Continue with Vector + Subspaces.

Column space + null space

Again what is Vector space:

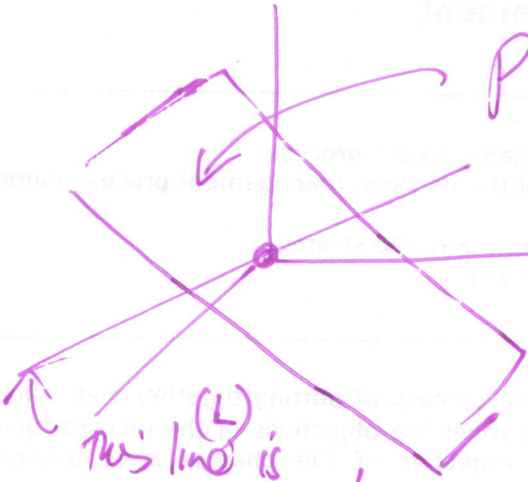
- Bunch vectors, where I can add any two vectors, and answer stay in space
- or multiply any vector by any constant and result stay in space.

∴ + and × ⇒ mean I can take linear combinations

∴ any multiple of $\underline{cv+dw}$ stay in space.

∴ say I have $R^3 \rightarrow$ 3D space

But if I have subspace inside $\mathbb{R}^3$
— this makes up vector space of itself



(P) Plane through is subspace of $\mathbb{R}^3$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(L) this line is subspace!

Note: all subspaces must contain origin.

∴ if I take (P) and (L) [two subspace] and put them together, (union)

$\Rightarrow$ $P \cup L$ = all vectors in P or L or both

→ is that subspace?

<u>No</u>

$\Rightarrow$ $P \cap L$ = we take intersection

meaning all vectors inside P and L

So in general:

Subspaces S and T.

intersection: $S \cap T$

Is a subspace

(vector 1) (vector 2)

sum : $v + w$

Scalar multiplication

and ~~intersection~~: cv

is allowed

that combine into linear combination $cv + dw$.

Column space of matrix: (A)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

∴ 3 columns

∴ Column space of A is subspace of R4

– what is R4?

$A = 4 \times 3$ matrix

4 → row → 4 dimensional

④

∴ Called C(A) ⇒ Column space of A.

↳ ~~de~~ Notation for subspace of A.

what else does it have as well? [i.e. don't have subspace if I only put in 3 vectors]

⇒ I also take their linear combinations.

∴ Col space of A is all linear combinations of ~~A~~ columns

requirements for subspace { 1) has 3 columns
2) and it (linear combinations }

, But how Big is that subspaces

∴ IS it the whole 4 Dim. space?

(No!)

⇒ But how much smaller?

Lets make critical connection with linear Equations.

∴ to understand ⇒ $Ax=b$

Does $Ax=b$ have a solution for every right hand side? No

⑤

- ~~which~~ $Ax = b$

What is it?

4 equations, 3 unknowns.

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

- Normally we can't solve 4 equations with only 3 unknowns.

But sometimes you can

for some right hand side you can solve it

But which right hand side(s)?

∴ Which b's allow this system to be solved here is one right hand side that will work:

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

another that will work:

$$Ax = \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}}_{\text{Column space}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ I can solve ~~b~~ $Ax=b$, Exactly when righthand b ~~(vector) is~~ is vector in Columnspace

$\therefore$ when b is combination of Columns / when in Column Space in $C(A)$

$\Rightarrow$ by its definition the Column Space Contains all the Combinations

— Since b is combination of column space,
* then that combination will tell
me what x will be.

⇒ Are those columns independent or
(~~do those columns~~) does each column
contribute something new?

i.e. does 1 column do nothing
i.e. if only have 2 column it
is total space?
∴ Can I throw away any columns
and have same column space

Yes
Can throw away Col. 3
why combination of Col 1 and 2.

also, Could I throw away Col. 1?

⇒ Can be described as a 2D sub space of R^4

(8)

$\boxed{n \times m}$

4

⇒ Lets look at null space:

Lets keep same matrix, But we going to have a totally different subspace [nullspace]

∴ Nullspace of A = all solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\underline{R}^{\overset{(n)}{3}}$ ~~3x~~ rows

to the $Ax = 0$

∴ righthand side is 0 $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

⇒ here we only interested in $\underline{x}$'s

⇒ So which solutions for x.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

∴ where is this nullspace [for this example] weights/or coefficients.

$$A = \begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 1 & 4\\ 4 & 1 & 5\end{bmatrix}\begin{bmatrix}1\\ \\ \\ \end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

N(A)

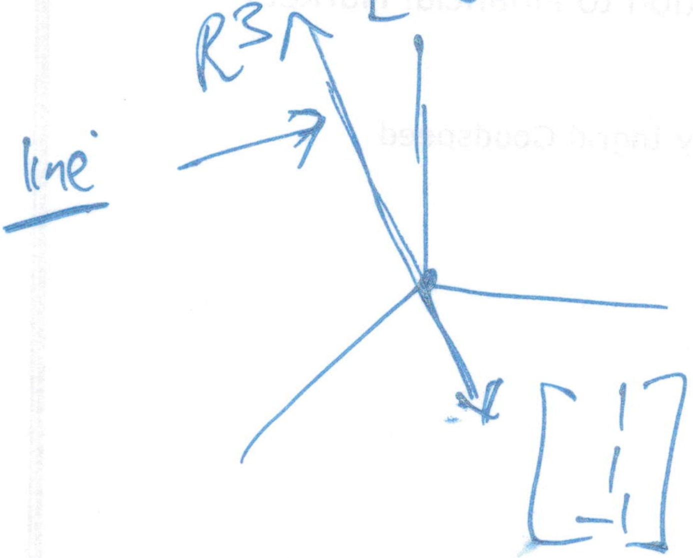
all those combinations.

Solutions ⇒

$$\therefore N(A) \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}c\\c\\-c\end{bmatrix} \text{ or } c\begin{bmatrix}1\\1\\-1\end{bmatrix} \leftarrow \underline{\text{line}}$$

, But couple vectors don't make subspace, what else do we need?

But what is $\begin{bmatrix}1\\1\\-1\end{bmatrix} \Rightarrow$ its a line



$\therefore \begin{bmatrix}7\\7\\-7\end{bmatrix}$ or $\begin{bmatrix}2\\2\\-2\end{bmatrix}$ etc

<u>all on this line</u>

∴ Check that the solutions to $Ax=0$ always give a <u>subspace</u>

∴ If $Av=0$ and $Aw=0$ then

sum is also in nullspace → $A(v+w)=0$

∴ $Av+Aw$