Chapter 3

The Solutions

3.1 Individual Speed Test Solutions

1. Evaluate $\dfrac{\sqrt{2}\cdot\sqrt{6}}{\sqrt{3}}$.

   Solution. The answer is $(\sqrt{2})^2 = \boxed{2}$.

2. If 6% of a number is 1218, what is 18% of that number?

   Solution. The answer is $1218 \cdot 3 = \boxed{3654}$.

3. What is the median of $\{42, 9, 8, 4, 5, 1, 13666, 3\}$?

   Solution. The answer is $\dfrac{5+8}{2} = \boxed{\dfrac{13}{2}}$. We can arrange these elements in increasing order: 1, 3, 4, 5, 8, 9, 42, 13666. Since there are an even number of values, the median is the average of the two middle numbers 5 and 8.

4. Define the operation $\heartsuit$ so that $i \heartsuit u = 5i - 2u$. What is $3 \heartsuit 4$?

   Solution. The answer is $5 \cdot 3 - 2 \cdot 4 = \boxed{7}$.

5. How many 0.2-inch by 1-inch by 1-inch gold bars can fit in a 15-inch by 12-inch by 9-inch box?

   Solution. The answer is $15 \cdot 5 \cdot 12 \cdot 9 = 3^4 \cdot 5^2 \cdot 2^2 = \boxed{8100}$.

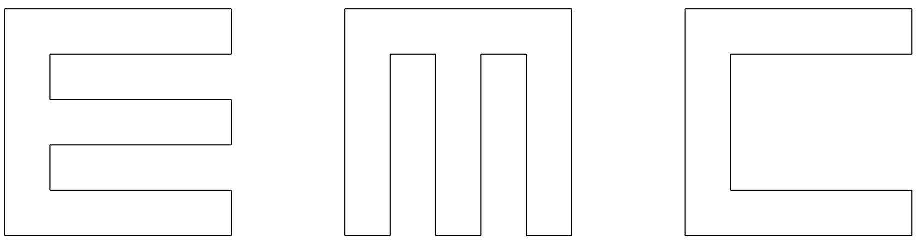
6. A tetrahedron is a triangular pyramid. What is the sum of the number of edges, faces, and vertices of a tetrahedron?

   Solution. The answer is $6 + 4 + 4 = \boxed{14}$.

7. Ron has three blue socks, four white socks, five green socks, and two black socks in a drawer. Ron takes socks out of his drawer blindly and at random. What is the least number of socks that Ron needs to take out to guarantee he will be able to make a pair of matching socks?

   Solution. The answer is $\boxed{5}$. There are only four different colors. Among any five socks, at least two of them are of the same color.

8. One segment with length 6 and some segments with lengths 10, 8, and 2 form the three letters in the diagram shown below. Compute the sum of the perimeters of the three figures.



   Solution. The answer is $30 \cdot 4 + 10 \cdot 8 = \boxed{200}$. For each letter, we can rearrange segments with lengths less than 8 to form a square of side 10. The sum of the perimeters of the three squares is 120. There are 10 additional segments of length 8.

9. How many integer solutions are there to the inequality $|x-6| \leq 4$?

   Solution. The answer is $\boxed{9}$. The possible values of $x-6$ are $0, \pm 1, \pm 2, \pm 3, \pm 4$.

10. In a land for bad children, the flavors of ice cream are grass, dirt, earwax, hair, and dust-bunny. The cones are made out of granite, marble, or pumice, and can be topped by hot lava, chalk, or ink. How many ice cream cones can the evil confectioners in this ice-cream land make? (Every ice cream cone consists of one scoop of ice cream, one cone, and one topping.)

    Solution. The answer is $5 \cdot 3 \cdot 3 = \boxed{45}$.

11. Compute the sum of the prime divisors of $245 + 452 + 524$.

    Solution. The answer is $11 + 3 + 37 = \boxed{51}$. Note that $245 + 452 + 524 = 222 + 444 + 555 = (2 + 4 + 5) \cdot 111 = 11 \cdot 3 \cdot 37$.

12. In quadrilateral $SEAT$, $SE = 2$, $EA = 3$, $AT = 4$, $\angle EAT = \angle SET = 90°$. What is the area of the quadrilateral?

    Solution. The answer is $5+6 = \boxed{11}$. The quadrilateral is the union of two right triangles EAT (with area 6 and $ET = 5$) and SET (with area 5).

13. What is the angle, in degrees, formed by the hour and minute hands on a clock at 10:30 AM?

    Solution. The answer is $\boxed{135}$.

14. Three numbers are randomly chosen without replacement from the set $\{101, 102, 103, \ldots, 200\}$. What is the probability that these three numbers are the side lengths of a triangle?

    Solution. The answer is $\boxed{1}$. Note that the sum of any two numbers in the set is greater than any number in the set.

15. John takes a 30-mile bike ride over hilly terrain, where the road always either goes uphill or downhill, and is never flat. If he bikes a total of 20 miles uphill, and he bikes at 6 mph when he goes uphill, and 24 mph when he goes downhill, what is his average speed, in mph, for the ride?

    Solution. The answer is $\dfrac{30}{\frac{20}{6} + \frac{10}{24}} = \boxed{8}$.

16. How many distinct six-letter words (not necessarily in any language known to man) can be formed by rearranging the letters in EXETER? (You should include the word EXETER in your count.)

    Solution. The answer is $\dfrac{6!}{3!} = \boxed{120}$.

17. A pie has been cut into eight slices of different sizes. Snow White steals a slice. Then, the seven dwarfs (Sneezy, Sleepy, Dopey, Doc, Happy, Bashful, Grumpy) take slices one by one according to the alphabetical order of their names, but each dwarf can only take a slice next to one that has already been taken. In how many ways can this pie be eaten by these eight persons?

    Solution. The answer is $8 \cdot 2^6 = 2^9 = \boxed{512}$. Snow White has 8 choices. Each of Bashful, Doc, Dopey, Grumpy, Happy, and Sleepy has two choices. Then, there is only one slice left for Sneezy to eat.

