Rec G/S orthogonalization:

we have matrix A with Colum, a,b,c

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

and we need to find orthogonal vectors q_1, q_2, q_3 from the above 3 Column

i.e a, b, c [Column of A]

Then write A as QR (Decompostion)

where Q is orthogonal matrix, and R is upper triangular)

Rem: orthogonal matrix, is a matrix whose columns are orthonormal vectors.

GS:

Consist of the following:
At each step you find your orthogonal vector (q_1, q_2, q_3) by taking the vector that you start with (a, b, c) and making it orthonormal to previous ones ($q_1 \ldots q_3$)

Find q_1: Start with a, and make it ~~q_1~~ orthonormal to the previous ones.
There is no previous ones, so that is very easy
⇒ direction of a is fine, so you just need to ensure that vector has length one (1)

$q_1 \neq 7$

a is vector (1, 0, 0)

$\therefore q_1 = \frac{a}{1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$q_2 \neq b$

to get q_2: will start with vector b, then make it orthogonal to what I already have, which is q_1

for that I am going to subtract off from b, the projection of b onto q_1.

$\therefore q_2 = b - (b \cdot q_1) q_1$

usually when you do the projection of vector onto another vector, you have to divide it by length of in this case q_1, but because q_1

has length 1, you don't need to do that.

So what will it be here:

$$q_2' = b - (b \cdot q_1) q_1$$

$$\therefore = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Ⓐ q_2' the vector is orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = q_1$

How do we check, by doing the dot product.

Ⓐ But we need this vector also to have a length 1.

→ Because we want these vectors to be orthonormal.

$$\therefore q_2 = \frac{q'^2}{3 \text{(length)}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$q_3 = \cancel{c}$ I start with my 3rd vector, c, then I want to subtract the projection of c onto q_1 and q_2, and that will give me a q_3 that's orthogonal to both q_1 and q_2.

$$q_3 = c - (c \cdot q_1)q_1 - (c \cdot q_2)q_2$$

$$= \begin{bmatrix}4\\5\\6\end{bmatrix} - \overset{(q_1)}{4}\begin{bmatrix}1\\0\\0\end{bmatrix} - \overset{(q_2)}{6}\begin{bmatrix}0\\0\\1\end{bmatrix} = \overset{B}{\begin{bmatrix}0\\5\\0\end{bmatrix}} q_3'$$

B vector is orthogonal to q_1 and q_2 but it is not orthonormal yet.

$$q_3 = \frac{q_3'}{5} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

<u>Note</u>: my vectors q_1, q_2, q_3 are very nice, usually when you perform GS orthogonalization, you end up with lots of square roots

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Since you dividing by the length.

in the case

In our case we have everything as integers, which is very lucky.

<u>Next:</u> we want to write the QR decomposition of the matrix A.

In Q, you want an orthogonal matrix
— such matrix have orthonormal vectors as its columns
— But we already have such a matrix

$$A = QR$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{3rd raw (A)} \\ \leftarrow \text{2nd raw of (A)} \end{array}$$

So how do we get R?

Lets look at matrix Q; it is simply a permutation matrix

Then we need an upper triangular matrix that makes the equality true

— so its very easy to come up with matrix for R

∴ What the permutation matrix (Q) does; it exchanges raw 2 and 3 of my matrix R, to give you A.

∴ we then know what R must be:

∴ and indeed R is upper triangular.

But lets see where these number in matrix R is Coming from.

$$
\begin{array}{c}
A \\
\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} \\
a \quad b \quad c
\end{array}
=
\begin{array}{c}
Q \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
q_1 \quad q_2 \quad q_3
\end{array}
\begin{array}{c}
R \\
\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \\
a \quad b \quad c
\end{array}
$$

∴ You knew from the way matrix multiplication work.

That $a = Q$ times First Column(R)

(R) first column

(First Column) $a = 1q_1 + 0q_2 + 0q_3$

(2nd Column) $b = 2q_1 + 3q_2 + 0q_4$

$c = 4q_1 + 6q_2 + 5q_4$

Lets see where these numbers are showing up. 8

$$\therefore q_1 = \frac{a}{1} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad a = 1q_1$$

$$3q_2 = q_2' = b - (\overset{2}{b \cdot q_1})q_1$$

$$q_3 = \frac{q_3'}{5} \qquad [q_3' = 5q_3]$$

$$5q_3 = q_3' = c - (\overset{4}{c \cdot q_1})q_1 - (\overset{6}{c \cdot q_2})q_2$$