Cover now Linear Transformation:
(Previously Course use to start with
This ~~lect~~ topic first)

∴ without coordinates: No matrix
(with coordinates ⇒ matrix)

Physicist like it this way, they don't
like coordinates.

Eg:

1. Projection → Can describe a
projection, without revealing
any matrix
(or anything about matrix)

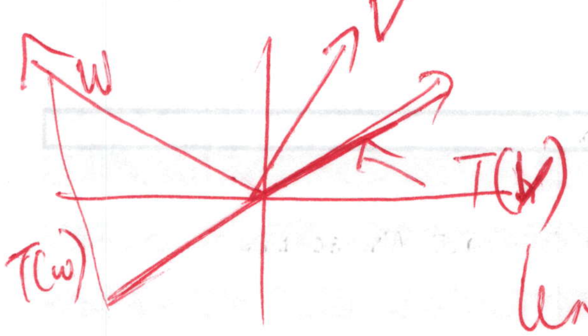
linear transformation

$$T: R^2 \longrightarrow R^2$$

"linear transformation that takes all of R^2
(every vector in plane), into
a vector in plane."

- this is a way people describe a mapping

By what rule?



- line and project every vector into that line

take ~~of the~~ the vector V, the linear transformation will produce the vector ~~of T(V)~~ as T(V) T of V

Exactly like a function.

- You give an input (V), the transformation produces output T(V)

- transformation: sometimes the word map is used map between inputs and outputs

eg 2 take another vector (w), what is T(w)

Here are rules for linear transformation:

$$T(V+w) = T(V) + T(w)$$

$$T(cV) = cT(V)$$

$$T(cv+dw) = cT(v) + dT(w)$$

∴ transformation does something special w.r.t. those operations.

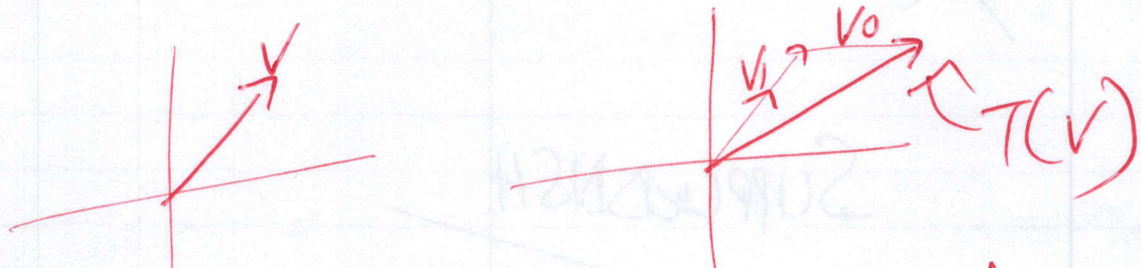
③

∴ Projection is Linear Transformation.

because if I want to take V to be twice as long, the projection will be twice as long.

Not difficult to show if Transformation is linear or ~~Not~~

~~suppose~~ Non-Example eg: Shift the whole plane by V_0



∴ take V and add to it (simple transformation)

But is it linear? **NO**

Which Laws Broken?

∴ if I double the length of V, does the transformation produce something double to I double T(v), NO

I am not adding 2 Vs, just 1 Vs ?

Aside:

that $T(0) = 0$

Another NonExample:

$$T(V) = \|V\|$$

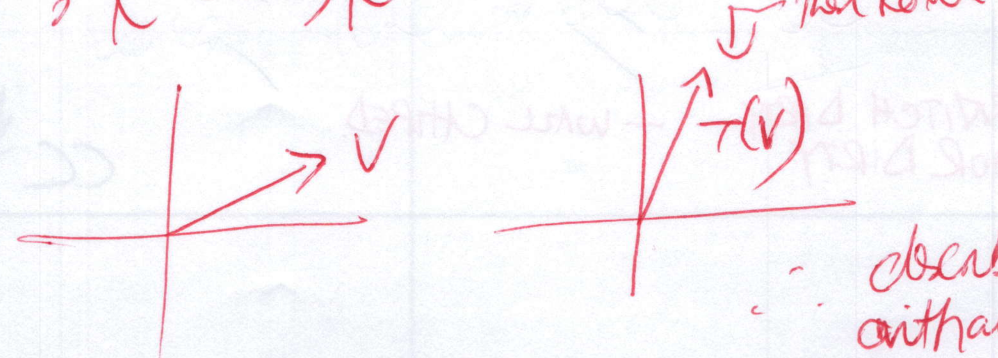
takes any v and produce R^3 : $T : R^3 \to R^1$

$\Rightarrow$ that is not linear.