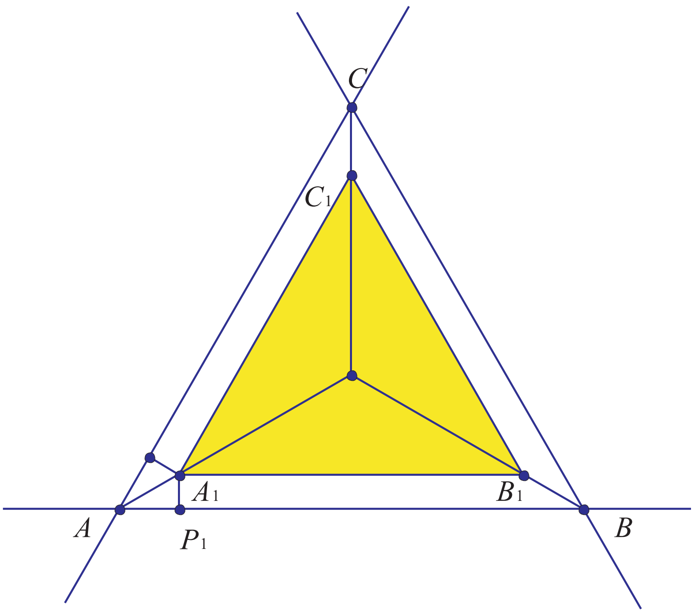
18. Assume that n is a positive integer such that the remainder of n is 1 when divided by 3, is 2 when divided by 4, is 3 when divided by 5, ..., and is 8 when divided by 10. What is the smallest possible value of n?

Solution. The answer is $\operatorname{lcm}(3,4\ldots,10)-2 = 5\cdot7\cdot8\cdot9-2 = \boxed{2518}$. Adding 2 to the desired number yields a number that is divisible by each of $3,4,\ldots,10$.

19. Find the sum of all positive four-digit numbers that are perfect squares and that have remainder 1 when divided by 100.

Solution. The answer is $49^2+51^2+99^2 = 2401+2601+9801 = \boxed{14803}$. Let $\overline{abcd}$ be such a number. Then $\overline{abcd} = (10x \pm 1)^2 = 100x^2 \pm 20x + 1$, implying that $x = 5$ or $x = 10$.

20. A coin of radius 1 cm is tossed onto a plane surface that has been tiled by equilateral triangles with side length $20\sqrt{3}$ cm. What is the probability that the coin lands within one of the triangles?



Solution. The answer is $\frac{(18\sqrt{3})^2}{(20\sqrt{3})^2} = \boxed{\frac{81}{100}}$. Assume that the center of the coin lies within the boundary of triangle ABC. In order for the coin to lie completely within the triangle, the center of the circle must lie in triangle $A_1B_1C_1$, where the distances between pairs of parallel lines A_1B_1 and AB, B_1C_1 and BC, and C_1A_1 and CA are all equal to 1. Let P_1 be the foot of the perpendicular from A_1 to line AB. It is not difficult to see that $A_1P_1 = 1$ and $AP_1 = \sqrt{3}$, from which it follows that $A_1B_1 = 18\sqrt{3}$ and the ratio between the areas of triangles $A_1B_1C_1$ and ABC is equal to $\frac{(18\sqrt{3})^2}{(20\sqrt{3})^2}$.

3.2 Individual Accuracy Test Solutions

1. Calculate $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)^2$.

   Solution. The answer is $\left(\frac{13}{12}\right)^2 = \boxed{\frac{169}{144}}$.

2. Find the 2010$^{\text{th}}$ digit after the decimal point in the expansion of $\frac{1}{7}$.

   Solution. The answer is $\boxed{7}$. Note that $\frac{1}{7} = 0.\overline{142857}$ and 2010 is divisible by 6.

3. If you add 1 liter of water to a solution consisting of acid and water, the new solutions will contain of 30% water. If you add another 5 liters of water to the new solution, it will contain $36\frac{4}{11}\%$ water. Find the number of liters of acid in the original solution.

   Solution. The answer is $\boxed{35}$. Assume that there are x liters of water and y liters of acid in the original solution. We have $\frac{x+1}{x+y+1} = 30\%$ and $\frac{x+6}{x+y+6} = 36\frac{4}{11}\%$, from which it follows that $x = 14$, $y = 35$.

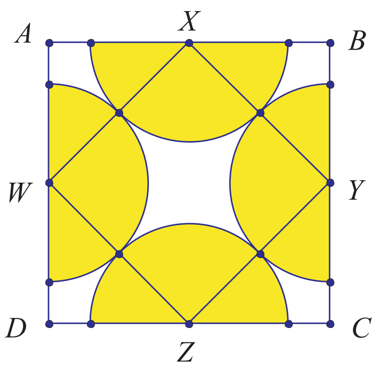
4. John places 5 indistinguishable blue marbles and 5 indistinguishable red marbles into two distinguishable buckets such that each bucket has at least one blue marble and one red marble. How many distinguishable marble distributions are possible after the process is completed?

   Solution. The answer is $4 \cdot 4 = \boxed{16}$. The first bucket can have x (with $x = 0, 1, 2,$ or 3) blue marbles and y (with $y = 0, 1, 2,$ or 3) red marbles, and the second bucket will have $4 - x$ blue marbles and $4 - y$ red marbles.

5. In quadrilateral $PEAR$, $PE = 21$, $EA = 20$, $AR = 15$, $RE = 25$, and $AP = 29$. Find the area of the quadrilateral.

   Solution. The answer is $\frac{21+15}{2} \cdot 20 = \boxed{360}$. Because $15^2 + 20^2 = 25^2$ and $21^2 + 20^2 = 29^2$, triangles APE and ERA are right triangles (with $\angle EAR = \angle PEA = 90^\circ$). Consequently, $PEAR$ is a right trapezoid with bases $PE = 21$ and $AR = 15$ and height $EA = 20$.

6. Four congruent semicircles are drawn within the boundary of a square with side length 1. The center of each semicircle is the midpoint of a side of the square. Each semicircle is tangent to two other semicircles. Region $\mathcal{R}$ consists of points lying inside the square but outside of the semicircles. The area of $\mathcal{R}$ can be written in the form $a - b\pi$, where a and b are positive rational numbers. Compute $a + b$.



Solution. The answer is $1 + \dfrac{1}{4} = \boxed{\dfrac{5}{4}}$. Let $ABCD$ denote the unit square, and let X, Y, Z, W be the midpoints of sides AB, BC, CD, DA, respectively. Then $AX = AW = \frac{1}{2}$ and $WX = \frac{\sqrt{2}}{2}$. The radii of the semicircles are equal to $r = \frac{WX}{2} = \frac{\sqrt{2}}{4}$. The area of the region covered by the semicircles is equal to $4(\frac{1}{2}\pi r^2) = \frac{\pi}{4}$ and the area of region $\mathcal{R}$ is equal to $1 - \frac{\pi}{4}$, implying that $(a, b) = \left(1, \frac{1}{4}\right)$.

