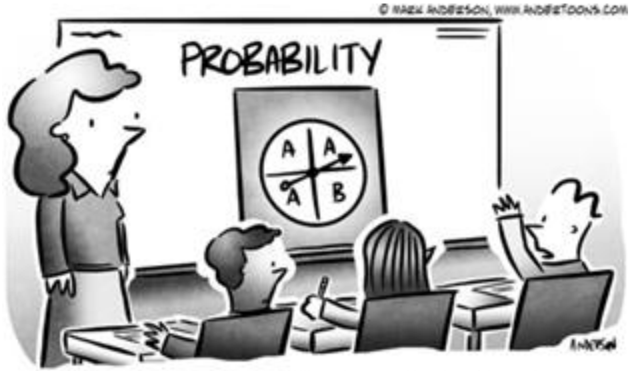


"I know mathematically that A is more likely, but I gotta say, I feel like B wants it more."

LESSON 7

Probability

Key concepts in probability and how to simulate probabilities using NumPy

CSCI 3022 @ CU Boulder

Maribeth Oscamou

Content credit: [Acknowledgments]

Course Logistics: Your Third Week At A Glance

Mon 1/27	Tues 1/28	Wed 1/29	Thurs 1/30	Fri 1/31
Attend & Participate in Class		Attend & Participate in Class	HW 3 Due 11:59pm via Gradescope	In Class Quiz 2 (beginning of class): Scope: Lessons 1-4; HW 2 Attend & Participate in Class
			HW 2 feedback/ grades posted	HW 4 released

Quiz 2 Details

- Pencil and paper quiz will take place the 15 minutes of class on Friday
- Scope: Lessons 1-4 and all concepts covered in HW 2
    - There will be some questions where you will be asked to either fill-in-the blank for code or select which code will lead to a given output
- Study Tips:
    - Review the Lesson Learning Objectives listed in the Course Schedule on Canvas and make sure you can meet each of the objectives
    - Review the key concepts covered in each HW question - could you explain/apply the concepts you learned in those questions?

- You are allowed a calculator
- You are allowed a 2-sided ***hand-written*** 8.5” x11” crib sheet
- You are also allowed to bring the Data Wrangling with Pandas Cheatsheet

Roadmap

Finish lesson 6:

EDA & Visualization Part 2 (starting at slide 32)

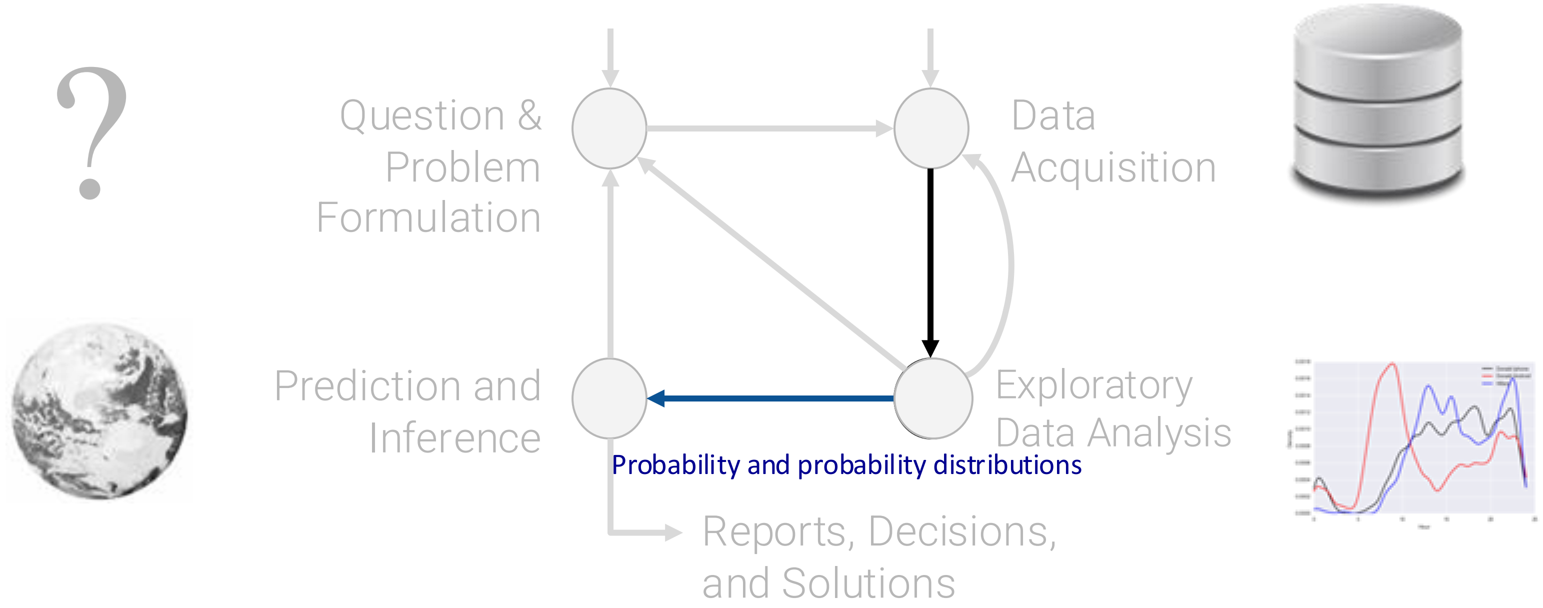
Start lesson 7: Probability

Lesson 7: Probability

- Probability
  - Philosophy & Axioms
  - Equally Likely Outcomes
  - Probability of OR
  - Probability of AND
  - Conditional Probability
  - General Problem Solving Techniques
  - Simulating Probabilities

- Supporting Materials
  - Set Theory Review (DeMorgan's Laws)

Plan for next few weeks



Probability and more specifically ***probability distributions*** are used to model data sets.

Philosophy and Axioms

- Probability
  - **Philosophy & Axioms**
  - Equally Likely Outcomes
  - Probability of OR
  - Probability of AND
  - Conditional Probability
  - General Problem Solving Techniques
  - Simulating Probabilities

- Supporting Materials
  - Set Theory Review (DeMorgan’s Laws)

Review from Discrete Structures: Probability Terminology

Experiment: A procedure that can be repeated, involves an element of chance, and has well defined outcomes.

Outcome: Result of an experiment

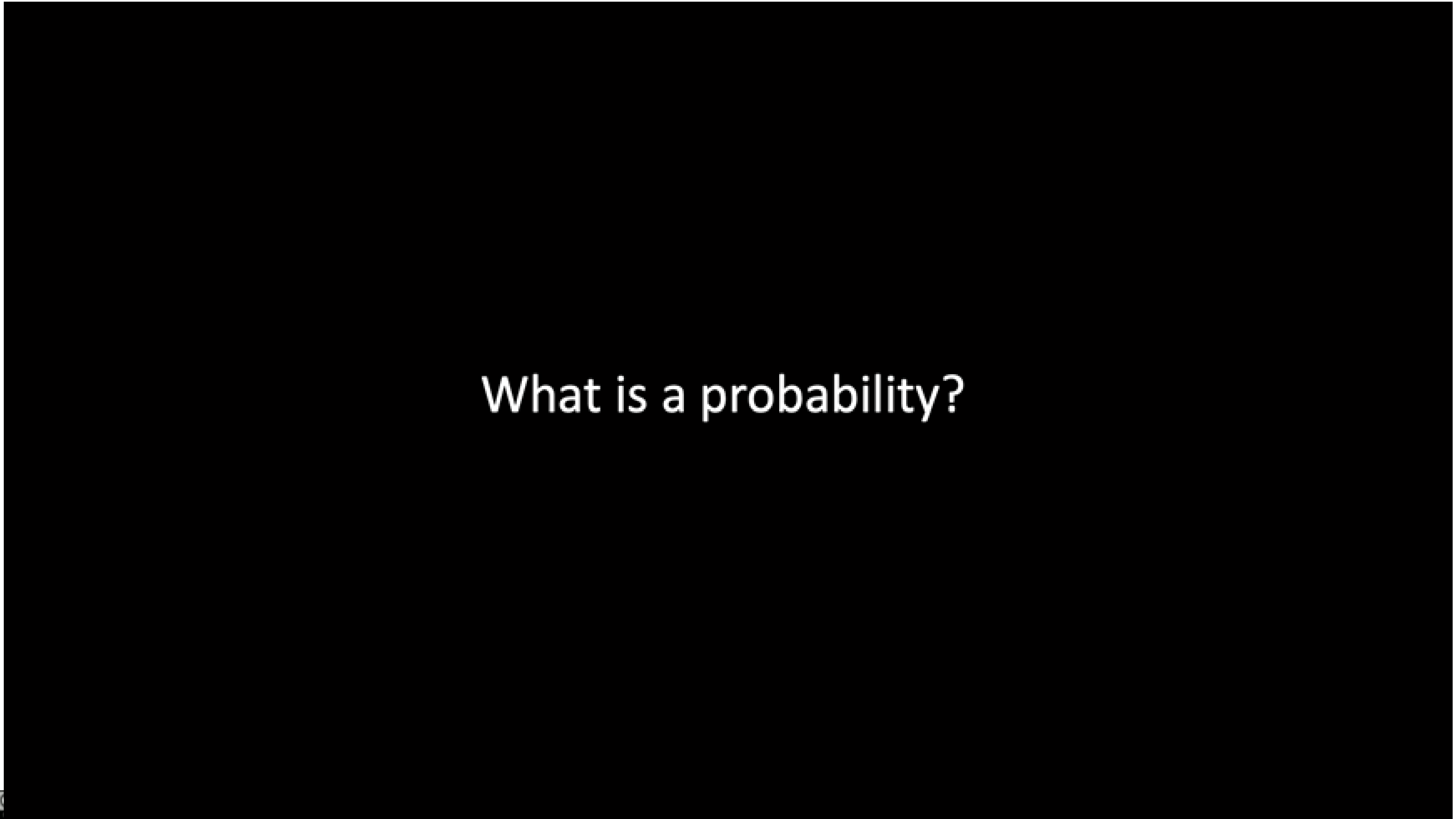
Sample Space, *S*: The **set** of all possible outcomes

