and $\angle ACB = 60°$. Let P and Q be points on segment BC, F a point on segment AB, and E a point on segment AC such that $FQ \parallel AC$ and $EP \parallel AB$. Let D be the foot of the altitude from A to BC. The lines AD, FQ, and PE form a triangle. Find the positive difference, in degrees, between the largest and smallest angles of this triangle.

21. [12pts] For real number x, $\lceil x \rceil$ is equal to the smallest integer larger than or equal to x. For example, $\lceil 3 \rceil = 3$ and $\lceil 2.5 \rceil = 3$. Let $f(n)$ be a function such that $f(n) = \left\lceil \frac{n}{2} \right\rceil + f\left(\left\lceil \frac{n}{2} \right\rceil\right)$ for every integer n greater than 1. If $f(1) = 1$, find the maximum value of $f(k) - k$, where k is a positive integer less than or equal to 2011.

1.4.8 Round 8

The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.

22. [14pts] Let W be the answer to problem 24 in this guts round. Let $f(a) = \frac{1}{1 - \frac{1}{1 - \frac{1}{a}}}$. Determine $|f(2) + \cdots + f(W)|$.

23. [14pts] Let X be the answer to problem 22 in this guts round. How many odd perfect squares are less than $8X$?

24. [14pts] Let Y be the answer to problem 23 in this guts round. What is the maximum number of points of intersections of two regular $(Y - 5)$-sided polygons, if no side of the first polygon is parallel to any side of the second polygon?

1.4.9 Round 9

25. [16pts] Cross country skiers $s_1, s_2, s_3, \dots, s_7$ start a race one by one in that order. While each skier skis at a constant pace, the skiers do not all ski at the same rate. In the course of the race, each skier either overtakes another skier or is overtaken by another skier exactly two times. Find all the possible orders in which they can finish. Write each possible finish as an ordered septuplet (a, b, c, d, e, f, g) where a, b, c, d, e, f, g are the numbers 1-7 in some order. (So a finishes first, b finishes second, etc.)

26. [16pts] Archie the Alchemist is making a list of all the elements in the world, and the proportion of earth, air, fire, and water needed to produce each. He writes the proportions in the form E:A:F:W. If each of the letters represents a whole number from 0 to 4, inclusive, how many different elements can Archie list? Note that if Archie lists wood as 2:0:1:2, then 4:0:2:4 would also produce wood. In addition, 0:0:0:0 does not produce an element.

27. [16pts] Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 12$. Let M be the midpoint of CD, and P be the point on BM such that $DP = DA$. Find the area of quadrilateral $ABPD$.

1.4.10 Round 10

28. [17pts] David the farmer has an infinitely large grass-covered field which contains a straight wall. He ties his cow to the wall with a rope of integer length. The point where David ties his rope to the wall divides the wall into two parts of length a and b, where $a > b$ and both are integers. The rope is shorter than the wall but is longer than a. Suppose that the cow can reach grass covering an area of $\frac{165\pi}{2}$. Find the ratio $\frac{a}{b}$. You may assume that the wall has 0 width.

29. [17pts] Let S be the number of ordered quintuples (a, b, x, y, n) of positive integers such that

$$\begin{aligned} \frac{a}{x} + \frac{b}{y} &= \frac{1}{n}, \\ abn &= 2011^{2011}. \end{aligned}$$

Compute the remainder when S is divided by 2012.

30. [17pts] Let n be a positive integer. An $n \times n$ square grid is formed by n^2 unit squares. Each unit square is then colored either red or blue such that each row or column has exactly 10 blue squares. A *move* consists of choosing a row or a column, and recolor each unit square in the chosen row or column – if it is red, we recolor it blue, and if it is blue, we recolor it red. Suppose that it is possible to obtain fewer than $10n$ blue squares after a sequence of finite number of moves. Find the maximum possible value of n.

Chapter 2

EMC^2 2011 Solutions

2.1 Individual Speed Test Solutions

1. Euclid eats $\frac{1}{7}$ of a pie in 7 seconds. Euler eats $\frac{1}{5}$ of an identical pie in 10 seconds. Who eats faster?

   Solution. The answer is $\boxed{\text{Euclid}}$. Euclid eats at $\frac{1}{49}$ pies per second; Euler eats at $\frac{1}{50}$ pies per second.

2. Given that $\pi = 3.1415926\ldots$, compute the circumference of a circle of radius 1. Express your answer as a decimal rounded to the nearest hundred thousandth (i.e. 1.234562 and 1.234567 would be rounded to 1.23456 and 1.23457, respectively).

   Solution. The answer is $\boxed{6.28319}$. The circumference is 2π.

3. Alice bikes to Wonderland, which is 6 miles from her house. Her bicycle has two wheels, and she also keeps a spare tire with her. If each of the three tires must be used for the same number of miles, for how many miles will each tire be used?

   Solution. The answer is $\boxed{4}$. The tires combined travel a total of $6 \cdot 2 = 12$ tire-miles. Alice has 3 tires, so the answer is $\frac{12}{3} = 4$ miles.

4. Simplify $\frac{2010 \cdot 2010}{2011}$ to a mixed number. (For example, $2\frac{1}{2}$ is a mixed number while $\frac{5}{2}$ and 2.5 are not.)

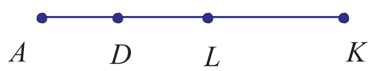
   Solution. The answer is $\boxed{2009\frac{1}{2011}}$.

   We have
   $$\frac{2010 \cdot 2010}{2011} = \frac{2010(2011-1)}{2011} = 2010 - \frac{2010}{2011} = \boxed{2009\frac{1}{2011}}.$$

5. There are currently 175 problems submitted for EMC^2. Chris has submitted 51 of them. If nobody else submits any more problems, how many more problems must Chris submit so that he has submitted $\frac{1}{3}$ of the problems?

   Solution. The answer is $\boxed{11}$. Suppose that Chris submits x more problems. Then,
   $$51 + x = \frac{(175 + x)}{3},$$
   implying that $x = 11$.

6. As shown in the diagram below, points D and L are located on segment AK, with D between A and L, such that $\frac{AD}{DK} = \frac{1}{3}$ and $\frac{DL}{LK} = \frac{5}{9}$. What is $\frac{DL}{AK}$?



Solution. The answer is $\boxed{\dfrac{15}{56}}$. Without loss of generality, let $AK = 1$. Then, we know that $AD = 1/4$ and $DK = 3/4$. It follows that

$$\frac{DL}{AK} = DL = \frac{5}{14} \cdot DK = \frac{5}{14} \cdot \frac{3}{4} = \frac{15}{56}.$$

7. Find the number of possible ways to order the letters G, G, e, e, e such that two neighboring letters are never G and e in that order.

   Solution. The answer is $\boxed{1}$. The only arrangement is $eeGGG$.

8. Find the number of odd composite integers between 0 and 50.

   Solution. The answer is $\boxed{10}$. The numbers are $9, 15, 21, 25, 27, 33, 35, 39, 45, 49$.

9. Bob tries to remember his 2-digit extension number. He knows that the number is divisible by 5 and that the first digit is odd. How many possibilities are there for this number?

   Solution. The answer is $\boxed{10}$. The first digit is one of $1, 3, 5, 7, 9$ and the last digit is one of $0, 5$ so there are $5 \cdot 2 = 10$ possible extensions.

10. Al walks 1 mile due north, then 2 miles due east, then 3 miles due south, and then 4 miles due west. How far, in miles, is he from his starting position? (Assume that the Earth is flat.)

    Solution. The answer is $\boxed{2\sqrt{2}}$. Al's final position is 2 miles south and 2 miles east of his initial position, so the distance is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.

11. When n is a positive integer, $n!$ denotes the product of the first n positive integers; that is, $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$. Given that $7! = 5040$, compute $8! + 9! + 10!$.

    Solution. The answer is $\boxed{4032000}$. The answer is

$$8! + 9! + 10! = 8!(1 + 9 + 9 \cdot 10) = 8! \cdot 100 = 7! \cdot 800 = 4032000.$$

12. Sam's phone company charges him a per-minute charge as well as a connection fee (which is the same for every call) every time he makes a phone call. If Sam was charged \$4.88 for an 11-minute call and \$6.00 for a 19-minute call, how much would he be charged for a 15-minute call?

    Solution. The answer is $\boxed{\$5.44}$. Suppose that he is charged m dollars per minute and n dollars for the initial connection fee. Then, we want

$$15m + n = \frac{30m + 2n}{2} = \frac{(11m + n) + (19m + n)}{2} = \frac{4.88 + 6.00}{2} = 5.44.$$

13. For a positive integer n, let s_n be the sum of the n smallest primes. Find the least n such that s_n is a perfect square (the square of an integer).

    Solution. The answer is $\boxed{9}$. It is routine to check that

$$(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9) = (2, 5, 10, 17, 28, 41, 58, 77, 100).$$

Note: The second least n such that s_n is a perfect square is $n = 2474$ with $s_{2474} = 25633969 = 5063^2$. Can you prove that no consecutive terms in the sequence $s_1, s_2, \dots$ are both perfect squares?

14. Find the remainder when 2011^{2011} is divided by 7.

    Solution. The answer is $\boxed{2}$. Because 2011 has remainder 2 when it is divided by 7, we need to find the remainder when 2^{2011} is divided by 7. It is easy to check that, when divided by 7, the respective remainders of $2^1, 2^2, 2^3, 2^4, 2^5, 2^6, \ldots$ are $2, 4, 1, 2, 4, 1, \ldots$. The remainder of 2^{2011} is the 2011^{th} term in this sequence and it is equal to 2.

