

Solving $Ax=b$

①

This is the lecture we we completely

One linear equation : $Ax=b$

(only if it has a solution)

And if there is a solution:

- is there only one solution

- or family of solutions.

So we going to use the same example
as we use for the null space:

$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$$

$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$

$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{matrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 3b_1 \end{matrix}$$


Augmented matrix = $[A, b]$

lets do Elimination

\therefore 1 is first Pivot.

2 is second Pivot.

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 & | & b_1 \\ 0 & 0 & \boxed{2} & 4 & | & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & | & b_3 - 3b_1 \end{bmatrix}$$


 Pivot columns

want this to be zero

More Elimination.

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 & | & b_1 \\ 0 & 0 & \boxed{2} & 4 & | & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & | & b_3 - b_2 - b_1 \end{bmatrix}$$

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$$\therefore 0 = b_3 - b_2 - b_1.$$

What if $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ \leftarrow This is ok as it allows for a solution

$$\therefore b_1 = 1$$

$$b_2 - b_1 = 3$$

$$b_3 - b_2 - b_1 = 0 \quad \checkmark \quad (\text{this is the main point})$$

Solvability: Condition on the righthand side $\underline{\quad (b) \quad}$
What is that Condition.

$Ax = b$ solvable when b is in $C(A)$
(Column space of A)

\therefore If a Combination of rows of A
gives zero row,

Then the same Combination of
Entries of b must give 0.

Again does this system have a solution?

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∴ What is the sequence of steps to find a solution?

To find a Complete Solution:

- let's find one Complete Solution:

↓ One particular solution

↓ guys that can be anything

① X particular: set all free variables to zero

⇒ Then solve $Ax=b$ for pivot variables

∴ Which are our free variables?

$$x_2 \text{ and } x_4 = 0.$$

∴ Pivot Equations: 

$$x_1 + 2x_3 = 1$$

$$2x_3 = 3.$$

$$\therefore x_1 + 2x_3 = 1 \quad \Rightarrow x_1 = -2. \quad \text{①}$$

$$2x_3 = 3 \quad \Rightarrow x_3 = \frac{3}{2}$$

\therefore Now we have the solution:

$$x_p \xrightarrow{\text{particular}} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

← One particular solution to be plugged into original system.

\Rightarrow Now we want all solutions

\therefore How do I find the rest.

② $x_{\text{nullspace}} \xrightarrow{\text{anything from null space}}$

← we know how to find vectors in null space.

\therefore Complete solution:

$$x = x_p + x_{\text{nullspace}} \quad \text{or } x_n \quad \xrightarrow{\text{any vector out of null space}}$$

But why this pattern:

⑥

$$Ax_p = b$$

$$Ax_n = 0 \quad \swarrow \text{lets add.}$$

$$\underline{A(x_p + x_n) = b} \rightarrow$$

\therefore if I have one solution (ie Ax_p), I
can add on anything from (i.e.) nullspace.

$$X_{\text{Complete}} = \begin{matrix} x_p \\ \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} \end{matrix} + \begin{matrix} x_n \\ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{matrix} + \begin{matrix} x_n \\ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \end{matrix}$$

these were special
solutions

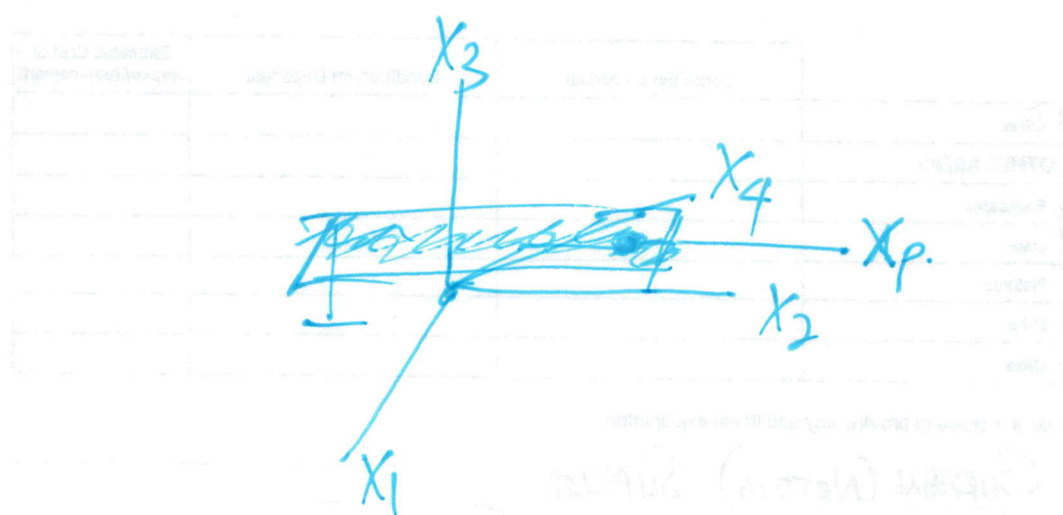
What do we
do to
get complete
solution.

