3.4 Guts Round Solutions

1. [5pts] Define the operation $\clubsuit$ so that $a \clubsuit b = a^b + b^a$. Then, if $2 \clubsuit b = 32$, what is b?

   Solution. The answer is $\boxed{4}$. Note that $2 \clubsuit b = 2^b + b^2 = 2^4 + 4^2$. (Why is $b = 4$ the unique solution?)

2. [5pts] A square is changed into a rectangle by increasing two of its sides by $p\%$ and decreasing the two other sides by $p\%$. The area is then reduced by 1%. What is the value of p?

   Solution. The answer is $\boxed{10}$. We have $(1-p\%)(1+p\%) = 1-1\%$; that is, $(p\%)^2 = 1\%$ or $p\% = 10\%$.

3. [5pts] What is the sum, in degrees, of the internal angles of a heptagon?

   Solution. The answer is $180 \cdot 5 = \boxed{900}$. The heptagon can be dissected into 5 triangles.

4. [5pts] How many integers in between $\sqrt{47}$ and $\sqrt{8283}$ are divisible by 7?

   Solution. The answer is $\boxed{13}$. Note that $\sqrt{47} < \sqrt{49} = 7 \cdot 1$ and $\sqrt{8283} > \sqrt{8281} = 91 = 7 \cdot 13$.

5. [8pts] Some mutant green turkeys and pink elephants are grazing in a field. Mutant green turkeys have six legs and three heads. Pink elephants have 4 legs and 1 head. There are 100 legs and 37 heads in the field. How many animals are grazing?

   Solution. The answer is $8 + 13 = \boxed{21}$. Assume that there are t turkeys and e elephants. Then $6t + 4e = 100$ and $3t + e = 37$, from which it follows that $(t, e) = (8, 13)$.

6. [8pts] Let $A = (0, 0)$, $B = (6, 8)$, $C = (20, 8)$, $D = (14, 0)$, $E = (21, -10)$, and $F = (7, -10)$. Find the area of the hexagon $ABCDEF$.

   Solution. The answer is $14 \cdot 18 = \boxed{252}$. The hexagon $ABCDEF$ can be dissected into two parallelograms $ABCD$ and $ADEF$. The area of parallelogram $ABCD$ is equal $14 \cdot 8$ and the area of parallelogram $ADEF$ is equal to $14 \cdot 10$.

7. [8pts] In Moscow, three men, Oleg, Igor, and Dima, are questioned on suspicion of stealing Vladimir Putin's blankie. It is known that each man either always tells the truth or always lies. They make the following statements:

   (a) Oleg: I am innocent!
   (b) Igor: Dima stole the blankie!
   (c) Dima: I am innocent!
   (d) Igor: I am guilty!
   (e) Oleg: Yes, Igor is indeed guilty!

   If exactly one of Oleg, Igor, and Dima is guilty of the theft, who is the thief?

   Solution. The answer is $\boxed{\text{Oleg}}$. Igor made two contradicting statements. Hence Igor always lies and Igor is not guilty, from which it follows that Oleg lies and he is guilty.

8. [8pts] How many 11-letter sequences of E's and M's have at least as many E's as M's?

Solution. The answer is $\dfrac{2^{11}}{2} = \boxed{1024}$. There are 2^{11} sequence of E's and M's. By symmetry, half them have more E's than M's. (Because 11 is odd, we cannot have equal number of E's and Ms in a 11-letter sequence of E's and M's.)

9. [11pts] John is entering the following summation $31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39$ in his calculator. However, he accidently leaves out a plus sign and the answer becomes 3582. What is the number that comes before the missing plus sign?

