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Optimal Execution of Portfolio Transactions

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1 Introduction

Among the problems that a trader may encounter is the liquidation of portfolios. To illustrate this, let's put ourselves in the shoes of a Broker (intermediate person whose role is to buy and sell shares for other people), and our role would be to execute transactions for a client, who asks us to sell 10 million shares of Google between now and in two hours, we are therefore faced with a high-frequency problem. We then have at our disposal a portfolio made up of 10 million shares of Google that we want to sell on the market between now and in two hours, so as to derive the maximum possible benefit.

First of all, the order book is made up of two parts, the Bid part which concerns buyers who want to buy shares for a price and for a quantity that they set. Then, the Ask part concerns sellers who want to sell shares for a price and for a quantity that they set as well.

Example:

Bid	Ask
2,803.01 $ / 200	2,803.05 $ / 831
2,803.00 $ / 350	2,803.04 $ / 941
2.802.99 $ / 400	2.803.03 $ / 31
...	...

Here, a buyer offers to buy 200 Google shares for 2,803.01 $ per unit, this is the best choice offered to us,but unfortunately he does not want to buy all 10 million shares available.

If we want to sell all of them immediately, then the approach that will bring us the most money will be to sell the first 200 shares at 2,803.01 $ then the 350 at 2,803.00 $ and so on down the Bid table until all the shares are liquidated.

In this case the average price per share will be the price of Bid weighted by their volume, we quickly notice that the higher the number of shares to sell, the more this approach will make us lose money.

At the time when our mission begins, we can imagine that the unit price of a Google share is around 2,803 $, so the client expects to receive in return 10 million * 2,803 $ whereas in our case he is offered way less. This naive approach is not going to please our client at all, and so we have to see another approach in order to make good use of the two hours at our disposal and make the most of it.

A second approach would be to take advantage of the two hours we have, while waiting for new buyers to appear on the front line of the order book (the maximum Bid) and to sell little by little, and if we don't succeed to sell all the shares after two hours, we set ourselves a constraint that 10 minutes from the end of the two hours we sell all that we have left in the same way as for the first approach. This approach seems more optimal, but one very important thing has not been taken into account is that the unit price of Google varies over time and we absolutely cannot predict its variation, Admittedly, we are always positioned at the front line of the order book, but if the price fluctuates a lot (decreases), it can be risky. The components that cause the price of a stock to fall are volatility,and market impact.

Market impact can be illustrated by the example where we notice that there is someone in the market who is constantly selling the shares of a company, other traders will see that there is something wrong and will also start selling, which will lead to a reduction in the share price (principle of supply and demand).

We will then see through this synthesis how to optimize our gains thanks to the algorithm of Robert Almgren and Neil Chriss.

2 Almgren and Chris Model

2.1 Setup

We consider we are selling one asset. We have X shares of this assets at $t_0 = 0$. We want everything to be sold at $t = T$. In order to do that we will split $[0, T]$ into N intervals of length $\tau = T/N$ and set $t_k = k\tau, k = 0, \ldots, N$. And the goal would be to find how many stocks we will sell in each of these intervals. We precise that the algorithm won't tell us how many stocks we should sell at a fixed time t but how many we need to sell in each interval of length τ. It gives us a solution of scheduling.

Then we define a trading strategy as a vector $(x_0, \ldots, x_N)$, with x_k the number of shares we still have at time t_k. Notice that $x_0 = X, x_N = 0$. Finally we define $n_k = x_{k-1} - x_k$ as the number of assets sold between t_{k-1} and t_k (and not exactly at time t_{k-1} or t_k), decided at time t_{k-1}.

2.2 Permanent impact component

As said in the introduction if market participants see us selling large quantities, they revise their prices down, it's called the market impact. Therefore, the "equilibrium price" of the asset is modified in a permanent way. Let S_k be the equilibrium price at time t_k :

$$S_k = S_{k-1} + \sigma\tau^{1/2}\xi_k - \tau g\left(n_k/\tau\right)$$

for $k = 1, \ldots, N$. Here σ represents the volatility of the asset, the ξ_k are iid standard Gaussians, and the permanent impact $g(v)$ is a function of the average rate of trading $v = n_k/\tau$ during the interval t_{k-1} to t_k. The second term of the equation is a gaussian fluctuation due to volatilty, whereas the last term is a fluctuation due to our market impact. Notice that since the market impact makes the price go down, it's going to be added with a negative sign to the equation. Notice also that S_k is the Mid price and not the Bid one, which is the average of the best Ask price and the best Bid price.

