Solving $Ax=0$

How do we describe or compute
spaces in vector space.
or those things

turning the idea into ~~a~~ algorithm.

talking about rectangular matrices

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

from:
Observation:

is it 'independent'

- Column 2 is multiple of col 1.
- row 3 is ~~multiple of~~ row 1 + row 2
- so row 3 is not independent.

- So use elimination, ~~can~~ and continue even if there is zero in pivot position.

Goal: do Elimination, without changing the null Space

remember, when we do elimination by subtracting / multiplying we not changing the Solution
i.e Same, we not changing null Space

⇒ But changing the Column Space

⇒ working on the left side, right side keeping up, But just goes

Lets do Elimination

$$\boxed{A = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

↳ First Pivot, make 2 zero

$-2R_1 + R_2 \to R_2$

$-3R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad \begin{array}{cccc} & x_2 & & x_4 \\ \end{array}$$

— use this as next pivot

$$\begin{array}{cccc} & x_2 & & x_4 \end{array}$$
$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

③

$\to -1R_2 + R_3 \Rightarrow R_3$

Pivot columns.

, But here we only have 2 pivots?

∴ this we call the rank of the matrix

∴ rank of A

$= \#$ of pivots

$= 2$.

always solved $Ax = 0$

now we can solve $Ux = 0$

∴ 2 pivot columns

∴ 3 free columns

Why do we use those words? (free)

– free columns – x_2

– x_4

Can assign values freely to x_2 and x_4

it solves $Ax=0$

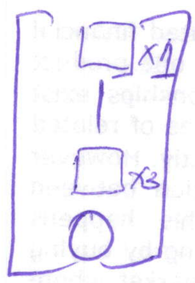
∴ Can solve the equations for x_1 and x_3

↳ ones with pivots.

I want find solution $Ux=0$

∴ Can assign anything to x_2 and x_4 ⇒ free columns.

and solve the equations for x_1 and x_3

$X=$ 

A

But what are my equations.

$$x_1+2x_2+2x_3+2x_4=0 \quad [\text{First}]$$

$$2x_3+4x_4=0 \quad [\text{Second}]$$

∴ I can find x_1 and x_3 by back substitution.

See A above

↓

A ⊗ New thing is that there are some free variables that we can give any number to

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So what is x_3 = $\begin{bmatrix} \boxed{}\,x_1 \\ 1 \\ \boxed{}\,x_3 \\ 0 \end{bmatrix} \begin{matrix} -2 & \boxed{C} \\ & \\ 0 & \boxed{B} \end{matrix} \Bigg\}$ (y)

⇒ Looking at last equation:

$$2x_3 + 4x_4 = 0$$

$$\therefore 2x_3 + 4(0) = 0$$

$$2x_3 = 0$$

$$\therefore x_3 = 0 \quad \boxed{B}$$

$$\therefore x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

$$x_1 + 2 + 0 + 0 = 0$$

$$\cancel{x_1 = 2}$$

$$x_1 + 2 = 0$$

$$x = -2. \quad \boxed{C}$$

But what does (y) above say?

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

this is solution for $Ax = 0$

and what does $x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ say?

$\therefore$ say -2 times (x) first column $+ 1$ times (x) second column is the 0

$$\therefore \begin{array}{c} (Col_1) \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{array} \times 2 + \begin{array}{c} (Col_2) \\ \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \end{array} \times 1 = 0$$

$\therefore$ we found 1 vector in null space $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

what are the others

$\therefore x = C \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

x can be any multiple of this

$\rightarrow$ this now describes a line.

, But what the whole null space

No.

- Remember we have 2 free variables

previous choice for x_2 and x_4 was 1 and 0

(Let's make another choice)

now $x_2 = 0$

$x_4 = 1$

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{bmatrix} \boxed{} \rightarrow 2. \\ 0 \\ \boxed{} \rightarrow -2. \\ 1 \end{bmatrix}$$

and we repeat calculated (complete solution)

