Pages 1–24 ①

Vector Spaces:

LA is study of linear maps on finite-dimensional vector spaces

The following will be covered here:

① Basic properties of Complex numbers

② R^n and C^n

③ Vector Spaces

④ Subspaces

⑤ Sums and Direct Sums of Subspaces

Complex numbers, what are they:

they were invented so we can take square roots of negative numbers

Same Basic properties of set $\mathbb{R}$ (of real numbers)

More technical definition of (complex number):

* ordered pair (a,b), where $(a,b) \in \mathbb{R}$, but we will write it as $\underline{a+bi}$

So we denote a set of complex numbers by $\mathbb{C}$

$$\therefore \mathbb{C} = \{a+bi : a,b \in \mathbb{R}\}$$

How do we denote addition and multiplication of $\mathbb{C}$?

$$(a+bi)+(c+di) = (a+c)+(b+d)i;$$

— Addition ——————

$$(a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i$$

$\therefore a,b,c,d \in \mathbb{R}$.

① sign change *
② loose i

$\boxed{i^2 = -1}$ NB

③

* i was used by Euler to denote $\sqrt{-1}$ (on the circle)

$\therefore \mathbb{R} \subseteq \mathbb{C}$

subset of $\mathbb{C}$.

$\therefore a + 0i$, the a is real number

$\therefore 0 + bi$, or simply just written as bi

$\therefore$ if $0 + 1i$, then we simply write i

Example for multiplication:

$$(a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i$$

Similarly.

$$(2+3i)\cdot(4+5i) = (8-15)+(10+12)i$$

$$= -7+22i$$

Properties of Complex arithmetic:

1. Commutative property ⇒ what does it mean? To commute ⇒ to move, to change

∴ $10+1 =$ (same) as $1+10$

or $x+y$ same as $y+x$

∴ $\alpha+\beta = \beta+\alpha$ $\quad [\alpha, \beta \in \mathbb{C}]$

or $\alpha\beta = \beta\alpha$

∴ get same result

∴ Commutative property of:

⇒ addition and

⇒ multiplication.

2. Associative Property

Eg. $1+2+3 = \underset{(1st)}{(1+2)} \underset{(2nd)}{+3}$

or $1+2+3 = \underset{2nd}{1+}\underset{1st}{(2+3)}$

produce same result

What does associative mean? to associate, i.e. form group

∴ group things together (using parens)

still get same result.

∴ $(\alpha+\beta)+\lambda$ = same = $\alpha+(\beta+\lambda)$

$\left[\alpha,\beta,\lambda \in \mathbb{C}\right]$

__or__ $(\alpha\beta)\lambda$ = same = $\alpha(\beta\lambda)$

∴ for:

⟹ addition and

⟹ multiplication.

③ Distributive property:

eg $2(3+4) = 2\cdot 3 + 2\cdot 4 = 14$

∴ Distribute the 2 and multiply the number with the individual parts inside

same as $2(3+4) = 2(7) = 14$

∴ Distribute the multiplication.

$\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$ [$\alpha \beta \lambda \in C$] 6

for: 1 addition 2 Multiplication

4 Identity Property:

$a + 0 = a$ [$\underline{a} \times 0 = 0$]

$a \times 1 = a$

a keeps it identity, does not change

[$\lambda \in C$]

$\lambda + 0 = \lambda$

or $\lambda 1 = \lambda$ (same as $\lambda \times 1 = \lambda$

[inverse/opposite]

5 Additive inverse

Add something to my source, can be number or expression to get 0.

N# 5. to make it zero

$5 - \underline{5} = 0$ (addition)

$\underline{\text{or}}$ expression:

$x+2$

$\times(-1)(x+2) = -x-2$ ← additive inverse

(multiplication)

∴ for every $\alpha \in \mathbb{C}$, there exist a unique $\beta \in \mathbb{C}$, such that $\alpha + \beta = 0$.

⑥ Multiplicative inverse.

Here we want the answer to be 1.

$2 \cdot \square = 1$.

$\square$ will be the reciprocal of 2

∴ $\frac{1}{2}$.

or if α is a fraction $\frac{3}{4}$, we flip the fraction $\frac{4}{3}$ — multiplicative inverse.