Solution. The answer is $\boxed{33}$. If the + in between the numbers $\overline{3a}$ and $\overline{3b}$ is left out, we entered $\overline{3a3b}$ and increased the sum by $\overline{3a3b} - (\overline{3a} + \overline{3b}) = 100 \cdot \overline{3a} + ol3b - (\overline{3a} + \overline{3b}) = 99 \cdot \overline{3a}$. Note that

$$3582 - (31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39) = 3582 - 315 = 3267 = 99 \cdot 33.$$

10. [11pts] Two circles of radius 6 intersect such that they share a common chord of length 6. The total area covered may be expressed as $a\pi + \sqrt{b}$, where a and b are integers. What is $a + b$?

Solution. The answer is $60 + 972 = \boxed{1032}$. The region covered by the two circles can be dissected into two congruent equilateral triangles of side length 6 and two congruent circular sectors of radius 6 and central angle 300°. Hence the total area covered is equal to

$$\frac{2 \cdot \frac{5}{6} \cdot 36\pi + 2 \cdot 36 \cdot \sqrt{3}}{4} = 60\pi + 18\sqrt{3} = 60\pi + \sqrt{972}.$$

11. [11pts] Alice has a rectangular room with 6 outlets lined up on one wall and 6 lamps lined up on the opposite wall. She has 6 distinct power cords (red, blue, green, purple, black, yellow). If the red and green power cords cannot cross, how many ways can she plug in all six lamps?

Solution. The answer is $\dfrac{6! \cdot 6!}{2} = \boxed{259200}$.

12. [11pts] Tracy wants to jump through a line of 12 tiles on the floor by either jumping onto the next block, or jumping onto the block two steps ahead. An example of a path through the 12 tiles may be: 1 step, 2 steps, 2 steps, 2 steps, 1 step, 2 steps, 2 steps. In how many ways can Tracy jump through these 12 tiles?

Solution. The answer is $\boxed{233}$. For $n \geq 1$, let f_n denote the number of ways Tracy can jump through through a line of n tiles. There are two choices for Tracy's first jump − jumping to the next block or jumping onto the block two steps ahead. In the former case, she has f_{n-1} ways to jump through the rest of $n - 1$ tiles. In the later case, she has f_{n-2} ways to jump through the rest of $n - 2$ tiles. Therefore, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. It is easy to compute that $f_1 = 1$, $f_2 = 2$, $f_3 = 3$, $f_4 = 5$, $f_5 = 8$, $f_6 = 13$, $f_7 = 21$, $f_8 = 34$, $f_9 = 55$, $f_{10} = 89$, $f_{11} = 144$, and $f_{12} = 233$. (This is the famous Fibonacci sequence.)

13. [14pts] What is the units digit of the number $(2^1 + 1)(2^2 - 1)(2^3 + 1)(2^4 - 1) \ldots (2^{2010} - 1)$?

Solution. The answer is $\boxed{5}$. All the multiplicands are odd and the units digit of $2^4 - 1$ is 5.

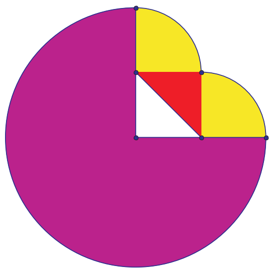
14. [14pts] Mr. Fat noted that on January 2, 2010, the display of the day is 01/02/2010, and the sequence 01022010 is a palindrome (a number that reads the same forwards and backwards). How many days does Mr. Fat need to wait between this palindrome day and the last palindrome day of this decade?

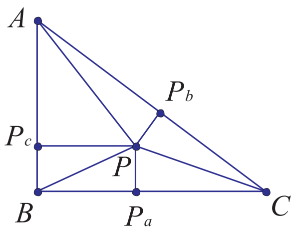
Solution. The answer is $2 \cdot 365 - 30 - 31 = \boxed{669}$. The last palindrome day of this decade is 11/02/2011, which is exactly 22 months (a November and December short of two full years) away from 01/02/2010.

15. [14pts] Farmer Tim has a 30-meter by 30-meter by $30\sqrt{2}$-meter triangular barn. He ties his goat to the corner where the two shorter sides meet with a 60-meter rope. What is the area, in square meters, of the land where the goat can graze, given that it cannot get inside the barn?

