Rec

1

Lets tackle a problem on Pseudo inverse

Given matrix $A = (1, 2)$ — not square

i) What is A^+ (pseudo inverse)

we define a Pseudo inverse using SVD

Since A is not square, the regular inverse does not exist, But we do know that the SVD exist for every matrix A, whether its ~~rectangular~~ square or not.

How do we compute the SVD matrix?

$$A = U \Sigma V^T$$

U, V orthogonal matrices

Σ matrix with positive values along diagonal

A: 1x2

U: 1x1

Σ: 1x2 always to be same as A

V^T: has to be square, 2x2

Because $U = [1]$

How do we get Σ and V

take:

$$A^TA = V(\Sigma^T\Sigma)V^T$$

will be a square matrix whose diagonal elements are squares of singular values

$$\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} = \begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$$

← second row is multiple of first row, ⇒ mean we then get 0 Evalue

↑ when square root it

$\lambda = 0 \rightarrow \sigma = 0$

$\lambda = 5$

Corresponding E'vectors:

$$\begin{pmatrix}-4 & 2\\ 2 & -1\end{pmatrix}u = 0$$

$u = \begin{bmatrix}1\\2\end{bmatrix}$ (not unit length)

so $u = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$

$\lambda = 0 = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\end{bmatrix}$

Σ

↓ order Evalues in correct order

$$A = \begin{bmatrix}1\end{bmatrix}\begin{bmatrix}\sqrt{5} & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}^T$$

So this gives us a representation for A, now that we have SVD of A

How do we compute A^+

Note!

Bad!

$$A^+ = V\Sigma^{-1}U^T$$

$$A^+ = V\begin{bmatrix}\frac{1}{\sqrt{5}}\\ 0\end{bmatrix}U^T$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}}\\ 0\end{bmatrix}$$

$$A^+ = \frac{1}{5}\begin{pmatrix}1\\ 2\end{pmatrix}$$

ii) AA^+ and A^+A

$$AA^+ = \begin{pmatrix}1 & 2\end{pmatrix}\frac{1}{5}\begin{pmatrix}1\\ 2\end{pmatrix} = 1$$

$$A^+A = \frac{1}{5}\begin{pmatrix}1\\ 2\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$$

iii) If x is in $N(A)$ what is A^+Ax?

Lets assum x in nullspace of A

$$N(A) = c\begin{bmatrix}-2\\1\end{bmatrix}$$

i.e $x = \begin{pmatrix}-2\\1\end{pmatrix}$

$$\therefore A^+\underbrace{Ax}_{=0 \text{ since}} = 0$$ will always end up with 0.

(iv) Q If x is in $C(A^T)$ what's A^+Ax.

$$C(A^T) = c\begin{bmatrix}1\\2\end{bmatrix}$$

$$x = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\therefore \underbrace{\tfrac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}}_{A^+A}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \rightarrow x$$

$\therefore A^+Ax = x \rightarrow$ we always recover x

what does mean?: If x is in the nullspace of A, it just kills it, If x is not in nullspace of A, we get x back.

$\therefore$ it is the identity matrix acting on x, in $C(A^T)$

If A is invertible, then A does not have null space

$\therefore$ $A^{+}A$ recovers the identity

$\therefore$ when multiply any vector we get vector back