

Optimization

Constrained optimization method
of Lagrange Multipliers

So we found out about GRAD, and that
 lets us sketch out a neat method for
 finding the minima / maxima for
 multivariable functions, which we called,

GRADIENT DESCENT.

In this session, we will look at what happens
 if we want to find the minima / maxima

subject to some constraint, e.g. that we
 want minima / ^{maxima / saddle point} on the line or something like that

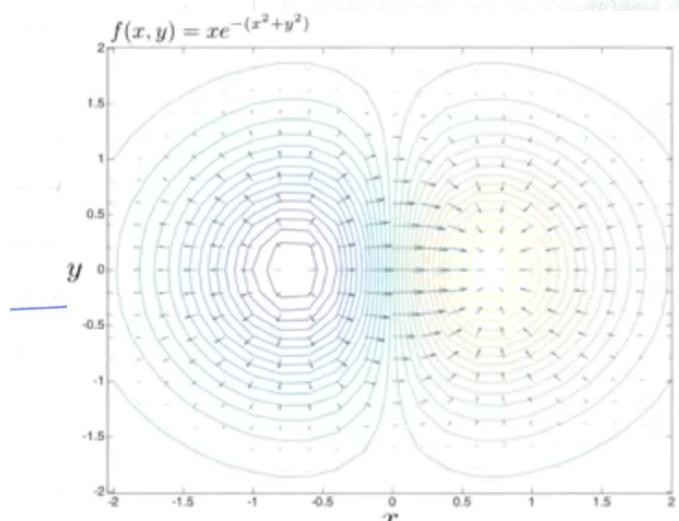
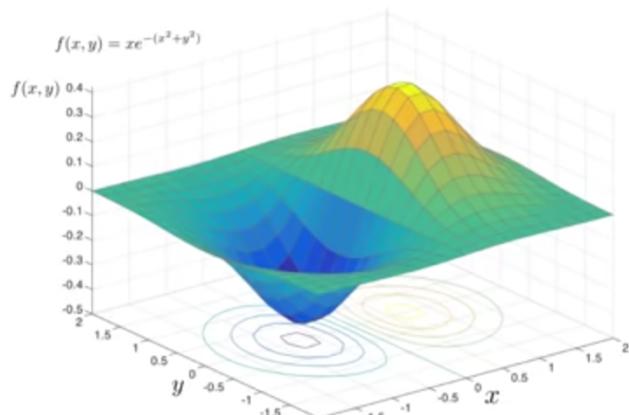
This is called (we studied it) the method using

LAGRANGE MULTIPLIERS

So what we did in the last ~~session~~^{lesson}, is we
 looked at functions like this one

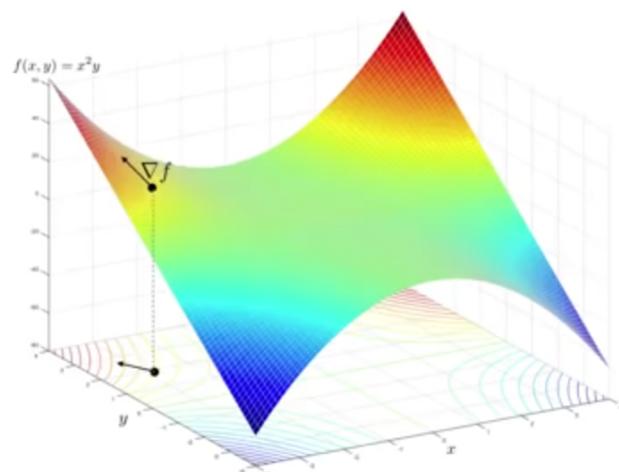
$$f(x,y) = xe^{-(x^2+y^2)}$$

And this is one of the standard MATLAB examples
 And plotted below it, is a contour map of values
 If I know plot, only the contour map, I can plot
 all the vectors of grad f
 As we said these vectors are perpendicular to the
 Contour lines (place here)

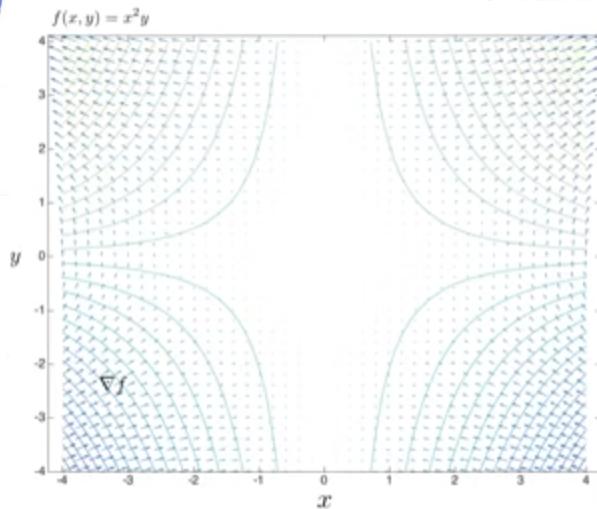


and key point on the steepest gradient
 Now let's return to the function we looked at
 last time $\nabla f(x, y) = x^2 y$
 Which is the Khan Academy standard example for
 this problem.