2.3 Temporary impact component

There are temporary impact costs that need to be considered in the model, it is due to the transaction costs. First of all, when we buy our stocks in the defined intervals of time, we buy them from the order book, which means that we buy them at the best Bid price as much as possible, but when there is not enough liquidity in the best Bid, we can consume the next lines of the order book, which will cost us more and needs to be included in the model. Secondly, since we buy with the Bid price and not the Mid one, we should include the difference between the Mid price and the bid price that we are going to loose for each stock. We will assume in the model this effect is temporary and the liquidity comes back after each period.

Let $\tilde{S}_k = \left(\sum n_{k,i}p_i\right)/n_k$, with $n_{k,i}$ the number of shares sold at price p_i between t_{k-1} and t_k. We set

$$\tilde{S}_k = S_{k-1} - h\left(n_k/\tau\right).$$

Which means that the average price we are going to sell at between t_{k-1} and t_k is the mid price at t_{k-1} impacted by the term $h\left(n_k/\tau\right)$ which is due the transaction costs. We call the function h the temporary impact.

2.4 Profit and Loss

Cost of Trading:
The result of the sell of the asset is:

$$\sum_{k=1}^{N} n_k \tilde{S}_k = XS_0 + \sum_{k=1}^{N} \left(\sigma \tau^{1/2} \xi_k - \tau g\left(n_k/\tau\right)\right) x_k - \sum_{k=1}^{N} n_k h\left(n_k/\tau\right)$$

Notice that XS_0 represents what we were hoping to gain if we didn't have other costs.
We define the trading cost as $\mathcal{C} = XS_0 - \sum_{k=1}^{N} n_k \tilde{S}_k$ which is equal to the difference between the amount of money that the client thinks is going to win before giving us this mission (called the Benchmark) and the real amount that we are going to gain according to the model, the goal would be then to minimize this trading cost to gain the maximum possible for our client.

2.5 Mean Variance analysis

We have:

$$\mathbb{E}[\mathcal{C}] = \underbrace{\sum_{k=1}^{N} \tau x_k g\left(n_k/\tau\right)}_{\text{Permanent impact component}} + \underbrace{\sum_{k=1}^{N} n_k h\left(n_k/\tau\right)}_{\text{Temporary impact component}} \quad , \operatorname{Var}[\mathcal{C}] = \sigma^2 \textstyle\sum_{k=1}^{N} \tau x_k^2$$

As for the Markowitz Portfolio Optimization, in order to build optimal trading trajectories, we will look for strategies minimizing:

$$\mathbb{E}[\mathcal{C}] + \lambda \operatorname{Var}[\mathcal{C}]$$

with λ a risk aversion parameter.
We didn't seek to optimize only the mean but we took into consideration the risk as well because we don't want to take the risk of having a bad performance for our client and lose him because it didn't work out well when he came the first time, if we were sure that he would come to us multiple times, the optimization of the mean only, would've been better, but it's not the case.

3 Naive Strategies

3.1 Permanent impact

We will choose a linear permanent impact : $g(v) = \gamma v$. it means that if we sell a volume v of stocks, we are going to move the price by γv.

Thus if we replace g by its expression we obtain:

$$S_k = S_0 + \sigma \sum_{j=1}^{k} \tau^{1/2} \xi_j - \gamma\left(X - x_k\right)$$

Moreover, in $\mathbb{E}[\mathcal{C}]$, the permanent impact component satisfies

$$\sum_{k=1}^{N} \tau x_k g\left(n_k/\tau\right) = \gamma \sum_{k=1}^{N} x_k\left(x_{k-1} - x_k\right) = \frac{1}{2}\gamma X^2 - \frac{1}{2}\gamma \sum_{k=1}^{N} n_k^2$$

3.2 Temporary impact

We will choose an affine temporary impact : $h\left(n_k/\tau\right) = \varepsilon + \eta\left(n_k/\tau\right)$. ε represents a fixed cost : fees and bid ask spread.

