3.2 Individual Accuracy Test Solutions

1. Calculate $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)^2$.

   Solution. The answer is $\left(\frac{13}{12}\right)^2 = \boxed{\frac{169}{144}}$.

2. Find the 2010$^{\text{th}}$ digit after the decimal point in the expansion of $\frac{1}{7}$.

   Solution. The answer is $\boxed{7}$. Note that $\frac{1}{7} = 0.\overline{142857}$ and 2010 is divisible by 6.

3. If you add 1 liter of water to a solution consisting of acid and water, the new solutions will contain of 30% water. If you add another 5 liters of water to the new solution, it will contain $36\frac{4}{11}\%$ water. Find the number of liters of acid in the original solution.

   Solution. The answer is $\boxed{35}$. Assume that there are x liters of water and y liters of acid in the original solution. We have $\frac{x+1}{x+y+1} = 30\%$ and $\frac{x+6}{x+y+6} = 44\frac{4}{11}\%$, from which it follows that $x = 14$, $y = 35$.

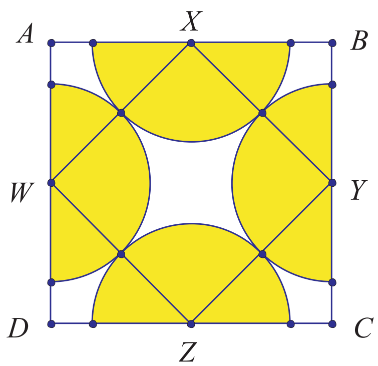
4. John places 5 indistinguishable blue marbles and 5 indistinguishable red marbles into two distinguishable buckets such that each bucket has at least one blue marble and one red marble. How many distinguishable marble distributions are possible after the process is completed?

   Solution. The answer is $4 \cdot 4 = \boxed{16}$. The first bucket can have x (with $x = 0, 1, 2,$ or 3) blue marbles and y (with $y = 0, 1, 2,$ or 3) red marbles, and the second bucket will have $4 - x$ blue marbles and $4 - y$ red marbles.

5. In quadrilateral $PEAR$, $PE = 21$, $EA = 20$, $AR = 15$, $RE = 25$, and $AP = 29$. Find the area of the quadrilateral.

   Solution. The answer is $\frac{21+15}{2} \cdot 20 = \boxed{360}$. Because $15^2 + 20^2 = 25^2$ and $21^2 + 20^2 = 29^2$, triangles APE and ERA are right triangles (with $\angle EAR = \angle PEA = 90°$). Consequently, $PEAR$ is a right trapezoid with bases $PE = 21$ and $AR = 15$ and hight $EA = 20$.

6. Four congruent semicircles are drawn within the boundary of a square with side length 1. The center of each semicircle is the midpoint of a side of the square. Each semicircle is tangent to two other semicircles. Region $\mathcal{R}$ consists of points lying inside the square but outside of the semicircles. The area of $\mathcal{R}$ can be written in the form $a - b\pi$, where a and b are positive rational numbers. Compute $a + b$.



Solution. The answer is $1 + \dfrac{1}{4} = \boxed{\dfrac{5}{4}}$. Let $ABCD$ denote the unit square, and let X, Y, Z, W be the midpoints of sides AB, BC, CD, DA, respectively. Then $AX = AW = \frac{1}{2}$ and $WX = \frac{\sqrt{2}}{2}$. The radii of the semicircles are equal to $r = \frac{WX}{2} = \frac{\sqrt{2}}{4}$. The area of the region covered by the semicircles is equal to $2\pi r^2 = \frac{\pi}{4}$ and the area of region $\mathcal{R}$ is equal to $1 - \frac{\pi}{4}$, implying that $(a, b) = \left(1, \frac{1}{4}\right)$.

7. Let x and y be two numbers satisfying the relations $x \geq 0$, $y \geq 0$, and $3x + 5y = 7$. What is the maximum possible value of $9x^2 + 25y^2$?

   Solution. The answer is $\boxed{49}$. We have $9x^2 + 25y^2 = (3x + 5y)^2 - 30xy = 49 - 30xy \leq 49$ for nonnegative numbers x and y.

