"Lecture 5:" Analysis of Language/ Quantifier:

⇒ "there exist" — "at least one"

⇒ for all

we look at following statement

"There is an object $\underline{x}$ having property $\underline{P}$"

eg The equation $x^2+2x+1=0$ has a real root.

we can rewrite it:

There is a real number x such that $x^2+2x+1=0$

OR

= There exist a real number x such that $x^2+2x+1=0$

symbol use of "is" or "exist"

is $\exists$ (Back to front E)

$\therefore\ \exists x\,[x^2+2x+1=0]$

" there exist an x such that $x^2+2x+1=0$ "

Aug '18

$\exists$ Called Existential Quantifier

$\exists x[x^2+2x+1=0]$ ← Existential Statement

∴ The simplest way to prove an existential statement of the form $[\exists x[x^2+2x+1=0]]$ is to find actual x that will satisfy the property

∴ To solve this statement, find x that solves the equation

take $x=-1$

then $x^2+2x+1=(-1)^2+(2(-1))+1=1-2+1$
$=0$

But, sometimes we can prove an existential statement without finding an object (x) that satisfies a property.

Eg

eg. $\exists x\,[x^3+3x+1=0]$ ③

∴ we wont find (or provide) an x to solve the existential equation

But I will show that there is a solution.

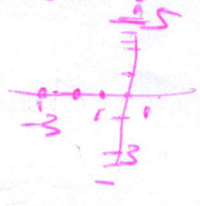
We will ~~rathe~~ start by looking at function:

$$y = x^3+3x+1$$

(this is a continuous function) — only thing we need to know about continuous function now.

" ↑ graph is smooth curve "

⇒ with no breaks or jumps ⇐

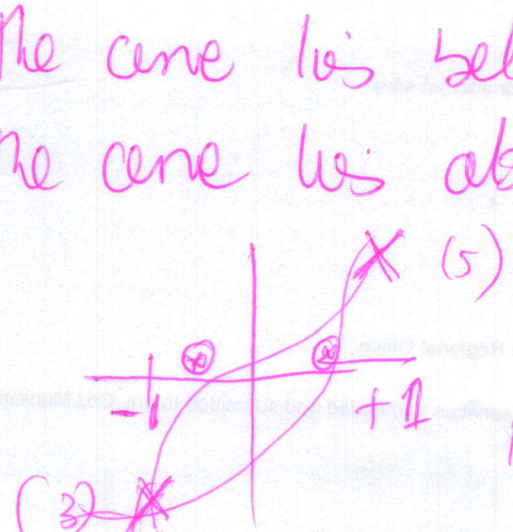


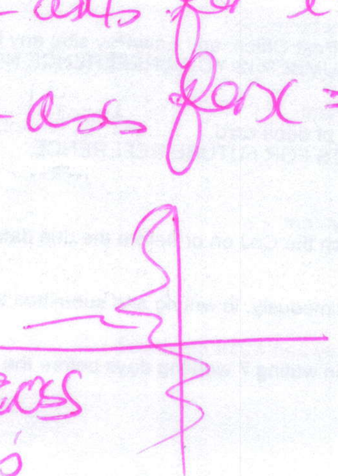
If $x=-1$ the curve function has value $y = -1 - 3 + 1 = -3$

If $x=+1$, the curve or function has value $y = 1+3+1 = 5$

∴ the curve lies below x-axis for $x=-1$

the curve lies above x-axis for $x=+1$





But must cross x-axis

So where it crosses the x-axis
the y value $= 0$

$\therefore$ where it crosses the x axis
x satisfies $x^3 + 2x + 1 = 0$

remember, we did not say which
x satisfies the equation
(whether neg or positive)
, But we do know that such a x existence.

