Technische Universität München

Department of Mathematics

Bachelor’s Thesis

The 27 Lines on a Cubic Surface

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I assure the single handed composition of this bachelor's thesis only supported by declared resources.

Garching,

Zusammenfassung

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In this Bachelor thesis I will prove in full detail the existence of 27 lines on an arbitrary smooth cubic in projective n-space over an algebraically closed field k.

Contents

1 General facts

1.1 projective space

In order to do geometry we need to explain what points, lines, surfaces are, and this can be done in several ways, synthetically or analytically.
There are several formulations of projective space, with varying degree of generality, and for our purposes I have chosen the one in terms of homogeneous coordinates.
Let k be an algebraically closed field and n a natural number. The projective space $\mathbb{P}^n_k$ is a topological space given as set by $k^{n+1} - \{0\}$ modulo the relation $X = (x_0, ..x_n) \sim Y = (y_0, ..y_n)$ if and only if X and Y are linearly dependent (as vectors in k^{n+1}). The equivalence class of X will be denoted $[x_0 : x_1 : .. : x_n]$.
The topology is given by the closed subsets

$$\mathcal{V}(I) = \left\{[x_0 : ..x_n] \in \mathbb{P}^n_k \mid \forall f \in I \text{ homogeneous}, f(x_0, ..x_n) = 0\right\} \tag{1}$$

for homogeneous ideals I of the polynomial ring $k[x_0, ..x_n]$. The topology is known as Zariski's topology. Because $k[x_0, ..x_n]$ is a Noetherian ring, every ideal is finitely generated and in particular every homogeneous ideal is finitely generated by homogeneous elements. So we may say that a closed set consists precisely of the points in projective space where a finite set of homogeneous ideals vanish.
In this framework I want to identify geometric objects with homogeneous ideals of $k[x_0, ..x_n]$. However, most of the time it is easier to speak of the closed subsets of projective space, instead of the ideal I, and I itself can be recovered by means of Hilberts's Nullstellensatz in the form of its radical.
A *hypersurface* is given by one equation $f = 0$ for $f \in k[x_0, ..x_n]$, and its set of points is $\mathcal{V}(f)$. In case of f being a linear form we call $\mathcal{V}(f)$ a *hyperplane*.
A line is determined by $n-1$ k-linearly independent linear forms and a point by n k-linearly independent linear forms.

1.2 partial derivatives

In this section we discuss partial derivatives – a useful algebraic tool to obtain some geometric properties.

Definition 1.1 (partial derivative)**.** *Let $D : k[x_0, ..x_n] \to k[x_0, ..x_n]$ be a derivation over k, i.e. a homomorphism of k-modules which satisfies the Leibniz rule $D(xy) = xDy + yDx$ for $x, y \in k[x_0, ..x_n]$. Furthermore let $D(h) \in k$ for every linear form h. We call such a derivation a* partial derivative.

Example 1.2. *The standard example for partial derivatives are of course the partial derivates with respect to one of the variables x_i, defined as $\partial_{x_i}(x_j) = \delta_{i,j} := \begin{cases} 1, & \text{for } i = j \\ 0, & \text{otherwise.} \end{cases}$*

Example 1.3. *For any monomial $\{h_i\}_{i=0}^n$ basis of $\bigoplus_{i=0}^n kx_i \subset k[x_0, ..x_n]$ we obtain a family of partial derivatives $\{\partial_{h_i}\}_{i=0}^n$ for which $\partial_{h_i}(h_j) = \delta_{i,j}$ holds. The construction goes as follows: Let $M = (a_{i,j}) \in k^{(n+1)\times(n+1)}$ be the base change matrix and $M^{-1} = (\tilde{a}_{i,j})$ be its inverse. This just* *means* $h_i = \sum_{j=0}^n a_{i,j} x_j$ *and hence* $\partial_{x_k}(h_i) = a_{i,k}$. *From* $\delta_{i,j} = \sum_{k=0}^n a_{i,k}\widetilde{a}_{k,j} = \sum_{k=0}^n \partial_{x_k}(h_i)\widetilde{a}_{k,j}$ *it is obvious that we need to define*

$$\partial_{h_j} = \sum_{k=0}^n \widetilde{a}_{k,j}\,\partial_{x_k} \tag{2}$$

Another write to write this would be

$$\begin{pmatrix}\partial_{h_0}\\ \vdots \\ \partial_{h_n}\end{pmatrix} = M^{-T}\begin{pmatrix}\partial_{x_0}\\ \vdots \\ \partial_{x_n}\end{pmatrix} \tag{3}$$