last time when we plotted gradf on both the 3D
Version and then brought that down to be corresponding
Contour map. (place here) 3



It will be easier if we just work with the
Contour map itself. Here's the contour map



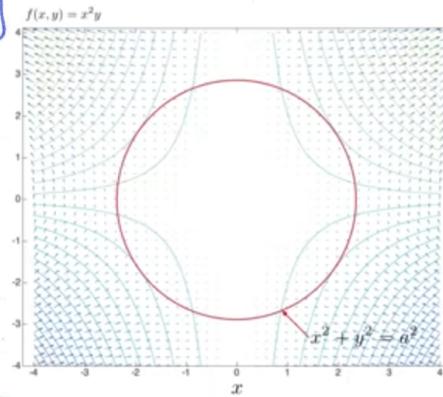
What I have (gradf) does use the gradient function
in matlab to just plot out the gradient
perpendicular to the Contours everywhere in space.

As the low sides are at negative y down here, and
high sides are at positive y way up here.
4
map

(Shaw pie)

So what happen if I want the maximum value
of this function: $f(x,y) = x^2y$

But constrain it to the Circle where the
highest point anywhere on that Circle (pic)



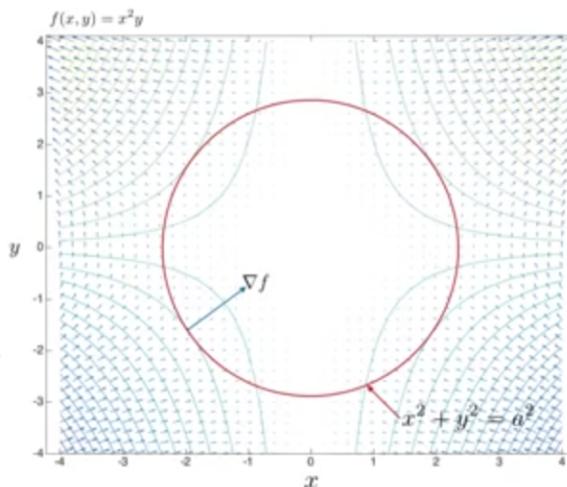
So if I have Circle with radius a^2 , and
want to find the minima/maxima on that
path as I go around that Circle

Now you can imagine, describing an equation
of perhaps by lines as eggs around arch, beginning
to be complete nightmare

but that's not the question we asking actually asked?
What we wanted to know was? What is the
maximum along that arch, not what the value
everywhere on the circle.

Now, Lagrange, was one of the French mathematicians,
that noticed was that when the contours just
touched the path, then you have found the
maximum/minima point

That is, when f is a little bit smaller, it won't
quite touch the path, see here in 3D, but when
it's a bit bigger it will cross a couple of times
(see contour map)



But when the Contours just touches, they would have found the minima/maxima of the function f as we go around the circle in this case.

And what Lagrange noticed is that when the Contours touch the path, then the vector perpendicular to Contour is in the same direction, upto to a minus sign, as the vector of the path itself, that perpendicular to path.

So, if we can find grad, we can find the minima/maxima points to solve problem

So if we can find grad perpendicular to Contour on both the path and function, we away / all good.

So we want to maximize the function $f(x,y)$

which is equal to x^2y

$$f(x,y) = x^2y$$

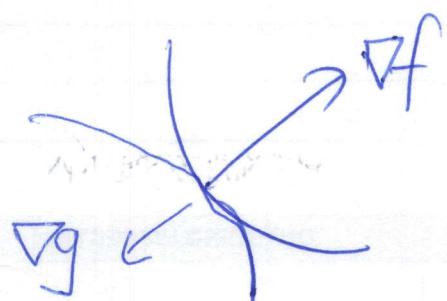
Subject to some constraint, we will call the constraint equation g . of (x,y) .

and what he ^{circle} equate ^{circle} $x^2 + y^2 = \text{some value}$

- maximize $f(x,y) = x^3y$

constant $g(x,y) = x^2 + y^2 - ?^2$

So what we saying is, if ~~we~~ we get function along something like this:



And has it grad ∇f , going ^{above} that way, and the circle with same path, and has it grad ∇g going that way.

