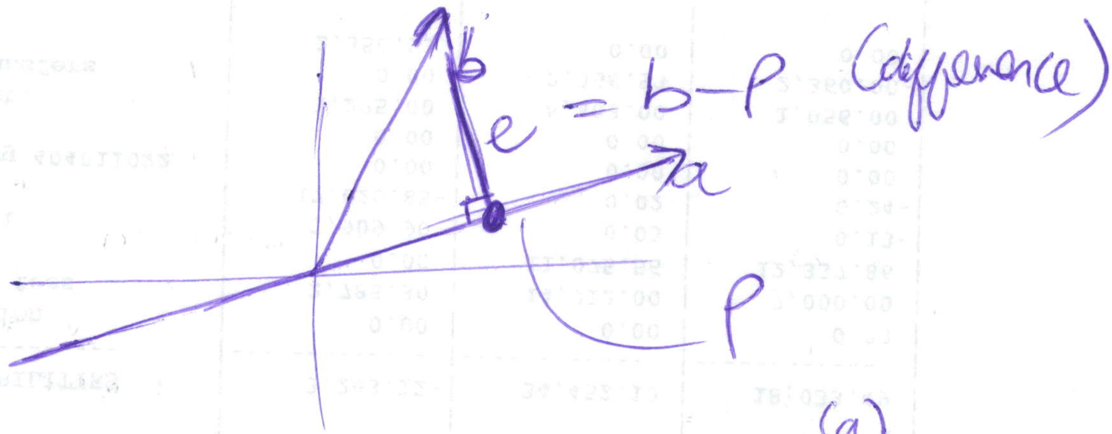
Important Lecture: About Projections

Lets start by projecting a vector b down on a vector A.



Like to find the point along this line (a), closest to b

⇒ Starting with one Dim,

Where is point closest to b (on line a)

It is p. [perpendicular to a]

e is how much I am wrong (or difference (b − p))

Whole pro

What do we know?

we know p (the projection) is some multiple of a [its on a's line]

its on the 1 Dim Subspace

$p = x\,a$ ← same multiple of a.

the number (x) I would like to find.

$\therefore \quad a^T(b - xa) = 0$

words: "a is perpendicular to $(b - xa)$ which equals zero"

Lets simplify:

$$a^T(b - xa) = 0$$

$$x\,a^Ta = a^Tb$$

$$x = \frac{a^Tb}{a^Ta}$$

(and) $p = ax$

Answer for x

$$\therefore\ p = a\frac{a^Tb}{a^Ta} \quad [\text{projection}]$$

Suppose, b is double, 2b

what happens to the projection?

projection will go twice as far

→ what happens if double a? ← that I am projecting onto.

projection does not change at all.

line does not change, still the same line.

⟹ projection is carried out by same matrix.

Called Projection Matrix.

$$p = a\frac{a^T b}{a^T a}$$

Proj(Matrix) $p = Pb$

Some matrix, that act on b. (input)

$$\therefore P = \frac{aa^T}{a^T a}$$

$\therefore$ = full scale $n \times n$ matrix

(Column space) $C(P)$ = line through a

$\text{rank}(P) = 1$

Is the matrix symmetric? Yes.

(x) $P^T = P$

What happens if I do a projection twice?

$\therefore$ will be at same point

(x) $\therefore$ $\overset{\text{(twice)}}{P^2} = P$ ← get same answer. project twice, as I did in first projection.

∴ Two properties that I am looking at a projection matrix: 5

(1) Its symmetric: $P^T = P$

(2) Its square is itself: $P^2 = P$

Why project?

Because we dealing with equations, $Ax=b$ that may have no solution.

Can't solve it.

So what do I do?

I solve the closest problem that I can solve

what is the closest one?

- Ax will always be in the column space of A. [problem]

- But b is probably not in columnspace

- So what do I change b to?

∴ Choose the closest vector in Column space.

∴ So instead, I will solve.

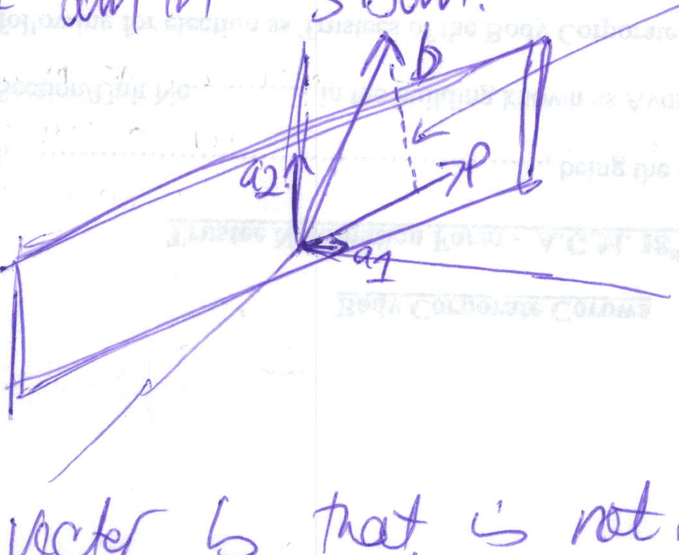
$Ax = p$

Projection of b onto Column space

good righthand side that's in Column space, that's as close as possible to b

⇒ then I know what to do.