Event, E: **Subset** of *S*

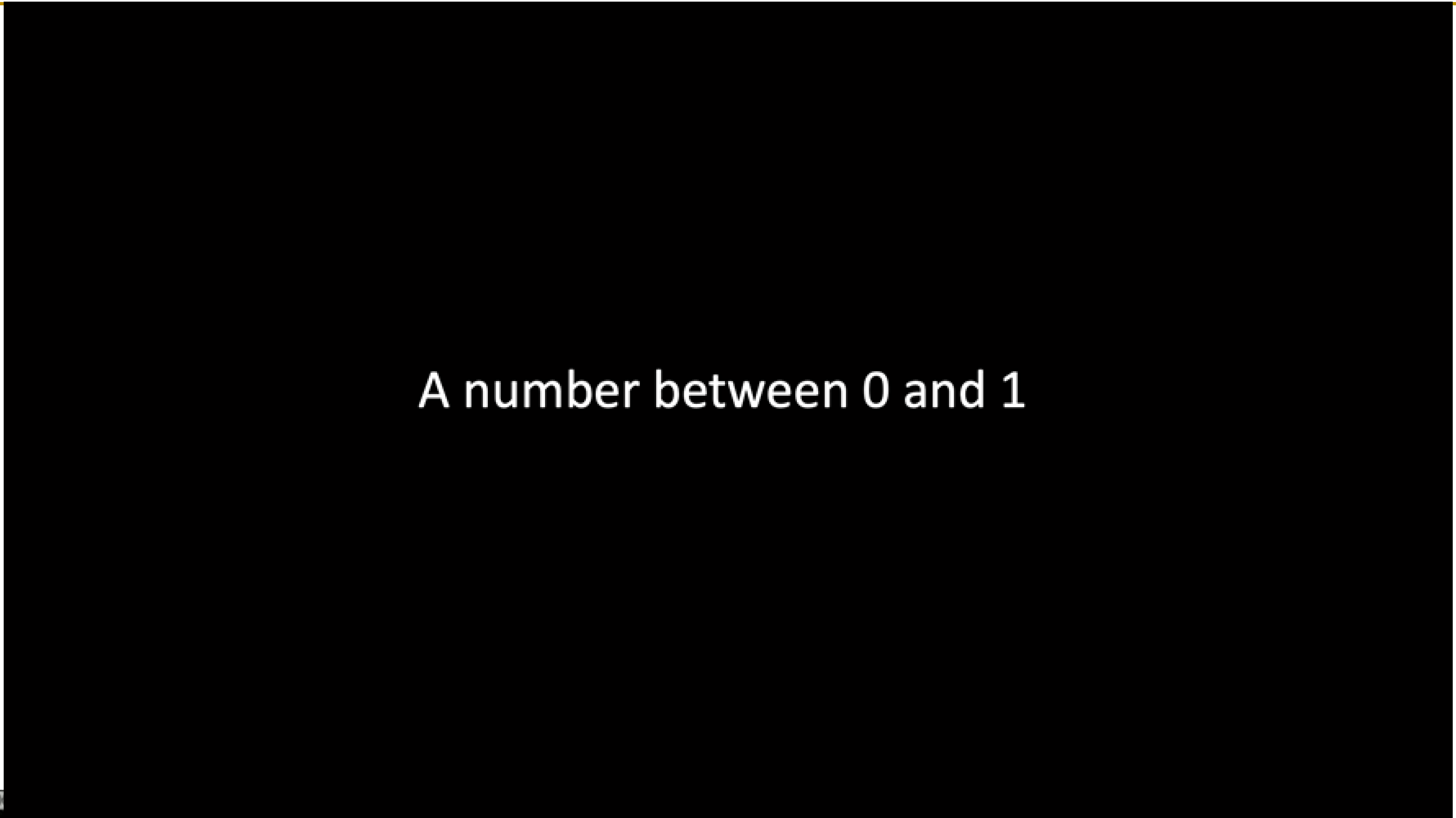
Sample Space (S) vs. Event Space (E)

Experiment	Sample Space	Event	Event Space
Flipping a coin	{Heads, Tails}	Getting heads	{Heads}
Rolling a dice	{1, 2, 3, 4, 5, 6}	At least 3	{3, 4, 5, 6}
Flipping two coins	{{H,H}, {H,T}, {T,H}, {T,T}}	One head	{{H,T}, {T,H}}
# inches of rain	$\{x \mid x \in \mathbf{Z}, x \geq 0\}$	Drought	$\{x \mid x \in \mathbf{Z}, 0 \leq x \leq 2\}$
# hours slept	$\{x \mid x \in \mathbf{R}, 0 \leq x \leq 24\}$	Good sleep	$\{x \mid x \in \mathbf{R}, 7 \leq x \leq 12\}$

Philosophy of Probability



Philosophy of Probability



But it's a number we ascribe meaning to!

$$P(E)$$

...represents our belief that event E occurs.

Philosophy of Probability

Objective (Frequentist view) Interpretation:
Defines probabilities as relative frequencies. So, what occurs in the long run is the probability.

$$\mathrm{P(Event)} = \lim_{n \to \infty} \frac{\mathrm{count(Event)}}{n}$$

Perform n trials of an "experiment" which could result in a particular “Event” occurring.

The probability of the event occurring, P(Event) , is the ratio of trials that result in the event, written as count(Event), to the number of trials performed, n.

In the limit, as your number of trials approaches infinity, the ratio will converge to the true probability.

Subjective (sometimes called Bayesian) Interpretation:
Defines probabilities as subjective degree of belief.

Philosophy of Probability

Objective (Frequentist view) Interpretation:

Defines probabilities as relative frequencies.
So, what occurs in the long run is the probability.

Subjective (sometimes called Bayesian) Interpretation:

Defines probabilities as subjective degree of belief.

$$P(\text{Event}) = \lim_{n\to\infty} \frac{\text{count}(\text{Event})}{n}$$

Ex: What does it mean when an insurance company says a new driver has a 20% probability of getting in at least an accident during their first year of driving?

Track all new drivers during the first year and record whether or not they get into an accident.. Over a long number of observations, the percent of those who do get in an accident converges to 20%.

Someone or some group (i.e. people who constructed insurance model) believe with 20% confidence a new driver will get in an accident during the first year

Intrigued? Check out STAT 4700: Philosophical and Ethical Issues in Statistics

Axioms of Probability

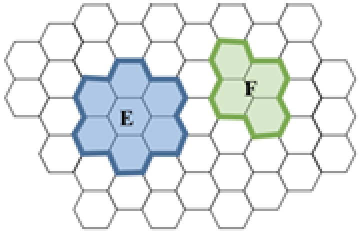
Here are some basic truths about probabilities that we accept as axioms:

Axiom 1: $0 \leq \mathrm{P}(E) \leq 1$ — All probabilities are numbers between 0 and 1.

Axiom 2: $\mathrm{P}(S) = 1$ — All outcomes must be from the Sample Space.

Axiom 3: If E and F are mutually exclusive, then $\mathrm{P}(E \text{ or } F) = \mathrm{P}(E) + \mathrm{P}(F)$ — The probability of "or" for mutually exclusive events

Two events, *E* and *F* are considered to be mutually exclusive if there are no outcomes that are in both events.



Example of two events: E, F, which are mutually exclusive.

