Module 2

Inner products of functions and random (~~vectors~~) variable

In previous sessions we looked at properties of inner product, to compute length, angles and distance

We focused on inner products of finite dim vector space.

In this session we will look at 2 examples of inner product of other type of vectors

i.e inner product of functions and inner product of random variable

The inner product we discussed so far we defined for vectors with finite number of entries, and we can think of these vectors as discrete functions with finite number of function values.

The concept of inner product can be generalized to continuous valued functions as well.

And then the sum of individual component of vectors turns into an integral.

The inner product between two functions is defined as follows:

"the inner product between two functions, u and v, is the integral of the interval from a to b, of u of x, times v of x, dx"

$$\langle u, v \rangle = \int_a^b u(x)v(x)\,dx.$$

And as with our normal inner product, we can define norms and orthogonality by looking at this inner product.

If that integral evaluates to 0, the functions u and v are orthogonal

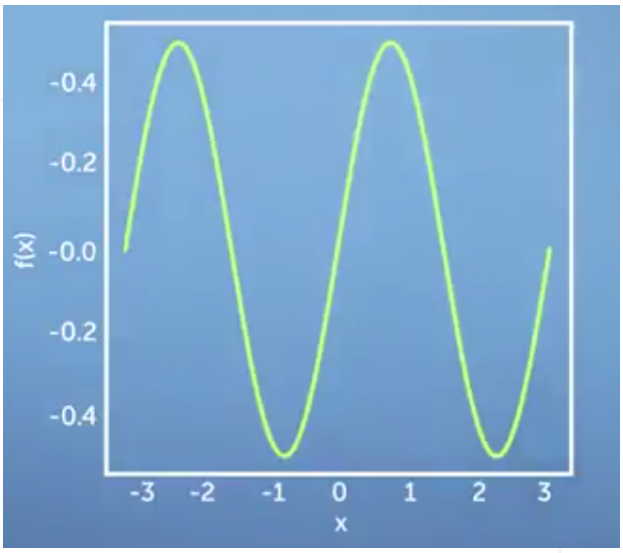
Lets have look at an example,

"If we choose, u of x equals sin x, and v of x is cos of x, and we define f of x to be u of x, times v of x

$$\therefore \quad u(x) = \sin(x)$$
$$v(x) = \cos(x)$$
$$f(x) = u(x)v(x)$$

Then we going to end up with this function

This function is sin(x) time cos(x) (see pic)



we see that this function is odd, which means that f of $-x$, ~~this~~ equals $-f$ of x.

If we choose the integral limit, to be minus π and plus π, then the integral of this product $\sin(x)$ times $\cos(x)$, evaluates to 0.

That means that sin and cos are orthogonal.

And actually holds, that if you "look at set of functions say 1, $\cos x$, $\cos 2x$, $\cos 3x$ and so on, that all of these functions are orthogonal to each other if we integrate from $-\pi$ to $+\pi$"

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots\}$$

Another example for defining an inner product, between unusual types are random variables or random vectors

If we have 2 random variables which are uncorrelated, then we know that the following relationship.

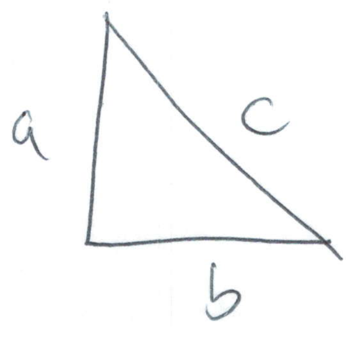
"we know that the variance of $x+y$ is variance of x, plus variance of y, where x and y are random variables"

$$\therefore Var[x+y] = var[x] + var[y]$$

If we remember that variance are measured in squared units, this looks very much like the pythagorean theorem for right triangle

that are states that "c squared equals a squared times b squared", if we look at triangle of this form

$$c^2 = a^2 + b^2$$



Lets see if we can find a geometric interpretation of variance relation of uncorrelated variable.

Random variable can be considered vectors in vector space if we can define inner product to obtain geometric properties of the random variables

"If we define the inner product between 2 random variables, between x and y, to be the covariance between x, y

$$\langle x, y \rangle = \mathrm{Cov}[x, y]$$

we see that the covariance is

- symmetric
- positive definite, and
- linear

So (~~linear~~) linearity would mean:

that "Covariance of λ times x plus y and z, where x, y, and z are random variable, and λ is real number, is λ time Covariance between x and z plus Covariance between y and z."

$$\mathrm{Cov}(\lambda x+y, z) = \lambda \mathrm{Cov}[x,z] + \mathrm{Cov}[y,z]$$

And of "length of random variable is square root of Covariance between x with itself which is square root of variance of x, this is standard deviation of random variable x"

$$\sqrt{\mathrm{Cov}[x,x]} = \sqrt{\mathrm{Var}[x]} = \sigma(x)$$

"This is the length of random variable"

$$\|X\| = \sqrt{\mathrm{Cov}[x,x]} = \sqrt{\mathrm{Var}[x]} = \sigma(x)$$

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Therefore the zero (0) vector is a vector that has no uncertainty, that means standard deviation is 0.

If we now look at the angle between random variables, we get the following relationship.

We get " the Cos of θ, which is the angle between two random variables, is by definition the inner product between the two random variable divided by the length of first random variable times length of second random variable

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \, \|y\|}$$

We can now write this out using the definition of our inner product,

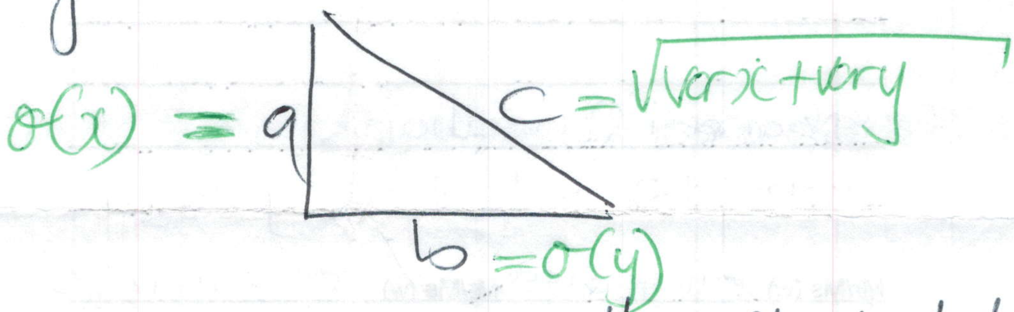
i.e $\langle x, y \rangle = \mathrm{Cov}[x, y]$

We get " Covariance between x and y divided by square root of variance of x, times variance of y.

$$= \frac{cov[x,y]}{\sqrt{var[x]var[y]}}$$

and this evaluates to 0, if and only if, the covariance between x and y is 0, and that is case when x and y are uncorrelated.

Coming back now to our geometric interpretation.



"we will now replace a with standard deviation (σ) of x, b is std dev of y, and c is square root of variance of x + variance of y"

And this is how we get our geometric interpretation of random variables.

~~In~~ this session, we looked at inner products of rather unusual object, functions and random variables

However, even with functions and random variables, the inner product allows us to think about length and angles between these objects

In case of random variables we saw that, the variance of sum of two uncorrelated random variables, can be geometrically interpreted along the pythagorean theorem.