le ①

Idea of Similar matrices.

What does similar mean?

Lets recap:

Pos. Def. matrices...

$x^T A x > 0$ — always positive (except for $x=0$)

quadratic form

⇒ But where does Pos. Def. matrices come from?

⇒ they come from (A^TA) least squares and all sort of physical problems...

∴ Start with rectangular matrix (A^TA is positive definite)

Suppose: matrix A is pos. def. (assuming it symmetric)

∴ also its inverse is pos. def.

If A, B are pos. def.

What about $A+B$. $\Rightarrow$ hope it would be true

But we do not know the E'values of $A+B$.

" " " " " " " pivots " "

$\therefore$ Best place to start is using $x^TAx > 0$.

$\therefore$ $x^T(A+B)x > 0$, so is $A+B$.

$\therefore$ $(x^TBx > 0)$

$\longrightarrow A$

Suppose A is rectangular.

$\therefore$ A m by n

No way, its pos. def $\Rightarrow$ its not symmetric.

— not even square.

But key for the rectangular ones:

was: A^TA 1) thats square
2) thats symmetric

But now, is it pos def.

Is A^TA – pos (or semi) def?

~~A x Ref~~

length squared

$$x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 \geq 0$$

But do I ensure Ax is never 0.
∴ no nullspace

∴ rank $\boxed{n}$ ← independent Column.

– truth pos. Def matrix, you never need to do row exchanges.

Similar Matrices start

no longer talking about symmetric

Suppose: A and B are similar ∴ 2 square matrices ($n \times n$ matrix)

what does similar mean.

for some matrix M:

$$B = M^{-1} A M$$ ← ~~why this~~ Why this combination

Example:

$$S^{-1}AS = \Lambda$$

Now we saying A is similar to Λ

Lets say $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ matrix Lambda $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ E'values $\lambda = 3, 1$ (A) E'vectors will be easy to find

E'matrix for that

Matrix A is similar to ~~A~~ ~~(mat~~ matrix Λ

take:

$$\begin{array}{cccc} M^{-1} & & M & \\ \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \begin{array}{c} B \\ \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix} \end{array}$$

took any M

$\lambda = 3, 1$ (A)

Similar Matrices:

(A) They have the same E'value $[\lambda's]$

why?

Eg $\begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$

all λ's same

$\therefore Ax = \lambda x$

[need to get B somewhere in picture]

$B = M^{-1}AM$

$(M^{-1}AM)M^{-1}x = \lambda M^{-1}x$

$B \quad M^{-1}x = \lambda M^{-1}x$

$\therefore$ ~~e'vector~~ e'vector of B is M^{-1} (e'vector of A)

→ Similar matrices have same λ's, and e'vectors just move around

— Do we have family of matrices with e'values that are (3, 1)

— But what happens if e'values are same?

$\therefore$ there may not be a full set of e'vectors and we may not be able to diagonalize.

BAD CASE $\lambda_1 = \lambda_2 = \overset{eg}{4}$

And we look at family of matrices with e'value (4, 4)?

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Too Loner

But now, top guy is not in same family as Bottom one?

$\therefore$ two families.

$$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

This is Big family ~~[illegible]~~ Boss

- By 'family' we mean similar

only matrix similar to $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ is itself

$$M^{-1}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

∴ get same matrix back, not getting any more members of family.

for Jordan Form.

But $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ Best one, But can't make it diagonal.

$\Rightarrow$ Not diagonalizable.

it only has 1 E'vector.

Called Jordan Form.

"Best looking" matrix in each Family"

now we covered all matrices

$\Rightarrow$ "including the non diagonalizable ones"

More members of family ⊕ ① Trace will be 8 ② det 16

$$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a & \text{an} \\ n & 8-a \end{bmatrix}$$

Example:

1

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

1

E'values = 0,0,0,0

Dim N(space) = 2

E'vectors will be in nullspace.

Since rank 2

indep Colum = 2

2 indep E'vectors

2 missing E'vectors.

change

$$\begin{bmatrix} & & 7 & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

rank = 2

3X3 Block

1X1 Block

Jordan Block has 1 E'vector only.

$$\begin{bmatrix} & & 0 & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

A Not similar to B

↑ As per Jordan.

1

$$B = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

1

E'values = 0,0,0,0

rank = 2

2 Evector

2 missing

2x2 Block

2x2 Block

⇒ Called Jordan Block

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

λ in diagonal

1's above

0's below

∴ Jordan's Theorem:

Every Square A (matrix) is similar to a Jordan matrix J

$$J = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_d} \end{bmatrix}$$

⇒ Consist of Jordan Blocks

Block = # E'vectors.

(GOOD CASE)

∴ $\boxed{J \text{ matrix is diagonalizable}}$

Good Case : J is Λ

But Jordan included everything:

— Repeated λ's

— Missing E'vectors.