Identity 1: $\mathrm{P}(E^{C}) = 1 - \mathrm{P}(E)$ — The probability of event E not happening

*Pf in supporting materials

Equally Likely Outcomes

- Probability
  - Philosophy & Axioms
  - **Equally Likely Outcomes**
  - Simulating Probabilities
  - Probability of OR
  - Probability of AND
  - Conditional Probability
  - General Problem Solving Techniques
- Supporting Materials
  - Set Theory Review (DeMorgan's Laws)

Equally Likely Outcomes

Some sample spaces have equally likely outcomes.

- Flipping one coin: S = {Head, Tails}
- Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
- Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}

Definition: Probability of Equally Likely Outcomes

If S is a sample space with equally likely outcomes, for an event E that is a subset of the outcomes in S:

$$\mathrm{P}(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}$$

Equally Likely Outcomes

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	Example 1
Experiment	Rolling a Fair 6-sided dice
Sample Space, S	{1, 2, 3, 4, 5, 6}
$\lvert S\rvert$	6
Event A: Event B: Event C:	Rolling a multiple of 3: Rolling a 5: Rolling an even number:

Ex 1:

a). What is P(A)?

b). What is P(B)?

c). What is P(C)?

Probability of “OR”

- Probability
  - Philosophy & Axioms
  - Equally Likely Outcomes
  - Simulating Probabilities
  - **Probability of OR**
  - Probability of AND
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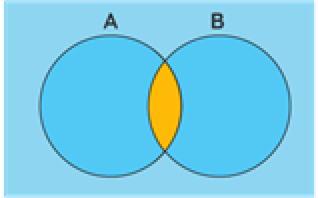
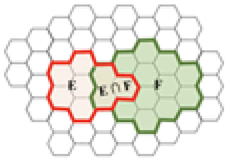
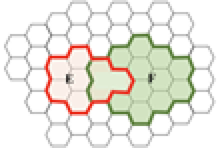
Notation: And vs Or



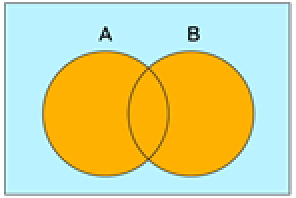
Shaded Portion in Orange is $A \cap B$



And i$\cap$tersection



Or $\cup$nion



Shaded Portion in Orange is $A \cup B$

$$P(E \text{ and } F)$$

$$P(E, F)$$

$$P(E \cap F)$$

$$P(E \text{ or } F)$$

$$P(E \cup F)$$

Definition: Inclusion Exclusion principle

For any two events: E, F:

$$\mathrm{P}(E \text{ or } F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \text{ and } F)$$

Equally Likely Outcomes

Definition: Probability of Equally Likely Outcomes

If S is a sample space with equally likely outcomes, for an event E that is a subset of the outcomes in S:

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	Example 1
Experiment	Rolling a Fair 6-sided dice
Sample Space, *S*	{1, 2, 3, 4, 5, 6}
|*S*|	6
Event *A*: Event *B*: Event *C*:	Rolling a multiple of 3: {3, 6} Rolling a 5: {5} Rolling an even number: {2, 4, 6}

Ex 1 cont'd:

c). What is P(AUB)?

d). What is P(AUC)? (Poll)

Inclusion Exclusion Principle

Definition: Inclusion Exclusion principle

For any two events: E, F:

$$\mathrm{P}(E \text{ or } F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \text{ and } F)$$

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$\|S\|$	6
Event A: Event B: Event C:	Rolling a multiple of 3: {3, 6} Rolling a 5: {5} Rolling an even number: {2, 4, 6}

Ex 1 cont'd:

d). What is $P(A \cup C)$?

Practice

Selecting Programmers

- P(student programs in Python) = 0.28
- P(student programs in C++) = 0.07
- P(student programs in Python and C++) = 0.05.

What is P(student does not program in (Python or C++))?

1. Define events & state goal
2. Identify known probabilities
3. Solve

Poll:

A). 0.95 **B). 0.70** **C). 0.6696**

D). 0.9894 **E). None of the above**

Practice

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A). 0.95 **B). 0.70** **C). 0.6696**
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A: Student programs in Python
B: Student programs in C++

Practice

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Poll:
A). 0.95 **B). 0.70** **C). 0.6696**
D). 0.9894 **E). None of the above**

1. Define events & state goal
2. Identify <u>known</u> probabilities
3. Solve

A: Student programs in Python
B: Student programs in C++

P(A) = 0.28
P(B) = 0.07
P(A, B) = 0.05

OPTION 1:

$$P((A \cup B)') = 1 - P(A \cup B)$$

Practice

Selecting Programmers

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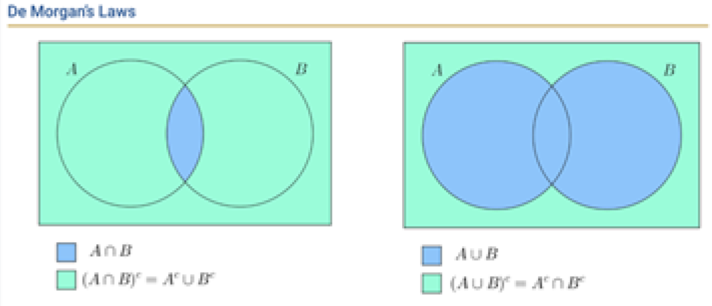
OPTION 2 - Recall DeMorgan's Laws:



$$P((A \cup B)') = P(A', B')$$

How do we find this!

Probability of "AND"

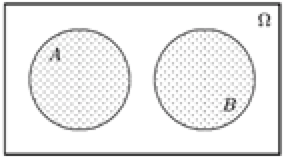
- Probability
  - Philosophy & Axioms
  - Equally Likely Outcomes
  - Simulating Probabilities
  - Probability of OR
  - **Probability of AND**
  - Conditional Probability
  - General Problem Solving Techniques
  - Simulating Probabilities
- Supporting Materials
  - Set Theory Review (DeMorgan's Laws)

Joint Probabilities

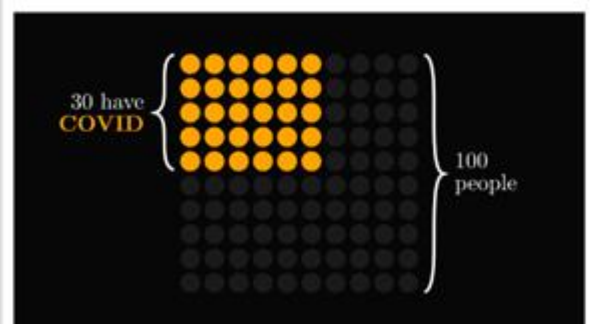
Definition: The probability that event A happens **AND** event B happens is called the **joint probability** of A and B:

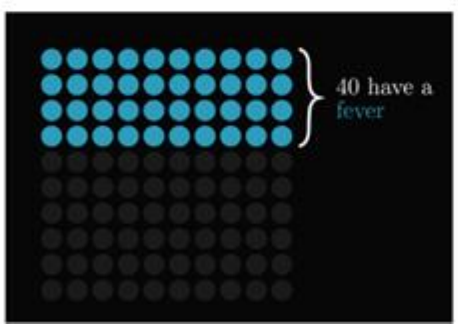
Notation: $P(A, B)$ *or* $P(A \cap B)$

Events A and B are called ***disjoint (or mutually exclusive)*** if and only if P(A, B) = 0



Disjoint sets A and B



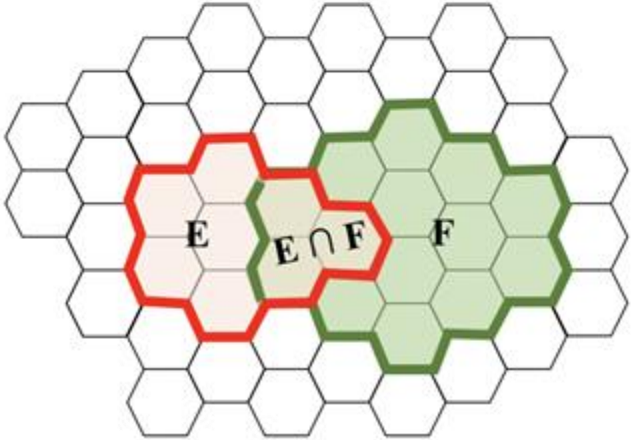


You randomly select a person from this group of 100 people.
What is the probability they have a fever **and** Covid?

