Module 1:

Product Rule

So we ~~ha~~ now have an exact mathematical formula for differentiation.

it became (clear) clear that even for relatively simple function, calculating derivative function can be quite ~~e~~ tedious

However, mathematicians have found a variety of convenient rules, that allow us to avoid wading through the limit of rise over run operation whenever possible

$$\therefore f'(x) = \lim_{\Delta x \to} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$$

So far we have met the:

1. Sum Rule
2. Power Rule

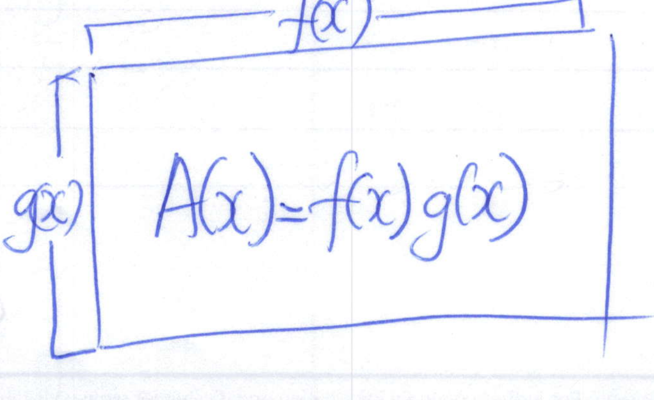
Here we will cover a convenient shortcut for differentiating the product of two functions, i.e PRODUCT Rule at all possible to derive the product rule, purely algebraically, but it does not really help you to develop much insight into whats going on.

For all topics in this Course, we must walk away with some intuitive understanding, rather just looking at examples.

Eg

Lets take rectangle, where length of one side is $f(x)$, and other side is $g(x)$

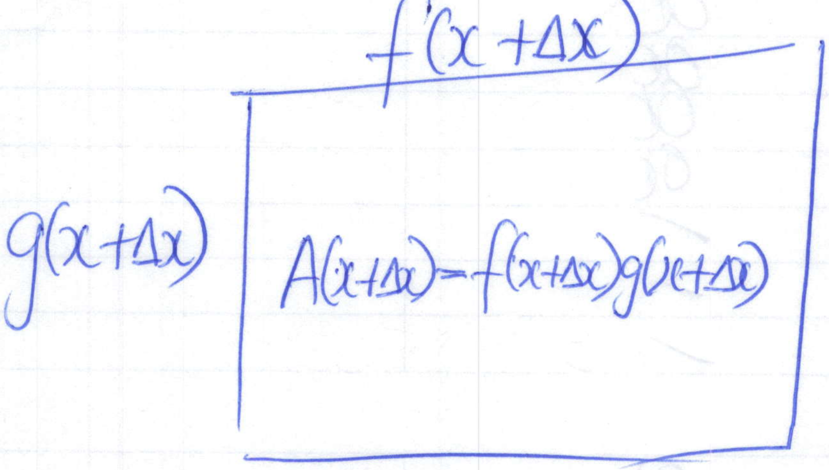
$\therefore$ product of these two functions, must give us the rectangle area, which we can call $A(x)$



Lets Consider: If we differentiate $f(x) \overset{\text{Times}}{\cdot} g(x)$

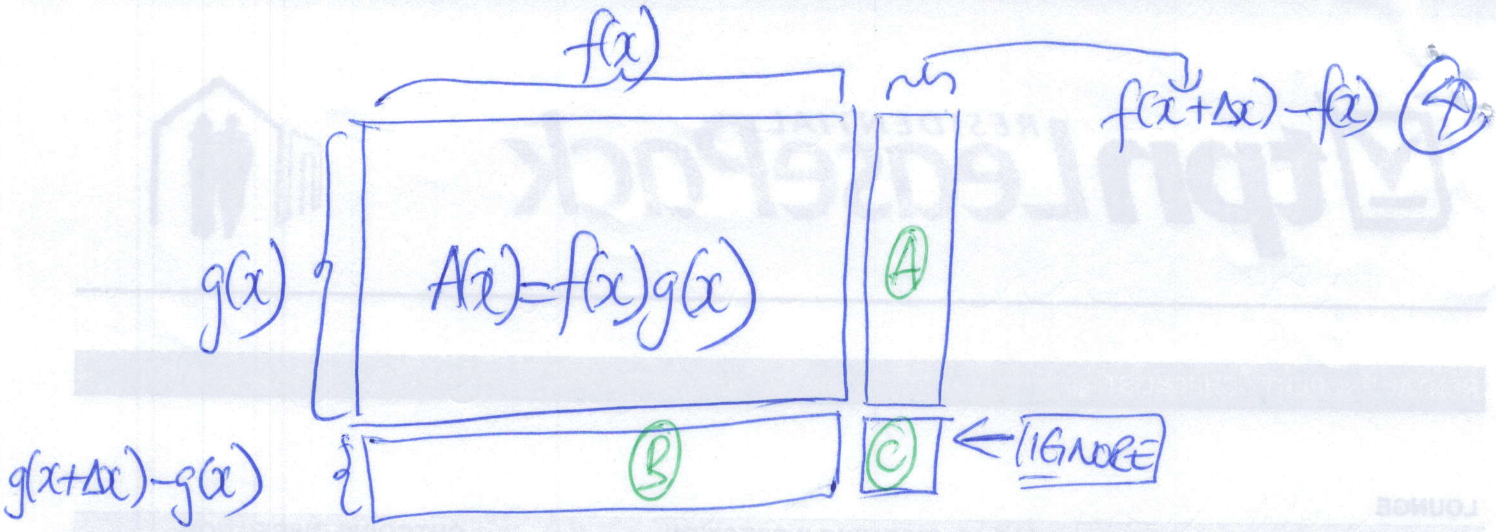
What we really looking for is the Change in area of our rectangle, as we vary (x)