15. Let a, b, c, and d be 4 positive integers, each of which is less than 10, and let e be their least common multiple. Find the maximum possible value of e.

    Solution. The answer is $\boxed{2520}$. The largest power of 2 possible is 2^3, and similarly the largest powers of 3, 5, and 7 are 3^2, 5^1, and 7^1, respectively. Thus, the answer is $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$, achieved with $\{a, b, c, d\} = \{5, 7, 8, 9\}$.

16. Evaluate $100 - 1 + 99 - 2 + 98 - 3 + \cdots + 52 - 49 + 51 - 50$.

    Solution. The answer is $\boxed{2500}$. We know that

    $$100-1+99-2+98-3+\cdots+52-49+51-50 = (100-50)+(99-49)+\cdots+(51-1) = 50\cdot 50 = 2500.$$

17. There are 30 basketball teams in the Phillips Exeter Dorm Basketball League. In how ways can 4 teams be chosen for a tournament if the two teams Soule Internationals and Abbot United cannot be chosen at the same time?

    Solution. The answer is $\boxed{27027}$. If neither team is chosen, then there are $\binom{28}{4} = 20475$ choices for the tournament. If one of them is chosen, then there are 2 choices for which one is chosen and $\binom{28}{3}$ choices for the other teams, resulting in 6552 choices. The answer is $20475 + 6552 = 27027$.

18. The numbers $1, 2, 3, 4, 5, 6$ are randomly written around a circle. What is the probability that there are four neighboring numbers such that the sum of the middle two numbers is less than the sum of the other two?

    Solution. The answer is $\boxed{1}$. We consider the neighboring numbers $a, b, 1, c, d$ (in that order). If $b > c$, then $(b, 1, c, d)$ has the property that $1 + c < d + b$; if $b < c$, then $(a, b, 1, c)$ has the property that $1 + b < a + c$.

19. What is the largest positive 2-digit factor of $3^{2^{2011}} - 2^{2^{2011}}$?

    Solution. The answer is $\boxed{97}$. Note that

    $$\begin{aligned}
    3^{2^{2011}} - 2^{2^{2011}} &= (3^{2^{2010}} + 2^{2^{2010}})(3^{2^{2010}} - 2^{2^{2010}}) \\
    &= (3^{2^{2010}} + 2^{2^{2010}})(3^{2^{2009}} + 2^{2^{2009}})(3^{2^{2009}} - 2^{2^{2009}}) \\
    &\vdots \\
    &= (3^{2^{2010}} + 2^{2^{2010}})(3^{2^{2009}} + 2^{2^{2009}})\cdots(3^{2^3} + 2^{2^3})(3^{2^2} + 2^{2^2})(3^{2^2} - 2^{2^2}).
    \end{aligned}$$

    Hence $3^{2^2} + 2^{2^2} = 97$ divides $3^{2^{2011}} - 2^{2^{2011}}$. On the other hand, because $3^{2^{2011}} - 2^{2^{2011}}$ is divisible by neither 2 nor 3, it is divisible by neither 98 nor 99.

20. Rhombus $ABCD$ has vertices $A = (-12, -4)$, $B = (6, b)$, $C = (c, -4)$ and $D = (d, -28)$, where b, c, and d are integers. Find a constant m such that the line $y = mx$ divides the rhombus into two regions of equal area.

Solution. The answer is $\boxed{-\frac{2}{3}}$. Let G be the intersection of diagonals AC and BD. Because G lies on AC, we know that $G = (x, -4)$ for some real number x. Note that the diagonals of a rhombus are perpendicular to each other. Because AC is parallel to the x-axis, it follows that BD is parallel to the y-axis. Hence $G = (6, -4)$. Note also that a rhombus has $180°$ rotational symmetry about G. Therefore, the line passing through the origin and G will divide the rhombus into two regions of equal area and $m = -\frac{4}{6} = -\frac{2}{3}$.

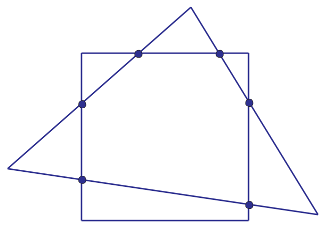
Note: Is such a line unique?

2.2 Individual Accuracy Test Solutions

Morning, January 29, 2011

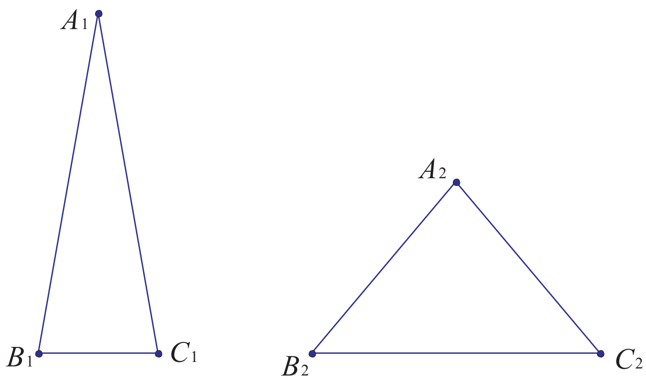
There are 10 problems, worth 9 points each, and 30 minutes to solve as many problems as possible.

1. What is the maximum number of points of intersection between a square and a triangle, assuming that no side of the triangle is parallel to any side of the square?



Solution. The answer is $\boxed{6}$. (See the proceeding diagram.) Each side of the triangle can intersect the square at no more than 2 points. This means there can be at most $3 \cdot 2 = 6$ intersections, which can be achieved.

2. Two angles of an isosceles triangle measure $80°$ and $x°$. What is the sum of all the possible values of x?



Solution. The answer is $\boxed{150}$. There are two possible triangles. (See the diagrams shown above.) In triangle $A_1B_1C_1$, $A_1B_1 = A_1C_1$, $\angle A_1 = 20°$, $\angle B_1 = \angle C_1 = 80°$. In triangle $A_2B_2C_2$, $A_2B_2 = A_2C_2$, $\angle A_2 = 80°$, $\angle B_1 = \angle C_1 = 50°$. Hence the possible values of x are 20, 50, and 80, with their sum equal to 150.

3. Let p and q be prime numbers such that $p + q$ and $p + 7q$ are both perfect squares. Find the value of pq.

Solution. The answer is $\boxed{4}$. Write $p + q = a^2$, $p + 7q = b^2$ so that $b^2 - a^2 = (b - a)(b + a) = 6q$. Since $(b + a) = (b - a) + 2a$, $b + a$ and $b - a$ have the same parity - in other words, $b - a$ and $b + a$ are either both even or both odd. Because $(b - a)(b + a) = 6q$ is even, $b - a$ and $b + a$ are both even. This means q is even; that is, $q = 2$. Then $(b - a)(b + a) = 12$ with $b - a = 2$ and $b + a = 6$, implying that $b = 4$, $a = 2$. This means $p = 2$ as well, so $pq = 4$.

4. Anna, Betty, Carly, and Danielle are four pit bulls, each of which is either wearing or not wearing lipstick. The following three facts are true:

   (1) Anna is wearing lipstick if Betty is wearing lipstick.

   (2) Betty is wearing lipstick only if Carly is also wearing lipstick.

   (3) Carly is wearing lipstick if and only if Danielle is wearing lipstick

   The following five statements are each assigned a certain number of points:

   (a) Danielle is wearing lipstick if and only if Carly is wearing lipstick. (This statement is assigned 1 point.)

   (b) If Anna is wearing lipstick, then Betty is wearing lipstick. (This statement is assigned 6 points.)

   (c) If Betty is wearing lipstick, then both Anna and Danielle must be wearing lipstick. (This statement is assigned 10 points.)

   (d) If Danielle is wearing lipstick, then Anna is wearing lipstick. (This statement is assigned 12 points.)

   (e) If Betty is wearing lipstick, then Danielle is wearing lipstick. (This statement is assigned 14 points.)

   What is the sum of the points assigned to the statements that *must* be true? (For example, if only statements (a) and (d) are true, then the answer would be $1 + 12 = 13$.)

   Solution. The answer is $\boxed{25}$. Statement (a) is true because it is equivalent to fact (3). By fact (1), if Betty did not wear lipstick, Anna still could be wearing lipstick. Hence statement (b) is not always true. If Betty wears lipstick, then Anna wears lipstick by fact (1) and Carly wears lipstick by fact (2), implying that Danielle wears lipstick by fact (3). Hence statement (c) is true, from which it follows that statment (e) is true. Finally, statement (d) is not always true. We can have Danielle wearing lipstick, Carly wearing lipstick, Betty not wearing lipstick, and Anna not wearing lipstick. Therefore statements (a), (c), (e) are true and the total points is $1 + 10 + 14 = 25$.

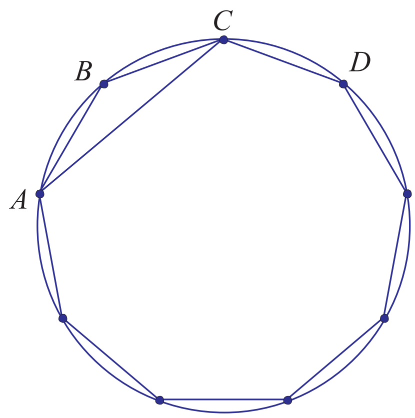
5. Let $f(x)$ and $g(x)$ be functions such that $f(x) = 4x + 3$ and $g(x) = \dfrac{x+1}{4}$. Evaluate $g(f(g(f(42))))$.

