Module 4

Power Series Derivation

Here we going to cover the derivation of Power Series representation of functions

Which, in short, is the idea that you can use a series of increasing powers of x to re-express functions

Just a warmup, to ~~fet~~ Blow our minds, we can take the function

$$e^x$$

and re-express it as series $1+x+\frac{x^2}{2}\cdots$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

Must agree, this is pretty incredible, and by end of session we will be able to build series like this for many other interesting functions

What the Taylor series method tells us

↳ that if we know everything about the function at some point

where by everything, I mean the function value, its its derivative, 2nd, 3rd derivative etc

then we can use this information to reconstruct the function ~~some~~ everywhere else.

So if I know everything about it at one (1) place, I also know everything about it everywhere

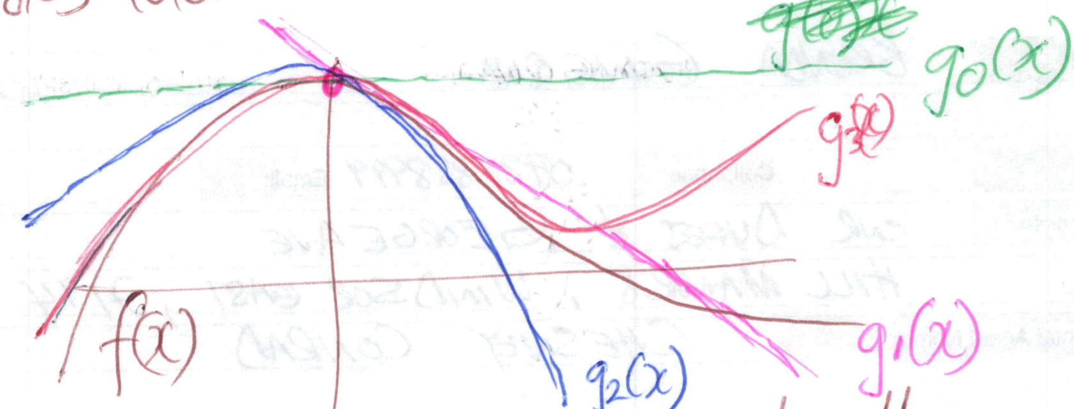
However, this is only true for certain type of function, we call WELL-BEHAVED

which means functions that are continuous, and you can differentiate as many times as you want.

That said, we know lots of functions like that.

So it turns out to be a very useful tool indeed.

So know we going to try to explain the concept graphically, by looking at graph with some arbitrary function f of x which looks like this:



and building a sequence of gradually improving approximations.

So, our first approximation, we can call g_0 we going to build ~~by just~~ using just one piece of information from our function $f(x)$ which is the value of the function at our point of interest, which in this case is just $x=0$

As you can imagine, if all we going to build our function of is ~~just~~ this one single value then our guess function isn't going to capture a very complicated shape like this!

In fact, all it can be is a horizontal line at point f at 0

So we can plot the first approximation, and also write an expression for calculating g_0 of x, which is just f of 0

$$g_0(x) = f(0)$$

We call the our zeroth order approximation, and clearly it is not very good.

But also notice that as the line is flat, it is not even a function of x

We can do better...

Lets find our 1st order approximation, and for this we going to use 2 pieces of information

- the value of the function, at $x=0$
- value of the gradient, at $x=0$, which we will call $x'(0)$

Using these we can build another straight line of the form $y = mx + c$

where the vertical axis intercept c, is just $f(0)$

we just substituting in for $f'(0)$, as the gradient of our line

we can now plot our first order approximation to the function, which has the same value of the gradient, and of the function $f(x)$

(and we can also)

$$g_1(x) = f(0) + f'(0)x$$

And we can also write down its expression g_1 of x is $f(0)$ plus $f'(0)$ times x.

