1.4 Guts Test

Afternoon, January 26, 2013

There are 24 problems, with varying point values, to be solved in 75 minutes.

1.4.1 Round 1

1. [6pts] Five girls and three boys are sitting in a room. Suppose that four of the children live in California. Determine the maximum possible number of girls that could live somewhere outside California.

2. [6pts] A 4-meter long stick is rotated $60°$ about a point on the stick 1 meter away from one of its ends. Compute the positive difference between the distances traveled by the two endpoints of the stick, in meters.

3. [6pts] Let
$$f(x) = 2x(x-1)^2 + x^3(x-2)^2 + 10(x-1)^3(x-2).$$
Compute $f(0) + f(1) + f(2)$.

1.4.2 Round 2

4. [8pts] Twenty boxes with weights $10, 20, 30, \ldots, 200$ pounds are given. One hand is needed to lift a box for every 10 pounds it weighs. For example, a 40 pound box needs four hands to be lifted. Determine the number of people needed to lift all the boxes simultaneously, given that no person can help lift more than one box at a time.

5. [8pts] Let ABC be a right triangle with a right angle at A, and let D be the foot of the perpendicular from vertex A to side BC. If $AB = 5$ and $BC = 7$, compute the length of segment AD.

6. [8pts] There are two circular ant holes in the coordinate plane. One has center $(0, 0)$ and radius 3, and the other has center $(20, 21)$ and radius 5. Albert wants to cover both of them completely with a circular bowl. Determine the minimum possible radius of the circular bowl.

1.4.3 Round 3

7. [10pts] A line of slope -4 forms a right triangle with the positive x and y axes. If the area of the triangle is 2013, find the square of the length of the hypotenuse of the triangle.

8. [10pts] Let ABC be a right triangle with a right angle at B, $AB = 9$, and $BC = 7$. Suppose that point P lies on segment AB with $AP = 3$ and that point Q lies on ray BC with $BQ = 11$. Let segments AC and PQ intersect at point X. Compute the positive difference between the areas of triangles APX and CQX.

9. [10pts] Fresh Mann and Sophy Moore are racing each other in a river. Fresh Mann swims downstream, while Sophy Moore swims $\frac{1}{2}$ mile upstream and then travels downstream in a boat. They start at the same time, and they reach the finish line 1 mile downstream of the starting point simultaneously. If Fresh Mann and Sophy Moore both swim at 1 mile per hour in still water and the boat travels at 10 miles per hour in still water, find the speed of the current.

1.4.4 Round 4

10. [12pts] The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. The first few terms of the Fibonacci sequence are $0, 1, 1, 2, 3, 5, 8, 13$. Every positive integer can be expressed as the sum of nonconsecutive, distinct, positive Fibonacci numbers; for example, $7 = 5 + 2$. Determine a set of nonconsecutive, positive Fibonacci numbers that sum to 121; if 121 were replaced by 7, then $\{5, 2\}$ would be a solution.

11. [12pts] There is a rectangular box of surface area 44 whose space diagonals have length 10. Find the sum of the lengths of all the edges of the box.

12. [12pts] Let ABC be an acute triangle, and let D and E be the feet of the altitudes to BC and CA, respectively. Suppose that segments AD and BE intersect at point H with $AH = 20$ and $HD = 13$. Compute $BD \cdot CD$.

1.4.5 Round 5

13. [14pts] In coordinate space, a *lattice point* is a point all of whose coordinates are integers. The lattice points (x, y, z) in three-dimensional space satisfying $0 \leq x, y, z \leq 5$ are colored in n colors such that any two points that are $\sqrt{3}$ units apart have different colors. Determine the minimum possible value of n.

14. [14pts] Determine the number of ways to express 121 as a sum of strictly increasing positive Fibonacci numbers.

15. [14pts] Let $ABCD$ be a rectangle with $AB = 7$ and $BC = 15$. Equilateral triangles ABP, BCQ, CDR, and DAS are constructed outside the rectangle. Compute the area of quadrilateral $PQRS$.

1.4.6 Round 6

Each of the three problems in this round depends on the answer to one of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.

16. [16pts] Let C be the answer to problem 18. Suppose that x and y are real numbers with $y > 0$ and

$$\begin{cases} x + y = C \\ x + \frac{1}{y} = -2. \end{cases}$$

Compute $y + \frac{1}{y}$.

17. [16pts] Let A be the answer to problem 16. Let PQR be a triangle with $\angle PQR = 90°$, and let X be the foot of the perpendicular from point Q to segment PR. Given that $QX = A$, determine the minimum possible area of triangle PQR.

18. [16pts] Let B be the answer to problem 17 and let $K = 36B$. Alice, Betty, and Charlize are identical triplets, only distinguishable by their hats. Every day, two of them decide to exchange hats. Given that they each have their own hat today, compute the probability that Alice will have her own hat in K days.

1.4.7 Round 7

19. [16pts] Find the number of positive integers a such that all roots of $x^2 + ax + 100$ are real and the sum of their squares is at most 2013.

20. [16pts] Determine all integers k between -2013 and 2013, inclusive, such that the system of equations

$$\begin{cases} y = x^2 - kx + 1 \\ x = y^2 - ky + 1 \end{cases}$$

has a real solution.

21. [16pts] Determine the minimum number of cuts needed to divide an $11 \times 5 \times 3$ block of chocolate into $1 \times 1 \times 1$ pieces. (When a block is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

1.4.8 Round 8

22. [18pts] A sequence that contains the numbers $1, 2, 3, \ldots, n$ exactly once each is said to be a *permutation* of length n. A permutation $w_1 w_2 w_3 \cdots w_n$ is said to be *sad* if there are indices $i < j < k$ such that $w_j > w_k$ and $w_j > w_i$. For example, the permutation $\mathbf{3142756}$ is sad because $7 > 6$ and $7 > 1$. Compute the number of permutations of length 11 that are not sad.

23. [18pts] Let ABC be a triangle with $AB = 39, BC = 56$, and $CA = 35$. Compute $\angle CAB - \angle ABC$ in degrees.

24. [18pts] On a strange planet, there are 2013 cities.Between any pair of cities, there can either be a one-way road, two one-way roads in different directions, or no road at all. Every city has a name, and at the source of every one-way road, there is a signpost with the name of the destination city. In addition, the one-way roads only intersect at cities, but there can be bridges to prevent intersections at non-cities. Fresh Mann has been abducted by one of the aliens, but Sophy Moore knows that he is in Rome, a city that has no roads leading out of it. Also, there is a direct one-way road leading from each other city to Rome. However, Rome is the secret police's name for the so-described city; its official name, the name appearing on the labels of the one-way roads, is unknown to Sophy Moore. Sophy Moore is currently in Athens and she wants to head to Rome in order to rescue Fresh Mann, but she does not know the value of n. Assuming that she tries to minimize the number of roads on which she needs to travel, determine the maximum possible number of roads that she could be forced to travel in order to find Rome. Express your answer as a function of n.