Example:

Rotation by $45°$

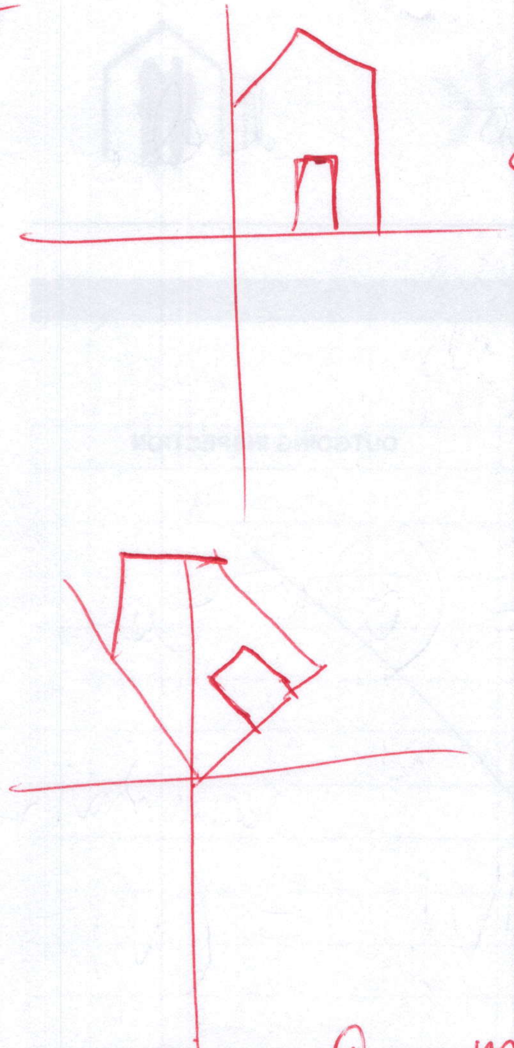
$$T : R^2 \to R^2$$



∴ describe it without coordinates

Eg

House $\xrightarrow{(in)} R^2$



← whole lot of points...

← idea here I can take all vectors, on outline of house and see where they all go

→ don't have to take 1 vector at time.

∴ Whole house will rotate.

Example 3. Coming from matrix A. ← this is linear Transformation

$$T(v) = Av$$

↓ check

$$A(v+w) = Av + Aw, \quad A(cv) = cAv$$

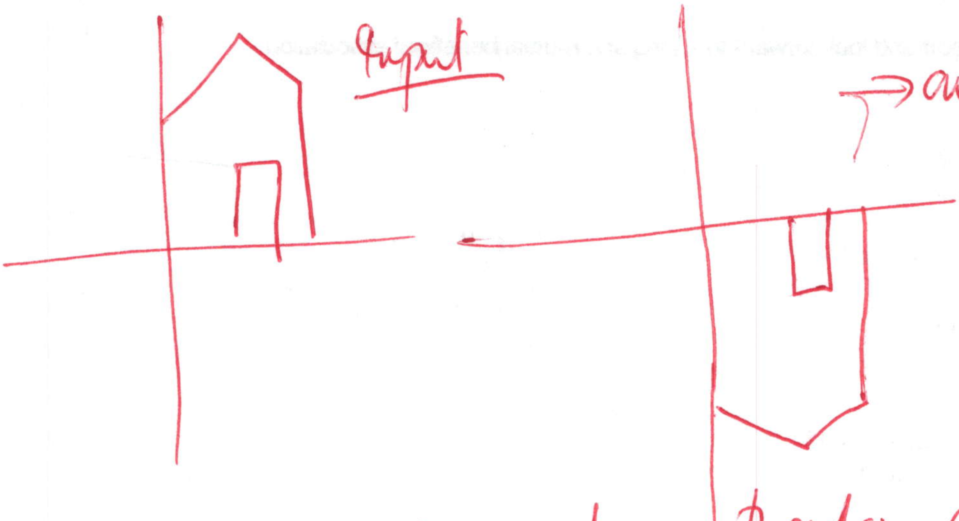
∴ whole plane is transformed, by matrix multiplication

— every vector in plane, gets multiplied by A.

Lets take Matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

→ applying this matrix? What happens to house.



goal: Find matrix that's defined By the linear Transformation

∴ we then have to bring in Coordinates

∴ Hence Choose Basis

Start: → Have linear Transformation.

$T : \mathbb{R}^3 \to \mathbb{R}^2$

$\mathbb{R}^3$ ← its inputs are vectors in $\mathbb{R}^3$

$\mathbb{R}^2$ ← outputs are vectors in $\mathbb{R}^2$

What's an example of such a transformation?

Example: $T(v) = A v$ ← shape matrix (2,3)

output in $\mathbb{R}^2$ ↑ ↓ input in $\mathbb{R}^3$

How much information is needed to know the transformation $T(v)$ for all input.

Can't take v_1 and tell what the transformation does to it (but thats only 1 vector)

$T(v_1)$, $T(v_2)$ ... up to → $T v$

, now you know what it does to v_1 and v_2

But knowing what it does to both, can you now know what ~~it does to the whole~~ what the transformation does

– (knowing only what it does to v_1 and v_2)

i.e. to larger bunch of vectors (know what transformation does.

- Now we know what it does to whole plane of vectors, v_1 & v_2.

assuming v_1 and v_2 are independent.

