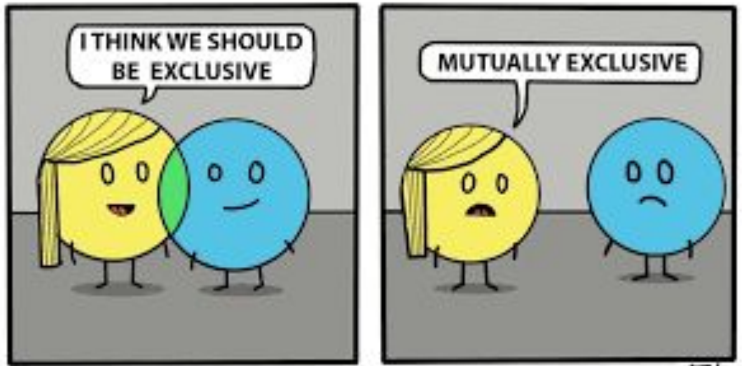


LESSON 9

Independence

CSCI 3022

Course Logistics: Your Fourth Week At A Glance

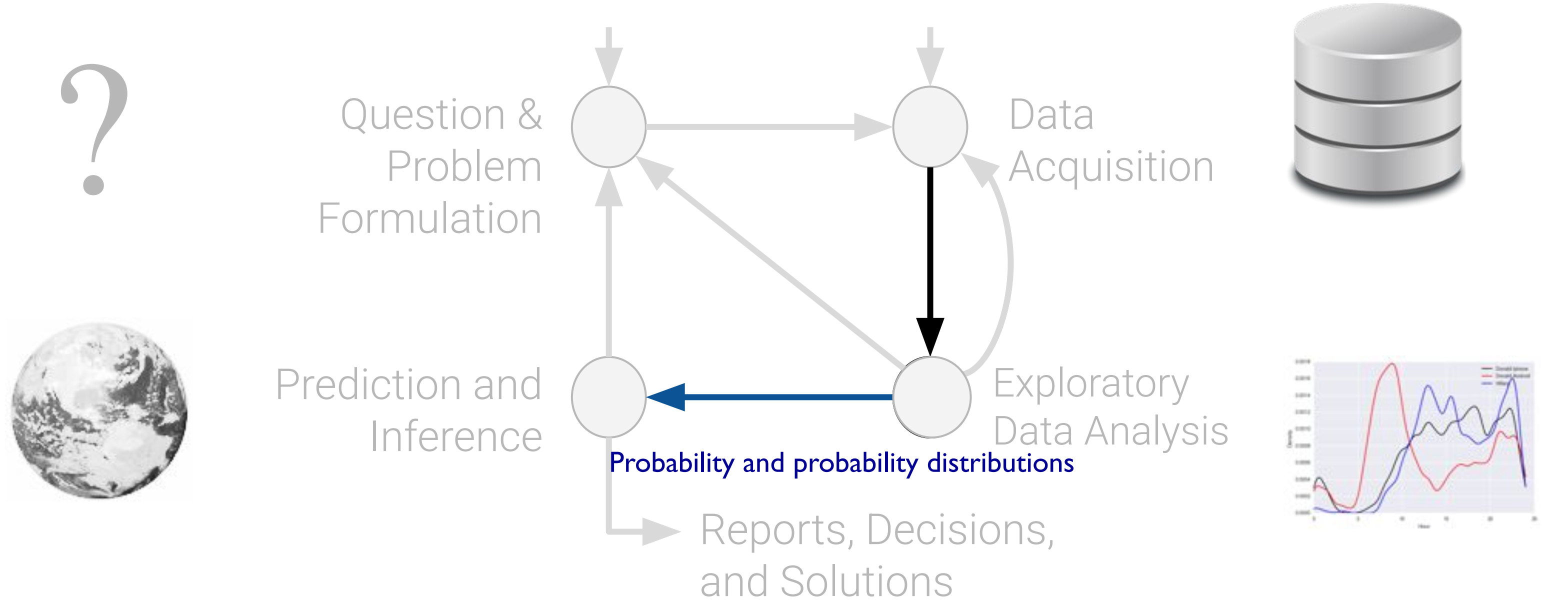
Mon 2/3	Tues 2/4	Wed 2/5	Thurs 2/6	Fri 2/7
Attend & Participate in Class		Attend & Participate in Class	HW 4 Due 11:59pm via Gradescope	In Class Quiz 3 (beginning of class): Scope: Lessons 1-6; HW 2 & HW 3 Attend & Participate in Class
	Quiz 2 grades posted		HW 3 feedback/ grades posted	HW 5 released

