Lec

Final Lectures Determinant
and its about its applications.

we got : 1) Properties
2) Formula

Now, to use the determinant
(or applications to use it)

Det packs all this information
into single number

and that number can give us
formula's for all sorts of
things we've been calculating
already /
WITHOUT formula.

Lets take: recognise? Co-factors?

— what was A^{-1}

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(for two by 2 easy)

what did I divide by:

— the determenant

looking for formula where it has 1 over the determinant.

⇒ times matrix of co-factors.

Big formula for A^{-1} — Cofactor matrix

$$A^{-1} = \frac{1}{\det A} C^T$$

But why is this formula correct?

Lets test other situations.

Lets look at (3x3)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1}$$

∴ looking for its inverse

∴ things with 3 factors go in $\rightarrow \dfrac{1}{\det A}$, ↑ products of n entries

things with "$n-1$ (factors)" C^T ← products of $n-1$ entries

Check:

$$A^{-1} = \frac{1}{\det A} C^T$$

$$AC^T = (\det A) I$$

But why is this true?

we have:

multiply by Cofactor matrix . T

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & & C_{n1} \\ C_{12} & & \\ \vdots & & \\ C_{1n} & & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \det A \end{bmatrix}$$

usually the 1 coming first that I am in row, But I have transposed, so those are Co-factors, those columns are cofactors of row 1

What do I get in the (1,1) entry

row 1, with Col 1st Co-factors.

what will I get here.

$a_{11} \times C_{11} + a_{12} \times C_{12} + \ldots$

$\Rightarrow$ sums up to ... (Co-factor formula for the Det)

First row with First Column always give det

But first row with say Col 2 will give you zero

Why?

Reason why answer zero?

∴ we really doing: $\det(2\times2)$

$$A_S = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$$

∴ Co-factor for the determinant steps:

— take a, times its co-factor (b) (1st row)

$+ \; b$ (top row) $\times (a)$ Bottom row.

$$\Rightarrow \det A_S = ab + b(-a) = 0$$

OK this is 2x2 picture.

Same here $(n\times n)$:

Second application:

$$Ax = b$$

$$\therefore x = A^{-1}b$$

But now I have a formula for A^{-1}

$$x = A^{-1} = \frac{1}{\det A} C^T b$$ multiplying cofactors by entries of b.

∴ now I have formula for answer (where we used elimination before)

Cramer's Rule a way of looking at this formula.
∴ formula for x.

Lets take x_1, ie the first component of this formula

$$x_j = \frac{\det B_j}{\det A}$$

Cramer realised what matrix it was, what B_1 and B_2 where.

What is B_1? , is matrix that has b in First Column, and rest is A.

$$B_1 = \left[\begin{array}{c|c} b & \begin{array}{c} n-1 \\ \text{Columns} \\ \text{of } A \end{array} \end{array}\right]$$

matrix A with Column 1 replaced by right hand side (b)

In general what's B_j

B_j = A with Column j replaced by b

But is Cramer's rule any good in practice?

But this takes forever to complete (the determinant)

when you can just do elimination...

But at least we have formula

So, advised not to use it.

$\Rightarrow$ Matlab won't do it

But will use elimination

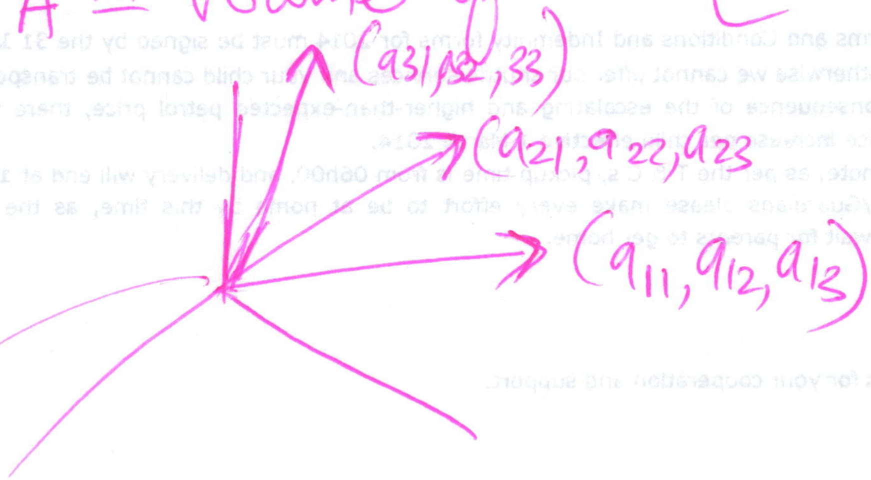
Applications 3

⇒ the determinant gives a volume

$|\mathrm{Det}| = $ volume of something.

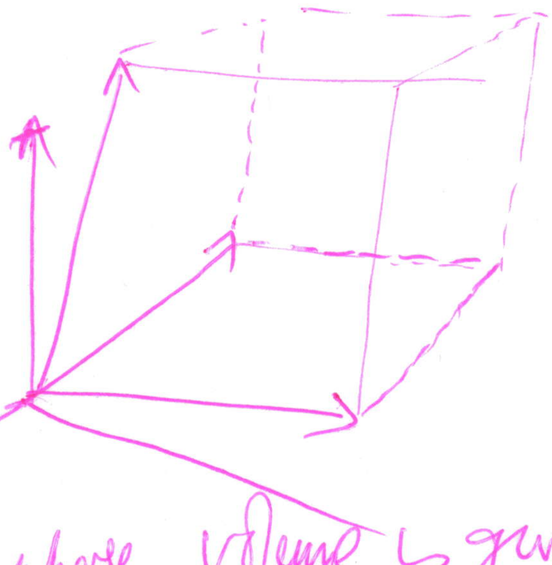
Lets look at special case (make following case:)

$\det \overset{(3\times3)}{A} =$ volume of Box [which Box?]



redrawn

this Box, whose volume is given by the determinant.



its that Box, whose volume is given by the determinant.