What we doing, \hookrightarrow we solving \hookrightarrow same number.

$$\nabla f = \lambda \nabla g$$

where λ is Lagrange's multiplier
That all we need to do

we just need to setup the equation and then solve them.

$$\therefore \nabla f = \nabla(x^2y) = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \lambda \nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

non differentiable

\therefore we got now two equations with 2 unknowns, and also have a third equation, which is the constant equation, which being is the actual value of the circle we particularly interested in

Do we just go to solve that

$$\therefore \nabla f = \nabla(x^2y) = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \lambda \nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\textcircled{1} \quad 2xy = 2x \Rightarrow y = 1$$

$$\textcircled{2} \quad x^2 = 2y \quad (\text{but } y=1) = 2y^2 =$$

$$x = \pm\sqrt{2y}$$

(3) now take the constant equation $x^2 + y^2 = a^2$

$$x^2 + y^2 = a^2 = 3y^2$$

$$y = \pm a/\sqrt{3}$$

These are my solutions

Solutions!

$$\frac{y-1}{x-1} = \sqrt{2}x + 1$$

$$\frac{a}{\sqrt{3}}$$

$$\left(\begin{matrix} \sqrt{2} \\ 1 \end{matrix}\right)$$

$$\frac{a}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$$

$$\frac{a}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

$$\frac{a}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} \\ -1 \end{bmatrix}$$

Now if I find the values of the function

$$f(x,y) = x^2y$$

$$\therefore f(x,y) = \frac{a^3}{3\sqrt{3}}$$

max

$$2 \quad \frac{-2a^3}{3\sqrt{3}}$$

min

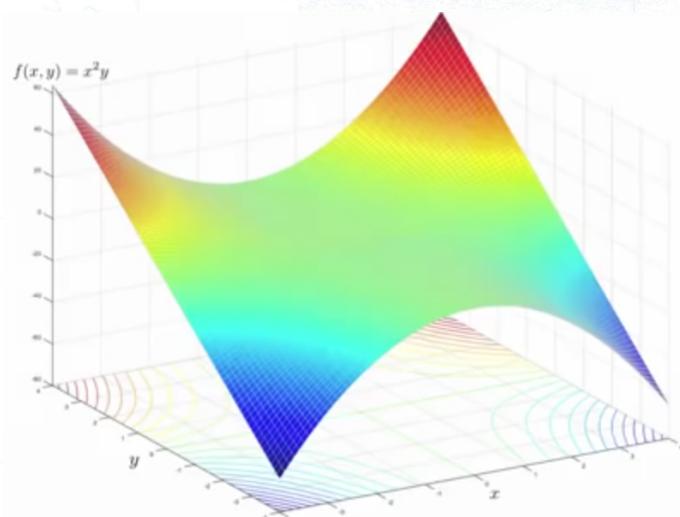
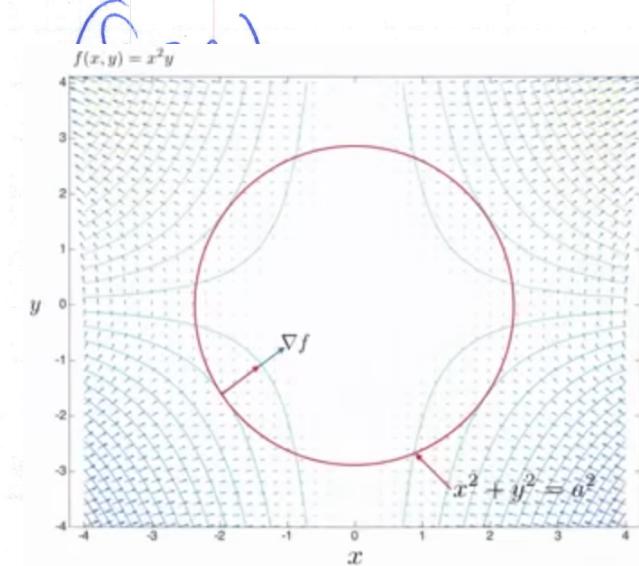
$$\frac{2a^3}{3\sqrt{3}}, \quad \frac{-2a^3}{3\sqrt{3}}$$

max

min

So I got 2 positive solutions, and I wanted to find the maximum, those are the max and we get the minima as well (for free)

Let's see how it looks like on graph:



(10)

So two of our plots are here^(*), and
2 here^(*)

When we switch to the 3D view, we can see,
that the 2 with positive y have maxima, and 2 with
negative y are the minima

So that's really neat

What we've done here, is use an understanding
of gradient to find minima/maxima

subject to some constraint equation, like
straight line, or circle

and very often we would want to make of that
one of variables in $f(x)$ that we want to plot as
fixed in relation to each other, they
have some fixed relationship like being in a circle
so this is handy thing to be able to do.
And very useful.