Module 4 2/4/07

Linearisation

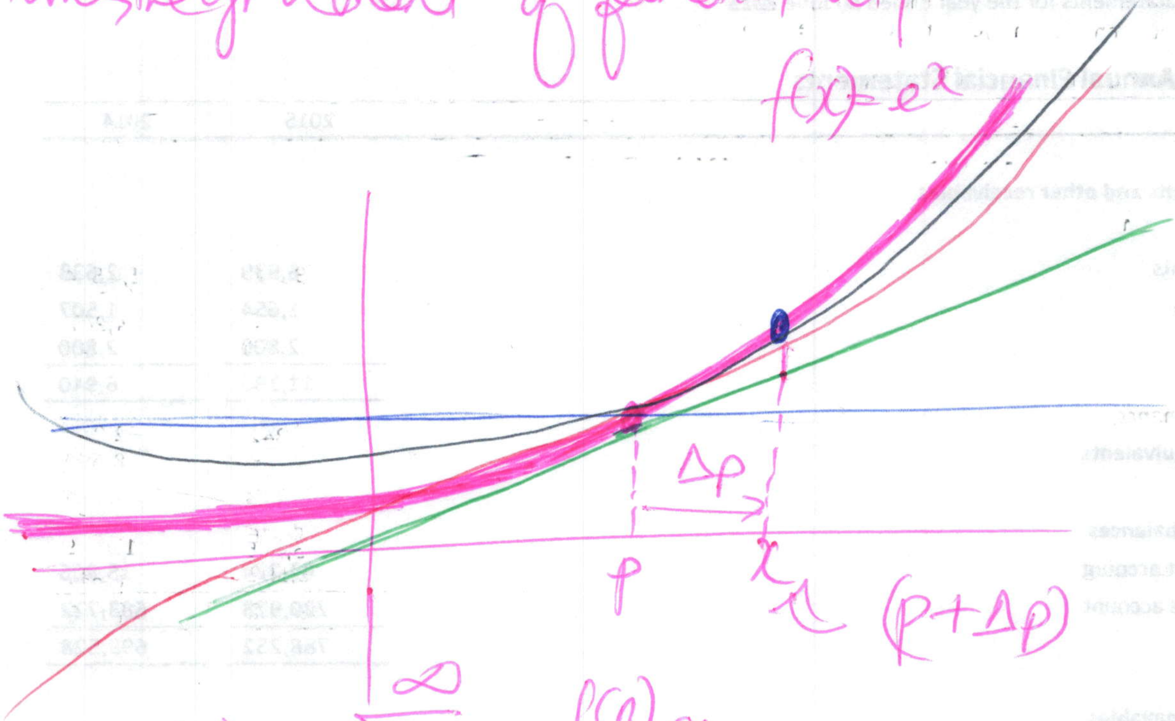
In this session we going to take what we have learned about Taylor Series, and reframe it to a form, that will help us understand the expected error in an approximation

As we have already seen, we can build up a sequence of gradually increasing approximation to function by adding higher power terms.

All we going to do now, is change the notation of some of the terms

So if we take our first order approximation, as an example, what this expression (A) pse2 is saying to us is: Starting from the height $f(p)$ as we move away from p our corresponding change in

height is equal to the distance away from p
times the gradient of function at p



$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(x-p)^n$$

(A) $g_1(x) = f(p) + f'(p)(x-p)$

So, in a sense, rather than using gradient $=\frac{rise}{run}$
we just rearranging:

RISE x GRADIENT = RISE

$\therefore (x-p) \times f'(p) = \text{Rise}$

You are ~~also~~ ultimately going to ~~use~~ use your approximation to evaluate ~~your~~ the function near p

as you most already know about it at p.

So, I am now going to say that our distance from p, currently called $x-p$, we will now call Δp, to ~~(represent a small)~~ signify a small step size away from p.

We can also re-cast x now w.r.t p

So x is just $p + \Delta p$

So its now time, to say goodbye to p

$$\therefore g_1(p+\Delta p) = f(p) + f'(p)(\Delta p)$$

as in our first order approximation, everything is actually w.r.t p

So I am going to do is swap all p's for x's

as this is the form people will normally want to see it.

going to do it all at once, to make it painless as possible:

$$g_1(x+\Delta x) = f(x) + f'(x)(\Delta x)$$

But have not change anything conceptually, I have just written a x, where we used to have a p.

And we can now also rewrite our Taylor Series expression in this new form:

$\Rightarrow$ with just x and Δx

$$\therefore f(x+\Delta x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \Delta x^n$$

So we in good shape to talk about approximations

what we now want to know, is, when we use the 1st order approximation, instead of evaluating the base function, how big should we expect the error to be

We can see, for example the gap, between the blue and green lines grows as we move along the x axis away from the point x.

well, one way to think about it is, what we know our function can be exactly represented by this infinitely long series

A

So although we may not be able to evaluate all the terms, we do know that the next term along i.e. the 1st term we ignore using our first 1st order of approximation has a Δx^2 in it. This means that if we can say that Δx's a small number, the Δx^2 must be a really small number

$$\text{A} \quad f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2}\Delta x^2 + \frac{f^{(3)}(x)}{6}\Delta x^3 + \dots$$

and Δx^3 must also be a ridiculously small number.