7. Let x and y be two numbers satisfying the relations $x \geq 0$, $y \geq 0$, and $3x + 5y = 7$. What is the maximum possible value of $9x^2 + 25y^2$?

   Solution. The answer is $\boxed{49}$. We have $9x^2 + 25y^2 = (3x + 5y)^2 - 30xy = 49 - 30xy \leq 49$ for nonnegative numbers x and y.

8. In the Senate office in Exie-land, there are 6 distinguishable senators and 6 distinguishable interns. Some senators and an equal number of interns will attend a convention. If at least one senator must attend, how many combinations of senators and interns can attend the convention?

   Solution. The answer is $\boxed{923}$. If there are i, $1 \leq i \leq 6$, senators attending the convention, then there are i interns attending the convention. Hence is the answer is

$$\binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 = 36 + 225 + 400 + 225 + 36 + 1 = 923.$$

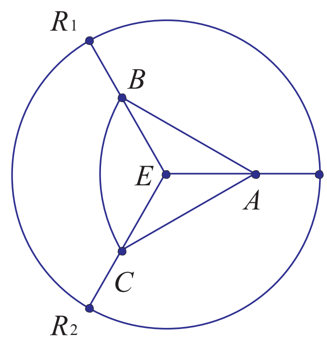
   This is a special case of Vandermonde's identity – the answer is equal to $\binom{12}{6} - 1$. Indeed, we can choose any 6 people (x senators interns and $6 - x$ interns) and we will then send the x chosen senators and x non-chosen interns to the convention. The only case we have to exclude is the case when 0 senators and 6 interns are chosen.

9. Evaluate $(1^2 - 3^2 + 5^2 - 7^2 + 9^2 - \cdots + 2009^2) - (2^2 - 4^2 + 6^2 - 8^2 + 10^2 - \cdots + 2010^2)$.

   Solution. The answer is $2008 + 2009^2 - 2010^2 = \boxed{-2011}$. The key fact is $n^2 - (n+1)^2 - (n+2)^2 + (n+3)^2 = 4$ for all numbers n. Thus,

$$\begin{aligned} & (1^2 - 3^2 + 5^2 - 7^2 + 9^2 - \cdots + 2009^2) - (2^2 - 4^2 + 6^2 - 8^2 + 10^2 - \cdots + 2010^2) \\ = \; & (1^2 - 2^2 - 3^2 + 4^2) + \cdots + (2005^2 - 2006^2 - 2007^2 + 2008^2) + 2009^2 - 2010^2. \end{aligned}$$

10. Segment EA has length 1. Region $\mathcal{R}$ consists of points P in the plane such that $\angle PEA \geq 120°$ and $PE < \sqrt{3}$. If point X is picked randomly from the region $\mathcal{R}$, the probability that $AX < \sqrt{3}$ can be written in the form $a - \frac{\sqrt{b}}{c\pi}$, where a is a rational number, b and c are positive integers, and b is not divisible by the square of a prime. Find the ordered triple (a, b, c).



Solution. The answer is $\boxed{\left(\frac{1}{2}, 3, 2\right)}$.

Let ω_1 denote the circle centered at E with radius $\sqrt{3}$. Points R_1 and R_2 lie on ω_1 with $\angle R_1EA = \angle R_2EA = 120^\circ$. Region $\mathcal{R}$ is the circular sector centered R_1ER_2. The area of region $\mathcal{R}$ is equal to π. Let ω_2 denote the circle centered at A with radius $\sqrt{3}$, and let ω_2 intersects segments ER_1 and ER_2 are B and C, respectively. Let $\mathcal{R}_1$ denote the region enclosed by arc $\widehat{BC}$ and segments EB and EC. Then a point X is in region $\mathcal{R}$ if and only if X lies in $\mathcal{R}_1$; that is, The probability p that X lies $\mathcal{R}_1$ is equal to ratio of the area of $\mathcal{R}_1$ to that of $\mathcal{R}$. It is not difficult to see that $EA = EB = EC = 1$, the area of circular sector BCA is $\frac{\pi}{2}$, the triangles EAB and EAC have the same area $\frac{\sqrt{3}}{4}$, the area of region $\mathcal{R}_1 = \frac{\pi}{2} - \frac{\sqrt{3}}{2}$, and

$$p = \frac{\frac{\pi}{2} - \frac{\sqrt{3}}{2}}{\pi} = \frac{1}{2} - \frac{\sqrt{3}}{2\pi}.$$

3.3 Team Round Solutions

1. A very large lucky number N consists of eighty-eight 8s in a row. Find the remainder when this number N is divided by 6.

   Solution. The answer is $\boxed{2}$. Because 888 is divisible by 6, $N - 8$ is divisible by 6.

2. If 3 chickens can lay 9 eggs in 4 days, how many chickens does it take to lay 180 eggs in 8 days?

   Solution. The answer is $\boxed{30}$. If 3 chickens can lay 9 eggs in 4 days, then 3 chickens can lay 18 eggs in 8 days, and then 30 chickens can lay 180 eggs in 8 days.

3. Find the ordered pair (x, y) of real numbers satisfying the conditions $x > y$, $x+y = 10$, and $xy = -119$.

   Solution. The answer is $\boxed{(17, -7)}$ We know that $(x - y)^2 = (x + y)^2 - 4xy = 100 - 4(-119) = 576$, so $(x - y) = 24$ and this gives us $(x, y) = (17, -7)$.

4. There is pair of similar triangles. One triangle has side lengths 4, 6, and 9. The other triangle has side lengths 8, 12 and x. Find the sum of two possible values of x.

   Solution. The answer is $\dfrac{16}{3} + 18 = \boxed{\dfrac{70}{3}}$. We have either $4 : 6 : 9 = x : 8 : 12$ or $4 : 6 : 9 = 8 : 12 : x$. In the former case, $x = \frac{16}{3}$. In the latter case, $x = 18$.

5. If $x^2 + \frac{1}{x^2} = 3$, there are two possible values of $x + \frac{1}{x}$. What is the smaller of the two values?

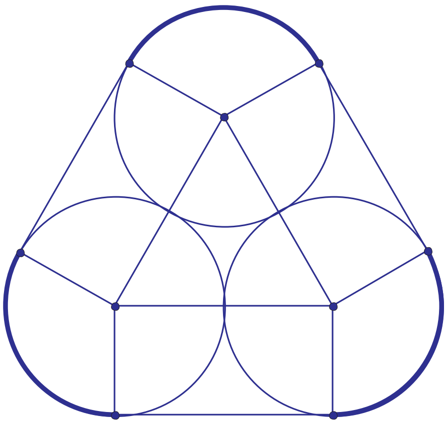
   Solution. The answer is $\boxed{-\sqrt{5}}$. We have $5 = x^2 + \frac{1}{x^2} + 2 = \left(x + \frac{1}{x}\right)^2$.