Let's use a visualization to get some intuition. Consider events E and F which have outcomes that are subsets of a sample space with 50 equally likely outcomes, each one drawn as a hexagon:

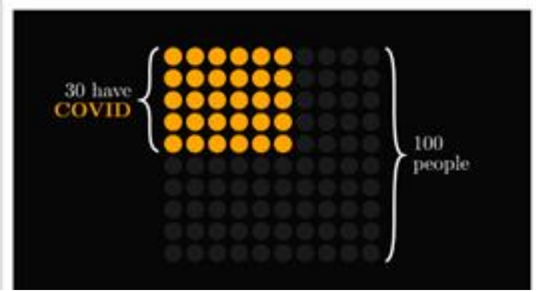
Law of Total Probability (Part 1):

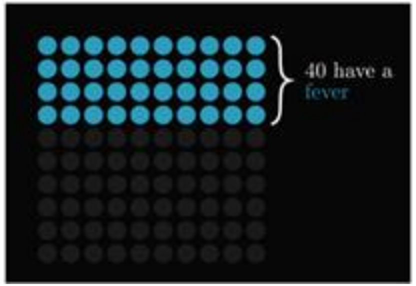
$$P(E) = P(E \text{ and } F) + P(E \text{ and } F^{C})$$



Even though the visual example (with equally likely outcome spaces) is useful for gaining intuition, the law of total probability applies regardless of whether the sample space has equally likely outcomes!

Joint Probabilities





You randomly select a person from this group of 100 people.

- Let F be the event that the person has a fever
- Let C be the event that the person has Covid

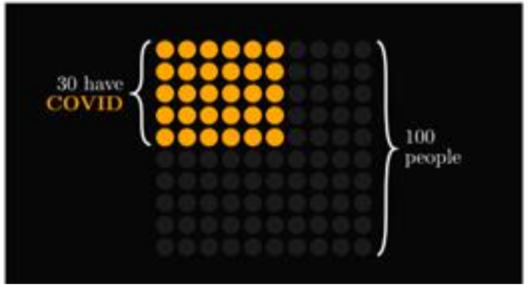
A **joint probability table** gives the joint probability of each possible outcome.

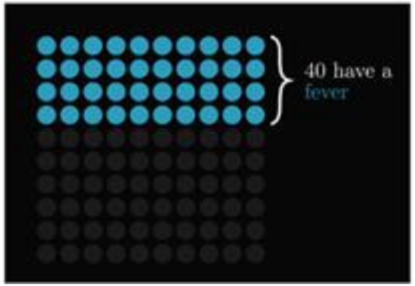
- Sum of all the joint probabilities (i.e. white boxes) in the table must add up to ______

Use the info above to fill in the joint probability table below.

	F	F’	Total
C	P(C, F)=	P(C, F’) =	
C’	P(C’, F) =	P(C’, F’)=	
Total			

Joint Probabilities





You randomly select a person from this group of 100 people.

- Let F be the event that the person has a fever
- Let C be the event that the person has Covid

A **joint probability table** gives the joint probability of each possible outcome.

- Sum of all the joint probabilities (i.e. white boxes) in the table must add up to 1.

Use the info above to fill in the joint probability table below.

	F	F'	Total
C	P(C, F)= 0.24	P(C, F') =0.06	P(C) = 0.30
C'	P(C', F) = 0.16	P(C', F')=0.54	P(C') = 0.70
Total	P(F) = 0.40	P(F') = 0.60	1

Back To Practice:

Selecting Programmers

- P(student programs in Python) = 0.28
- P(student programs in C++) = 0.07
- P(student programs in Python and C++) = 0.05.

What is P(student does not program in (Python or C++))?

1. Define events & state goal
2. Identify known probabilities
3. Solve

OPTION 2 - Recall DeMorgan's Laws:

$$P((A \cup B)') = P(A', B')$$

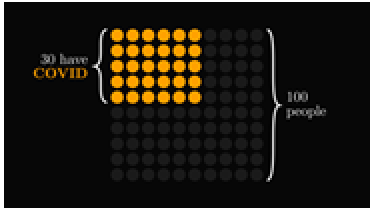
A: Student programs in Python
B: Student programs in C++

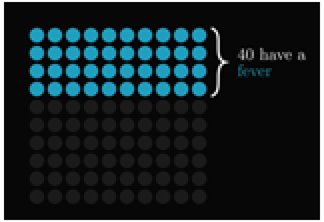
P(A) = 0.28
P(B) = 0.07
P(A, B) = 0.05

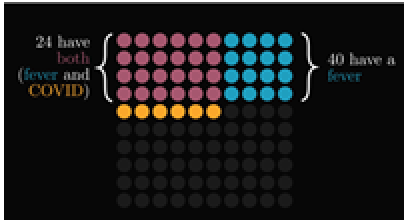
Fill in the joint probability table:

	A	A'	Total
B	P(B, A)=	P(B, A') =	P(B) =
B'	P(B', A) =	P(B', A')=	P(B') =
	P(A) =	P(A') =	1

Example







	F	F'
C	0.24	0.06
C'	0.16	0.54

You randomly select a person from this group of 100 people.

a). What's the probability the person has covid?

b). What's the probability that the person has covid **and** a fever?

c). ***Given*** *that the person has a fever*, what's the probability the person has covid?

Simulating Probabilities

- Probability
  - Philosophy & Axioms
  - Equally Likely Outcomes
  - **Simulating Probabilities**
  - Probability of OR
  - Probability of AND
  - Conditional Probability
  - General Problem Solving Techniques
- Supporting Materials
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Simulating Probabilities - Jupyter Demo

Demo Slides

Conditional Probability

- Probability
  - Philosophy & Axioms
  - Equally Likely Outcomes
  - Simulating Probabilities
  - Probability of OR
  - Probability of AND
  - **Conditional Probability**
  - General Problem Solving Techniques
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Joint Probabilities vs Conditional Probabilities

JOINT PROBABILITY:
The probability that event A happens **AND** event B happens is called the **joint probability** of A and B:

Notation: $P(A,B)$ *or* $P(A \cap B)$

Ex: P(CS, Third year) = *chance of a* ***random student*** *being a CS major* ***and*** *a third year*

CONDITIONAL PROBABILITY:
The probability that event A happens **GIVEN** that event B has already happened is called the **conditional probability** of A given B:

Notation: $P(A \mid B)$

$(the\ stuff\ after\ \mid\ \ is\ given)$

Ex: P(CS **|** Third year) = *chance of a* ***random third year student*** *being a CS major*

Poll: **Let A and B be non-disjoint events where 0<P(A)< 1 and 0< P(B) < 1.**
Assume P(B)<P(S)
Which is larger?

A). P(A, B) **B). P(A | B)** **C). Depends on A and B**

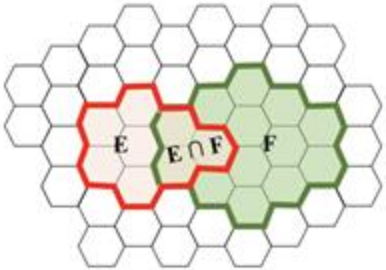
Conditional Probability

Definition: Conditional Probability.

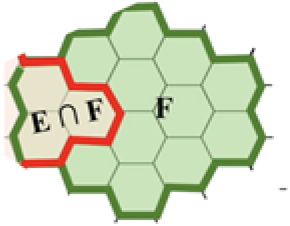
The probability of E given that (aka conditioned on) event F already happened:

$$P(E|F) = \frac{P(E \text{ and } F)}{P(F)}$$

Let's use a visualization to get an intuition for why the conditional probability formula is true. Again consider events E and F which have outcomes that are subsets of a sample space with 50 equally likely outcomes, each one drawn as a hexagon:



$$P(E|F) = \frac{P(E \text{ and } F)}{P(F)} = \frac{3/50}{14/50} = \frac{3}{14} \approx 0.21$$



Even though the visual example (with equally likely outcome spaces) is useful for gaining intuition, conditional probability applies regardless of whether the sample space has equally likely outcomes!

