Rec

Lets tackle a problem on Pseudo inverse

Given matrix $A = \begin{pmatrix} 1 & 2 \end{pmatrix}$ ← not square

i) What $\rightarrow A^+$ (pseudo inverse)

we define a Pseud inverse using SVD

Since A is not square, the regular inverse does not exist. But we do know that the SVD exist for every matrix A, whether its ~~rectangular~~ square or not.

∴ How do we compute the SVD matrix?

$$A = U \Sigma V^T$$

U, V orthogonal matrices.

Σ matrix with positive values along diagonal

1×2 ↙ A

1×2 ↓ always be same as A

Σ 1×2

V^T ↓ has to be square ↓ 2×2

1×1 ↓ ? Because

$$U = [1]$$

How do we get Σ and V

tade:

$$A^T A = V (\Sigma^T \Sigma) V^T$$

$\curvearrowright$ will be a square matrix, & one diagonal elements are squares of single values

$$\binom{1}{2}(1\ 2) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$\leftarrow$ second row, is multiple of first row, mean we only get 1 $\heartsuit$ Σvalue

$\downarrow$ when square root it

$\lambda 1 = 0 \longrightarrow \sigma = 0$

$\lambda = 5$

Corresponding E'vectors:

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} u = 0$$

$$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (\text{not unit length})$$

$$u = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = 0 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\longrightarrow$ order Σ values in correct order

$$A = [\ 1\ ]\begin{bmatrix} \sqrt{5} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

Σ

V^T

So this gives us a representation
for A, now that we have SVD of A

How do we compute A^+

Note! $\qquad$ Bad!

$$A^+ = V \Sigma^{-1} U^T$$

$$A^+ = V \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} U^T$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$A^+ = \frac{1}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

i) AA^+ and A^+A

$$AA^+ = \begin{pmatrix} 1 & 2 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1$$

$$A^+A = \frac{1}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

iii) If x is in $N(A)$ what is A^+Ax?

Let assume x in nullspace of A

$$N(A) = c \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

i.e $x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\therefore \underbrace{A^+A x}_{=0 \in \text{ since,}} = 0$ will always end up with 0.

iv) If x is in $C(A^T)$ what is A^+Ax.

$$C(A^T) = c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \underbrace{\frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{A^+A} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow x$$

$\therefore A^+Ax = x \rightarrow$ we always recover x

what does mean?: If x is in the nullspace of A, it just kills it,
if x is not in nullspace of A, we get x back.
$\therefore$ it is the identity matrix acting on x, in $C(A^T)$