Lec 1

The lecture on Positive Definite Matrices

this topic brings the whole course together.

this who guy to watch in this lecture:

$$\left(x^T A x > 0 \right)$$

Goal: How can I tell if a matrix is Positive Definite.

What is the test?

and what does it mean, why are we so interested in it?

Lets begin by 2x2, all matrices symmetric

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$ → it's symmetric, but is it positive definite

if a knew the eigenvalues are they all positive

1. $\lambda_1 > 0$ $\lambda_2 > 0$

1x1 det   2x2 det

(2) $a > 0$ $ac - b^2 > 0$] determinant test (2)

∴ a looking at it as a 1x1 determinant.

∴ the 2x2 determinant test.

— are they positive

det divided by a

(3) pivots $a > 0$ $\frac{ac - b^2}{a} > 0$

new idea:

(4) (x) $x^T A x > 0$

But how do we know the e'value test and determinant test pick out the same matrices?

Eg

$$\begin{bmatrix} 2 & 6 \\ 6 & \boxed{} \end{bmatrix}$$

what do I need to put here to be positive definite?

Here we would use the det test. [not use the e'value test]

19 YES, 18 — borderline

→ positive semi definite

And what are the e'value? 18

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∴ we have a singular matrix.

$\lambda = 0, 20$

(from the trace)

e'values ≥ 0

= (equal to) that brought the word "semi" definite in → for 18.

what are pivots of matrix?

pivots ⇒ 2 — singular matrix, so only have 1 pivot.

rank 1 matrix

∴ $A^T A x \geq 0$ .... x is any vector now

$[x_1\ x_2] \begin{bmatrix} \cancel{x_1} \\ \cancel{x_2} \end{bmatrix}$ ↓ Back to real matrics

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix}$$

A X

$ax^2 + 2bxy + cy^2$

$= 2x_1^2 + 12x_1 x_2 + 18x_2^2$

∴ ~~Not~~ No longer linear, ⇒ quadratic now

∴ degree 2.

∴ is that quantity positive or not $\cancel{>}0$

$$ax^2+2bxy+cy^2>0$$

∴ then matrix is positive definite →

Lets say we need 7.

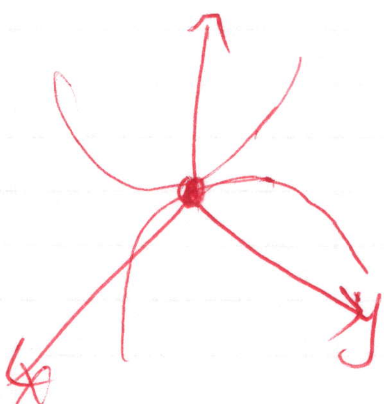
$$\begin{bmatrix} 2 & 6 \\ 6 & \boxed{7} \end{bmatrix}$$

, then we get pivots that are negative,

∴ pivots will be 2, −11

Graph of $f(x,y)=x^TAx$

$$=ax^2+2bxy+cy^2$$



Saddle

∴ $2xy^2+12xy+7y^2$ (what's graph)

∴ non positive definite

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Now, lets make it 20

$$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} = 2x_1^2 + 12x_1x_2 + 20x_2^2 \quad (A)$$

det 4

we know:

Why? → $\lambda_1 \times \lambda_2$ is positive

it's the determinant

$2 \cdot 20 = 40 - (6\times6) = 40-36 = 4$

trace (sum down the diagonal) = 20+2 = 22

(they multiply to get 4)

∴ they either Both positive or Both negative.

But if they both negative the trace, cannot be 22?

