Lec1 ①

Second Lecture on Eigenvalue / vectors

$Ax = \lambda x$ ⇐ key Equation

now how to use it?

L = values
C = vectors.

Job 1 — find E'V_L & E'V_C

~~after~~ we find them, what do we do with it

we need to diagonalize the matrix.

$S^{-1}AS = \Lambda$ ⇐ key Equation

matrix A, put its E'vectors in
columns of matrix S, S is
the E'vector matrix

look at magic combination.

$S^{-1}AS$.

we need to invert the E'vector
matrix S.
, hence S^{-1}

So for that we need $\underline{n}$ independent E'vectors.

Suppose: we have n linearly independent E'vectors of A.

Put them in the column of S (matrix)

$\therefore$ what happens when you multiply $A \times S$.

First E'vector in 1st Colum

nth Evector in nth Colum.

$$AS = A \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

S.

$\therefore$ (A) do matrix multiplication, 1 column at time (A)

$$\Rightarrow = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$$

(A) A time first column (x_1), its an E'vector,

$= $ to $\lambda_1 x_1$

$=$ Next step separate out the E'values (multiplying numbers) from X's

∴ How do I separate it out? (matrix multiplication) (Big deal)

$$\Rightarrow \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = S\Lambda$$

E'value matrix

Λ – Capital Lamba

$\downarrow$ for matrices

∴ $AS = S\Lambda \Rightarrow$

∴ multiply on the left by its inverse

$$S^{-1}AS = \Lambda$$

(provided S is invertible, i.e. n independent E'vectors)

[Note: there are small number of matrices that does not have n independent E'vectors]

or

Can be written as $A = S\Lambda S^{-1}$ (diagonal matrix)

$\rightarrow$ new factorization

Replacement for: $\Rightarrow LU$ (for Elimination)

or $\Rightarrow QR$ (Gram Schmidt)

∴ this is the Combination we will see further now...

Lets use it

For eg What's the EV_c and EV_L for A^2

∴ Started with $Ax = \lambda x$

to get A^2, ~~we~~ lets ~~can~~ multiply both sides with A.

∴ If $Ax = \lambda x$ — Number (scalar, so put on left).

multiply by A $A^2x = \lambda Ax = \lambda^2 x$ Ⓐ

substitute $Ax = \lambda x = \lambda^2 x$

⇒

∴ EValues of $A^2 = \lambda^2$ (vectors)

OK, now lets see how A^2 looks like from the formula $A = S\Lambda S^{-1}$

$$A = S\Lambda S^{-1}$$

$$A^2 = S\Lambda \cancel{S^{-1}S}\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

∴ tells me same thing as per Ⓐ

But in matrix form

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∴ E'Vectors are Same, which is S

But E'Values are squared Λ^2

little lamba λ

E'V and E'V

∴ Gives opportunity to see what is happening inside a matrix.

Lets take A kth power → E'V will be to kth power, whereas E'Ve will be Same.

$$A^k = S\Lambda^k S^{-1}$$

E'Value tells you something about the powers of matrix (in a way we had no way to approach previously)

∴ When does powers of matrix goes to 0 (zero)

Theorem:

$A^k \to 0$ as $k \to \infty$

if all $|\lambda_i| < 1$

(⇒ How can I tell for matrix, if its powers go to zero)
somewhere inside matrix, that information
– not present in pivots, present in eigenvalue

~~A~~ Λ ⇐ diagonalization

: which matrices are diagonalizable.

A is sure to have n indep. eigenvectors
(and be diagonalizable)
if all the λ's are different (distinct)
(no repeated λ's)

If repeated eigenvalues, may or may not have
n independent eigenvectors

: 10x10 (Identity matrix), what eigenvalues of that matrix:
– they all 1's, 1's ~~rep~~
repeated 10 times

∴ But there's no shortage of e vector for I matrix, infact every vector is E vector; 10 independent vectors

Suppose if its triangular

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$ Trouble

Evalue? → 2 and 2

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} \rightarrow \lambda = 2,\ \lambda = 2$$

∴ E vectors:

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

∴ $x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ can't find 2 independent E vectors

Solve Equation:

Start with given vector u_0

$\cancel{u_k = A u_k}$

$$u_{k+1} = A u_k$$

multiply by A every step.

$u_1 = Au_0, \; u_2 = A^2 u_0, \; \rightarrow u_k = A^k u_0$

∴ To solve it.

$u_0 = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \Rightarrow Sc$ really

$A^{100} u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + \dots + c_n \lambda_n^{100} x_n$

$= \Lambda^{100} S c \dots$

Fibonacci example: $0, 1, 1, 2, 3, 5, 8, 13, \dots F_{100} = ?$

(what formula for 100th fib no.

→ and how fast are they growing

How fast they are growing lies in the E'values

∴ $F_{k+2} = F_{k+1} + F_k$

+ $F_{k+1} = F_{k+1}$

∴ $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ $\quad u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ or u_k

matrix will be:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \text{is symmetric} \therefore$$

e'values will come out real. (no complex numbers) and e'vectors will be $\perp$ orthogonal

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 + \sqrt{1+4}}{2}$$

$$\lambda_1 = \tfrac{1}{2}(1+\sqrt{5}) \simeq 1.618$$

$$\lambda_2 = \tfrac{1}{2}(1-\sqrt{5}) \simeq .618$$

($\Rightarrow$ is matrix diagon)

$$F_{100} \simeq c_1 \left(\frac{1+\sqrt{5}}{2}\right)^{100}$$

what's controlling the growth of the Fibonacci numbers?

– the e'values and which e'values are controlling the growth?

– the big ones

Couple process by finding E'vector.

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow 0$$

$$X_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$\therefore u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore c_1 X_1 + c_2 X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Summary: when things are evolving in time, by 1st order system, starting from original u_0, the key is to find the E'values and E'vectors of A

– You then need to take your u_0 and write it out the combination of E'vectors, and then follow each E'vector separately