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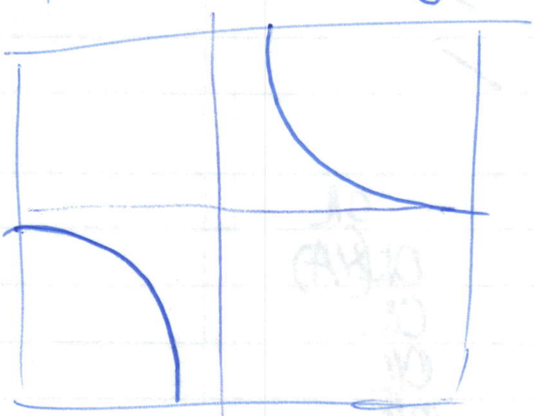
Module 4:

Differentiation examples and special cases.

We're going to investigate 3 (three) core functions which will give us interesting results when differentiated.

First Function: $\boxed{f(x) = \frac{1}{x}}$

which is plotted as follows



Note: gradient of this function is negative everywhere, except at point $x=0$.

⇒ where we can't see what it is and something interesting must be happening at this point

Somehow on negative side the function ②
drops down, presumably toward -negative
infinity, But it somehow re-emerges
from above from positive side

⇒ This sudden break in our otherwise
smooth function is what we refer
to as a $\boxed{\text{discontinuity}}$.

we mentioned already that the operation
divide by 0, is undefined
- which means that this function does
not have value at point $x=0$

But what about gradient? Lets "sub"
our function into differentiation expression
to investigate

$$f(x) = \frac{1}{x}$$

$$f'(x) = \lim_{\Delta x \to 0} \left(\frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \right)$$

Lets make top raw, i.e. numerator. have a single fraction

Combine two fractions

which means we need to make the denominator the same

multiply top and bottom by x

$$= \lim_{\Delta x \to 0} \left(\frac{\frac{\cancel{x}}{x(x+\Delta x)} - \frac{\cancel{x}+\Delta x}{x(x-\Delta x)}}{\Delta x} \right)$$

now we have the same denominator

we can do subtraction.

$$= \lim_{\Delta x \to 0} \left(\frac{\dfrac{-\cancel{\Delta x}}{x(x+\Delta x)}}{\cancel{\Delta x}} \right)$$

$$= \lim_{\Delta x \to} \left(\frac{-1}{x^2 + x\Delta x} \right)$$

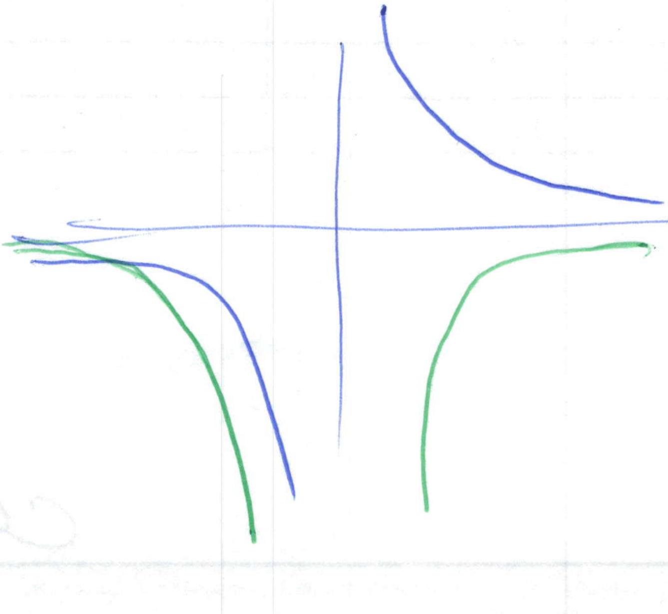
This is where the magic of limits come into play.

– ($x\Delta x$ has delta x in it, this means that as Δx becomes very small, this term itself will become very small

– and hence the will become very irrelevant.

