3.3 Team Round Solutions

1. A very large lucky number N consists of eighty-eight 8s in a row. Find the remainder when this number N is divided by 6.

   Solution. The answer is $\boxed{2}$. Because 888 is divisible by 6, $N - 8$ is divisible by 6.

2. If 3 chickens can lay 9 eggs in 4 days, how many chickens does it take to lay 180 eggs in 8 days?

   Solution. The answer is $\boxed{30}$. If 3 chickens can lay 9 eggs in 4 days, then 3 chickens can lay 18 eggs in 8 days, and then 30 chickens can lay 180 eggs in 8 days.

3. Find the ordered pair (x, y) of real numbers satisfying the conditions $x > y$, $x+y = 10$, and $xy = -119$.

   Solution. The answer is $\boxed{(17, -7)}$.

4. There is pair of similar triangles. One triangle has side lengths 4, 6, and 9. The other triangle has side lengths 8, 12 and x. Find the sum of two possible values of x.

   Solution. The answer is $\dfrac{16}{3} + 18 = \boxed{\dfrac{70}{3}}$. We have either $4 : 6 : 9 = x : 8 : 12$ or $4 : 6 : 9 = 8 : 12 : x$. In the former case, $x = \frac{16}{3}$. In the latter case, $x = 18$.

5. If $x^2 + \frac{1}{x^2} = 3$, there are two possible values of $x + \frac{1}{x}$. What is the smaller of the two values?

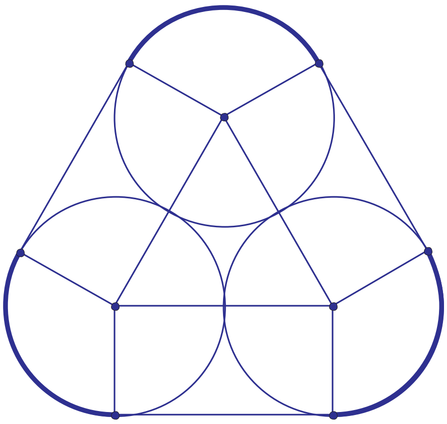
   Solution. The answer is $\boxed{-\sqrt{5}}$. We have $5 = x^2 + \frac{1}{x^2} + 2 = \left(x + \frac{1}{x}\right)^2$.

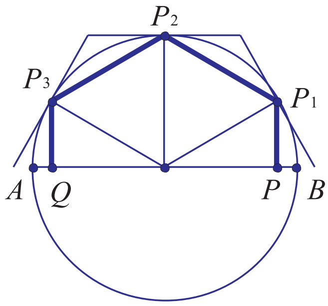
6. Three flavors (chocolate strawberry, vanilla) of ice cream are sold at Brian's ice cream shop. Brian's friend Zerg gets a coupon for 10 free scoops of ice cream. If the coupon requires Zerg to choose an even number of scoops of each flavor of ice cream, how many ways can he choose his ice cream scoops? (For example, he could have 6 scoops of vanilla and 4 scoops of chocolate. The order in which Zerg eats the scoops does not matter.)

   Solution. The answer is $\boxed{21}$. Assume that Zerg has $2x$ scoops of chocolate ice cream, $2y$ scoops of strawberry ice cream, and $2z$ scoops of vanilla ice cream, where x, y, z are nonnegative integers. Then $2x + 2y + 2z = 10$ or $x + y + z = 5$, which has $\binom{5+3-1}{3-1} = \binom{7}{2} = 21$ ordered triples of solutions in nonnegative integers.

7. David decides he wants to join the West African Drumming Ensemble, and thus he goes to the store and buys three large cylindrical drums. In order to ensure none of the drums drop on the way home, he ties a rope around all of the drums at their mid sections so that each drum is next to the other two. Suppose that each drum has a diameter of 3.5 feet. David needs m feet of rope. Given that $m = a\pi + b$, where a and b are rational numbers, find sum $a + b$.

   Solution. The answer is $3.5 + 10.5 = \boxed{14}$. The rope band is formed by six parts – three circular parts and three linear parts. Because three drums are congruent to each other, by symmetry, the three circular parts are congruent to each other and three linear parts are congruent to each other. (See the left-hand side figure shown below.) Moving along this band exactly once, one turns around 360 degrees. Because there is no changing of direction along the linear parts, each circular part of the band is one third of a full circle with diameter 3.5 feet. The remaining three parts are line segments. The length of each of the linear parts of the band is equal to the distance between the center of the drums. Therefore, the total length of the rope is $10.5 + 3.5\pi$.

