Sect C: Beginning of "Real analysis"

"Real number system"

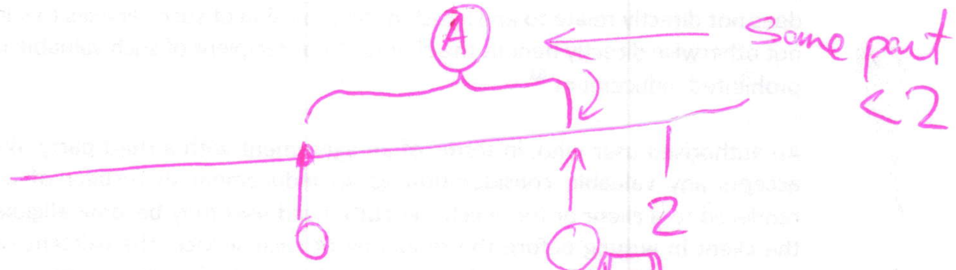
Theorem: The rational line is not complete

[Completeness: If $A \subseteq \mathbb{R}$ has an upper bound, then it has a "lub", in $\mathbb{R}$]

what this says, is that this property does not hold for rational numbers.

Does hold for $\mathbb{R}$ numbers.

Proof: Let $A = \{ r \in \mathbb{Q} \mid r > 0 \wedge r^2 < 2 \}$



here we are talking about rational [Not reals]

A is bounded above.

e.g. 2 is an upper bound,

(we only need to find one)

[I will show that A has no least upperbound (lub)]

I.e that will mean that A is a set of rationals, which has an upperbound, but no lub, and hence "the rational line is not complete"

How will I show there is no lub?

Let $x \in \mathbb{Q}$ be any upper bound of A, and show there is a smaller one (in $\mathbb{Q}$)

aside: why do we use $\mathbb{Q}$ for rationals; $\mathbb{Q}$ = quotients! ☺

rationals are Quotients of integers

Let $x = p/q$, where $p, q \in \mathbb{N}$)

③

Suppose $x^2 < 2$ [~~...~~ or $x \geq 2$, has to be one]

Then $2q^2 > p^2$

As n gets larger, $n^2/2n+1$ increases without bound,

so we can pick an $n \in \mathbb{N}^+$ so large that

$$\frac{n^2}{2n+1} > \frac{p^2}{2q^2 - p^2}$$

that means this number is positive number

i.e $2n^2q^2 > (n+1)^2 p^2$

Hence $\left(\frac{n+1}{n}\right)^2 \frac{p^2}{q^2} < 2$

Let $y = \left(\frac{n+1}{n}\right)\frac{p}{q}$.

Thus $y \in \mathbb{Q}$ and $y^2 < 2$

so $y \in A$

But $y > x$

(Contradiction), since x is an upper bound of A

So $x^2 \geq 2$

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Now we'll show ~~the~~ ~~is~~ a smaller upperbound, (using this extra information)

and no chance of ~~x~~ any x being a lub.

Recap:

$A = \{ r \in \mathbb{Q} \mid r \geq 0 \wedge r^2 < 2 \}$,

x is an upperbound of A,

(in form) $x = p/q$

Goal: Show that A has upperbound smaller than x.

We just showed that $x^2 \geq 2$, hence

since $\sqrt{2}$ is irrational, $x^2 > 2$

Thus (since $x = p/q$). $p^2 > 2q^2$

Pick ~~a~~ n so large that: $\frac{n^2}{2n+1} > \frac{2q^2}{p^2 - 2q^2}$

(rearrange) i.e. $p^2 n^2 > 2q^2 (n+1)^2$

rearrange

i.e $\frac{p^2}{q^2}\left(\frac{n}{n+1}\right)^2 > 2$

Let $y = \left(\frac{n}{n+1}\right)\frac{p}{q}$, then $y \in \mathbb{Q}$

and $y^2 > 2$.

Since $\left(\frac{n}{n+1}\right) < 1$

But, for any $a \in A$, $a^2 < 2 < y^2$, so $a < y$.

Hence y is an upper bound of A, smaller than x

Thus A does not have a lub.

This proves the theorem. $\boxed{}$

Real number sequences ⇒

[they are big deal in analysis, but also in Calculus!

What is a sequence? List: $a_1, a_2, a_3, \ldots$

$$\{a_n\}_{n=1}^{\infty} \quad \text{infinite sequen}$$

eg $1, 2, 3, \ldots = \{n\}_{n=1}^{\infty}$ (infinite sequence)

or $7, 7, 7, \ldots = \{7\}_{n=1}^{\infty}$

or $3, 1, 4, 1, 5, 9, \ldots$ = the decimal digits of π

or $\{(-1)^{n+1}\}_{n=1}^{\infty} = +1, -1, +1, -1, \ldots$ (alternating sequen)

Look at $\{\frac{1}{n}\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ numbers get closer to zero (0) (arbitrarily)

$\{1 + \frac{1}{2^n}\}_{n=1}^{\infty} = 1\frac{1}{2}, 1\frac{1}{4}, 1\frac{1}{8}, 1\frac{1}{16} \ldots$ get arbitrarily close to $\underline{1}$

$3, 3.1, 3.14, 3.141, 3.145 \ldots$ get close to π

If the numbers in a sequence $\{a_n\}_{n=1}^{\infty}$ get arbitrarily closer to some fixed number a, we say $\{a_n\}_{n=1}^{\infty}$ tend to the limit a, and write $a_n \to a$ as $n \to \infty$ OR

$$\lim_{n \to \infty} a_n = a$$