   Solution. The answer is $\boxed{44}$. We have

$$g(f(x)) = g(4x+3) = \frac{4x+3}{4} + \frac{1}{4} = \frac{4x+4}{4} = x + 1.$$

   It then follows that $g(f(g(f(42)))) = g(f(43)) = 44$.

6. Let A, B, C, and D be consecutive vertices of a regular polygon. If $\angle ACD = 120^\circ$, how many sides does the polygon have?



Solution. The answer is $\boxed{9}$. In isosceles triangle ABC, we set

$$\angle BAC = \angle BCA = x°.$$

Then it follows that

$$\angle ABC = 180° - 2x°$$

and

$$\angle BCD = 120° + x°$$

. Because this is a regular polygon, we have

$$\angle ABC = \angle BCD$$

or

$$180° - 2x° = 120° + x°$$

; that is, $x = 20°$. It follows that each interior angle of the polygon is $140°$; that is, each exterior angle is $40°$. Because the sum of the exterior angles of a polygon is $360°$, we conclude that this polygon has 9 sides.

7. Fred and George have a fair 8-sided die with the numbers $0, 1, 2, 9, 2, 0, 1, 1$ written on the sides. If Fred and George each roll the die once, what is the probability that Fred rolls a larger number than George?

Solution. The answer is $\boxed{\dfrac{23}{64}}$. The probability of both getting a 0 is equal to $(\left(\dfrac{1}{4}\right)^2 = \dfrac{1}{16}$. The probability of both getting a 1 is equal to $\left(\dfrac{1}{4}\right)^2 = \dfrac{1}{16}$. The probability of both getting a 2 is equal to $\left(\dfrac{3}{8}\right)^2 = \dfrac{9}{64}$. The probability of both getting a 9 is equal to $\left(\dfrac{1}{8}\right)^2 = \dfrac{1}{64}$. Hence the probability of both getting the same number is

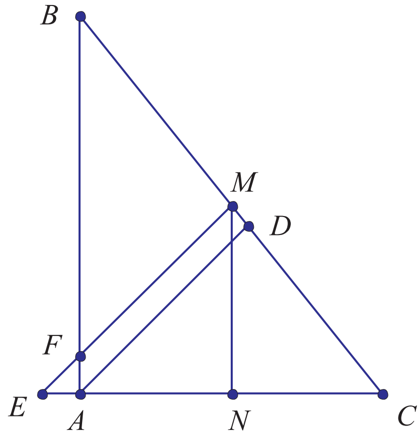
$$\frac{1}{16} + \frac{1}{16} + \frac{9}{64} + \frac{1}{64} = \frac{9}{32}.$$

It follows that the probability of them getting distinct numbers is $1 - \dfrac{9}{32} = \dfrac{23}{32}$. If they do not get the same number then Freds number will be larger one-half of the time. Therefore the answer is $\left(\frac{23}{32}\right) \cdot \left(\frac{1}{2}\right) = \frac{23}{64}$.

8. Find the smallest *positive* integer t such that $(23t)^3 - (20t)^3 - (3t)^3$ is the square of an integer.

   Solution. The answer is $\boxed{115}$. Note that $(23t)^3 - (20t)^3 - (3t)^3 = t^3(23^3 - 20^3 - 3^3)$. Because $(x+y)^3 = x^3+y^3+3xy(x+y)$, we have $23^3-20^3-3^3 = (20+3)^3-20^3-3^3 = 3\cdot20\cdot3\cdot(20+3) = 2^2\cdot3^2\cdot5\cdot23$. Hence
$$(23t)^3 - (20t)^3 - (3t)^3 = t^3(23^3 - 20^3 - 3^3) = t^2 \cdot 2^2 \cdot 3^2 \cdot 5 \cdot 23 \cdot t.$$
   Therefore, $(23t)^3 - (20t)^3 - (3t)^3$ is a perfect square if and only if $5 \cdot 23 \cdot t$ is a perfect square. The minimum positive value of such t is clearly $t = 5 \cdot 23 = 115$.

9. In triangle ABC, $AC = 8$ and $AC < AB$. Point D lies on side BC with $\angle BAD = \angle CAD$. Let M be the midpoint of BC. The line passing through M parallel to AD intersects lines AB and AC at F and E, respectively. If $EF = \sqrt{2}$ and $AF = 1$, what is the length of segment BC?

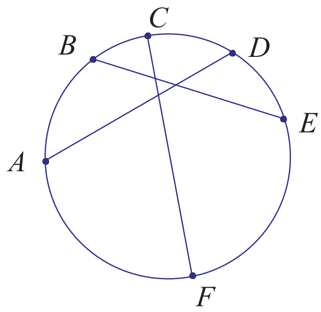


   Solution. The answer is $\boxed{2\sqrt{41}}$. (See the proceeding diagram.)Because $EM \parallel AD$, $\angle AEF = \angle CAD$ and $\angle FAD = \angle AFE$. By the given condition, we have $\angle CAD = \angle BAD = \angle FAD$. Combining the last two equations yields $\angle AEF = \angle AFE$; that is, AEF is an isosceles triangle with $AE = AF = 1$. In addition, because $AE = AF = 1$ and $EF = \sqrt{2}$, we conclude that $\angle AEF$ is an isosceles right triangle with $\angle EAF = 90^\circ$ and $\angle AEF = \angle AFE = 45^\circ$. Let N be the midpoint of segment AC. Then MN is the midline of triangle ABC. In particular, $MN \parallel AB$, from which it follows that MNE is an isosceles triangle with $MN = NE = NA + AE = 4 + 1 = 5$. It follows that $MC = \sqrt{NC^2 + MN^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$ and $BC = 2MC = 2\sqrt{41}$.

10. There are 2011 evenly spaced points marked on a circular table. Three segments are randomly drawn between pairs of these points such that no two segments share an endpoint on the circle. What is the probability that each of these segments intersects the other two?

    Solution. The answer is $\boxed{\dfrac{1}{15}}$. We note that it does not matter *which* six points on the circle are used as endpoints; what matters is the manner in which segments are drawn between them. Let our 6 endpoints be A, B, C, D, E, F around the circle in that order. (See the following diagram.)

    We will show that A must be connected to D, B to E, and C to F. If A is connected to B, C, E, or F, then there are three points on one side of the segment with an endpoint at A. It follows that there must be a segment among this points; however, this segments would not intersect the one passing through A, which is a contradiction. Hence, A must be connected to D. Similarly, B must be connected to E and C to F.



The probability that A is connected to D is $\frac{1}{5}$. If A is connected to D, then the probability that B is connected to E is $\frac{1}{3}$. If both those connections occur, then C is automatically connected to F. Thus, the probability of getting a set of segments which pairwise intersect is $\frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$.

2.3 Team Test Solutions

1. Velociraptor A is located at $x = 10$ on the number line and runs at 4 units per second. Velociraptor B is located at $x = -10$ on the number line and runs at 3 units per second. If the velociraptors run towards each other, at what point do they meet?

   Solution. The answer is $\boxed{-\frac{10}{7}}$. Suppose the velociraptors collide at a. The distances from a to the starting points of velociraptors A and B are $10 - a$ and $a + 10$, respectively. Because the time it takes for the velociraptors to reach this point is the same, we know that

   $$\frac{a+10}{3} = \frac{10-a}{4},$$

   so $a = \frac{-10}{7}$.

2. Let n be a positive integer. There are n non-overlapping circles in a plane with radii $1, 2, \ldots, n$. The total area that they enclose is at least 100. Find the minimum possible value of n.

   Solution. The answer is $\boxed{5}$. If $n = 4$, then the total area is $\pi \cdot (1+4+9+16) = 30\pi < 30 \cdot 3.2 = 96 < 100$. If $n = 5$, the total area is $\pi \cdot (1 + 4 + 9 + 16 + 25) = 55\pi > 100$.

3. How many integers between 1 and 50, inclusive, are divisible by 4 but not 6?

   Solution. The answer is $\boxed{8}$. These integers are $4, 8, 16, 20, 28, 32, 40, 44$.

4. Let $a \star b = 1 + \frac{b}{a}$. Evaluate $((((((1 \star 1) \star 1) \star 1) \star 1) \star 1) \star 1) \star 1$.

   Solution. The answer is $\boxed{\frac{34}{21}}$.

   $$\begin{aligned} 1 \star 1 &= 2 \\ 2 \star 1 &= \frac{3}{2} \\ \frac{3}{2} \star 1 &= \frac{5}{3} \\ \frac{5}{3} \star 1 &= \frac{8}{5} \\ \frac{8}{5} \star 1 &= \frac{13}{8} \\ \frac{13}{8} \star 1 &= \frac{21}{13} \\ \frac{21}{13} \star 1 &= \frac{34}{21} \end{aligned}$$

   Alternatively, note that if $x = \frac{F_n}{F_{n-1}}$, where F_n is the n^{th} Fibonacci number (that is, $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$), then

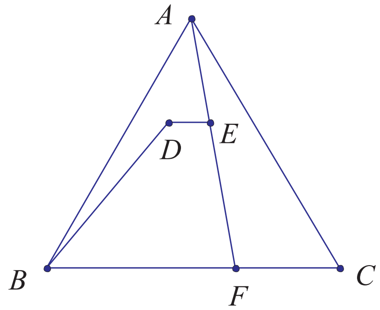
   $$x \star 1 = 1 + \frac{F_{n-1}}{F_n} = \frac{F_{n-1} + F_n}{F_n} = \frac{F_{n+1}}{F_n}.$$

Note also that

$$1 \star 1 = \frac{2}{1} = \frac{F_3}{F_2},$$

from which it follows that the number we want is $\dfrac{F_9}{F_8} = \dfrac{34}{21}$.