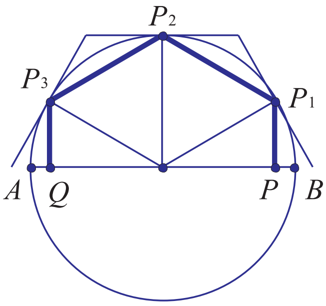
6. Three flavors (chocolate strawberry, vanilla) of ice cream are sold at Brian's ice cream shop. Brian's friend Zerg gets a coupon for 10 free scoops of ice cream. If the coupon requires Zerg to choose an even number of scoops of each flavor of ice cream, how many ways can he choose his ice cream scoops? (For example, he could have 6 scoops of vanilla and 4 scoops of chocolate. The order in which Zerg eats the scoops does not matter.)

   Solution. The answer is $\boxed{21}$. Assume that Zerg has $2x$ scoops of chocolate ice cream, $2y$ scoops of strawberry ice cream, and $2z$ scoops of vanilla ice cream, where x, y, z are nonnegative integers. Then $2x + 2y + 2z = 10$ or $x + y + z = 5$, which has $\binom{5+3-1}{3-1} = \binom{7}{2} = 21$ ordered triples of solutions in nonnegative integers.

7. David decides he wants to join the West African Drumming Ensemble, and thus he goes to the store and buys three large cylindrical drums. In order to ensure none of the drums drop on the way home, he ties a rope around all of the drums at their mid sections so that each drum is next to the other two. Suppose that each drum has a diameter of 3.5 feet. David needs m feet of rope. Given that $m = a\pi + b$, where a and b are rational numbers, find sum $a + b$.

   Solution. The answer is $3.5 + 10.5 = \boxed{14}$. The rope band is formed by six parts – three circular parts and three linear parts. Because three drums are congruent to each other, by symmetry, the three circular parts are congruent to each other and three linear parts are congruent to each other. (See the left-hand side figure shown below.) Moving along this band exactly once, one turns around 360 degrees. Because there is no changing of direction along the linear parts, each circular part of the band is one third of a full circle with diameter 3.5 feet. The remaining three parts are line segments. The length of each of the linear parts of the band is equal to the distance between the center of the drums. Therefore, the total length of the rope is $10.5 + 3.5\pi$.

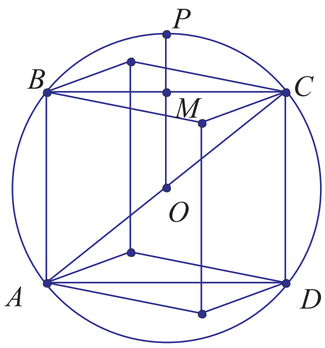


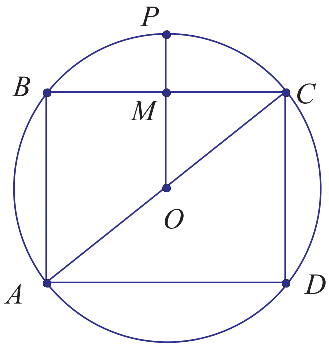


8. Segment AB is the diameter of a semicircle of radius 24. A beam of light is shot from a point $12\sqrt{3}$ from the center of the semicircle, and perpendicular to AB. How many times does it reflect off the semicircle before hitting AB again?

   Solution. The answer is $\boxed{3}$. Note that the beam of light will trace a half of a regular hexagon. (See the right-hand side figure shown above. The path of the light beam is $P \to P_1 \to P_2 \to P_3 \to Q$.

9. A cube is inscribed in a sphere of radius 8. A smaller sphere is inscribed in the same sphere such that it is externally tangent to one face of the cube and internally tangent to the larger sphere. The maximum value of the ratio of the volume of the smaller sphere to the volume of the larger sphere can be written in the form $\frac{a-\sqrt{b}}{36}$, where a and b are positive integers. Find the product ab.





   Solution. The answer is $9 \cdot 75 = \boxed{675}$. Let s denote the side length of the cube. The length of a face diagonal BC is $s\sqrt{2}$ and the length of an interior diagonal AC is $s\sqrt{3}$. Note that AC is also a diameter of the sphere. Hence, $s\sqrt{3} = 16$ or $s = \frac{16}{\sqrt{3}}$. Let M be the midpoint of segment BC. The maximum value of the diameter of the smaller sphere is equal to $PM = OP - OM = r - \frac{s}{2} = 8 - \frac{8}{\sqrt{3}}$, and the maximum value of the ratio of the volume of the smaller sphere to the volume of the larger sphere is equal to

$$\left(\frac{4 - \frac{4}{\sqrt{3}}}{8}\right)^3 = \left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)^3 = \frac{6\sqrt{3}-10}{24\sqrt{3}} = \frac{9-5\sqrt{3}}{36} = \frac{9-\sqrt{75}}{36}.$$

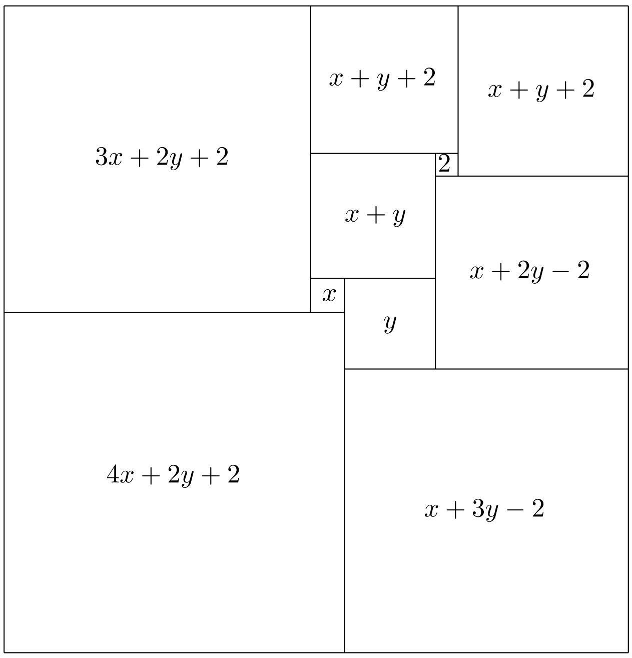
10. How many ordered pairs (x, y) of integers are there such that $2xy + x + y = 52$?

