1.3 Team Test

Morning, January 28, 2012

There are 15 problems, worth 20 points each, to be solved in 45 minutes.

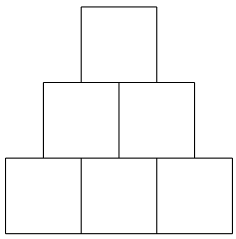
1. The longest diagonal of a regular hexagon is 12 inches long. What is the area of the hexagon, in square inches?

2. When Al and Bob play a game, either Al wins, Bob wins, or they tie. The probability that Al does not win is $\frac{2}{3}$, and the probability that Bob does not win is $\frac{3}{4}$. What is the probability that they tie?

3. Find the sum of the $a+b$ values over all pairs of integers (a, b) such that $1 \le a < b \le 10$. That is, compute the sum
$$(1+2)+(1+3)+(1+4)+\cdots+(2+3)+(2+4)+\cdots+(9+10).$$

4. A 3×11 cm rectangular box has one vertex at the origin, and the other vertices are above the x-axis. Its sides lie on the lines $y = x$ and $y = -x$. What is the y-coordinate of the highest point on the box, in centimeters?

5. Six blocks are stacked on top of each other to create a pyramid, as shown below. Carl removes blocks one at a time from the pyramid, until all the blocks have been removed. He never removes a block until all the blocks that rest on top of it have been removed. In how many different orders can Carl remove the blocks?



6. Suppose that a right triangle has sides of lengths $\sqrt{a+b\sqrt{3}}$, $\sqrt{3+2\sqrt{3}}$, and $\sqrt{4+5\sqrt{3}}$, where a, b are positive integers. Find all possible ordered pairs (a, b).

7. Farmer Chong Gu glues together 4 equilateral triangles of side length 1 such that their edges coincide. He then drives in a stake at each vertex of the original triangles and puts a rubber band around all the stakes. Find the minimum possible length of the rubber band.

8. Compute the number of ordered pairs (a, b) of positive integers less than or equal to 100, such that $a^b - 1$ is a multiple of 4.

9. In triangle ABC, $\angle C = 90^\circ$. Point P lies on segment BC and is not B or C. Point I lies on segment AP. If $\angle BIP = \angle PBI = \angle CAB = m^\circ$ for some positive integer m, find the sum of all possible values of m.

10. Bob has 2 identical red coins and 2 identical blue coins, as well as 4 distinguishable buckets. He places some, but not necessarily all, of the coins into the buckets such that no bucket contains two coins of the same color, and at least one bucket is not empty. In how many ways can he do this?

11. Albert takes a 4×4 checkerboard and paints all the squares white. Afterward, he wants to paint some of the square black, such that each square shares an edge with an odd number of black squares. Help him out by drawing one possible configuration in which this holds. (Note: the answer sheet contains a 4×4 grid.)

12. Let S be the set of points (x, y) with $0 \leq x \leq 5, 0 \leq y \leq 5$ where x and y are integers. Let T be the set of all points in the plane that are the midpoints of two distinct points in S. Let U be the set of all points in the plane that are the midpoints of two distinct points in T. How many distinct points are in U? (Note: The points in T and U do not necessarily have integer coordinates.)

13. In how many ways can one express 6036 as the sum of at least two (not necessarily positive) consecutive integers?

14. Let a, b, c, d, e, f be integers (not necessarily distinct) between -100 and 100, inclusive, such that $a+b+c+d+e+f = 100$. Let M and m be the maximum and minimum possible values, respectively, of
$$abc + bcd + cde + def + efa + fab + ace + bdf.$$
Find $\dfrac{M}{m}$.

15. In quadrilateral $ABCD$, diagonal AC bisects diagonal BD. Given that $AB = 20$, $BC = 15$, $CD = 13$, $AC = 25$, find DA.