5. In acute triangle ABC, D and E are points inside triangle ABC such that $DE \parallel BC$, B is closer to D than it is to E, $\angle AED = 80°$, $\angle ABD = 10°$, and $\angle CBD = 40°$. Find the measure of $\angle BAE$, in degrees.



Solution. The answer is $\boxed{50°}$. Extend AE to meet BC at F. The angles of $\triangle ABF$ sum to $180°$. Therefore,

$$\begin{aligned}\angle BAE &= 180° - \angle ABF - \angle AFB = 180° - (\angle ABD + \angle DBC) - \angle AED \\ &= 180° - (10° + 40°) - 80° = 50°.\end{aligned}$$

6. Al is at $(0,0)$. He wants to get to $(4,4)$, but there is a building in the shape of a square with vertices at $(1,1)$, $(1,2)$, $(2,2)$, and $(2,1)$. Al cannot walk inside the building. If Al is not restricted to staying on grid lines, what is the shortest distance he can walk to get to his destination?

Solution. The answer is $\boxed{\sqrt{5}+\sqrt{13}}$. One possible shortest path is to go from $(0,0)$ to $(1,2)$ and then to $(4,4)$. From $(0,0)$ to $(1,2)$, the distance is $\sqrt{1^2+2^2} = \sqrt{5}$. From $(1,2)$ to $(4,4)$, the distance is $\sqrt{3^2+2^2} = \sqrt{13}$. The total length is $\sqrt{5}+\sqrt{13}$.

(To show that this is actually the shortest path: If we put a rubber band around nails at the points $(0,0)$ and $(4,4)$ and we stretch the rubber band so that it does not pass through the building, then if we let go of the rubber band, it will hit the building at $(1,2)$. Hence the distance is composed of two straight line segments as computed above.)

7. Point $A = (1, 211)$ and point $B = (b, 2011)$ for some integer b. For how many values of b is the slope of AB an integer?

Solution. The answer is $\boxed{72}$. The slope $m = \dfrac{2011-211}{b-1} = \dfrac{1800}{b-1}$ is an integer, so $(b-1)$ divides 1800. If x is a positive factor of 1800, then $1+x$ and $1-x$ are solutions for b, so each positive factor of 1800 should be counted twice. Because, $2^3 \cdot 3^2 \cdot 5^2 = 1800$, we have $\cdot(4 \cdot 3 \cdot 3) = 36$ positive factors of 1800 and therefore $2 \cdot 36 = 72$ possible values of b.

8. A palindrome is a number that reads the same forwards and backwards. For example, $1, 11$ and 141 are all palindromes. How many palindromes between 1 and 1000 are divisible by 11?

Solution. The answer is $\boxed{17}$. We have three cases; the palindrome has either 1, 2 or 3 digits.

- *Case 1* The palindrome has 1 digit. There are no one-digit palindromes that are divisible by 11.
- *Case 2* The palindrome has 2 digits. There are 9 two-digit palindromes, and all are divisible by 11.
- *Case 3* The palindrome has 3 digits. Note that for any given hundreds digit, there is at most one palindrome divisible by 11 (because if a is a palindrome divisible by 11, then the next palindrome divisible by 11 is at least $a + 121$). Thus, we can try to find the palindrome for each hundreds digit. The three-digit palindromes are $979, 858, 737, 616, 484, 363, 242, 121$, for a total of 8.

The total number of palindromes is therefore $9 + 8 = 17$.

9. Suppose x, y, z are real numbers that satisfy:

$$\begin{aligned} x + y - z &= 5 \\ y + z - x &= 7 \\ z + x - y &= 9. \end{aligned}$$

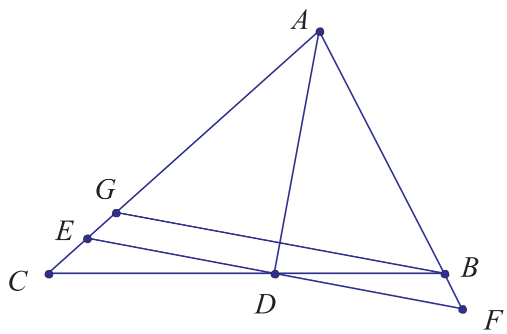
Find $x^2 + y^2 + z^2$.

Solution. The answer is $\boxed{149}$. Adding the equations gives $x + y + z = 21$, and subtracting each original equation from this yields

$$2z = 16 \quad 2x = 14 \quad 2y = 12.$$

Hence $(x, y, z) = (7, 6, 8)$ and $x^2 + y^2 + z^2 = 7^2 + 6^2 + 8^2 = 149$.

10. In triangle ABC, $AB = 3$ and $AC = 4$. The bisector of angle A meets BC at D. The line through D perpendicular to AD intersects lines AB and AC at F and E, respectively. Compute $EC - FB$. (See the diagram shown below.)



Solution. The answer is $\boxed{\frac{1}{7}}$. Let G be the reflection of B over line AD. Then, $AG = AB = 3$ and $BF = GE$, so we find that $EC + EG = AC - AG = 4 - 3 = 1$. Because $GB \parallel EF$ and AD is the angle bisector of $\angle A$, we have

$$\frac{CE}{EG} = \frac{CD}{DB} = \frac{CA}{AB} = \frac{4}{3}.$$

Then, $EC = \frac{4}{7}$ and $BF = GE = \frac{3}{7}$, so $EC - FB = \frac{1}{7}$.

11. Bob has a six-sided die with a number written on each face such that the sums of the numbers written on each pair of opposite faces are equal to each other. Suppose that the numbers 109, 131, and 135 are written on three faces which share a corner. Determine the maximum possible sum of the numbers on the three remaining faces, given that all three are positive primes less than 200.

Solution. The answer is $\boxed{39}$. Let p, q, r be the primes opposite 109, 131, 135, respectively. Because 109, 131, and 135 have distinct remainders when divided by 3, it follows that p, q, and r have distinct remainders when divided by 3. We conclude that exactly one of p, q, r is 3. Because r is opposite the largest number, we have $r = 3$. This yields $p = 29$, and $q = 7$, so $p + q + r = 29 + 7 + 3 = 39$.

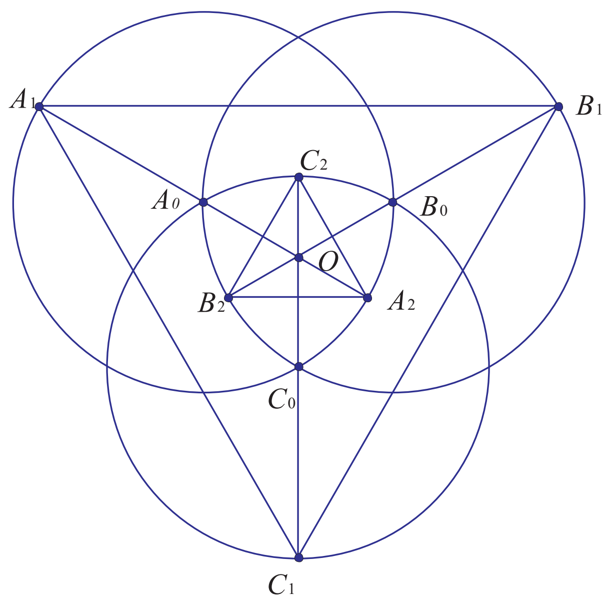
12. Let d be a number chosen at random from the set $\{142, 143, \ldots, 198\}$. What is the probability that the area of a rectangle with perimeter 400 and diagonal length d is an integer?

Solution. The answer is $\boxed{\frac{29}{57}}$. Let x, y be the side lengths of the rectangle, and K be the area. Hence

$$\begin{aligned} xy &= K \\ x + y &= 200 \\ x^2 + y^2 &= d^2 \end{aligned}$$

Squaring the second equation gives $200 = x^2 + y^2 + 2xy = d^2 + 2K$. It follows that K is an integer if and only if d is even. The set $\{142, 143, \ldots, 198\}$ has 29 even elements out of 57 total elements, so the probability that the area of the rectangle is an integer is $\frac{29}{57}$.

13. There are 3 congruent circles such that each circle passes through the centers of the other two. Suppose A, B, and C are points on the circles such that each circle has exactly one of A, B, or C on it and triangle ABC is equilateral. Find the ratio of the maximum possible area of ABC to the minimum possible area of ABC.



Solution. The answer is $\boxed{7+4\sqrt{3}}$. Without loss of generality, let the radius of each circle be 1. Let A_0, B_0, C_0 be the centers of the circles, $A_1B_1C_1$ be the largest triangle, and $A_2B_2C_2$ be the smallest triangle, with A_0 the center of the circle containing A_1 and A_2, and similarly for B_0 and C_0. Note that both triangles share a circumcenter. Let this be O. Note that $A_0B_0C_0$ is an equilateral triangle with sidelength 1 and circumradius $\dfrac{\sqrt{3}}{3}$. To find the ratio of the areas of the triangles, we can square the ratio of their circumradii. This is equal to:

$$\frac{(1-C_0O)^2}{(1+C_0O)^2} = \frac{(1+(\frac{\sqrt{3}}{3}))^2}{(1-(\frac{\sqrt{3}}{3}))^2} = 7+4\sqrt{3}.$$

14. Let k and m be constants such that for all triples (a, b, c) of positive real numbers,

$$\sqrt{\frac{4}{a^2}+\frac{36}{b^2}+\frac{9}{c^2}+\frac{k}{ab}} = \left|\frac{2}{a}+\frac{6}{b}+\frac{3}{c}\right| \quad \text{if and only if} \quad am^2+bm+c=0.$$

Find k.