Joint vs Conditional Probabilities

	F	F'	Total
C	P(C, F)= 0.24	P(C, F') =0.06	P(C) = 0.30
C'	P(C', F) = 0.16	P(C', F')=0.54	P(C') = 0.70
Total	P(F) = 0.40	P(F') = 0.60	1

$$P(C, F) = \quad 0.24$$

$$P(C \mid F) = \frac{0.24}{0.40} = \frac{P(C, F)}{P(F)} = 0.60$$

Multiplication Rule

Notice:

$$P(C \mid F) = \frac{P(C, F)}{P(F)}$$

	F	F'	Total
C	0.24	0.06	0.30
C'	0.16	0.54	0.70
Total	0.40	0.60	1

$$P(F \mid C) = \frac{P(C, F)}{P(C)}$$

	F	F'	Total
C	0.24	0.06	0.30
C'	0.16	0.54	0.70
Total	0.40	0.60	1

Rearranging terms leads us to the **Multiplication Rule for Joint Probabilities:**

$$P(C, F) = P(F)\,P(C \mid F) = P(C)\,P(F \mid C)$$

Multiplication Rule

The multiplication rule:

$$P(A,B) = P(A)P(B|A) = P(B)P(A|B)$$

In words:

Chance that two events *A* and *B* both happen= P(*A* happens) x P(*B* happens given that *A* has happened)

= P(B happens) x P(A happens given that B has happened)

- The answer is *less than or equal to* each of the two chances being multiplied
- The more conditions you have to satisfy, the less likely you are to satisfy them all

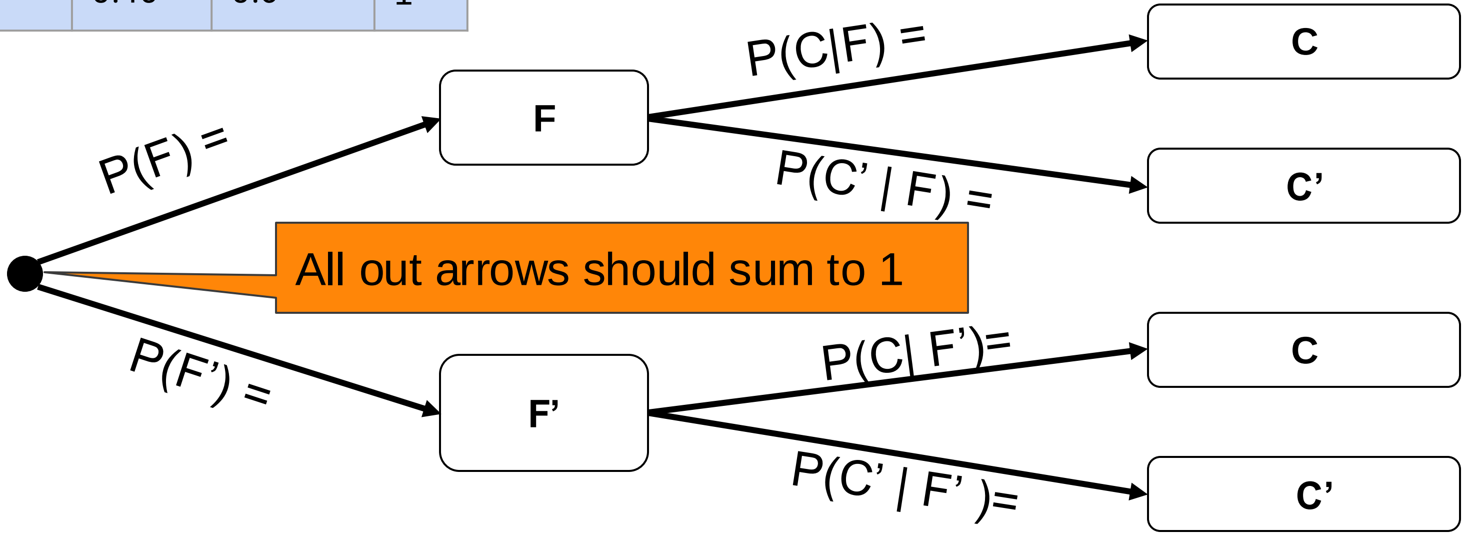
This generalizes to more than 2 events:

$$P(A,B,C,D) = P(A)P(B\,|A)\,P(C\,|\,A,B)\,P(D|\,A,B,C)$$

Tree Diagram

	F	F'	Total
C	0.24	0.06	0.3
C'	0.16	0.54	0.7
Total	0.40	0.6	1

Tree Diagram: Depicts each event and the probability of the next event **given** the current event.



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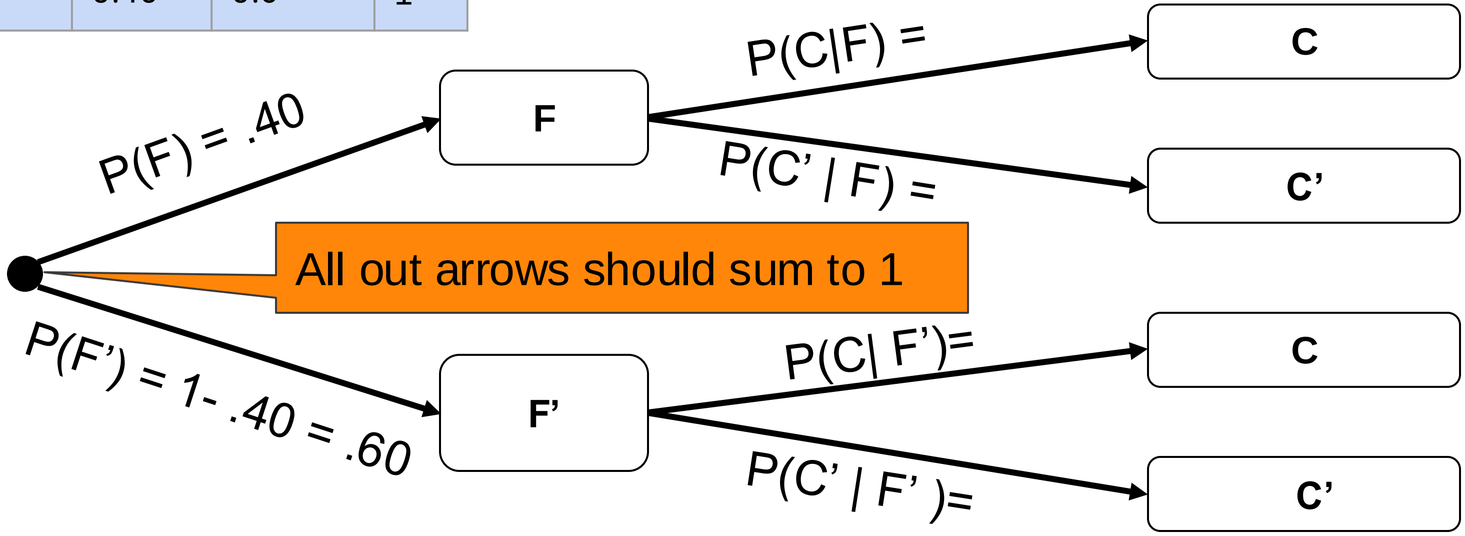

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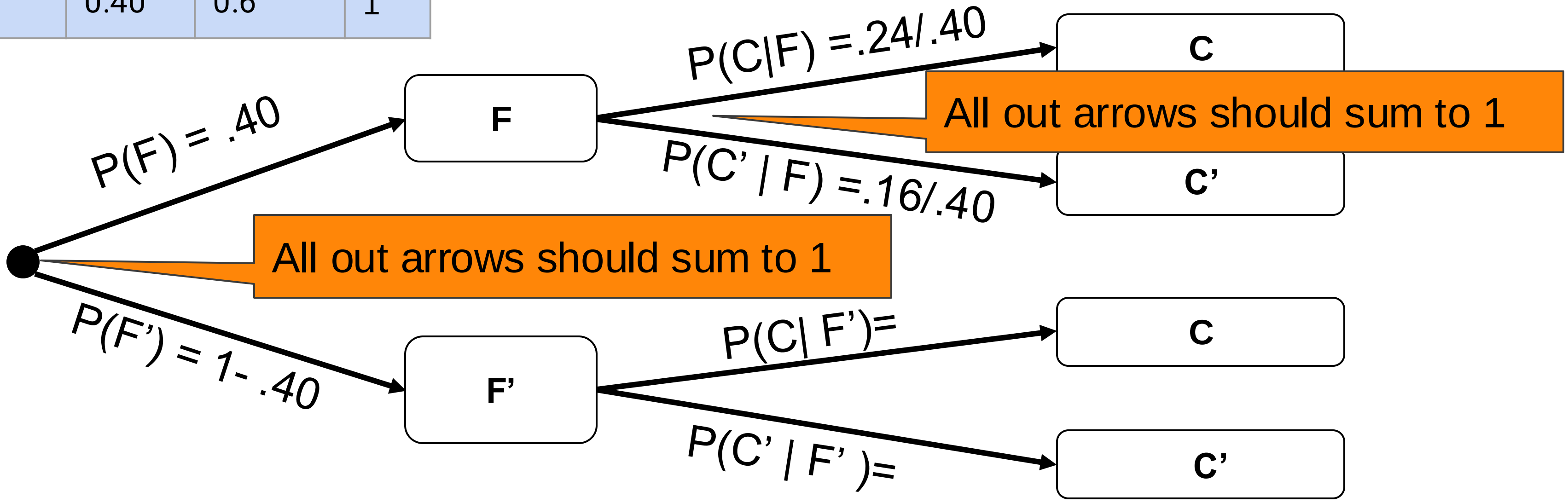

