Lect

Now Covering - Eigenvalues, and Eigenvectors.

Remember, matrices are square, now we looking for special numbers, Eigenvalues, - vectors

Here we will cover what are these numbers ⇒ other vectors we have do we use it - and why do we want it.

What's Eigen Vectors?

Matrix:

A = ... what does matrix do?

It acts on vectors, it multiplies vectors x.

In goes vector x, and out comes vector Ax

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like a function.

vectors we specifically interested are?

— the one that come out in same direction, as the went in

, But that won't be typical:

Most vectors, Ax point in some different directions, but there are certain vectors where Ax, comes out parallel to x.

Those are Eigenvectors

Eigenvectors

Ax parallel to x

$$Ax = \lambda x$$

same multiple (lambda) of x.

Big equation.

(3)

- we look for special vectors, But remember most vectors won't be Eigenvectors

- when saying same direction, all I allow it to be the very opposite direction
- I allow λ to be negative, or 0.

x (Eigenvector x)
λ (Eigenvalue)

- Eigenvalue 0 ⇒ not special deal ⇒

$$Ax = 0x$$

- then what's Eigenvectors, if $EV = 0$.
- they are guys in null space

$$Ax = 0$$

- eg A is singular [ie it takes some vector x into 0]
- $\lambda = 0$ is Eigen value

How do we find the x's and λ's

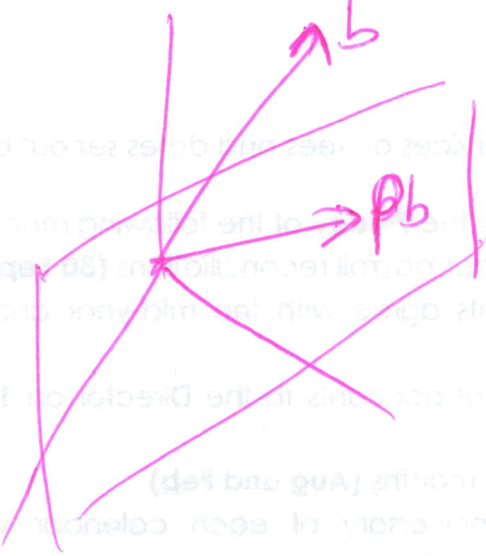
we don't have Equation $Ax=b$

So we can't use elimination any more.

λ, and x, are both unknowns

we need a good idea on how to find them.

First Lets take a few matrices:



⇒ Suppose we have a plane, what are the evalues of a projection matrix?

What are the x's and λ's for a projection matrix.

⇒ we take a vector b, what matrix does it down to Pb

is b an eigenvector? in this picture <u>No</u>

∴ b is not an E'vector, because Pb, its projection, is in a different direction.

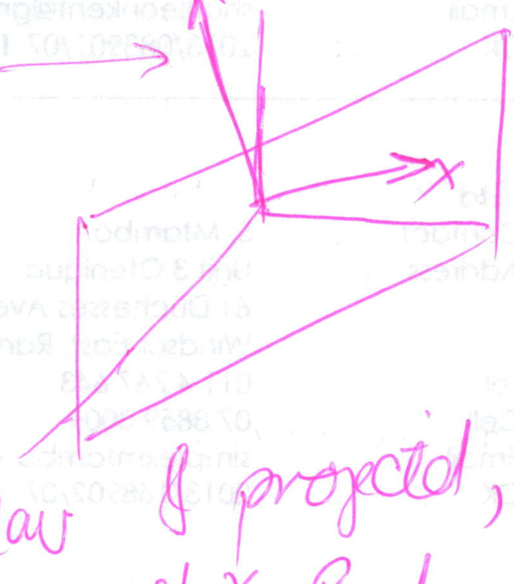
∴ What vectors are E'vectors of P

∴ What vectors do get projected in same direction as they start?

∴ If vector is already in plane.

∴ Any x in plane: Px

Not in plane, But ⊥



∴ vector that is in the plane

, and now if projected, what now do I get Back?
get x Back.

∴ it does not move.

∴ any x in plane is unchanged by P

$Px = x$

∴ that tells me that x is E'vector

and ~~that~~ also letting me ~~that~~ whats the E'value with multipliers.

E'value = 1

$\lambda = 1$.

∴ we have a whole plane of E'vectors.

– are there any other E'vectors? Yes

3Dim ⇒ so hopefully 3 independent E'vectors.

