Lec graphs and networks

Here we will cover applications of linear algebra:

Most of the examples given (null space / row space), a matrix is created, which is invented on the fly.

Truth: Real "LA" uses matrices that come from somewhere, they just not randomly invented.

They come from applications

they have definite structure.

Eg. Chemistry ~~now~~ reducing matrices. matrices tells them how much of each element / molecule goes into a reaction, and what comes out

∴ By row reduction we get a clearer picture of complicated reaction.

∴ the most important model in applied math:

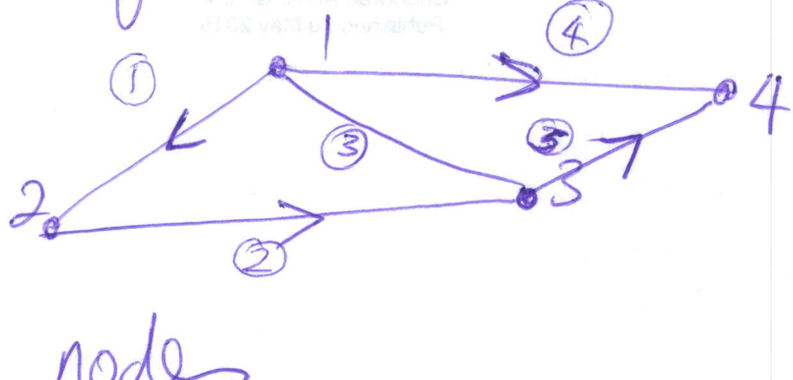
⇒ graph.

∴ Draw a graph, write down a matrix that is associated with it

∴ (great source of matrices)

∴ graph has nodes and edges ("line")

e.g. graph of all websites ...., telephones



$n=4$ nodes

m = row for every edges = 5 edges

∴ Same number of row

3

∴ need to give a direction (+ or −)
for every edge

∴ by using an arrow.

with arrow
against arrow
+ve or −ve

∴ what will be the matrix?

∴ words I will use will be − potential difference
− currents...

∴ other words, thinking of
an electrical network.

, But that is just one possibility

, can be hydraulic network, − flow water
− flow oil.

Lets call it the incidents matrix:
(use the current example)..

$$
\begin{array}{c|cccc}
\text{Node} & 1 & 2 & 3 & 4 \\
\hline
 & & & & \\
\end{array}
\qquad
A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}
$$

Edge 1 (node 1 to 2)

Edge 2 (Node 2 to 3)

Edge 3 (Node 1 to 3)

Edge 4 (Node 1 to 4)

Edge 5 (Node 3 to 4)

Every raw Correspond to Edge.

How to read the first line:

Edge 1 (raw) ⇒ leave node 1 (−1) and goes into ~~raw~~ node 2 (1), and does not touch Node 3 and 4

Edge 1, 2, 3, ⇒ Form a loop

look at these 3 raws are they independent or not?

Yes they are dependent add raw 1 and 2 will give you raw 3

Loops: Correspond to linearly dependent raws

- What questions can we ask about matrices?

- If I have a big matrix, it will have a lot of zero's
  → (*) every row only has two non-zero's
  ∴ very sparse matrix
  ∴ no# of non-zero's, is exactly
  (*) 2x5 (2xm)

- This is its structure
  ∴ real graphs (in real world) of genuine problems has structure

∴ Because of the structure, we can answer the main question about matrices!

- nullspace?
  ∴ looking at cols of matrix
  and asking are those cols independent.
  ∴ if independent, then what is in the null space?
  - only zero vector.

the nullspace tells ~~us~~ what (~~combination of~~) how to combine columns to get zero

• is there anything in the nullspace of this matrix, other than just the zero vector

$\Rightarrow$ are those 4 column independent or dependent.

we can do it properly (finding nullspace):

$$Ax = 0.$$

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \begin{array}{c} \text{Edge} \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{array}$$

$$Ax = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

← want to find out when it equals zero.

See what matrix Ax is doing:

⇒ it computes the difference across every edge.

∴ difference in potential.

∴ Lets introduce some terminology:

$x = x_1, x_2, x_3, x_4$

Potentials at nodes.

∴ When I multiply by A, I get:

$x_2 - x_1$, etc

∴ they potential differences Across edges.

∴ When are these differences all zero?

A ∴ Columns in A are dependent ?

∴ (I can find solutions to that equation)

Lets get 1 vector in nullspace

$x = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \Rightarrow$ Constant potential.

∴ If potential are constant, all the ~~att~~ potential difference are ~~Constant~~ zero, then x is in the nullspace.

∴ Whatelse is in the null space.

Lets get Basis for Null space.

good one is $x = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$

and here's whole null space: $x = c\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$

Any multiple of (1,1,1,1,1)

(9)

∴ Dim $N(A) = 1$

What does it mean physically or in the application?

