Lec

Singular Value Decomposition = SVD

*** Final and BEST FACTORIZATION ***
of Matrix

factors will be

$$A = U \Sigma V^T$$

U = orthogonal

$\Sigma \Rightarrow$ diagonal

$V^T \Rightarrow$ orthogonal

things we've seen before, these good matrices

New Point: we need two (2) orthogonal matrices

A = can be any matrix, different

But need A to do it

Note: This factorization has jumped into importance

⇒ bringing together everything in this course.

Bring together.

S.P.D ⇒ Symmetric Positive Definite.

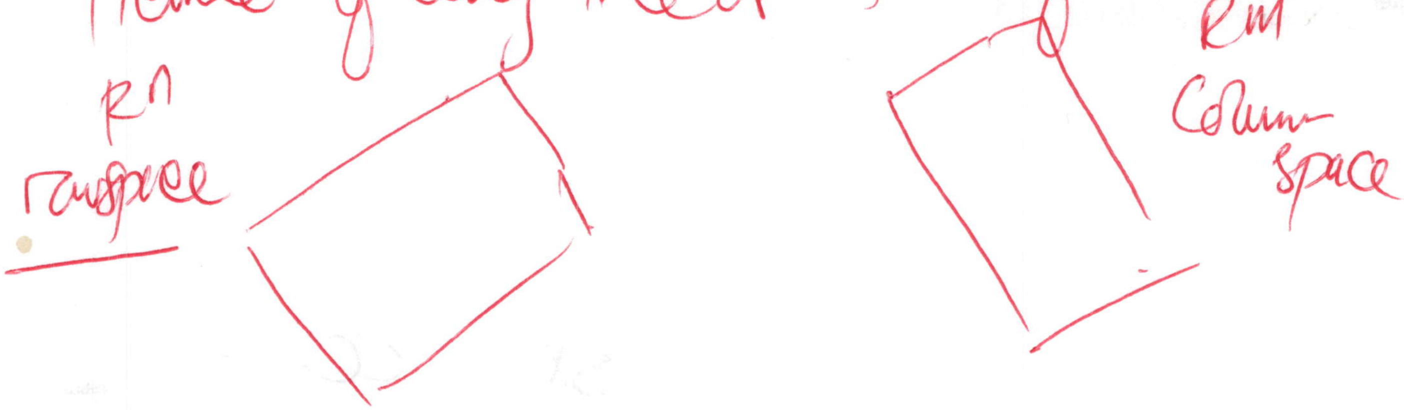
$$A = Q\Lambda Q^T$$ (usual one)

$$\cancel{A = S\Lambda S^{-1}}$$ (~~eigenvalues~~ not good eign)

