Exeter Math Club Contest
January 28, 2012

Contents

Organizing Information

- *Tournament Directors* Yong Wook (Spencer) Kwon, Abraham Shin
- *Tournament Supervisor* Zuming Feng
- *System Administrator and Webmaster* Albert Chu
- *Head Problem Writers* Ravi Bajaj, Dai Yang
- *Problem Committee* Ravi Bajaj, Ravi Jagadeesan, Yong Wook (Spencer) Kwon, Ray Li, Abraham Shin, Alex Song, Dai Yang, David Yang
- *Solutions Editors* Ravi Bajaj, Ravi Jagadeesan, Yong Wook (Spencer) Kwon, Ray Li, Alex Song, Dai Yang
- *Problem Reviewers* Zuming Feng, Chris Jeuell, Richard Parris
- *Problem Contributors* Ravi Bajaj, Priyanka Boddu, Mickey Chao, Bofan Chen, Jiapei Chen, Vivian Chen, Albert Chu, Trang Duong, Mihail Eric, Claudia Feng, Chelsea Ge, Mark Huang, Ravi Jagadeesan, Spencer Kwon, Jay Lee, Ray Li, Calvin Luo, Abraham Shin, Alex Song, Elizabeth Wei, Dai Yang, David Yang, Grace Yin, Will Zhang, Joy Zheng
- *Treasurer* Harlin Lee
- *Publicity* Jiexiong (Chelsea) Ge, Claudia Feng
- *Primary Tournament Sponsor* We would like to thank Jane Street Capital for their generous support of this competition.
- *Tournament Sponsors* We would also like to thank the Phillips Exeter Academy Math Department and Art of Problem Solving for their support.

Contest day information

- *Proctors* Emery Real Bird, Shi-fan Chen, Geoffrey Cheng, Yoon-Ho Chung, Claudia Feng, Ellen Gao, Andrew Holzman, Daniel Kim, Katie Kimberling, Max Le, Christina Lee, Francis Lee, Jay Lee, Scott Lu, Antong Liu, Jack Qiu, Tiffany Tuedor, Linh Tran, Andy Wei, Grace Yin, Sam Yoo, Nicole Yoon, Abdullah Yousufi, Henry Zhao
- *Head Graders* Albert Chu, Yong Wook (Spencer) Kwon,
- *Graders* Ravi Bajaj, Ravi Charan, Bofan Chen, In Young Cho, Albert Chu, Ravi Jagadeesan, Spencer Kwon, Ray Li, David Rush, Alex Song, David Xiao, Dai Yang, Allen Yuan, Will Zhang, Joy Zheng
- *Judges* Zuming Feng, Greg Spanier
- *Runners* Michelle (Hannah) Jung *(Head Runner)*, Mickey Chao, Max Vachon, Elizabeth Wei

Chapter 1

EMC^2 2012 Problems

1.1 Individual Speed Test

Morning, January 28, 2012

There are 20 problems, worth 3 points each, to be solved in 20 minutes.

1. Evaluate
$$\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}.$$

2. A regular hexagon and a regular n-sided polygon have the same perimeter. If the ratio of the side length of the hexagon to the side length of the n-sided polygon is $2:1$, what is n?

3. How many nonzero digits are there in the decimal representation of $2\cdot 10\cdot 500\cdot 2500$?

4. When the numerator of a certain fraction is increased by 2012, the value of the fraction increases by 2. What is the denominator of the fraction?

5. Sam did the computation $1-10\cdot a+22$, where a is some real number, except he messed up his order of operations and computed the multiplication last; that is, he found the value of $(1-10)\cdot(a+22)$ instead. Luckily, he still ended up with the right answer. What is a?

6. Let $n! = n\cdot(n-1)\cdots 2\cdot 1$. For how many integers n between 1 and 100 inclusive is $n!$ divisible by 36?

7. Simplify the expression $\sqrt{\dfrac{3\cdot 27^3}{27\cdot 3^3}}$.

8. Four points A, B, C, D lie on a line in that order such that $\dfrac{AB}{CB}=\dfrac{AD}{CD}$. Let M be the midpoint of segment AC. If $AB=6, BC=2$, compute $MB\cdot MD$.

9. Allan has a deck with 8 cards, numbered $1,1,2,2,3,3,4,4$. He pulls out cards without replacement, until he pulls out an even numbered card, and then he stops. What is the probability that he pulls out exactly 2 cards?

10. Starting from the sequence $(3,4,5,6,7,8,\ldots)$, one applies the following operation repeatedly. In each operation, we change the sequence
$$(a_1,a_2,a_3,\ldots,a_{a_1-1},a_{a_1},a_{a_1+1},\ldots)$$
to the sequence
$$(a_2,a_3,\ldots,a_{a_1},a_1,a_{a_1+1},\ldots)\,.$$
(In other words, for a sequence starting with x, we shift each of the next $x-1$ term to the left by one, and put x immediately to the right of these numbers, and keep the rest of the terms unchanged. For example, after one operation, the sequence is $(4,5,3,6,7,8,\ldots)$, and after two operations, the sequence becomes $(5,3,6,4,7,8,\ldots)$. How many operations will it take to obtain a sequence of the form $(7,\ldots)$ (that is, a sequence starting with 7)?

11. How many ways are there to place 4 balls into a 4×6 grid such that no column or row has more than one ball in it? (Rotations and reflections are considered distinct.)

12. Point P lies inside triangle ABC such that $\angle PBC=30^\circ$ and $\angle PAC=20^\circ$. If $\angle APB$ is a right angle, find the measure of $\angle BCA$ in degrees.

13. What is the largest prime factor of $9^3 - 4^3$?

14. Joey writes down the numbers 1 through 10 and crosses one number out. He then adds the remaining numbers. What is the probability that the sum is less than or equal to 47?

15. In the coordinate plane, a *lattice point* is a point whose coordinates are *integers*. There is a pile of grass at every lattice point in the coordinate plane. A certain cow can only eat piles of grass that are at most 3 units away from the origin. How many piles of grass can she eat?

16. A book has 1000 pages numbered $1, 2, \ldots, 1000$. The pages are numbered so that pages 1 and 2 are back to back on a single sheet, pages 3 and 4 are back to back on the next sheet, and so on, with pages 999 and 1000 being back to back on the last sheet. How many pairs of pages that are back to back (on a single sheet) share no digits in the same position? (For example, pages 9 and 10, and pages 89 and 90.)

17. Find a pair of integers (a, b) for which $\dfrac{10^a}{a!} = \dfrac{10^b}{b!}$ and $a < b$.

18. Find all ordered pairs (x, y) of real numbers satisfying

$$\begin{cases} -x^2 + 3y^2 - 5x + 7y + 4 &= 0 \\ 2x^2 - 2y^2 - x + y + 21 &= 0 \end{cases}$$

19. There are six blank fish drawn in a line on a piece of paper. Lucy wants to color them either red or blue, but will not color two adjacent fish red. In how many ways can Lucy color the fish?

20. There are sixteen 100-gram balls and sixteen 99-gram balls on a table (the balls are visibly indistinguishable). You are given a balance scale with two sides that reports which side is heavier or that the two sides have equal weights. A weighing is defined as reading the result of the balance scale: For example, if you place three balls on each side, look at the result, then add two more balls to each side, and look at the result again, then two weighings have been performed. You wish to pick out two different sets of balls (from the 32 balls) with equal numbers of balls in them but different total weights. What is the minimal number of weighings needed to ensure this?

1.2 Individual Accuracy Test

Morning, January 28, 2012

There are 10 problems, worth 9 points each, to be solved in 30 minutes.

1. An 18oz glass of apple juice is 6% sugar and a 6oz glass of orange juice is 12% sugar. The two glasses are poured together to create a cocktail. What percent of the cocktail is sugar?

2. Find the number of positive numbers that can be expressed as the difference of two integers between -2 and 2012 inclusive.

3. An *annulus* is defined as the region between two concentric circles. Suppose that the inner circle of an annulus has radius 2 and the outer circle has radius 5. Find the probability that a randomly chosen point in the annulus is at most 3 units from the center.

4. Ben and Jerry are walking together inside a train tunnel when they hear a train approaching. They decide to run in opposite directions, with Ben heading towards the train and Jerry heading away from the train. As soon as Ben finishes his 1200 meter dash to the outside, the front of the train enters the tunnel. Coincidentally, Jerry also barely survives, with the front of the train exiting the tunnel as soon as he does. Given that Ben and Jerry both run at 1/9 of the train's speed, how long is the tunnel in meters?

5. Let ABC be an isosceles triangle with $AB = AC = 9$ and $\angle B = \angle C = 75^\circ$. Let DEF be another triangle congruent to ABC. The two triangles are placed together (without overlapping) to form a quadrilateral, which is cut along one of its diagonals into two triangles. Given that the two resulting triangles are incongruent, find the area of the larger one.

6. There is an infinitely long row of boxes, with a Ditto in one of them. Every minute, each existing Ditto clones itself, and the clone moves to the box to the right of the original box, while the original Ditto does not move. Eventually, one of the boxes contains over 100 Dittos. How many Dittos are in that box when this first happens?

7. Evaluate
$$26 + 36 + 998 + 26 \cdot 36 + 26 \cdot 998 + 36 \cdot 998 + 26 \cdot 36 \cdot 998.$$

8. There are 15 students in a school. Every two students are either friends or not friends. Among every group of three students, either all three are friends with each other, or exactly one pair of them are friends. Determine the minimum possible number of friendships at the school.

9. Let $f(x) = \sqrt{2x + 1 + 2\sqrt{x^2 + x}}$. Determine the value of
$$\frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \cdots + \frac{1}{f(24)}.$$

10. In square $ABCD$, points E and F lie on segments AD and CD, respectively. Given that $\angle EBF = 45°$, $DE = 12$, and $DF = 35$, compute AB.

1.3 Team Test

Morning, January 28, 2012

There are 15 problems, worth 20 points each, to be solved in 45 minutes.

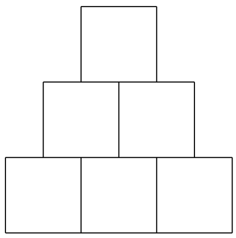
1. The longest diagonal of a regular hexagon is 12 inches long. What is the area of the hexagon, in square inches?

2. When Al and Bob play a game, either Al wins, Bob wins, or they tie. The probability that Al does not win is $\frac{2}{3}$, and the probability that Bob does not win is $\frac{3}{4}$. What is the probability that they tie?

3. Find the sum of the $a + b$ values over all pairs of integers (a, b) such that $1 \le a < b \le 10$. That is, compute the sum
$$(1 + 2) + (1 + 3) + (1 + 4) + \cdots + (2 + 3) + (2 + 4) + \cdots + (9 + 10).$$

4. A 3×11 cm rectangular box has one vertex at the origin, and the other vertices are above the x-axis. Its sides lie on the lines $y = x$ and $y = -x$. What is the y-coordinate of the highest point on the box, in centimeters?

5. Six blocks are stacked on top of each other to create a pyramid, as shown below. Carl removes blocks one at a time from the pyramid, until all the blocks have been removed. He never removes a block until all the blocks that rest on top of it have been removed. In how many different orders can Carl remove the blocks?



6. Suppose that a right triangle has sides of lengths $\sqrt{a + b\sqrt{3}}$, $\sqrt{3 + 2\sqrt{3}}$, and $\sqrt{4 + 5\sqrt{3}}$, where a, b are positive integers. Find all possible ordered pairs (a, b).

7. Farmer Chong Gu glues together 4 equilateral triangles of side length 1 such that their edges coincide. He then drives in a stake at each vertex of the original triangles and puts a rubber band around all the stakes. Find the minimum possible length of the rubber band.

8. Compute the number of ordered pairs (a, b) of positive integers less than or equal to 100, such that $a^b - 1$ is a multiple of 4.

9. In triangle ABC, $\angle C = 90°$. Point P lies on segment BC and is not B or C. Point I lies on segment AP. If $\angle BIP = \angle PBI = \angle CAB = m°$ for some positive integer m, find the sum of all possible values of m.

10. Bob has 2 identical red coins and 2 identical blue coins, as well as 4 distinguishable buckets. He places some, but not necessarily all, of the coins into the buckets such that no bucket contains two coins of the same color, and at least one bucket is not empty. In how many ways can he do this?

11. Albert takes a 4×4 checkerboard and paints all the squares white. Afterward, he wants to paint some of the square black, such that each square shares an edge with an odd number of black squares. Help him out by drawing one possible configuration in which this holds. (Note: the answer sheet contains a 4×4 grid.)