A useful fact, that allows us to recover a homogeneous polynomial by its partial derivatives is

Proposition 1.4 (Euler's formula)**.** *For any* $f \in k[x_0,..x_n]$ *homogeneous of degree* d *we have the equality*

$$df = \sum_{i=0}^n \partial_{x_i}(f)x_i$$

Proof. By linearity we only need to prove the monomial case $f = \prod_{i=0}^n x_i^{a_i}$, a_i being integers such that $\sum_{i=0}^n a_i = d$.

$$\sum_{i=0}^n \partial_{x_i}(f)x_i = \sum_{i=0}^n \left\{\begin{matrix}\left(\prod_{j\neq i} x_j^{a_j}\right) a_i x_i^{a_i-1}x_i, & \text{for } a_i > 0\\ 0, & \text{for } a_i = 0\end{matrix}\right\} = \sum_{i=0}^n a_i f = df \tag{4}$$

□

Corollary 1.5. $\mathcal{V}(\partial_{x_0}(f),..\partial_{x_n}(f),df) = \mathcal{V}(\partial_{x_0}(f),..\partial_{x_n}(f))$

Lemma 1.6. *Let* ∂_1,∂_2 *be partial derivatives, then* $\partial_1.\partial_2 = \partial_2.\partial_1$.

Proof. We only need to prove this for monomials, and we'll perform an induction on the degree. If f is a monomial of degree less than 2, then $\partial_1(\partial_2(f)) = 0 = \partial_2(\partial_1(f))$. Now suppose $f = x_i f'$ and $\partial_1(\partial_2(f')) = \partial_2(\partial_1(f'))$.

$$\partial_1.\partial_2(x_i f') = \partial_1(x\partial_2(f') + f'\partial_2(x)) = \partial_1(x)\partial_2(f') + \partial_1(f')\partial_2(x) + x\partial_1(\partial_2(f')) + \underbrace{f'\partial_1(\partial_2(x))}_{=0} \tag{5}$$

The last term is symmetric in ∂_1,∂_2 (by assumption). □

Corollary 1.7 (Taylor's formula)**.** *Let* $f \in k[x_0,..x_n]$ *be a polynomial and* $f = \sum_{i=0}^d f_i$ *its decomposition into homogeneous parts. Then* $f_i = \frac{1}{i!}\sum_{|\alpha|=\mathbf{i}} \partial^\alpha(f)(0)x^\alpha$ *in multi-index notation.*

Proof. □

1.3 singularities and the tangent plane

1.3.1 Invariance of singularities under a change of the system of partial derivatives

Oftentimes it is more convenient to perform base changes to simplify equations and sometimes one can do without, bnut in exchange one needs to introduce more flexible tools. Remember, in order to study singularities we introduced partial derivatives $\partial_{x_i} : k[x_0,..x_n] \to k[x_0,..x_n]$ which satisfied $\partial_{x_i}(x_j) = \delta_{i,j}$.

1.4 plane conics – the question of degeneracy

A plane conic is a algebraic variety $\mathcal{V}(f)$ given by a quadratic form $f \in k[x_0, x_1, x_2]$. One might ask the question whether the conic is a union of two lines (in which case the conic is called *degenerate*), or in algebraic terms, whether f factors into two linear forms or whether it is irreducible. Let's turn our attention the an easier question: When is a conic singular?
Assume that the characteristic of our base field k is not 2, then the conic can be written, for appropriate coefficients a, b, c, d, e, f as:

$$f = ax_0^2 + 2bx_0x_1 + cx_1^2 + 2dx_0x_2 + 2ex_1x_2 + fx_2^2 \tag{6}$$

The singular points are given by the system of equations

$$\partial_{x_0} f = \partial_{x_1} f = \partial_{x_2} f = 0 \tag{7}$$

which written out in matrix notation, amounts to

$$2\underbrace{\begin{pmatrix} a & b & e \\ b & c & d \\ e & d & f \end{pmatrix}}_{=:M}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0 \tag{8}$$

We call the matrix M. A singular point $[s_0 : s_1 : s_2] \in \mathbb{P}_k^2$ would of course be a non-zero solution of above equation and as such can only exist precisely if the determinant of M vanishes. So far we have obtained

Corollary 1.8. *Let k be a field of characteristic not 2 and $f \in k[x_0, x_1, x_2]$ be a quadratic form. The conic $\mathcal{V}(f) \subset \mathbb{P}_k^2$ is singular if and only if*

$$\det\begin{pmatrix} a & b & e \\ b & c & d \\ e & d & f \end{pmatrix} = acf + 2bde - ce^2 - ad^2 - b^2 f = 0 \tag{9}$$