$\therefore$ is an eg of an indirect Proof

" a lot of mathematical Proofs
are of this Nature!
$\Rightarrow$ Pretty cool

Quiz

same eg as before: $\exists x[x^3+3x+1=0]$

(5)

Which one of the following is it:

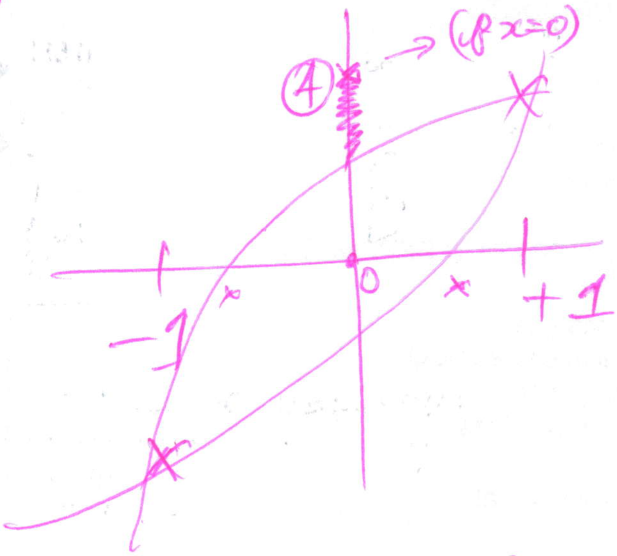
✓ 1) $(\exists x<0)[x^3+3x+1=0]$

2) $(\exists x>0)[x^3+3x+1=0]$

(there is or exist an $x<0$, such that $x^3+3x+1=0$)

(there is or exist an $x>0$, such that $x^3+3x+1=0$)

But how did we get it..?



Prove it (1):

First attempt was to go -1 and $+1$ (for x)

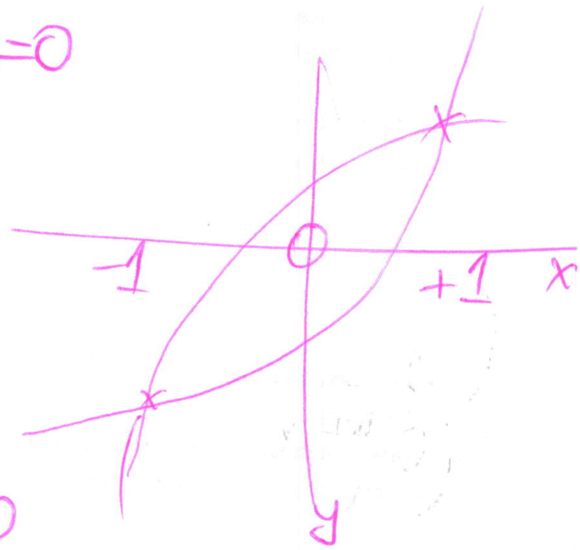
But instead we can go between -1 and 0

∴ Note if $x=0$, then $y=x^3+3x+1=0+0+1=1$ (4)

∴ y is negative, if $x=-1$

y is positive, if $x=0$ $\quad x$

so it somewhere between -1 and 0 where since the curve lies below the axis at $x=-1$, and above axis at $x=0$, it (y) must cross axis between $x=-1$ and $x=0$



Or Prove 2

We can simply observe that $x \geq 0$, then $x^3+3x+1>0$ ($\text{so} \neq 0$)

So proving it this way is false, which leaves us with Proof 1

$\Longrightarrow$ Example of Indirect Proof !!

Wobbly table Theorem

∴ If you have a wobbly table, you can find piece paper and put it under one leg.

<u>or</u>

you can rotate the table until it balances (??)

[last thinking out box]

⇒ [Come up with mathematical solution] or confirm the solution "experimentally"

go search for table

Sometimes it is <u>not</u> <u>obvious</u> that a statement is an "<u>existence assertion</u>"

But many statements in fact turn out to be <u>so</u>!

<u>eg.</u> statement that x is rational is an <u>existence</u> statement.

⇒ here is why:

<u>Lets take:</u>

$\sqrt{x^2}$ is rational

(we know this is false)

(But we just use it as an example of statement)

→ from face of it, it does <u>not</u> look like an existence statement.