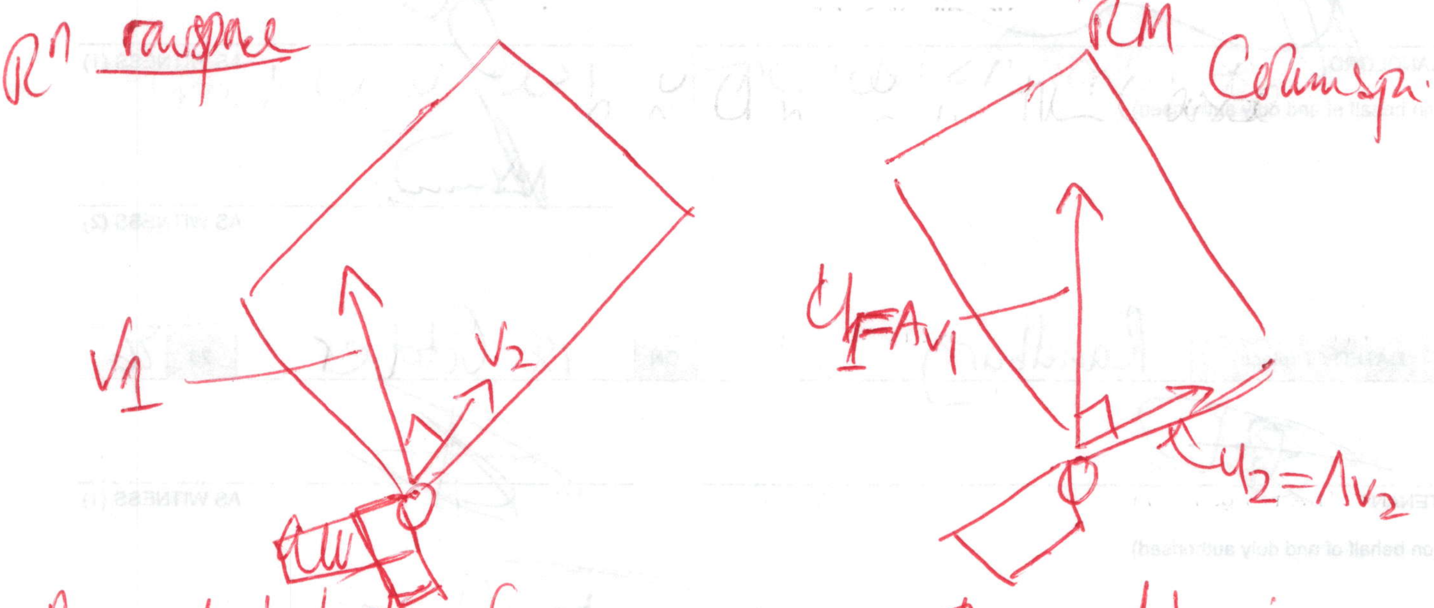
we looking now for [ortho. diag. ortho.]

⇒ what does that mean and where does it come from →

Picture of any linear transformation



typical vector in the rowspace v_1, is taken over to same vector in column space.



Now what looking for now is orthogonal basis in rowspace to get knocked over to orthogonal basis ~~nex~~ in column space

∴ that is pretty special.

∴ another vector in rowspace — into another vector in column space

∴ v_2 into u_2

But can I find a orthogonal basis in rowspace? — Yes.

Graham Schmidt tells me how to do it.

But if I take any ortho. basis

and multiply by A, there is

no reason why it should be

orthogonal over (on right hand graph)

Let stick in nullspace in both graph's

$$A\begin{bmatrix} v_1 & v_2 & \cdots \end{bmatrix}$$ (for rowspace)

Instead of making it orthogonal, let

make it orthonormal,

i.e. make it unit vectors

$v_1 \ldots v_2$ may be unit vectors

and go over onto multiples of unit

vectors on column space

$$\underset{\text{multiple}}{\sigma_1} u_1 = Av_1 \qquad \sigma_2 u_2 = Av_2$$

$\Rightarrow$ this is goal, and express it

into matrix language

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$$A = [v_1, v_2 \cdots v_r]$$

Explain what I want to do with figure!

$$= [u_1, u_2 \cdots u_r]\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ & & \ddots \end{bmatrix}$$

$AV = U\Sigma$ ← goal

orthonormal rowspace

orthonormal colspace

R^2 ← lost after.

$A\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$: v_1, v_2 in rowspace R^2

u_1, u_2 in Colspace R^2

$\sigma_1 > \sigma_2 > 0$ ← scalars

But not diagonal!

Wait:

$$Av_1 = \sigma_1 u_1$$

$$Av_2 = \sigma_2 u_2.$$

Using

$$AV = U\Sigma$$

$$A = U\Sigma V^{-1} = U\Sigma V^T$$

I have two, orthogonal matrix here,
But I don't want it all at once

I need some expression that will make U's disappear,
and leave me only with V's

We can use: Collapse into I

$$A^TA = V\Sigma^T U^T U \Sigma V^T$$

and $\Sigma^T\Sigma$ are diagonal.

$$= V \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix} V^T$$

A^TA

∴ U is out of picture

∴ V is E'vectors

σ^2 is E'values ≥ 0

How do we find the U's

— Can look at A^TA.

⑦

V's are e'vectors of A^TA

U's are e'vectors of AA^T ← (different)

Σ's are square roots of σ (e'values)

→ ~~Ademonstrate~~ demonstrate above

Lets take:

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

What's first step?