Solution. The answer is $\boxed{20}$. We square both sides of the first equation and simplify to get

$$\frac{4}{a^2}+\frac{36}{b^2}+\frac{9}{c^2}+\frac{k}{ab} = \frac{4}{a^2}+\frac{36}{b^2}+\frac{9}{c^2}+\frac{24}{ab}+\frac{36}{bc}+\frac{12}{ca}$$

or

$$0 = \frac{24-k}{ab}+\frac{12}{ac}+\frac{36}{bc}.$$

Multiplying both sides of the last equation by abc yields $36a+12b+(24-k)c=0$. Solving this equation in c gives $c = \dfrac{12(3a+b)}{24-k}$.

Hence, for all triples (a, b, c) of positive real numbers,

$$c = \frac{12(3a+b)}{24-k} \quad \text{if and only if} \quad c = -am^2 - bm;$$

that is,

$$\frac{12(3a+b)}{24-k} = -am^2 - bm$$

or

$$a\left(\frac{36}{24-k} - m^2\right) + b\left(\frac{12}{24-k} - m\right) = 0.$$

Because this equation holds for all real numbers a and b, the coefficients of a and b must be 0; that is

$$\frac{36}{24-k} = m^2 \quad \text{and} \quad \frac{12}{24-k} = m.$$

Dividing the first equation by the second equation yields $m=3$, from which it follows that $k=20$.

15. A bored student named Abraham is writing n numbers $a_1, a_2, \ldots, a_n$. The value of each number is either 1, 2, or 3; that is, a_i is 1, 2 or 3 for $1 \le i \le n$. Abraham notices that the ordered triples

$$(a_1, a_2, a_3), \quad (a_2, a_3, a_4), \quad \ldots, \quad (a_{n-2}, a_{n-1}, a_n), \quad (a_{n-1}, a_n, a_1), \quad (a_n, a_1, a_2)$$

are distinct from each other. What is the maximum possible value of n? Give the answer n, along with an example of such a sequence. Write your answer as an ordered pair. (For example, if the answer were 5, you might write $(5, 12311)$.)

Solution. A possible answer is $\boxed{(27, 122322213211131123331332312)}$. Note that there are $3^3 = 27$ distinct possible sequences of 1, 2, 3. If we use n numbers, the resulting sequence has n ordered triples, which must be distinct, so $n \leq 27$. For $n = 27$, we consider the following sequence:

$$122322213211131123331332312$$

which gives no repeated ordered triples.

Note: Can you find other examples (that are not circular permutations of the current example) to support the conclusion $n = 27$?

2.4 Guts Test Solutions

2.4.1 Round 1

1. [3pts] In order to make good salad dressing, Bob needs a 0.9% salt solution. If soy sauce is 15% salt, how much water, in mL, does Bob need to add to 3 mL of pure soy sauce in order to have a good salad dressing?

   Solution. The answer is $\boxed{47}$. Let x be the amount of water Bob adds to the solution. Because the amount of salt remains the same, we have $3 \cdot 0.15 = (3 + x) \cdot .009$, implying that $x = 47$.

2. [3pts] Alex the Geologist is buying a canteen before he ventures into the desert. The original cost of a canteen is \$20, but Alex has two coupons. One coupon is \$3 off and the other is 10% off the entire remaining cost. Alex can use the coupons in any order. What is the least amount of money he could pay for the canteen?

   Solution. The answer is $\boxed{\$15}$. If he uses the first coupon and then the second he pays $(20 - 3) \cdot 0.9 = \$15.30$. However, if he uses the second and then the first, he only has to pay $(20 \cdot 0.9) - 3 = \$15$.

3. [3pts] Steve and Yooni have six distinct teddy bears to split between them, including exactly 1 blue teddy bear and 1 green teddy bear. How many ways are there for the two to divide the teddy bears, if Steve gets the blue teddy bear and Yooni gets the green teddy bear? (The two do not necessarily have to get the same number of teddy bears, but each teddy bear must go to a person.)

   Solution. The answer is $\boxed{16}$. We know that Steve gets the blue teddy bear and Yooni gets the green one. Then Yooni and Steve have to divide the 4 remaining teddy bears. Now, each bear can go to two possible people (Yooni or Steve), and we have four bears, so we have $2 \cdot 2 \cdot 2 \cdot 2 = 16$ ways to distribute them.

2.4.2 Round 2

4. [5pts] In the currency of Mathamania, 5 wampas are equal to 3 kabobs and 10 kabobs are equal to 2 jambas. How many jambas are equal to twenty-five wampas?

   Solution. The answer is $\boxed{3}$. The exchange rate for wampas to kabobs is $\frac{3}{5}$, and the rate for kabobs to jambas is $\frac{2}{10} = \frac{1}{5}$, so 25 wampas is equivalent to $25 \cdot \frac{3}{5} \cdot \frac{1}{5} = 3$ jambas.

5. [5pts] A sphere has a volume of 81π. A new sphere with the same center is constructed with a radius that is $\frac{1}{3}$ the radius of the original sphere. Find the volume, in terms of π, of the region between the two spheres.

   Solution. The answer is $\boxed{78\pi}$. The ratio of the radius of the new sphere to that of the original sphere is 1:3, so the ratio of their volumes is $1 : 3^3$, or $1 : 27$. Therefore the volume of the new sphere is $\frac{1}{27} \cdot 81\pi = 3\pi$. The volume between the two spheres is thus $81\pi - 3\pi = 78\pi$.

6. [5pts] A frog is located at the origin. It makes four hops, each of which moves it either 1 unit to the right or 1 unit to the left. If it also ends at the origin, how many 4-hop paths can it take?

   Solution. The answer is $\boxed{6}$. Since the frog ends at the origin, it must move an equal number of right and left steps. Thus, it must choose 2 steps out of 4 to go to the right, so there are $\binom{4}{2} = 6$ paths total.

2.4.3 Round 3

7. [6pts] Nick multiplies two consecutive positive integers to get $4^5 - 2^5$. What is the smaller of the two numbers?

Solution. The answer is $\boxed{31}$. We see that $4^5 - 2^5 = (2^5) \cdot (2^5 - 1) = 32 \cdot 31$.

8. [6 pts] In rectangle $ABCD$, E is a point on segment CD such that $\angle EBC = 30°$ and $\angle AEB = 80°$. Find $\angle EAB$, in degrees.

Solution. The answer is $\boxed{40°}$. We note that $\angle ABE = 90° - 30° = 60°$. It follows that $\angle EAB = 180° - 60° - 40° = 40°$.

9. [6pts] Mary's secret garden contains clones of Homer Simpson and WALL-E. A WALL-E clone has 4 legs. Meanwhile, Homer Simpson clones are human and therefore have 2 legs each. A Homer Simpson clone always has 5 donuts, while a WALL-E clone has 2. In Mary's secret garden, there are 184 donuts and 128 legs. How many WALL-E clones are there?

Solution. The answer is $\boxed{17}$. Let H be the number of Homer clones, and let W be the number of WALL-E clones. By the given conditions, we have $4W + 2H = 128$ legs and $2W + 5H = 184$ donuts. Solving these equations yields $H = 30$ and $W = 17$.

2.4.4 Round 4

10. [7pts] Including Richie, there are 6 students in a math club. Each day, Richie hangs out with a different group of club mates, each of whom gives him a dollar when he hangs out with them. How many dollars will Richie have by the time he has hung out with every possible group of club mates?

Solution. The answer is $\boxed{80}$. Since each person other than Richie can either be in a group or not be in a group, there are $2^5 = 32$ groups. An individual student (other than Richie) is equally likely to be or not to be in one of these groups; that is, an individual student (other than Richie) is in exactly $\frac{32}{2} = 16$ of these groups, and thus will contribute \$16. There are five students, so the answer is 80.

11. [7pts] There are seven boxes in a line: three empty, three holding \$10 each, and one holding the jackpot of $\$1,000,000$. From the left to the right, the boxes are numbered 1, 2, 3, 4, 5, 6 and 7, in that order. You are told the following:

- No two adjacent boxes hold the same contents.
- Box 4 is empty.
- There is one more \$10 prize to the right of the jackpot than there is to the left.

Which box holds the jackpot?

Solution. The answer is $\boxed{\text{Box 3}}$. There are two \$10 boxes are to the right of the jackpot, and one to the left, so the jackpot cannot be in boxes 1, 6, or 7. We know that box 4 is empty so the jackpot is in box 2, 3, or 5. If the jackpot is in box 5, then boxes 6 and 7 would both hold \$10 prizes, which is impossible. If the jackpot is in box 2, then box 1 must hold \$10, but then there is no way to arrange the other two empty boxes without violating the first condition. Hence the jackpot must be in box 3 with arrangement (\$10, empty, Jackpot, empty, \$10, empty, \$10).

12. [7pts] Let a and b be real numbers such that $a + b = 8$. Let c be the minimum possible value of $x^2 + ax + b$ over all real numbers x. Find the maximum possible value of c over all such a and b.