Let $\tilde{\eta} = \eta - \frac{1}{2}\gamma\tau$, we get:

$$\mathbb{E}[\mathcal{C}] = \frac{1}{2}\gamma X^2 + \varepsilon X + \frac{\tilde{\eta}}{\tau}\sum_{k=1}^{N} n_k^2.$$

3.3 Regular strategy

The first naive strategy is to sell the same amount of stocks regularly in each interval.In other meaning we take $n_k = X/N, x_k = (N-k)X/N, k = 1, \ldots, N$. We easily get

$$\mathbb{E}[\mathcal{C}] = \frac{1}{2}\gamma X^2 + \varepsilon X + \tilde{\eta}\frac{X^2}{T},$$
$$\operatorname{Var}[\mathcal{C}] = \frac{\sigma^2}{3}X^2 T\left(1 - \frac{1}{N}\right)\left(1 - \frac{1}{2N}\right).$$

It can be shown that this strategy has the smallest expectation. However the variance can be very big if T is large.

3.4 Selling everything at t_0

A second naive strategy would be to sell everything at t_0 which means that we take $n_1 = X$, $n_2 = \ldots = n_N = 0, x_1 = \ldots = x_N = 0$.

We get:

$$\mathbb{E}[\mathcal{C}] = \varepsilon X + \frac{\eta X^2}{\tau}$$
$$\operatorname{Var}[\mathcal{C}] = 0$$

This strategy has the smallest variance since we sell everything at t_0. However, if τ is small, the expectation can be very large.

4 Optimal Strategies

4.1 Optimization Program

As explained before, the trader wants to minimize

$$U(\mathcal{C}) = \mathbb{E}[\mathcal{C}] + \lambda \operatorname{Var}[\mathcal{C}]$$

$U(\mathcal{C})$ is equal to

$$\frac{1}{2}\gamma X^2 + \varepsilon X + \frac{\tilde{\eta}}{\tau}\sum_{k=1}^{N}\left(x_{k-1} - x_k\right)^2 + \lambda\sigma^2\sum_{k=1}^{N}\tau x_k^2$$

4.2 Derivation

in order to do that, we will need to derive U with respect to all x_j for $j = 1, \dots, N-1$,

$$\frac{\partial U}{\partial x_j} = 2\tau \left(\lambda \sigma^2 x_j - \tilde{\eta} \frac{(x_{j-1} - 2x_j + x_{j+1})}{\tau^2} \right)$$

Therefore

$$\frac{\partial U}{\partial x_j} = 0 \Leftrightarrow \frac{(x_{j-1} - 2x_j + x_{j+1})}{\tau^2} = \tilde{K} x_j$$

with $\tilde{K} = \lambda\sigma^2/\tilde{\eta}$. Notice that the sequence (x_k) verifies a recursive relation of second order that can be easily solved.

4.3 Solution

We can show that the solution can be written $x_0 = X$ and for $j = 1, \dots, N$:

$$x_j = \frac{\sinh\left(K\left(T - t_j\right)\right)}{\sinh(KT)} X$$
$$n_j = \frac{2\sinh(K\tau/2)}{\sinh(KT)} \cosh(K(T - j\tau + \tau/2))X$$

where K satisfies $\frac{2}{\tau^2}(\cosh(K\tau) - 1) = \tilde{K}$. We now know, how to choose our intervals and how many stocks we should sell in each interval.

5 Conclusion

The last fomulas obtained for x_j and n_j show that the solution depends on λ (which is specific to the trader), σ and the market impact parameters.
This shows the importance of knowing how to estimate those parameters in order to gain the maximum possible amount of money. the estimation of the market parameters is going to be something that could be inferred by experience of the trader and which is generally stable, compared to the volatility whom the estimation will be very crucial in this model. Notice that if $\lambda = 0$ (no risk aversion) then $\tilde{K} = K = 0$ and so $n_j = \tau/T = X/N$. We retrieve the strategy with minimal expected cost.

References

[1] *Execution of Portfolio Transactions*, Robert Almgren and Neil Chriss, December 2000