So we can now rewrite our first order approximation to include an error term, which we just say is on the order of Δx^2

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + O(\Delta x^2)$$

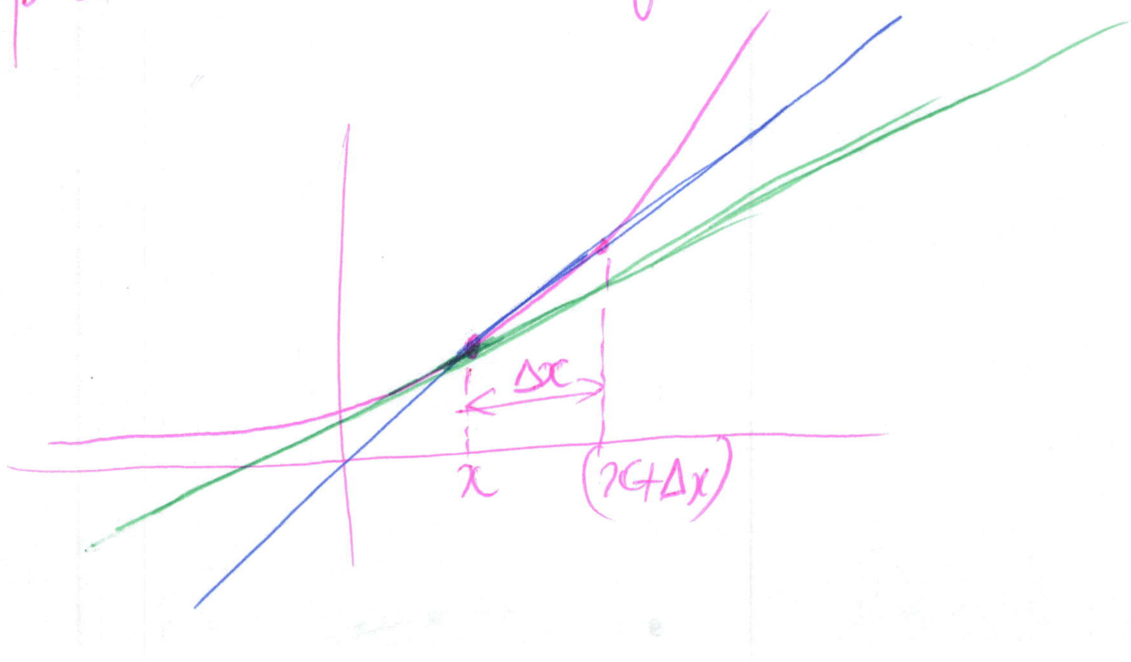
Or equally that it is second order accurate

This process of taking a function and ignoring the terms above Δx, is referred to as linearisation.

And hope this is <u>now Clear</u> to us, why this is the case. We've taken some potentially very nasty function and just approximated it with a straight line

The last interesting idea of this session, is the most interesting, as it brings us right back to our $\frac{\text{rise}}{\text{run}}$ approximation at beginning of course.

The green line is our 1st order taylor series approximation to the function at the point x



which is off course also the Tangent to the curve of x

But lets add another line (Blue), which is the approximation to the gradient at x, using the $\frac{\text{rise}}{\text{run}}$ method with second point Δx away

we use this forward difference method, to help us build a definition of the derivative at the beginning of course

And we notice as Δx goes to 0, the approximation becomes exact.

However, what happens if Δx, does not go to 0? ~~then~~ and the second point remains some finite distance away from x?

well, the calculated gradient will now contain an error, and once again we can rearrange the full taylor series to work out how big ~~the~~ we expect~~ed~~ the error to be with some fiddly algebra, we can rearrange the expression, such that the gradient $f'(x)$ is stated on left hand side, ~~be~~ (A)

Because the series is infinite this is still an exact expression for the gradient of x

But what we get on the right hand side is something that looks suspiciously like the $\frac{MSE}{ran}$ expression + collection of higher order terms

(A) $$f'(x) = \frac{f(x+\Delta x)-f(x)}{\Delta x} - f''(x)\Delta x - f^{(3)}(x)\Delta x^2 - \dots$$

If we notice the first of the high order term has a Δx in it, we know we can lump them all together ~~and~~ (A) page 9

And say that using $\frac{MSE}{ras}$ method between two points with a finite separation, will give us an approximation to the gradient that contains an error ~~that~~ proportional to Δx.

or more simply put, the forward difference method Ⓐ is 1st order accurate:

Ⓐ $$\therefore f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x)$$

It may seem a little odd to go to all that trouble just to get an idea of the error in an approximation.

But it turns out to be a hugely important concept when, as is typical, we ask computers to solve numerically, rather than analytically.