(9)

is it a right handed box or left handed box.

$A = I$ ← is that equal to volume of box.

then Box is a cube

det = 1.

Suppose I have an orthogonal matrix.

Now

$A = Q$ [orthogonal matrix]

matrix whose column were orthonormal.

i.e. Colums were 1 unit vectors.

if shape matrix is an orthogonal matrix then shape of box, is a cube (as well)

→ How is it different from the I cube? just rotated. (turned in space)

But is the det = 1?

if we have $A = Q$

$Q^T Q = I$

why is det = 1? or ± 1

take det's of both sides

$\det Q^TQ = 1 \; (I)$

whats the dete. of that product.

determ of product: $|Q^T| \cdot |Q| = 1$

Whats det of Q^T, same as det of Q

Lets push it to rectangular boxes? (non cubes)

of stretch / double first edge

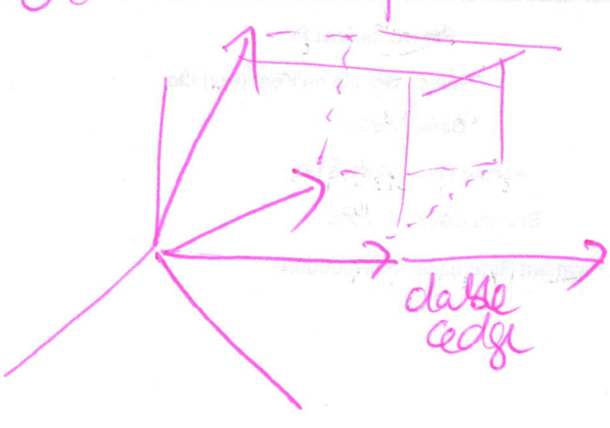
$|Q|^2 = 1$

→ Lets push this to non cubes

rectangular boxes.

keep ⊥ and stretch the edges

double the first edge



(1c)

- Keeping the other edges the same ?
  - what happens to the volume?
    - it volume double.

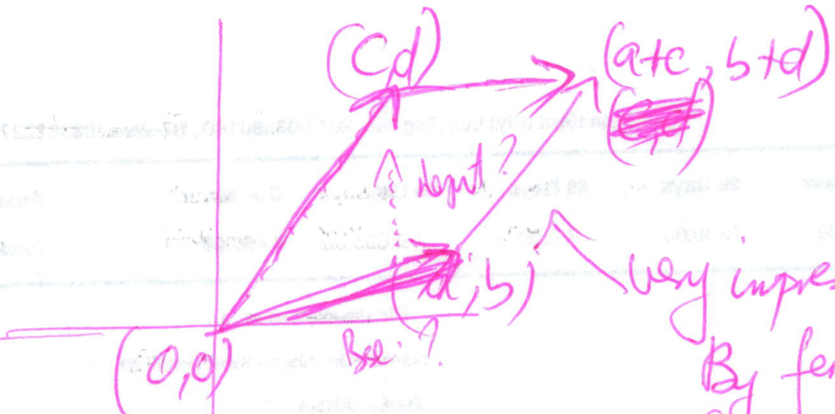
What happens to the det?
  - it also doubles (rule 3A)

$$|\det A| = \text{volume of Box}$$

has properties: 1, 2, 3a, 3b.

What's prop 3b:

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$



very impressed

By formula for area of parallelogram

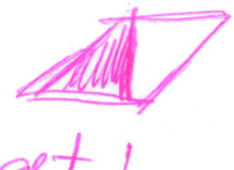
∴ area of parallelogram: Base x height
(does not nice to do on graph)

But rather use det

∴ $\text{area} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$= ad - bc$

No square roots, totally rememberable

∴ if you know the coordinates of box, area of volume
then you have great formula for volume
– that does not involve any length or
any angle, or any heights
– just involves the coordinates.

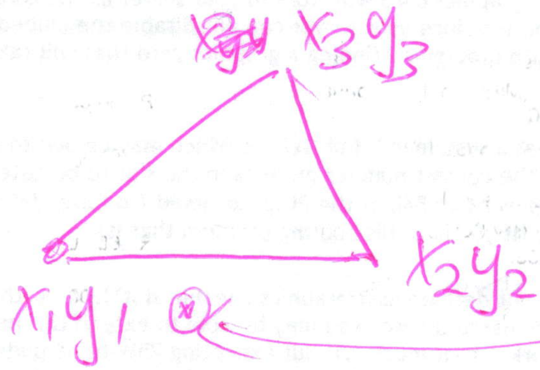


∴ Now we can half Box, and get
area of triangle.
everyone would say: Half base x height.
– that's fine
But here, we know the coordinates.

– we know the vertices

area triangle $= \frac{1}{2}(ad-bc)$ for triangle.

But if what if triangle did not start at (0,0)



we know area of this ⊗ is zero!

∴ we have any 3 corners.

∴ these 6 numbers must determine the area.

∴ what's the formula:

$$\text{area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$