Lets say I am in 3 Dim. $e = b - p$



But vector b that is not in plane

, But want to project b down into the plane.

⑦

∴ what's formula for projecting b onto plane?
(remember, right angle will be crucial)

But, firstly what's the plane?
— what information do we need

1) Basis
2) 2 vectors, a_1 and a_2

a_1 and a_2 don't have to perpendicular
But they must be independent

$\Rightarrow$ Plane of a_1, a_2

∴ plane is Column Space of (what matrix)

$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$

, but can be 2 columns or n columns
These 2 columns describe the plane
$\Rightarrow$ and to project.

and given vector b that probably not in column space

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, But if b is in Column Space
then projection is simple, its just b

But mostly I have error $e = b - p$ (is perpendicular to plane)

∴ What is P?

$$P = \hat{x}_1 a_1 + \hat{x}_2 a_2$$

Some multiple of ↑

But don't want to write out

Rather write:

$$p = A\hat{x}$$

∴ find the right combination of Column, so that the Error (e) vector is perpendicular to plane

$\therefore$ $p = A\hat{x}$, Find $\hat{x}$

Key: $\overbrace{b - A\hat{x}}^{e}$ is perpedicular to plane

$\therefore$ perp. to a_1 and a_2.

$\therefore$ $a_1^T(b - A\hat{x}) = 0$, $a_2^T(b - A\hat{x}) = 0$

$\Rightarrow$ These are my two equations.

But I need this in matrix form.

put two equations together as a matrix equation.

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T \underbrace{(b - A\hat{x})}_{e} = 0$$

$\therefore$ what subspace is e in?

$\therefore$ e in $N(A^T)$

What do we know about $N(A^T)$:

this says $e \perp C(A)$

∴ Our Equation:

$$A^TA\hat{x} = A^Tb$$ ⇒ will give us x.

⇒ 1 Dim ~~Cas~~ Case.

∴ What's

1) $\hat{x}$
2) Projection
3) Projection matrix

⇒ now we ready for n-Dim Case.

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$p = A\hat{x} = A(A^TA)^{-1}A^Tb$$

matrix $P = A(A^TA)^{-1}A^T$

$P^T = P$

$P^2 = P$

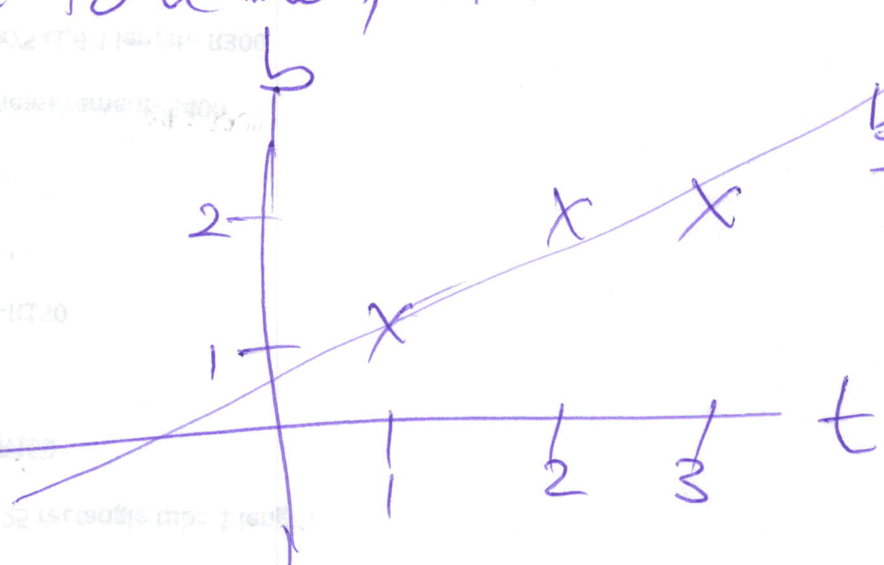
Least Squares

Fitting "a line" } Terms

Lets Start problem:

⇒ given Bunch of data points, they lie close to a line, but not on a line

$b = C + Dt$



- given 3 points and fit them by line:

$(1,1), (2,2), (3,2)$

∴ looking for best line to "fit them"

∴ looking for $b = C + Dt$

∴ looking C and D

$$\left.\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2 \end{aligned}\right\}$$ write down the equations

∴ What is the equation that we can't solve

∴ What Matrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$A \quad x = b$$

Note: No Solution, but looking for best Solution.

Note Can't Solve $Ax=b$

But we Can Solve: $A^T A \hat{x} = A^T b$