Returning to our initial question we want to establish the fact that the conic given by the quadratic form f is irreducible if and only if it is non-singular. For that assume reducibility, that is $f = gh$ for 1-forms g and h. Then $\partial_{x_i} f = H\alpha + \beta G$ for $\alpha := \partial_{x_i} G \in k$ and $\beta := \partial_{x_i} H \in k$. Assume now that a point P lies in the intersection of $\mathcal{V}(H)$ and $V(G)$, then $\partial_{x_i} f(P) = H(P)\alpha + \beta G(P) = 0$, so the intersection is a singularity. Of course, in the projective plane there always exists an intersection for any two lines. (This is just a consequence of linear algebra: Say $H = a_0x_0 + a_1x_1 + a_2x_2, G = b_0x_0 + b_1x_1 + b_2x_2$, then the kernel of the matrix $\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$ is nontrivial and if $(s_0, s_1, s_2) \neq 0 \in k^3$ lies in the kernel, then $P := [s_0 : s_1 : s_2]$ lies in the intersection.)
The converse can be seen as follows. Let $P = [p_0 : p_1 : p_2]$ be a singularity and $P' = [p'_0 : p'_1 : p'_2]$ is any other point on the conic (for instance any intersection point of $\mathcal{V}(f) \cap \mathcal{V}(x_i)$). For f to contain the line through P and P' means that $f(\lambda P + \mu P') = 0 \in k[\lambda, \mu]$.

Again, Euler's equality shows itself to be quite useful in the calculation

$$0 = 2f(\lambda P') = \sum_{i=0}^{2} \lambda p'_i \partial_{x_i} f(\lambda P') + \sum_{i=0}^{2} \lambda p'_i \underbrace{\partial_{x_i} f(\mu P)}_{=0} \tag{10}$$

$$= \sum_{i=0}^{2} \lambda p'_i \partial_{x_i} f(\lambda p' + \mu p) \tag{11}$$

$$=2f(\lambda p' + \mu p') - \sum_{i=0}^{2} \mu p_i \partial_{x_i} f(\lambda P' + \mu P) \tag{12}$$

Finally I claim that the last sum disappears due to the equality

$$\sum_{i=0}^{2} p_i \partial_{x_i} f = 0 \in k[x_0, x_1, x_2] \tag{13}$$

The calculation is straight-forward:

$$\sum_{i=0}^{2} p_i \partial_{x_i} f = \sum_{i=0}^{2} p_i \sum_{j=0}^{2} \partial_{x_j} \partial_{x_i} f x_j \tag{14}$$

$$\overset{\text{rearrange}}{=} \sum_{j=0}^{2} x_j \sum_{i=0}^{2} \partial_{x_i} \partial_{x_j} f p_i = \sum_{j=0}^{2} x_j \partial_{x_j} f(P) = 0 \tag{15}$$

Now that we have shown the sum to disappear, we obtain $0 = 2f(\lambda P' + \mu P)$ (in characteristic not 2), hence the conic contains a line. This proves

Theorem 1.9. *Let k be a field of characteristic not 2 and $\mathcal{V}(f) \subset \mathbb{P}^2_k$ a conic given by a quadratic form $f = ax_0^2 + 2bx_0x_1 + cx_1^2 + 2dx_0x_2 + 2ex_1x_2 + fx_2^2$. Then the following are equivalent:*

1. *The conic is degenerate.*

2. *The quadratic form f factors into two linear forms, $f = gh$.*

3. *$\mathcal{V}(f)$ is a union of two lines.*

4. *$\mathcal{V}(f)$ is singular.*

5. $\det \begin{pmatrix} a & b & e \\ b & c & d \\ e & d & f \end{pmatrix} = acf + 2bde - ce^2 - ad^2 - b^2 f = 0$

1.5 the restriction of equations

One useful operation one may perform is to take the intersection of a hypersurface $S = \mathcal{V}(f) \subset \mathbb{P}^n_k$ with a hyperplane $\Pi = \mathcal{V}(h)$. Now one expects from geometric intuition that the d-form f 'restricts' to a d-form on Π where we think of Π as $\mathbb{P}^{n-1}_k$, unless S contains Π in which case one would expect the equation to 'restrict' to the zero polynomial. The goal of this section is to make this intuition precise, by defining a homomorphism $\mathrm{res} : k[x_0, ..x_n] \to k[x_0, ..x_{n-1}]$ and an isomorphism $\mathbb{P}^{n-1} \xrightarrow{\sim} \Pi \hookrightarrow \mathbb{P}^n$.

