REC

Today problem is about Change of Basis

The vector space of all polynomials in x of degree ≤ 2 has basis $1, x, x^2$

Let w_1, w_2, w_3 be a different basis of polynomials whose values at $x = -1, 0, 1$ are given by:

we don't know it explicitly

But we know their values at x given by table

x	w_1	w_2	w_3
-1	1	0	0
0	0	1	0
1	0	0	1

4) Express $y(x) = -x + 5$ in this Basis (w_1, w_2, w_3)

5) Find the change of basis matrix ~~(1, x, x^2)~~

$(1, x, x^2) \rightleftarrows (w_1, w_2, w_3)$

which is linear map of this space

6) Find the matrix of "taking derivative" both of these Basis

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a) Find ~~y~~ α, β, γ

(A) $y(x) = \alpha \cdot w_1(x) + \beta \cdot w_2(x) + \gamma \cdot w_3(x)$

- one way to do, is look at tables of values ~~explicitly~~ explicitly find w_1, w_2 and w_3

<u>But better way?</u>

try to do it, without finding w_1, w_2 and w_3 (explicitly)

Lets see the values of y at these points

$\therefore y = -x + 5$

x	w_1	w_2	w_3	y
-1	1	0	0	6
0	0	1	0	5
1	0	0	1	4

$\therefore$ Let evaluate (A), with $x=-1, x=0, x=1$

(B)
$$\begin{cases} y(-1) = \alpha \cdot w_1(-1) + \beta \cdot w_2(-1) + \gamma \cdot w_3(-1) \\ (0) = \ldots \\ (1) = \ldots \end{cases}$$

this of this as linear system that has unknowns α, β and γ.

"Coefficients" →

∴ into in matrix, and read coeffs from A B

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

$y = 6w_1 + 5w_2 + 4w_3$

∴ no matter what values we put in B, left hand side above will remain the same, only right hand side will change.

So if we given any other polynomial now: to express Basis w_1, w_2, w_3 we dont have to do anything, or any computation

— What we do we go back to our table in beginning and read off y values 6, 5, 4.

⇒ Hint to next question (b)

b) Change of Basis : ∴ Expressing one basis i.t.o. another.

X	w_1	w_2	w_3		1	X	X^2
-1	1	0	0	6	1	-1	1
0	0	1	0	5	1	0	0
1	0	0	1	4	1	1	1

∴ $1 = w_1 + w_2 + w_2$

$X = -w_1 + w_3$

$X^2 = w_1 + w_3$

∴ we can write one change of basis matrix.

$1, x, x^2$ expressed i.t.o. w_1, w_2, w_3

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \end{array}$$

$$A^{-1} = \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ x \\ x^2 \end{array}$$

c)

$$D_x = \begin{array}{c|ccc} & 1 & x & x^2 \\ \hline 1 & & & \\ x & & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} & \\ x^2 & & & \end{array}$$

$$D_w = A \cdot D \cdot A^{-1}$$

$$\begin{bmatrix} -3/2 & 2 & -1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & -2 & 3/2 \end{bmatrix}$$