12. Let S be the set of points (x, y) with $0 \leq x \leq 5, 0 \leq y \leq 5$ where x and y are integers. Let T be the set of all points in the plane that are the midpoints of two distinct points in S. Let U be the set of all points in the plane that are the midpoints of two distinct points in T. How many distinct points are in U? (Note: The points in T and U do not necessarily have integer coordinates.)

13. In how many ways can one express 6036 as the sum of at least two (not necessarily positive) consecutive integers?

14. Let a, b, c, d, e, f be integers (not necessarily distinct) between -100 and 100, inclusive, such that $a+b+c+d+e+f = 100$. Let M and m be the maximum and minimum possible values, respectively, of
$$abc + bcd + cde + def + efa + fab + ace + bdf.$$
Find $\dfrac{M}{m}$.

15. In quadrilateral $ABCD$, diagonal AC bisects diagonal BD. Given that $AB = 20$, $BC = 15$, $CD = 13$, $AC = 25$, find DA.

1.4 Guts Test

Afternoon, January 28, 2012

There are 24 problems, with varying point values, to be solved in 75 minutes.

1.4.1 Round 1

1. [6pts] Ravi has a bag with 100 slips of paper in it. Each slip has one of the numbers 3, 5, or 7 written on it. Given that half of the slips have the number 3 written on them, and the average of the values on all the slips is 4.4, how many slips have 7 written on them?

2. [6pts] In triangle ABC, point D lies on side AB such that $AB \perp CD$. It is given that $\frac{CD}{BD} = \frac{1}{2}$, $AC = 29$, and $AD = 20$. Find the area of triangle BCD.

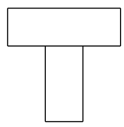
3. [6pts] Compute $(123 + 4)(123 + 5) - 123 \cdot 132$.

1.4.2 Round 2

4. [8pts] David is evaluating the terms in the sequence $a_n = (n + 1)^3 - n^3$ for $n = 1, 2, 3, \ldots$ (that is, $a_1 = 2^3 - 1^3, a_2 = 3^3 - 2^3, a_3 = 4^3 - 3^3$, and so on). Find the first composite number in the sequence. (An positive integer is *composite* if it has a divisor other than 1 and itself.)

5. [8pts] Find the sum of all positive integers strictly less than 100 that are not divisible by 3.

6. [8pts] In how many ways can Alex draw the diagram below without lifting his pencil or retracing a line? (Two drawings are different if the order in which he draws the edges is different, or the direction in which he draws an edge is different).



1.4.3 Round 3

7. [10pts] Fresh Mann is a 9th grader at Euclid High School. Fresh Mann thinks that the word *vertices* is the plural of the word *vertice*. Indeed, vertices is the plural of the word *vertex*. Using all the letters in the word *vertice*, he can make m 7-letter sequences. Using all the letters in the word *vertex*, he can make n 6-letter sequences. Find $m - n$.

8. [10pts] Fresh Mann is given the following expression in his Algebra 1 class:

$$101 - 102 = 1.$$

Fresh Mann is allowed to move some of the digits in this (incorrect) equation to make it into a correct equation. What is the minimal number of digits Fresh Mann needs to move?

9. [10pts] Fresh Mann said, "The function $f(x) = ax^2+bx+c$ passes through 6 points. Their x-coordinates are consecutive positive integers, and their y-coordinates are $34, 55, 84, 119, 160$, and 207, respectively." Sophy Moore replied, "You've made an error in your list," and replaced one of Fresh Mann's numbers with the correct y-coordinate. Find the corrected value.

1.4.4 Round 4

10. [12pts] An assassin is trying to find his target's hotel room number, which is a three-digit positive integer. He knows the following clues about the number:

    (a) The sum of any two digits of the number is divisible by the remaining digit.

    (b) The number is divisible by 3, but if the first digit is removed, the remaining two-digit number is not.

    (c) The middle digit is the only digit that is a perfect square.

    Given these clues, what is a possible value for the room number?

11. [12pts] Find a positive real number r that satisfies

$$\frac{4+r^3}{9+r^6} = \frac{1}{5-r^3} - \frac{1}{9+r^6}.$$

12. [12pts] Find the largest integer n such that there exist integers x and y between 1 and 20 inclusive with

$$\left|\frac{21}{19} - \frac{x}{y}\right| < \frac{1}{n}.$$

1.4.5 Round 5

13. [14pts] A unit square is rotated $30°$ counterclockwise about one of its vertices. Determine the area of the intersection of the original square with the rotated one.

14. [14pts] Suppose points A and B lie on a circle of radius 4 with center O, such that $\angle AOB = 90°$. The perpendicular bisectors of segments OA and OB divide the interior of the circle into four regions. Find the area of the smallest region.

15. [14pts] Let $ABCD$ be a quadrilateral such that $AB = 4$, $BC = 6$, $CD = 5$, $DA = 3$, and $\angle DAB = 90°$. There is a point I inside the quadrilateral that is equidistant from all the sides. Find AI.

1.4.6 Round 6

The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.

16. [16pts] Let C be the answer to problem 18. Compute
$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{C^2}\right).$$

17. [16pts] Let A be the answer to problem 16. Let $PQRS$ be a square, and let point M lie on segment PQ such that $MQ = 7PM$ and point N lie on segment PS such that $NS = 7PN$. Segments MS and NQ meet at point X. Given that the area of quadrilateral $PMXN$ is $A - \frac{1}{2}$, find the side length of the square.

18. [16pts] Let B be the answer to problem 17 and let $N = 6B$. Find the number of ordered triples (a, b, c) of integers between 0 and $N - 1$, inclusive, such that $a + b + c$ is divisible by N.

1.4.7 Round 7

19. [16pts] Let k be the units digit of $\underbrace{7^{7^{7^{7^{7^{7^{7}}}}}}}_{\text{Seven 7s}}$. What is the largest prime factor of the number consisting of k 7's written in a row?

20. [16pts] Suppose that $E = 7^7$, $M = 7$, and $C = 7 \cdot 7 \cdot 7$. The characters E, M, C, C are arranged randomly in the following blanks.

$$\underline{\quad} \times \underline{\quad} \times \underline{\quad} \times \underline{\quad}$$

Then one of the multiplication signs is chosen at random and changed to an equals sign. What is the probability that the resulting equation is true?

21. [16pts] During a recent math contest, Sophy Moore made the mistake of thinking that 133 is a prime number. Fresh Mann replied, "To test whether a number is divisible by 3, we just need to check whether the sum of the digits is divisible by 3. By the same reasoning, to test whether a number is divisible by 7, we just need to check that the sum of the digits is a multiple of 7, so 133 is clearly divisible by 7." Although his general principle is false, 133 is indeed divisible by 7. How many three-digit numbers are divisible by 7 and have the sum of their digits divisible by 7?

1.4.8 Round 8

22. [18pts] A *look-and-say sequence* is defined as follows: starting from an initial term a_1, each subsequent term a_k is found by reading the digits of a_{k-1} from left to right and specifying the number of times each digit appears consecutively. For example, 4 would be succeeded by 14 ("One four."), and 31337 would be followed by 13112317 ("One three, one one, two three, one seven.")

If a_1 is a random two-digit positive integer, find the probability that a_4 is at least six digits long.

23. [18pts] In triangle ABC, $\angle C = 90°$. Point P lies on segment BC and is not B or C. Point I lies on segment AP, and $\angle BIP = \angle PBI = \angle CAB$. If $\dfrac{AP}{BC} = k$, express $\dfrac{IP}{CP}$ in terms of k.

24. [18pts] A subset of $\{1, 2, 3, \ldots, 30\}$ is called *delicious* if it does not contain an element that is 3 times another element. A subset is called *super delicious* if it is delicious and no delicious set has more elements than it has. Determine the number of super delicious subsets.

1.5 Puzzle Test

Afternoon, January 28, 2012

There are a total of 100 boxes to be filled in, worth 1 point each, to be completed in 25 minutes.

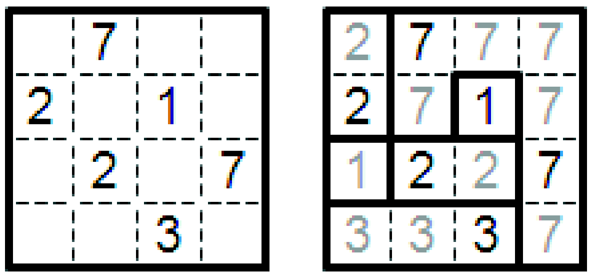
1.5.1 The Grid

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

1.5.2 Fillomino

This puzzle round is a **fillomino puzzle** with clues presented in the form of math problems.

A fillomino is a puzzle on an $n \times n$ grid. The goal is to fill a positive integer into each square of the grid such that, if one were to draw a boundary between every pair of neighboring squares with distinct numbers, we would partition the grid into polyominoes (i.e. connected groups of adjacent cells). The number in each square should tell how many cells are in the polyomino that contains it. Below, we present a fillomino puzzle (from http://mellowmelon.wordpress.com/) along with its only solution.



As you can see, in a conventional fillomino, some clues are given, and the puzzle solver must use logical reasoning to determine what numbers go in the other squares. However, for this round, you are initially given a blank grid, and the clues are given in the form of math problems.

1.5.3 The Format

The clues are split into two categories. To determine the unique solution to the fillomino puzzle, your team will only need to solve the problems from section 1.5.4. However, if you get stuck or cannot solve all of those problems, additional clues are available in section 1.5.5. Note that even if you solve all of the problems in the two sections, there will still be empty spaces in the grid that will need to be filled in with logical deduction. Your score for this section will be the number of correct squares that you have filled in.

Teamwork and speed are critical during this round. You are not expected to solve all the problems, and it is important to find a balance between solving problems and logically deducing the puzzle. As always, it is important to have fun.

1.5.4 Clues

Each problem will be assigned a *clue-value*, and the answer to each problem will be an integer between 0 and 99 inclusive. Each answer corresponds to one of the 100 boxes in the grid shown on the previous page. The clue-value for a given problem is the number that goes inside the box corresponding to the problem's answer. For example if the answer to a problem with clue-value 2 is 27, then the number in box 27 is a 2. No two problems will have the same answer, ensuring that no clues contradict each other.

Clue-value 1 Problems

- What is the smallest even two-digit positive integer whose tens and units digits are equal?
- A class has 5 boys and 5 girls. How many ways are there to chose two presidents from the class, one boy and one girl?
- What is the smallest composite integer that is 1 more than the 4th power of a positive integer?

Clue-value 2 Problems

- What is the smallest 2-digit prime number?
- What is the smallest prime number whose digits sum to 10?
- How many ways are there to arrange Al, Bob, Carl, and David in a line such that at least one person is standing in front a person whose name comes later in the alphabet?
- What is the average of the five largest two-digit multiples of 3?
- Let N be the largest 3 digit perfect square. Find $\frac{N-21}{10}$.
- Evaluate $3^4 + 2^4 + 1^4$.

Clue-value 3 Problems

- A square is inscribed in a circle of area $\frac{\pi}{2}$. What is the area of the square?
- Two lattice points with different x and y-coordinates are chosen, one on the line $y = x + 1$, the other on the line $y = x$. If the slope of the line through these two points is an integer, what is the slope?
- Pete has 3 slips of paper. For each slip, he randomly chooses one of $1, 2$, or 3, each with equal probability, and writes it on the slip. (The numbers are chosen independently for each slip.) Let p be the probability that the sum of the numbers on the slips equals 9. What is $\frac{1}{p}$?
- A bug is moving along a number line at a constant rate. At time 0, the bug is at 3. At time 2, the bug is at 13. Where is the bug at time 10?
- Let G denote the greatest common divisor and L the least common multiple of 6 and 159. What is the remainder when $L \cdot G$ is divided by 100?

Clue-value 4 Problems

- In rectangle $ABCD$, $AB = 1$ and $BC = 2$. Let M be the midpoint of segment AD. Compute the remainder when the degree measure of $\angle BMD$. is divided by 100.
- Let x be the answer to this problem. Compute $\dfrac{x+86}{3}$.
- How many ways are there to chose a two person team from a group of 10 people?
- How many ways are there to select a president and vice-president from a group of 9 people containing Newt and Rom, if Newt cannot be vice president when Rom is president?

Clue-value 5 Problems

- Find the smallest positive odd number that leaves a remainder of 2 upon division by 3 and a remainder of 4 upon division by 5.
- A rectangular box has sides $9, 12$, and 36. Compute the length of the longest diagonal of the box.
- What is the 7^{th} smallest positive integer with an odd number of positive integer factors?
- How many integers are between $\dfrac{444}{239}$ and $\dfrac{239}{4}$?

