Key Lecture: ①

→ Idea of independence [Bunch of vectors, not matrix]

All related to vectors { — The space they span
— Basis for vector space

Some numb. → Dimension of that subspace

- Now we will get clear meaning for those terms.

(Earlier clarity)
Suppose I have matrix (A) which ~~m~~ m by n [pivots, n — variables]
with m < n.

Then ~~there~~ there are <u>non zero</u> Solutions to Ax=0

∴ Matrix has lots of Columns

∴ More unknowns (x_n) than Equations

∴ Conclusion Ax=0 [Null space]

∴ Here there will be some free Columns / variables
⇒ or at least one.

, and those variables I can assign non zero values to.
, and then solve the pivot variables.

- This is an important point we will use in this lecture.

- What does it mean for vector to be independent?

When vectors $x_1, x_2, x_3, \ldots, x_n$ are independent

iff no combination gives zero vector (except zero combination : all $c_i = 0$)

$c_1 x_1 + c_2 x_2 + c_n x_n \neq 0$

- Lets say we in 2 Dim space

← force vector.

those are dependent

one vector is force the other.

$\therefore 2V_1 - V_2 = 0$

Another Example

, Again they dependent

$\therefore V_1 = 0 + V_2$.

or put another way:

How many V_1 + V_2 should I take to get zero vector

$0 V_1 + 0 V_2 = 0$

- Some combination will give 0 vector

- Basically if one has zero vector, independence is out of window

this is independent (only V_1 and V_2)

any combination of V_1 or V_2 will not give you zero

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Practical definition of independence:

When $v_1, v_2, \dots, v_n$ are columns of A matrix

of & are in m dimensional space, &

Can answer the dependence / independence question directly, by putting those vectors ($v_1, v_2 \dots v_n$) in columns of A (matrix) ... $A(n)$

- Then they are independent if nullspace A is only zero vector [zero vector]

- they are dependent if $Ac=0$ for some nonzero c [in nullspace]

$\Rightarrow$ they dependent

Ac says some combination (c) of A (columns) give me zero.

∴ the rank r of matrix in case of independent
Column is [∴ all columns will be pivot]
(all Columns in)

Recap:

They independent: if null space of A is zero vector
and rank = n, no free variables, $N(A) = \{0\}$

They are dependent: if $Ac = 0$ for some nonzero c.
and rank < n, yes free variables

What does it mean for Vectors to span a space:

~~Mean~~: Vectors $v_1, v_2, \ldots, v_l$ span a space,
means: space consist of all combinations
of those Vectors.

∴ Columns of matrix spans Col. space

, But we lo[...] interested in Vectors that are
independent **AND** span space.

Basis for a space is a sequence of Vectors $v_1, v_2 \dots v_d$, that has two properties:

1. independent
2. Span Space

Lets look at example: 3 Dim Space:

Space is $\mathbb{R}^3$ ← [we looking for Basis for $\mathbb{R}^3$]

when ask Basis, I ask for Vectors or list of Vectors

one Basis is

— the three that come to mind:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$ for (x, y, z axis)

But that is not the only Basis:

these two don't span, we need another vector.

$$\begin{array}{ccccc} \begin{bmatrix}1\\1\\2\end{bmatrix} & , & \begin{bmatrix}2\\2\\5\end{bmatrix} & , & \begin{bmatrix}3\\3\\8\end{bmatrix} \\ x & & y & & \end{array}$$

← Can be any vector that is not in the plane

$\begin{bmatrix}3\\3\\7\end{bmatrix}$ will make it depender as it is the sum of x

How do we know now if we have basis:

- we will put them in columns of matrix
- and do elementary row reduction.
- and see if get any free variable
- or all column pivot columns.

But the matrix would be 3x3, so what's test on matrix

∴ in our case where space is $\mathbb{R}^3$

and we have 3 vectors

- my matrix is square
- and what am I asking ABOUT matrix
  for THOSE columns to be basis

∴ $\mathbb{R}^n$ ~~n vectors given~~ n vectors give basis
if $n \times n$ matrix with those columns
is invertible [SQUARE matrix]

But is there space for these two vectors to be basis?

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} \Rightarrow$$ they independent
they will be a plane
inside $\mathbb{R}^{\cancel{3}2}$

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Lets go Back to:

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$$

Remember Basis are not unique [there are many]

But what have all bases in common (for R^3)

⇒ they all have same number of vectors

R^{3} ⇒ 3 vector elements.

~~Every Bas~~

Given a space: Cols [R^n]

⇒ Every basis for the basis has the same number of vectors

<u>or</u> how many vectors I need to have to have basis

⇒ What do we call this number ⇒ <u>dimension</u> of that space

Examples:

Column space of matrix $C(A)$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$ ∴ 4 vectors

∴ Do they ~~space~~ span the Column space of matrix, Yes

Are they independent or Basis?

— No

Tell me vector that is in the null space ~~of~~ matrix?

— Looking for some vector that combines these Columns and produces zero Column.

∴ looking for Solutions $Ax=0$

∴ Vector in nullspace $N(A)$

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

-1 of ~~Col~~ Col 1
-1 of Col 2
1 of Col 3
0 of Col 4 ⇒ equal zero Vector!

⇒ They span, But ~~th~~ they not independent!

what is basis for that vector. [Column space] (10)

Column 1 and 2 ← this we can put in Basis (pivot col.)

∴ rank = 2.

Column space.

∴ rank (A) = # pivot columns = dimension of Col(A)

↓ Not dimension of A (matrix)

– its dimension of subspace, Column space.

⇒ Also, I dont talk about rank of subspace, we talk about rank of Matrix.

– What is another basis for Column space:

Col. 1 and 3. or Col. 2 and 3

or Col. 2 & 4, or Basis not made up of any of Column at all.?

∴ another Basis for Column space for C(A):

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$$

← 2 and independent so they must span space.

∴ Key point dim C(A) = 2

What about the null space?
We have other vectors in null space

Yes

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

→ then figured out these two.

} took free variables

(X) these vectors in null space are telling me the combination of columns that give zero

— they telling me in what way the columns are dependent

(X) and are these vectors a basis for null space?

→ first what's the dimension (2)

Yes

$\therefore$ Dimension of nullspace = # free variables

dim $N(A)$ = # free variables

$= n - r$