Solution. The answer is $\boxed{3150\pi + 450}$. The grazing area consists of one 270° circular sector region of radius 60 meters, two 90° circular sector regions of radius 30 meters, and one 30-meter by 30-meter by $30\sqrt{2}$-meter triangular region. (See the left-hand side figure shown below.) The total area is equal to

$$\frac{3}{4} \cdot 3600\pi + 2 \cdot \frac{1}{4}900\pi + 450 = 3150\pi + 450.$$





16. [14pts] In triangle ABC, $AB = 3$, $BC = 4$, and $CA = 5$. Point P lies inside the triangle and the distances from P to two of the sides of the triangle are 1 and 2. What is the maximum distance from P to the third side of the triangle?

Solution. The answer is $\boxed{\frac{2}{5}}$. Let P_a, P_b, P_c be the feet of the perpendiculars from P to sides BC, CA, AB, respectively. Set $x = PP_a$, $y = PP_b$, and $z = PP_c$. Because the sum of the areas of triangles PAB, PBC, PCA is equal to the area of triangle ABC, we have $4x + 5y + 3z = 12$. Two of x, y, z take the values 1 and 2, and we want to maximize the third. If $y = 2$, then $4x + 5y + 3z \geq 13$, which is not possible. If $y = 1$, then $4x + 3z = 7$, implying that $z = 2$ and $x = \frac{1}{4}$. If y is unknown, then $5y = 12 - (4x + 3z)$ is equal to either 2 (with $(x, z) = (1, 2)$) or 1 (with $(x, z) = (2, 1)$), implying that $y = \frac{2}{5}$ or $y = \frac{1}{5}$.

17. [17pts] Let Z be the answer to the third question on this guts quadruplet. If $x^2 - 2x = Z - 1$, find the positive value of x.

Solution. The answer is $\boxed{17}$. We can rearrange the given equation to get $x^2 - 2x + 1 = Z$, or $(x - 1)^2 = Z$, so $x = \sqrt{Z} + 1$.

18. [17pts] Let X be the answer to the first question on this guts quadruplet. To make a FATRON2012, a cubical steel body as large as possible is cut out from a solid sphere of diameter X. A TAFTRON2013 is created by cutting a FATRON2012 into 27 identical cubes, with no material wasted. What is the length of one edge of a TAFTRON2013?

Solution. The answer is $\boxed{\frac{17\sqrt{3}}{9}}$. We see that each side of the FATRON2012 has side length $\frac{X}{\sqrt{3}}$. Thus, each side of the TAFTRON2013 is $\frac{1}{3} \cdot \frac{X}{\sqrt{3}} = \frac{X\sqrt{3}}{9}$.

19. [17pts] Let Y be the smallest integer greater than the answer to the second question on this guts quadruplet. Fred posts two distinguishable sheets on the wall. Then, Y people walk into the room. Each of the Y people signs up on 0, 1, or 2 of the sheets. Given that there are at least two people in the room other than Fred, how many possible pairs of lists can Fred have?

Solution. The answer is $\boxed{256}$.

We know that each person can sign up on neither of the sheets, only the first sheet, only the second sheet, or both sheets. Thus, there are $Z = 4^Y$ possible sets of lists.

By the definitions of X and Y, we obtain that

$$Y - 1 \le \frac{X\sqrt{3}}{9} = \frac{(2^Y + 1)\sqrt{3}}{9} = \frac{2^Y + 1}{3\sqrt{3}} \le Y$$

or

$$3\sqrt{3}Y - 3\sqrt{3} - 1 \le 2^Y \le 3\sqrt{3}Y - 1, \tag{3.1}$$

implying that $Y = 4$ (and consequently, $Z = 4^4 = 256$, $X = 17$).

