1.3 Team Test

Morning, January 26, 2013

There are 10 problems, worth 30 points each, to be solved in 45 minutes.

1. Determine the number of ways to place 4 rooks on a 4×4 chessboard such that:

   (a) no two rooks attack one another, and

   (b) the main diagonal (the set of squares marked X below) does not contain any rooks.

X			
	X		
		X	
			X

   The rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

2. Seven students, numbered 1 to 7 in counter-clockwise order, are seated in a circle. Fresh Mann has 100 erasers, and he wants to distribute them to the students, albeit unfairly. Starting with person 1 and proceeding counter-clockwise, Fresh Mann gives i erasers to student i; for example, he gives 1 eraser to student 1, then 2 erasers to student 2, et cetera. He continues around the circle until he does not have enough erasers to give to the next person. At this point, determine the number of erasers that Fresh Mann has.

3. Let ABC be a triangle with $AB = AC = 17$ and $BC = 24$. Approximate $\angle ABC$ to the nearest multiple of 10 degrees.

4. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = \frac{3}{5}$ and for $n \geq 1$, $x_{n+1} = 2 - \dfrac{1}{x_n}$. Compute the product $x_1 x_2 x_3 \cdots x_{2013}$.

5. In equilateral triangle ABC, points P and R lie on segment AB, points I and M lie on segment BC, and points E and S lie on segment CA such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PR = 2$, $IM = 3$, and $ES = 5$, compute the area of hexagon $PRIMES$.

6. Let $f(a, b) = \frac{a^2}{a+b}$. Let A denote the sum of $f(i, j)$ over all pairs of integers (i, j) with $1 \leq i < j \leq 10$; that is,

   $$A = (f(1,2) + f(1,3) + \cdots + f(1,10)) + (f(2,3) + f(2,4) + \cdots + f(2,10)) + \cdots + f(9,10).$$

   Similarly, let B denote the sum of $f(i, j)$ over all pairs of integers (i, j) with $1 \leq j < i \leq 10$; that is,

   $$B = (f(2,1) + f(3,1) + \cdots + f(10,1)) + (f(3,2) + f(4,2) + \cdots + f(10,2)) + \cdots + f(10,9).$$

   Compute $B - A$.

7. Fresh Mann has a pile of seven rocks with weights 1, 1, 2, 4, 8, 16, and 32 pounds and some integer X between 1 and 64, inclusive. He would like to choose a set of the rocks whose total weight is exactly X pounds. Given that he can do so in more than one way, determine the sum of all possible values of X. (The two 1-pound rocks are indistinguishable.)

8. Let $ABCD$ be a convex quadrilateral with $AB = BC = CA$. Suppose that point P lies inside the quadrilateral with $AP = PD = DA$ and $\angle PCD = 30°$. Given that $CP = 2$ and $CD = 3$, compute CA.

9. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = 2$, $x_2 = \frac{13}{2}$, and for $n \geq 1$,

$$x_{n+2} = 3 - \frac{3}{x_{n+1}} + \frac{1}{x_n x_{n+1}}.$$

Compute x_{100}.

10. Ten prisoners are standing in a line. A prison guard wants to place a hat on each prisoner. He has two colors of hats, red and blue, and he has 10 hats of each color. Determine the number of ways in which the prison guard can place hats such that among any set of consecutive prisoners, the number of prisoners with red hats and the number of prisoners with blue hats differ by at most 2.