So lets see what happens to the area, when we increase x by some small amount, Δx



But take Note: For our case here, we have picked a very friendly pair of functions where they both happen to increase with x, However this won't necessary be the case

We can now divide up our rectangle into 4 regions, one of which was our original area, $A(x)$

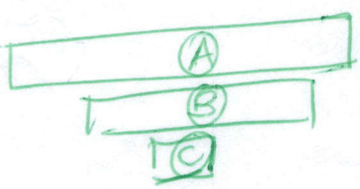


As the total edge length from top, is now $f(x) + \Delta x$, that means the width of new region, must be the difference between original width and new width $[f(x+\Delta x) - f(x)]$

$\Rightarrow$ that means, the same logic applies to height

and we can also write an expression for just the extra area, which we will call, $\Delta A(x)$

$\therefore$ This is the sum of area of the three (3) new rectangles: A B C

$\Delta A(x) =$ 

(5)

$$\Delta A(x) = f(x)(g(x+\Delta x) - g(x)) + g(x)(f(x+\Delta x) - f(x)) + (f(x+\Delta x) - f(x))(g(x+\Delta x) - g(x))$$

As Δx goes to 0, although all the new rectangles will shrink, its the smallest rectangle that will shrink the fastest.

$$\therefore \lim_{\Delta x \to 0} (\Delta A(x)) = \lim_{\Delta x \to 0} (\ldots\ldots\ldots\ldots\ldots)$$

Using this intuition, we can justify how we can ignore the small rectangle and leave its contribution to the area out of our differential expression all together.

Now that we have our expression for approximating Δx, we return to our original question?

What is the derivative of A w.r.t x.

$$\therefore \lim_{\Delta x \to 0}\left(\Delta A(x)\right) = \lim_{\Delta x \to 0}\Big(f(x)\big(g(x+\Delta x)-g(x)\big) + g(x)\big(f(x+\Delta x)-f(x)\big)\Big)$$

So, want the limit of ΔA divided by x, i.e rise over run.

which means we also need to divide the right hand side by Δx

$$\therefore \lim_{\Delta x \to 0}\left(\frac{\Delta A(x)}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{f(x)\big(g(x+\Delta x)-g(x)\big) + g(x)\big(f(x+\Delta x)-f(x)\big)}{\Delta x}\right)$$

Okay, rearrange: splitting it into 2 fractions

$$= \lim_{\Delta x \to 0}\left(\frac{f(x)\big(g(x+\Delta x)-g(x)\big)}{\Delta x} + \frac{g(x)\big(f(x+\Delta x)-f(x)\big)}{\Delta x}\right)$$

and move $f(x)$ and $g(x)$ out of numerator

$$= \lim_{\Delta x \to 0}\left(\underbrace{f(x)\frac{\big(g(x-\Delta x)-g(x)\big)}{\Delta x}}_{\substack{g'(x) \\ \text{1st part}}} + \underbrace{g(x)\frac{\big(f(x+\Delta x)-f(x)\big)}{\Delta x}}_{\substack{f'(x) \\ \text{2nd part}}}\right)$$

So first part contains the:

— the definition of the derivative of $g'(x)$

and second part contains the

— the definition of the derivative $f'(x)$

This will lead us to an final expression for Derivative of A w.r.t. x.

$$A'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

$\boxed{\text{PRODUCT RULE}}$

if $A(x) = f(x)\,g(x)$

then $A'(x) = f(x)\,g'(x) + g(x)\,f'(x)$

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~~Sag~~ So if we want to differentiate the product of 2 functions $f(x)$ and $g(x)$, we simply find the sum of $f(x)g'(x)$ and $g(x)f'(x)$

This will come in handy in later sessions.