Example: Reflecting in a plane

Lets put all this together... use of our transformation and basis knowledge in order to do something .... TRICKY... ☺

and in same process make our life a lot simpler...

what we want to do here is know what a vector looks like when we reflect it in some FUNNY PLANE

Eg ~~Board~~ Reflection example, mirrored effect

But mirror is in some funny angle.

First challenge: we do not know the plane of the mirror very well.

But we do know 2 vectors in mirror

(1, 1, 1) and (2, 0, 1)

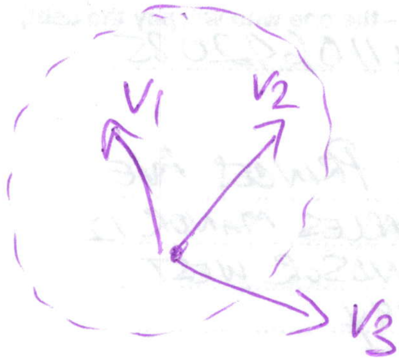
and third vector outside plane of mirror

(3, 1, -1)

in plane

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{bmatrix}2\\0\\1\end{bmatrix} \quad \begin{bmatrix}3\\1\\-1\end{bmatrix}$$

$$v_1 \qquad v_2 \qquad v_3$$



Lets do Gram-Schmidt, to get orthonormal vectors to describe ~~our~~ plane and its "normal" v_3.

$$e_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$ (e_1 & will just be the normalized version on v_1)

$$u_2 = v_2 - (v_2 \cdot e_1)e_1 = \begin{bmatrix}2\\0\\1\end{bmatrix} - \left(\begin{bmatrix}2\\0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\right)\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\0\\1\end{bmatrix} - \frac{1}{\cancel{3}}\cancel{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$\cancel{e_2}\; e_2 = \frac{u_2}{|u_2|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

③

$u_2 = v_2 -$ Some number of e_1, which is the projection of v_2 onto $\cancel{e_2}$ e_1 times e_1
$(v_2 \cdot e_1)$

$u_2 \cdot e_2 =$ If I want to normalize u_2, e_2 is equal to normalized version of $u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$

Now we need to find $\cancel{v_3}$ u_3

$u_3 = \cancel{v_3}\ v_3 -$ projection of v_3 onto e_1, minus the projection v_3 onto e_2

$$u_3 = v_3 - (v_3 \cdot e_1)e_1 - (v_3 \cdot e_2)e_2 =$$

$$\begin{bmatrix}3\\1\\-1\end{bmatrix} - \left[\begin{pmatrix}3\\1\\-1\end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\right]\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} -$$

$$\left[\begin{bmatrix}3\\1\\-1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}\right]\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$= \begin{bmatrix}3\\1\\-1\end{bmatrix} - \frac{1}{3}\,3\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\,2\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

- then normalize this to get e_3

$$e_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

∴ New Transformation Matrix E:

$$E = \begin{bmatrix} \begin{bmatrix} e_1 \end{bmatrix} & \begin{bmatrix} e_2 \end{bmatrix} & \begin{bmatrix} e_3 \end{bmatrix} \end{bmatrix}$$

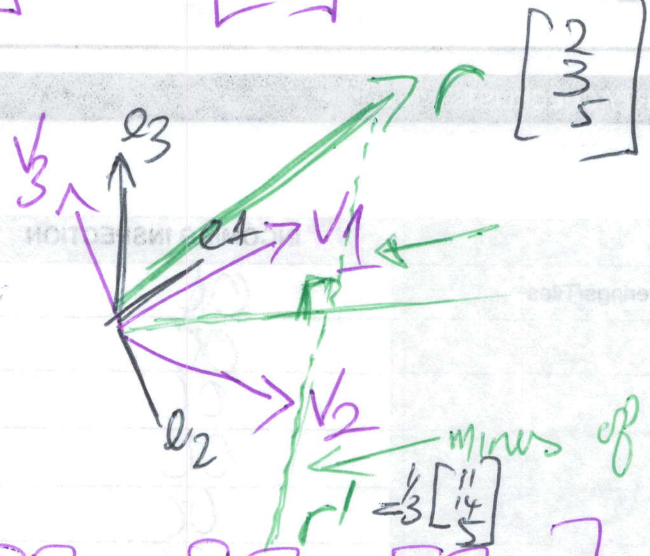
↑ all column vectors

e_1 & e_2 (plane)

e_3 (normal to plane)