Clue-value 6 Problems

- When $\dfrac{1}{93} + \dfrac{1}{186}$ is simplified and reduced to lowest terms, what is the denominator of the resulting fraction?
- Find the remainder when $3 \cdot 6 \cdot 37$ is divided by 100.
- Let A be the surface area of a cube with side length 5 units. Compute $\frac{A}{2}$.
- The probability that it will be sunny on a given day is twice the probability that it will be hot. If the probability that it will not be hot that day is 57%, then the probability that it will be sunny is $x\%$. What is x?

Clue-value 9 Problems

- When 4 standard, six-sided dice are tossed, how many possible values are there for their sum?
- If $4^{x-1} = 2$, compute $4^{x+1} - 1$.
- Two sides of a right triangle are 9 and 40, and all the sides are integers. What is the length of the third side?
- When a number n is increased by 50%, the result is 35 more than when n is decreased by $33\frac{1}{3}\%$. Find n.

Clue-value 16 Problems

- When evaluated as a decimal number, the quantity $(2^2)(3^3)(4^4)(5^5)(6^6)(7^7)(8^8)(9^9)(10^{10})$ ends with a string of n consecutive zeroes. Determine n.
- Let S be the set of all nonnegative integers whose base-3 representation contains no twos. How many elements of S are less than $2+3+3^2+3^3+3^4$?

Clue-value 27 Problems

- In my (rather large) drawer, I have 2012 red socks, and 2012^{2012} blue socks. How many socks do I need to draw in order to guarantee that I have two socks of the same color?
- If $a@b = ab - a$ and $a\&b = 2a + b$, evaluate $3@(3\&14)$
- A frog is located on a number line. Each second it jumps either 1 unit backwards or 3 units forward. At how many positions could it be after 96 jumps?

1.5.5 Optional Clues

Clue-value 1 Problems

- Suppose Albert bikes x mph when going uphill, and y mph when going downhill, where x and y are positive integers with $y > x$. During one trip, Albert biked uphill for some distance and then back downhill that same distance. His average speed on this trip was more than 15 miles per hour. For another trip, he biked uphill for some amount of time and then back downhill for the same amount of his time. His average speed was an integer between 1 and 14 miles per hour inclusive. For how many pairs of positive integers (x, y) could this have happened?

Clue-value 2 Problems

- Find the smallest positive integer d greater than 1 such that $2d-1$ and $5d-1$ are both perfect squares.
- What is the largest two-digit factor of $\underbrace{11...11}_{18\text{ 1's}}$?

Clue-value 3 Problems

- Let $f(x) = \frac{x+1}{6}, g(x) = 3x - 2$ and $h(x) = 2x + 3$. Evaluate $h(g(f(h(g(f(17))))))$.
- Al has an 8 by 8 chessboard. Initially all the squares are white. First, he colors all the squares on the two main diagonals black. Next, he picks a row and colors all white squares in that row black. Finally, he picks a column and colors all white squares in that column black. What is the maximum possible number of remaining white squares?
- Let N be a two-digit prime number such that when the digits of N are reversed, the resulting number is a multiple of 5. There are two possible values of N, and let the smaller possibility be x. Find $x - 1$.

Clue-value 4 Problems

- The Fibonacci Sequence is defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. Let F_k be the smallest term greater than 1 in the Fibonacci Sequence that is a perfect square. Determine the remainder when F_k is divided by 100.

- A palindrome is a number that reads the same forwards and backwards. The product of the digits of a 3-digit palindrome, N, is 36. It is known that N is not a perfect cube, and the remainder when N is divided by 100 is not a perfect square. What is the remainder when N is divided by 100?

Clue-value 6 Problems

- Al had 4 test scores whose average was 81 points. Recently he took another test, and now the average of his 5 tests is 80 points. What was his score on the last test?

- A number is called *almost-prime* if it is not prime but not divisible by 2,3, or 5. There are three 2-digit almost-prime numbers. What is the second-smallest one?

- How many paths of length 6 are there from the point $(0, 0, 0)$ to the point $(2, 2, 2)$ in space if one can only move between lattice points separated by a distance of 1?

Clue-value 9 Problems

- Five circles with distinct radii are drawn in the plane. What is the maximum possible number of distinct points of intersection between the circles?

Clue-value 16 Problems

- Let $ABCD$ be a rectangle with $AB = 1, BC = 42$. Let E, F, G, H be the midpoints of segments AB, BC, CD, DA, respectively. Let P be the intersection of DE and BH and Q be the intersection of DF and BG. Compute the area of quadrilateral $BPDQ$.

- How many ways are there to choose an ordered quadruple of positive integers (a, b, c, d) such that $a + b + c + d = 9$?

- An ordered quintuple of nonnegative integers (a, b, c, d, e) is called *good* if it satisfies $2 \geq a \geq b \geq c \geq d \geq e \geq 0$. Define the *goodness* of a good quintuple as the product of all its elements. What is the sum of the goodness of all good quintuples?

- A *lucky number* is a number whose digits are all either 4 or 7 (ignoring leading zeros). The smallest lucky number is 4. What is the remainder when the 2013$^{\text{th}}$ smallest lucky number is divided by 100?

Clue-value 27 Problems

- How many ways are there to make 39 cents using pennies, nickels, and dimes, given that at least one of each coin must be used, and the total value of the dimes used is strictly less than the total value of the other coins used?

- Evaluate the sum

$$(1) + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + 4 + 5 + 6 + 7)$$

- Let $ABCD$ be a quadrilateral with $AB = AC = AD = 12$, $BC = CD$, and $\angle BAC = \angle CAD = 30°$. What is the area of $ABCD$?

- Let N be a two-digit multiple of 5 such that when the digits of N are reversed, the resulting number is a two digit prime. There are two possible values of N. What is the larger one?

Chapter 2

EMC^2 2012 Solutions

2.1 Individual Speed Test Solutions

1. Evaluate
$$\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}.$$

   Solution. The answer is $\boxed{\frac{7}{120}}$.

   Combining the fractions gives
$$\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}=\frac{5}{2\cdot 3\cdot 4\cdot 5}+\frac{2}{2\cdot 3\cdot 4\cdot 5}=\frac{5+2}{2\cdot 3\cdot 4\cdot 5}=\frac{7}{2\cdot 3\cdot 4\cdot 5}=\frac{7}{120}.$$

2. A regular hexagon and a regular n-sided polygon have the same perimeter. If the ratio of the side length of the hexagon to the side length of the n-sided polygon is $2:1$, what is n?

   Solution. The answer is $\boxed{12}$.

   Let the side length of the regular hexagon be $2x$. Then the side length of the n-sided polygon is x. Because the regular hexagon and the regular n-sided polygon have the same perimeter, we have $nx = 6\cdot 2x$ or $n = 12$.

3. How many nonzero digits are there in the decimal representation of $2\cdot 10\cdot 500\cdot 2500$?

   Solution. The answer is $\boxed{2}$.

   We have $2\cdot 10\cdot 500\cdot 2500 = 2\cdot 5\cdot 25\cdot 100000 = 25000000$.

4. When the numerator of a certain fraction is increased by 2012, the value of the fraction increases by 2. What is the denominator of the fraction?

   Solution. The answer is $\boxed{1006}$.

   Let x be the numerator and y be the denominator of the original fraction. We have
$$\frac{x}{y}+2=\frac{x+2012}{y}\quad\text{or}\quad\frac{x+2y}{y}=\frac{x+2012}{y}$$
   from which it follows that $y = 1006$.

5. Sam did the computation $1-10\cdot a+22$, where a is some real number, except he messed up his order of operations and computed the multiplication last; that is, he found the value of $(1-10)\cdot(a+22)$ instead. Luckily, he still ended up with the right answer. What is a?

   Solution. The answer is $\boxed{221}$.

   Simplifying both sides of the equation yields $23-10a = -9a-198$. Adding $10a+198$ to both sides, we have $a = 221$.

6. Let $n! = n\cdot(n-1)\cdots 2\cdot 1$. For how many integers n between 1 and 100 inclusive is $n!$ divisible by 36?

   Solution. The answer is $\boxed{95}$.

   Note that $6!$ divides $n!$ for all $n\geq 6$. Because 36 does not divide $5! = 120$ and 36 does divide $6! = 720$, the possible values of n are $6,\dots,100$.

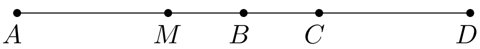
7. Simplify the expression $\sqrt{\dfrac{3 \cdot 27^3}{27 \cdot 3^3}}$.

   Solution. The answer is $\boxed{9}$.

   We have
$$\sqrt{\frac{3 \cdot 27^3}{27 \cdot 3^3}} = \sqrt{\frac{27^2}{3^2}} = \sqrt{9^2} = 9.$$

8. Four points A, B, C, D lie on a line in that order such that $\dfrac{AB}{CB} = \dfrac{AD}{CD}$. Let M be the midpoint of segment AC. If $AB = 6, BC = 2$, compute $MB \cdot MD$.

   Solution. The answer is $\boxed{16}$.



   We have $\frac{AB}{CB} = 3$. Let the length of segment CD be x. Hence
$$3 = \frac{AB}{CB} = \frac{AD}{CD} = \frac{8+x}{x} \quad \text{or} \quad x = 4.$$
   We conclude that $AM = MC = \frac{AC}{2} = 4$, $MB = AB - AM = 2$, $MD = MC + CD = 8$, and $MB \cdot MD = 2 \cdot 8 = 16$.

9. Allan has a deck with 8 cards, numbered $1, 1, 2, 2, 3, 3, 4, 4$. He pulls out cards without replacement, until he pulls out an even numbered card, and then he stops. What is the probability that he pulls out exactly 2 cards?

   Solution. The answer is $\boxed{\dfrac{2}{7}}$.

   If Allan pulls out exactly 2 cards, his first card must be odd and his second card must be even. The probability that the first card is odd is $\frac{1}{2}$, and given that the first card is odd, the probability that the second card is even is $\frac{4}{7}$. Thus the probability that Allan stops after the second draw is $\frac{1}{2} \cdot \frac{4}{7} = \frac{2}{7}$.

10. Starting from the sequence $(3, 4, 5, 6, 7, 8, \dots)$, one applies the following operation repeatedly. In each operation, we change the sequence
$$(a_1, a_2, a_3, \dots, a_{a_1-1}, a_{a_1}, a_{a_1+1}, \dots)$$
    to the sequence
$$(a_2, a_3, \dots, a_{a_1}, a_1, a_{a_1+1}, \dots).$$
    (In other words, for a sequence starting with x, we shift each of the next $x-1$ term to the left by one, and put x immediately to the right of these numbers, and keep the rest of the terms unchanged. For example, after one operation, the sequence is $(4, 5, 3, 6, 7, 8, \dots)$, and after two operations, the sequence becomes $(5, 3, 6, 4, 7, 8, \dots)$. How many operations will it take to obtain a sequence of the form $(7, \dots)$ (that is, a sequence starting with 7)?

    Solution. The answer is $\boxed{7}$.

    We obtain the following sequences in order: $(4, 5, 3, 6, 7, 8, \dots)$, $(5, 3, 6, 4, 7, 8, \dots)$, $(3, 6, 4, 7, 5, 8, \dots)$, $(6, 4, 3, 7, 5, 8, \dots)$, $(4, 3, 7, 5, 8, 6, \dots)$, $(3, 7, 5, 4, 8, 6, \dots)$, $(7, 5, 3, 4, 8, 6, \dots)$.

11. How many ways are there to place 4 balls into a 4×6 grid such that no column or row has more than one ball in it? (Rotations and reflections are considered distinct.)

    Solution. The answer is $\boxed{360}$.

    Note there must be exactly 1 ball in each row. There are six ways to place a ball in the first row, and after that is placed, five in the second row. Likewise there are four ways for placement in the third row and three in the fourth row. Thus there are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ ways in total.

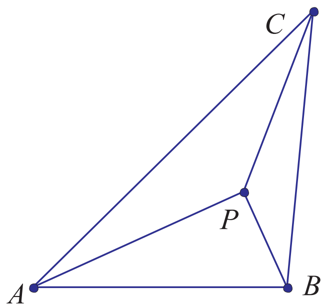
12. Point P lies inside triangle ABC such that $\angle PBC = 30°$ and $\angle PAC = 20°$. If $\angle APB$ is a right angle, find the measure of $\angle BCA$ in degrees.

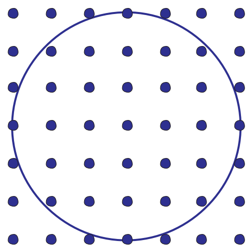
    Solution. The answer is $\boxed{40°}$.