Solution. The answer is $\boxed{9}$. By substituting $b = 8 - a$ and completing the square (for x), we have

$$x^2 + ax + b = x^2 + ax + 8 - a = \left(x + \frac{a}{2}\right)^2 + 8 - a - \frac{a^2}{4} \geq 8 - a - \frac{a^2}{4}.$$

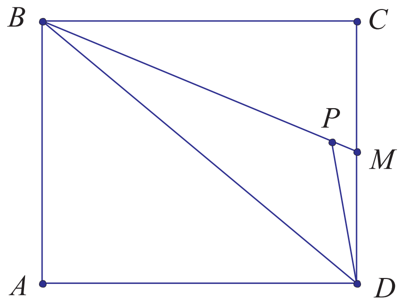
Hence for a fixed pair (a, b), $c = 8 - a - \frac{a^2}{4}$. Completing the square (for a) yields

$$c = 8 - a - \frac{a^2}{4} = -\frac{(a+2)^2}{4} + 9,$$

from which it follows that the maximum possible value of c is 9 with $a = -2$ and $b = 10$.

2.4.5 Round 5

13. [9pts] Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 12$. Let M be the midpoint of CD, and P be the point on BM such that $BP = BC$. Find the area of $ABPD$.



Solution. The answer is $\boxed{\frac{1140}{13}}$. (See the diagram shown above.) We will use $[\mathcal{R}]$ to denote the area of a region $\mathcal{R}$. (*This notation also applies to later solutions.*) We have $[APDB] = [ABCD] - [MBC] - [DPM]$. The area of MBC is $\frac{1}{2} \cdot \frac{10}{2} \cdot 12 = 30$. Note that MBC is a 5-12-13 triangle, so $BM = 13$, $BP = 12$, and $PM = 1$. Because triangles DPM and DBM share a common altitude from D, the area of DPM is $\frac{PM}{BM} \cdot [DMB] = \frac{1}{13}[DMB]$. In addition, $[DMB] = \frac{1}{2}[DBC]$, because both triangles share a common altitude. Finally, because $[DBC] = \frac{1}{2} \cdot 10 \cdot 12 = 60$, it follows that $[DMB] = 30$ and $[DPM] = \frac{30}{13}$. Thus, the area of quadrilateral $APDB$ is

$$[ABCD] - [DPM] - [MBC] = 120 - \frac{30}{13} - 30 = \frac{1140}{13}.$$

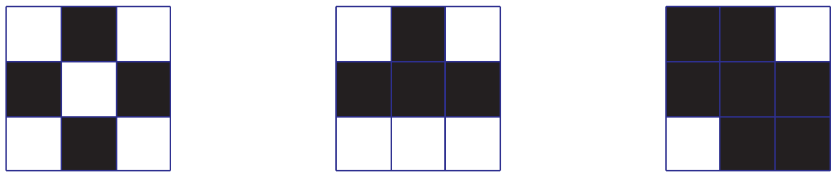
14. [9pts] The number 19 has the following properties:

- It is a 2-digit positive integer.
- It is the two leading digits of a 4-digit perfect square, because $1936 = 44^2$.

How many numbers, including 19, satisfy these two conditions?

Solution. The answer is $\boxed{65}$. The numbers $51^2, 52^2, \ldots, 99^2$ each have distinct leading digit pairs, because $(k+1)^2 - k^2 = 2k+1 > 100$ for $k > 50$, so the difference between consecutive perfect squares in this set is greater than 100. The numbers $32^2, 33^2, ..., 50^2$ cover each of the leading digit pairs 10 through 25, inclusive (because the difference between consecutive perfect squares here is less than 100), so we have $16 + 49 = 65$ numbers with the requisite properties.

15. [9pts] In a 3×3 grid, each unit square is colored either black or white. A coloring is considered "nice" if there is at most one white square in each row or column. What is the total number of nice colorings? Rotations and reflections of a coloring are considered distinct. (For example, in the three squares shown below, only the rightmost one has a nice coloring.)



Solution. The answer is $\boxed{34}$. Consider a few cases based on the number of white squares. There is one configuration for zero white squares. If we add one white square, we always satisfy the condition, so there are 9 ways to do this. For two white squares, there are four possible squares where the second white square could be placed. Because the order of the white squares doesn't matter, the number of configurations with two white squares is then $\frac{9 \cdot 4}{2!} = 18$. Similarly, the number of configurations for three white squares is $\frac{9 \cdot 4}{3!} = 6$, since the order doesn't matter and there is only possible space for the last white square after the first two have been fixed. The total is thus $1 + 9 + 18 + 6 = 34$ nice colorings.

2.4.6 Round 6

16. [11pts] Let $a_1, a_2, \ldots, a_{2011}$ be a sequence of numbers such that $a_1 = 2011$ and $a_1 + a_2 + \cdots + a_n = n^2 a_n$ for $n = 1, 2, \ldots, 2011$. (That is, $a_1 = 1^2 \cdot a_1$, $a_1 + a_2 = 2^2 \cdot a_2$, etc.) Compute a_{2011}.

Solution. The answer is $\boxed{\frac{1}{1006}}$. Because $a_1 + a_2 + \cdots + a_{n-1} + a_n = n^2 a_n$, it follows that $(n-1)^2 a_{n-1} + a_n = n^2 a_n$, from which it follows that $a_n = \frac{n-1}{n+1} \cdot a_{n-1}$. Thus,

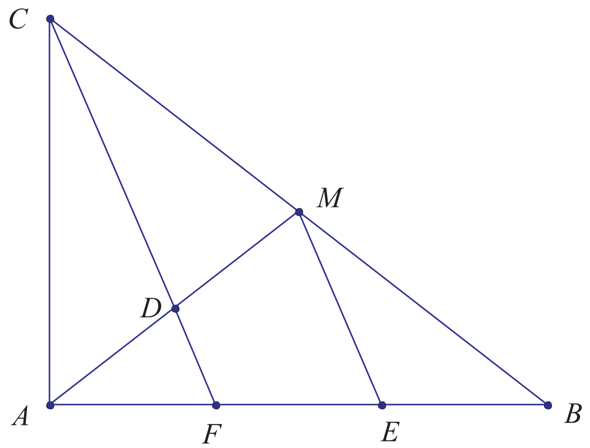
$$a_n = a_1 \cdot \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdots \frac{n-1}{n+1}.$$

Most of the terms in this product cancel, leaving $a_{2011} = 2011 \cdot \frac{1 \cdot 2}{2011 \cdot 2012} = \frac{1}{1006}$.

17. [11pts] Three rectangles, with dimensions 3×5, 4×2, and 6×4, are each divided into unit squares which are alternately colored black and white like a checkerboard. Each rectangle is cut along one of its diagonals into two triangles. For each triangle, let m be the total black area and n the total white area. Find the maximum value of $|m - n|$ for the 6 triangles.

Solution. The answer is $\boxed{\frac{1}{2}}$. Each of the three rectangles has either odd by odd dimensions or even by even dimensions. Whenever a diagonal is drawn, the two triangles produced are congruent in shape and color. Thus, because an even by even rectangle has the same number of black and white squares total, $|m-n|$ for triangles produced from even by even rectangles will always equal 0. Meanwhile, because an odd by odd rectangle has 1 more square of one color than of the other, $|m-n| = \frac{1}{2}$ for triangles produced from odd by odd rectangles.

18. [11pts] In triangle ABC, $\angle BAC = 90°$, and the length of segment AB is 2011. Let M be the midpoint of BC and D the midpoint of AM. Let E be the point on segment AB such that $EM||CD$. What is the length of segment BE?



Solution. The answer is $\boxed{\frac{2011}{3}}$. (See the diagram shown above.) Let CD intersect AB at F. We can see because $AD = DM$ and triangles AFD and AEM are similar to each other, it follows that $AF = FE$. Similarly, because $BM = MC$ and triangles BEM and BFC are similar to each other, it follows that $FE = EB$. Because $AF = FE = EB$, we have $EB = \frac{2011}{3}$.

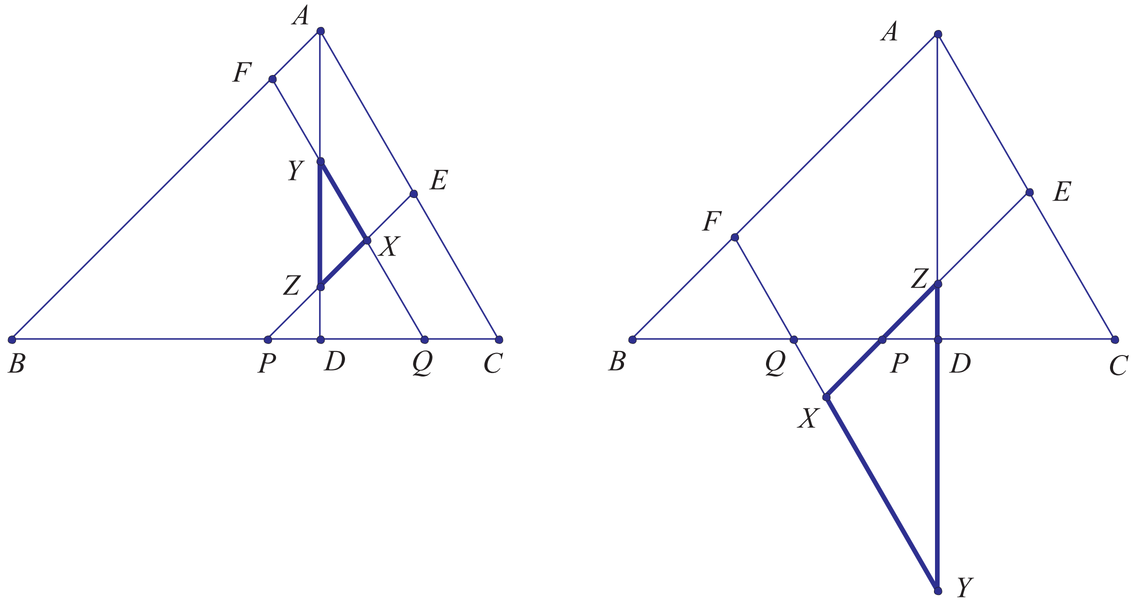
2.4.7 Round 7

19. [12pts] How many integers from 1 to 100, inclusive, can be expressed as the difference of two perfect squares? (For example, $3 = 2^2 - 1^2$).