$x\Delta x \Rightarrow$ ignore this term.

$= \frac{-1}{x^2}$, which looks like

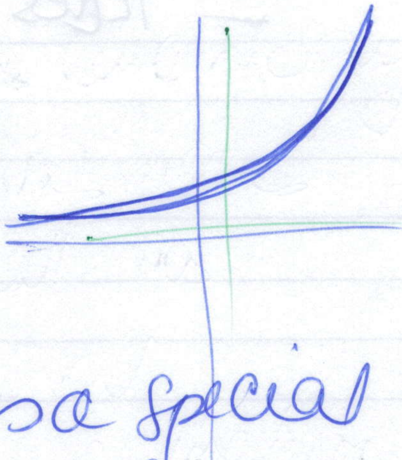


$$\boxed{f(x) = \frac{-1}{x^2}}$$

∴ this's derivative function is negative everywhere

and like our base function, the derivative is also undefined, (at $x=0$)

Second Function: $\boxed{f(x) = e^x}$



What does it do? Its function that has a special property that value of function $f(x)$ is always equal to its value of own gradient $f'(x)$

Another function ('boring') that has this property is $f(x)=0$ — since its a horizontal line, clearly the function and gradient is 0 everywhere

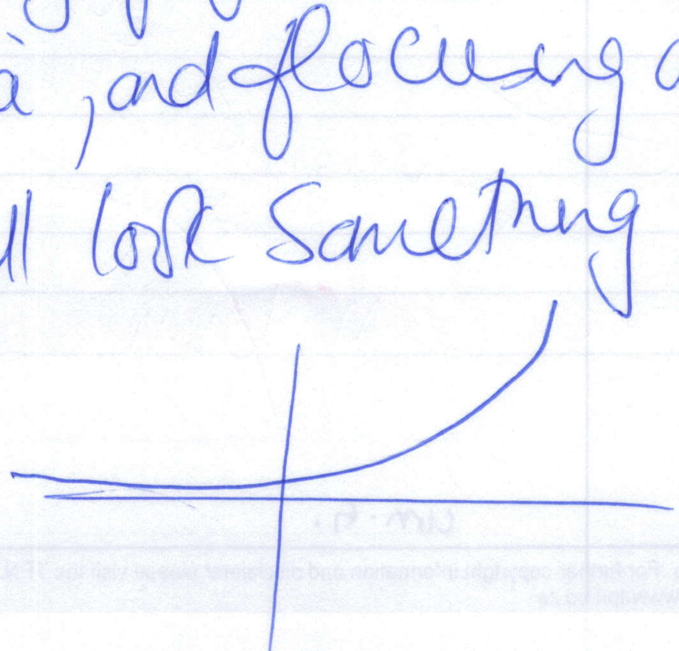
But there's a much more interesting case....

That means our mystery function must always either be positive, or always be negative, as if it ever try to cross the horizontal axis, then both the function and gradient will be 0 (so it get stuck)

So it will just be at our boring function 0 again

By virtue of always increasing or always decreasing, it can never return to the same value again

But plenty of functions can fit those criteria, and focusing on the positive case they all look something like this



But Besides the 0 function (i.e $f(x)=0$)
there is only one function that will satisfy
our demands and that's

$$\boxed{f(x)=e^x}$$

the exponential function

where e is Euler's number
$e = 2.71828....$
(this number is very important in study of calculus)

But more than that, e like π turns
up all over maths
and is written all over fabric of universe.