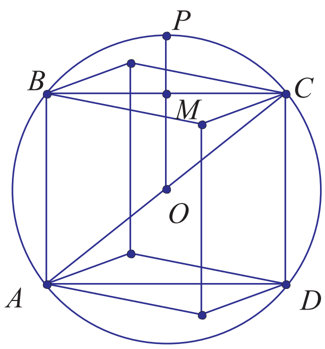


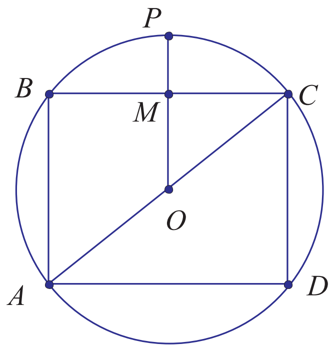


8. Segment AB is the diameter of a semicircle of radius 24. A beam of light is shot from a point $12\sqrt{3}$ from the center of the semicircle, and perpendicular to AB. How many times does it reflect off the semicircle before hitting AB again?

 Solution. The answer is $\boxed{3}$. Note that the beam of light will trace a half of a regular hexagon. (See the right-hand side figure shown above. The path of the light beam is $P \to P_1 \to P_2 \to P_3 \to Q$.

9. A cube is inscribed in a sphere of radius 8. A smaller sphere is inscribed in the same sphere such that it is externally tangent to one face of the cube and internally tangent to the larger sphere. The maximum value of the ratio of the volume of the smaller sphere to the volume of the larger sphere can be written in the form $\frac{a-\sqrt{b}}{36}$, where a and b are positive integers. Find the product ab.





 Solution. The answer is $9 \cdot 75 = \boxed{675}$. Let a denote the side length of the cube. The length of a face diagonal BC is $a\sqrt{2}$ and the length of an interior diagonal AC is $a\sqrt{3}$. Note that AC is also a diameter of the sphere. Hence, $a\sqrt{3} = 16$ or $a = \frac{16}{\sqrt{3}}$. Let M be the midpoint of segment BC. The maximum value of the diameter of the smaller sphere is equal to $PM = OP - OM = r - \frac{a}{2} = 8 - \frac{8}{\sqrt{3}}$, and the maximum value of the ratio of the volume of the smaller sphere to the volume of the larger sphere is equal to

$$\left(\frac{4 - \frac{4}{\sqrt{3}}}{8}\right)^3 = \left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)^3 = \frac{6\sqrt{3}-10}{24\sqrt{3}} = \frac{9-5\sqrt{3}}{36} = \frac{9-\sqrt{75}}{36}.$$

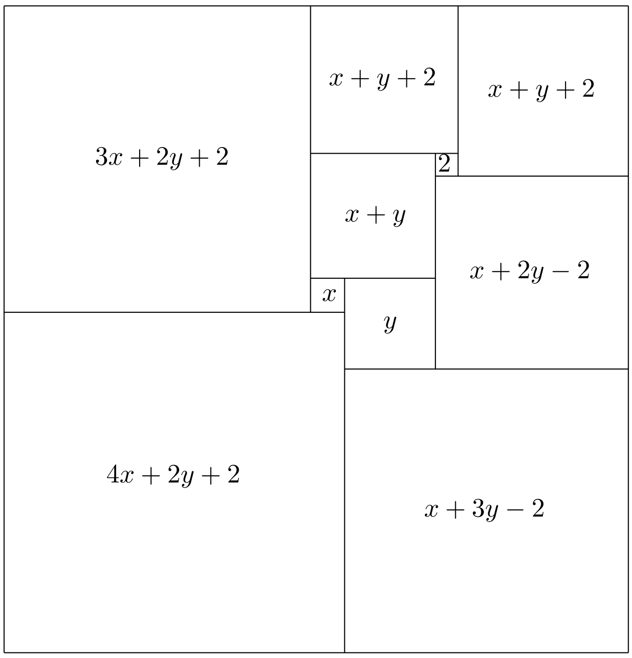
10. How many ordered pairs (x, y) of integers are there such that $2xy + x + y = 52$?

    Solution. The answer is $\boxed{16}$. We can write the given function as $4xy + 2x + 2y + 1 = 105$ or $(2x+1)(2y+1) = 105$. Note that $105 = 3 \cdot 5 \cdot 7$ has a total of $2 \cdot 2 \cdot 2 \cdot 2 = 16$ (positive and negative) integer divisors.

11. Three musketeers looted a caravan and walked off with a chest full of coins. During the night, the first musketeer divided the coins into three equal piles, with one coin left over. He threw it into the ocean and took one of the piles for himself, then went back to sleep. The second musketeer woke up an hour later. He divided the remaining coins into three equal piles, and threw out the one coin that was left over. He took one of the piles and went back to sleep. The third musketeer woke up and divided the remaining coins into three equal piles, threw out the extra coin, and took one pile for himself. The next morning, the three musketeers gathered around to divide the coins into three equal piles. Strangely enough, they had one coin left over this time as well. What is the minimum number of coins that were originally in the chest?

    Solution. The answer is $\boxed{79}$. Assume that there are x coins in the beginning. The first musketeer threw away a coin, took $\frac{x-1}{3}$ coins and left $\frac{2x-2}{3}$ coins. The second musketeer threw away one coin, took $\frac{2x-5}{9}$ coins and left $\frac{4x-10}{9}$ coins. The third musketeer threw away one coin, took $\frac{4x-19}{27}$ coins and left $\frac{8x-38}{27}$ coins. They then threw away one coin and each got $\frac{8x-65}{81}$ coins. Thus, $8x - 65$ must be a multiple of 81. Because $8x - 65 = 8x + 16 - 81 = 8(x+2) - 81$, the minimum value of x for $8x - 65$ being a multiple of 81 is $x = 79$.