    Construct segment CP. In triangles ACP and BCP, we have $\angle ACP + \angle CPA = 180° - \angle PAC = 160°$ and $\angle BCP + \angle CPB = 180° - \angle PBC = 150°$. Adding the last two equations gives

$$\begin{aligned} 310° &= \angle ACP + \angle CPA + \angle BCP + \angle CPB = (\angle ACP + \angle BCP) + (\angle CPA + \angle CPB) \\ &= \angle ACB + (360° - \angle APB), \end{aligned}$$

    from which it follows that $\angle ACB = 310° - (360° - 90°) = 40°$. (Please see the left-hand side diagram shown below.)





13. What is the largest prime factor of $9^3 - 4^3$?

    Solution. The answer is $\boxed{19}$.

    We have $9^3 - 4^3 = 729 - 64 = 665 = 5 \cdot 133 = 5 \cdot 7 \cdot 19$.

14. Joey writes down the numbers 1 through 10 and crosses one number out. He then adds the remaining numbers. What is the probability that the sum is less than or equal to 47?

    Solution. The answer is $\boxed{\dfrac{3}{10}}$.

    Let the number that Joey crossed out be x. Therefore sum of the numbers which remain is equal to $(1 + 2 + \cdots + 10) - x = 55 - x$. We want $55 - x \leq 47$, or $8 \leq x$, which happens with probability $\frac{3}{10}$.

15. In the coordinate plane, a *lattice point* is a point whose coordinates are *integers*. There is a pile of grass at every lattice point in the coordinate plane. A certain cow can only eat piles of grass that are at most 3 units away from the origin. How many piles of grass can she eat?

Solution. The answer is $\boxed{29}$.

The points in the coordinate plane the cow can reach are $(\pm 3, 0), (0, \pm 3)$, and (x, y) with the possible values of x and y being $-2, -1, 0, 1, 2$. (Please see the right-hand side diagram shown above.)

16. A book has 1000 pages numbered $1, 2, \ldots, 1000$. The pages are numbered so that pages 1 and 2 are back to back on a single sheet, pages 3 and 4 are back to back on the next sheet, and so on, with pages 999 and 1000 being back to back on the last sheet. How many pairs of pages that are back to back (on a single sheet) share no digits in the same position? (For example, pages 9 and 10, and pages 89 and 90.)

Solution. The answer is $\boxed{23}$.

From 1 to 10, there are 5 pairs of pages that do not share any digits: $(1, 2), (3, 4), (5, 6), (7, 8), (9, 10)$. Any other pairs of back-to-back pages that share no digits in the same position must differ in their tens digit. There are 99 such pairs, i.e. $19 - 20, 29 - 30, \ldots, 999 - 1000$. However, among these 99 pairs, 81 have a repeat in the hundreds digit: $(\overline{k09}, \overline{k10}), (\overline{k29}, \overline{k30}), \ldots, (\overline{k89}, \overline{k90})$, where $1 \le k \le 9$ ($\overline{kab}$ denotes the integer with hundreds' digit k, tens' digit a and units' digit b). Thus, the total number of pairs of back-to-back pages with no shared digits in the same position is $5 + (99 - 81) = 23$.

17. Find a pair of integers (a, b) for which $\dfrac{10^a}{a!} = \dfrac{10^b}{b!}$ and $a < b$.

Solution. The answer is $\boxed{(9, 10)}$.

We have

$$\frac{10!}{9!} = 10 = \frac{10^{10}}{10^9}.$$

18. Find all ordered pairs (x, y) of real numbers satisfying

$$\begin{cases} -x^2 + 3y^2 - 5x + 7y + 4 &= 0 \\ 2x^2 - 2y^2 - x + y + 21 &= 0 \end{cases}$$

Solution. The answer is $\boxed{(3, -4)}$.

Adding the two given equations, we get $x^2 + y^2 - 6x + 8y + 25 = 0$ or $(x - 3)^2 + (y + 4)^2 = 0$, by completing the squares. Hence $(x, y) = (3, -4)$ is the only possible solution of the system, and it is easy to check that it indeed is a solution.

19. There are six blank fish drawn in a line on a piece of paper. Lucy wants to color them either red or blue, but will not color two adjacent fish red. In how many ways can Lucy color the fish?

Solution. The answer is $\boxed{21}$.

Let R_n denote the number of ways to color n fish such that the last fish is colored red, and let B_n denote the number of ways to do so with the last fish colored blue. Then, we have $R_1 = 1$, $B_1 = 1$, and for $n \ge 1$,

$$\begin{aligned} R_{n+1} &= B_n \\ B_{n+1} &= R_n + B_n. \end{aligned}$$

(From the first equation, we have $R_n = B_{n-1}$. Substituting the last equation into the second equation in the above system gives $B_{n+1} = R_n + B_n = B_{n-1} + B_n$, implying that the sequence B_n is the *Fibonacci* sequence.) For our current problem, we can simply evaluate $B_2 = R_1 + B_1 = 2$, $R_2 = B_1 = 1$, $B_3 = R_2 + B_2 = 3$, $R_3 = B_2 = 2$, $B_4 = R_3 + B_3 = 5$, and so on, to obtain $B_7 = 21$.

20. There are sixteen 100-gram balls and sixteen 99-gram balls on a table (the balls are visibly indistinguishable). You are given a balance scale with two sides that reports which side is heavier or that the two sides have equal weights. A weighing is defined as reading the result of the balance scale: For example, if you place three balls on each side, look at the result, then add two more balls to each side, and look at the result again, then two weighings have been performed. You wish to pick out two different sets of balls (from the 32 balls) with equal numbers of balls in them but different total weights. What is the minimal number of weighings needed to ensure this?

    Solution. The answer is $\boxed{1}$.

    It is clear that one has to use at least one weighing. We claim that it is also enough.

    Split the thirty-two balls into three sets S_1, S_2, and S_3 of 11, 11, and 10 balls, respectively. Weigh sets S_1 and S_2 against each other. If the total weights are not equal, we are done. Otherwise, discard one ball from S_1 to form a new set T_1 of 10 balls. We claim that T_1 and S_3 have different weights. If not, then they have the same number of 10-gram balls, say, n. Then S_1 and S_2 either each had n 10-gram balls or each had $n + 1$ 10-gram balls. This would imply that 16 equals $3n$ or $3n + 2$, both of which are impossible.

2.2 Individual Accuracy Test Solutions

1. An 18oz glass of apple juice is 6% sugar and a 6oz glass of orange juice is 12% sugar. The two glasses are poured together to create a cocktail. What percent of the cocktail is sugar?

   Solution. The answer is $\boxed{7.5\%}$.

   There is $0.06 \cdot 18 = 1.08$oz of sugar in the glass of apple juice and $0.12 \cdot 6 = 0.72$oz of sugar in the glass of orange juice. Hence, there is 1.80oz of sugar in the cocktail, which yields an answer of $\frac{1.80}{24} = 7.5\%$.

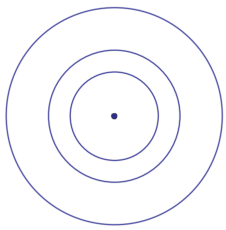
2. Find the number of positive numbers that can be expressed as the difference of two integers between -2 and 2012 inclusive.

   Solution. The answer is $\boxed{2014}$.

   Every positive integer between $2012 - 2011 = 1$ and $2012 - (-2) = 2014$ inclusive can be expressed as a difference between two integers in our set, and no positive integer greater than 2014 can be expressed.

3. An *annulus* is defined as the region between two concentric circles. Suppose that the inner circle of an annulus has radius 2 and the outer circle has radius 5. Find the probability that a randomly chosen point in the annulus is at most 3 units from the center.

   Solution. The answer is $\boxed{\dfrac{5}{21}}$.

   The total area of the annulus (the region between the inner circle and the outer circle in the diagram shown above) is $\pi \cdot 5^2 - \pi \cdot 2^2 = 21\pi$. The region of the annulus that is within 3 units from the center is another annulus; that is, the region between the two concentric circles of radius 2 (the inner circle) and radius 3 (the middle circle). It has area $\pi \cdot 3^2 - \pi \cdot 2^2 = 5\pi$. It follows that the probability that a randomly chosen point is within 3 units of the center is $\frac{5\pi}{21\pi} = \frac{5}{21}$.

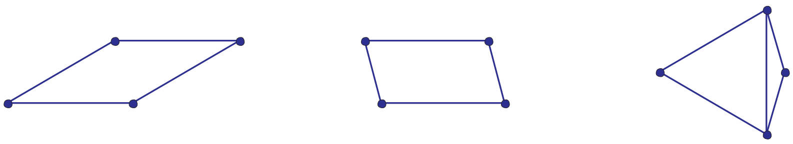
4. Ben and Jerry are walking together inside a train tunnel when they hear a train approaching. They decide to run in opposite directions, with Ben heading towards the train and Jerry heading away from the train. As soon as Ben finishes his 1200 meter dash to the outside, the front of the train enters the tunnel. Coincidentally, Jerry also barely survives, with the front of the train exiting the tunnel as soon as he does. Given that Ben and Jerry both run at 1/9 of the train's speed, how long is the tunnel in meters?

   Solution. The answer is $\boxed{2700}$.

   Suppose that the tunnel has length ℓ meters. In the time that the train takes to reach the north entrance, Ben runs 1200 meters, so the train was $1200 \cdot 9 = 10800$ km away from the north entrance. Therefore, in the time that Jerry takes to run $\ell - 1200$ meters to the south entrance, the train moves $10800 + \ell$ meters. Thus, we have $9(\ell - 1200) = 10800 + \ell$, implying that $\ell = 2700$.

5. Let ABC be an isosceles triangle with $AB = AC = 9$ and $\angle B = \angle C = 75^\circ$. Let DEF be another triangle congruent to ABC. The two triangles are placed together (without overlapping) to form a quadrilateral, which is cut along one of its diagonals into two triangles. Given that the two resulting triangles are incongruent, find the area of the larger one.

   Solution. The answer is $\boxed{\dfrac{81\sqrt{3}}{4}}$.

   The two isosceles triangles are congruent to each other. To stack the two triangles together to form a quadrilateral, we must match one side of a triangle with its corresponding side in the other triangle. There are three ways to do so. (Please see the diagrams shown above.) Two of the resulting quadrilaterals are parallelograms and the other is a kite (shown in the above right-hand side diagram). In this kite, one of its diagonals cuts the quadrilateral into two incongruent triangles, with the larger triangle being an equilateral triangle with side length 9 and area $\frac{81\sqrt{3}}{4}$.

6. There is an infinitely long row of boxes, with a Ditto in one of them. Every minute, each existing Ditto clones itself, and the clone moves to the box to the right of the original box, while the original Ditto does not move. Eventually, one of the boxes contains over 100 Dittos. How many Dittos are in that box when this first happens?

   Solution. The answer is $\boxed{126}$.

   Call the box in which the single Ditto starts out box 0 and call the initial time 0. Now, let $a_{b,t}$ denote the number of Dittos in box b and time t. It is not difficult to see that

$$a_{b,t} = a_{b-1,t-1} + a_{b,t-1},$$

   because the first term is the number of new Dittos that come from the box to the left of box b and the second term is the number of Dittos that stay in box b. We also know that $a_{b,0} = 1$ if $b = 0$ and $a_{b,0} = 0$ otherwise. Thus, we notice that the terms of this sequence are the entries on Pascal's triangle: $a_{b,t}$ satisfies the same recursion as the binomial coefficients (Pascal's Identity), and it clearly satisfies the initial conditions as well. It follows that $a_{b,t} = \binom{t}{b}$. Now, the first value of t for which there is a b such that $\binom{t}{b} > 100$ is $t = 9$, so we have $\binom{9}{4} = \binom{9}{5} = 126$, as claimed.

7. Evaluate

$$26 + 36 + 998 + 26 \cdot 36 + 26 \cdot 998 + 36 \cdot 998 + 26 \cdot 36 \cdot 998.$$

   Solution. The answer is $\boxed{998000}$.

   Let p denote the desired quantity. We note that p resembles the following identity:

$$(1 + a)(1 + b)(1 + c) - 1 = a + b + c + ab + bc + ca + abc.$$

   Setting $(a, b, c) = (26, 36, 998)$ in the above identity gives

$$p + 1 = (1 + 26)(1 + 36)(1 + 998) = 27 \cdot 37 \cdot 999 = 999^2.$$

   Hence $p = 999^2 - 1 = (999 + 1)(999 - 1) = 1000 \cdot 998 = 998000.$

8. There are 15 students in a school. Every two students are either friends or not friends. Among every group of three students, either all three are friends with each other, or exactly one pair of them are friends. Determine the minimum possible number of friendships at the school.

