Example

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} : \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots$$

Prove $\frac{n}{n+1} \to 1$ as $n \to \infty$

Proof: $(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(\forall m \geq n)\left[\left|\frac{m}{m+1} - 1\right| < \varepsilon\right]$

assume we given ε

then find n dependent on ε

that makes the thing true

$\therefore$

Let $\varepsilon > 0$ be given (arbitrarily). We need to find an n such that for all $m \geq n$:

$$\left|\frac{m}{m+1} - 1\right| < \varepsilon$$

Pick n so large that $n > \frac{1}{\varepsilon}$

Then for any $m \geq n$:

$$\left|\frac{m}{m+1} - 1\right| = \left|\frac{-1}{m+1}\right| = \frac{1}{m+1} < \frac{1}{m} \leq \frac{1}{n} < \varepsilon$$

Done!

Example:

③

we know: $\quad$ tends

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \Rightarrow \frac{1}{n} \to 0 \text{ as } n \to \infty \quad (A)$$

Lets prove this rigorously w.r.t definition of limit we just gave.

Proof: $(\forall \varepsilon>0)(\exists n \in \mathbb{N})(\forall m \geq n)\left[\left|\frac{1}{m}-0\right|<\varepsilon\right]$

i.e Lets simplify

$$(\forall \varepsilon>0)(\exists n)(\forall m \geq n)\left[\frac{1}{m}<\varepsilon\right] \Leftarrow$$ need to verify this in order to prove (A)

Let $\varepsilon>0$ be given (Let $\varepsilon>0$ be arbitrary)

We need to find an n such that

$$(\forall m \geq n)\left(\frac{1}{m}<\varepsilon\right)$$

Pick any n such that $n>\frac{1}{\varepsilon}$ [uses Archimedean Property]

See Assignment 10.2

Then, if $m \geq n$, $\frac{1}{m} \leq \frac{1}{n}<\varepsilon$ Done

n depends on the ε

$\therefore$ Quantifiers order matter!

This definition absolutely crucial in ~~to~~ real analysis, which absolutely crucial to calculus, many, crucial to science physics, technology etc.

$\epsilon \Rightarrow$ epsilon $\Rightarrow$ small and positive $\Rightarrow$

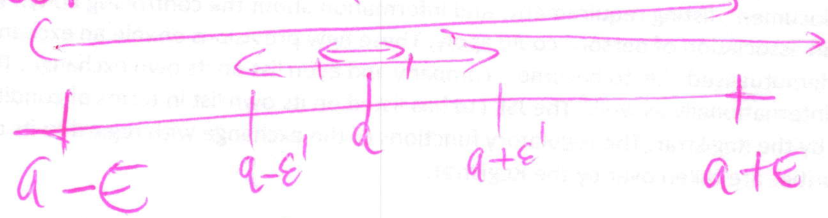
Consider the part $(\exists n)(\forall m \geq n)[|a_m - a| < \epsilon]$

From some ~~around~~ point n onwards [all ~~numbers~~ numbers in sequence are some distance from epsilon], all the numbers in $\{a_n\}_{n=1}^{\infty}$, are within a distance of ϵ from a.

Intuition → that we can take $\epsilon > 0$ as small as we want.

→ all elements of sequence are in here

if we take smaller epsilon, then smaller.



notice that n depends on the ϵ (Epsilon)

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Lets get more formal:

$\{a_n\}_{n=1}^{\infty}$ a

also written as:

$a_n \to a$ as $n \to \infty$

members of sequence tend to limit a, as n tends to infinity

that ≈ "corresponds"

$|a_n - a|$ becomes arbitrarily close to 0

(Now for formal definition)

(Epsilon) natural

$a_n \to a$ as $n \to \infty$ iff $\cancel{(\forall \varepsilon > 0)(\exists n \in \mathbb{N})}$

(A) iff $(\forall \overset{\text{epsilon}}{\varepsilon} > 0)(\exists n \in \mathbb{N})(\forall m \geq n)[|a_m - a| < \varepsilon]$

for any Epsilon greater than zero → This thing holds

Now we can see why we spend so much time on quantifiers and particular order they appear.