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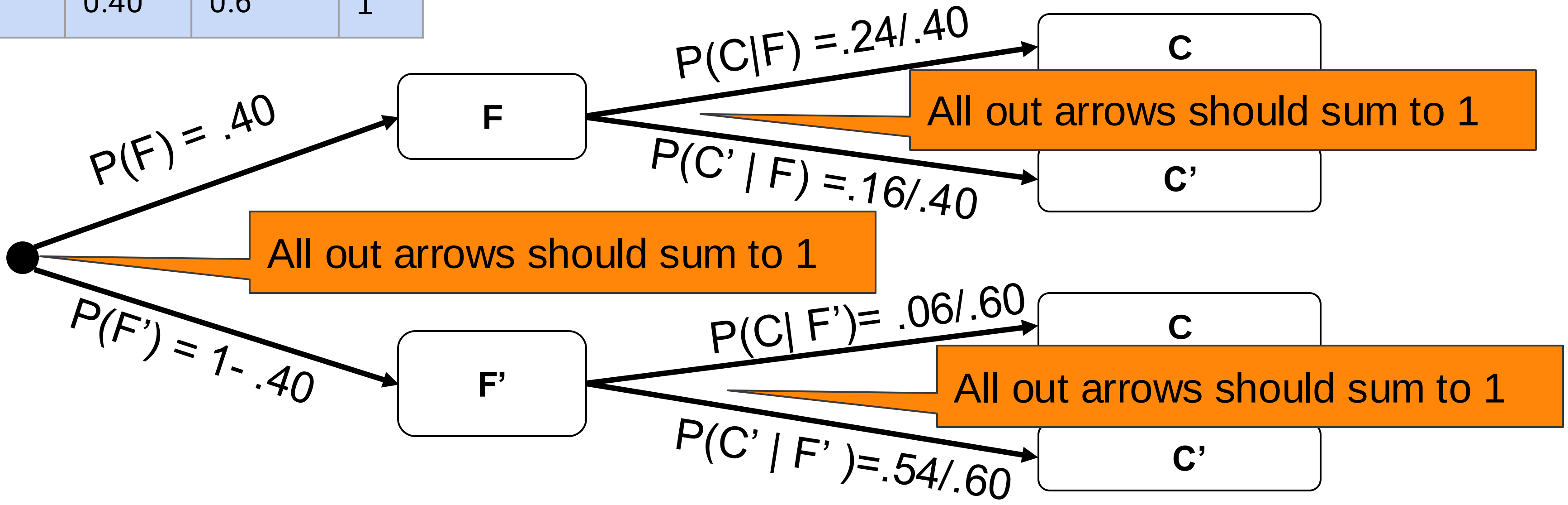
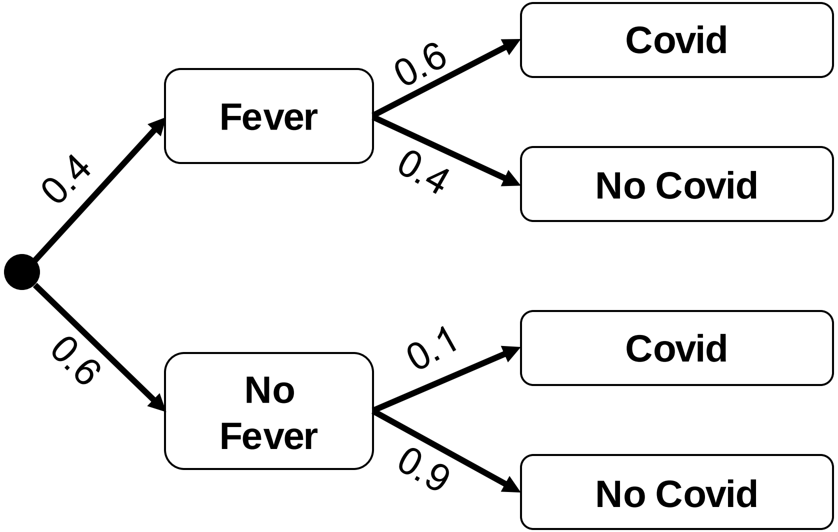


These two representations contain the same information.



joint probability table:

	Fever	No Fever
Covid	0.24	0.06
No Covid	0.16	0.54

Probability Problem Solving Techniques

- Probability
  - Philosophy & Axioms
  - Equally Likely Outcomes
  - Simulating Probabilities
  - Probability of OR
  - Probability of AND
  - Conditional Probability
  - **General Problem Solving Techniques**

- Supporting Materials
  - Set Theory Review (DeMorgan's Laws)

General Problem Solving Technique

Calculating Probabilities Involving Trials of Events:

Ask yourself what event must happen on the first trial:

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Ask yourself what event must happen on the first trial:

- If there's a clear answer (e.g. "not a six") whose probability you know, you can most likely use the **multiplication rule**.

General Problem Solving Technique

The multiplication rule:

$$P(A,B) = P(A)P(B|A) = P(B)P(A|B)$$

In general for more than 2 events:

$$P(A,B,C,D) = P(A)P(B\,|A)\,P(C\,|\,A,B)\,P(D|\,A,B,C)$$

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- If there's <u>no clear answer</u> (e.g. "could be King or Queen, but then the next one would have to be Queen or King …"), list all the **distinct** ways your event could occur and **add** up their chances

General Problem Solving Technique

The **multiplication rule**: $P(A,B) = P(A)P(B\|A) = P(B)P(A\|B)$ In general for more than 2 events: $P(A,B,C,D) = P(A)P(B\|A)\,P(C\|A,B)\,P(D\|A,B,C)$	**Calculating Probabilities Involving Trials of Events:** *Ask yourself what event must happen on the first trial:* • If there's a clear answer (e.g. "not a six") whose probability you know, you can most likely use the **multiplication rule**.
Definition: Inclusion Exclusion principle For any two events: E, F: $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$	• If there's no clear answer (e.g. "could be King or Queen, but then the next one would have to be Queen or King …"), list all the **distinct** ways your event could occur and **add** up their chances

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General Problem Solving Technique

	Calculating Probabilities Involving Trials of Events: *Ask yourself what event must happen on the first trial:*
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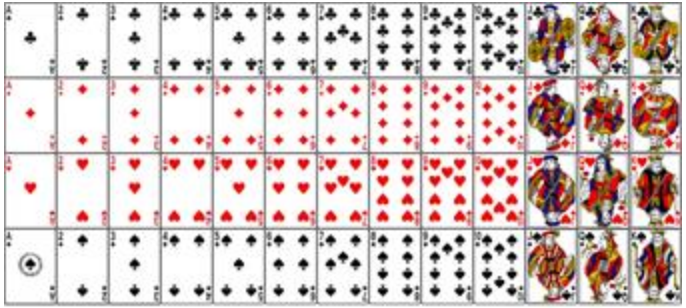
The complement rule: $P(A) = 1 - P(A')$	● If the <u>list above is long</u> and complicated, look at the **complement**. If the complement is simpler (e.g. the complement of "at least one" is "none"), you can find its chance and subtract that from 1.

Practice Question 1:



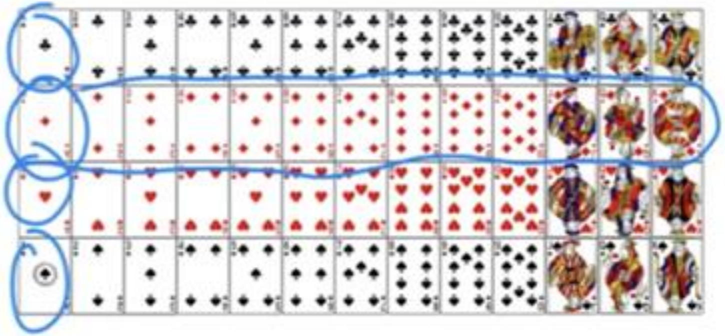
A standard 52-card deck consists of 13 cards in each of four *suits*: clubs, diamonds, hearts and spades

Suppose you draw a single card at random from a standard 52-card deck.

a). What is the probability that the card is **the** Ace of Diamonds?

b). What is the probability that the card is an Ace or a Diamond?

Practice Question 1:



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Suppose you draw a single card at random from a standard 52-card deck.

a). What is the probability that the card is **the** Ace of Diamonds?

E: Ace of Diamonds $\quad |E| = 1$

S: $\quad |S| = 52$

$$P(E) = \frac{|E|}{|S|} = \frac{1}{52}$$

equally likely outcomes

b). What is the probability that the card is an Ace or a Diamond?

