49:28

Module 2

10:05

Angles and orthogonality

Previously we have looked at lengths of vectors, and distance ~~of~~ between vectors.

In this session we'll introduce angles as a second important geometric concept that will allow us to define orthogonality.

Orthogonality is central to projections and Dimensionality reduction.

Similar to lengths and distances, the angle between two vectors is defined through the inner product.

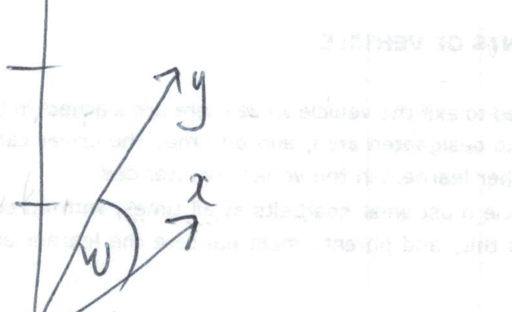
We have two vectors x and y, and we want to determine the angle between them, we can use the following relationship "the cos of the angle between two vectors, is given by the inner product between the two vectors, divided by the norm of x times norm of y"

$$\cos w = \frac{\langle x, y \rangle}{\|x\| \, \|y\|}$$

Let us have a look at an example, and let us compute the angle between 2 vectors x, which is $\begin{bmatrix}1\\1\end{bmatrix}$ and y which is $\begin{bmatrix}1\\2\end{bmatrix}$

$$x = \begin{bmatrix}1\\1\end{bmatrix}, \quad y \begin{bmatrix}1\\2\end{bmatrix}$$

Lets draw the



we interested in angle w

If we use the dot product, as the inner product, we get "Cos of omega is x transpose y, divided by squareroot of x transpose x times y transpose y"

$$\cos w = \frac{x^T y}{\sqrt{x^T x \, y^T y}}$$

"which is 3 divided by squareroot of 10

$$= \frac{3}{\sqrt{10}}$$

This means the angle is approximately (w) 0.32 radians or 18 degrees

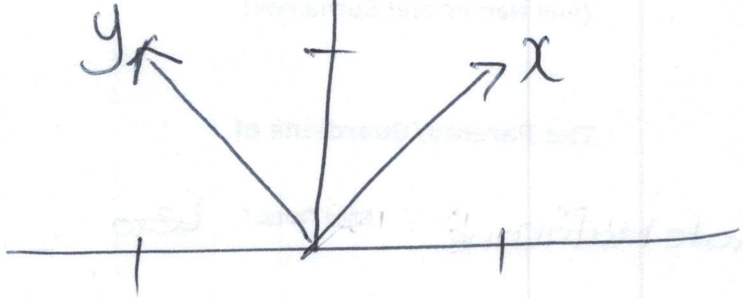
$$w \approx 0.32 \text{ rad} \approx 18^\circ$$

Intuitively the angle between two vectors tells us how similar their orientations are.

lets look at another example in 2D, again with dot product as the inner product.

we look at same vector x as we had before, $\begin{bmatrix}1\\1\end{bmatrix}$, and now we choose y to be $\begin{bmatrix}-1\\1\end{bmatrix}$

and here is drawing.



Now we going to compute the angle, between these two vectors

"the cos of the angle between x, y, is with dot product x transpose y, divided by the norm of x, times norm of y"

$$\cos \omega = \frac{x^T y}{\|x\| \|y\|}$$

$$= 0$$

"This means that omega(ω) is π over 2 in radians or 90 degrees"

$$\Rightarrow \omega = \frac{\pi}{2} \text{rad} = 90^{\circ}$$

This is an example where two vectors are orthogonal. Generally the inner product allows us to characterise orthogonality*

Two vectors x and y, where x, y are non zero vectors, are orthogonal, if and only if, the inner product is 0

This also means that orthogonality is defined w.r.t inner product and vectors that are orthogonal w.r.t one inner product, do not have to be orthogonal w.r.t another inner product.

Let take here the two vectors that we just had, whose dot product between them gave that they are orthogonal

But we going to choose a different inner product.

In particular, we going to choose, "the inner product between x, y, to be x transpose the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, times y"

$$\langle x, y \rangle = x^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} y$$

and if we choose this inner product, it follows that the inner product between x, y is $= -1$

$$\Rightarrow \langle x, y \rangle = -1$$

This means that the two vectors are not orthogonal w.r.t. this particular inner product.

From a geometric point of view, we can think of 2 orthogonal vectors, as two vectors that are most dissimilar, and have nothing in common beside the origin

We can also find a basis of a vector space, such that the basis vectors are all orthogonal to each other

i.e. we get the inner product between b_i and b_j is 0, if i is not the same index as j

$$\langle b_i, b_j \rangle = 0$$

$$\text{if } i \neq j$$

And we can also use the inner product to normalize the basis vectors

i.e. we can make sure that every b_i

has length 1"

$\therefore \|b_i\| = 1$

then we call this an orthonormal basis

Here we discussed how to compute angles between vectors using inner product

We also introduced the concept of orthogonality, and so that vectors may be orthogonal w.r.t to one inner product, but not necessarily if we change the inner product

We will be exploiting orthogonality later on in course e.g. If we have vector and we want to compute the smallest difference vector to any point on line that does not contain vector, then we will find a point on the line, such that the segment between point and original vector is orthogonal to that line 9