12. The diagram shows a rectangle that has been divided into ten squares of different sizes. The smallest square is 2×2 (marked with $*$). What is the area of the rectangle (which looks rather like a square itself)?



Solution. The answer is $55 \cdot 57 = \boxed{3135}$. We assume that the two center squares have side lengths x and y, respectively. We can then express all the side lengths in terms of x and y. (See the figure shown above.) Because the opposite sides of the rectangle have the same lengths, we obtain the system of equations

$$\begin{cases} (3x+2y+2)+(x+y+2)+(x+y+4) = (4x+2y+2)+(x+3y-2), \\ (3x+2y+2)+(4x+2y+2) = (x+y+4)+(x+2y-2)+(x+3y-2) \end{cases}$$

or

$$\begin{cases} 5x+4y+8 = 5x+5y, \\ 7x+4y+4 = 3x+6y \end{cases}$$

from which it follows that $(x, y) = (3, 8)$ and that it is a 55×57 rectangle.

13. Let $A = (3, 2)$, $B = (0, 1)$, and P be on the line $x + y = 0$. What is the minimum possible value of $AP + BP$?

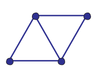
Solution. The answer is $\boxed{2\sqrt{5}}$. Set $C = (-1, 0)$. Note that B and C are reflections of each other across the line $x + y = 0$ and that points A and C lie on opposite sides of the line. Hence $AP + BP = AP + PC \le AC = 2\sqrt{5}$.

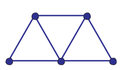
14. Mr. Mustafa the number man got a $6 \times x$ rectangular chess board for his birthday. Because he was bored, he wrote the numbers 1 to $6x$ starting in the upper left corner and moving across row by row (so the number $x + 1$ is in the 2^{nd} row, 1^{st} column). Then, he wrote the same numbers starting in the upper left corner and moving down each column (so the number 7 appears in the 1^{st} row, 2^{nd} column). He then added up the two numbers in each of the cells and found that some of the sums were repeated. Given that x is less than or equal to 100, how many possibilities are there for x?

Solution. The answer is $\boxed{14}$. Let (i, j) denote the cell in the i^{th} row and j^{th} column. Let $a_{i,j}$ denote the number written in the cell (i, j) in the first scheme, and let $b_{i,j}$ denote the number written in the cell (i, j) in the second scheme. Then we have $a_{i,j} = (i-1)x + j$ and $b_{i,j} = 6(j-1) + i$. The sum of the numbers written in the cell (i, j) is equal to $a_{i,j} + b_{i,j} = (i-1)x + j + 6(j-1) + i = i(x+1) + 7j - x - 6$. Assume that the sums of the numbers written in cells (i_1, j_1) and (i_2, j_2) are equal. We have $i_1(x+1)+7j_1-x-6 = i_2(x+1)+7j_2-x-6$ or $(i_1-i_2)(x+1) = 7(j_2-j_1)$. Because i_1 and i_2 are row numbers, their positive difference is less than 6. In particular, $i_1 - i_2$ is not divisible by 7. Hence $x + 1$ must be a multiple of 7. The possible values of x are $6, 13, 20, \ldots, 97$. It is easy to check that each of these values indeed works.

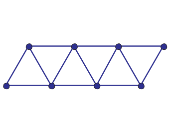
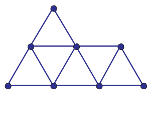
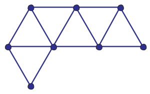
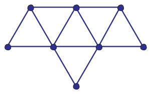
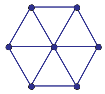
15. Six congruent equilateral triangles are arranged in the plane so that every triangle shares at least one whole edge with some other triangle. Find the number of distinct arrangements. (Two arrangements are considered the same if one can be rotated and/or reflected onto another.)

Solution. The answer is $\boxed{12}$. For an arrangement of this six equilateral triangles, we define its *diameter* as the maximum number of triangles appeared in row in t his arrangement. In any arrangement, some two triangles must be in a row, and by symmetry, a third triangle must be attached to these two triangle to form a row of three triangle.

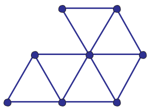
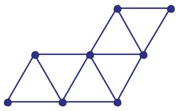
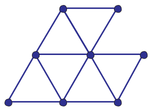
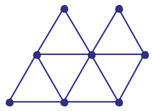




Hence the diameter is equal to 3, 4, 5, or 6. If the diameter is equal to 3, 5, or 6, it is not difficult to see the following 5 arrangements:

If the diameter is equal to 4, we have 7 arrangements. There are 4 arrangements with diameter 4 and two additional triangle on the same side of the diameter.

There are 3 arrangements with diameter 4 and two additional triangle on the opposite sides of the diameter.