E: Ace or Diamond: $\quad |E| = 16 \quad \Rightarrow \quad P(E) = \frac{16}{52} = \frac{4}{13}$

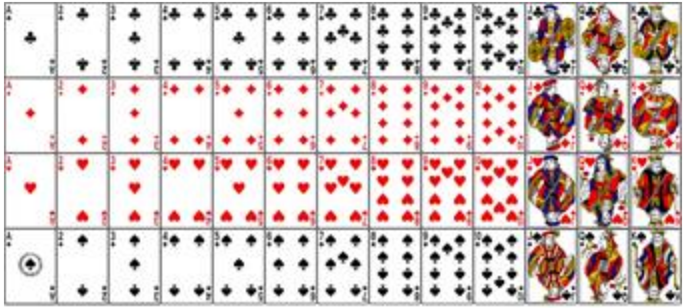
$|S| = 52$

Option 2: $P(A \cup D) = P(A) + P(D) - P(A,D) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

Practice Question 2:

A standard 52-card deck consists of 13 cards in each of four *suits*: clubs, diamonds, hearts and spades

I shuffle them and draw one card (without replacement) and then draw a 2nd card.



a). What is the chance that I get an Ace followed by a King?

b). What is the chance that one of the cards I draw is a King and the other is an Ace?

Practice Question 2:

A standard 52-card deck consists of 13 cards in each of four *suits*: clubs, diamonds, hearts and spades

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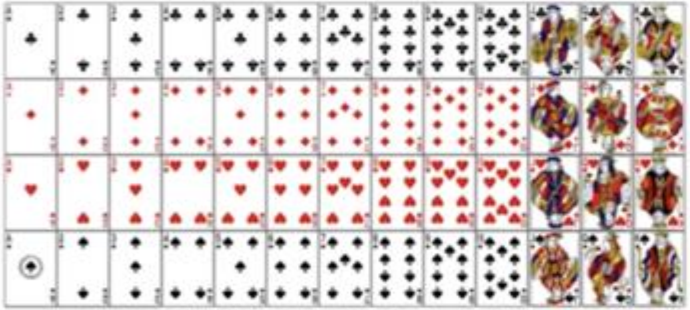

a). What is the chance that I get an Ace followed by a King?

$$P(1st\ A, 2nd\ K) = P(1st\ A) \cdot P(2nd\ K \mid 1st\ A)$$
$$= \left(\frac{4}{52}\right)\left(\frac{4}{51}\right) = \frac{16}{52 \cdot 51} \approx .006$$

b). What is the chance that one of the cards I draw is a King and the other is an Ace?

option 1:

$$P(1st\ A, 2nd\ K) = \frac{16}{52 \cdot 51}$$
$$+\ P(1st\ K, 2nd\ A) = \left(\frac{4}{52}\right)\cdot\left(\frac{4}{51}\right) = \frac{16}{52 \cdot 51}$$
$$\Rightarrow P(\text{one King one Ace}) = \frac{2 \cdot 16}{52 \cdot 51} \approx .012$$

option 2:

E: 1 Ace, 1 King unordered

S: 2 cards out of 52 unordered

$|S|: \binom{52}{2}$ $\quad |E| = \binom{4}{1}\cdot\binom{4}{1}$

Practice Question 5:

Suppose you have a biased coin, with probability of heads given by **q**

a). You flip the coin 5 times. What is the probability that you get the sequence HTTHT?

b). You flip the coin 5 times. What is the probability that you get 2 heads?

Practice Question 5

Suppose you have a biased coin, with probability of heads given by **q**

a). You flip the coin 5 times. What is the probability that you get the sequence HTTHT?

$$
\begin{aligned}
P(\text{you get the sequence } HTTHT) &= P(H_1, T_2, T_3, H_4, T_5) \text{ (because this is a joint probability)} \\
&= P(H_1)P(T_2|H_1)P(T_3|H_1, T_2)P(H_4|H_1, T_2, T_3)P(T_5|H_1, T_2, T_3, H_4) \text{ (by the multiplication rule)} \\
&= q(1-q)(1-q)q(1-q) \\
&= q^2(1-q)^3
\end{aligned}
$$

b). You flip the coin 5 times. What is the probability that you get 2 heads?

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Ways you can get 2 heads:

$$
\begin{aligned}
E1 &= HTTHT \\
E2 &= HHTTT \\
E3 &= HTTTH \\
E4 &= HTHTT \\
E5 &= THHTT \\
E6 &= THTHT \\
E7 &= THTTH \\
E8 &= TTHHT \\
E9 &= TTHTH \\
E10 &= TTTHH
\end{aligned}
$$

Practice Question 5

Suppose you have a biased coin, with probability of heads given by **q**

a). You flip the coin 5 times. What is the probability that you get the sequence HTTHT?

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b). You flip the coin 5 times. What is the probability that you get 2 heads?

Ways you can get 2 heads:

$$\begin{aligned} E1 &= HTTHT \\ E2 &= HHTTT \\ E3 &= HTTTH \\ E4 &= HTHTT \\ E5 &= THHTT \\ E6 &= THTHT \\ E7 &= THTTH \\ E8 &= TTHHT \\ E9 &= TTHTH \\ E10 &= TTTHH \end{aligned}$$

For this problem we're interested in different possible ways we can get exactly 2 heads when flipping a coin 5 times.

We've provided the full list, but we don't actually need the list, just the number of ways.
We can count this using combinations. The number of ways is equivalent to $\binom{5}{2} = \frac{5!}{2!3!} = 10$

$$\begin{aligned} P(\text{exactly two heads}) &= P(E1 \cup E2 \cup E3 \cup E4 \ldots \cup E10) \\ &= P(E1) + P(E2) + P(E3) + \ldots P(E10) \text{ (by the addition rule for disjoint events)} \\ &= q^2(1-q)^3 + q^2(1-q)^3 + q^2(1-q)^3 \ldots q^2(1-q)^3 \text{ (because the probability for EACH of these events is } q^2(1-q)^3\text{)} \\ &= \binom{5}{2} q^2(1-q)^3 \\ &= 10q^2(1-q)^3 \end{aligned}$$

Practice Question 6:

- There population of CU undergraduates is n=31,000 students
- Suppose you are friends with r=100 people.
- You walk into a classroom and you see k=160 random people.
- Assume each group of k CU undergrads is equally likely to be in the room.

What is the probability that you see at least one friend in the room?

Practice Question 6:

- There population of CU undergraduates is n=31,000 students
- Suppose you are friends with r=100 people.
- You walk into a classroom and you see k=160 random people.
- Assume each group of k CU undergrads is equally likely to be in the room.

What is the probability that you see at least one friend in the room?

Define

- S (unordered)
- E: ≥ 1 friend in the room

What strategy would you use?

A. $P(\text{exactly } 1) + P(\text{exactly } 2)$
$P(\text{exactly } 3) + \cdots$

B. $1 - P(\text{see no friends})$

Practice Question:

- There population of CU undergraduates is n=31,000 students
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- You walk into a classroom and you see k=160 random people.
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What is the probability that you see at least one friend in the room?

Define

- S (unordered)
- E: ≥ 1 friend in the room

It is often much easier to compute $P(E^c)$.

Practice Question 3:

You roll a fair 6-sided die 4 times. What is the probability that you roll at least one 6?

Practice Question 3

You roll a fair 6-sided die 4 times. What is the probability that you roll at least one 6?

Define the event A: Roll at least 1 six

Tip: Much easier to calculate $P(A')$, so use complement rule!

$$P(A) = 1 - P(A')$$

$P(A') = P(\text{no sixes in 4 rolls})$

Let N_i be the event not rolling a 6 in ith roll

$$= P(N_1, N_2, N_3, N_4)$$

$$= P(N_1)P(N_2|N_1)P(N_3|N_1,N_2)P(N_4|N_1,N_2,N_3)$$ By multiplication rule

$$= (5/6)\cdot(5/6)\cdot(5/6)\cdot(5/6)$$

$$= \left(\frac{5}{6}\right)^4$$

$$\Rightarrow P(A) = 1 - \left(\frac{5}{6}\right)^4 = \boxed{\frac{671}{1296}}$$

Practice Question 4:

Cats and sharks (note: stuffed animals)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

A. $\frac{3}{7}$

B. $\frac{1}{4} \cdot \frac{2}{3}$

C. $\frac{4}{7} + 2 \cdot \frac{3}{6}$

D. $\frac{12}{35}$

E. 0

Practice Question 4

Cats and sharks (ordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

Cats and sharks (unordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

Define

- S = Pick 3 distinct items out of 7, unordered
- E = 1 distinct cat, 2 distinct sharks, unordered

$$|E| = \binom{4}{1} \cdot \binom{3}{2}$$

$$|S| = \binom{7}{3}$$

$$\Rightarrow P(1 \text{ cat}, 2 \text{ sharks}) = \frac{|E|}{|S|} = \frac{\binom{4}{1}\cdot\binom{3}{2}}{\binom{7}{3}} = \frac{12}{35}$$

Cats and sharks (unordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

Define

- $S =$ Pick 3 distinct items
- $E =$ 1 distinct cat, 2 distinct sharks

Practice Question 4

Cats and sharks (ordered solution)

$$P(E) = \frac{|E|}{|S|}$$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

$$P(C_1, S_2, S_3) = \left(\frac{4}{7}\right)\cdot\left(\frac{3}{6}\right)\cdot\left(\frac{2}{5}\right)$$

$$= P(C_1)\cdot P(S_2 | C_1)\cdot P(S_3 | C_1, S_2)$$

All possible orderings

$$P(1 \text{ cat} + 2 \text{ sharks}) =$$

$$P(C_1, S_2, S_3)$$

$$+ P(S_1, C_2, S_3)$$

$$+ P(S_1, S_2, C_3)$$

$$= 3\cdot\left(\frac{4\cdot 3\cdot 2}{7\cdot 6\cdot 5}\right) = \frac{12}{35}$$

Practice Question 7:

You take a random survey of CU students and ask them if they use Facebook and/or X.
80% of the students report they use X, 40% of the students report that they use Facebook and 3% of students report they don't use either.

