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1.10.52.

module 2

Jacobian applied

Here we going to extend the concept of the Jacobian from vectors up to <u>matrices</u> which will allow us to describe the rate of change of a vector valued function.

However, before we do it, we first going to recap, why applying what we have learned so far, about Jacobian to another simple system.

Consider the function:

$$\boxed{f(x,y) = e^{-(x^2+y^2)}}$$

Using our knowledge of partial differentiation:

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its fairly straightforward to find the Jacobian (of this) vector

$$J = \left[-2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)}\right]$$

We now going to do the reverse of our approach of last time, and start by looking at the vector view/field of the Jacobian of f.

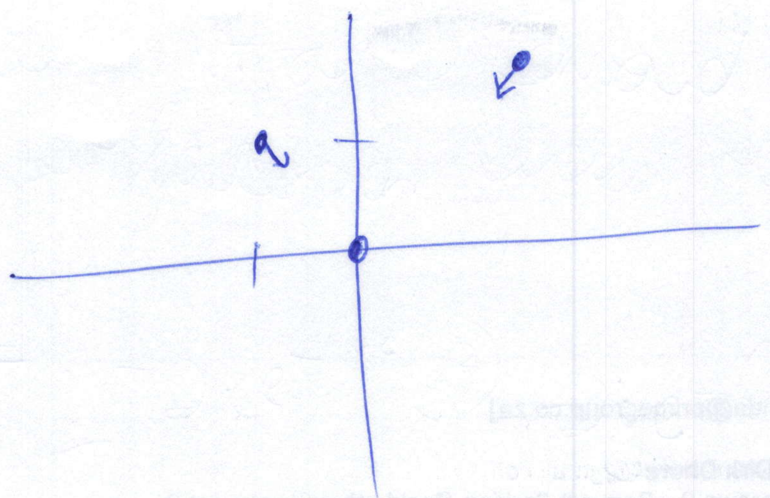
Then see if we can understand how function must look.

Lets start by finding the Jacobian at a few specific points:

Firstly Point: $(1,-1)$

$$J(1,-1) = \left[-2(-1)e^{-((-1)^2+1^2)}, -2(1)e^{-1((-1)^2+2^2)}\right]$$

$$= \left[2e^{-1}, 2e^{-2}\right] = \left[-0{,}27, 0{,}27\right]$$



Substituting these coordinates into our Jacobian expression, and simplifying it,

We can see a vector pointing directly towards the origin

If we move further out to point (2,2)

$J(2,2) = [-0.001, -0.001]$, much smaller vector, But still pointing toward origin

$J(0,0)$, lets look at the origin itself

$J(0,0) = [0,0]$, suggesting the function is flat

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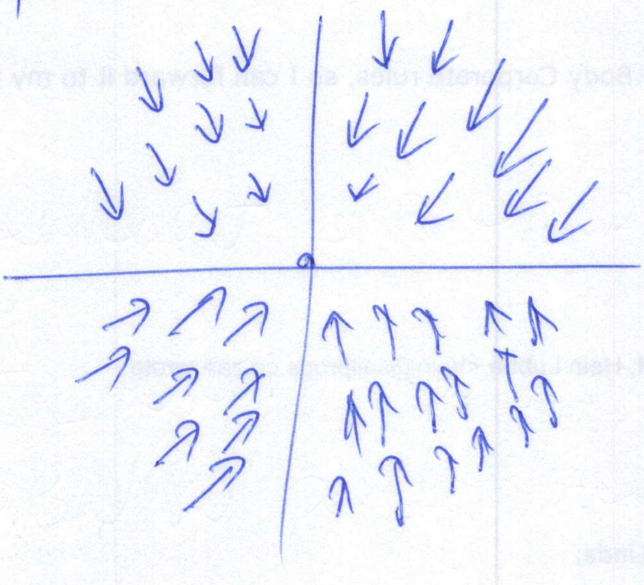
Which must mean 1 of 3 things:

- either points is a maximum, minimum or/or something call a saddle

However, if we reveal the rest of the Jacobian vector field, it becomes clear that the origin, must be the maximum of the system