Compute A^TA since I want its e'vectors.

$$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}}_{A} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

So e'vectors will be V's

, e'values will be squares of σ.

e'vectors. e'value for A^TA

take square root for Sigma ~~sigma~~ ~~σ~~

1st e'vector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 32 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 18 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(E.V.

But need to normalize E'vectors

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$$A^TA\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} = 32$$
$$ATA=\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} = 18$$

Stay same. V's

Here: A: $U \quad \Sigma \quad V^T$

$$\begin{bmatrix}4&4\\-3&3\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\sqrt{32}&0\\0&\sqrt{18}\end{bmatrix}\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}^{T}$$

Find U's: u_1, u_2 (using AA^T)

$$AA^T = U\Sigma \underbrace{V^T V}_{I}\Sigma^T U^T$$

$$= U\Sigma\Sigma^T U^T$$

Comes out DIAGONAL

AA^T

$$AA^T = \underbrace{\begin{bmatrix}4&4\\-3&3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}4&-3\\4&3\end{bmatrix}}_{A^T}=\begin{bmatrix}32&0\\0&18\end{bmatrix}$$

E'Vector → $AA^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 32 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Filled Above

$AA^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 18 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

But got 32 and 18 again

E'values of AB = E'values of BA.

Same issue with answer in Y

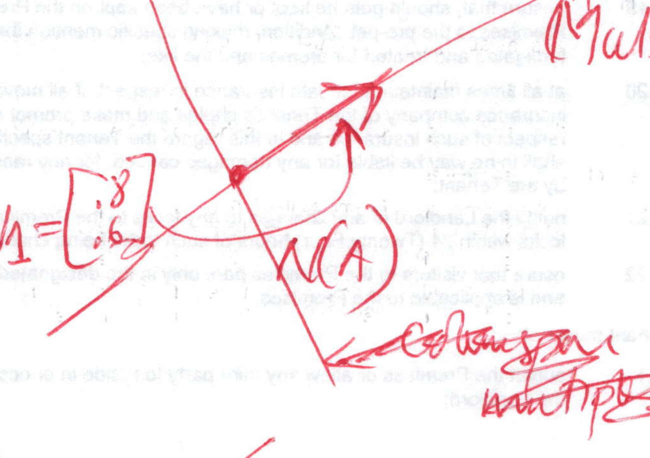
Come Back

take another example (2x2) rank=1

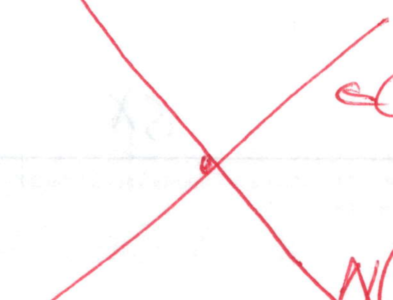
$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

row space

Multiples of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$



← ~~Column space multiples of $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$~~



← Column space is multiples of $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

⇒ Using the orthogonal basis for rowspace and column space ⇒ no problem.
⇒ They only 1 dimensional.

i) V_1 (one dimensional) will be the vector will be $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ But made into unit vector

$$v_1 = \begin{bmatrix} .8 \\ .6 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} .6 \\ -.8 \end{bmatrix}$$

ii) u_1 will be the unit vector.

$$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Let Complete the Singular Value Decomposition for this matrix

need to take square root — rank 1

$$\underset{A}{\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}}$$

all to produce ⟸ ⊢ Columns —| ⊢ diagonal —| ⊢ rows —|

diagonal   orthogonal

$$\Sigma = A^TA = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

rank 1, so every row is multiple of 4, 3

ϵ Values $= 0, 125$ → which is Sigma Square

$$U = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

orthogonal

Dim Rowspace and Columnspace is rank r

$v_1 \dots v_2$ orthonormal Basis for rowspace

$u_1 \dots u_2$ " " " Columnspace

Dim: $(n-r)$ $v_{r+1} \dots v_n$ " " nullspace

Dim: $(m-r)$ $u_{r+1} \dots u_m$ " " " $N(A^T)$

these Bases make the matrix diagonal.

$$Av_i = \sigma_i u_i$$