Redrawn

$$V_1=\begin{bmatrix}1\\1\\1\end{bmatrix},\ V_2=\begin{bmatrix}2\\0\\1\end{bmatrix},\ V_3=\begin{bmatrix}3\\1\\-1\end{bmatrix}$$



$$E=\begin{bmatrix}(e_1)&(e_2)&(e_3)\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}&\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}&\frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}\end{bmatrix}$$

Note: Have orignal vectors V_1 and V_2

- we defined e_1 to be the normalized version of V_1
- we defined e_2 to be the perpendicular part of V_2, to e_1 (normalized to unit length)
  - so they all in plane
- e_3 is normal to that plane, the bit of V_3 (~~that we can't~~) that we can't make by projection on V_1 or V_2, of unit length.

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Lets say we have vector r, so we want to reflect r down to the plane, drop r down down, and on the other side to get vector r'

Set say r has numbers $(2, 3, 5)$

⇒ But now everything is fenny and awkward (with), (and how can't do projection, will be a lot of trigonometry)

<u>But</u> we can think of r as been composed of vector that is in the plane, that is same vector that normal to e_3

— same means but, to get top.....

Then we can write:

$T_E =$ keep the e_1 bit the same, keep e_2 bit the same, and reflect the e_3 bit from been up, to been down.

a reflection matrix in e_3

reflection in plane

$$T_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

so this T_E is in the basis of plane, not in my basis, but in the Basis plane.

$\Rightarrow$ So if I can get r vector defined in the plane Basis vector set (e), I can then do the reflection, and put it back to my basis vector set.
(and have a complete transformation)

$$\begin{tikzcd}
r \arrow[r, "\text{through some transformation}", "\text{hard}"'] \arrow[d, "E^{-1}"'] & r' \\
r_E \arrow[r, "T_E"'] & r'_E \arrow[u, "E"']
\end{tikzcd}$$

, But I can transform it into Basis of plane, and do it using E^{-1}
then can do reflection, and get r'_E, that is transformed, in basis of plane, then read Back into my basis by doing E.

$$\therefore E T_E E^{-1} r = r'$$

$$\underbrace{r = E\, T_E\, E^T r}_{\text{Transformation we going to do.}}$$ [Now lets do calc]

Note: we have carefully constructed E through Gram-Schmidt to be orthonormal, we know

$$E^T = E^{-1}$$

$$E\, T_E\, E^T = \begin{bmatrix} \frac{1}{3}+\frac{1}{2}-\frac{1}{6} & \frac{1}{3}-\frac{1}{2}-\frac{1}{6} & \frac{1}{3}+0+\frac{5}{6} \\ \frac{1}{3}-\frac{1}{2}-\frac{1}{6} & \frac{1}{3}+\frac{1}{2}-\frac{1}{6} & \frac{1}{3}+0+\frac{2}{6} \\ \frac{1}{3}+0+\frac{2}{6} & \frac{1}{3}+0+\frac{2}{6} & \frac{1}{3}+0-\frac{4}{6} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix} = T$$

$$r_T = T \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 11 \\ 14 \\ 5 \end{bmatrix}$$

This is something that would be very difficult to do with trigonometry

Recap:

- Here we have put all that we know of matrices together to describe how to do something fun, such as to reflect point in space in a mirror
- ideal for transforming face facing in one direction to facing it in another direction.
- (then afterwards use NN to recognise the face)
- next different lecture ☹