First we may assume without loss of generality that $h = \alpha x_n + \widetilde{h}$ where $0 \neq \alpha \in k$ and $\widetilde{h} \in k[x_0, ..x_{n-1}]$.
Then we define

$$\text{res}: \begin{cases} x_i \mapsto x_i, & \text{for } i \neq n \\ x_n \mapsto -\frac{1}{\alpha}\widetilde{h}. \end{cases} \tag{16}$$

and furthermore we extend it to

$$\widetilde{\text{res}}: k[x_0, ..x_n] \xrightarrow{\text{res}} k[x_0, ..x_{n-1}] \hookrightarrow k[x_0, ..x_n] \tag{17}$$

The isomorphism $\vartheta : \mathbb{P}_k^{n-1} \to \Pi$ we define as

$$\vartheta : [x_0 : .. : x_{n-1}] \mapsto [x_0 : .. : x_{n-1} : -\frac{1}{\alpha}\widetilde{h}(x_0, .., x_{n-1})] \tag{18}$$

One can easily see by evaluation that h vanishes on the image of ϑ, so ϑ maps into Π. To confirm, that we indeed defined an isomorphism we construct an inverse. A left-inverse of couse has to look like this:

$$\vartheta^{-1} : [x_0 : .. : x_n] \mapsto [x_0 : .. : x_{n-1}] \tag{19}$$

To show well-definedness, one needs to prove that $[x_0 : .. : x_{n-1}] = [0 : ... : 0]$ is not in the image. Suppose it were, then $0 = h(x_0, .., x_{n-1}) = \alpha x_n + \widetilde{h}(0)$ and therefore $x_n = -\frac{1}{\alpha}\widetilde{h}(0) = 0$ as well, so the preimage would have to be $[x_0 : .. : x_n] = [0 : .. : 0]$ which cannot happen. What we've also seen in the calculation so far is that the coordinate x_n depends uniquely on the other ones, hence ϑ^{-1} as defined above is a right-inverse.
Having everything defined we obtain the following relation of a form f and its restriction to the plane res(f): Let $P = [p_0 : .. : p_n] \in \Pi$ be a point on the plane, so $p_n = -\frac{1}{\alpha}\widetilde{h}(p_0, .., p_{n-1}) = -\frac{1}{\alpha}\widetilde{h}(\vartheta^{-1}(P))$.

$$f(P) = 0 \text{ iff } f(p_0, .., p_{n-1}, -\frac{1}{\alpha}\widetilde{h}(\vartheta^{-1}(P))) = 0 \tag{20}$$

$$\text{iff res}(f)(\vartheta^{-1}(P)) = 0 \tag{21}$$

$$(\text{ iff } \widetilde{\text{res}}(f)(P) = 0) \tag{22}$$

This allows us to understand the projective variety $\Pi \cap \mathcal{V}(f)$ in terms of the equation $\text{res}(f) = 0$ by $\Pi \cap \mathcal{V}(f) \simeq \mathcal{V}(\text{res}(f)) \subset \mathbb{P}_k^{n-1}$. Iterating this process we may consider further restrictions to $\mathbb{P}_k^{n-2}$ etc. (we will make this statement precise in corollary 1.11.
Another thing I want to point out is that the endomorphism $\widetilde{\text{res}}$ is idempotent (i.e. $\widetilde{\text{res}}.\widetilde{\text{res}} = \widetilde{\text{res}}$) and the kernel is the ideal generated by h: On one hand $\text{res}(h) = 0$, on the other hand if res maps a form f to 0, then f vanishes on Π, so h divides f (say, by Hilbert's Nullstellensatz).
In particular we obtain

Proposition 1.10. *Let k be algebraically closed. For any d-form $f \in k[x_0, .., x_n]$ and $\widetilde{\text{res}}$ as defined before there exists a $(d-1)$-form $\widetilde{f}$ such that*

$$f = \widetilde{\text{res}}(f) + h\widetilde{f} \tag{23}$$

TODO

- We can formulate the result in a slightly higher level of generality. So first we replace algebraic subsets of $\mathbb{P}^n_k$ by ideals $J \subset k[x_0,..x_n]$. Furthermore, algebraic subsets of $J = \mathcal{V}(h)$ correspond to ideals $I \subset k[x_0,..x_n]/J$.