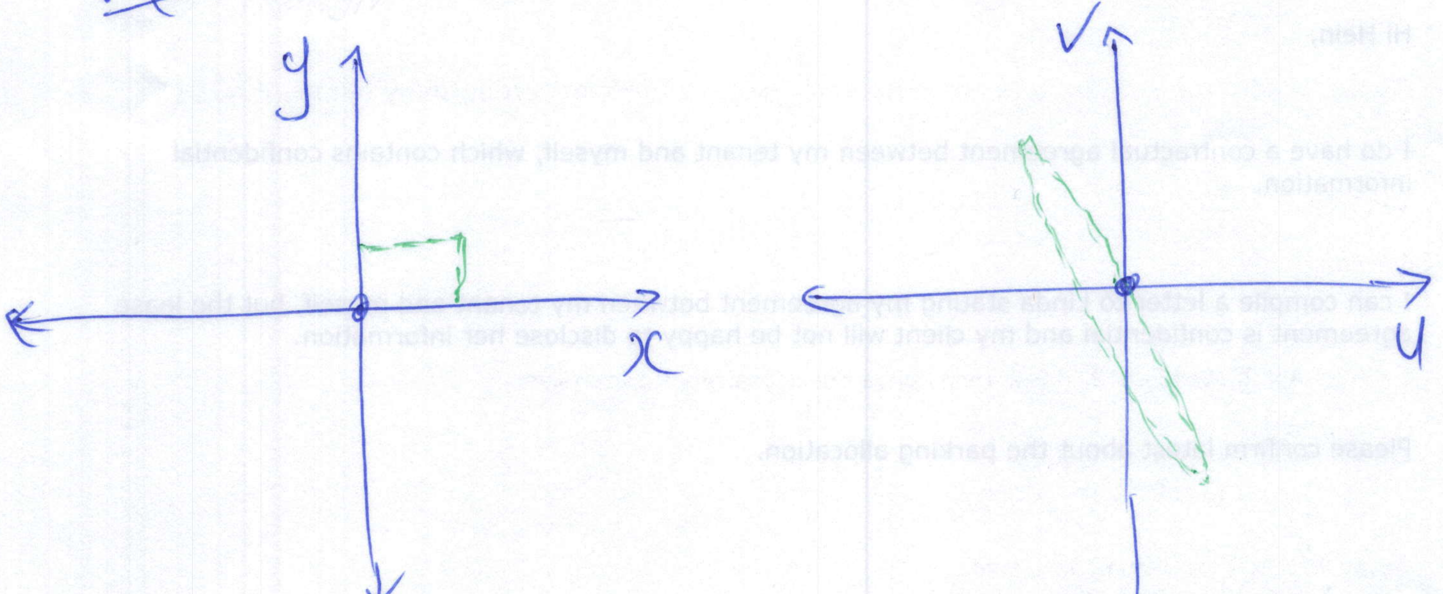


Graphically showing remaining the vector field and observe the function in 3D. (and should match up with our expectations)

Next: Build a Jacobian matrix, which describes functions that takes a vector as input, But unlike our previous examples, also give a vector as the output

Lets Consider the two functions:

$\boxed{u(x,y) = x - 2y}$ $\boxed{v(x,y) = 3y - 2x}$



We can think of these as two(2) vector spaces, one containing vectors with coordinate system in (u,v) and other coordinates in (x,y)

Each point in (x,y) has a corresponding
location in (u,v)

As we move around in (x,y) space we would
expect our corresponding path in (u,v) to
be quite different

Those familiar with LA, will see where this
is going..., we can make separate
row vectors Jacobians for u and v,
However, as we considering u and v
to be components of a single vector it
make more sense to extend our Jacobian
by stacking the vectors as rows
of a matrix:) see...

$$J_u = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \end{bmatrix}$$

$$J_v = \begin{bmatrix} \partial v/\partial x & \partial v/\partial y \end{bmatrix}$$

$$J = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}$$

So, that we have the structure and motivation for building a Jacobian ~~vector~~ matrix for vector valued functions, Lets apply this to our example function and see what we get:

$$u(x,y) = x - 2y \qquad v(x,y) = 3y - 2x$$

$$J = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \begin{array}{l} \leftarrow u \text{ from above} \\ \leftarrow v \text{ from above} \end{array}$$

our Jacobian ~~vector~~ matrix no longer even contains any variables, which is what we should expect when we consider that clearly both u and v are linear functions of x and y so the gradient must be constant everywhere.