Both det's, pivots, e'values ≥ 0 (are positive)

(A) Believe above the function is positive everywhere, except at 0,0

↳ $x^TAx > 0$, except at $x=0$

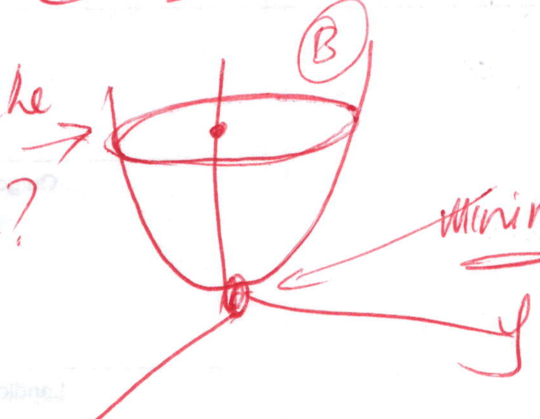
How does its graph look like?

— it does not have a saddle point.

$$f(x,y) = 2x^2 + 12xy + 20y^2$$

⇒ when the squares (x, y) overwhelmed (xy), ~~then~~ which could be +ve or −ve, then it makes the total thing positive

where I do, when I move away from origin, → if go up



what will be the equation for this curve?

minimum

derivative

we know:

— 1st derivis = 0 (But not enough for minimum)

— It's the 2nd derivis that control everything.

How can we tell this function is always positive?

— If we can express it to square.

∴ If we squaring something, we will not be negative

∴ Our headaches are coming from

$\underline{12xy}$

$xy \rightarrow$ ~~we are~~ if

we can get this in a square.

$= 2(x + \underline{3y})^2$?

$+ 2y^2$ ← now everything ∴ squares ⇒ ⇒ now we positive

∴ (B) will be an ellipse

∴ $2(x+3y)^2+2y = \underline{1}$

gives me cross section, which will be an ellipse

∴ 2, 3, 2 above numbers — they actually came from eliminating

(Do Elimination)

$$\underbrace{\begin{bmatrix}2 & 6\\ 6 & 20\end{bmatrix}}_{A} \Rightarrow \underbrace{\begin{bmatrix}2 & 6\\ 0 & 2\end{bmatrix}}_{U}$$

$$L = \begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}$$

Completing the square is Elimination.

— Nicely coming together.

But what are outside these square: (circled)

$$2(x+3y^2 + 2y^2$$

— they are the pivots.

& that's why positive pivots, give me sum of squares

, and graph goes up

, a minimum at the origin

, all connected together

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Lets do 3X3 matrix?

[favorite matrix]

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

1 — Is the matrix pos. def?

2 — whats the function associated with this matrix? $A^T A x > 0$

3 — Do we have minimum for this function

4 — whats the graph

Calculate 1, But taking the det ① + ② + ③ above

det ⇒ 2, 3, 4

what are pivots? ⇒ 2, $\frac{3}{2}$, $\frac{4}{3}$

[i] $2 \cdot \frac{3}{2} = 3$

[ii] $2 \cdot \frac{3}{2} \cdot \frac{4}{3} = 4$

product of [i] gives ~~me~~ det = 3

product of [ii] gives me det = 4.

what are eivalues;

⇒ they are positive ; $2 - \sqrt{2}, 2, 2 + \sqrt{2}$

— 2 —

$$f = X^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 \quad 2 \quad 2 \quad -2x_2x_3 > 0$$

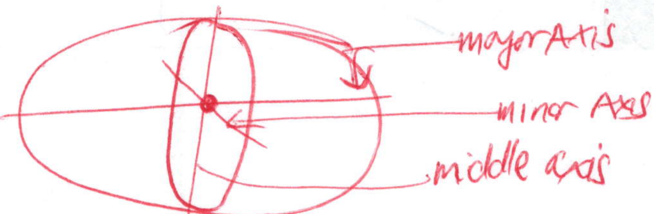
1 1 1 → those came from the diagonal.

, But I can also complete the squares,
write it as sum of 3 squares
, and 3 squares will be multiplied
by 3 pivots.

Now lets take all points at height 1

— 2 — = 1

(it gave an ellipse in 2x2 case,
in this case it will be an ellipsoid)
football (lopsided)



- those 3 axis, will be in the direction of the E'vectors.
- length of axis, will be determined by E'values.

write A as

$\underline{Q \Lambda Q^T}$ ← ~~[illegible]~~ principal axis

from the diagonal!