– It means, the potential can only be determined up to a constant.

⇒ Remember potential difference is what makes current flow, what make things happen. (e.g. between x_1 and x_2)

∴ nothing will move if all potentials are the same

∴ there is an arbitrary constant (c) that raises and drops all the potentials.

e.g. Looking at temp: – flow heat from higher to lower, if temp equal free – no flow

①

To see what will happen, we fix one of the potentials, like last one:

ground node 4, set $x_4 = 0$

- So if we fix it
$\Rightarrow$ Not a node any more

then that column disappear

- So grounding a node, is a way of getting rid of matrix.

We Back:

What is rank of matrix?

5×4 matrix

- we have located its null space (1 dim)

- how many independent columns?

rank = 3.

$\Rightarrow$ any 3 columns will be independent.

Columnspace: all combinations of those columns

Nullspace of A^T [left hand side nullspace]

~~Note: the equation $Ax=0$~~ ← most fundamental equation of applied mathematics

$A^T y = 0$

now I am finding the Nullspace of A^T

$N(A^T)$

Dim: $N(A^T)$

general formula of $N(A^T)$

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

$A^T = n \times m$

4×5

Columns will become rows

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\dim N(A^T) = m - r = 5 - 3 = 2.$

- What does ~~Aty~~ A transpose y actually mean?
  - Why am I interested?

⇒ Back to our terminology:

there is matrix that I will call C that connect potential difference to current

Currents on edges: y_1, y_2, y_3, y_4, y_5

and this relation between current and potential differences $x_2 - x_1$, etc

is called Ohms Law!

it says current on edge is some number times the potential drop

∴ Change in ~~current~~ potential, that make some ~~potential~~ current happen. (13)

and Ohms law, that says how much current happens

Final Step of the Equation.

$A^T y = 0$ = Kirchoff's CL — Current law.

Back! → nullspace of A^T?

$$A^T \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- we know it dim: = 2.
- what is a vector in it?
- what is been asked? give me 5 currents that satisfy Kirchoff's CL
- What does KCL say?

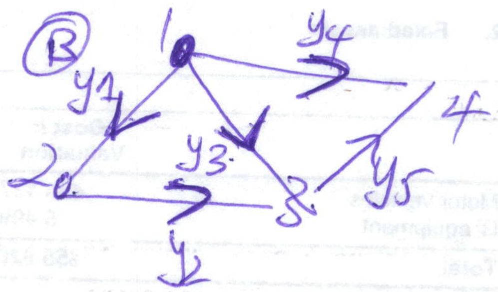
∴ What does law say in First row of AT ∴ 14

A

$-y_1 - y_3 - y_4 = 0$ [where did Equation come from]

B

Let's redraw graph



∴ we have currents (y's)

A

What is this equation telling me? (~~Kirchhoff equation~~)

∴ first Equation of K's (current Law)

– What does this Equation mean for this graph.

I can see y_1, y_3 and y_4 and current leaving Node 1.

∴ this equation refers to node 1.

∴ KCL is a balance equation

∴ the equation says that the net flow is zero

∴ KCL / equation ⇒ a balance equation
∴ Conservation law

Lets take second row:

$$y_1 - y_2 = 0$$

∴ Current y_1 goes into node 2
Current y_2 leaves node 2.

Third row:

$$y_2 + y_3 - y_5 = 0$$

∴ Current y_2 goes into node 3
y_3 goes into node 3
y_5 leaves node 3

(fourth) 4th row:

$$y_4 + y_5 = 0$$

Current y_4 goes into node 4
y_5 goes into node 5

∴ Charges does not accumulate at node, it travels around

∴ What's vector y that solves these equations? 16

$$-y_1 - y_3 - y_4 = 0$$
$$y_1 - y_2 = 0$$
$$y_2 + y_3 - y_5 = 0$$
$$y_4 + y_5 = 0$$

C

∴ Can I figure out what nullspace is of this matrix by looking at graph?

⇒ we can do elimination, we know how to do it.
(on matrix)

→ we then get in reduced echelon form. and special solution will pop right out.

→ But would like to do it without it.

A - see page 13

, But if I do elimination of this matrix what would the last row be? will be all zero? why?

∴ Cause rank = 3, we only going to have 3 pivots, and 4th row will be all zeros

- Elimination will tell us what we spotted earlier... (what nullspace, dependence...)

But here in real example, we can find them by <u>THINKING</u>.

- What is a solution y.
- How can current travel around this network (graph) (B) page 14 without collecting any charge at nodes

⇒ what is a y.

- Basis for $N(A^T)$ → how many vectors looking for?

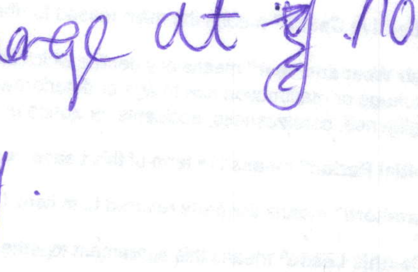
  Basis = 2 vectors [2 dim space]

<u>One vector?</u> one set currents.