Corollary 1.11. *Let $h_1,..h_m \in k[x_0,..x_n]$ be m k-linearly independent linear forms. Then there exists an isomorphism $k[x_0,..x_n]/(h_1,..h_m) \simeq k[x_0,..x_{n-m}]$. Furthermore the ideal $(h_1,..h_m) \subset k[x_0,..x_n]$ is radical.*

Proof. We prove the first assertion. The case $m = 1$ has been covered already. The inductive step goes as follows: Let $I = (h_1,..h_{m-1}), I+(h_m) = (h_1,..h_m)$ be ideals of $k[x_0,..x_n]$ and assume without loss of generality that the x_n coefficient of h_m is non-zero. The restriction homomorphism for the hyperplane $\mathcal{V}(h_m)$ gives us a short exact sequence

$$0 \to (h_m) \overset{\ker(\text{res})}{\to} k[x_0,..x_n] \overset{\text{res}}{\to} k[x_0,..x_{n-1}] \to 0 \tag{24}$$

Because res is an epimorphism, we get isomorphisms

$$\frac{k[x_0,..x_{n-1}]}{\text{res}((h_m))} \simeq \frac{\text{res}^{-1}(k[x_0,..x_{n-1}])}{\text{res}^{-1}(\text{res}((h_m))} = \frac{k[x_0,..x_n]}{I+(h_m)} = \frac{k[x_0,..x_n]}{(h_1,..h_m)} \tag{25}$$

Because the linear forms are k-linear independent, none of the forms $h_0,..h_{m-1}$ lie in the kernel I of the restriction homomorphism. Obviously the restriction homomorphism maps the linear forms h_i to linear forms $\text{res}(h_i) \neq 0$ which are linearly independent. Too see this, choose $\beta_i \in k$ such that $h_i = \text{res}(h_i) + \beta_i h_m$. Any non-trivial linear combination $0 = \sum_{i=0}^{m-1} \alpha_i \text{res}(h_i)$ of the $\text{res}(h_i)$, induces a non-trivial linear combination $0 = \sum_{i=0}^{m-1} \alpha_i (h_i - \beta_i h_m) = \sum_{i=0}^{m-1} \alpha_i h_i - (\sum_{i=0}^{m-1} \alpha_i \beta_i) h_m$ of the h_i. Hence we can apply the induction hypothesis on $\text{res}(I) = (\text{res}(h_0),..\text{res}(h_{m-1})) \subset k[x_0,..x_{n-1}]$, which concludes our proof.
The second claim is an easy corollary, as $k[x_0,..x_n]/(h_0,..h_m) \simeq k[x_0,..x_{n-m}]$ is reduced which is equivalent to the ideal $(h_0,..h_m)$ being radical.

□

1.6 lines on surfaces

So far we have defined a line to be an intersection of hyperplanes, but of course we also want to understand a line as the unique intersection of such hyperplanes containing two distinct points.
Let's fix a projective space $\mathbb{P}^n_k$ and let $P = [p_0 : ..p_n]$ and $Q = [q_0 : ..q_n]$ be distinct points in $\mathbb{P}^n_k$. Now consider the homomorphism

$$L : \begin{cases} k[x_0,..x_n] & \to k[\lambda,\mu] \\ f & \mapsto f(\lambda P + \mu Q) := f(\lambda p_0 + \mu q_0,..\lambda p_n + \mu q_n) \end{cases} \tag{26}$$

Its kernel contains those polynomials, which vanish on $\lambda P + \mu Q$ and in particular on any point $\lambda_0 P + \mu_0 Q$ for $[\lambda_0 : \mu_0] \in \mathbb{P}^1_k$, such as P and Q. These were the points we expected to be contained on the line anyhow. Consider the linear map $\bigoplus_{i=0}^{n} kx_i \to k\lambda \oplus k\mu$ defined by the $2\times(n+1)$-matrix

$$M = \begin{pmatrix} p_0 & \dots & p_n \\ q_0 & \dots & q_n \end{pmatrix} \tag{27}$$

Because P and Q are distinct points in projective space, the matrix has full rank 2, and hence the kernel is spanned by $(n-1)$ linear forms $h_0,..h_{n-2}$. By our choice of the matrix M, these linear forms lie in the kernel of L: $h_i(\lambda P+\mu Q) = h_i(P)\lambda + h_i(Q)\mu = 0\lambda+0\mu$. In fact, the kernel of L is spanned by these linear forms already, which follows from the *projective Nullstellensatz* which I will just mention here without proof:

TODO

- put the Nullstellensatz into the introductory section for projective space, because I reference it before this

Theorem 1.12 (Projective Nullstellensatz)**.** *For a set X of points in projective space $\mathbb{P}_k^n$ define its ideal $\mathscr{I}(X) := \left\{f \in k[x_0,..x_n] \mid f(x_0,..x_n) = 0\,\forall [x_0:..x_n] \in X\right\}$. Then $\mathscr{I}$ and $\mathscr{V}$ are inclusion reversing and for ideals J with $\mathscr{V}(J) \neq \emptyset$ we have $\mathscr{I}(\mathscr{V}(J)) = \sqrt{J}$.*

Assume now that $f \in \ker(L)$, then $\mathscr{V}(f) \supset \left\{\lambda P+\mu Q \mid \lambda,\mu \in k\right\} = \mathscr{V}(h_0,..h_{n-2})$ and hence $(f) \subset \sqrt{(f)} \subset \sqrt{(h_0,..h_n)} = (h_0,..h_n)$.

TODO

- Show that $\mathscr{V}(h_0,..h_{n-2}) = \{\lambda P+\mu Q\}$

2 proof of the main theorem

2.1 (7.1) bis Step 2

$S = \mathrm{V}^+(f) \subset \mathbb{P}^3$ sei eine nicht-singuläre Kubik mit $f \in k[x,y,z,t]$.

2.1.1 Behauptung: $l = T_P(l) \subset T_P(S)$

Wir erinnern uns an die Definition des Tangentialraumes $T_P(\mathrm{V}^+(I)) = \cap_{f\in I}\mathrm{V}^+(f_P)$. Weiters sei $l = \mathrm{V}^+(H_1,H_2) \subset S := \mathrm{V}^+(f)$ die Gerade auf der Kubik. Dann erhalten wir $l = \mathrm{V}^+(H_1) \cap \mathrm{V}^+(H_2) = \cap_{\alpha,\beta}\mathrm{V}^+(\alpha H_1+\beta H_2) \overset{\text{def}}{=} T_P(l)$. Die zweite Gleichung folgt aus der Tatsache, dass $\mathrm{I}^+(l) \supset \mathrm{I}^+(S)$, der Schnitt also über eine größere Menge stattfindet.

2.1.2 Behauptung: Sei $\mathrm{V}^+(H) \subset \mathbb{P}^3$ eine Ebene. (Diese Ebene soll nicht in der Kubik enthalten sein). Es gibt eine 3-Form $h \in k[x,y,z]$ und eine Inklusion $\iota : \mathbb{P}^2 \hookrightarrow \mathbb{P}^3$ sodass $\mathrm{V}^+(H) = \mathbb{P}^2$ und $S \cap \mathbb{P}^2 = \mathrm{V}^+(h) \subset \mathbb{P}^2$.

oBdA sei $H = t-\alpha x-\beta y-\gamma z$ mit $\beta,\gamma \in k$.
Mit homogenen Koordinaten:

(1) $\mathrm{V}^+(H)$ **ist isomorph zu** $\mathbb{P}^2$**.** Ein Punkt $[x_0 : x_1 : x_2 : x_3]$ liegt auf $\mathrm{V}^+(H)$ genau dann, wenn $x_3 = \alpha x_0 + \beta x_1 + \gamma x_2$. Hier kann man auch anmerken, dass x_0, x_1, x_2 nie gleichzeitig verschwinden können, da sonst auch $x_3 = 0$ folgen würde. Damit ist die Projektion $\pi : [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_1 : x_2]$ wohldefiniert. Es liegt nahe, eine Inverse $\mathbb{P}^2 \to \mathrm{V}^+(H)$ wie folgt zu definieren: $\pi^{-1} : [x_0 : x_1 : x_2] \mapsto [x_0 : x_1 : x_2 : \alpha x_0 + \beta x_1 + \gamma x_2]$. Die Abbildung ist wohldefiniert, denn x_1, x_2 verschwinden nicht simultan und $H(\pi^{-1}([x_0 : x_1 : x_2])) = 0$.