    Solution. The answer is $\boxed{16}$. We can write the given function as $4xy + 2x + 2y + 1 = 105$ or $(2x + 1)(2y + 1) = 105$. Note that $105 = 3 \cdot 5 \cdot 7$ has a total of $2 \cdot 2 \cdot 2 \cdot 2 = 16$ (positive and negative) integer divisors.

11. Three musketeers looted a caravan and walked off with a chest full of coins. During the night, the first musketeer divided the coins into three equal piles, with one coin left over. He threw it into the ocean and took one of the piles for himself, then went back to sleep. The second musketeer woke up an hour later. He divided the remaining coins into three equal piles, and threw out the one coin that was left over. He took one of the piles and went back to sleep. The third musketeer woke up and divided the remaining coins into three equal piles, threw out the extra coin, and took one pile for himself. The next morning, the three musketeers gathered around to divide the coins into three equal piles. Strangely enough, they had one coin left over this time as well. What is the minimum number of coins that were originally in the chest?

    Solution. The answer is $\boxed{79}$. Assume that there are x coins in the beginning. The first musketeer threw away a coin, took $\frac{x-1}{3}$ coins and left $\frac{2x-2}{3}$ coins. The second musketeer threw away one coin, took $\frac{2x-5}{9}$ coins and left $\frac{4x-10}{9}$ coins. The third musketeer threw away one coin, took $\frac{4x-19}{27}$ coins and left $\frac{8x-38}{27}$ coins. They then threw away one coin and each got $\frac{8x-65}{81}$ coins. Thus, $8x - 65$ must be a multiple of 81. Because $8x - 65 = 8x + 16 - 81 = 8(x + 2) - 81$, the minimum value of x for $8x - 65$ being a multiple of 81 is $x = 79$.

12. The diagram shows a rectangle that has been divided into ten squares of different sizes. The smallest square is 2×2 (marked with $*$). What is the area of the rectangle (which looks rather like a square itself)?



Solution. The answer is $55 \cdot 57 = \boxed{3135}$. We assume that the two center squares have side lengths x and y, respectively. We can then express all the side lengths in terms of x and y. (See the figure shown above.) Because the opposite sides of the rectangle have the same lengths, we obtain the system of equations

$$\begin{cases} (3x+2y+2)+(x+y+2)+(x+y+4)=(4x+2y+2)+(x+3y-2), \\ (3x+2y+2)+(4x+2y+2)=(x+y+4)+(x+2y-2)+(x+3y-2) \end{cases}$$

or

$$\begin{cases} 5x+4y+8=5x+5y, \\ 7x+4y+4=3x+6y \end{cases}$$

from which it follows that $(x,y)=(3,8)$ and that it is a 55×57 rectangle.

13. Let $A=(3,2)$, $B=(0,1)$, and P be on the line $x+y=0$. What is the minimum possible value of $AP+BP$?

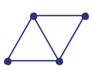
    Solution. The answer is $\boxed{2\sqrt{5}}$. Set $C=(-1,0)$. Note that B and C are reflections of each other across the line $x+y=0$ and that points A and C lie on opposite sides of the line. Hence $AP+BP=AP+PC \le AC = 2\sqrt{5}$.

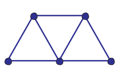
14. Mr. Mustafa the number man got a $6 \times x$ rectangular chess board for his birthday. Because he was bored, he wrote the numbers 1 to $6x$ starting in the upper left corner and moving across row by row (so the number $x+1$ is in the 2nd row, 1st column). Then, he wrote the same numbers starting in the upper left corner and moving down each column (so the number 7 appears in the 1st row, 2nd column). He then added up the two numbers in each of the cells and found that some of the sums were repeated. Given that x is less than or equal to 100, how many possibilities are there for x?

    Solution. The answer is $\boxed{14}$. Let (i,j) denote the cell in the i^{th} row and j^{th} column. Let $a_{i,j}$ denote the number written in the cell (i,j) in the first scheme, and let $b_{i,j}$ denote the number written in the cell (i,j) in the second scheme. Then we have $a_{i,j}=(i-1)x+j$ and $b_{i,j}=6(j-1)+i$. The sum of the numbers written in the cell (i,j) is equal to $a_{i,j}+b_{i,j}=(i-1)x+j+6(j-1)+i=i(x+1)+7j-x-6$. Assume that the sums of the numbers written in cells (i_1,j_1) and (i_2,j_2) are equal. We have $i_1(x+1)+7j_1-x-6=i_2(x+1)+7j_2-x-6$ or $(i_1-i_2)(x+1)=7(j_2-j_1)$. Because i_1 and i_2 are row numbers, their positive difference is less than 6. In particular, i_1-i_2 is not divisible by 7. Hence $x+1$ must be a multiple of 7. The possible values of x are $6, 13, 20, \ldots, 97$. It is easy to check that each of these values indeed works.

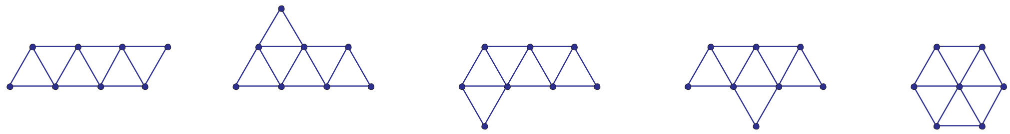
15. Six congruent equilateral triangles are arranged in the plane so that every triangle shares at least one whole edge with some other triangle. Find the number of distinct arrangements. (Two arrangements are considered the same if one can be rotated and/or reflected onto another.)

    Solution. The answer is $\boxed{12}$. For an arrangement of this six equilateral triangles, we define its *diameter* as the maximum number of triangles appeared in row in t his arrangement. In any arrangement, some two triangles must be in a row, and by symmetry, a third triangle must be attached to these two triangle to form a row of three triangle.

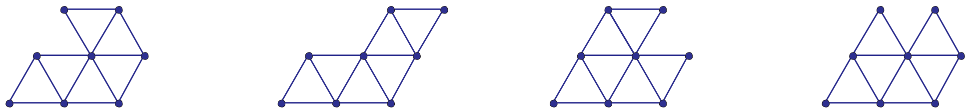




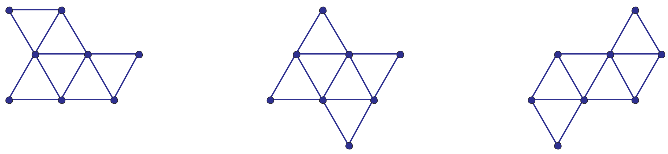
Hence the diameter is equal to 3, 4, 5, or 6. If the diameter is equal to 3, 5, or 6, it is not difficult to see the following 5 arrangements:



If the diameter is equal to 4, we have 7 arrangements. There are 4 arrangements with diameter 4 and two additional triangles on the same side of the diameter.