Solution. The answer is $\boxed{75}$. A difference of integer squares a^2-b^2 can be factored as $(a+b)(a-b)$, the product of two integers of the same parity. Thus, for an even integer to be a difference of integer squares, it must be a product of two even integers, and therefore must be divisible by 4. Conversely, any integer multiple of 4 may be expressed as the difference of two integer squares, because $4n = (n+1)^2-(n-1)^2$. Any odd number may be expressed as the difference of two integer squares, because $2n+1 = (n+1)^2-n^2$. There are 25 multiples of 4 from 1 to 100 and 50 odd numbers. The answer is therefore $25 + 50 = 75$.

20. [12pts] In triangle ABC, $\angle ABC = 45°$ and $\angle ACB = 60°$. Let P and Q be points on segment BC, F a point on segment AB, and E a point on segment AC such that $FQ \parallel AC$ and $EP \parallel AB$. Let D be

the foot of the altitude from A to BC. The lines AD, FQ, and PE form a triangle. Find the positive difference, in degrees, between the largest and smallest angles of this triangle.



Solution. The answer is $\boxed{75^\circ}$. Let FQ and EP intersect at X. Then $\angle PXQ = A = 180-60-45 = 75$. Then the largest angle of the triangle formed is the angle adjacent to $\angle PXQ$, or $\angle 180^\circ - \angle PXQ = 105^\circ$. The other angles are $90^\circ - \angle B = 45^\circ$ and $90^\circ - 60^\circ = 30^\circ$, so the difference is $105^\circ - 30^\circ = 75^\circ$. (See the diagrams above for two of the few possible configurations of the triangle. It is important to note that the any resulting triangles XYZ are always similar to each other, because the angles between respective parallel lines in different configurations remain the same.)

21. [12pts] For real number x, $\lceil x \rceil$ is equal to the smallest integer larger than or equal to x. For example, $\lceil 3 \rceil = 3$ and $\lceil 2.5 \rceil = 3$. Let $f(n)$ be a function such that $f(n) = \left\lceil \frac{n}{2} \right\rceil + f\left(\left\lceil \frac{n}{2} \right\rceil\right)$ for every integer n greater than 1. If $f(1) = 1$, find the maximum value of $f(k) - k$, where k is a positive integer less than or equal to 2011.

Solution. The answer is $\boxed{10}$. Let $g(n) = f(n) - n$. We see that

$$f(n) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \cdots + \left\lceil \frac{n}{2^j} \right\rceil,$$

where 2^j is the smallest power of 2 such that $2^j > n$, and that

$$n = \left(\frac{n}{2} + \frac{n}{4} + \cdots + \frac{n}{2^{j-1}} + \frac{n}{2^j}\right) + \frac{n}{2^j}.$$

Then,

$$g(n) = f(n) - n = \left[\left(\left\lceil \frac{n}{2} \right\rceil - \frac{n}{2}\right) + \left(\left\lceil \frac{n}{4} \right\rceil - \frac{n}{4}\right) + \cdots + \left(\left\lceil \frac{n}{2^j} \right\rceil - \frac{n}{2^j}\right)\right] - \frac{n}{2^j}$$

Note that the quantity

$$\left\lceil \frac{n}{k} \right\rceil - \frac{n}{k}$$

is maximized if and only if n is congruent to 1 modulo k. We will show that the maximum value of $g(n)$ over $[2^i, 2^{i+1}-1]$ is achieved at $n = 2^i + 1$, for all i. Therefore, to maximize $g(n) + \frac{n}{2^j}$, we want

$$n \equiv 1 \pmod 2, \quad n \equiv 1 \pmod 4, \quad n \equiv 1 \pmod 8, \quad \ldots, \quad n \equiv 1 \pmod{2^j}.$$

As long as $\frac{n}{2^i}$ is minimized, $g(n)$ is maximized over $[2^i, 2^{i+1}-1]$ at $n = 2^i + 1$. Thus, we only have to check $n = 2^i$ for a potentially larger value of $g(n)$. Let $a_k = f(2^k)$. Then, we have $a_0 = 1$, $a_k = 2^{k-1} + a_{k-1}$. We can prove by induction that $a_k = 2^k$ for all k. So, $g(2^k) = 0$, for all k. Let $b_k = g(2^k + 1)$. Then, we have $b_0 = 0$,

$$b_k = 2^{k-1} + 1 + (b_{k-1} + 2^{k-1} + 1) - 2^k - 1 = 2^k + 1 - 2^k + b_{k-1} = 1 + b_{k-1}.$$

Thus, $b_k = k$ for all $k > 0$, by induction on k. Because $g(2^k + 1) = k$, we have proved that the maximum value of $g(n)$ over $[2^i, 2^{i+1}-1]$ is $g(2^i + 1) = i$. In addition, because $2^{10} < 2011 < 2^{11}$, the maximum value of $g(n)$ is achieved when $n = 1025 = 2^{10} + 1$, in which case $g(1025) = 10$.

2.4.8 Round 8

The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.

22. [14pts] Let W be the answer to problem 24 in this guts round. Let $f(a) = \frac{1}{1 - \frac{1}{1-\frac{1}{a}}}$. Determine $|f(2) + \cdots + f(W)|$.

Solution. The answer is $\boxed{66}$. We have

$$f(a) = \frac{1}{1 - \frac{1}{1-\frac{1}{a}}} = \frac{1}{1 - \frac{a}{a-1}} = 1 - a.$$

Hence

$$|f(1) + f(2) + \cdots + f(W)| = |W - (1 + 2 + \cdot + W)| = |W - \frac{W(W+1)}{2}| = \frac{W(W-1)}{2}.$$

For $W = 12$, the answer is 66.

23. [14pts] Let X be the answer to problem 22 in this guts round. How many odd perfect squares are less than $8X$?

Solution. The answer is $\boxed{11}$. It is given from problem 22 that $X = \frac{W(W-1)}{2}$, so $8X = 4W(W-1)$. Note that $(2W-2)^2 < 4W(W-1) < (2W-1)^2$, so the number of odd perfect squares less than 8X is $W - 1$. For $W = 12$, the answer is 11.

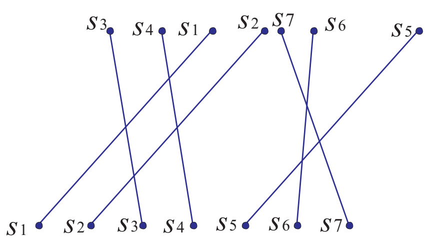
24. [14pts] Let Y be the answer to problem 23 in this guts round. What is the maximum number of points of intersections of two regular $(Y-5)$-sided polygons, if no side of the first polygon is parallel to any side of the second polygon?

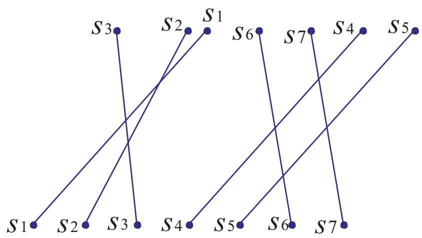
Solution. The answer is $\boxed{12}$. You are given that $Y = W - 1$ from problem 23. The maximum number of intersections of two regular k-sided polygons is $2(Y-5) = 2(W-6) = W$ or $W = 12$.

2.4.9 Round 9

25. [16pts] Cross country skiers $s_1, s_2, s_3, \ldots, s_7$ start a race one by one in that order. While each skier skis at a constant pace, the skiers do not all ski at the same rate. In the course of the race, each skier either overtakes another skier or is overtaken by another skier exactly two times. Find all the possible orders in which they can finish. Write each possible finish as an ordered septuplet (a, b, c, d, e, f, g) where a, b, c, d, e, f, g are the numbers 1-7 in some order. (So a finishes first, b finishes second, etc.)

Solution. The answers are $\boxed{(s_3, s_2, s_1, s_6, s_7, s_4, s_5)}$ and $\boxed{(s_3, s_4, s_1, s_2, s_7, s_6, s_5)}$. Clearly the s_7 must finish 5$^{\text{th}}$, and s_1 must finish 3$^{\text{rd}}$. If the s_6 finishes 7$^{\text{th}}$, then he has only been passed by s_7, so he must finish 6$^{\text{th}}$ or 4$^{\text{th}}$. In both cases s_5 must finish 7$^{\text{th}}$, and similarly s_3 must finish 1$^{\text{st}}$. If he finishes 6$^{\text{th}}$, we must only determine the place of s_4 and s_2. Because s_4 cannot be overtaken by s_7, s_6, s_5 (because these have already overtaken or been overtaken twice), s_4 cannot place 4$^{\text{th}}$, because it has not overtaken or been overtaken. Then s_4 must place 2nd and s_22 must place 4$^{\text{th}}$ - so $(s_3, s_2, s_1, s_6, s_7, s_4, s_5)$ is a solution. When s_6 places 4$^{\text{th}}$, s_4 must place 6$^{\text{th}}$, or another skier will have to be overtaken more than two times. This means, s_2 places 2nd, so $(s_3, s_4, s_1, s_2, s_7, s_6, s_5)$ is the other solution. (Please see the diagrams shown below for a geometric interpretation of the solution.)