Quiz 3 Details

- Pencil and paper quiz will take place the 15 minutes of class on Friday
- Scope: Lessons 1-6 and all concepts covered in HW 2 & HW 3
  - There will be some questions where you will be asked to either fill-in-the blank for code or select which code will lead to a given output
- Study Tips:
  - Review the Lesson Learning Objectives listed in the Course Schedule on Canvas and make sure you can meet each of the objectives
  - Review the key concepts covered in each HW question - could you explain/apply the concepts you learned in those questions?

- You are allowed a calculator
- You are allowed a 2-sided 8.5” x11” crib sheet
  - (in addition you are also allowed to bring the Data Wrangling with Pandas Cheatsheet for any quiz/exam)

Plan for next few weeks



Probability and more specifically ***probability distributions*** are used to model data sets.

Roadmap

- Finish Lesson 8: (start at slide 20)

This Lesson:

- Independence

The Core Probability Toolkit

The Law of Total Probability

$$P(E) = P(E \text{ and } F) + P(E \text{ and } F^C)$$

$$P(E) = P(E|F)\,P(F) + P(E|F^C)\,P(F^C)$$

$$P(E) = \sum_{i=1}^{n} P(E \text{ and } B_i)$$

$$= \sum_{i=1}^{n} P(E|B_i)\,P(B_i)$$

Bayes' Theorem

$$P(B|E) = \frac{P(E|B) \cdot P(B)}{P(E)}$$

$$P(B|E) = \frac{P(E|B) \cdot P(B)}{P(E|B) \cdot P(B) + P(E|B^C) \cdot P(B^C)}$$

Definition of Conditional Probability

$$P(E|F) = \frac{P(E \text{ and } F)}{P(F)}$$

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

$$P(E^C) = 1 - P(E)$$

De Morgan's Laws

$(A \text{ or } B)^C = A^C \text{ and } B^C$

$(A \text{ and } B)^C = A^C \text{ or } B^C$

Axiom 3: If E and F are mutually exclusive, then $P(E \text{ or } F) = P(E) + P(F)$

Otherwise, use Inclusion-Exclusion:

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

Multiplication Rule

$$P(E \text{ and } F) = P(E|F) \cdot P(F)$$
$$= P(F|E) \cdot P(E)$$

Independence

$$P(E|F) = P(E)$$

$$P(E \text{ and } F) = P(E)\,P(F)$$

Learning Objectives:

- **State the definition of independent events**
- **Determine whether two events are independent.**

Independent Events

Independence:

- Definition
- Examples

Two events E and F are defined as independent if:

Otherwise E and F are called dependent events.

If E and F are independent, then:

Intuition via Proof

Statement:

If E and F are independent, then $P(E|F) = P(E)$.

Proof:

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of conditional probability

$$= \frac{P(E)P(F)}{P(F)}$$ Independence of E and F

$$= P(E)$$ Taking the bus to cancellation city

Knowing that F happened does not change our belief that E happened.

Example 4

Independent events E and F ⟺ $P(EF) = P(E)P(F)$, $P(E|F) = P(E)$

Suppose you flip a fair coin twice.

Define event H_1: coin lands on heads on the 1st flip
event H_2: coin lands on heads on the 2nd flip

Are H_1 and H_2 independent or dependent?

Example 4

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

Suppose you flip a fair coin twice.

Define event H_1: coin lands on heads on the 1st flip
event H_2: coin lands on heads on the 2nd flip

Are H_1 and H_2 independent or dependent?

