* Proofs.

- Now we going to use or put that "precision" ("previous lessons") to proof mathematical statements.

In natural science, truth is established by empirical means, involving:

- observation
- measurement, and
- Experiment (gold standard)

In math, truth is established/determined By constructing a <u>proof</u>:

→ logical sound argument that Establishes the truth of statement

"argument: — probing and looking for flaws.

Here:

→ gain some understanding ~~in proofing~~ of what it means to proof a Mathematical statement,
and why mathematicians make such a big deal about proofs

(1) what is a proof and (2) why do we use them?

"to establish truth and communicate with others"

⇒ proof: convincing ourselves that some statement is true

e.g. I might have intuition that some statement is true, but until I have proved it, I cannot be sure.

⇒ But I also need to convince someone else:

↳ that is second purpose of proof.

⇒ argument must be logically sound

⇒ But proof has to be explained in such way, that other person can understand it

→ Some proofs are so complex, that only experts in field can understand it.

3x39x8
27x8

$2^4 + 3^4 =$
$16 + 81$

eg. if $n \geq 3$, the equation

$$x^n + y^n = z^n$$ has no solutions

for positive whole numbers (x, y, z)

(Fermat's Last Theorem)

or (Leonard Euler)
Every even number greater than 2 can be expressed as a sum of two primes

⇒ Goldbach Conjecture

⇒ Most mathematicians believe it to be true.

⇒ But it has not yet been proved!

Theorem: $\sqrt{2}$ is irrational [Proof it]

"a result that is significant is called theorem"

Lemma → significant, but does not merit statement to be called theorem.

∴ little theorem.

Assume (on contrary), that $\sqrt{2}$ were rational

Then there are natural numbers p, q with no common factors, such that

$\sqrt{2} = p/q$ [means

Remember a rational number is the result of dividing two integers or ratio of two integers

1.5 is rational number because it the result of $3/2 = (1.5)$

⇒ 3 and 2 are integers

p, q [we do not know it]

$\sqrt{2}$ is rational ∴ $\sqrt{2} = p/q$

$\therefore$ $3/2 = 1.5$ → rational number ↳ the quotient of two integers.

dividend, divisor, quotient

But 3, 2 have no common factors, ~~e~~ i.e. they smallest / have to be cancelled at.

$\therefore$ $\sqrt{2} = p/q$

Let's square the equation: $\Rightarrow$ get rid of square root sign. (↳ "to power of 2")

$\therefore$ $2 = p^2/q^2$

Rearrange: $2q^2 = p^2$

$\frac{2}{1} \times \frac{p^2}{q^2}$

$\therefore 2q^2 = p^2$

or

$2 = \frac{p^2}{q^2}$

$(q^2)2 = (q^2)\left(\frac{p^2}{q^2}\right)$

$2q^2 = p^2$

So p^2 is even, since it is equal to $2 \times$ (something) $(2(q^2))$

$\therefore$ p is even

, since square of even number (p^2) is even number.

Same square of odd number is odd number

remember: square of something is number multiplied by itself.

$\therefore$ $3 \times 3 = 9 = 3^2$ (odd)

$2 \times 2 = 4 = 2^2 =$ (even)

$7 \times 7 = 49 = 7^2 =$ (odd)

Lets make r not number we do not know (what p is!)

$\therefore$ $p = 2r$ (for some r)

Lets now use $p = \underline{2r}$ to substitute back into our equation

Substitution: $2q^2 = p^2$

$2q^2 = (2r)^2$

$\therefore$ $2q^2 = 4r$

Cancelling: $q^2 = 2r^2$

$\therefore$ q is even, since it is equal to 2 × (something) 2×(r)

So then q is even.

$\therefore$ p, q both even

Lets Step Back: what is factor

eg 12, 16

factors for 12 : 1, 2, 3, 6 (and 12)

factors for 16 : 1, 2, 4, 8 (and 16)

∴ Common factors for 12 and 16

is (1, 2, 3)

They have these Common Factors.

eg. 3, 2 they have no Common facts!

remember we said ; p, q has no Common factors.

