Module3

Projections onto 1 Dim subspace.

In this session we look at orthogonal projections of vectors onto 1 Dim subspaces

Lets look at an illustration,

"we given vector x in 2 Dim, and x can be represented as ~~an~~ a linear combination of basis vectors of $\mathbb{R}^2$ (see pic)

We also have a 1 Dim subspace U, with basis vector b.

That means that all vectors in U can be represented as λ times b for some λ.

Now we interested in finding a vector in U that is closest to x

Lets have ~~a~~ a look at this (see pic)

When I compute the length of difference of all vectors in U in vector x, I am getting the graph on right

It turns out that we can find the vector in U, that's closest to x, by an orthogonal projection of x onto U, i.e the difference vector of x and its projection is orthogonal to U.

(see pic)

Overall we looking at the orthogonal projection of x onto u, 3
and we'll denote this "projection by πu of x".

$$\pi_u(x)$$

The projection has two important properties:

First, "since πu of x is in u it follows that there exists a λ in $\mathbb{R}$ such that π_u of x can be written as λ times b, the multiple of basis vector that spans u"

$$\pi_u(x) \in u \Rightarrow \exists \lambda \in \mathbb{R} : \pi_u(x) = \lambda b$$

Then λ is the coordinate of the projection w.r.t. basis b of subspace u

Second, "the difference vector of x and its projection onto u is orthogonal to u", i.e. its orthogonal to basis vector that spans u, so second property...

∴ "the inner product between b and difference between Π_u of x and x, is zero (0)*, this is the orthogonality condition."

$$\langle b, \Pi_u(x) - x \rangle = 0$$

These properties generally hold for any x in $\mathbb{R}^D$ and 1 Dim subspace U

Lets exploit these properties to find Π_u of x.

<u>Here we have:</u> two dim vector x, and 1 Dim subspace U, which is spanned by vector b, and we interested in finding the orthogonal projection of x onto U, which we call Π_u of x (see pic)

And we have 2 conditions for Π_U of x:

<u>Firstly</u>, "since Π_U of x is an element of U, we can write it as a scaled version of vector b, so there must be a λ in $\mathbb{R}$ such that Π_U of x is λ times b"

$$\exists \lambda \in \mathbb{R} : \Pi_U(x) = \lambda b \quad (\text{as } \Pi_U(x) \in U)$$

<u>Second condition</u>: "is the orthogonality condition that the difference vector between x and Π_U of x, is orthogonal to U, i.e. it's orthogonal to spanning vector b"

$$\langle b, \Pi_U(x) - x \rangle = 0 \quad (\text{orthogonality})$$

Now, let's exploit these two properties to find Π_U of x.

First, "we start writing, we use the condition that b and $\Pi_U x$ minus x, inner product is 0, which is equivalent to that the inner product of b and $\Pi_U(x)$ minus the inner product of b and x is zero"

$$\langle b, \Pi_U(x) - x \rangle = 0$$

$$\Leftrightarrow \langle b, \Pi_U(x) \rangle - \langle b, x \rangle = 0$$

Now we going to rewrite $\Pi_U x$ as λ times b, which is equivalent to, "b times λb inner product, minus the inner product b and x must be zero"

$$\langle b, \lambda b \rangle - \langle b, x \rangle = 0$$

Now we can move the λ out again, because of the linearity of inner product...

"which is λ times squared norm of b minus the inner product of b and x must be 0, and that's equivalent to λ is inner product of b with x divided by square norm of b."

$$\Longleftrightarrow \lambda \|b\|^2 - \langle b, x \rangle = 0$$

$$\Longleftrightarrow \lambda = \frac{\langle b, x \rangle}{\|b\|^2}$$

Now we found λ, which is the coordinate of our projection w.r.t the basis b

that means, " that our projection using ~~second~~ First Condition is λ times b which is now the inner product of b with x, divided by squared norm of b times b"

$$\Longrightarrow \pi_U(x) = \lambda b = \frac{\langle b, y \rangle}{\|b\|^2} b$$

If we choose the dot product as the inner product
we can rewrite this in a slightly different way.

"we will get b Transpose times x times b, divided by squared norm of b"

$$\frac{b^T x b}{\|b\|^2}$$

So given that the one ($b^T x$) is a scalar we can just move it over here, "this is equivalent to saying ($b^T x b$) b times b transpose divided by squared norm of b times x, is our projected point

$$\frac{b^T x b}{\|b\|^2} = \underbrace{\frac{b b^T}{\|b\|^2}}_{\text{PROJECTION MATRIX}} x = \pi_u(x)$$

If we look at this, this is a matrix.

and this matrix is a projection matrix that projects any point in two dimensions onto a one dim subspace.

So, if we look at special case of b having norm 1, we get a much simpler result, we will get:

"So if norm of b equals 1, then we will get that λ is b transpose x, and $\pi_u(x)$ is b times b transpose times x

$$\|b\| = 1$$

$$\Rightarrow \lambda = b^T x \quad (*)$$

$$\pi_u(x) = \underline{bb^T} x \quad (+)$$

So, we will get the coordinate of the projected point (*), wrt to basis b, just by looking at dot product of b with x, and the projection matrix is simply given by b times b transpose (+)

Let's make a comment at end: Our projection $\Pi_U(x)^{\circ}$ is still a vector in $\mathbb{R}^D$, however we no longer require d coordinates to represent it but we only need a single one, which is λ

In this session, we discussed orthogonal projection onto one dimensional subspace

we arrived at the solution by making two observations:

1) we must be able to represent the projected point using a multiple of the basis vector that spans the subspace

2) and difference vector between the original vector and its projection is orthogonal to the subspace.

In next session we will look at an example.