Recall: a **joint probability table** gives the joint probability of each possible outcome

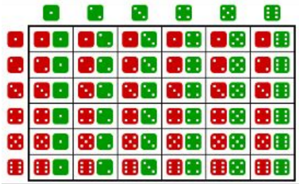
	H_1	T_1
H_2		
T_2		

$S = \{ (H_1,H_2), (T_1,H_2), (H_1,T_2), (T_1,T_2) \}$

$P(H_1,H_2) =$

Example 5

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

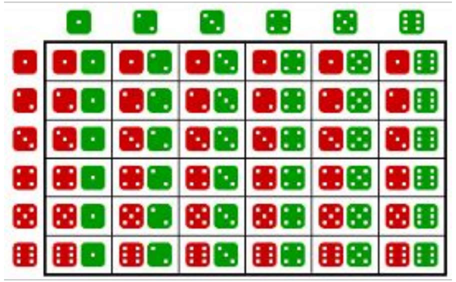


- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 5$

1. Are E and F independent?
2. Are E and G independent?

Example 5

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$



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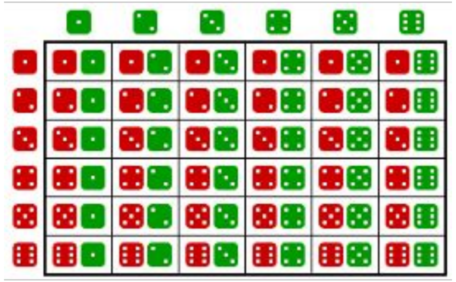


- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 5$

1. Are E and F independent?

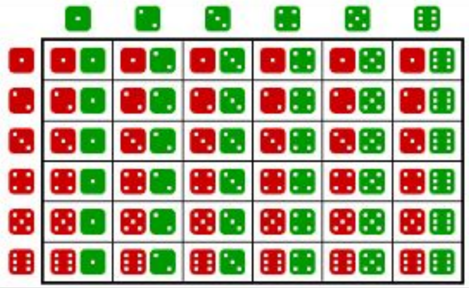
   Recall: a **joint probability table** gives the joint probability of each possible outcome

	E	E’
F		
F’		

2. Are E and G independent?

Example 5

Independent events E and F ⟺ $P(EF) = P(E)P(F)$, $P(E|F) = P(E)$



- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 5$

1. Are E and F independent?

Recall: a **joint probability table** gives the joint probability of each possible outcome

	E	E'
F	1/36	5/36
F'	5/36	25/36

Example 5

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
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1. Are E and F independent?

Recall: a **joint probability table** gives the joint probability of each possible outcome

	E	E'
F	1/36	5/36
F'	5/36	25/36

$P(E) = 1/6$
$P(F) = 1/6$
$P(EF) = 1/36$

✅ independent

Example 5

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
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- Roll two 6-sided dice, yielding values D_1 and D_2.
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Recall: a **joint probability table** gives the joint probability of each possible outcome

	E	E’
F	1/36	5/36
F’	5/36	25/36

$P(E) = 1/6$
$P(F) = 1/6$
$P(EF) = 1/36$

✅ independent

2. Are E and G independent?

	E	E’
G		
G’		

Example 5

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
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- Roll two 6-sided dice, yielding values D_1 and D_2.
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	E	E'
F	1/36	5/36
F'	5/36	25/36

$P(E) = 1/6$
$P(F) = 1/6$
$P(EF) = 1/36$
✅ independent

2. Are E and G independent?

	E	E'
G	1/36	3/36
G'	5/36	27/36

Example 5

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

- Roll two 6-sided dice, yielding values D_1 and D_2.
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	E	E'
F	1/36	5/36
F'	5/36	25/36