Intuitively, because the exponential function 2^Y grows much faster than the liner function $3\sqrt{3}Y - 1$, there are very few integer values of Y satisfying (3.1). It is not difficult to check that $Y = 4$ is the only integer answer to (3.1) such that Y is at least 2.

Rigorously, we can show that $2^n > 6n > 6n - 1 > 3\sqrt{3}n - 1$ for $n \ge 6$. (Hence the only possible values of Y satisfying (3.1) are $Y = 1, 2, 3, 4, 5$. By a quick check, we can conclude that $Y = 4$ is the only solution for Y at least 2.) To show that $2^n > 6n$ for $n \ge 6$, we induct on n. For $n = 6$, this is clearly true. Assume that $2^n > 6n$ is true for some integer $n = k \ge 6$; that is, $2^k > 6k$ and $k \ge 6$. Then for $n = k + 1$, we have $2^{k+1} = 2 \cdot 2^k > 2 \cdot 6k > 6(k + 1)$.

20. [17pts] Let A, B, C, be the respective answers to the first, second, and third questions on this guts quadruplet. At the Robot Design Convention and Showcase, a series of robots are programmed such that each robot shakes hands exactly once with every other robot of the same height. If the heights of the 16 robots are 4, 4, 4, 5, 5, 7, 17, 17, 17, 34, 34, 42, 100, A, B, and C feet, how many handshakes will take place?

Solution. The answer is $\binom{3}{2} + \binom{2}{2} + \binom{4}{2} + \binom{2}{2} = \boxed{11}$. We know that $A = 17$, $B = \frac{17\sqrt{3}}{9}$, and $C = 256$. Thus, we have 3 robots of height 4, 2 robots of height 5, 4 robots of height 17, and 2 robots of height 34.

21. [20pts] Determine the number of ordered triples (p, q, r) of primes with $1 < p < q < r < 100$ such that $q - p = r - q$.

Solution. The answer is $10 + 22 + 14 = \boxed{46}$. Let $d = q - p = r - q$ be the common difference. Because both r and q must be odd, d must be even.

If d is not a multiple of 3, then one of p, q, r is a multiple of 3. Because $p < q < r$ are primes, we must have $p = 3$. It is not difficult to list all 10 such triples, namely, $(3, 5, 7)$, $(3, 7, 11)$, $(3, 11, 19)$, $(3, 13, 23)$, $(3, 17, 31)$, $(3, 23, 46)$, $(3, 31, 59)$, $(3, 37, 71)$, $(3, 41, 79)$, $(3, 43, 83)$,

If d is a multiple of 3, then d is a multiple of 6. We classify the primes greater than 3 and less than 100 modulo 6:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$6k-1$	5	11	17	23	29		41	47	53	59		71		83	89	
$6k+1$	7	13	19		31	37	43			61	67	73	79			97

Note that the primes $6k_1 \pm 1, 6k_2 \pm 1, 6k_3 \pm 1$ form an arithmetic progression if and only if the numbers k_1, k_2, k_3 form an arithmetic progression.

To count the 3-term arithmetic progressions of primes in the from $6k-1$, we count all the 3-term arithmetic progressions in the set $\{1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 14, 15\}$. There are 22 such triples, namely,

$$\begin{aligned}&(1,2,3),(1,3,5),(1,4,7),(1,5,9),(1,8,15);(2,3,4),(2,5,8),(2,7,12),(2,8,14);\\&(3,4,5),(3,5,7),(3,9,15);(4,7,10),(4,8,12),(4,9,14);(5,7,9),(5,10,15);\\&(7,8,9);(8,9,10),(8,10,12);(9,12,15);(10,12,14).\end{aligned}$$

To count the 3-term arithmetic progressions of primes in the from $6k+1$, we count all the 3-term arithmetic progressions in the set $\{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 16\}$. There are 14 such triples, namely,

$$\begin{aligned}&(1,2,3),(1,3,5),(1,6,11),(1,7,13);(2,6,10),(2,7,12);\\&(3,5,7),(3,7,11);(5,6,7);(6,11,16);(7,10,13);(10,11,12),(10,13,16);(11,12,13).\end{aligned}$$

22. [20pts] For numbers a, b, c, d such that $0 \le a, b, c, d \le 10$, find the minimum value of $ab + bc + cd + da - 5a - 5b - 5c - 5d$.

