

# Vector Space

①

Why do we need it?

It has properties:

- Commutative

- Associative

- Identity

- Every element has additive inverse

- Distributive

Addition, Scalar multiplication:

~~Addition~~ Elements  $u+v \in V$

multiplication

~~Scalar~~  $Av \in V$

$\vdots$   $a \in F, v \in V$

Fundamental Properties:

Commutative

$$u+v = v+u$$

Associative

$$(u+v)+w = u+(v+w)$$

Additive Identity:

$$V+0=V$$

Additive inverse:

$$V+(-V) = 0$$

multiplicative inverse

$$AV+A0=0$$

$$IV+0=V$$

Distributive:

$$a(u+v)=au+av$$

∴ all properties of Complex numbers  
applies to elements of vector space.  
(No need to repeat) ②

When we think geometrically:  
Elements in vector space are called  
vectors or points

Scalar multiplication in vector space  
depends on  $F$

remember correctly: "over"  $F$   
 $V$  is vector space in  $F$

or  $\mathbb{R}^n$  is vector space over  $\mathbb{R}$

$C^n$  is vector space over  $C$

Definition:

Vector space over  $R$   
is called Real vector space

Vector space over  $C$   
is called a Complex vector space

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Example:

$F^\infty$  is list or sequence of all  
Elements of  $F$

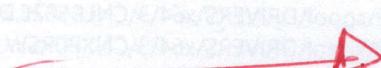
$$F^\infty = \{(x_1, x_2, \dots) : x_j \in F \text{ for } j=1, 2, \dots\}.$$

Addition and Scalar multiplication of  $F^\infty$

$\therefore$  (adding the corresponding coordinate)

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots)$$

& Scalar follow distributive property:

$$d \underbrace{(x_1, x_2, \dots)}_{\rightarrow} = (dx_1, dx_2, \dots)$$


Notation FS.

## Elementary properties of vector spaces

④

Vector space has a unique additive identity:

Suppose  $0$  and  $0'$  are both additive

identities in vector space  $\checkmark$

$$\text{Then } 0' = 0' + \underline{0} = 0 + \underline{0'} = 0 \quad ! \underline{0' = 0}$$

$\therefore$  has only 1 additive identity

Also, each element  $\overset{\otimes}{\in} V$  in vector space has an additive inverse only 1 additive inverse.

Proof

Suppose  $V$  is vector space. Let  $v \in V$ , and

Suppose  $w$  and  $w'$  are additive inverses of  $v$

$$w = w + 0 = \underline{w + (v + w')} - (w + v) + w' = 0 + w' = w'$$

$$\therefore w = w'$$

The following notation now makes sense:

$$v, w \in V$$

$$\overline{v}(-v) = 0$$

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①  $-v$  denotes additive inverse of  $v$

②  $w-v$  is defined as  $w + (-v)$   $\Rightarrow$  ②



Number 0 times a vector:

$$\boxed{0} \times i = i$$

$$0v = 0$$

for every  $v \in V$

$$ax^i = a.$$

A number times a vector 0

$$\underline{a}0 = 0$$

for every  $a \in F$

The number -1 times a vector

$$(-1)v = -v \text{ for every } v \in V$$

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## Subspace:

A subset  $U$  of  $V$  called a subspace  
of  $V$  if  $U$  is also a vector space  
(using same addition and multiplication)  
as on  $V$

### Example:

$$\{(x_1, x_2, 0) : x_1, x_2 \in F\} \text{ is a subspace of } F^3$$

## Conditions for subspace:

Subset  $U$  of  $V$  is subspace of  $V$ , if and  
only if,  $U$  satisfies the  
following 3 conditions:

additive identity:  $0 \in U$  must be an element of  $U$

### Closed under addition:

$u, w \in U$ , implies  $u + w \in U$

### Closed under scalar multiplication:

$a \in F$  and  $u \in U$  implies  $au \in U$

After Condition we also need to  
check if Commutativity and associativity  
 $\Rightarrow$  But if true  $V$ , then so also  $U$ .

only the 3 above conditions  $\Rightarrow$  need  
to be satisfied.

Example:

$$b \in F, \text{ then } \{(x_1, x_2, x_3, x_4) : x_3 = x_4 + b\}$$

, then  $b$  is subspace of  $F^4$ , if and only  
 $Q \quad b = 0$

we must have the additive  
closure in  $set U$

Property ①

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## Direct sum:

Sum  $U_1 + \dots + U_m$ , where each element  
 Can only be written in only one way  
 as sum  $u_1 + \dots + u_m$ , where  $u_j \in U_j$

### (+) Direct sum sign

Example:

$$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F^3 : x, y \in F \right\} \text{ and}$$

$$W = \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \in F^3 : z \in F \right\}$$

Both  $U$  and  $W$  are subspaces of  $F^3$

:  $U$  and  $W$  are a direct sum,

Can only be written in one way.

How can we check if sum is a direct sum.  
 $\Rightarrow$  what's the condition:

(a)

Suppose  $U_1, \dots, U_m$  are subspaces of  $V$ ,

then  $U_1 + U_2 + \dots + U_m$ , is a direct sum

if and only if the only way to write 0  
as a sum  $u_1 + \dots + u_m$ , by ~~not taking~~  
~~any~~  $u_j$  equal to 0.

⑦  
Review:  $U \rightarrow$  subset of  $V$ .

Sums of Subspaces:

$\therefore U_1, \dots, U_m$ , subset of  $V$

$$U_1 + U_2 + \dots + U_m = \{u_1 + u_m : u_i \in U_1, \dots, u_m \in U_m\}$$

so

