Solving $Ax=b$

This is the lecture we we completely solve linear equation : $Ax=b$
(only if it has a solution)
and if there is a solution:
- is there only one solution
- or family of solution.

So we going to use the same example as we use for the null space:

$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$$
$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$
$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

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$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{array}\right]$$

Augmented matrix $= [A, b]$

Lets do Elimination

$\therefore$ 1 is first Pivot.

2 is second Pivot.

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{array}\right]$$

want this to be zero

Pivot columns

More Elimination.

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{array}\right]$$

$\therefore \; 0 = b_3 - b_2 - b_1.$

what if $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ ← this is OK as it allows for a solution

$\therefore \; b_1 = 1$

$b_2 - 2b_1 = 3$

$b_3 - b_2 - b_1 = 0$ ✓ (this is the main point)

Solvability: Condition on the right hand side (b)

what is that condition.

$Ax = b$ solvable when b in $C(A)$
(Column space of A)

$\therefore$ If a combination of rows of A
gives zero row,
then the same combination of
entries of b must give 0.

Again does this system have a solution

∴ What is the sequence of steps to find a solution?

To find a complete solution:

- lets find one complete solution

one particular solution

guys that can be anything

① $x_{particular}$: set all free variables to zero

⇒ then solve $Ax=b$ for pivot variables

∴ Which one are free variables?

$$x_2 \text{ and } x_4 = 0.$$

∴ Pivot Equations:

$$x_1 + 2x_3 = 1$$

$$2x_3 = 3.$$

$\left[x_1 + 2\left(\frac{3}{2}\right) = 1 - 3 = -2\right]$ 5

$\therefore \; x_1 + 2x_3 = 1 \Rightarrow x_1 = -2.$

$2x_3 = 3 \Rightarrow x_3 = 3/2$

$\therefore$ Now we have the solution:

$$\underbrace{X_p}_{\text{particular}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

One particular solution to be plugged into original system.

$\Rightarrow$ Now we want all solutions

$\therefore$ How do I find the rest.

② $X_{\text{nullspace}}$ anything from null space

we know how to find vectors in null space.

$\therefore$ Complete solution:

$$X = X_p + \underbrace{X_{\text{nullspace}}}_{\text{any vector out of null space}} \text{ or } X_n$$

But why this pattern:

$$A x_p = b$$
$$A x_n = 0$$ lets add.

$$A(x_p + x_n) = b$$

$\therefore$ if I have one solution (ie A_p), I can add on anything from (in?) nullspace.

$$X_{complete} = \begin{array}{ccccc} x_p & & x_n & & x_n \\ \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} & + & \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} & + & \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \end{array}$$

those were special solution

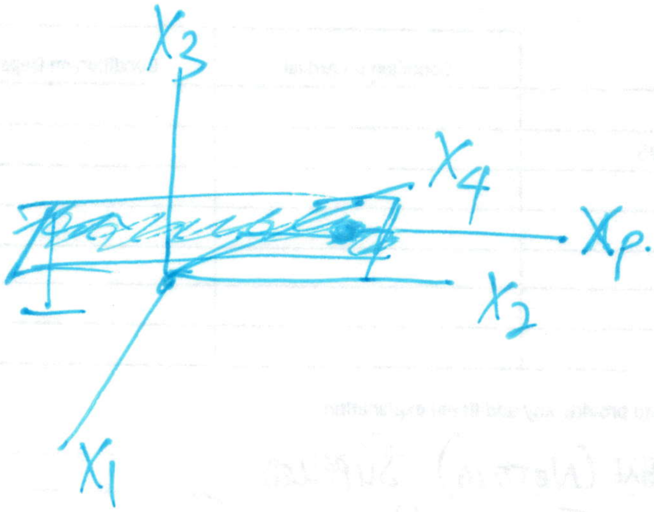
what do we do to get complete solution.

- How do I get the complete solution now?

$\therefore$ I multiply with anything $[c_1, c_2]$

eg $c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

∴ Plot all Solutions x in Dimensions? $\mathbb{R}^4$ (7)



— only the null space is sub space, not the x_p (particular Solution)

So now what is the bigger picture?

Lets think about: m by n matrix A of rank r

definition of rank: # of pivots

What is the relationship between r and m

∴ if have m rows in matrix and r pivots, then we conclude:

∴ $r \le m$, $r \le n$

must have ≤ pivots than rows

column cannot have more than one pivot.

(Know: $r \leq m$, $r \leq n$)

But very interested in Full rank.

∴ biggest r can be.

∴ Full Column rank means $r = n$ $\Longrightarrow$: No free variables

– What does that imply about our Solution ?

– What does it tell us about Null Space ?

– What does it tell us about Complete Solution.

∴ pivot in every Column ⟹

∴ no free variables

∴ what does mean Null Space:

∴ $N(A) = \{$ only Zero Vector $\}$

∴ and what Solution to $Ax = b$:

∴ Solution: $x = x_p$ ← only one Solution, unique Solution. (if it exist)

$\boxed{\text{from } r=n}$

∴ (either 0 or 1 solution)

Now I need an example: → tall and thin.

e.g.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$$

∴ what is rank matrix.
i.e. how many pivots will I find = 2.

∴ RREF for above

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

∴ this is a case of full pivot rank:

- two columns, with 2 pivots
- there is nothing in the null space
- no combination of those column will give 0 column.
- except zero combination

But is there always a solution to $Ax=b$?

∴ I have 4 equations, but only 2 x's

Full Row Rank means $\boxed{r=m}$ ∴ m pivots →

∴ every row has a pivot.

which right hand side

- Can solve $Ax=b$ for every b [Exists]

∴ how many free variable?

∴ Left with $n-r$ free variable

∴ $n-m$ free variable.

∴ Let transpose the previous variable:

$$A=\begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

- What is its rank? 2 pivots = rank.

RREF

$$R=\begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix}$$

F

Fulldrank [Column and rows]

$\therefore r = m = n$ ⇐ Square matrix

(1)

∴ invertible matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$\therefore r = m = n \Rightarrow$ invertible matrix

$RREF = R = I$ [reduce matrix them all the identity matrix]

what is the nullspace for this matrix?

⇒ only the zero vector.

- to solve $Ax = b$ for this example?

$$\therefore A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

⇒ what are conditions for b_1 and b_2

⇒ None at all.

$R = RREF$

Lets summarise:

$r = m = n$

$R = I$

Square invertible solution (Chapter 2)

$\Rightarrow$ only 1 solution

$r = \cancel{m}\, n < m$

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

$\Rightarrow$ 0 or 1 solution ($Ax=b$)

$r = m \le n$

$R = \begin{bmatrix} I & F \end{bmatrix}$ or $\begin{bmatrix} I \; F \end{bmatrix}$ F mixed in with I

$\Rightarrow$ always a solution [No zero rows]

∞ solutions

$r < m,\ r < n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

0 or ∞ solutions

The rank tells you everything about the # of solutions