Rec: Four Fundamental Subspaces:

Suppose: — Product of lower triangular matrix and upper triangular matrix.

$$B = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Find a basis for and compute the dimension of each of the 4 fundamental subspace of (B)

$\longleftrightarrow$ LU Decomposition of B.

$$B = LU$$

Lets do one space at time:

Column Space:

What is the dimension of Column space

$\dim C(B) = r = 2$ pivots.

Let look at U matrix, how many pivots do we have : 2 pivots

How do we find basis for Column space:

~~$C(B)$~~

One way:

take Matrix B, do elimination on it, then take pivot column of original matrix

another way:

Can also take the pivot column in L matrix. (will amt to same thing)

$$\begin{pmatrix} 1 & & \\ 2 & 1 & \\ -1 & 0 & 1 \end{pmatrix}$$

$C(B)$

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \Leftarrow$$ Basis for Column space of B.

Null space:

Dim $N(B)$ =

No of Column less No of Pivots

$\Rightarrow$ No free variables $= 1$

How do we find the 1 vector in the null space. ③

A basis for $N(B)$ is

$$\left\{\begin{pmatrix} -3/5 \\ -1 \\ 1 \end{pmatrix}\right. \qquad \text{using} \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

plug in 1 for free variable, and Back solve to get the other 2

<u>Row Space:</u>

What's the Dim of row space

$$C(B^T) = 2. \circledast$$

— Same as Dim of Column Space
∴ just No # of pivots. ⊛

Basis for row space?

<u>one way</u>: we got U (matrix) from B doing elimination, and elimination does not change the rowspace

∴ Can just use the two (2) pivot rows of Matrix (U).

Basis for rowspace $C(B^T)$ is
$$\begin{bmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Left nullspace or nullspace of B^T

$$\text{Dim } N(B^T) = \overset{3}{\#\text{rows}} - \overset{2}{\#\text{pivots}}$$
$$= 3 - 2 = 1$$

take $B = LU$

and invert L

and take $EB = U$

Move L over to left hand side

inverse of L matrix

$$\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 1 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- Now that L is on the left hand side, now I can read off the vectors in Left null space

- Not my pivot, but my free variables

But want to look at this E matrix

third row of E matrix

$$\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Corresponds to free row

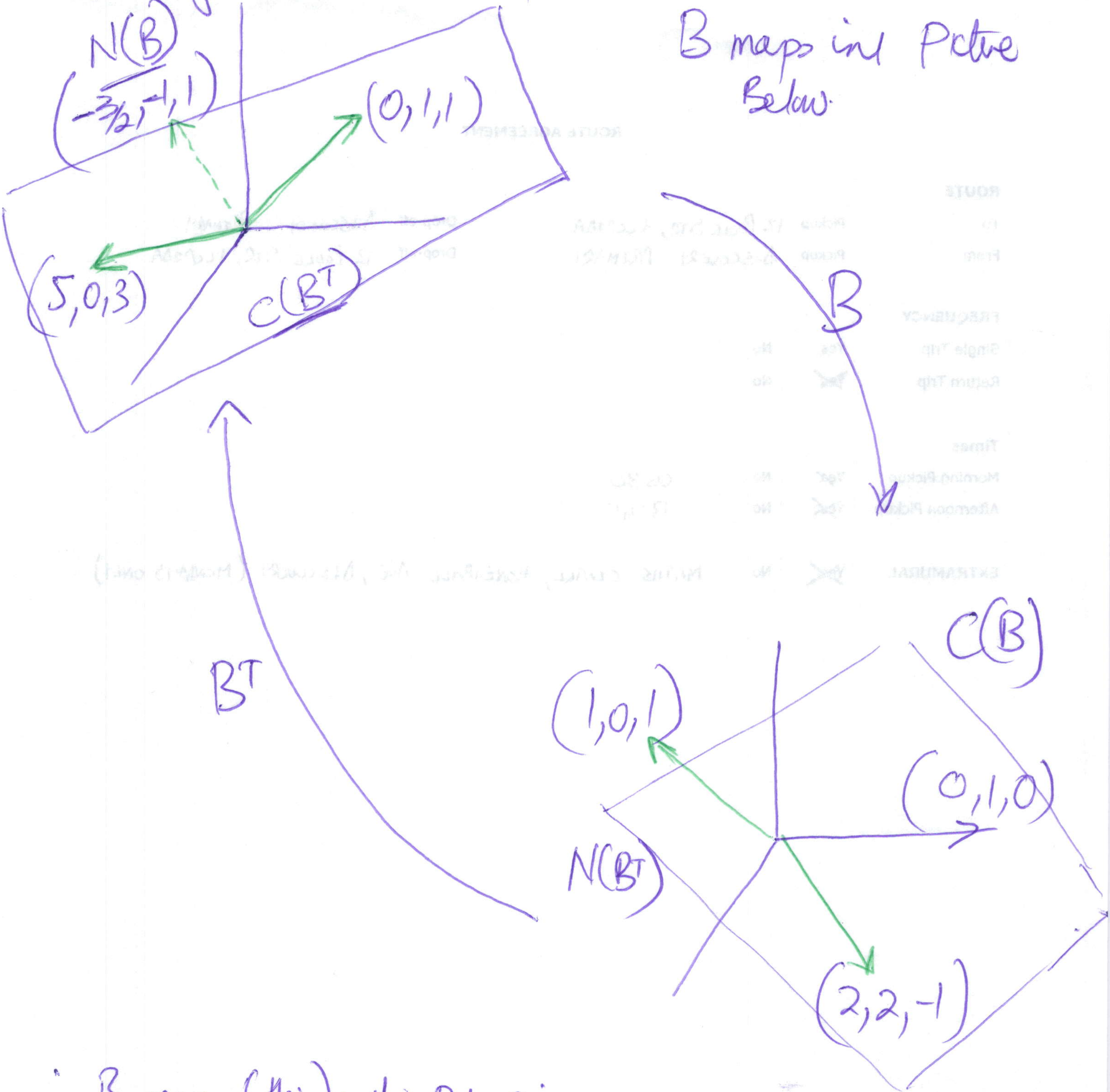
⇒ and multiply it with B, so just get zero's

Basis for $N(B^T)$ is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

⇒ Now lets recall which of the L or U matrix we used for each of the subspace?

- Column space we used the pivot column of L matrix $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
- for nullspace we looked at U matrix
- row space we also looked at U matrix
- left null space we inverted the L matrix.

Last thing is the draw a picture:



- ∴ B maps (this) into Below Pic
- the Nullspace(B) all go to Pic Below (zero)
  B takes everything else including rowspace in Column Space (Pic Below)
- B^T kills $(1,0,1)$ this vector, and takes everything else into rowspace, Column Space of ~~B~~ $C(B^T)$