But we concluded that p, q ~~however~~ are both even.

↳ ~~this means~~ Both statements can't be true!

This means our initial assumption

$\sqrt{2}$ is rational

— so we reached a false conclusion using this assumption

— using a valid argument

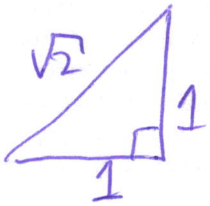
∴ we must then have a false assumption.

$\sqrt{2}$ are rational is false ∴ $\sqrt{2}$ is irrational (QED)

What does this mean:

– that quotients of integers are not enough to measure all lengths in geometric figures

eg



$\sqrt{2}$ is length of a diagonal of right handed triangle whose sides measure 1.

$\therefore$ quotient $= \sqrt{2}$ cannot be written as a ratio of two integers

This is what you call proof by contradiction

1. You want to prove some statement ϕ
2. You start by assuming $\neg \phi$.
3. You reason until you reach a conclusion that is false
   - By reducing both ψ and $\neg \psi$ for some ψ

   (eg) statement: p, q have no common factors

   $\neg$ negation: p, q are both even.
4. A true assumption cannot lead to false conclusion
5. Hence the assumption $\neg \phi$ must be false
6. In other words, ϕ must be true

Contradiction proof

we can look at proof by contradiction using truth tables.

What can we conclude from a proof $\Phi \Rightarrow \Psi$ $\Rightarrow$ (True) where Ψ is false?

Now we look

θ	Ψ	$\theta \Rightarrow \Psi$
T	T	T
T	F	F
F	T	T
F	F	T

So if $\Phi \Rightarrow \Psi$, need to be true

So we not interested in:

~~T F F~~ [not interested]

But we need Ψ to be false

So the only line in truth table where this is so

F F T

$\therefore$ θ is false

Conditional Proof

we want to proof the conditional $\Phi \Rightarrow \psi$ (1)

we know this is true if Φ is false (using previous truth table)

, So we can assume Φ is true

To prove it, we assume Φ, and deduce ψ

Eg.

Let x, y be variables for real numbers, and ~~prove~~ prove:

$$[x, y \text{ are rational}] \Rightarrow [x+y \text{ is rational}]$$

Assume x, y are rational

Then there are integers p, q, n, m ← unpack the assumption using some useful information

such that $x = p/m$ ~~such~~

$y = q/n$

then $x+y = p/m + q/n$

$$= \frac{pn + qm}{mn}$$

Hence $x+y$ is rational

Adding fractions:

$$\frac{p \times n}{m \times n} + \frac{q \times m}{n \times m}$$

$$\frac{pn}{mn} + \frac{qm}{mn}$$

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For conditionals involving quantifiers are sometimes best handled by ~~proofing~~ proving the contrapositive

- What is the contrapositive

e.g. To prove $\theta \Rightarrow \psi$,
prove $(\neg\psi) \Rightarrow (\neg \phi)$

$\Rightarrow$ i.e reverse ϕ and ψ
and put negation in front of them.

eg Prove: $(\sin\theta \neq 0) \Rightarrow (\forall n \in \mathbb{N})(\theta \neq n\pi)$

the statement is equivalent to: (swap)

$$\neg (\forall n \in \mathbb{N})(\theta \neq n\pi) \Rightarrow \neg(\sin\theta \neq 0)$$

in positive form:

$$(\exists n \in \mathbb{N})(\theta = n\pi) \Rightarrow (\sin\theta = 0)$$

($\neg$ moves inside, and negation it making it positive)

better know this is true

$\Rightarrow$ whenever we have a whole number π its $\sin = 0$

another word on proofs conditional;

To prove a biconditional $\phi \Leftrightarrow \psi$,
we construct two proofs:

$$\phi \Rightarrow \psi \quad , \quad \psi \Rightarrow \phi$$

since the biconditional ($\Leftrightarrow$) is just a conjunction of the two conditionals,
that amounts to proof of biconditional.