(2) varDer Schnitt $\mathrm{V}^+(H) \cap S$ **wird via** π **auf eine Varietät** $\mathrm{V}^+(h) \subset \mathbb{P}^2$ **transportiert, wobei** h **eine 3-Form ist.** Sei $[x_0 : x_1 : x_2] \in \mathbb{P}^2$ ein Punkt. $[x_0 : x_1 : x_2] \in \pi(\mathrm{V}^+(H) \cap S) \Leftrightarrow f(\pi^{-1}([x_0 : x_1 : x_2])) = 0 \Leftrightarrow f(x_0, x_1, x_2, \alpha x_0 + \beta x_1 + \gamma x_3) = 0$, d.h. $\pi(\mathrm{V}^+(H) \cap S) = \mathrm{V}^+(g)$ mit $g = f(x, y, z, \alpha x + \beta y + \gamma z) = \mathrm{eval}(—, (x, y, z, \alpha x + \beta y + \gamma z))(f)$ eine 3-Form.

2.1.3 Behauptung: Es gibt eine Darstellung von $f = h + HB$ mit h die obige "Restriktion" von f auf $\mathrm{V}^+(H)$ sowie B eine 2-Form.

Wir können die Auswertung an $\theta := (x, y, z, \alpha x + \beta y + \gamma z)$ ergänzen zu einem varHomomorphismus $k[x, y, z, t] \to k[x, y, z] \hookrightarrow k[x, y, z, t]$ mit Kern H. Weiters gilt $\mathrm{eval}(—,\theta)(f) = \mathrm{eval}(—,\theta).(\mathrm{eval}(—,\theta))($ also $f = \mathrm{eval}(f,\theta) + p = g + p$ wobei $p \in \ker(\mathrm{eval}(—,\theta)) = (H)$. Es gibt also eine Darstellung $p = HB$ für ein Polynom B. Nun nutzen wir aus, dass f eine $3 - Form$ ist und erhalten $f = g + HB$ wobei B eine quadratische Form ist.

2.1.4 Behauptung: Für $f = h + HB$ wie oben hat h nicht die Form $g^2 A$ mit g, A 1-Formen.

Angenommen dies wäre der Fall, also $f = g^2 A + HB$. Wir wollen zeigen, dass dann ein singulärer Punkt auf S existiert. Dieser erfüllt genau $f_x = f_y = f_z = f_t = 0$ (vgl. Shafarevich, BAG1). Wir brauchen eine verallgemeinerte Aussage.

2.1.5 LEMMA 1, LEMMA 1b, LEMMA 2

LEMMA 1: Sei R eine k-Algebra erzeugt durch $y_0, ..y_n$ algebraisch unabhängig. Dann gibt es varDerivationen $D_i : R \to k$ sodass $D_i(y_i) = 1$ und $D_i(y_j) = 0$ für $j \neq i$. Bezeichne diese varDerivationen als "partielle Ableitungen".
LEMMA 1b: Angenommen $V = \mathrm{span}(y_i) = \mathrm{span}(y'_i)$ als k-Vektorräume, dann gibt es eine Basiswechselmatrix M mit $(\vec{y}') = M(\vec{y})$ und $\vec{D}f = M\vec{D}f'$.
LEMMA 2: Sei f eine d-Form und $S = \mathrm{V}^+(f)$ eine Hyperfläche. Die singulären Punkte von S sind gegeben durch $S^{sing} = \mathrm{V}^+(f_{y_0}, ..f_{y_n})$. Hier bezeichnet f_{y_i} die Ableitung von f nach y_i, also $f_{y_i} := D_i(f)$.

(1) Siehe (Matsumura, 26.F).

(1b) Ich behaupte es gibt einen Iso von k-Vektorräumen $\mathrm{Hom}_k(V, k) \xrightarrow{\sim} \mathrm{Der}_k(R, k)$. Weiters induziert M als lineare Abbildung $V \to V$ einen Basiswechsel der Dualbasen: $\vec{y}'^* = M(\vec{y}^*)$.

(2) Sei $R = k[x_0, ..x_n]$ ein Polynomring über einem Körper k und d eine natürliche Zahl mit $char(k)$ teilt nicht d. Seien weiters $y_0, ..y_n \in R$ eine Basis der 1-Formen (notwendigerweise sind die y_i linear). Seien weiters $y'_0, ..y'_n \in R$ eine Basis der 1-Formen (notwendigerweise sind die y'_i linear). Anwendung von LEMMA 1 gibt und partielle Ableitungen D_i, D'_i. Ich behaupte, dass ein Punkt ist singulär bzgl der D_i, genau dann, wenn er singulär bzgl der D'_i ist. Dies folgt direkt aus LEMMA 1b. LEMMA 2 folgt dann aus (Shafarevich, BAG1) mit $y'_i = x_i$.

