Lec

Determinant is a facinating, small topic inside LA

→ determinants used to be big thing

→ and LA was small

But they have changed (???)

Goal: find formula for det – will be messy (so now appropriate time)

and also look at formula for Co-factors.

Remember we have these simple properties:

1. det $I = 1$
2. Sign reverse with Row exchange
3. det is linear in each row separately.

⇒ From those, we got an additional 7 more properties.

Let's take determinant of (2x2) matrix

But let's get out of 3 properties

$\cancel{\begin{vmatrix} a & b \end{vmatrix}}$

Looking for a (det) formula for these det.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Prop. 1 $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, Prop 2 $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

Know this = 1

Know one exchange rows, and det = −1

⇒ Now use prop 3 to get 'everybody'

— How will I do that?

(3)

If keep second raw the same, allowed to use linearity! (linearity)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \leftarrow \text{(Second raw)}$$

write a b as (a 0) and (0 b)

one step using property (3)

linearity in the first raw when the second raw is the same.

now keep first raw fixed and split second raw

split again

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}}_{\text{det} = ad} + \underbrace{\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

(A) det nothing? ⇒ has column of zeros singular matrix

det = ad

— Flip first
— sign chang
— det bc

same reasoning as (A), columns as zeros

$$\boxed{ab - bc}$$

See the method I was using

Using the same above method, can do 3x3, etc any size.

Lets do 3x3 mentally:

1. Keep row 2 and 3 the same
2. Split first row into 3 pieces
   (a 0), (0 b) and (0 c)
   ↳ first row
3. Previously with (2x2), went from:
   ⇒ 1 piece, to
   ⇒ 2 pieces, to
   ⇒ 4 pieces (go from)

Now with (3x3) will split into:

⇒ 1 piece, to

⇒ 3 pieces, to

⇒ 9 pieces, then have another row to straighten out

⇒ 27 pieces

(But a lot of them would be zero)

, But when would they not be zero.
— lets get the non zero.

Eg —
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

∴ what are survivors and how many of them.
⇒ and when do I get a survivor?

Here's survivor:
$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$$
its det: $= a_{11} a_{22} a_{33}$

another survivor
$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix}$$
its dets $= \boxed{-} a_{11} a_{23} a_{32}$

what sign; will be (-ve): as there is <u>one</u> row exchange

∴ the survivor has <u>1</u> entry from <u>each</u> <u>row</u>, and <u>each column</u>

More parts:

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} = \det = (-ve)\, a_{12}\, a_{21}\, a_{33}$$

exchange raw 1 and 3 from (+) to (-ve)
and raw 2 and 3 from -ve to (+)

$$\begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} = \underline{\text{Det}} \; + a_{12}\, a_{23}\, a_{31}$$

$$\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} = + a_{13}\, a_{21}\, a_{32}$$

One exchange (raw 1 and 3)

$$\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} = (-ve)\, a_{13}\, a_{22}\, a_{31}$$

But now we have formula for the det
– which come from the 3 properties.

Big jump
(n×n Now)

∴ Let's get the general formula:

$$\det A = \sum_{n!\ \text{terms}} \pm\, a_{1\alpha}\, a_{2\beta}\, a_{3\gamma} \cdots a_{n\omega}$$

$(\alpha, \beta, \gamma, \ldots \omega)$ = Perm of $(1, 2, \ldots n)$

∴ Where does $n!$ terms come from:
Bec the guy in First row can be
Chosen n ways,
and after he's chosen, that has used
up that column,
then one in second row can be
Chosen $n-1$ ways
∴ after that, that 2nd column has
been used.
– etc

∴ 3rd raw is then chosen $n-2$ ways

Best:

∴ The one in first raw ($a_{1\alpha}$) came from some column alpha (α), then multiply it ($a_{2\beta}$) with somebody from second raw from some different column (β), and multiply that ($a_{3\gamma}$) with somebody in 3rd raw with some other column (γ)

($\alpha, \beta, \gamma \ldots \omega$)

∴ the whole point, those column numbers are different, and some permutation of $(1 \ldots\ldots \text{to } n)$

∴ $(\alpha, \beta, \gamma \ldots \omega) =$ perm of $(1, 2 \ldots n)$

∴ n column numbers ~~are each~~ are used once

∴ gives us $n!$ terms

∴ choosing somebody from every column.

∴ from perm example

$= a_{11}\, a_{22}\, a_{33}$

$\alpha \quad \beta \quad \gamma$

Example (4×4) matrix ∴ whats the |det|

∴ 24 (n!) terms.

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

take (underline) (4, 3, 2, 1) (Permutation)
→ +ve sign

take (×) (3, 2, 1, 4)
→ −ve sign

take ○ → ______

→ Can start with elimination...

→ From above, the matrix must be singular.

→ ↳ here another way to see if singular?

∴ find something in its nullspace

or find combination of rows that give zero.

∴ What Combination of raws does give zero.

- Add raw 1 and 3. — get raw of all 1's

- add raw 2 and 4 → "

∴ raw 1 − 2, + row 3 − raw 4

⇒ zero raw

∴ singular matrix.

Shows: - that we get 24 terms

∴ - great advantage of having a lot of zero's in it.

Lets now look at Co-factors:

∴ Way of breaking up (Det) Big formula

⇒ That connect the nxn determenant

to determenant one smaller

∴ we looking at Co-factors: and use ~~13~~ 3X3.

Looking at Prev example:

$$|det| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Co-factors in Parens

$$|det| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(\quad) + a_{13}(\quad) + c(\quad)$$

∴ Choose Column 1 (a_{11}) and take possibilities, called Co-factors in Parens

∴ What is this Co-factor, what is there that is multiplying a_{11}

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}$$

∴ a_{11} is determinant of smaller guy (a_{22} a_{23} a_{32} a_{33})

this are used, so can't use this Column again

— what correct sign. (12)

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} \quad -a_{12}(a_{21}a_{33} - a_{23}a_{31})$$

$$a_{12}(-a_{21}a_{33} + a_{23}a_{31})$$

$$\begin{vmatrix} 0 & 0 & a_{13} \\ & & \\ & \dots & \end{vmatrix} \quad +$$

Co-factors either + or − the determinant

Cofactor of a_{ij} = C_{ij}

a of any number ij

$\pm \det\begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \\ \text{col } j \text{ erased} \end{pmatrix}$

is +ve if $i+j$ is even

is −ve if $i+j$ is odd

$$\therefore \begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$$

Sign given by pattern rule

Cofactor rule?

What is the Cofactor formula? (along row 1)

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{1n}C_{1n}$$

Eg. What is the Cofactor formula for (2×2)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + b(-c)$$

Note: the determinant is a product of the pivots

One more Q of chapter

(4×4)

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \boxed{1} & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

But real idea is to do $(n\times n)$ → subtract row 3 from 2 then use co-factor formula along row 2. $\square$ above

OK just do a_1, a_2, a_3, a_4

$|A_1| = 1$ $\quad |A_2| = 0$ $\quad |A_3| = ?\ -1$

use co-factors?

$|A_4| = 1 \cdot |A_3| - 1|A_2| = -1$

$|A_5| = 0$ — m

$|A_6| = 1$

$|A_7| = 1$

∴ formula for anything:

$$|A_n| = |A_{n-1}| - |A_{n-2}|$$