$y_1 = 1$
$y_2 = 2$
$y_3 = 0$
$y_4 = 0$

[using graph on P13]

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

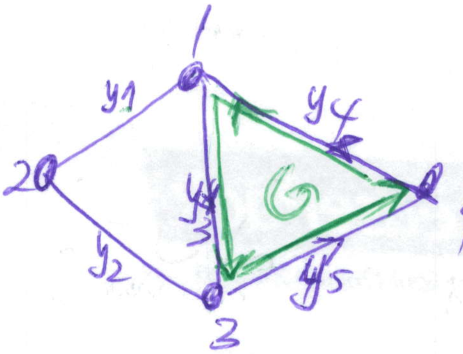


↳ satisfy KCL.

↳ (C) on page 16

Another set vects?

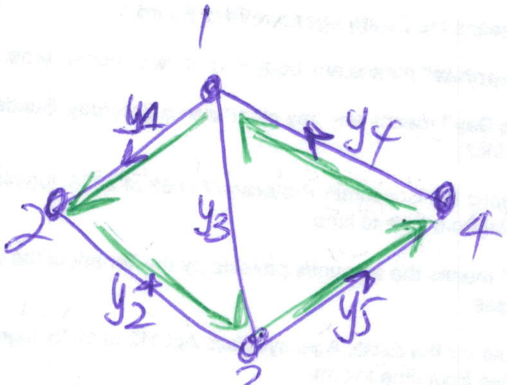
$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$



18

What about sending current around big loop?

(A) $$\begin{bmatrix} 1 \\ \cancel{1} \\ 0 \\ -1 \\ 1 \end{bmatrix}$$



But is this vector in nullspace of A^T. (yes)

, But ~~know~~ now we have (A) third vector in basis

~~But is~~ and it is not independent

- this vector (A) is the sum of first two vectors

What is the rowspace (Column space A^T)

What is Dim (rowspace)

original A, 5 rows, ~~what~~ how many were independent?

Asking rank again $r = 3$

∴ 3 independent rows.

and when it is transposed,

3 independent columns.

└ But is it, No not independent.

, they not independent, because they

came from Loop.

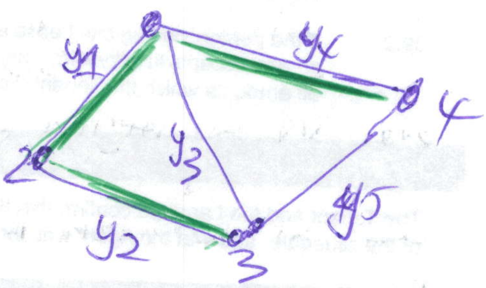
first column of A will be

↓ 1st, ↓ 2nd, , ↓ 4th ,

∴ Edge 1

Edge 2

Edge 4



(they have NO Loop.

* name for graph without Loop!

∴ 4 nodes

3 edges

But I put another edge in, I will get Loop

⟹ TREE : NO LOOPS

formula for Dim. $N(A^T) = m - r$

(independent) #Loops

m = # of Edges − (#nodes − 1)

r = (rank in r

∴ #Loops = # Edges − (# nodes − 1)

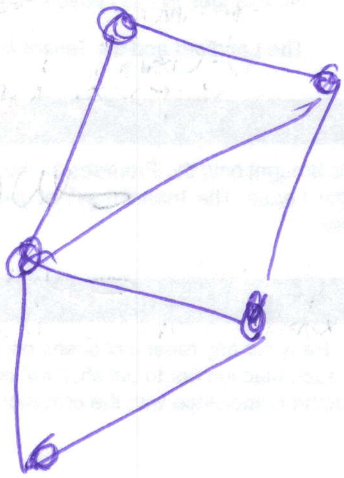
or #nodes − #edges + # Loops = 1

Euler's formula

∴ Linear algebra proves Euler's Formula

∴ great topology fact...

Lets draw any graph



∴ #nodes − #Edges + # Loops

$5 - 7 + 3 = 1$

∴ To close off:

$$\cancel{e = Ax} \quad \text{~~potential differences~~} \; (x_1 - \overset{e}{x_2}, \ldots)$$

$$\cancel{y = Ce} \quad \text{~~Current~~}$$

∴ Let $e = x_1 - x_2, \ldots$

$e = Ax$ [equation]

$y = Ce$: current come from potential differences

$A^T y = 0$ [and currents satisfy Kirchoff Law]

If current come from outside we can change.

$$A^T y = f$$

Basic equation of applied math.

$A^T C A x = f$ → Equation is always symmetric

⇒ Balanced Equation.

⇒ always balanced equation to look for.