Lec ①

: Leflaff on Differential Equations ?

How to solve a system of 1st order

first derivative, Constant Coefficient

linear equations

: If we do it right, it turns

directly into linear algebra.

: Key ideas: the solutions to

Constant Coefficient linear equation

are exponentials

: If you look for an exponential,

then all you have to find is

what up here in the exponent,

and what multiply the exponential

→ and that's the linear algebra

∴ Only one thing that we will find, is its
Completely parallel to powers of
a matrix

⇒ last time was how to compute
A^k or A^{100}

how to compute high powers
of a matrix

Now its not powers any more, it's
Exponentials →

thats the natural thing for
differential Equations.

∴ How would I solve 2 diff. Equations:

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$\rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$

$\therefore$ Same initial conditions:

Suppose: $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\leftarrow$ suppose everything is in U_1

$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$ what's EV$_L$ and EV$_C$?

$\therefore$ matrix is singular.

if singular, what does it tell me about 1 of the E'value?

$\lambda = 0$, other $\lambda = -3$.

$\therefore \left| A - \lambda I \right| = \begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 3\lambda = 0$

$\therefore \lambda(\lambda + 3)$ and we get 2 E'values above.

$\therefore$ E'vectors:

$\lambda = 0 \quad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\qquad \lambda = -3 \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in nullspace

$Ax_1 = 0x_1$

$Ax_2 = 3x_2$

Solution: $u(t) = \underbrace{C_1 e^{\lambda_1 t} x_1}_{A} + C_2 e^{\lambda_2 t} x_2$ two pure exponential special solution

Check: $\frac{du}{dt} = Au$ Plug in $e^{\lambda_1 t} x_1$

$$\Rightarrow \lambda_1 \cancel{e^{\lambda_1 t}} x_1 = A \cancel{e^{\lambda_1 t}} x_1$$

$$C_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

∴ how do we get C_1 and C_2? Comes from the initial condition

Use $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$ to find C_1 and C_2

∴ At $t = 0$ $C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution

$C_1 = 1/3$, $C_2 = 1/3$

$$\therefore \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

: Steady State / Stability

$$u(\infty) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

: When do we get stability $u(t) \to 0$

: when will the solution go to zero, no matter what the initial condition is?

: $-$'ve E'values.

Suppose the E'values are complex nos (since they could be

so we need $e^{\lambda t} \to 0$

eg $\left|e^{(-3+6i)t}\right|$ ← how big is that number?

: does the imaginary part play a roll.

$= e^{-3t}$

$(\text{as } |e^{6it}| = 1)$

Steady State : when will we have a steady state ends in same direction?

$\lambda_1 = 0$. and other e'values have real part $\lambda < 0$

and we blow up if any Real part $\lambda > 0$

Comment

2X2 Stability

$\underline{Re}\ \lambda_1 < 0 \quad \underline{Re}\ \lambda_2 < 0$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\underline{trace}\ a + d = \lambda_1 + \lambda_2 < 0$

is a negative trace enough to make matrix stable?

Look trace < 0, but still blow up.

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

∴ we need another condition

det > 0 $(\lambda_1 \cdot \lambda_2)$

Come Back

$$c_1 \begin{bmatrix} c \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$S \quad C = u(0)$$

Lets link 1.to. S and λ ⑦

$$\frac{du}{dt} = Au$$

matrix A couples them, point of Eigvectors is to uncouple them!

diagonalising it

Set $u = Sv$

$$S\frac{dv}{dt} = ASv$$

diagonal matrix

$$\frac{dv}{dt} = S^{-1}ASv = \Lambda v$$

$$\frac{dv_1}{dt} = \lambda_1 v_1$$

$$\vdots$$

$v = c_1$

$$v(t) = e^{\Lambda t} v(0)$$

$$u(t) = S e^{\Lambda t} S^{-1} u(0) = e^{At} u(0)$$

$$e^{At} = S e^{\Lambda t} S^{-1}$$

Matrix Exponential e^{At}

How do we define exponential of something

there's a power series for exponentials.

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \dots + \frac{(At)^n}{n!}$$

Aside:

Taylor series $e^{x} = \sum_{0}^{\infty} \frac{x^n}{n!}$

other $\frac{1}{1-x} = \sum_{0}^{\infty} x^n$ (power series)

Back

$$(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \dots$$

$$|\lambda(At)| < 1$$

Matrix Exponential $e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \dots$

$\Rightarrow I + SAS^{-1}t + \frac{SA^2S^{-1}t^2}{2} + \dots$

∴ pull out S from everything

$\Rightarrow$ & rewrite $I = SS^{-1}$

$\Rightarrow Se^{At}S^{-1}$ } matrix exponential.

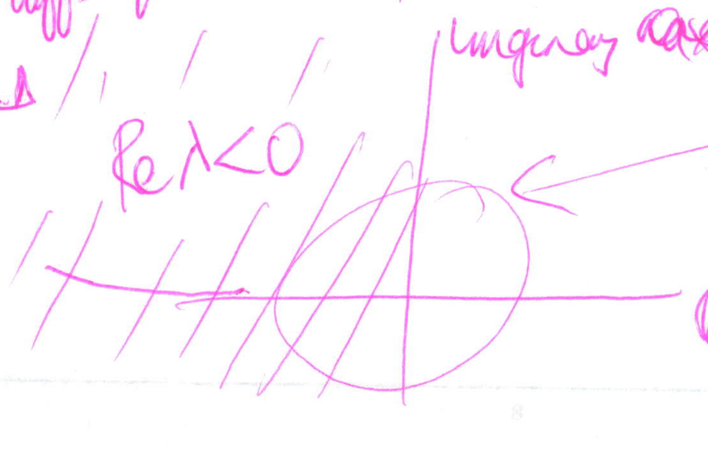
$\Rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$ where $\mathrm{Re}\,\lambda < 0$

∴ What is exponential of diagonal matrix.

$\Rightarrow \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$

λ values must be here for stability of diff. Equations

Complex plane



Example

$$y'' + by' + ky = 0$$

How do I change one 2nd order Eqn into 2x2 1st order systems

Let $u = \begin{bmatrix} y' \\ y \end{bmatrix}$ $u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$ 2