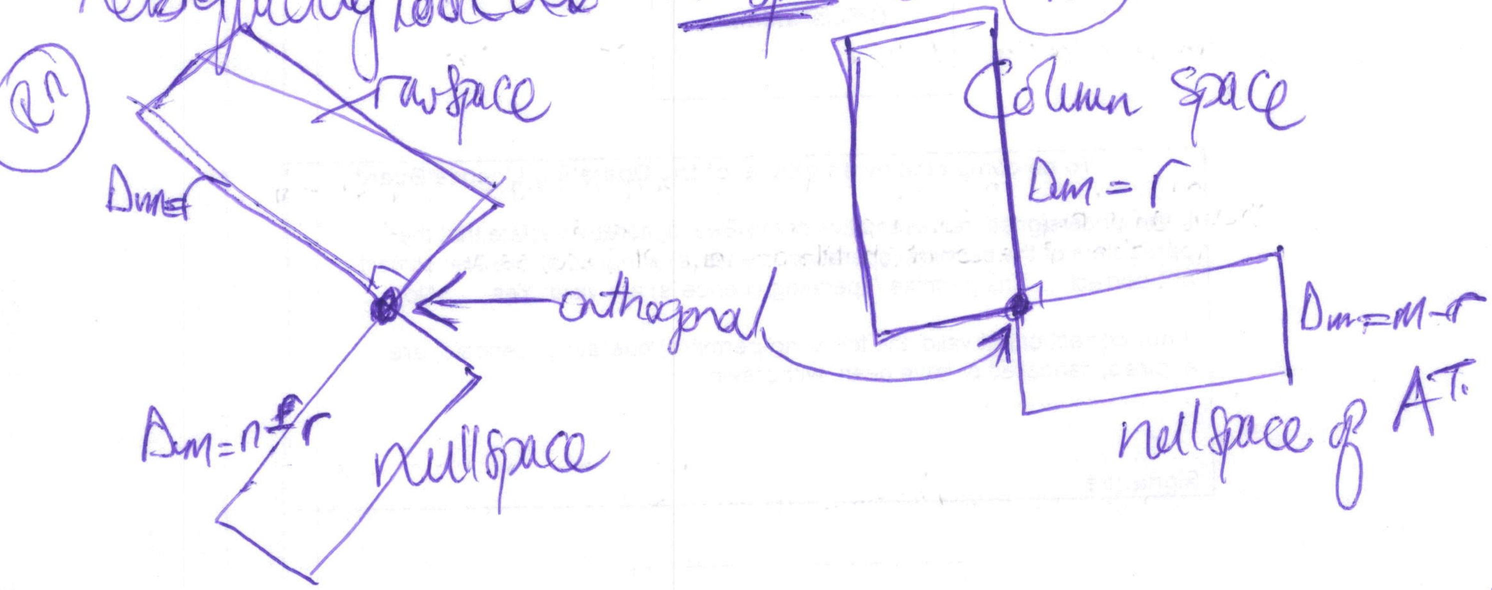
~~Rec~~ Lec

What does it mean for Vectors & Subspaces/Basis to be Orthogonal

⇒ This is 90% Chapter

Lets quickly look at subspace: R^m



– we know alot ~~abo~~ about that picture (previous lessons)

∴ angle between subspace is 90°

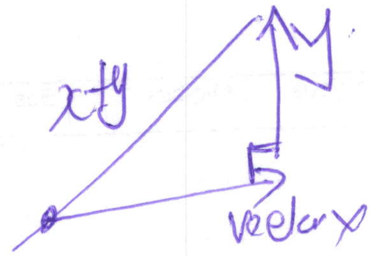
⇒ What does it mean for subspaces to be orthogonal?

what does it mean for 2 vectors to be orthogonal?

orthogonal = perpendicular

$\therefore$ in n dim space the angle between them are $90°$

form a right triangle



(Dot product)

Pythagoras: $x^T y \Rightarrow$ row $\times$ column

if $\boxed{x^T y = 0}$ [test] orthogonality.

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$

Let:

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad y \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$\|x\|^2 = 14$ $\quad \|y\|^2 = 5$ $\quad \|x+y\|^2 = 19$

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$

$$\cancel{x^Tx} + \cancel{y^Ty} = (x+y)^T(x+y)$$

same thing

$$= \cancel{x^Tx} + \cancel{y^Ty} + x^Ty + y^Tx$$

$$0 = 2x^Ty$$

make 2 as they are same.

∴ Dot product of orthogonal vectors are zero ⟹

— what if one of these guys are zero vector?

eg. x zero vector
y is whatever.
are they orthogonal, Yes!

∴ zero vector is orthogonal to everyone.

What does it mean if I say one subspace is orthogonal to other subspace?

∴ subspace S is orthogonal to subspace T.

- What natural "extension" from orthogonal vectors to orthogonal subspaces?

⇒ Means: every vector in S is orthogonal to every vector in T.

No they are not.

rowspace is orthogonal to the nullspace (x in nullspace)

why? we know the nullspace solves vectors for $Ax=0$ $= \begin{bmatrix} \text{row 1 of } A \\ \text{row 2 of } A \\ \text{row 3} \ldots \\ \ldots \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

∴ the equation is telling me x is orthogonal to all rows $\begin{bmatrix} \text{row 1} \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

∴ x is orthogonal to each separate row

" " to combination of rows

$c(\text{row}^1)^T, c = 0$

$c(\text{row}^2)^T, c = 0$

⇒ What are (few) orthogonal subspace in Three(3) dimension?

⇒ Couple of orthogonal lines.

↳ Can this be the rowspace of nullspace

∴ in 3 Dim, have rowspace as line

in " " nullspace as line

NO, Dimensions are not right.

Eg.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

rowspace are dimensional

Dim rowspace: 1

Dim nullspace: 2.

$n = 3 \quad r = 1$

N(A) its a plane, which is perpendicular to (1, 2, 5)

∴ nullspace and row space are orthogonal Complement in $\mathbb{R}^n$

∴ Nullspace contains all vectors that are perpendicular to rowspace.

: "solve" $Ax=b$ when there's no solution (in column space)

"typical A is rectangular."

and $m > n$

equations, unknowns

⇒ The matrix that will play a key role ⇒

$$\overset{n\times m}{A^T}\overset{m\times n}{A}$$

Matrix: its $n\times n$ square

– its symmetric

$$(A^TA)^T = A^TA$$

good equation comes from:

$$Ax=b$$

$$\downarrow$$

$$A^TA\hat{x} = A^Tb$$ ⇐ this will be the central location in chapter.

When is equation invertible?

eg $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$ ∴ $m \times n$

$\underset{\text{(rows)}}{3}$ $\underset{\text{(column)}}{2}$

∴ rank = 2 (columns are independent)

∴ we want with A^T.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

invertible

not always invertible

∴ Null space $N(A^TA) = N(A)$

∴ rank of A^TA = rank of A

A^TA is invertible exactly if ~~its nullspace~~

ie A has independent column

A has independent columns.

A