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

$$2x_3 + 4x_4 = 0$$

$$2x_3 + 4(1) = 0$$
$$2x_3 = -4$$
$$x_3 = \frac{-4}{2}$$
$$x_3 = -2.$$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$x_1 + 2x_2 + 2(-2) + 2(1) = 0$$
$$x_1 + 2x_2 - 4 + 2 = 0$$
$$x_1 + 0 + -2 = 0$$
$$x_1 = 2.$$

$$\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

what does $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ say → [find another vector in null space]

$$\underset{\text{Col. 1}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \times 2 + \underset{\text{Col 3}}{\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}} \times (-2) + \underset{\text{Col 4}}{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}} = 0 \checkmark$$

$$\begin{matrix} 2 \\ 4 \\ 6 \end{matrix} - \begin{matrix} 4 \\ 12 \\ 16 \end{matrix} + \begin{matrix} 2 \\ 8 \\ 10 \end{matrix}$$

$$= \begin{matrix} 4 \\ 12 \\ 16 \end{matrix} - \begin{matrix} 4 \\ 12 \\ 16 \end{matrix} = 0 \checkmark$$

But what are all the solutions to $Ax=0$

now can also take any multiple of $d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} + c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Special + sign

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∴ Færan Solving:

$Ax=0$ or $Ux=0$

So how many special solution;

one for every free variable

Remember rank $r=2$ (pivot variable)

∴ 2 free variable.

(rows) (pivot)

∴ $n-2 = 4-2 = 2$ free variables

This is a Complete Solution for all the solutions to $Ax=0$

But lets Clean up this matrix

→ make it as good as it can be.

- The above solution was in Echelon Form.

To make it better:

Reduced Echelon Form.

R = Reduced Row Echelon Form.

What does this mean:
I can work harder on U

Zeros → Above
and → BELOW
the pivots

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

remember ~~r3~~ r_3 was combination of R_1 and R_2
– and elimination found this out.

To clean up further:
– I can do elimination upwards
– and get zero's above the pivots

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-1R_1 + R_2 = R_1$

- I can clean it up even more:
  - we can make the pivots $\rightarrow = 1$.

(Division)

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

- Divide 2nd Equation by 2

$\therefore \frac{1}{2}[R_2]$

- Now we can execute the whole algorithm

'row reduced echelon form of (A)'

$\Rightarrow rref(A)$

And what information does this now provide. (12)

$$\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- pivot rows: r_1 and r_2.
- pivots column: r_1 and r_3
  x_1 and x_3

and has 2x2 identity matrix sitting in x_1 and x_3 columns

i.e $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

- free columns:

$$x_1 + 2x_2 - 2x_4 = 0$$
$$x_3 + 2x_4 = 0$$

$Ax=0$ $\quad Ux=0$ $\quad$ Now $Rx=0$

Pivot Column   Free Column

$$\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}$$

I → (pivot columns) · ← F (free columns)

(0 0 0 0) ← Which Dont need.

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$ ← pivot ~~cols~~ rows

pivot cols · m-r free cols

$$Rx = 0 \qquad RN = 0$$

null space matrix

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$Rx = 0 \quad \left[x_{pivot} = -F x_{free} \right]$$

Another example: [lets take transpose of A] (14)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

How many pivot variable do you expect here

- how many colums will have pivots
- we have 3 colums, will there be 3 pivots?

No!

- 3rd col is same first 2.

Expecting x_1 and x_2 to be pivot cols as they are independent.

∴ 3rd col will be free as it is dependent on x_1 and x_2 3rd

⇒ Elimination discovered it.

row echelon

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

Swap

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

U

free variable

(rank) $r = 2$

2 pivot columns

1 free column

whats in null space? set free variable to some convenient value eg. 1 [not 0]

$$x \begin{bmatrix} \square \\ \square \\ 1 \end{bmatrix} \begin{matrix} \rightarrow -1 \\ \rightarrow -1 \\ \\ \end{matrix}$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$2x_2 \times 2x_3 = 0$$

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$$2x_2 + 2(x_3) = 0$$

$$2x_2 + 2 = 0$$

$$2x_2 = -2$$

$$x_2 = -1$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 + 2(-1) + 3(1) = 0$$

$$x_1 - 1 + 3 = 0$$

$$x_1 + 2 = 0$$

$$x_1 = -2$$

whole null space

$$x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \text{ (multiply by } c\text{)}$$

Keep going to R

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{cc} R & F \\ \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -F \\ I \end{bmatrix}$$

↑ null space