   Solution. The answer is $\boxed{49}$.

   This can be achieved by forming a set of 7 students who are all friends with each other, and separate set of 8 students who are all friends with each other, with no pairs of friends between these two sets. It is not difficult to see that this construction satisfies the conditions of the problem, and it consists of $\binom{7}{2} + \binom{8}{2} = 49$ pairs of friends. We will now show that there must be at least 49 friendships.

   Let $s_1, \ldots, s_{15}$ denote these 15 students. There must a nonzero number of friendships, so assume that s_1 has friends and his friends are $s_2, \ldots, s_a$ for some $2 \leq a \leq 15$. Consider three students s_1 and his friend s_m, s_n (that is, $2 \leq m, n \leq a$). We already have two pairs of friends (s_1, s_m) and (s_1, s_n). Hence s_m and s_n must also be friends. Therefore, all of $s_1, \ldots, s_a$ are friends of each other.

   If $a \geq 14$, then we have at least 14 students who are all friends with each other, and thus at least $\binom{14}{2} > 49$ pairs of friends.

   Now we assume that $a < 14$; that is, at least two people (namely, $s_{a+1}, \ldots, s_{15}$) are not friends with s_1. We consider three students s_1, s_i and s_j that are not friends with s_1 (that is, $a < i, j \leq 15$). Because there is at least one pair of friends among these three students, s_i and s_j must be friends with each other. Hence all the students that are not friends with s_1 are friends with one another. Therefore, we have at least

   $$\binom{j}{2} + \binom{15-j}{2} = \frac{a(a-1) + (15-a)(14-a)}{2} = a^2 - 15a + 105$$

   pairs of friends. For integers a, the quadratic expression $a^2 - 15a + 105$ obtains its minimum value of 49 when $a = 7$ or $a = 8$.

   Note: This problem and its solution are motivated by graph theory. Every student can be treated as a vertex, and every friendship can be treated as an edge. Then we are given the condition that every set of three vertices has an odd number of edges among them, and we want to minimize the number of edges in the graph. The example given is the disjoint union of two complete graphs with sizes as equal as possible.

9. Let $f(x) = \sqrt{2x + 1 + 2\sqrt{x^2 + x}}$. Determine the value of

   $$\frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \cdots + \frac{1}{f(24)}.$$

   Solution. The answer is $\boxed{4}$.

   Note that, for $x \geq 0$,

   $$\begin{aligned} f(x) &= \sqrt{2x + 1 + 2\sqrt{x^2 + x}} = \sqrt{x + (x+1) + 2\sqrt{x(x+1)}} \\ &= \sqrt{(\sqrt{x} + \sqrt{x+1})^2} = \sqrt{x} + \sqrt{x+1}. \end{aligned}$$

   Then, we have

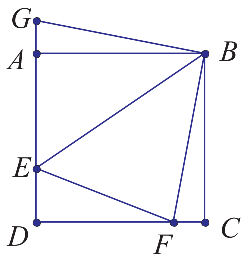
   $$\frac{1}{f(x)} = \frac{1}{\sqrt{x} + \sqrt{x+1}} = \sqrt{x+1} - \sqrt{x}$$

by rationalizing the denominator. Therefore, we can evaluate the desired expression as a telescoping sum, which yields

$$\begin{aligned}\frac{1}{f(1)} + \frac{1}{f(2)} + \cdots + \frac{1}{f(24)} &= (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + \cdots + (\sqrt{25}-\sqrt{24}) = \sqrt{25}-\sqrt{1} \\ &= 5-1=4.\end{aligned}$$

10. In square $ABCD$, points E and F lie on segments AD and CD, respectively. Given that $\angle EBF = 45^\circ$, $DE = 12$, and $DF = 35$, compute AB.

Solution. The answer is $\boxed{42}$.

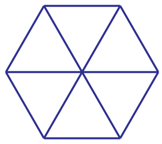


Extend segment DA through A to G such that $AG = CF$. Then right triangles BAG and BCF are congruent to each other. It follows that $BG = BF$ and $\angle GBA = \angle FBC$, which implies that $\angle GBE = \angle GBA + \angle ABE = \angle FBC + \angle ABE = \angle ABC - \angle EBF = 45^\circ$. We conclude that triangles BEG and BEF are congruent to each other by SAS conditions ($BG = BF$, $\angle GBE = \angle FBE = 45^\circ$, $BE = BE$). In right triangle DEF, we have $EF = 37$. Hence $DG = DE + EG = DE + EF$ and $DG = AD + AG = AD + CF = AD + CD - DF = 2AD - DF$, from which it follows that $2AD - DF = DE + EF$ or $AD = \frac{DE+EF+FD}{2} = 42$.

2.3 Team Test Solutions

1. The longest diagonal of a regular hexagon is 12 inches long. What is the area of the hexagon, in square inches?

   Solution. The answer is $\boxed{54\sqrt{3}}$.

   As shown in the diagram above, we can divide the given regular hexagon into six equilateral triangles, each of which has side length 6. The area of the hexagon is $6 \cdot 9\sqrt{3} = 54\sqrt{3}$.

2. When Al and Bob play a game, either Al wins, Bob wins, or they tie. The probability that Al does not win is $\dfrac{2}{3}$, and the probability that Bob does not win is $\dfrac{3}{4}$. What is the probability that they tie?

   Solution. The answer is $\boxed{\dfrac{5}{12}}$.

   The probability that Al wins is $\frac{1}{3}$ and the probability that Bob wins is $\frac{1}{4}$. Hence, the probability that one of them wins is $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. Thus, the probability that neither of them wins is $\frac{5}{12}$.

3. Find the sum of the $a+b$ values over all pairs of integers (a, b) such that $1 \le a < b \le 10$. That is, compute the sum

   $$(1+2) + (1+3) + (1+4) + \cdots + (2+3) + (2+4) + \cdots + (9+10).$$

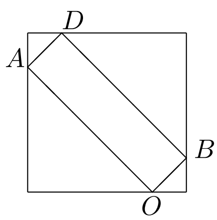
   Solution. The answer is $\boxed{495}$.

   Each number between 1 and 10 appears a total of 9 times over all the pairs (a, b). Therefore, $(1+2) + (1+3) + (1+4) + \cdots + (2+3) + (2+4) + \cdots + (9+10) = 9(1+2+\cdots+10) = 9 \cdot 55 = 495$.

4. A 3×11 cm rectangular box has one vertex at the origin, and the other vertices are above the x-axis. Its sides lie on the lines $y = x$ and $y = -x$. What is the y-coordinate of the highest point on the box, in centimeters?

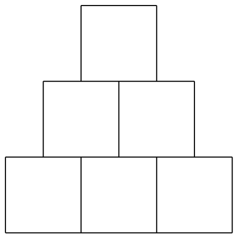
   Solution. The answer is $\boxed{7\sqrt{2}}$.

   Suppose a bug starts at the origin and walks up the edges of the box until it reaches the highest point. For each centimeter that the bug travels up an edge, its height increases by $\frac{1}{\sqrt{2}}$ centimeters. Since the bug travels a total of 14 centimeters before it goes down, its final height is $\frac{14}{\sqrt{2}} = 7\sqrt{2}$.

In the coordinate plane, one possible configuration, with $O = (0, 0)$, $A = \left(-\frac{11}{\sqrt{2}}, \frac{11}{\sqrt{2}}\right)$, $B = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, and $C = \left(-\frac{8}{\sqrt{2}}, \frac{14}{\sqrt{2}}\right)$, is shown below. It is not difficult to find the second possible configuration.

5. Six blocks are stacked on top of each other to create a pyramid, as shown below. Carl removes blocks one at a time from the pyramid, until all the blocks have been removed. He never removes a block until all the blocks that rest on top of it have been removed. In how many different orders can Carl remove the blocks?



Solution. The answer is $\boxed{16}$.

In the first move, Carl must take the top block. Consider the next two moves that Carl makes. He can either remove both blocks in the second row or remove two blocks from the same side to obtain a 3-block pyramid. We now check these two cases separately.

- *Case 1:* If he removes the second row, he can do this in 2 ways. He can remove the remaining 3 blocks in the bottom row in $3! = 6$ ways, for a total of $2 \cdot 6 = 12$ orders in which he can remove the blocks.
- *Case 2:* If he removes two blocks from the same side, he can do this in 2 ways, either by removing the right side or the left side. For the remaining 3-block pyramid, he must remove the top block first, and then remove the remaining two blocks in $2! = 2$ ways for a total of $2 \cdot 2 = 4$ orders.

Adding the results of these two cases, we see that there are $12 + 4 = 16$ orders possible.

6. Suppose that a right triangle has sides of lengths $\sqrt{a + b\sqrt{3}}$, $\sqrt{3 + 2\sqrt{3}}$, and $\sqrt{4 + 5\sqrt{3}}$, where a, b are positive integers. Find all possible ordered pairs (a, b).

Solution. The answers are $\boxed{(1, 3) \text{ and } (7, 7)}$.

Because $\sqrt{4 + 5\sqrt{3}} > \sqrt{3 + 2\sqrt{3}}$, either $\sqrt{4 + 5\sqrt{3}}$ or $\sqrt{a + b\sqrt{3}}$ is the hypotenuse.

If $\sqrt{4 + 5\sqrt{3}}$ is the hypotenuse, we have, by Pythagorean Theorem,

$$a + b\sqrt{3} + 3 + 2\sqrt{3} = (a + 3) + (b + 2)\sqrt{3} = 4 + 5\sqrt{3}$$

which gives $(a, b) = (1, 3)$.

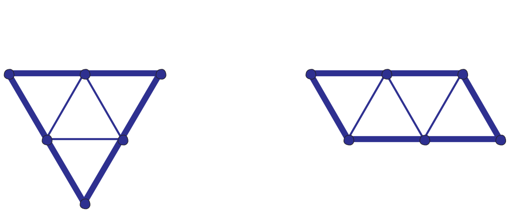
If $\sqrt{a + b\sqrt{3}}$ is the hypotenuse, we have

$$3 + 2\sqrt{3} + 4 + 5\sqrt{3} = 7 + 7\sqrt{3} = a + b\sqrt{3},$$

which gives $(a, b) = (7, 7)$, so the only ordered pairs of (a, b) are $(1, 3)$ and $(7, 7)$.

7. Farmer Chong Gu glues together 4 equilateral triangles of side length 1 such that their edges coincide. He then drives in a stake at each vertex of the original triangles and puts a rubber band around all the stakes. Find the minimum possible length of the rubber band.

   Solution. The answer is $\boxed{4+\sqrt{3}}$.

   Let T be the polygon consisting of the union of the 4 equilateral triangles. As shown in the diagrams above, there are three possible shapes of T. The left-hand side and the middle polygons can be surrounded by a rubber band of length 6. However, the surrounding rubber band for the right-hand polygon is a pentagon with four sides of length 1 and a fifth side of length $\sqrt{3}$, which gives the minimum perimeter of $4+\sqrt{3}$. (As a side note, the polygon the rubber band forms is called the *convex hull* of polygon T.)

8. Compute the number of ordered pairs (a, b) of positive integers less than or equal to 100, such that $a^b - 1$ is a multiple of 4.

   Solution. The answer is $\boxed{3750}$.

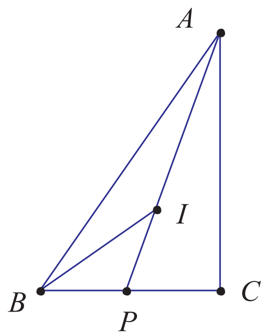
   We consider the cases a is even, a is of the form $4n+1$, and a is of the form $4n+3$.

   - *Case 1:* If a is even, $a^b - 1$ is always odd. Hence there are no solutions when a is even.
   - *Case 2:* If a is of the form $4n+1$, where n is an integer, 4 is a factor of $a^b - 1$ for all integers b, which gives $25 \cdot 100 = 2500$ pairs.
   - *Case 3:* If a is of the form $4n+3$, where n is an integer, 4 is a factor of $a^b - 1$ only for even b, which gives $25 \cdot 50 = 1250$ pairs.

   This gives us a total of $2500 + 1250 = 3750$ pairs.

9. In triangle ABC, $\angle C = 90°$. Point P lies on segment BC and is not B or C. Point I lies on segment AP. If $\angle BIP = \angle PBI = \angle CAB = m°$ for some positive integer m, find the sum of all possible values of m.