Weiter im Beweis Da f nicht komplett auf der Ebene $\mathrm{V}^+(H)$ verschwindet, sind g und H teilerfremd, bzw. k-linear unabhängig. Wir ergänzen zu einer Basis von $\mathrm{span}_k(x, y, z, t) = \mathrm{span}(H, g, r_1, r_2)$. Dann gelten

$$\begin{aligned} D_{r_i}(f) =& g^2 D_{r_i}(A) + HD_{r_i}(B) \\ D_H(f) =& g^2 D_H(A) + HD_H(B) + B \\ D_g(f) =& 2gA + g^2 D_g(A) + HD_g(B) \end{aligned}$$

Wir schränken die Ebene $\mathrm{V}^+(H)$ weiter ein auf die Gerade $\mathrm{V}^+(H, g)$, sodass die singulären Punkte von S auf der Geraden definiert sind durch $B = 0$. Da B aber entweder auf der geraden verschwindet oder eine 2-Form auf ihr ist, hat sie für k algebraisch abgeschlossen Nullstellen und es existieren somit singuläre Punkte, konträr zur Behauptung.

2.1.6 Behauptung: Sei C der Schnitt von $P \in S$ nicht-singulär mit der Tangentialebene $T_P(S)$, $S = \mathrm{V}^+(f)$ irreduzibel und nicht die Ebene C selbst, dann ist C singulär bei P.

Sei $H = f_x(P)x + f_y(P)y + f_z(P)z + f_t(P)t$ die Gleichung der Tangentialebene und oBdA $f_t(P) \neq 0$ (P war als nicht-singulär angenommen). Wir restringieren H via $\mathrm{eval}(—) := \mathrm{eval}(—, (x, y, z, -\frac{f_x(P)}{f_t(P)}x - \frac{f_y(P)}{f_t(P)}y - \frac{f_z(P)}{f_t(P)}z))$ und erhalten $f' = \mathrm{eval}(f)$. Demonstrativ leite ich nach x ab, aber dieselbe Rechnung funktioniert aus Symmetriegründen natürlich auch mit y, z:

$$f'_x = f_x D_x x + f_y D_x y + f_z D_x z + f_t D_x\left(-\frac{f_x(P)}{f_t(P)}x - \frac{f_y(P)}{f_t(P)}y - \frac{f_z(P)}{f_t(P)}z\right) = f_x - \frac{f_x(P)}{f_t(P)} f_t \tag{28}$$

Offensichtlich verschwindet f'_x in P: $f'_x(P) = f_x(P) - \frac{f_x(P)}{f_t(P)} f_t(P) = 0$.

2.2 (7.1) bis (7.2) Step 2

.

2.3 (7.2) Step 3

.

2.4 (7.2) Step 4

2.4.1 Behauptung: Sei $M \in R^{n\times n}$ eine Matrix über dem Polynomring R. Es gilt $\mathrm{det}(\mathrm{lt}(M)) = \mathrm{lt}(\mathrm{det}(M))$ g.d.w. $\mathrm{det}(\mathrm{lt}(M)) = \mathrm{lt}(\mathrm{det}(\mathrm{lt}(M)))$.

Das bedeutet natürlich, dass wir nur $\mathrm{lt}(\mathrm{det}(\mathrm{lt}(M))) = \mathrm{lt}(\mathrm{det}(M))$ zeigen brauchen. Lüge: Summen $x + y$ und Produkte xy in R vertauschen im Folgenden Sinne mit der lt-Operation: $\mathrm{lt}(x+y) = \mathrm{lt}(\mathrm{lt}(x)+\mathrm{lt}(y))$ und $\mathrm{lt}(xy) = \mathrm{lt}(\mathrm{lt}(x)\mathrm{lt}(y))$. Die Leibnitzformel liefert nun:

$$\begin{aligned}
\mathrm{lt}(\mathrm{det}(M)) = \quad & \mathrm{lt}\left(\sum_\sigma (-1)^\sigma \prod_i M_{i,\sigma(i)}\right) = \mathrm{lt}\left(\sum_\sigma \mathrm{lt}\left((-1)^\sigma \prod_i M_{i,\sigma(i)}\right)\right) \\
= \quad & \mathrm{lt}\left(\sum_\sigma \mathrm{lt}\left((-1)^\sigma \prod_i \mathrm{lt}(M_{i,\sigma(i)})\right)\right) = \mathrm{lt}\left(\sum_\sigma (-1)^\sigma \prod_i \mathrm{lt}(M_{i,\sigma(i)})\right) = \mathrm{lt}\left(\mathrm{det}(\mathrm{lt}(M))\right)
\end{aligned}$$