Suppose you choose a name at random from the list you surveyed.

a). What's the probability that this person uses Facebook **or** X?

b). What's the probability that this person uses Facebook **and** X?

c). What's the probability that this person does not use Facebook **given** that they do use X?

Practice Question 7:

You take a random survey of CU students and ask them if they use Facebook and/or X. 80% of the students report they use X, 40% of the students report that they use Facebook and 3% of students report they don't use either.

Tip: Use this info to make a joint probability table

Suppose you choose a name at random from the list you surveyed.

Define following events:

F: Use Facebook X: Use X

Fill in table!

	F	F'	Total
X			0.80
X'		0.03	
Total	0.40		1

	F	F'	Total
X	0.23	0.57	0.80
X'	0.17	0.03	0.20
Total	0.40	0.60	1

a). What's the probability that this person uses Facebook **or** X?

$$P(F \cup X) = P(F) + P(X) - P(F, X)$$
$$= 0.40 + 0.80 - 0.23 = 0.97$$

b). What's the probability that this person uses Facebook **and** X?

$P(F, X) = 0.23$ (see joint prob table)

c). What's the probability that this person does not use Facebook **given** that they do use X?

$$P(F' \mid X) = \frac{P(F', X)}{P(X)} = \frac{0.57}{0.80} = \frac{57}{80}$$

Supporting Materials

Set Theory Review

Proof of Identity 1:

Where does $P(E^c) = 1 - P(E)$ come from?

$$P(E \cup E^c) = P(E) + P(E^c)$$

Axiom 3: E and E^c are mutually exclusive

$$P(S) = P(E) + P(E^c)$$

Since all outcomes are either in E or E^c

$$1 = P(E) + P(E^c)$$

Axiom 2: $P(S) = 1$

$$P(E^c) = 1 - P(E)$$

Subtract $P(E)$ from both sides

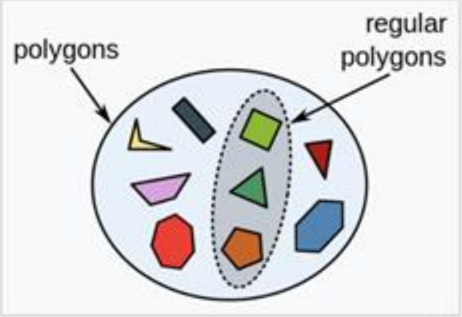
Set Notation

A **set** is a collection of well-defined, unordered objects called elements or members.

$$A = \{1,2,3,3\} \quad B = \{1,1,2,3\}$$

If two sets contain precisely the same elements, regardless of the order or possible repetition, then they are considered equal.

A set *B* is a **subset** of a set *A* if all elements of *B* are also elements of *A*

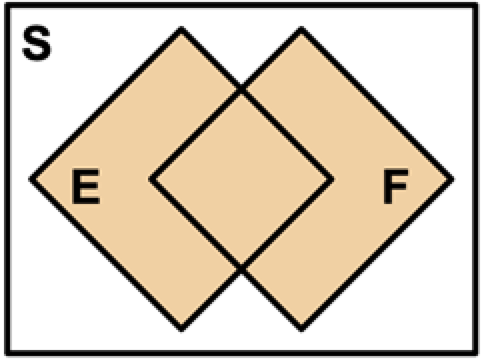


The regular polygons form a subset of the polygons.

Empty Set

$\{\ \}$ or $\varnothing$

Calc



E and F are events in S.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

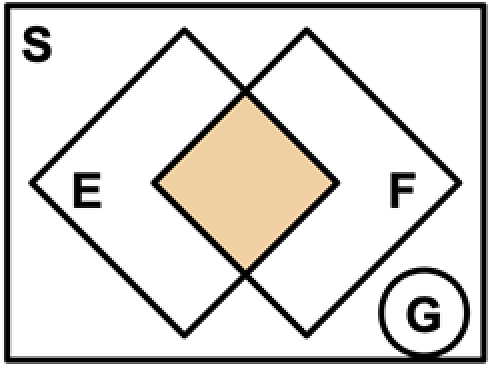
Let $E = \{1, 2\}$, and $F = \{2,3\}$

def Union of events, $E \cup F$

The event containing all outcomes in E or F.

$$E \cup F = \{1,2,3\}$$

Set Notation - Intersection



E and F are events in S.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

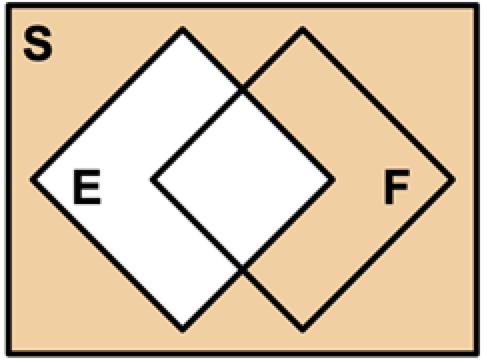
def Intersection of events, $E \cap F$

The event containing all outcomes in E *and* F.

$$E \cap F = EF = \{2\}$$

def Mutually exclusive events F and G means that $F \cap G = \emptyset$

Set Notation - Complement



E and F are events in S.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

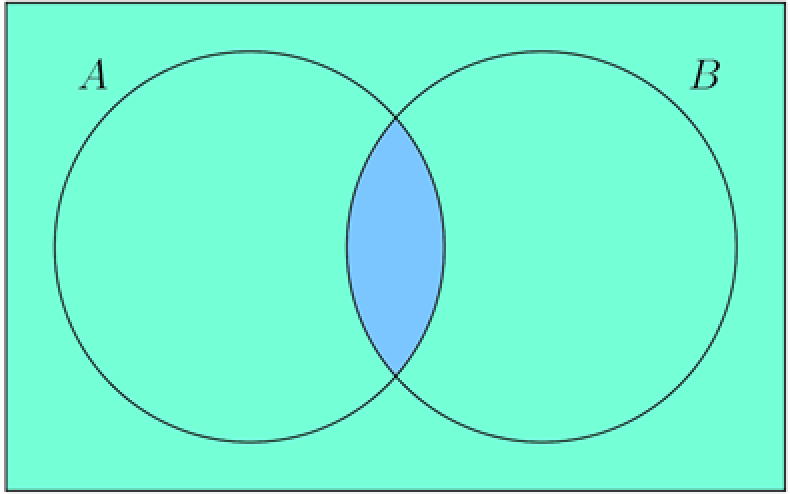
Let $E = \{1, 2\}$, and $F = \{2,3\}$

def Complement of event E, E^C

The event containing all outcomes in that are *not* in E.

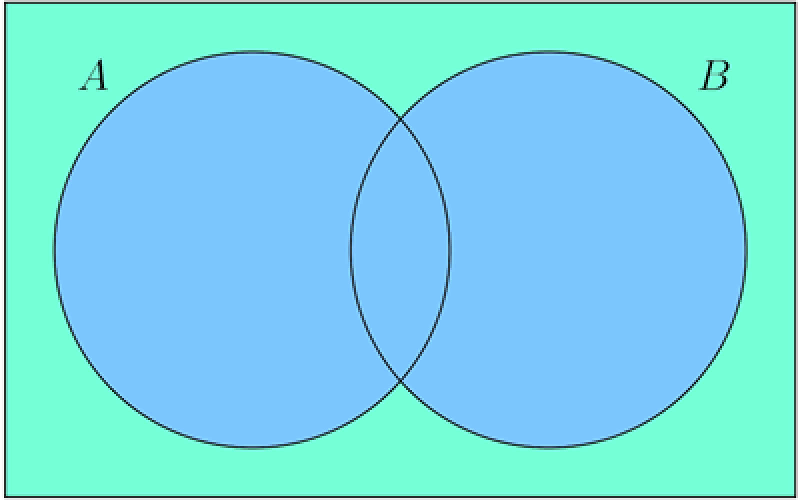
$$E^C = \{3, 4, 5, 6\}$$

De Morgan’s Laws



$A \cap B$

$(A \cap B)^c = A^c \cup B^c$



$A \cup B$

$(A \cup B)^c = A^c \cap B^c$