∴ If I know what the transformation does to every vector in a basis, then I know everything

∴ $T(v_1), T(v_2)$ up to $\ldots T(v_n)$
for any basis $v_1 \ldots v_n$
[This is all I need to be told]
to know $T(v)$ for all v's
then I know the Lin. Trans.

same combination

Since Every $v = c_1v_1 + c_2v_2 \ldots + c_nv_n$
(that's what basis is, it spans the space)

Now I will know:
$$T(v) = c_1T(v_1) + \ldots + c_nT(v_n)$$

So now what, the step that takes us from Lin. Trans free of coordinates to matrix? (w.r.t. coordinates)

So where does the Coordinates come from?
- we know once we decide on Basis, then every vector $(c_1, c_2 \ldots)$, these are coordinates in that basis

What are Coordinates?
- they come from a basis

(2) Coordinates of $V = c_1 v_1 + \ldots + c_n v_n$

we assumed, given

$$V = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix},$$

x 1st coordinate ~~1st co~~ — times 2nd coordinate — times 3rd coor

(A) that it is $3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

we assumed the Standard basis

But I might have had ~~of~~ another basis in mind.

eg. Evectors of matrix to be a basis

so (A) is an example but, (2) is real thing.

Construct matrix A that represents lin trans. T. [Projection or Rotation ... etc]

$T: R^n \to R^m$

transform from R^n to R^m
Dim Space Dim Space

I need two Basis
- Input Basis (Describe input)
- Output Basis (To give me coordinate)

Choose basis $V_1 \dots V_n$ for inputs from R^n

" " $W_1 \dots W_m$ " outputs from R^m

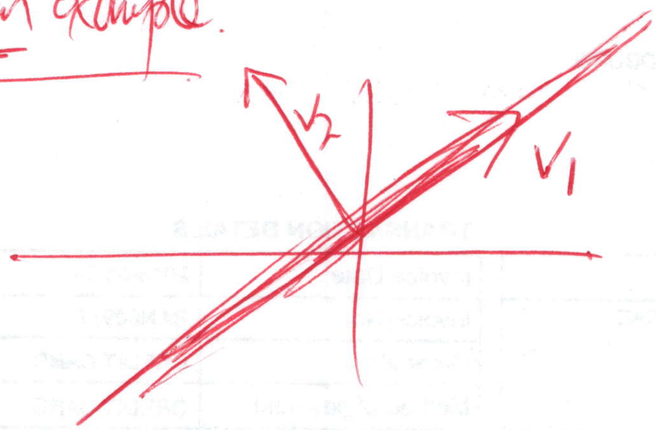
- Once I have chosen basis, that settles the matrix.

In words: I take vector V, I express it in its basis, so I gets its coordinates, then multiply those coordinates, and multiply those coordinates by the right matrix A and that will give me the coordinates of output in output basis

WANT:

- Matrix A

Take Projection Example:



But I will choose a basis that is better than Standard Basis ... Choose the same basis for inputs and outputs ...

the first Basis will be right on the line

v_1

and second will be perpendicular to line

v_2

and choose as the output Basis as well

$v_1 = w_1$

$v_2 = w_2$

What's matrix now?

Rule:

Take any V, which is some combination of first basis vector and second basis vector

$$V = C_1 V_1 + C_2 V_2$$

What is $T(V) =$

Lets say V_1 is input, whats output V_1
(Since projection leaves ~~this~~ it allone
⇒ on line)

What does the projection do to the second basis vector V_2
(it kills it, sends it to 0)

– and what does the projection do to a combination

C_2V_2 (got killed)

$$T(V) = C_1 V_1$$

(Now need find matrix)

B

⇒ which take c_1 and c_2 (c_1, c_2)
and gives output c_1 and 0 $(c_1, 0)$

Matrix

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}c_1\\ 0\end{bmatrix}$$

A · input coords   output coordin

Rem; Here the input Base, was same as output Basis

∴ matrix came out diagonal
as ~~Λ~~ Lamda Λ

Good Matrix
Eigenvector Basis leads to diagonal matrix of E'values (Λ)

∴ Good Coords are E'vectors

Suppose I did the projection in Standard Basis

Eg Projecting onto 45° Line and we use Standard basis $v_1 = \begin{bmatrix}1\\0\end{bmatrix} = w_1$, $v_2 = \begin{bmatrix}0\\1\end{bmatrix} = w_2$

Chapter 4 Uses

matrix $P = \frac{aa^T}{a^Ta} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$ ← (not diagonal)

∴ Rule to find Matrix A, given basis $v_1 - v_n$, $w_1 - w_m$

1st Column of A: How do I find this Column (tells me what happens to first basis vector)

↳ apply Lin Tran to v_1 ⇒ $T(\overset{\text{INPUT}}{v_1}) = a_{11}w_1 + a_{21}w_2 \dots a_{m1}w_m$

2nd Column of A: $T(v_2) = a_{12}w_1 + \dots + a_{m2}w_m$