"rewrite":

There exist natural numbers p, q
such that $\sqrt{2} = p/q$

(A) $\exists p \exists q [\sqrt{2} = p/q]$

(remember when we prove, there will be no value for p, q that satisfies this equation)

- we need to know in advance that p, q denote natural numbers.

But (A) equation does not say this!
(from the above it can be real or complex no's)

We need to make it more specific:
By writing it in the following way:

$$(\exists p \in \mathbb{N})(\exists q \in \mathbb{N})[\sqrt{2} = p/q]$$

$\mathbb{N}$ (double line)

where $\mathbb{N}$ denotes a set of natural numbers

" there exist a p in $\mathbb{N}$, and there exist a q in $\mathbb{N}$, such that $\sqrt{2} = p$ over q "

or

$$(\exists p, q \in \mathbb{N})[\sqrt{2} = p/q]$$

(in aside):

⇒ rational number can take form p/q where
p and q are integers and
q is not zero.

Come back: ~~proof~~
prove $\sqrt{2}$ is irrational
we proof it by contradiction!

Next Language Construct:

Universal quantifier ⇒

$\forall x$ means "for all x, it is the case that..."

upside down letter A, which means "for all" eg – "the square (root) of any real number is greater or equal to zero" — any, all, ~~(exist)~~

$$\forall x \; (x^2 \geq 0)$$

"for all x, $(x^2 \geq 0)$" — (Compared to existence assertion)

→ "for all assertion"

→ this is the universal quantifier

⇒ how do I know what the x means

I need to be explicit

$$(\forall x \in \mathbb{R}), (x^2 \geq 0)$$

Lets look at combinations of quatifiers — [Most consist of two or more quantifiers combined]

⇒ There is no largest natural numbers

$$: (\forall m \in \mathbb{N})(\exists n \in \mathbb{N})(n > m)$$

"For all m in set Natural Numbers, there is n in set Natural Numbers, such that $n > m$"

Note: order of the quantifiers are important

If we swap the qualifier around e.g. (11)

$(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(n > m)$

which says "there is 'one' a natural number n, which has property for all natural numbers m, $n > m$

"which says" There is a natural number bigger than all natural numbers" ⇒ which is **false!**

↳ ~~tru~~ turn true sentence into false sentence

(wrong):

melanoma association: "One American dies of melanoma almost every hour"

$\exists A \forall H [A \text{ dies in hour } H]$

"there is an american, ~~and~~ every hour, such A dies in hour H"

⇒ One guy dies every hour, of melanoma

↳ then gets "reborn"

Correct:

$\forall H \exists A [A \text{ dies in hour } H]$ — different americans for different hours

" for - every hour there exists an American, such that A dies in hour h."

- "all"
- "any"
- "is"

Quiz: State licence sentence:

"Do you have a licence from more than one state"

Which ~~stat~~ formulas ~~eg~~ express the literal meaning of this statement:

① $(\exists L)(\exists S_1)(\exists S_2)[(S_1 \neq S_2) \wedge From(L, S_1) \wedge From(L, S_2)]$

"there is/exist licence, for which there are two distinct state, where that licence comes from"

→ literal meaning, But it is false

_ licence are issued by state

- you can't get a licence issued by two separate stats.

② $(\exists L_1)(\exists L_2)(\exists S)[(L_1 \neq L_2) \wedge From(L_1, S) \wedge From(L_2, S)]$

"there two licenses, that issued by one state"

(~~But~~ not, possibly true, But not what sentence says)

③ $(\exists L_1)(\exists L_2)(\exists S_1)(\exists S_2)[(S_1 \neq S_2) \wedge (L_1 \neq L_2) \wedge from(L_1, S_1) \wedge From(L_2, S_2)]$

" there are two licenses, and two state, different licences and different state, one licence from one state, other from other state"

this correct one!

Literal meaning 1,
No 3 intended meaning!

"ϕ implies truth of ψ"

$\Rightarrow$ read how to read mathematical
(Suppl. Tutorials) Formulas