

Key lecture:

①

- Idea of independence [Bunch of vectors, ~~not~~ matrix]
- All related to vectors — the space they span
- Basis for vector space
- Some number → Dimension of that subspace

— Now we will get clear meanings for these terms.

(Earlier clarity)

Suppose I have matrix (A) which ^{pivots} ~~is~~ ^{is} n by n with $m < n$. — variables

Then there are non zero solutions to $Ax=0$

∴ Matrix has lots of columns

more unknowns ^(x_n) than Equations

∴ Conclusion: $Ax=0$ [Null space]

∴ Here there will be some free columns / variables
⇒ or at least one.

, and these variables, I can assign non-zero values to,
and then solve the pivot variables.

- This is an important point we will see in this lecture.

(2)

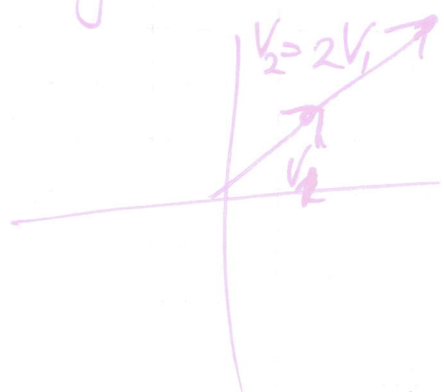
What does it mean for vector to be independent?

When vectors $x_1, x_2, x_3, \dots, x_n$ are independent

\therefore If no combination gives zero vector (except zero combination: all $c_i = 0$)

$$c_1 x_1 + c_2 x_2 + \dots + c_n x_n \neq 0$$

Let's say we in 2D space



two vectors.
these are dependent
one vector is twice
the other.

$$\therefore 2V_1 - V_2 = 0$$

Another
example



, Again they dependent

\therefore ~~that~~

$$\therefore V_1 = 0 + V_2.$$

or put another way:

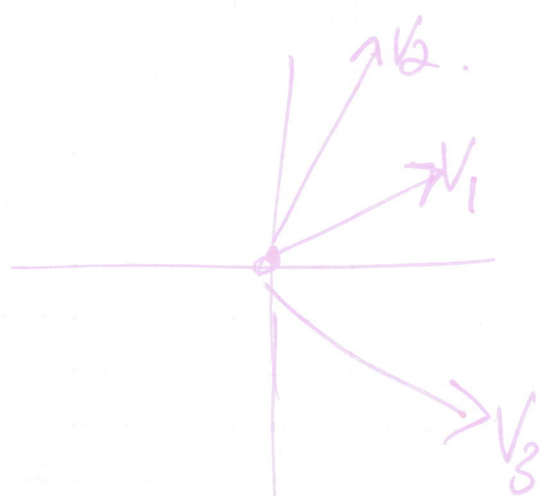
③

How many $V_1 + -V_2$ should I take
to get zero vector

$$\textcircled{0} V_1 + \textcircled{0} V_2 = 0$$

- Same combination will give 0 vector

Basically if one has 2 vectors, independence
is out of window



^{only}
(V_1 and V_2)
this is independent
any combination of
 V_1 or V_2 will
not give you zero

Practical definition of Independence:

When v_1, v_2, \dots, v_n are columns of A

↓ matrix

if I am in n dimensional space, I
can answer the dependence / independence
question directly, by putting

these vectors (v_1, v_2, \dots, v_n) in

columns of A (matrix)

$A(n)$

— Then they are independent if nullspace A
is only zero vector [zero vector]

— They are dependent if $Ac=0$ for
some nonzero c [in nullspace]

\Rightarrow they dependent

A says some combination (c)
of A (columns) give
me zero.

the rank r of matrix in case of independent ^⑤
Column is [all columns will be pivot]
(all columns in)

Recap:

They independent: if null space of A is zero vector
and rank $= n$, no free variables, $N(A) = \{0\}$

They are dependent: if $Ac = 0$ for some nonzero c .
and rank $< n$, yes free variables

What does it mean for vectors to span a space:

Mean: Vectors v_1, v_2, \dots, v_e span a space, ~~if~~

means: space consist of all combinations
of those vectors.

