

## Module 3

### Multivariate Chain Rule Extra

In the last session we defined an expression for the multivariate chain rule.

Here, we going to start by picking up one more little detail, and follow the op, by adding another link to our chain

If we have function  $f(x)$ , where  $x$  is a vector, and in which each term in  $x$  depends on  $t$ , which we can compactly write as

$$f(x(t))$$

We also have our compact form of our multivariate chain rule to go with it:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt}$$

(2)

From last time, we notice that a vector of partial derivative,  $\frac{\partial f}{\partial x}$  by  $dx$  is just the same as the Jacobian vector which we saw in last module, except we write it as a column, instead of row vector.

$$\frac{f}{J_f} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} =$$

So far as knowledge of LA, we can say that  $\frac{\partial f}{\partial x}$  by  $dx$  must be the transpose of the Jacobian of  $f$

$$= (J_f)^T$$

Relating to realize is that taking the dot product of two column vectors

is the same operation as multiplying  
the row vector by column vector. ③

So finally we can see that over all  
that, the Jacobian, offers us perhaps  
the most convenient representation of  
the multivariate Chain rule.

$$\frac{df}{dt} = f \frac{dx}{dt}$$

Next we going to see that the Chain rule  
still works for more than 2 links

Satoshi: we will work through quick  
outvariant example, where we have  
to add in another function separating  $f$   
from  $t$ .

(4)

$$f(x) = 5x$$

~~$$f$$~~

$$x(u) = 1 - u$$

$$u(t) = t^2$$

Do we have 3 functions, and we separating  
f from t, by extra step.

Of course we can just sub in each step into  
each other and find an expression for

$$f(t) = 5(1-t^2) = 5 - 5t^2$$

~~f~~ what's x what's u? you're one go  $\rightarrow$   
what's t

$$\frac{df}{dt} = -10t$$

or we can apply a 2 step chain rule.

(5)

$$\left. \begin{array}{l} f(x) = 5x \\ x(u) = 1-u \\ u(t) = t^2 \end{array} \right\} \frac{df}{dt} = \frac{df}{dx} \frac{dx}{du} \frac{du}{dt}$$

: multiplying of each of our term

$$\frac{df}{dt} = 5(-1)(2t)$$

$$= -10t \quad (\text{same answer})$$

We can see this approach works for Chain  
of univariate functions.

And we can extend it out to as many  
intermediate functions between point  
as we like.

But what about the multivariate case?

The Chain rule does work here too  
But we have to pay attention to a few  
extra details

(6)

Let's start by considering:  
 $f(x(u(t)))$  (read as  
 $f$  of  $x$  of  $u$  of  $t$ )

The function  $f$  takes vector  $x$  as input, but this time  $x$  is a vector valued function, which also takes vector  $u$  as its input:

$$f(x) = f(x_1, x_2) \quad \textcircled{A}$$

$$x(u) = \begin{bmatrix} x_1(u_1, u_2) \\ x_2(u_1, u_2) \end{bmatrix} \quad \textcircled{B}$$

(Remember, from last session, the bold symbols indicate vectors)

and  $u$  is again a vector valued function,  
 and it takes the scalar  $t$  as its input

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad \textcircled{C}$$

(7)

Ultimately we still relating the scalar input  $t$ , to a scalar output  $f$ ,

But through two intermediate vector valued functions  $x$  and  $y$ .

Once again we would like to know the derivative of  $f$  w.r.t  $t$ , we can essentially the same expression as the constant case, except now several of our terms are

in bold

$$\left. \begin{array}{l} A \\ B \\ C \end{array} \right\} \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} \frac{\partial u}{\partial t}$$

We have already seen that differentiating the scalar valued  $f$  w.r.t its input vector  $x$ , gives us the Jacobian row vector

$$= \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right]$$

We've also seen differentiating, a vector valued function of w.r.t. scalar variable gives us a column vector of derivatives

$$? \quad \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix}$$

But what about the middle term (?)  $\frac{dx}{dy}$ ? Well for the function  $x$ , we need to find the derivative of each of the two output variables with respect to the input variables.

So we end up with 4 terms in total, which as we saw in last module, can be conveniently arranged as a matrix

? (above)

$$\begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}$$

We will refer to this object as the Jacobian. ⑨

So we can now say that the derivative of  $f$  w.r.t vector  $x$  is the product of the Jacobian of  $f$ , with the Jacobian of  $x$ , and the derivative vector of  $u$ .

Note that the dimensions of ~~the~~ vectors and matrices as shown here, are such that this operation are possible, and that they will return as Scalar just as we expected.

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} \frac{du}{dt} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{du_1}{dt} \\ \frac{\partial x_2}{\partial u_1} & \frac{du_2}{dt} \end{bmatrix} = \frac{du_1}{dt} \frac{\partial f}{\partial x_1} + \frac{du_2}{dt} \frac{\partial f}{\partial x_2}$$

$$(1 \times 1) = (1 \times 2)(2 \times 2)(2 \times 1)$$

Slabs

⑩

We now see the various threads of LA,  
in multivariant Calculus weave together.

1. Linear Algebra

2. Multivariate Calculus

3. Differential Equations

4. Numerical Methods

5. Optimization

6. Probability & Statistics

7. Machine Learning

8. Financial Mathematics

9. Cryptocurrency

10. Quantum Computing

11. Game Theory

12. Economics

13. Physics

14. Engineering

15. Computer Science

16. Biology

17. Chemistry

18. Geology

19. Meteorology

20. Climatology

21. Oceanography

22. Atmospheric Science

23. Earth Sciences

24. Environmental Science

25. Space Science

26. Materials Science

27. Nanoscience

28. Condensed Matter Physics

29. High Energy Physics

30. Nuclear Physics

31. Particle Physics

32. Cosmology

33. Astrophysics

34. Planetary Science

35. Geophysics

36. Seismology

37. Volcanology

38. Glaciology

39. Tectonics

40. Paleontology

41. Astrobiology

42. Planetary Protection

43. Space Exploration

44. Space Weather

45. Space Settlement

46. Space Mining

47. Space Tourism

48. Space Defense

49. Space Colonization

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