$\therefore$ the matrix is just the linear transformation from x, y space to u, v space

So if we would apply the x, y vector $(2,3)$ we'll get the following:

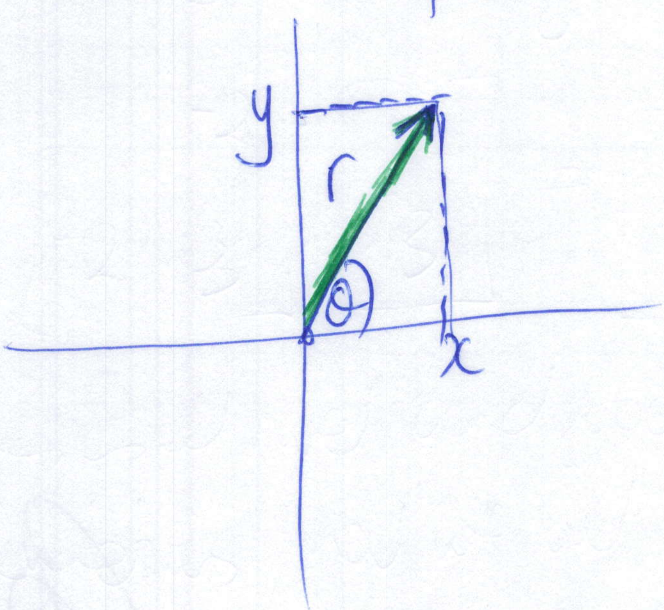
$$\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

This is all well and good, but many of the functions that we will encounter, will be highly non linear, and generally much more complicated, than this simple linear example.

However, often they may still be smooth which means if we zoom in close enough, we can consider each little region of space, to be approximately linear

and therefore by adding up all the contributions of the Jacobian determinant at each point in space, we can still calculate the change in the size of region after the transformation

A classical example:



$$x(r,\theta) = r\cos(\theta)$$
$$y(r,\theta) = r\sin(\theta)$$

where we transform between cartesian and polar coordinate systems

So, if we have a vector, expressed into a radius r, and angle from x axis θ, But we would like it to be expressed into x, y instead. we can write the following expression, just by thinking about trigonometry. (see above)

Now, we can build the Jacobian matrix, and take its determinant:

$$J = \begin{bmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{bmatrix}$$

$$J = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

$$|J| = r(\cos^2(\theta) + \sin^2(\theta))$$

$$|J| = r$$

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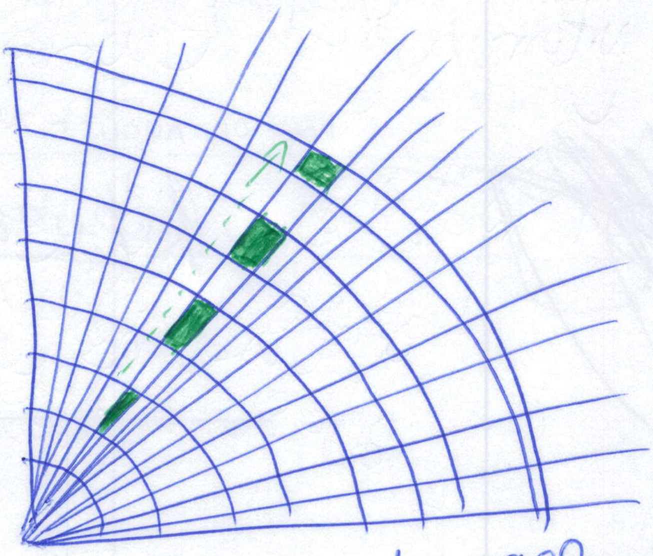
the results simply the r, and not function θ, tell as that as we move along r, away from the origin, small regions of space will scale as a function of r



as depicted by this diagram...