\therefore Columns of matrix spans Col. space

But we today are interested in vectors that are
independent AND span space.

⑥

Basis for a space is a sequence of vectors v_1, v_2, \dots, v_d that has two properties:

- ① Independent
- ② Span space

Let's look at example: 3D space:

Space is \mathbb{R}^3 \leftarrow [we look for Basis for \mathbb{R}^3]

When ask Basis, I ask for vectors or list of vectors

One Basis is

— the three that came to mind:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for (x, y, z axis)

But that is not the only Basis:

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$$

two two don't span, we need another vector.

\leftarrow Can be any vector that is not in the plane

$\begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$ will make it dependent as it is the sum of x

⑦

How do we know row if we have basis:

- we will put them in Columns of matrix
- and do elimination / row reduction.
- and see if get any free variable
- or all Columns pivot Columns.

But the matrix would be 3×3 , so what's test on matrix

\therefore in our case where space $\subset \mathbb{R}^3$

and we have 3 vectors

- my matrix is square
- and what am I asking ABOUT matrix if for THOSE Columns to be basis

\therefore \mathbb{R}^n ~~n vectors~~ n vectors give basis
if $n \times n$ matrix with those Columns
is invertible [square matrix]

But is there space for these two vectors to be basis?

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

\Rightarrow they independent
 \therefore they will be a plane
inside \mathbb{R}^3



Let's go back to:

⑧

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Remember Bases are not unique [there are many]

But what have all bases in common (for \mathbb{R}^2)

\Rightarrow they all have same number of vectors

$\mathbb{R}^2 = 2$ vector elements.

~~\Rightarrow every basis~~

Given a space: \mathbb{R}^n

\Rightarrow Every basis for the basis has the

same number of vectors

or how many vectors I need
to have to have basis \rightarrow

\Rightarrow What do we call this number. \Rightarrow dimension
of that space

Examples:

⑨

Column space of matrix $C(A)$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

\therefore 4 vectors

\therefore Do they ~~space~~ span the Column space of matrix, Yes
Are they independent or Basis?

— No

Tell me vector that is in null space of matrix?

— Looking for same vector that combines these
Columns and produces zero Column.

\therefore looking for Solutions $AX=0$

Vector in null space $N(A)$

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

-1 of Col 1

-1 of Col 2

1 of Col 3

0 of Col 4

\Rightarrow equal & opposite!

\Rightarrow they span, But they not independent!

what is basis for that vector. [Column space] (C)

Column 1 and 2 \leftarrow this we can put in Basis (pivot col.)

$$\therefore \text{rank} = 2.$$

$$\therefore \text{rank}(A) = \# \text{ pivot columns} = \text{dimension of Col}(A)$$

\downarrow Not dimension of A (matrix)

- it's dimension of subspace, Column space.

\Rightarrow also, I don't talk about rank of subspace, we talk about rank of Matrix.

What is another basis for Column space:

Col. 1 and 3. or Col. 2 and 3

or Col. 2 & 4, or Basis not made up of any of Columns at all?

another Basis for Column space for $C(A)$:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$$

\leftarrow 2 are independent so they must span space.

\therefore Key point $\dim C(A) = 2$

what about the null space?

are there other vectors in null space

Yes

(X) $\begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow$ Ken figured out the two.
} Took free variables

• These vectors in null space are
telling me the combinations
of columns that give zero

— they telling me in what way the
columns are dependent

• And are these vectors a basis for
null space?

→ for what's the dimension (2)

Yes

(2)

\therefore Dimension of nullspace = #free variables

$$\dim N(A) = \text{\#free variables}$$

$$\longrightarrow = n - r$$