2 in plane, and 1 not in plane

∴ any $x \perp$ to plane: $Px = 0x$

∴ $\lambda = 0$

what are E'~~vecto~~ (values)s for a projection matrix: (P)

$\lambda = 1$

$\lambda = 0$

What about a Permutation Matrix:

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, what can be vector x?

what vector can I multiply by and end up in same direction?

which vector will have E'value 1

$x =$

if I permute it, it won't change:

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (A) (E'vector)

if $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda = 1$ $[Ax = x]$

(A) and (B) are $\perp$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (B)

$Ax = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ (B) $= \lambda = -1$ $[Ax = -x]$

Special fact about E'values:

nxn matrices, will have n e'values.

But not so easy to find them.

But here is nice fact:

sum of e'values = sum down the diagonal

Called trace →

Sum of λ's $= a_{11} + a_{22} + \dots a_{nn}$

→ the trace tells you what e'value is, then next one is easy to find.

How to solve $Ax = \lambda x$ [two unknowns both in equation]

Rewrite $(A - \lambda I)x = 0$

don't know λ or x, but do know:

This matrix $(A - \lambda I)x$ must be singular

and what do we know about SINGULAR matrices; ... the determinant is 0.

$\therefore \det(A-\lambda I)=0$

$\therefore$ got an equation for λ

$\therefore$ called the characteristic equation.

or eigenvalue equation.

$\therefore$ Find λ first [or n different λ's] or repeated.

$\therefore$ after we found the λ, then finding x is just by elimination.

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take another Example:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

→ made it 2x2
2) made it symetric
3) made it constant down diagonal.

$$\therefore \det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$\therefore$ it's matrix with λ removed from the diagonal

diagonal is shifted and I am taking the determinant.

want this to be zero

$$\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$$

set to 0 and solve, quadratic equation, and use factorization.

$= (\lambda-4)(\lambda-2)$ (A)

what's the number 6 here : it's a trace

What's the number 8 here : it's the det

(A) $\lambda_1 = 4,\ \lambda_2 = 2$ (Eigenvalue)

How to get the e'vectors?

Let's find the e'vector for 4 first.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

(subtract 4)

$\lambda = 4$

$$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \leftarrow \text{Singular matrix}$$

what's the x now, the x's in the nullspace?

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda = 4$$

E'vector

E'value

E'vector that goes with E'value.

$\lambda = 2$

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \text{Singular matrix}$$

What vector(s) is in the nullspace?

$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \lambda = 2$$

E'vector

E'value

2

What is the relationship:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \lambda = 1 \quad x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ B} \quad \lambda = -1 \quad x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ A}$$

one is just $3I$ more than other one $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \lambda = 4 \quad x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ B} \quad \lambda = 2 \quad x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ A}$$

λ E'values got now 3 added

$\lambda = 1 + 3 = \underline{4}$ $\qquad$ $\lambda = -1 + 3 = \underline{2}$

But E'vectors stayed the same

A = A B = B ⇒ above.

iff I add $3I$ to matrix,

⇒ its e'vectors don't change

⇒ and e'values are 3 bigger

Suppose: If $Ax = \lambda x$

3 added

then $(A+3I)x = \lambda x + 3x = (\lambda+3)x$

↑ x stays same for Both matrices.

~~tako~~ But what is Not so GREAT: $A+B$ or $A \times B$

If $Ax = \lambda x$, B has e'value

⇒ If we have e'value of one matrix, there is no reason to believe that its ~~e~~'value (x) is also a e'value of B

B has e'value, but it has different e'vectors.

i.e. $By = \alpha y$ Compare

Eg: Rotation Matrix: [matrix that rotates every vector by $90°$] ①

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ its orthogonal matrix.

$\therefore$ trace $= 0 + 0 = \lambda_1 + \lambda_2$.

$\det = \underline{1} = \lambda_1 \lambda_2$

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

det

$\lambda_1 = \underline{i} \quad \lambda_2 = \underline{-i}$

But they not real numbers, even though the matrix was real.

Now we need to start using Complex numbers

Even worse: (that can happen)

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$ $\leftarrow$ what are E'v$_s$ E'v$_e$ [E'values E'vectors]

$\therefore$ Matrix is triangular.

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda)$$

$\therefore \lambda_1 = 3 \quad \lambda_2 = 3$

$\therefore$ But problem here is the E'vectors:

$$(A-\lambda I)x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

looking for x

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

independent (NOT)

$\therefore$ only has 1 line e'vectors; instead of 2.

Shortage of E'vectors

x_2 = there is no second independent vecter for x_2.