- How do I get the complete solution now?

\therefore I multiply with anything $[c_1, c_2]$

$$\text{eg } c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

\therefore Plot all solutions x in Dimensions? \mathbb{R}^4 ⑦



— only the null space is subspace, not the x_p (particular solution)

So now what is the bigger picture?

lets think about: m by n matrix A of rank r
 definition of rank: # of pivots

What is the relationship between r and m

\therefore if have m rows in matrix and r pivots, then we can know

$$\therefore r \leq m, \quad r \leq n$$

\uparrow must have \leq pivots than rows
 \uparrow Column cannot have more than one pivot.

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(know: $r \leq m, r \leq n$)

But very interested in Full rank.

\therefore biggest r can be.

\therefore Full Column Rank means $\underline{r = n}$: No free variables

- What does that imply about our solution?
- What does it tell us about Null space?
- What does it tell us about Complete solution.

\therefore pivot in every column \Rightarrow

no free variables

What does mean Null space:

$N(A) = \left\{ \begin{matrix} \text{only} \\ \text{Zero} \\ \text{vector} \end{matrix} \right\}$

and what solution to $Ax = b$:

\therefore solution $x = x_p$ \rightarrow Only one solution, ^{unique} solution.
(if it exist)

\therefore (either 0 or 1 solution) from $r=A$ (9)

Now I need an example: \rightarrow tall and thin.

eg. $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$

\therefore what is rank matrix:
i.e. how many pivots will I find = 2.

\therefore RREF for above

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- \therefore this is a case of Full pivot rank:
- two columns; with 2 pivots
 - there is nothing in the null space
 - no combination of these columns will give 0 column.
 - except zero zero combination

But is there always a solution to $Ax=b$? (10)

\therefore I have 4 equations, but only 2x's

Full Row Rank means $\boxed{r=m}$ \therefore m pivots \rightarrow

\therefore every row has a pivot.

- Can solve $Ax=b$ for every b [exists] \rightarrow which right hand side

\therefore how many free variables:

\therefore Left with $n-r$ free variable

$\therefore n-m$ free variables.

\therefore Lets transpose the previous variable:

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

- What is its rank? 2 pivots = rank.

RREF

$$R = \begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix}$$

Full rank [column & rows]

$$\therefore r = m = n \leftarrow \text{Square matrix}$$

①

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad \therefore \text{invertible matrix}$$

$$\therefore r = m = n \Rightarrow \text{invertible matrix}$$

$$RREF = R = I \quad \left[\begin{array}{l} \text{reduce matrix} \\ \text{then all the identity matrix} \end{array} \right]$$

What is the nullspace for this matrix?

\Rightarrow only the zero vector.

to solve $Ax = b$ for this example?

$$\therefore A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

\Rightarrow What are conditions for b_1 and b_2

\Rightarrow None at all.

$$R = R_{REF}$$

(k)

lets summarize:

$$r = m = n$$

$$R = I$$

Square invertible
Sridhar (Chapter 2)

\Rightarrow only 1 solution

$$r = \overset{n}{m} < m$$

$$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

\Rightarrow 0 or 1 solution ($Ax=b$)

$$r = m \leq n$$

$$R = \begin{bmatrix} I & F \end{bmatrix} \text{ or } \begin{bmatrix} I & F \end{bmatrix} \begin{matrix} F_{m \times (n-m)} \\ F_{(n-m) \times (n-m)} \end{matrix}$$

F mixed in with I

\Rightarrow Always a solution [no zero rows]

~~no~~ solutions

$$r < m, r < n$$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

0 or ∞ solutions

The rank tells
you everything
about the # of
solutions