There are 3 arrangements with diameter 4 and two additional triangles on the opposite sides of the diameter.



3.4 Guts Round Solutions

1. [5pts] Define the operation $\clubsuit$ so that $a \clubsuit b = a^b + b^a$. Then, if $2 \clubsuit b = 32$, what is b?

   Solution. The answer is $\boxed{4}$. Note that $2 \clubsuit b = 2^b + b^2 = 2^4 + 4^2$. (Why is $b = 4$ the unique solution?)

2. [5pts] A square is changed into a rectangle by increasing two of its sides by $p\%$ and decreasing the two other sides by $p\%$. The area is then reduced by 1%. What is the value of p?

   Solution. The answer is $\boxed{10}$. We have $(1-p\%)(1+p\%) = 1-1\%$; that is, $(p\%)^2 = 1\%$ or $p\% = 10\%$.

3. [5pts] What is the sum, in degrees, of the internal angles of a heptagon?

   Solution. The answer is $180 \cdot 5 = \boxed{900}$. The heptagon can be dissected into 5 triangles.

4. [5pts] How many integers in between $\sqrt{47}$ and $\sqrt{8283}$ are divisible by 7?

   Solution. The answer is $\boxed{13}$. Note that $\sqrt{47} < \sqrt{49} = 7 \cdot 1$ and $\sqrt{8283} > \sqrt{8281} = 91 = 7 \cdot 13$.

5. [8pts] Some mutant green turkeys and pink elephants are grazing in a field. Mutant green turkeys have six legs and three heads. Pink elephants have 4 legs and 1 head. There are 100 legs and 37 heads in the field. How many animals are grazing?

   Solution. The answer is $8 + 13 = \boxed{21}$. Assume that there are t turkeys and e elephants. Then $6t + 4e = 100$ and $3t + e = 37$, from which it follows that $(t, e) = (8, 13)$.

6. [8pts] Let $A = (0, 0)$, $B = (6, 8)$, $C = (20, 8)$, $D = (14, 0)$, $E = (21, -10)$, and $F = (7, -10)$. Find the area of the hexagon $ABCDEF$.

   Solution. The answer is $14 \cdot 18 = \boxed{252}$. The hexagon $ABCDEF$ can be dissected into two parallelograms $ABCD$ and $ADEF$. The area of parallelogram $ABCD$ is equal $14 \cdot 8$ and the area of parallelogram $ADEF$ is equal to $14 \cdot 10$.

7. [8pts] In Moscow, three men, Oleg, Igor, and Dima, are questioned on suspicion of stealing Vladimir Putin's blankie. It is known that each man either always tells the truth or always lies. They make the following statements:

   (a) Oleg: I am innocent!
   (b) Igor: Dima stole the blankie!
   (c) Dima: I am innocent!
   (d) Igor: I am guilty!
   (e) Oleg: Yes, Igor is indeed guilty!

   If exactly one of Oleg, Igor, and Dima is guilty of the theft, who is the thief?

   Solution. The answer is $\boxed{\text{Oleg}}$. Igor made two contradicting statements. Hence Igor always lies and Igor is not guilty, from which it follows that Oleg lies and he is guilty.

8. [8pts] How many 11-letter sequences of E's and M's have at least as many E's as M's?

Solution. The answer is $\frac{2^{11}}{2} = \boxed{1024}$. There are 2^{11} sequence of E's and M's. By symmetry, half them have more E's than M's. (Because 11 is odd, we cannot have equal number of E's and Ms in a 11-letter sequence of E's and M's.)

9. [11pts] John is entering the following summation $31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39$ in his calculator. However, he accidently leaves out a plus sign and the answer becomes 3582. What is the number that comes before the missing plus sign?

Solution. The answer is $\boxed{33}$. If the + in between the numbers $\overline{3a}$ and $\overline{3b}$ is left out, we entered $\overline{3a3b}$ and increased the sum by $\overline{3a3b} - (\overline{3a} + \overline{3b}) = 100 \cdot \overline{3a} + ol3b - (\overline{3a} + \overline{3b}) = 99 \cdot \overline{3a}$. Note that

$$3582 - (31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39) = 3582 - 315 = 3267 = 99 \cdot 33.$$

10. [11pts] Two circles of radius 6 intersect such that they share a common chord of length 6. The total area covered may be expressed as $a\pi + \sqrt{b}$, where a and b are integers. What is $a + b$?

Solution. The answer is $60 + 972 = \boxed{1032}$. The region covered by the two circles can be dissected into two congruent equilateral triangles of side length 6 and two congruent circular sectors of radius 6 and central angle 300°. Hence the total area covered is equal to

$$\frac{2 \cdot \frac{5}{6} \cdot 36\pi + 2 \cdot 36 \cdot \sqrt{3}}{4} = 60\pi + 18\sqrt{3} = 60\pi + \sqrt{972}.$$

11. [11pts] Alice has a rectangular room with 6 outlets lined up on one wall and 6 lamps lined up on the opposite wall. She has 6 distinct power cords (red, blue, green, purple, black, yellow). If the red and green power cords cannot cross, how many ways can she plug in all six lamps?

Solution. The answer is $\frac{6! \cdot 6!}{2} = \boxed{259200}$.

12. [11pts] Tracy wants to jump through a line of 12 tiles on the floor by either jumping onto the next block, or jumping onto the block two steps ahead. An example of a path through the 12 tiles may be: 1 step, 2 steps, 2 steps, 2 steps, 1 step, 2 steps, 2 steps. In how many ways can Tracy jump through these 12 tiles?

Solution. The answer is $\boxed{233}$. For $n \geq 1$, let f_n denote the number of ways Tracy can jump through through a line of n tiles. There are two choices for Tracy's first jump − jumping to the next block or jumping onto the block two steps ahead. In the former case, she has f_{n-1} ways to jump through the rest of $n - 1$ tiles. In the later case, she has f_{n-2} ways to jump through the rest of $n - 2$ tiles. Therefore, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. It is easy to compute that $f_1 = 1$, $f_2 = 2$, $f_3 = 3$, $f_4 = 5$, $f_5 = 8$, $f_6 = 13$, $f_7 = 21$, $f_8 = 34$, $f_9 = 55$, $f_{10} = 89$, $f_{11} = 144$, and $f_{12} = 233$. (This is the famous Fibonacci sequence.)