8. In the Senate office in Exie-land, there are 6 distinguishable senators and 6 distinguishable interns. Some senators and an equal number of interns will attend a convention. If at least one senator must attend, how many combinations of senators and interns can attend the convention?

   Solution. The answer is $\boxed{923}$. If there are i, $1 \leq i \leq 6$, senators attend the convention, there there are i interns attend the convention. Hence is the answer is

$$\binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 = 36 + 225 + 400 + 225 + 36 + 1 = 923.$$

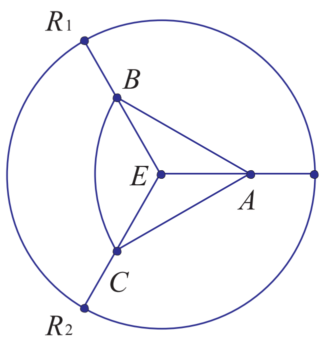
   This is a special case of Vandermonde's identity – the answer is equal to $\binom{12}{6} - 1$. Indeed, we can choose any 6 people (x senators interns and $6 - x$ interns) and we will then send the x chosen senators and x non-chosen interns to the convention. The only case we have to exclude is the case when 0 senators and 6 interns are chosen.

9. Evaluate $(1^2 - 3^2 + 5^2 - 7^2 + 9^2 - \cdots + 2009^2) - (2^2 - 4^2 + 6^2 - 8^2 + 10^2 - \cdots + 2010^2)$.

   Solution. The answer is $2008 + 2009^2 - 2010^2 = \boxed{-2011}$. The key fact is $n^2 - (n+1)^2 - (n+2)^2 + (n+3)^2 = 4$ for all numbers n. Thus,

$$\begin{aligned} & (1^2 - 3^2 + 5^2 - 7^2 + 9^2 - \cdots + 2009^2) - (2^2 - 4^2 + 6^2 - 8^2 + 10^2 - \cdots + 2010^2) \\ = \; & (1^2 - 2^2 - 3^2 + 4^2) + \cdots + (2005^2 - 2006^2 - 2007^2 + 2008^2) + 2009^2 - 2010^2. \end{aligned}$$

10. Segment EA has length 1. Region $\mathcal{R}$ consists of points P in the plane such that $\angle PEA \geq 120^\circ$ and $PE < \sqrt{3}$. If point X is picked randomly from the region $\mathcal{R}$, the probability that $AX < \sqrt{3}$ can be written in the form $a - \frac{\sqrt{b}}{c\pi}$, where a is a rational number, b and c are positive integers, and b is not divisible by the square of a prime. Find the ordered triple (a, b, c).



Solution. The answer is $\boxed{\left(\frac{1}{2}, 3, 2\right)}$.

Let ω_1 denote the circle centered at E with radius $\sqrt{3}$. Points R_1 and R_2 lie on ω_1 with $\angle R_1EA = \angle R_2EA = 120°$. Region $\mathcal{R}$ is the circular sector centered R_1ER_2. The area of region $\mathcal{R}$ is equal to π. Let ω_2 denote the circle centered at A with radius $\sqrt{3}$, and let ω_2 intersects segments ER_1 and ER_2 are B and C, respectively. Let $\mathcal{R}_1$ denote the region enclosed by arc $\widehat{BC}$ and segments EB and EC. Then a point X is in region $\mathcal{R}$ if and only if X lies in $\mathcal{R}_1$; that is, The probability p that X lies $\mathcal{R}_1$ is equal to ratio of the area of $\mathcal{R}_1$ to that of $\mathcal{R}$. It is not difficult to see that $EA = EB = EC = 1$, the area of circular sector BCA is $\frac{\pi}{2}$, the triangles EAB and EAC have the same area $\frac{\sqrt{3}}{4}$, the area of region $\mathcal{R}_1 = \frac{\pi}{2} - \frac{\sqrt{3}}{2}$, and

$$p = \frac{\frac{\pi}{2} - \frac{\sqrt{3}}{2}}{\pi} = \frac{1}{2} - \frac{\sqrt{3}}{2\pi}.$$