   Solution. The answer is $\boxed{525}$.

Let $\angle CAP = x^\circ$ and $\angle PAB = y^\circ$. Then $x + y = m$. We have $\angle IBP = \angle BIP = (x + y)^\circ$. Note that $\angle BIP$ is an exterior angle of triangle ABI and hence it is equal to the sum of its two remote angles ABI and BAI. It follows that $\angle ABI = \angle BIP - \angle BAI = m^\circ - y^\circ = x^\circ$, and so $\angle ABC = \angle ABI + \angle IBP = (2x + y)^\circ$. Because $\angle C = 90^\circ$, we have $90^\circ = \angle ABC + \angle CBA = (2x + y)^\circ + (x + y)^\circ = (3x + 2y)^\circ$ or

$$90 = 3x + 2y = 2m + x.$$

Because m is an integer, $x = 90 - 2m$ is an even integer and $2y = 90 - 3x$ is a multiple of 3. Hence we can write $x = 2x_1$ and $y = 3y_1$ for some positive integers x_1 and y_1. Substituting into $90 = 3x + 2y$ yields $90 = 6x_1 + 6y_1$ or $15 = x_1 + y_1$, with possible values for y_1 being $1, 2, \ldots, 14$. Note that $m = x + y = 2x_1 + 3y_1 = 30 + y_1$. The possible values of m are $31, 32, \ldots, 44$ and their sum is equal to $31 + 32 + \cdots + 44 = \frac{31+44}{2} \cdot 14 = 525$.

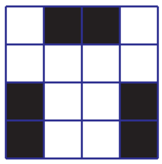
10. Bob has 2 identical red coins and 2 identical blue coins, as well as 4 distinguishable buckets. He places some, but not necessarily all, of the coins into the buckets such that no bucket contains two coins of the same color, and at least one bucket is not empty. In how many ways can he do this?

Solution. The answer is $\boxed{120}$.

Notice that the placement of the red coins does not affect the placement of the blue coins and vise versa. Thus, we can consider the placement of the red coins independently of the blue coins. There is $\binom{4}{0} = 1$ way to place 0 red coins so that no two are in the same bucket. There are $\binom{4}{1} = 4$ ways to place 1 red coin so that no two are in the same bucket. There are $\binom{4}{2} = 6$ ways to place 2 red coins so that no two are in the same bucket. Thus, there are a total of $1 + 4 + 6 = 11$ valid ways to place red coins, and symmetrically, 11 valid ways to place blue coins. Consequently, there are $11 \cdot 11 = 121$ total valid configurations, but we must exclude the case in which all the buckets are empty, which yields a total of 120 ways.

11. Albert takes a 4×4 checkerboard and paints all the squares white. Afterward, he wants to paint some of the square black, such that each square shares an edge with an odd number of black squares. Help him out by drawing one possible configuration in which this holds. (Note: the answer sheet contains a 4×4 grid.)

Solution. One possible answer is the following:



There are two ways to motivate the above configuration. One is to note that every black square must be adjacent to another black square. Thus, if we color two adjacent squares black, and color none of their other neighbors, each of the black squares is adjacent to exactly one other black square, which satisfies the problem conditions. Therefore, it makes sense to tile the board with 1×2 black dominoes instead of unit squares, in such a way that every unit square is either contained in a domino or adjacent to exactly one domino. From there, we can easily check that there are four ways to place dominoes on a 4×4 board in this manner, which are the picture above and its rotations.

The other approach is to divide the large square into two sets of 8 unit squares each in the manner of a chessboard, such that no two adjacent squares belong to the same set. Marking the unit squares in one set only affects the unit squares in the other set, so we only have to find a valid way to color half of the unit squares, then reflect our coloring horizontally across the midline to complete the picture. After this simplification, it is easier to find the above configuration through experimentation.

Query: Prove that any $2n \times 2n$ board can be colored in this manner.

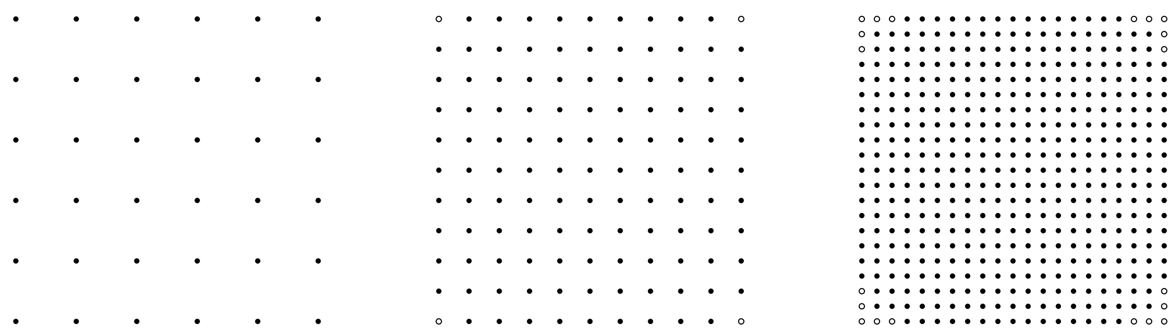
12. Let S be the set of points (x, y) with $0 \leq x \leq 5, 0 \leq y \leq 5$ where x and y are integers. Let T be the set of all points in the plane that are the midpoints of two distinct points in S. Let U be the set of all points in the plane that are the midpoints of two distinct points in T. How many distinct points are in U? (Note: The points in T and U do not necessarily have integer coordinates.)

Solution. The answer is $\boxed{421}$.

Initially, the points in S form a 6×6 grid. (refer to diagrams below for S, T, U) After we replace these points by all their possible midpoints to get T, we obtain an 11×11 grid of points, but with one point missing from each corner, because these points are not midpoints of distinct points in S.

Now we replace these points by all their possible midpoints again to get U. Note that U is a 21×21 grid with 5 points missing from each corner of U. Thus, there are $4 \cdot 5 = 20$ total points missing from the 21×21 grid. Hence, the total number of points in U is $441 - 20 = 421$.

We have included a diagram of S, T, and U in that order to help visualize the problem. Note that the white circles indicate the points in the corner excluded from the entire grid after each iteration.



13. In how many ways can one express 6036 as the sum of at least two (not necessarily positive) consecutive integers?

Solution. The answer is $\boxed{7}$.

The sum of b consecutive integers starting from a is

$$a + (a + 1) + (a + 2) + \cdots + (a + b - 1) = \frac{(2a + b - 1)(b)}{2} = 6036$$

Note that $b \geq 2$, and $2a + b - 1$ and b differ by an odd number $2a - 1$, so one of b and $2a + b - 1$ is odd. It follows that one of $2a + b - 1$ and b must be an odd factor of $6036 \cdot 2 = 8 \cdot 3 \cdot 503$. Thus, one of them must be a factor of $3 \cdot 503$. There are 4 factors of $3 \cdot 503$, so there are $4 \cdot 2$ ways to pick $(2a + b - 1, b)$. However, because b must be at least 2, we must exclude the case where $(2a + b - 1, b) = (12072, 1)$, yielding a total of 7 solutions.

14. Let a, b, c, d, e, f be integers (not necessarily distinct) between -100 and 100, inclusive, such that $a+b+c+d+e+f = 100$. Let M and m be the maximum and minimum possible values, respectively, of

$$abc + bcd + cde + def + efa + fab + aec + bdf.$$

Find $\dfrac{M}{m}$.

Solution. The answer is $\boxed{-\dfrac{1}{9}}$.

Note that

$$S = abc + bcd + cde + def + efa + fab + ace + bdf = (a+d)(b+e)(c+f).$$

Let $a+d = x$, $b+e = y$, $c+f = z$ and we have $S = xyz$, $x+y+z = 100$, and $-200 \le x, y, z \le 200$.

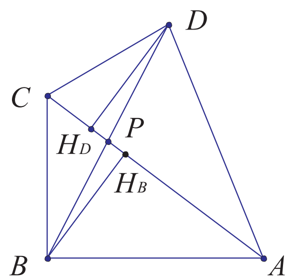
First, we will minimize xyz. The sum of x, y, z is positive, but their product is minimized at a negative value. Consequently, xyz is minimized when one of x, y, z is negative and the other two are positive. Assume without loss of generality x is negative. For any given x, we can maximize yz by setting $y = z$, hence we can minimize xyz by setting $y = z$. Then xyz becomes $x \cdot \left(\frac{100-x}{2}\right)^2$. When $x \le 0$, the expression $\left(\frac{100-x}{2}\right)^2$ increases as x decreases, so xyz decreases as x decreases. Thus, xyz is minimized when x is minimized at -200. This gives $y = z = 150$, for a minimum of $m = (-200) \cdot 150 \cdot 150$.

Now, we will maximize xyz. If x, y, z are all positive, $xyz \le 33 \cdot 33 \cdot 34$ by the AM-GM inequality. If x and y are negative and z is positive, xyz is maximized when $x = -50$, $y = -50$, and $z = 200$ by a reasoning similar to the minimization argument, giving $xyz = (-50) \cdot (-50) \cdot 200$, which is clearly larger than $33 \cdot 33 \cdot 34$. Thus, $M = (-50) \cdot (-50) \cdot 200$.

Therefore, $\frac{M}{m} = \frac{(-50)\cdot(-50)\cdot 200}{-200\cdot 150\cdot 150} = -\frac{1}{9}$.

15. In quadrilateral $ABCD$, diagonal AC bisects diagonal BD. Given that $AB = 20$, $BC = 15$, $CD = 13$, $AC = 25$, find DA.

Solution. The answer is $\boxed{4\sqrt{34}}$.



As shown in the diagram above, P is the intersection of diagonals AC and BD, and H_B and H_D are the feet of the perpendiculars from B and D to line AC, respectively. Because $BP = PD$, it follows that right triangles BPH_B and DPH_D are congruent. In particular, we have $BH_B = DH_D$. Because $AB^2 + BC^2 = 15^2 + 20^2 = 25^2 = CA^2$, ABC is a right triangle with area 150. It follows that $\frac{BH_B \cdot AC}{2} = 150$, which yields that $DH_D = BH_B = 12$. In right triangle CDH_D, we have $CD = 13$ and $DH_D = 12$, and thus $CH_D = 5$. Hence $AH_D = AC - CH_D = 20$. Therefore, in right triangle ADH_D, we have $AD^2 = DH_D^2 + AH_D^2 = 12^2 + 20^2$, from which it follows that $AD = 4\sqrt{34}$.

2.4 Guts Test Solutions

2.4.1 Round 1

1. [6pts] Ravi has a bag with 100 slips of paper in it. Each slip has one of the numbers 3, 5, or 7 written on it. Given that half of the slips have the number 3 written on them, and the average of the values on all the slips is 4.4, how many slips have 7 written on them?

   Solution. The answer is $\boxed{20}$.

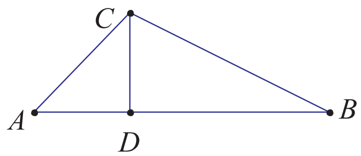
   Let x be the number of slips with 7 written on them. Then, there are $50 - x$ slips with 5 written on them. The average of the numbers on the slips is

   $$4.4 = \frac{50 \cdot 3 + (50 - x) \cdot 5 + x \cdot 7}{100} = \frac{400 + 2x}{100}.$$

   This yields $x = 20$, and it follows that there are 20 slips with 7 written on them.

2. [6pts] In triangle ABC, point D lies on side AB such that $AB \perp CD$. It is given that $\frac{CD}{BD} = \frac{1}{2}$, $AC = 29$, and $AD = 20$. Find the area of triangle BCD.

   Solution. The answer is $\boxed{441}$.



   In right triangle ADC, we have $CD = 21$. The area of triangle BCD is equal to

   $$\frac{BD \cdot CD}{2} = CD^2 = 441.$$

3. [6pts] Compute $(123 + 4)(123 + 5) - 123 \cdot 132$.

   Solution. The answer is $\boxed{20}$.

   Expanding the expression, we have

   $$\begin{aligned}(123+4)(123+5) - 123 \cdot 132 &= 123 \cdot 123 + 4 \cdot 123 + 5 \cdot 123 + 4 \cdot 5 - 123(123 + 9) \\ &= 123 \cdot 123 + 9 \cdot 123 + 20 - 123 \cdot 123 - 9 \cdot 123 = 20.\end{aligned}$$

2.4.2 Round 2

4. [8pts] David is evaluating the terms in the sequence $a_n = (n+1)^3 - n^3$ for $n = 1, 2, 3, \ldots$ (that is, $a_1 = 2^3 - 1^3, a_2 = 3^3 - 2^3, a_3 = 4^3 - 3^3$, and so on). Find the first composite number in the sequence. (An positive integer is *composite* if it has a divisor other than 1 and itself.)