Solution. The answer is $\boxed{-100}$. We have

$$\begin{aligned}& ab+bc+cd+da-5a-5b-5c-5d = (a+c)(b+d)-5(a+c)-5(b+d)\\ =\ & (a+c-5)(b+d-5)-25 \ge (-5)\cdot 15-25=-100.\end{aligned}$$

23. [20pts] Daniel has a task to measure 1 gram, 2 grams, 3 grams, 4 grams , ..., all the way up to n grams. He goes into a store and buys a scale and six weights of his choosing (so that he knows the value for each weight that he buys). If he can place the weights on either side of the scale, what is the maximum value of n?

Solution. The answer is $\frac{3^6-1}{2} = \boxed{364}$. We see that we can put each weight on either the right side of the scale, the left side of the scale, or neither, for 3^6 possible combinations. However, we need to exclude the case where there are no weights on the scale, leaving 3^6-1 ways. Without loss of generality, we always want to weight our object on the right side of the scale, so we always want the placement of the weights to leave the left side heavier; by symmetry, this only happens in half the cases we are currently counting, so we have at most $\frac{3^6-1}{2}$ different masses we can weight. We can check that using the set $\{3^0, 3^1, 3^2, 3^3, 3^4, 3^5\}$ will let us weight any mass between 1 and 364: The weight 3^0 will let us weight objects of mass 1. Then, adding the weight 3^1 will then let us weight objects from $3^1 - 3^0$ to $3^1 + 3^0$ (from 2 to 4). Adding the weight of 3^2 will then let us weight objects from $3^2 - (3^1 + 3^0)$ to $3^2 + 3^1 + 3^0$ (from 5 to 13), and so on.

24. [20pts] Given a Rubik's cube, what is the probability that at least one face will remain unchanged after a random sequence of three moves? (A Rubik's cube is a 3 by 3 by 3 cube with each face starting as a different color. The faces (3 by 3) can be freely turned. A move is defined in this problem as a 90 degree rotation of one *face either clockwise or counter-clockwise. The center square on each face–six in total–is fixed.)*

Solution. The answer is $\boxed{\dfrac{2}{9}}$.

For each move, we can select one of the six faces and make either a 90 degree clockwise rotation or a 90 degree counter-clockwise rotation, implying that there are 12 distinct options. Thus, there are 12^3 distinct sequence of three moves.

We consider those sequences of three moves that leave at least one face unchanged. We consider three pairs of parallel faces A_1 and A_2, B_1 and B_2, and C_1 and C_2. The key observation is that, after three moves, if one face in a pair is unchanged, then so does the other face in the pair. Indeed, by symmetry, we may assume without loss of generality that A_1 is unchanged. We categorize the moves into two types:

- moves on faces A_1 and A_2. There are 4 such moves, and they are all denoted by T;
- moves on the other four faces. There are 8 such moves, and they are all denoted by S.

For A_1 to be left unchanged, we have either three T moves or a T move followed by a S move and then its inverse move $S{-}1$ or a S move, its inverse move S^{-1}, and then a T move. In each case, face A_2 is also unchanged.

In summary, we can choose one of the three pairs of parallel faces to be unchanged. For each chosen pair of parallel faces, there are $4^3 + 4 \cdot 8 \cdot 1 + 8 \cdot 1 \cdot 4 = 2 \cdot 4^3$ distinct sequence of three moves to have the chosen pair of faces unchanged. Hence the desired probability is equal to

$$\frac{3 \cdot 2 \cdot 4^3}{12^3} = \frac{3 \cdot 2}{3^3} = \frac{2}{9}.$$