This thing clearly does a better job than $g(0)$ at approximating $f(x)$ near the point f equals 0

But it is still not great

Moving to our 2nd order approximation $g_2(x)$ we going to use 3 pieces of information.

- $f(0)$ — f of 0
- $f'(0)$ — f dash of 0
- $f''(0)$ — f double dash of 0

Now to have a function that can make use of these 3 pieces of information, we going to need a quadratic equation

$$y = ax^2 + bx + c$$

Differentiation this twice.

$$y' = 2ax + b$$

$$y'' = 2a$$

What we wants for this function to be

Same as f of x $f(x)$

(when we sub in point $x=0$ $f(0)$)

$\therefore$ at $x=0$

$\therefore y'' = 2a = \cancel{f''(x)}\ f''(0)$

at $x=0$, we want this equation to be

Clearly our coefficient a is just going to equal

$a = \dfrac{f''(0)}{2}$

$\therefore y' = 2ax + b = f'(0)$

(and subbing in 0 for x)

$\therefore 2ax$ just disappears

$b = f'(0)$

$\therefore y = ax^2 + bx + c = f(0)$

$c = f(0)$

So now we have all our coefficients for our equation.

And we can say, lets go back to our graph, and add our 2nd order approximation (as it is a x^2 term, it will be a parabola)

$$g_2(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2$$

Notice that each time we update our approximation the region in which it matches up with $f(x)$ grows a little

Lets now find the 3rd order approximation

$$y = ax^3 + bx^2 + cx + d$$

Based what we seen from the first three steps, its only the coefficient a, that we need to find, as b, c, d will be the same as we found for $g_2(x)$

Lets differentiate this now 3 time

$$y = ax^3 + bx^2 + cx + d$$

$$y' = 3x + 2bx + c$$

$$y'' = 6ax + 2b$$

$$y^{(3)} = 6a.$$

Clearly we want this thing & 3rd derivative approximation function, to equal our function f of x, when we differentiate it 3 times, and evaluating it at point $x(0)$ / $x=0$ ⑨

So its setting this x to 0, But theres no 0.

$$\therefore y^{(3)} = 6a = f^{(3)}(0)$$

$$\Rightarrow a = \frac{f^{(3)}(0)}{6}$$

So we can add the third order approximation to or graph and write at its expression

$$g_3(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f^{(3)}(0)x^3$$

We can see that not only has the approximation improved significantly, but also we can add higher order terms piecewise, and in each case the lower order terms remain the same

What we now want to build is the general expression, the lower order terms remain just write down the nth order approximation without working through it

10

Notice that the $\frac{1}{6}$ coefficient in front of the quebit term was the result of having to differentiate ~~qu~~ qubit term three times, so when we differentiate x^4 3 times we going to get $4 \times 3 \times 2$ in front.

~~(Do a coefficient of 1)~~

Leading to a coefficient of 1, divided by 4 times 3×2 which is 1 divided by 24

$$g_4(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + \tfrac{1}{6}f^{(3)}(0)x^3 + \tfrac{1}{24}f^{(4)}(0)x^4$$

we have a name for the operation $4 \times 3 \times 2$, called $4!$ (factorial)

All the terms can be thought of as having factorial in front of them. Even 0, $0!$, which is in fact $= 1$

$$g_4(x) = \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

So with the last piece of puzzle in place, we can now say that the nth term in the approximation is just the nth derivative of f evaluated at 0, divided by $n!$ multiplied by x power of n.

Therefore complete power series can be written as follow

$$\frac{1}{n!} f^{(n)}(0) x^n$$

$$\therefore g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n$$

which is the sum of $n=0$ to ∞ of these terms

Although what we have written certainly does count as Taylor series, because we specifically looking at point $x=0$, we often refer to this case as a

Maclaurin series

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

In the rest of this module we going to apply the concept of Power Series to some interesting cases as well as generalizing to higher dimension. Where rather than building approximation curves, we will be constructing approximating <u>HYPERSURFACES</u>