   Solution. The answer is $\boxed{91}$.

   The first four terms of the sequence, $a_1 = 7, a_2 = 19, a_3 = 37, a_4 = 61$ are all prime. The fifth term $a_5 = 91 = 13 \cdot 7$ is the first composite number in the sequence.

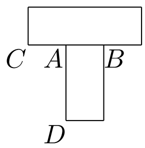
5. [8pts] Find the sum of all positive integers strictly less than 100 that are not divisible by 3.

   Solution. The answer is $\boxed{3267}$.

   Each term n can be paired with $99 - n$, because if n is a multiple of 3, so is $99 - n$, and vice versa. There are 66 positive integers strictly less than 100 that are not divisible by 3. Hence, their sum is

$$1 + 2 + 4 + 5 + 7 + 8 + \cdots + 97 + 98 = \frac{99 \cdot 66}{2} = 3267.$$

6. [8pts] In how many ways can Alex draw the diagram below without lifting his pencil or retracing a line? (Two drawings are different if the order in which he draws the edges is different, or the direction in which he draws an edge is different).



   Solution. The answer is $\boxed{12}$.

   Such a drawing must either start at corner A and end at corner B or start at B and end at A. By symmetry, half of the drawings start at A and half start at B. There are three possible ways to go from A to B without retracing: (1) via segment AB; (2) go around the upper rectangle via corner C; (3) go around the lower rectangle via corner D. A drawing starting from A is a permutation of (1), (2), and (3). Hence the answer is $2 \cdot 3! = 12$.

   Note: These drawings are called *Eulerian paths/cycles* in the subject of *Graph Theory*, first studied by one of the most celebrated mathematician Euler when he was walking on the bridges in the city of Konigsburg in 1736.

2.4.3 Round 3

7. [10pts] Fresh Mann is a 9th grader at Euclid High School. Fresh Mann thinks that the word *vertices* is the plural of the word *vertice*. Indeed, vertices is the plural of the word *vertex*. Using all the letters in the word *vertice*, he can make m 7-letter sequences. Using all the letters in the word *vertex*, he can make n 6-letter sequences. Find $m - n$.

   Solution. The answer is $\boxed{2160}$

   For 7-letter sequences, we can place the five letters excluding e in $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$ ways. After having placed these letters, there is only one way to place the e's, so $m = 2520$. Similarly, we can place the four letters excluding e in the six-letter sequence in $6 \cdot 5 \cdot 4 \cdot 3$ ways, so $n = 360$. This yields $m - n = 2160$.

8. [10pts] Fresh Mann is given the following expression in his Algebra 1 class:

$$101 - 102 = 1.$$

   Fresh Mann is allowed to move some of the digits in this (incorrect) equation to make it into a correct equation. What is the minimal number of digits Fresh Mann needs to move?

Solution. The answer is $\boxed{1}$.

Moving the 2, we have $101 - 10^2 = 1$.

9. [10pts] Fresh Mann said, "The function $f(x) = ax^2+bx+c$ passes through 6 points. Their x-coordinates are consecutive positive integers, and their y-coordinates are $34, 55, 84, 119, 160$, and 207, respectively." Sophy Moore replied, "You've made an error in your list," and replaced one of Fresh Mann's numbers with the correct y-coordinate. Find the corrected value.

Solution. The answer is $\boxed{32}$.

We define

$$g(x+1) = f(x+1) - f(x) = a(x+1)^2 + b(x+1) + c - ax^2 - bx - c = 2ax + a + b.$$

We then define

$$h(x+1) = g(x+1) - g(x) = 2a(x+1) + a + b - 2ax - a - b = 2a.$$

Notice that $h(x)$ is constant.

Let the x-values that correspond to $35, 55, 84, 119, 160$, and 207 be $n, n+1, n+2, n+3, n+4$, and $n+5$, respectively. Then we have $h(n+3) = h(n+4) = h(n+5) = 6$, and $h(n+2) = 9$. Consequently, there are no errors in $f(n+1), f(n+2), f(n+3), f(n+4)$, and $f(n+5)$. For $h(n+2)$ to equal 6, we must have

$$g(n+2) - g(n+1) = 29 - (55 - f(n)) = 6$$

Solving for $f(n)$ yields $f(n) = 32$.

2.4.4 Round 4

10. [12pts] An assassin is trying to find his target's hotel room number, which is a three-digit positive integer. He knows the following clues about the number:

(a) The sum of any two digits of the number is divisible by the remaining digit.

(b) The number is divisible by 3, but if the first digit is removed, the remaining two-digit number is not.

(c) The middle digit is the only digit that is a perfect square.

Given these clues, what is a possible value for the room number?

Solution. The answer is either $\boxed{213}$ or $\boxed{246}$.

It is easy to check that all three conditions are satisfied for both values.

11. [12pts] Find a positive real number r that satisfies

$$\frac{4+r^3}{9+r^6} = \frac{1}{5-r^3} - \frac{1}{9+r^6}.$$

Solution. The answer is $\boxed{\sqrt{2}}$.

Simplifying, we have

$$\frac{5+r^3}{9+r^6} = \frac{1}{5-r^3},$$

and cross-multiplying yields, $25 - r^6 = 9 + r^6$ or $16 = 2r^6$. It follows that $r^6 = 8$, or $r = \sqrt[6]{8} = \sqrt{2}$.

12. [12pts] Find the largest integer n such that there exist integers x and y between 1 and 20 inclusive with
$$\left|\frac{21}{19} - \frac{x}{y}\right| < \frac{1}{n}.$$

Solution. The answer is $\boxed{189}$.
When $x = 11$ and $y = 10$, we have
$$\left|\frac{21}{19} - \frac{11}{10}\right| = \frac{1}{190} < \frac{1}{189},$$
and hence $n = 189$ satisfies the problem conditions. We will show that
$$\frac{1}{190} \le \left|\frac{21}{19} - \frac{x}{y}\right| = \left|\frac{21y - 19x}{19y}\right|.$$
for all integers x, y between 1 and 20 inclusive by casework. Because $0 < x < 21$, $19x$ is not a multiple of x. In particular, we have $21y - 19x \neq 0$. We consider the cases $|21y - 19x| \ge 2$, $21y - 19x = 1$, and $21y - 19x = -1$.

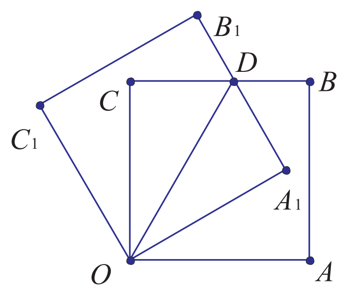
- *Case 1:* If $|21y - 19x| \ge 2$, then we have
$$\left|\frac{21}{19} - \frac{x}{y}\right| = \left|\frac{21y - 19x}{19y}\right| \ge \frac{2}{19y} \ge \frac{2}{19 \cdot 20} = \frac{1}{190}.$$
- *Case 2:* If $21y - 19x = 1$, then 21 divides $19x + 1$ from which it follows that 21 divides $-2x + 1$. The only possible value for x between 1 and 20 inclusive is $x = 11$, which yields $y = 10$ and $\left|\frac{21}{19} - \frac{x}{y}\right| = \frac{1}{190}$.
- *Case 3:* If $21y - 19x = -1$, then 21 divides $19x - 1$, from which it follows that 21 divides $-2x - 1$. The only possible value for x in the required range is $x = 10$, which yields $y = 9$ and $\left|\frac{21}{19} - \frac{x}{y}\right| = \frac{1}{171}$.

Combining the above, we have $\left|\frac{21}{19} - \frac{x}{y}\right| \ge \frac{1}{190}$ for all integers x,y between 1 and 20 inclusive.

2.4.5 Round 5

13. [14pts] A unit square is rotated 30° counterclockwise about one of its vertices. Determine the area of the intersection of the original square with the rotated one.

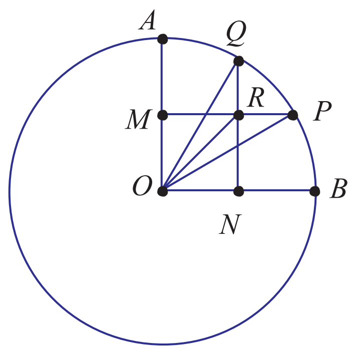
Solution. The answer is $\boxed{\dfrac{\sqrt{3}}{3}}$.



The diagram has reflective symmetry over line OD. We have $\angle A_1OA = 30°$, and hence we have $\angle COA_1 = 60°$. By symmetry, we have $\angle COD = 30°$, and thus triangle COD is a $30° - 60° - 90°$ right triangle. We have $CD = \frac{\sqrt{3}}{3}$, which yields that the area of triangle COD is $\frac{\sqrt{3}}{6}$. By symmetry the area of triangle DOA_1 is also $\frac{\sqrt{3}}{6}$, and it follows that the area of intersection is $\frac{\sqrt{3}}{3}$.

14. [14pts] Suppose points A and B lie on a circle of radius 4 with center O, such that $\angle AOB = 90°$. The perpendicular bisectors of segments OA and OB divide the interior of the circle into four regions. Find the area of the smallest region.

Solution. The answer is $\boxed{\dfrac{4\pi}{3} + 4 - 4\sqrt{3}}$.



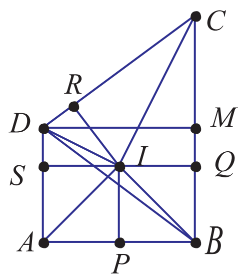
Because Q lies on the perpendicular bisector of segment OB, we have $BQ = QO$ and because Q lies on the circle, we have $QO = OB$. Thus triangle QOB is equilateral, and hence we have $\angle QOB = 60°$. Similarly, we have $\angle AOP = 60°$. Furthermore, we have $QN = QO\frac{\sqrt{3}}{2} = 2\sqrt{3}$. Because $\angle AOB = 90°$, we have $\angle QOP = 30°$, from which it follows that the area of sector QOP is $\frac{16\pi}{12} = \frac{4\pi}{3}$. We have $OM = ON = 2$, and because $\angle RMO = \angle MON = \angle ONR = 90°$, quadrilateral $RMON$ is a square. Hence, we have $RN = 2$ and it follows that $QR = 2\sqrt{3} - 2$. Thus, the area of triangle QOR is

$$\frac{ON \cdot QR}{2} = \frac{2(2\sqrt{3} - 2)}{2} = 2\sqrt{3} - 2,$$

which by symmetry equals the area of triangle ROP. The area of the desired region is the area of sector QOP minus the sum of the areas of triangles QOR and ROP, so the answer is $\frac{4\pi}{3} + 4 - 4\sqrt{3}$.

15. [14pts] Let $ABCD$ be a quadrilateral such that $AB = 4$, $BC = 6$, $CD = 5$, $DA = 3$, and $\angle DAB = 90°$. There is a point I inside the quadrilateral that is equidistant from all the sides. Find AI.

Solution. The answer is $\boxed{2\sqrt{2}}$.



Let the distance from I to each side be r. Because triangle ABD is right, we have $BD = 5$, which yields that triangle BCD is isosceles. Let M be the midpoint of segment BC, and we have $BM = MC = 3$, from which it follows that $DM = 4$. Hence, $ADMB$ is a rectangle, so lines AD and BC are parallel. Because I is equidistant from AD and BC and $AB = 4$, we have that $r = 2$.

Let P, Q, R, S be the feet of the perpendiculars from I to AB, BC, CD, DA, respectively. We have that $ASIP$ is a square with side length r. Hence, we have $AI = r\sqrt{2} = 2\sqrt{2}$.

2.4.6 Round 6

The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.

16. [16pts] Let C be the answer to problem 18. Compute

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{C^2}\right).$$

Solution. The answer is $\boxed{\frac{37}{72}}$. For every $t \neq 0$, we have that

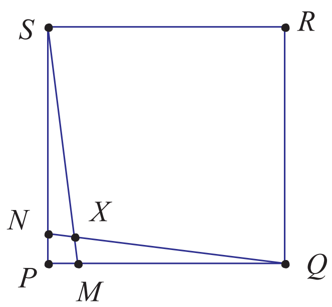
$$1-\frac{1}{t^2} = \frac{t^2-1}{t} = \frac{t-1}{t}\cdot\frac{t+1}{t}.$$

This yields

$$\begin{aligned}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{C^2}\right) &= \left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\cdots\left(\frac{C-1}{C}\cdot\frac{C+1}{C}\right)\\ &= \frac{1}{2}\left(\frac{3}{2}\cdot\frac{2}{3}\right)\left(\frac{4}{3}\cdot\frac{3}{4}\right)\cdots\left(\frac{C}{C-1}\cdot\frac{C-1}{C}\right)\frac{C+1}{C}\\ &= \frac{1}{2}\cdot\frac{C+1}{C} = \frac{C+1}{2C}.\end{aligned}$$

17. [16pts] Let A be the answer to problem 16. Let $PQRS$ be a square, and let point M lie on segment PQ such that $MQ = 7PM$ and point N lie on segment PS such that $NS = 7PN$. Segments MS and NQ meet at point X. Given that the area of quadrilateral $PMXN$ is $A - \frac{1}{2}$, find the side length of the square.