13. [14pts] What is the units digit of the number $(2^1 + 1)(2^2 - 1)(2^3 + 1)(2^4 - 1) \ldots (2^{2010} - 1)$?

Solution. The answer is $\boxed{5}$. All the multiplicands are odd and the units digit of $2^4 - 1$ is 5.

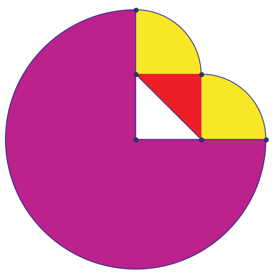
14. [14pts] Mr. Fat noted that on January 2, 2010, the display of the day is 01/02/2010, and the sequence 01022010 is a palindrome (a number that reads the same forwards and backwards). How many days does Mr. Fat need to wait between this palindrome day and the last palindrome day of this decade?

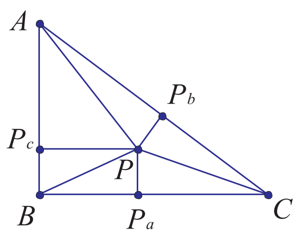
Solution. The answer is $2 \cdot 365 - 30 - 31 = \boxed{669}$. The last palindrome day of this decade is 11/02/2011, which is exactly 22 months (a November and December short of two full years) away from 01/02/2010.

15. [14pts] Farmer Tim has a 30-meter by 30-meter by $30\sqrt{2}$-meter triangular barn. He ties his goat to the corner where the two shorter sides meet with a 60-meter rope. What is the area, in square meters, of the land where the goat can graze, given that it cannot get inside the barn?

Solution. The answer is $\boxed{3150\pi + 450}$. The grazing area consists of one 270° circular sector region of radius 60 meters, two 90° circular sector regions of radius 30 meters, and one 30-meter by 30-meter by $30\sqrt{2}$-meter triangular region. (See the left-hand side figure shown below.) The total area is equal to

$$\frac{3}{4} \cdot 3600\pi + 2 \cdot \frac{1}{4} 900\pi + 450 = 3150\pi + 450.$$





16. [14pts] In triangle ABC, $AB = 3$, $BC = 4$, and $CA = 5$. Point P lies inside the triangle and the distances from P to two of the sides of the triangle are 1 and 2. What is the maximum distance from P to the third side of the triangle?

Solution. The answer is $\boxed{\frac{2}{5}}$. Let P_a, P_b, P_c be the feet of the perpendiculars from P to sides BC, CA, AB, respectively. Set $x = PP_a$, $y = PP_b$, and $z = PP_c$. Because the sum of the areas of triangles PAB, PBC, PCA is equal to the area of triangle ABC, we have $4x + 5y + 3z = 12$. Two of x, y, z take the values 1 and 2, and we want to maximize the third. If $y = 2$, then $4x + 5y + 3z \geq 13$, which is not possible. If $y = 1$, then $4x + 3z = 7$, implying that $z = 2$ and $x = \frac{1}{4}$. If y is unknown, then $5y = 12 - (4x + 3z)$ is equal to either 2 (with $(x, z) = (1, 2)$) or 1 (with $(x, z) = (2, 1)$), implying that $y = \frac{2}{5}$ or $y = \frac{1}{5}$.

17. [17pts] Let Z be the answer to the third question on this guts quadruplet. If $x^2 - 2x = Z - 1$, find the positive value of x.

Solution. The answer is $\boxed{17}$. We can rearrange the given equation to get $x^2 - 2x + 1 = Z$, or $(x - 1)^2 = Z$, so $x = \sqrt{Z} + 1$.

18. [17pts] Let X be the answer to the first question on this guts quadruplet. To make a FATRON2012, a cubical steel body as large as possible is cut out from a solid sphere of diameter X. A TAFTRON2013 is created by cutting a FATRON2012 into 27 identical cubes, with no material wasted. What is the length of one edge of a TAFTRON2013?

Solution. The answer is $\boxed{\frac{17\sqrt{3}}{9}}$. We see that each side of the FATRON2012 has side length $\frac{X}{\sqrt{3}}$. Thus, each side of the TAFTRON2013 is $\frac{1}{3} \cdot \frac{X}{\sqrt{3}} = \frac{X\sqrt{3}}{9}$.

19. [17pts] Let Y be the smallest integer greater than the answer to the second question on this guts quadruplet. Fred posts two distinguishable sheets on the wall. Then, Y people walk into the room. Each of the Y people signs up on 0, 1, or 2 of the sheets. Given that there are at least two people in the room other than Fred, how many possible pairs of lists can Fred have?

Solution. The answer is $\boxed{256}$.

We know that each person can sign up on neither of the sheets, only the first sheet, only the second sheet, or both sheets. Thus, there are $Z = 4^Y$ possible sets of lists.

By the definitions of X and Y, we obtain that

$$Y - 1 \le \frac{X\sqrt{3}}{9} = \frac{(2^Y + 1)\sqrt{3}}{9} = \frac{2^Y + 1}{3\sqrt{3}} \le Y$$

or

$$3\sqrt{3}Y - 3\sqrt{3} - 1 \le 2^Y \le 3\sqrt{3}Y - 1, \qquad (3.1)$$

implying that $Y = 4$ (and consequently, $Z = 4^4 = 256$, $X = 17$).

Intuitively, because the exponential function 2^Y grows much faster than the liner function $3\sqrt{3}Y - 1$, there are very few integer values of Y satisfying (3.1). It is not difficult to check that $Y = 4$ is the only integer answer to (3.1) such that Y is at least 2.

Rigorously, we can show that $2^n > 6n > 6n - 1 > 3\sqrt{3}n - 1$ for $n \ge 6$. (Hence the only possible values of Y satisfying (3.1) are $Y = 1, 2, 3, 4, 5$. By a quick check, we can conclude that $Y = 4$ is the only solution for Y at least 2.) To show that $2^n > 6n$ for $n \ge 6$, we induct on n. For $n = 6$, this is clearly true. Assume that $2^n > 6n$ is true for some integer $n = k \ge 6$; that is, $2^k > 6k$ and $k \ge 6$. Then for $n = k + 1$, we have $2^{k+1} = 2 \cdot 2^k > 2 \cdot 6k > 6(k + 1)$.

20. [17pts] Let A, B, C, be the respective answers to the first, second, and third questions on this guts quadruplet. At the Robot Design Convention and Showcase, a series of robots are programmed such that each robot shakes hands exactly once with every other robot of the same height. If the heights of the 16 robots are 4, 4, 4, 5, 5, 7, 17, 17, 17, 34, 34, 42, 100, A, B, and C feet, how many handshakes will take place?

