Lect

1

Last chapter on Orthogonality

- we did ortho. vectors

subspace [rowspace nullspace]

Now we will cover

⇒ ortho basis

ortho matrix

rather we want orthonormal vectors.

we letter q for orthogonal things

vectors are $q_1, \ldots, q_n$

every q_i is orthogonal to every other

natural idea

Ortho-normal vectors:

$$q_i^T q_j = \begin{cases} 0 & \text{if } i=j \\ 1 & \text{if } i=j \end{cases} \quad (*)$$

q_i is not orthogonal to itself [But take best guess then]

How does having an orthonormal basis make things nice.

- a lot of LA is built around working with orthonormal vectors

- they never get at hand, i.e. overflow or underflow.

put them into matrix Q

Suppose Basis/Column A are not orthonormal

- How do I make them so?

two names associated with that idea?

Gram-Schmidt

∴ we have a basis like this (*),
lets put it into Column of matrix

∴ Matrix Q with orthonormal vectors:

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

and put $q_i^T q_j = 0$ in matrix form

→ Could be square or rectangle

we will then look at:

now has vectors in its rows.

$$Q^TQ = \begin{bmatrix} \text{---} & q_1^T & \text{---} \\ & \vdots & \\ \text{---} & q_n^T & \text{---} \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q^T \quad Q = I$$

⇒ Can Q be called orthonormal matrices?
Sounds like best name.

Only use this name when square

~~If Square matrix~~

If Q is square then $Q^TQ = I$
tells us $Q^T = Q^{-1}$

eg Any permutation matrix

$$\text{perm } Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ ⇒ has unit vectors in its column ⇒ vectors are ⊥

Let take it Q^T.

$$Q^T \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = I$$

Another Example:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ ⇒ is this ortho.? ⇒ unit vectors!

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Is this ortho. matrix (not quite)

- Have ortho columns
- How to fix to be ortho. matrix?

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$$Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ (Now orthogonal)

length squared $= 2$, so need to divide by $\sqrt{2}$

<u>another:</u>

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Adam R

$\rightarrow$ to make unit vector.

Why is it good to have orthogonal matrices

$\Rightarrow$ what calculation is made easy

Lets first look at rectangular matrix:

Squared 1 4 4

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

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Best to question ~~what makes~~ what formulas are made easy?

Suppose:

Q has orthonormal columns

(in future, always mean that Q has orthonormal columns)

Project onto its column space

what projection matrix → just using Q now instead of A

$$P = Q(Q^TQ)^{-1}Q^T$$ → $= I$ if Q is square.

$$= QQ^T$$ (projection matrix)

What properties of projection matrix:

1) It's symmetric

2) $(QQ^T)(QQ^T) = QQ^T$ ← (true)

If matrix is square and has orthonormal columns, then what is column space

~~its~~ its whole space.

⇒ and what is the projection matrix onto whole space.

⇒ Identity matrix.

Remember:

$$A^T A \hat{x} = A^T b$$

Now A is Q

$$\therefore Q^T Q \hat{x} = Q^T b$$

⇒ Good about this is matrix on left is identity.

$$\therefore \hat{x} = Q^T b$$

Second half:

Now, we don't start with orthogonal matrix/ orthonormal vector, we start with independent vectors and we want to make them orthonormal

Here's § (GS): (gram schmidt)

Here's a calculation (similar to elimination)

, But remember the goal with elimination was to make the matrix triangular.

— Now goal is to make matrix orthogonal.

$\Rightarrow$ i.e. make columns orthonormal.

: Start with 2 columns

[independent] Vectors a, b

Maybe 12 Dim or 2 Dim space

But they independent

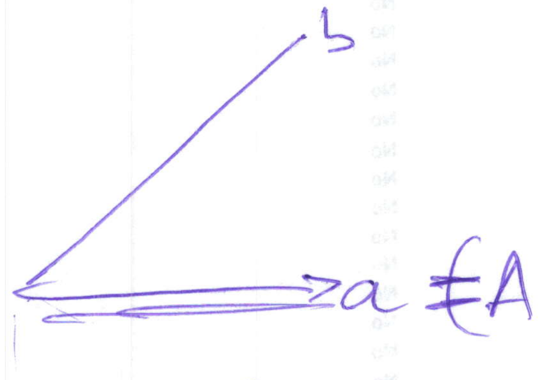
Now from these want to produce q_1 and q_2 — from any 2 vectors

Goal: to get to orthogonal vectors, call, A, B.

then After that I'll get orthonormal vectors

$$q_1 = \frac{A}{\|A\|} \qquad q_2 = \frac{B}{\|B\|}$$

Divide by length.



~~a is fine~~

a is fine, but that direction is fine,

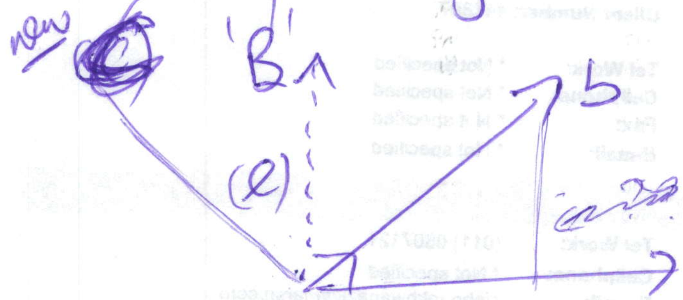
But b, second direction is not fine, as it is not orthogonal to a

Looking for vector that starts with b and make it orthogonal

— How do I get it?

Remember the vectors may be in 2 Dim or 12 Dim.

remember previously I could get "error part"



which was B' (e) and it was $B = b - p$ so instead of wanting the p (projection) that is part I want to throw away

formula $$B = b - \frac{A^T b}{A^T A} A$$

how do I check if the B ⊥ to A.

$A \perp B$.

or check (A) matrix above is ~~perpe~~ ⊥ to A.

multiply by A^T and get 0

$$A^T B = 0$$

$$= A^T\left(b - \frac{A^T b}{A^T A} A\right) = 0$$

So now need to take another vector to see we got it right

a, b, c

Now A B C — the new problem is it's perpendicular ⊥ to A & B.

$$q_1 = \frac{A}{\|A\|} \quad q_2 = \frac{B}{\|B\|} \quad q_3 = \frac{C}{\|C\|}$$

$$C - \frac{A^T c}{A^T A} A - \frac{B^T C}{B^T B} B$$

removed its projection in B direction

removed it's projection in A direction

Lets do (GS) again

$$a = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad b = \begin{bmatrix}1\\0\\2\end{bmatrix}$$

Same as ↳ A

whats B?

Substract same multiple of A

$$B = \begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{3}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

is original B

$$\frac{A^T b}{A^T A}$$

— How do I know B is right?

Check $A \perp B$.

what is q_1 and q_2?

when you take q you need to normalise it, make it unit vectors

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

B but divide by square root 2

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

B

How is the column space of A
related to column space of Q

⇒ it's plane

, they are and same, it's same column space

⇒ remember I am working in same,
just getting 90° degree angles in it.

⟶

∴ Remember Elimination in Matrix
Language:

$$A = LU$$

Need to do the same for ("GS")

$$A = QR$$

$$\begin{bmatrix} (a) & (b) \\ a_1 & a_2 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} & \\ 0 & \end{bmatrix}$$

$R \rightarrow$ turns out to be upper triangular.

$\downarrow$

these the Connection