∴ $\frac{3}{4} \cdot \boxed{\frac{4}{3}} = 1$

if $-$ve, keep sign the SAME

$-2 \cdot -\frac{1}{2} = 1$

∴ $\frac{-2}{1} \times \frac{1}{-2} = \frac{-2}{-2} = -1$

$0.2 \times \square = 1$

$\therefore 5$

for every $\alpha \in C$, there exist $\beta \in C$, such that $\alpha\beta = \underline{1}$

Proof:

Show $\alpha\beta = \beta\alpha$ [Commutetive property]

Suppose $\alpha = (a + bi)$
$\beta = (c + di)$

$\alpha\beta = (a + bi)\cdot(c + di)$
$= (ac - bd) + (ad + bc)i$

$\beta\alpha = (c + di)(a + bi)$
$= (ca - db) + (cb + da)i$

$ac = ca$ (Commutive)
$bd = db$ Commutetive

$\alpha\beta = \beta a$

Simple Definitions

To denote the additive inverse of α.

we use $-\alpha$

$$\therefore \alpha + (-\alpha) = 0$$

To denote subtraction of $\mathbb{C}$ (complex nums)

$$\beta - \alpha = \beta + (-\alpha)$$

To denote the multiplicative inverse of α (where $\alpha \neq 0$) will be $\frac{1}{\alpha}$

$$\alpha \cdot \frac{1}{\alpha} = 1$$

To denote Division of $\mathbb{C}$

$$\beta/\alpha = \beta(1/\alpha)$$

Throughout the Book we use F to denote $\mathbb{C}$ or $\mathbb{R}$

Elements of F are called scalars

i.e fancy word for a NUMBER

to emphasize an object is a NUMBER, (as opposed to vector)

α^m, simply means: $\underbrace{a \times a \times a}_{m \text{ times}}$

then follows,

$(\alpha^m)^n = \alpha^{mn}$

or $(\alpha\beta)^m = \alpha^m \beta^m$

[m, n positive integers]

What does R^2 mean? — must be 2

— a plane, with ordered PAIR of real numbers

$R^2 = \{(x, y) : x, y \in \mathbb{R}\}$

(11)

what does R^3 mean?

⇒ an "ordinary space" of ordered set of triples of real numbers (3's)

$$R^3 = \{(x, y, z) ; x, y, z \in R\}$$

what is a list?

Suppose n non-negative integer,

~~so list is ordered collection of n element~~

∴ a list (we don't know yet what it is) of length (1) n (2) is an ordered collection of n element.

⇒ separated by commas

⇒ inside parens

list of x's — (now we know)

$$(x_1, x_2 \ldots \ldots x_n)$$

length.

lists are equal if they have same length and same element.

But list has a finite length, ∴ the following is Not a list

$(x_1, x_2, \ldots\ldots)$

↑ infinite length.

But list may also have 0 length

⇒ nothing inside paren

$(\ )$

How does list differ from set;

list	Set
① order matters	order is irrelevant
② Repetition have meaning	repetition have no meaning

Example

list	Set
① $(3,5) \neq (5,3)$	$\{3,5\} = \{5,3\}$
② $(4,4) \neq (4,4,4)$ they don't have same length	$\{4,4\} = \{4,4,4\}$ Both Equal to set $\{4\}$

~~Remembe~~

Remember Now, we said we will use

F to denote C or R.

$\therefore$ F^n is set of all lists of length n of elements of F

$\therefore$ $F^n = \{(x_1, \ldots x_n) : x_j \in F$ for $j = 1, \ldots n\}$

we say x_j is j^{th} coordinate of $(x_1, \ldots x_n)$

Note F^n, where 2 or $3 = n$

$\therefore$ F^2 same as R^2

$\therefore$ (2 pairs of numbers)

$\therefore$ $C^4 = 4$ complex number lists

$C^4 = \{(z_1, z_2, z_3, z_4) : z_1, \ldots, z_4 \in C\}$

(where 4 comes in

$\Rightarrow$ list, order and repetition matters

How do we define <u>addition</u> in F^n

$\therefore (x_1, x_2, x_3 \dots x_n) + (y_1, y_2, y_3 \dots y_n)$

$= (x_1+y_1, x_2+y_2, \dots x_n+y_n)$

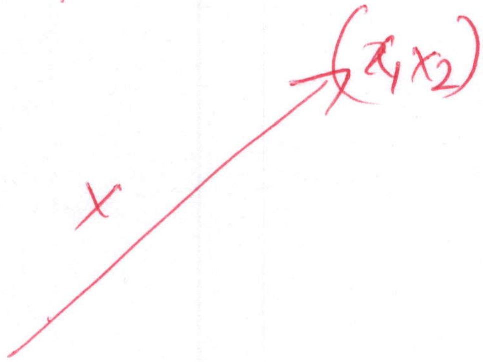
$\therefore$ we add the CORRESPONDING <u>coordinates</u>

Sometimes better to <u>avoid explicit</u> coordinate

just use x or y.