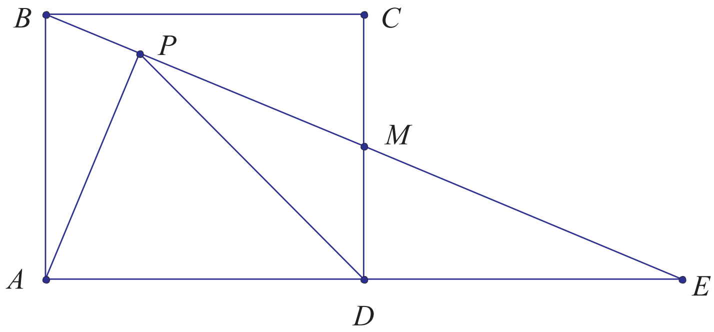




26. [16pts] Archie the Alchemist is making a list of all the elements in the world, and the proportion of earth, air, fire, and water needed for producing each. He writes the proportions in the form E:A:F:W (E for Earth, A for Air, F for Fire, and W for water). If each of the letters represents a whole number from 0 to 4, inclusive, how many different elements can Archie list? Note that if Archie lists Wood as 2:0:1:2, then 4:0:2:4 would also produce wood, and that 0:0:0:0 does not produce an element.

Solution. The answer is $\boxed{529}$. Note that, except for $0:0:0:0$, there are $5^4 - 1 = 624$ proportions if we ignore the condition that multiples of a proportion produce the same element. However, there are two cases of overcounting. Firstly, doubling: $2:0:1:2$ and $4:0:2:4$ produce the same element. In this case, there are $3^4 - 1 = 80$ overcounts. Secondly, tripling and quadrupling: $(1:0:1:1 = 4:0:4:4 = 3:0:3:3)$. Here, there are $2^4 - 1 = 15$ overcounts from the tripling. The quadruples have already been counted by the doubling case. Hence, there are $624 - 80 - 15 = 529$ proportions.

27. [16pts] Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 12$. Let M be the midpoint of CD, and P be the point on BM such that $DP = DA$. Find the area of quadrilateral $ABPD$.



Solution. The answer is $\boxed{\frac{11640}{169}}$. Extend segment BM through M to meet ray AD at point E. It is easy to see that triangle BCM is congruent to triangle EDM. In particular, we conclude that $DE = BC = AD = DP$. This means that, in triangle APE, median PD is equal to half of the side AE; that is, APE is a right triangle with $\angle APE = 90°$. Note that both triangles APB and APE are similar to triangle BCM, which is a 5-12-13 triangle. Note also that $[APD] = \frac{[APE]}{2}$ because $AD = DE$. Thus,

$$[ABP] = \left(\frac{AB}{BM}\right)^2 \cdot [BCM] = \frac{100[BCM]}{169}$$

and

$$[APD] = \frac{[APE]}{2} = \frac{1}{2} \cdot \left(\frac{AE}{BM}\right)^2 \cdot [BCM] = \frac{288[BCM]}{169}.$$

It follows that $[ABPD] = \frac{388[BCM]}{169} = \frac{388 \cdot 30}{169} = \frac{11640}{169}$.

2.4.10 Round 10

28. [17pts] David the farmer has an infinitely large grass-covered field which contains a straight wall. He ties his cow to the wall with a rope of integer length. The point where David ties his rope to the wall divides the wall into two parts of length a and b, where $a > b$ and both are integers. The rope is shorter than the wall but is longer than a. Suppose that the cow can reach grass covering an area of $\frac{165\pi}{2}$. Find the ratio $\frac{a}{b}$. You may assume that the wall has 0 width.

Solution. The answer is $\boxed{\frac{9}{2}}$. The cow eats the grass in front of the wall with the area of $\frac{1}{2} \cdot \pi \cdot \ell^2$, and eats the grass behind the wall with the area of

$$\frac{1}{2} \cdot \pi \cdot (\ell - a)^2 + \frac{1}{2} \cdot \pi \cdot (\ell - b)^2.$$

Therefore $\ell^2 + (\ell - a)^2 + (\ell - b)^2 = 165$. Note that $165 = 10^2 + 7^2 + 4^2$ or $165 = 10^2 + 8^2 + 1^2$; that is, $(\ell, a, b) = (10, 6, 3)$ or $(\ell, a, b) = (10, 9, 2)$. Because $a + b > \ell$, we must have $(\ell, a, b) = (10, 9, 2)$ and $\frac{a}{b} = \frac{9}{2}$.

29. [17pts] Let S be the number of ordered quintuples (a, b, x, y, n) of positive integers such that

$$\begin{aligned} \frac{a}{x} + \frac{b}{y} &= \frac{1}{n}, \\ abn &= 2011^{2011}. \end{aligned}$$

Compute the remainder when S is divided by 2012.

Solution. The answer is $\boxed{1006}$. Because 2011 is a prime, by the second given equation, we may set $n = 2011^m, a = 2011^k, b = 2011^{2011-m-k}$ for some nonnegative integers m, k. Clearly, the possible values of m are $0, 1, 2, \ldots, 2011$. For every $0 \le m \le 2011$, there are $2012 - m$ possible values (namely, $0, 1, \ldots, 2011 - m$) for k; that is, for each fixed $n = 2011^m$, there are $2012 - m$ triples $(a, b, n) = \left(2011^k, 2011^{2011-m-k}, 2011^m\right)$ of positive integers satisfy the second given equation. For a fixed such triple (a, b, n), we claim that there are $2012 + m$ pairs (x, y) of positive integers satisfy the first given equation. With this claim, we have established the fact that for each $0 \le m \le 2011$, there are $(2012 - m)(2012 + m) = 2012^2 - m^2$ quintuples (a, b, x, y, n) of positive integers satisfy the given system of the equation; that is,

$$\begin{aligned} S &= \left(2012^2 - 0^2\right) + \left(2012^2 - 1^2\right) + \cdots + \left(2012^2 - 2011^2\right) = 2012^3 - \left(1^2 + 2^2 \cdots + 2011^2\right) \\ &= 2012^3 - \frac{2011 \cdot 2012 \cdot (2 \cdot 2011 + 1)}{6}. \end{aligned}$$

It is easy to check that S has remainder 2006 when divided by 2012.

To complete our solution, it suffices to establish our claim. Multiplying both sides of the first given equation by abn yields $ayn + bxn = xy$ or $xy - bnx - any = 0$. Adding abn^2 to both sides the last equation gives $xy - bnx - any + abn^2 = abn^2$ or

$$(x - an)(y - bn) = 2011^{2011+m}.$$

Because 2011 is a prime, we may set $(x - an, y - bn) = \left(2011^\ell, 2011^{2011+m-\ell}\right)$ or

$$(x, y) = \left(2011^\ell + an, 2011^{2011+m-\ell} + bn\right).$$

Clearly, there are exactly $2012 + m$ possible values for ℓ, from which our claim follows.

30. [17pts] Let n be a positive integer. An $n \times n$ square grid is formed by n^2 unit squares. Each unit square is then colored either red or blue such that each row or column has exactly 10 blue squares. A *move* consists of choosing a row or a column, and recolor each unit square in the chosen row or column – if it is red, we recolor it blue, and if it is blue, we recolor it red. Suppose that it is possible to obtain fewer than $10n$ blue squares after a sequence of finite number of moves. Find the maximum possible value of n.

Solution. The answer is $\boxed{39}$. First, we prove $n < 40$. Assume that we make moves on a rows and b columns. For each unit square, its final color depends only on the parity of the moves on the row and the column it lies on. Hence the order of the moves does not matter, and we may assume that each row or column is chosen at most once. It follows that there are $f(a, b) = (n - a)b + (n - b)a$ squares have changed their color, and we call these squares *chameleons*. Because $f(a, b) = f(n - a, n - b)$, we may assume that $a + b \le n$. For the final resulting board to have fewer than $10n$ blue squares, more than half of the chameleons were blue at the beginning stage. On the other hand, at the beginning stage, at most $10a + 10b$ blue squares lie on the chosen rows and columns where the moves were made, and only they can be the candidates for the chameleons that were blue at the beginning stage. Therefore, we have

$$\frac{1}{2}[(n-a)b+(n-b)a] < 10(a+b)$$

Multiplying both sides of the last inequality by 2 and rearranging the terms gives

$$n(a+b) < 20(a+b)+2ab \quad \text{or} \quad 2n(a+b) < 40(a+b)+4ab$$

Because $(a-b)^2 \geq 0$ or $(a+b)^2 \geq 4ab$, we have

$$2n(a+b) < 40(a+b)+4ab \leq 40(a+b)+(a+b)^2 \quad \text{or} \quad 2n < 40+(a+b).$$

Because $a+b \leq n$, we conclude that $2n < 40+(a+b) \leq 40+n$, implying that $n < 40$.

For $n = 39$, we consider the 39×39 board with a 19×29 all red sub-board on the top left, a 19×10 all blue sub-board on the top right, a 20×10 all blue sub-board on the bottom left, and a 20×19 all red sub-board on the bottom right. We can make moves on each of the bottom 20 rows and right 19 columns to obtain a board with $19 + 20 \times 10 = 371$ blue unit squares.