$P(E) = 1/6$
$P(F) = 1/6$
$P(EF) = 1/36$

✅ independent

2. Are E and G independent?

	E	E'
G	1/36	3/36
G'	5/36	27/36

$P(E) = 1/6$
$P(G) = 4/36 = 1/9$
$P(EG) = 1/36 \neq P(E)P(G)$

❌ dependent

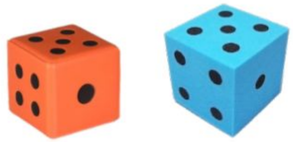
Example 5

- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$

  event F: $D_2 = 6$

  event G: $D_1 + D_2 = 5$

  event H: $D_1 + D_2 = 7$



$G = \{(1,4), (2,3), (3,2), (4,1)\}$

1. Are E and F independent?

   $P(E) = 1/6$

   $P(F) = 1/6$

   $P(E,F) = 1/36$

   ✅ <u>independent</u>

2. Are E and G independent?

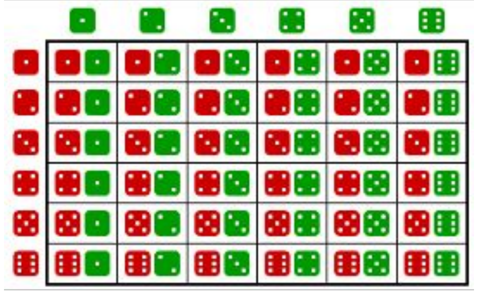
   $P(E) = 1/6$

   $P(G) = 4/36 = 1/9$

   $P(E,G) = 1/36 \neq P(E)P(G)$

   ❌ <u>dependent</u>

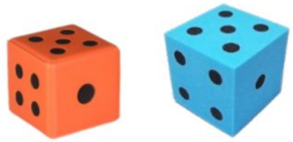
3. **Are E and H independent?**



Example 5

- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 5$ $\quad G = \{(1,4), (2,3), (3,2), (4,1)\}$

  event H: $D_1 + D_2 = 7$



1. Are E and F independent?

$$P(E) = 1/6$$
$$P(F) = 1/6$$
$$P(E,F) = 1/36$$

✅ independent

2. Are E and G independent?

$$P(E) = 1/6$$
$$P(G) = 4/36 = 1/9$$
$$P(E,G) = 1/36 \neq P(E)P(G)$$

❌ dependent

3. **Are *E* and *H* independent?**

We often are interested in experiments consisting of n independent trials.

- n trials, each with the same set of possible outcomes
- n-way independence: an event in one subset of trials is independent of events in other subsets of trials

Examples:

- Flip a coin n times
- Roll a die n times
- Send a multiple-choice survey to n people
- Send n web requests to k different servers

Generalizing Independence

Three events E, F, and G are independent if:

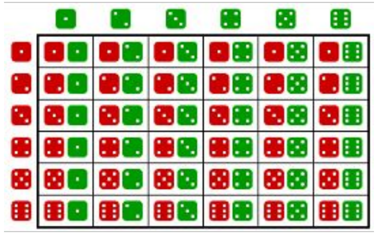
$P(E,F,G) = P(E)P(F)P(G)$, and
$P(E,F) = P(E)P(F)$, and
$P(E,G) = P(E)P(G)$, and
$P(FG) = P(F)P(G)$

n events $E_1, E_2, \dots, E_n$ are independent if:

for $r = 1, \dots, n$:
    for every subset $E_1, E_2, \dots, E_r$:
        $P(E_1 E_2, \dots, E_r) = P(E_1)P(E_2) \cdots P(E_r)$

Example 6

Pairwise independence is not sufficient to prove independence of >2 events!



- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event H $D_1 + D_2 = 7$

$H = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

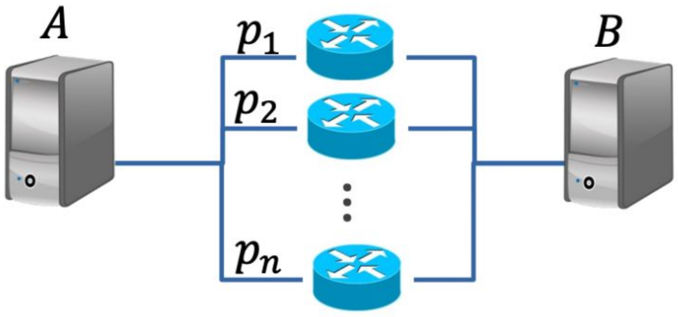
1. Are E and F independent? ✅
2. Are E and H independent?
3. Are F and H independent?
4. Are E, F, H independent?

Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- $E =$ functional path from A to B exists.

What is $P(E)$?



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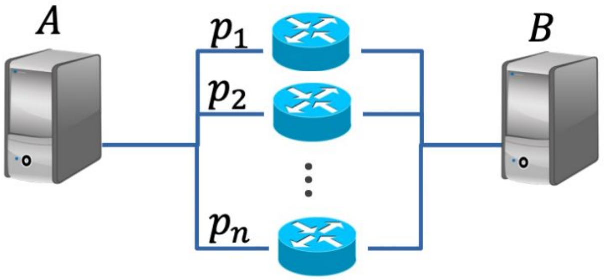


What is $P(E)$?

$$\begin{aligned} P(E) &= P(\geq 1 \text{ one router works}) \\ &= 1 - P(\text{all routers fail}) \\ &= 1 - (1 - p_1)(1 - p_2)\cdots(1 - p_n) \\ &= 1 - \prod_{i=1}^{n}(1 - p_i) \end{aligned}$$

≥ 1 with independent trials: take complement

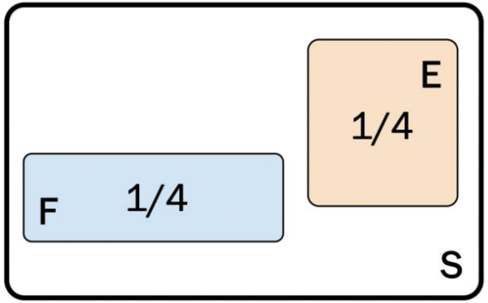
Example 8

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

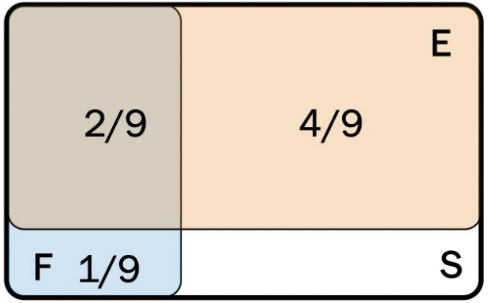
1. True or False? Two events E and F are independent if:
   A. Knowing that F happens means that E can't happen.
   B. Knowing that F happens doesn't change probability that E happened.

2. Are E and F independent in the following pictures?

A.



B.



Be careful:
- Independence is NOT mutual exclusion.
- Independence is difficult to visualize graphically.

Example 8

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   A. Knowing that F happens means that E can't happen.
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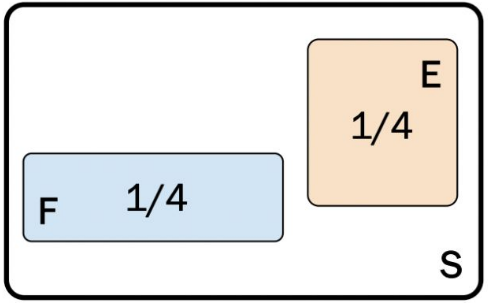
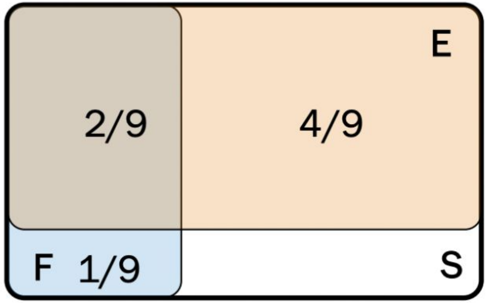


B.



Be careful:
- Independence is NOT mutual exclusion.
- Independence is difficult to visualize graphically.

Let P(E)>0, P(F)>0	Independent	Dependent
Disjoint	Not possible	If disjoint, then dependent
Not Disjoint	Possible! Need to test mathematically	Possible! Need to test mathematically

Sally Clark:

https://www.mcgrayne.com/disc.htm