∴ Differentiating $f(x)=e^x$, gives us

$$f'(x)=e^x$$

and thus we can differentiate it many times

$f'(x)=e^x$, $f''(x)=e^x$, $f^{(3)}(x)=e^x$, $f^{(4)}(x)=e^x$

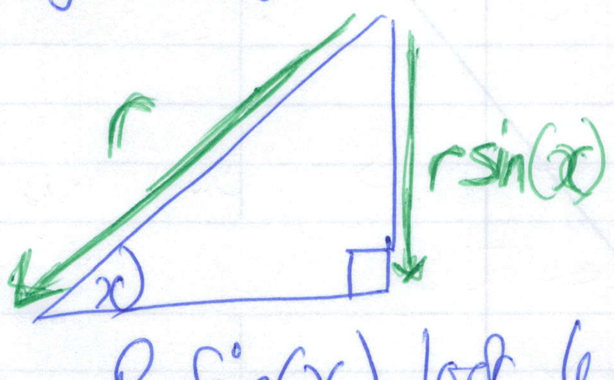
.....

and nothing is going to change...
(this will come in very handy later)

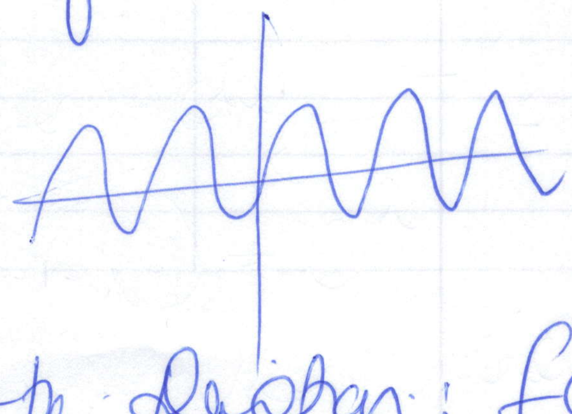
Thereal Function:

Trigonometric Functions: Sin and Cosine

Recall, For a right angle triangle, sin of angle x, multiplied by hypotenuse r, gives you the length of the opposite side ~~of~~ to angle

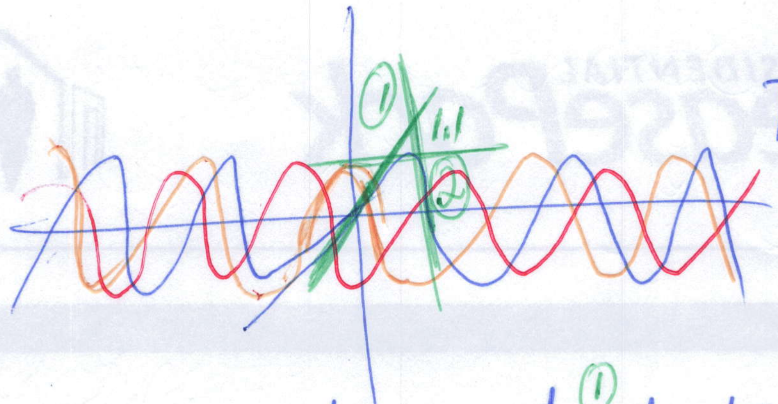


and graph of $\sin(x)$ look like this:



Lets look at this function: $f(x) = \sin(x)$ and see if we can work out what slope its ~~deg~~ derivative would be by:

$f(x)=\sin(x)$



Start with positive gradient, (1) which gently decreases, hits zero (1.1) at top bump, start (2) negative again

derivative of Sin(x), is just Cosine(x)

$$f(x)=\sin(x)$$

$$f'(x)=\cos(x)$$

orange = Cos(x)

and if we differentiate a 2nd time, we get $f''(x)=-\sin(x)$ ⇐ red

and after 3th time we get

$$f^{(3)}(x)=-\cos(x)$$

then 4th time $\boxed{f^{(4)}(x)=\sin(x)}$ ⇐ Back to original

then pattern repeats; This Self Similarity is similar to the exponential function discussed above.

Since these trigonometric functions are actually just exponentials in disguise

$$\Rightarrow \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Most of the functions we've discussed here were skimmed over very quickly, but the thing we need to understand is that differentiation is fundamentally quite a simple concept,

- we just looking for the "rise over run" gradient at each point