Solution. The answer is $\binom{3}{2} + \binom{2}{2} + \binom{4}{2} + \binom{2}{2} = \boxed{11}$. We know that $A = 17$, $B = \frac{17\sqrt{3}}{9}$, and $C = 256$. Thus, we have 3 robots of height 4, 2 robots of height 5, 4 robots of height 17, and 2 robots of height 34.

21. [20pts] Determine the number of ordered triples (p, q, r) of primes with $1 < p < q < r < 100$ such that $q - p = r - q$.

Solution. The answer is $10 + 22 + 14 = \boxed{46}$. Let $d = q - p = r - q$ be the common difference. Because both r and q must be odd, d must be even.

If d is not a multiple of 3, then one of p, q, r is a multiple of 3. Because $p < q < r$ are primes, we must have $p = 3$. It is not difficult to list all 10 such triples, namely, $(3, 5, 7)$, $(3, 7, 11)$, $(3, 11, 19)$, $(3, 13, 23)$, $(3, 17, 31)$, $(3, 23, 46)$, $(3, 31, 59)$, $(3, 37, 71)$, $(3, 41, 79)$, $(3, 43, 83)$,

If d is a multiple of 3, then d is a multiple of 6. We classify the primes greater than 3 and less than 100 modulo 6:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$6k-1$	5	11	17	23	29		41	47	53	59		71		83	89	
$6k+1$	7	13	19		31	37	43			61	67	73	79			97

Note that the primes $6k_1 \pm 1, 6k_2 \pm 1, 6k_3 \pm 1$ form an arithmetic progression if and only if the numbers k_1, k_2, k_3 form an arithmetic progression.

To count the 3-term arithmetic progressions of primes in the from $6k - 1$, we count all the 3-term arithmetic progressions in the set $\{1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 14, 15\}$. There are 22 such triples, namely,

$$(1,2,3),(1,3,5),(1,4,7),(1,5,9),(1,8,15);(2,3,4),(2,5,8),(2,7,12),(2,8,14);$$
$$(3,4,5),(3,5,7),(3,9,15);(4,7,10),(4,8,12),(4,9,14);(5,7,9),(5,10,15);$$
$$(7,8,9);(8,9,10),(8,10,12);(9,12,15);(10,12,14).$$

To count the 3-term arithmetic progressions of primes in the from $6k + 1$, we count all the 3-term arithmetic progressions in the set $\{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 16\}$. There are 14 such triples, namely,

$$(1,2,3),(1,3,5),(1,6,11),(1,7,13);(2,6,10),(2,7,12);$$
$$(3,5,7),(3,7,11);(5,6,7);(6,11,16);(7,10,13);(10,11,12),(10,13,16);(11,12,13).$$

22. [20pts] For numbers a, b, c, d such that $0 \le a, b, c, d \le 10$, find the minimum value of $ab + bc + cd + da - 5a - 5b - 5c - 5d$.

Solution. The answer is $\boxed{-100}$. We have

$$\begin{aligned} & ab + bc + cd + da - 5a - 5b - 5c - 5d = (a+c)(b+d) - 5(a+c) - 5(b+d) \\ = \; & (a+c-5)(b+d-5) - 25 \ge (-5) \cdot 15 - 25 = -100. \end{aligned}$$

23. [20pts] Daniel has a task to measure 1 gram, 2 grams, 3 grams, 4 grams , ..., all the way up to n grams. He goes into a store and buys a scale and six weights of his choosing (so that he knows the value for each weight that he buys). If he can place the weights on either side of the scale, what is the maximum value of n?

Solution. The answer is $\frac{3^6-1}{2} = \boxed{364}$. We see that we can put each weight on either the right side of the scale, the left side of the scale, or neither, for 3^6 possible combinations. However, we need to exclude the case where there are no weights on the scale, leaving $3^6 - 1$ ways. Without loss of generality, we always want to weight our object on the right side of the scale, so we always want the placement of the weights to leave the left side heavier; by symmetry, this only happens in half the cases we are currently counting, so we have at most $\frac{3^6-1}{2}$ different masses we can weight. We can check that using the set $\{3^0, 3^1, 3^2, 3^3, 3^4, 3^5\}$ will let us weight any mass between 1 and 364: The weight 3^0 will let us weight objects of mass 1. Then, adding the weight 3^1 will then let us weight objects from $3^1 - 3^0$ to $3^1 + 3^0$ (from 2 to 4). Adding the weight of 3^2 will then let us weight objects from $3^2 - (3^1 + 3^0)$ to $3^2 + 3^1 + 3^0$ (from 5 to 13), and so on.

24. [20pts] Given a Rubik's cube, what is the probability that at least one face will remain unchanged after a random sequence of three moves? (A Rubik's cube is a 3 by 3 by 3 cube with each face starting as a different color. The faces (3 by 3) can be freely turned. A move is defined in this problem as a 90 degree rotation of one *face either clockwise or counter-clockwise. The center square on each face–six in total–is fixed.)*

Solution. The answer is $\boxed{\frac{2}{9}}$.

For each move, we can select one of the six faces and make either a 90 degree clockwise rotation or a 90 degree counter-clockwise rotation, implying that there are 12 distinct options. Thus, there are 12^3 distinct sequence of three moves.

We consider those sequences of three moves that leave at least one face unchanged. We consider three pairs of parallel faces A_1 and A_2, B_1 and B_2, and C_1 and C_2. The key observation is that, after three moves, if one face in a pair is unchanged, then so does the other face in the pair. Indeed, by symmetry, we may assume without loss of generality that A_1 is unchanged. We categorize the moves into two types:

- moves on faces A_1 and A_2. There are 4 such moves, and they are all denoted by T;
- moves on the other four faces. There are 8 such moves, and they are all denoted by S.

For A_1 to be left unchanged, we have either three T moves or a T move followed by a S move and then its inverse move $S{-}1$ or a S move, its inverse move S^{-1}, and then a T move. In each case, face A_2 is also unchanged.

In summary, we can choose one of the three pairs of parallel faces to be unchanged. For each chosen pair of parallel faces, there are $4^3 + 4 \cdot 8 \cdot 1 + 8 \cdot 1 \cdot 4 = 2 \cdot 4^3$ distinct sequence of three moves to have the chosen pair of faces unchanged. Hence the desired probability is equal to

$$\frac{3 \cdot 2 \cdot 4^3}{12^3} = \frac{3 \cdot 2}{3^3} = \frac{2}{9}.$$