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Module 2

The Sandpit

We've seen that the Jacobian describe the gradient of a multivariable system, and if you calculate it for a scalar value multivariate system, you get a new vector pointing up the direction of greater slope, with length proportional to local steepness.

Here we going to introduce a gradient playground.

To further develop our intuition of the Jacobian.

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In everyday language we use the word optimization to describe the process of trying to make something as good as it can be

In maths optimization basically means the same thing

as much of the research's dedicated to finding the input values of functions that corresponds to either a maximum or minimum of system.

Examples of ~~mathe opti~~ mathematical optimization in action in real world includes:

- planning of routs through busy street
- Scheduling of production in factory
- Strategy for selecting stock when trading

If we go back to the simplest function we saw in last section, and we said that we wanted to find the location of the maximum, we can simply solve the system analytically by first building the Jacobian and then finding the values of x, y, which make it equal to 0

$$f(x,y) = e^{-(x^2+y^2)}$$

$$J = \left[-2xe^{-(x^2+y^2)} \quad -2ye^{-(x^2+y^2)}\right] = 0$$

$$\Rightarrow x = 0, y = 0$$

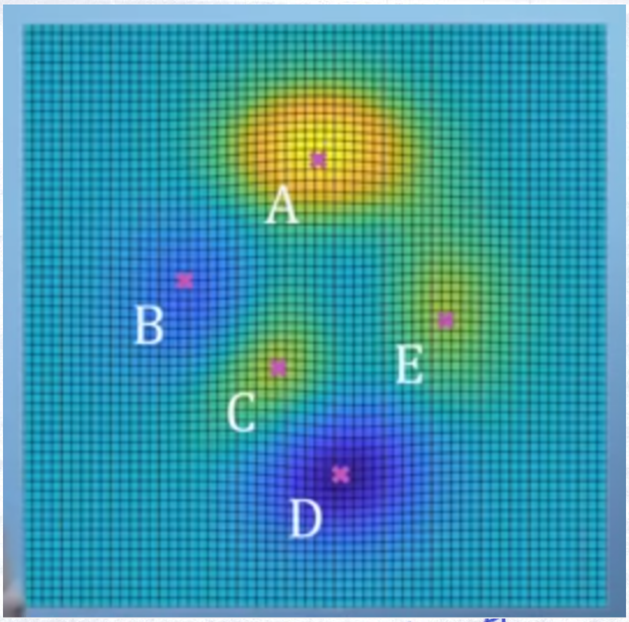
However, when the function get a lot more complicated finding the ~~minimum~~ maximum ~~can be~~ can be tricky

$$z(x,y) = 3(1-x^2)e^{-x^2-(y+1)^2} - 10\left(\frac{x}{5} - x^3 - y^5\right)e^{-x^2-y^2} - \frac{1}{3}e^{-(x+1)^2-y^2}$$

If, as in this case, we still have an analytical expression, then we can at least still find general expression for the Jacobian

But now, just simply setting it to 0, is not much more complicated, but it also is not enough, as the function has multiple locations with 0 gradient

If we assume that all of the maxima and minima can be seen in the region we are plotting (graph)...,
then just look at the surface plot of our function, makes it very clear where the tallest peak and deepest trough of are
we refer to all peaks as maxima,
But in this case, if have a single tallest peak A, which we will call the global maximum, as well as several local maximum, at C, and E.



Similarly we refer to all troughs as minima, and we also have a single deepest point D, which we call global minimum, as well as local minimum at point B.

all fairly straightforward, however, there is a very important point here, that is so obvious, we may have missed it...

Imagine standing on surface, with its hills and valleys and we trying to climb to top of highest peak

that no problem, we look around, spot tallest mountain, and walk towards it

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But what if we walking at night, perhaps similar, where we did not have an analytical expression for a function so we simply aren't able to plot the whole function and look around.

Perhaps each data point is a result of a week long simulation on a supercomputer, or maybe be outcome of actual real world experiment.

These night time scenarios very commonly arise in optimization, and can be very challenging to solve.

However, if we lucky we might find that using a torch, we can see the Jacobian vectors painted on road signs all around us.

But note, although the Jacobian's all point uphill, they don't necessarily point to top of tallest hill.

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and you can find yourself walking up to one of local maxima, or even worse, when you get there you can find that all of road signs are pointing to directly at you.

This night time hill walking analogy is often used when discussing a problem of optimization.

However, it does have some misleading features, such a aspect that when you are really evaluating a function, its no problem to effectively transport all over the map by teleportation, as you try the function at many different places, but there is no need to evaluate everywhere in between and calculation cost the same, no matter how far the points are

So, we not really walking, instead we'll switch to the analogy of sandpit, with an uneven base.

In next exercise ~~(without)~~ we will try to find the deepest point of a sandpit, by measuring the depth at various points, using a long stick.

This is very deep sandpit so if you push the stick down to bottom, there's no way to move it around sideways, just have to pull it out, and try somewhere else.

Also, crucially like walking scenario, you have no idea what peaks and troughs look like, at bottom of pit, because you can't see, the sand is in the way.

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As we work through exercises, we will hopefully pick up some of the subtleties of optimization