Solution. The answer is $\boxed{1}$.



By symmetry, triangles XNP and XPM have the same area, and hence they must each have area $\frac{2A-1}{4}$. The ratio of the area of triangle XNP to the area of triangle XSP is $\frac{NP}{SP} = \frac{1}{8}$, and it follows that triangle XSP has area $4A-2$. Hence, triangle MSP has area $4A-2+\frac{2A-1}{4} = \frac{18A-9}{4}$. Because $\frac{PN}{PS} = \frac{1}{8}$, the area of triangle QSP is $36A-18$, from which it follows that the area of square $PQRS$ is $72A-36$. Thus, the side length of the square is $\sqrt{72A-36}$.

18. [16pts] Let B be the answer to problem 17 and let $N = 6B$. Find the number of ordered triples (a, b, c) of integers between 0 and $N-1$, inclusive, such that $a+b+c$ is divisible by N.

    Solution. The answer is $\boxed{36}$. For every choice of integers a, b between 0 and $N-1$ inclusive, there is a unique integer c between 0 and $N-1$ inclusive such that $a+b+c$ is divisible by N. Because there are N choices for each of a and b, the number of ordered triples (a, b, c) with $a+b+c$ divisible by N is $N^2 = 36B^2$.

    Solution (to the system). We have

$$\begin{aligned} A &= \frac{C+1}{2C} \\ B &= \sqrt{72A-36} \\ C &= 36B^2. \end{aligned}$$

Substituting the expression for A into the expression for B, we have

$$B = \sqrt{72\left(\frac{C+1}{2C} - 36\right)} = \sqrt{72 \cdot \frac{1}{2C}} = \sqrt{\frac{36}{C}}.$$

and hence we have

$$C = 36B^2 = \frac{36^2}{C}.$$

This yields that $C = 36$, from which it follows that $A = \frac{37}{72}$ and $B = 1$.

2.4.7 Round 7

19. [16pts] Let k be the units digit of $\underbrace{7^{7^{7^{7^{7^{7^{7}}}}}}}_{\text{Seven 7s}}$. What is the largest prime factor of the number consisting of k 7's written in a row?

    Solution. The answer is $\boxed{37}$.

    Because $7^{7^{7^{7^{7}}}}$ is odd, the remainder of $7^{7^{7^{7^{7^{7}}}}}$ on division by 4 is 3. We have that 7^4 has a units digit of 1, and hence we have that $7^{7^{7^{7^{7^{7^{7}}}}}}$ has the same units digit as $7^3 = 343$, and it follows that $k = 3$. We have that $777 = 3 \cdot 7 \cdot 37$, and thus the largest prime divisor is 37.

20. [16pts] Suppose that $E = 7^7$, $M = 7$, and $C = 7 \cdot 7 \cdot 7$. The characters E, M, C, C are arranged randomly in the following blanks.

$$\_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_$$

Then one of the multiplication signs is chosen at random and changed to an equals sign. What is the probability that the resulting equation is true?

Solution. The answer is $\boxed{\frac{1}{6}}$.

Suppose that the side of the equation containing E also contains one of the other letters. Then, that side will be at least $7E = 7^8$ while the other side will be at most $C^2 = 7^6$, so the two sides cannot be equal. Thus, the only way in which the equation can be true is if E is at one of the ends and the $=$ sign is right next to it, and it is clear that any such configuration is a true equation. Because the probability that E is placed at one of the ends is $\frac{1}{2}$ and the probability that the $=$ sign is next to it is $\frac{1}{3}$, the answer is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

21. [16pts] During a recent math contest, Sophy Moore made the mistake of thinking that 133 is a prime number. Fresh Mann replied, "To test whether a number is divisible by 3, we just need to check whether the sum of the digits is divisible by 3. By the same reasoning, to test whether a number is divisible by 7, we just need to check that the sum of the digits is a multiple of 7, so 133 is clearly divisible by 7." Although his general principle is false, 133 is indeed divisible by 7. How many three-digit numbers are divisible by 7 and have the sum of their digits divisible by 7?

    Solution. The answer is $\boxed{18}$.

    Suppose that n is a three-digit number with hundreds digit a, tens digit b, and units digit c. The first condition is that $a + b + c$ is divisible by 7; and the condition that n is divisible by 7 is equivalent to $2a + 3b + c$ being divisible by 7. The two conditions are equivalent to $a + 2b$ being divisible by 7 and $b - c$ being divisible by 7.

    We consider the cases $a = 1$ or 8, $a = 2$ or 9, $a = 3$, $a = 4$, $a = 5$, $a = 6$, and $a = 7$.

    - *Case 1:* We have $a = 1$ or $a = 8$. Then, we must have $b = c = 3$, which yields 2 solutions.
    - *Case 2:* We have $a = 2$ or $a = 9$. Then, we must have $b = c = 6$, which yields 2 solutions.
    - *Case 3:* We have $a = 3$. Then, we can have $b = 2$ or $b = 9$ and $c = 2$ or $c = 9$, which yields 4 solutions.
    - *Case 4:* We have $a = 4$. Then, we must have $b = c = 5$, which yields 1 solution.
    - *Case 5:* We have $a = 5$. Then, we can have $b = 1$ or $b = 8$ and $c = 1$ or $c = 8$, which yields 4 solutions.
    - *Case 6:* We have $a = 6$. Then, we must have $b = c = 4$, which yields 1 solution.
    - *Case 7:* We have $a = 7$. Then, we can have $b = 0$ or $b = 7$ and $c = 0$ or $c = 7$, which yields 4 solutions.

    There are a total of $2 + 2 + 4 + 1 + 4 + 1 + 4 = 18$ solutions.

2.4.8 Round 8

22. [18pts] A *look-and-say sequence* is defined as follows: starting from an initial term a_1, each subsequent term a_k is found by reading the digits of a_{k-1} from left to right and specifying the number of times each digit appears consecutively. For example, 4 would be succeeded by 14 ("One four."), and 31337 would be followed by 13112317 ("One three, one one, two three, one seven.")

    If a_1 is a random two-digit positive integer, find the probability that a_4 is at least six digits long.

Solution. The answer is $\boxed{\dfrac{89}{90}}$.

We claim that $a_1 = 22$ is the only starting value such that a_4 has less than six digits. If the starting value is 22, then it is easy to see that $a_2 = a_3 = a_4 = 22$. Suppose that a_1 has a tens digit of x and a units digit of y. We write $a_1 = \overline{xy}$.

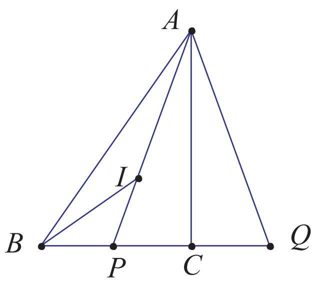
We consider the cases $x = y = 1$, $x = y \neq 1$ or 2, $x \neq y$ and $x, y \neq 1$, $x = 1$ and $y \neq 1$, and $x \neq 1$ and $y = 1$.

- *Case 1:* We have $x = y = 1$. Then, the sequence is $a_2 = 21$, $a_3 = 1211$ and $a_4 = 111221$, which has 6 digits.
- *Case 2:* We have $x = y \neq 1, 2$. Then, the sequence is $a_2 = \overline{2x}$, $a_3 = \overline{121x}$ and $a_4 = \overline{1112111x}$, which has 8 digits.
- *Case 3:* We have $x \neq y$ and $x, y \neq 1$. Then, the sequence is $a_2 = \overline{1x1y}$, $a_3 = \overline{111x111y}$ and $a_4 = \overline{311x311y}$, which has 8 digits.
- *Case 4:* We have $x = 1$ and $y \neq 1$. Then, the sequence is $a_2 = \overline{111y}$, $a_3 = \overline{311y}$ and $a_4 = \overline{13211y}$, which has 6 digits.
- *Case 5:* We have $x \neq 1$ and $y = 1$. Then, the sequence is $a_2 = \overline{1x11}$, $a_3 = \overline{111x21}$. If $x = 2$, then we have $a_4 = 312221$, which has 6 digits; if $x \neq 2$, then we have $a_4 = \overline{311x1211}$, which has 8 digits.

Combining the above, we have that 22 is the only choice of a_1 such that a_4 has less than 6 digits, and it follows that the probability that a_4 has at least 6 digits is $\frac{89}{90}$.

23. [18pts] In triangle ABC, $\angle C = 90°$. Point P lies on segment BC and is not B or C. Point I lies on segment AP, and $\angle BIP = \angle PBI = \angle CAB$. If $\dfrac{AP}{BC} = k$, express $\dfrac{IP}{CP}$ in terms of k.

Solution. The answer is $\boxed{\dfrac{2-k}{k-1}}$.



Let $\angle PBI = \theta$ and let Q be the reflection of point P over line AC. We have $QA = AP$, which yields that $\angle AQP = \angle APQ$. By exterior angles on triangle PIB, we have $\angle APQ = \angle PBI + \angle BIP = 2\theta$. Additionally, we have

$$\angle QBA = \angle CBA = 90° - \angle BAC = 90° - \theta.$$

Because the angles of a triangle sum to $180°$, we have

$$\angle BAQ = 180° - 2\theta - (90° - \theta) = 90 - \theta = \angle QBA,$$

from which it follows that $BQ = QA = AP$. Because $\angle BIP = \angle PBI$, we have $BP = PI$.

Let $BP = x$ and $PC = y$; we have $CQ = PC = y$. This yields that $BC = x + y$ and $BQ = x + 2y$. We have

$$\frac{x+2y}{x+y} = \frac{BQ}{BC} = \frac{AP}{BC} = k.$$

Hence, we have $x + 2y = kx + ky$ or $y(2 - k) = x(k - 1)$ or $\frac{2-k}{k-1} = \frac{x}{y}$. It follows that

$$\frac{IP}{CP} = \frac{BP}{PC} = \frac{x}{y} = \frac{2-k}{k-1}.$$

24. [18pts] A subset of $\{1, 2, 3, \ldots, 30\}$ is called *delicious* if it does not contain an element that is 3 times another element. A subset is called *super delicious* if it is delicious and no delicious set has more elements than it has. Determine the number of super delicious subsets.

Solution. The answer is $\boxed{96}$.

We partition the set $\{1, 2, 3, \cdots, 30\}$ into subsets:

$$\{1,3,9,27\}, \{2,6,18\}, \{4,12\}, \{5,15\}, \{7,21\}, \{8,24\}, \{10,30\}, \{11\}, \{13\}, \{14\}, \{16\}, \{17\}, \{19\}, \{20\}, \{22\}, \{23\}, \{25\}, \{26\}, \{28\}, \{29\}.$$

If A is delicious, then it can contain at most 1 member of each listed 1 or 2-element subset and at most 2 members of each listed 3 or 4-element subset. Thus A contains at most $2 \cdot 2 + 18 \cdot 1 = 22$ elements. If A is super delicious, then A must contain the given maximum possible number of elements for each listed subset. There are 3 ways to choose 2 elements of the 4-element listed subset: $\{1, 9\}, \{1, 27\}, \{3, 27\}$, and there are 2 ways to choose the an element of each 2-element subset. There is only 1 way to choose 2 elements of the 3-element subset; we must choose 2 and 18. Thus there are $3 \cdot 2^5 = 96$ super delicious sets A.

2.5 Puzzle Round Answer

1	3	3	27	27	27	27	27	27	2
2	2	3	2	16	16	16	3	27	2
9	9	1	2	16	1	16	3	27	5
9	9	16	16	16	4	16	3	27	5
9	9	9	4	4	4	16	27	27	5
9	9	3	3	3	16	16	27	5	5
6	6	6	16	16	16	6	27	27	27
6	4	27	27	16	6	6	6	6	27
6	4	1	27	27	27	6	27	27	27
6	4	4	2	2	27	27	27	2	2