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Studying of real numbers

Numbers arose from two distinct "cognitive" human conceptions,

- Counting, and [discrete numbers] (A)
- Measurement [continuous, real] (B)

The connection between these two numbers (A) & (B) until 19th Century.

Path : integers → rationals → real.

(ratio of two integers) → constructed from integers

Constructing real from rationals is more difficult.

Here we will look at constructing and using the real numbers
- by looking at properties of rationals.

Properties of rationals:

- with rationals you have a system of numbers that is adequate for all real world measurement.
- this is captured by the following property of rational numbers.

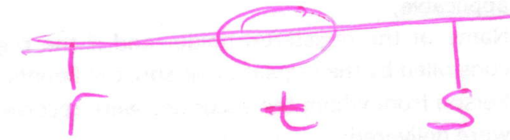
$r < s$ ($4 < 8$), t (7)

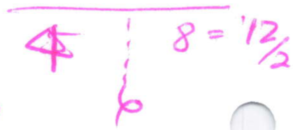
Give as theorem:

If r, s are rationals, $r < s$, then there is a rational t such that $r < t < s$.

[This property is called <u>density</u>. The rational line is dense]

we can find rational number t between r and s.





Proof: Let $t = \frac{1}{2}(r+s)$

Clearly $r < t < s$

Can be read:
r less t and t less s <u>or</u>
t less s and t greater r

$\therefore$ is t a rational number.

It obviously is, but lets prove it.

Let $r = \frac{m}{n}$, $s = \frac{p}{q}$, where $m, n, p, q \in \mathbb{Z}$ integers,

then $t = \frac{1}{2}\left(\frac{m}{n} + \frac{p}{q}\right) = \frac{mp}{2nq}$,

(~~So as mq+np~~)

So as $mq + np,\ 2nq \in \mathbb{Z}$, so $t \in \mathbb{Q}$ [rational]

denote rational

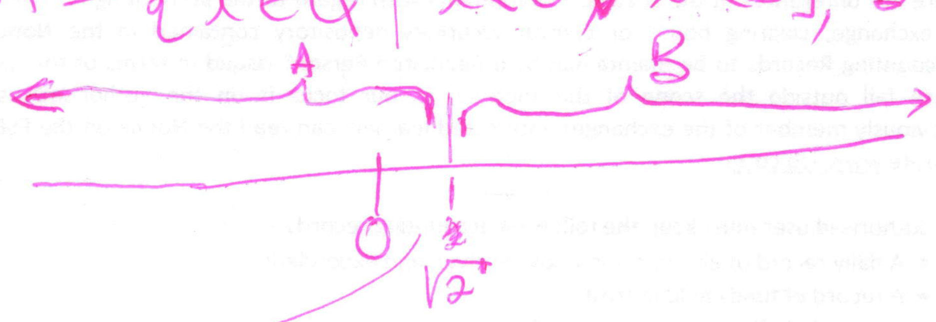
Density means rationals are good for measurement, i.e. we get numbers close to any particular length.

$\Rightarrow 7.21\ldots$

Density does not mean there are holes in the rational line (eg $\sqrt{2}$)

set rationals　　either negative

Let $A = \{x \in \mathbb{Q} \mid x < 0 \vee x^2 < 2\}$, $B = \{x \in \mathbb{Q} \mid x > 0 \wedge x^2 > 2\}$



$A \cup B = \mathbb{Q}$, But A has no greatest member and B has no smallest member

"A union B is rational line"

Hence, the rationals are inadequate to do mathematics

— in $\mathbb{Q}$ we cannot solve the equation $x^2 - 2 = 0$

- The way out of problem.

including naturals, integers, rational

we have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$, then $\subset \mathbb{R}$

negatives

Real numbers were used to fill in the gaps/holes in rational line.

But once real numbers were discovered, there were some mindblowing elements to it!

e.g. however you can't read it, there are more holes/gaps in rational line, than there are rational numbers.

$\therefore$ real numbers (as set) are more, than rational numbers (in infinite sets)

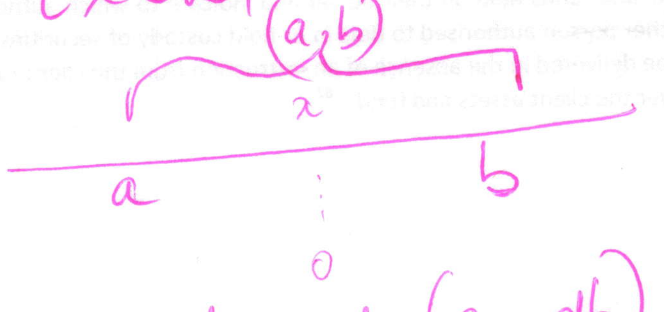
Intervals of real line

written this way.

Let $a, b \in \mathbb{R}$, $a < b$. Then open interval (a,b) is the set

$a < b$
10 20
$x = 15$

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$$

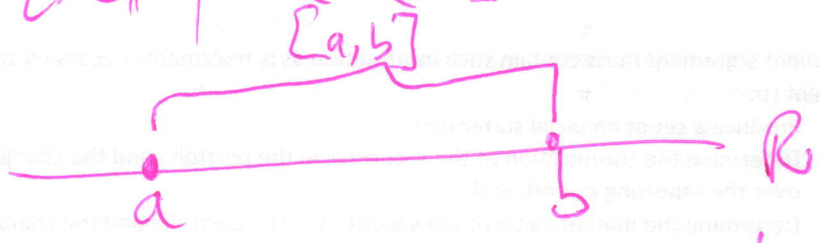


[real line]

But interval exclude, (a and b) the two endpoints.

the closed interval [a,b] is the set — written this way

$$[a,b]=\{x\in\mathbb{R} \mid a\le x\le b\}$$



this interval will include the two endpoints [a,b]

$\therefore$ $a,b \in [a,b]$, but $a,b \notin (a,b)$

$\therefore$ this is a big <u>Distinction</u>

Some variations on notation:

half-open (or half closed) intervals

$[a,b) = \{x\in\mathbb{R} \mid a\le x<b\}$ — left closed, right open

$(a,b) = \{x\in\mathbb{R} \mid a< x\le b\}$ — left open, right closed

$(-\infty,a) = \{x\in\mathbb{R} \mid x<a\}$ — everything to left a

$(-\infty,a] = \{x\in\mathbb{R} \mid x\le a\}$

$(a,\infty) = \{x\in\mathbb{R} \mid x>a\}$

$[a,\infty) = \{x\in\mathbb{R} \mid x\ge a\}$

We dont have: $(-,\infty]$ or $[\infty,-)$ — denotes same thing (∞ not real number)

Key property that $\mathbb{R}$ have that $\mathbb{Q}$ doesn't have:

Completeness property

Given a set A of reals, a number b such that $(\forall a \in A)[a \leq b]$ is said be an upper bound of A

we say b is a least upper bound of A if, in addition, for any upper bound c of A, we have $b \leq c$

Notation we use of least upper bounds:

$\mathrm{lub}(A)$ or "lub" A.

we can make the same definition for $(\mathbb{N}, \mathbb{Z}, \mathbb{Q})$

The completeness property of the real number says that every non empty set of reals that has an upper bound, has a least upper bound (in $\mathbb{R}$)

↳ this is the key to number systems and most modern maths.