$\therefore$ let x denote $(x_1, x_2 \dots x_n)$

The following represent a vector:



— where we think of x as an arrow, and then referred to as a vector

Note $x_1 \dots x_{500}$ → Cannot be geometrically represented,

But we can do it algebraically

VIA through linear algebra

→ Δ

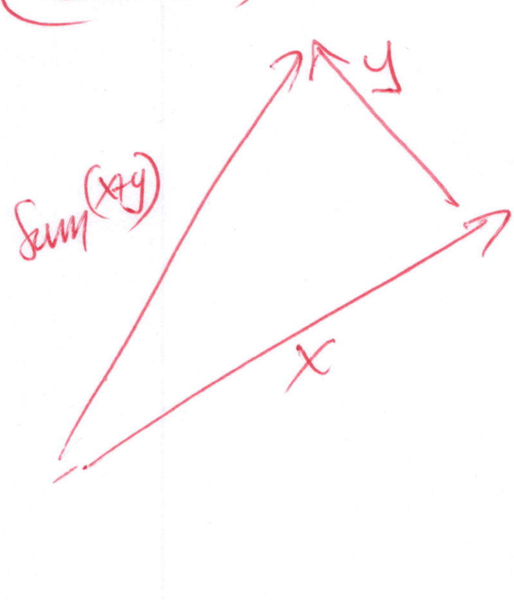
$(2, -3, 17, \pi, \sqrt{2})$ ~ element of $\mathbb{R}^5$

— Can be represented a point/~~[illegible]~~ in $\mathbb{R}^5$

— OR vector in $\mathbb{R}^5$

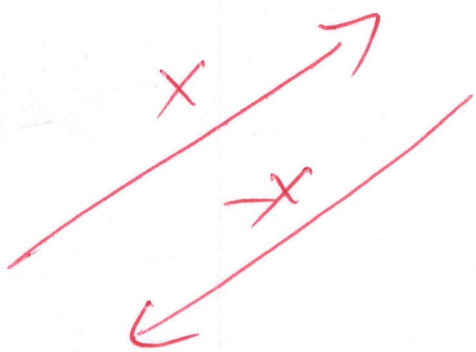
(without worrying about representing it geometrically)

Remember sum of Elements (groups coordinate) is obtained by adding corresponding coordinate of (~~[illegible]~~)



What is the additive inverse in F^n

$x \in F^n$, additive inverse is $-x$, is vector



$\therefore X = (x_1, x_2, x_3 \dots x_n)$ Additive inverse.

$\therefore -x = (-x_1, -x_2, -x_3 \dots -x_n)$

$\therefore$ additive inverse is same as parallel vector of x.

How do we deal with multiplication of corresponding coordinate in F^n?

⇒ Here will look at scalar multiplication.

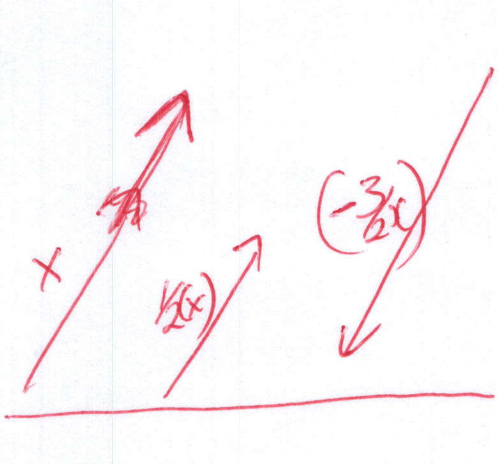
∴ Product of a λ (number) (scalar) and vector in F^n, by multiplying the coordinate of each vector by the λ (scalar)

$$\lambda(x_1, x_2 \ldots x_n) = (\lambda x_1, \lambda x_2 \ldots \lambda x_n)$$

where $\lambda \in F$ (or R or C) and $(x_1, x_2 \ldots x_n) \in F^n$

Geometrically: Scalar multiplication gives us the following nice property [geometric]

⇒ in R^2 space, x is vector.



λx (if λ positive), will point in same direction;

if negative will be opposite direction,

∴ x vector will shrink or grow by factor of λ

What is a field?

Set containing at least two distinct elements called 0 and 1, including

- operations of addition and multiplication

$\mathbb{R}$ & $\mathbb{C}$ are fields .....

In this book we will only look at Fields $\mathbb{R}$ & $\mathbb{C}$.