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Eigen Problems:

Calculating Eigen vectors

We now have a reasonable idea of how an Eigen problem look like geometrically.. ☺

⊛ Now formalize the problem into algebraic expression (to allow us to calculate eigen value/vectors whenever they exist)

— Once we know this, we will be glad that it can be done by a computer program

Let consider transformation A, of the Eigen vectors, these are vectors that are / stay on same span (as before) or following the transformation.

They can change length or even point in opposite direction, but if they remain in same span, they are Eigen vectors

If we call Eigenvector x, then we can say;

$$Ax = \lambda x$$

A = transformation matrix
x = vector
} left side

x = vector
λ = stretching vector x by scalar λ, just some number
} right side

∴ we try to find values of λ to make the two side equal.

∴ for our eigenvectors, having A applied to them, scales their length, or does nothing at all, or same as scaling them by factor of 1

∴ A is n dimensional transform, meaning it must be $n \times n$ square matrix

(2)

- these eigenvector x, must be an n-dimensional vector.

To find solution to this expression, we can rewrite it, by putting all terms on one side, then factoring

$$\underbrace{(A - \lambda I)}_{\circ}x = 0$$

where does I come from? Just an $n \times n$ Identity matrix, matrix same size as A, but 1's along the leading diagonal, and zero's everywhere else.

we did not need this in first expression we wrote, as multiplying ~~vectors~~ scalars by vectors is defined.

However subtracting scalars from matrices is not defined (Not possible), so the I just tidies up maths, without changing ~~mean~~ meaning

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So now, for the left hand side to equal 0, either:

- Content of brackets must be 0, or
- vector x is zero.

But we not interested in case where vector x is zero, (that's where it has no direction or length) which is a trivial solution.

Instead we must find where the term in brackets is 0.

Referring back to previous lectures: we can test ~~the operator~~ that a matrix operator will result in a 0 output, by calculating its determinant.

$$\det(A - \lambda I) = 0$$

(Calculating the determinant manually is a lot of work for high dim. matrices, so let apply it to an arbitrary 2x2 matrix.)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\therefore \det(A - \lambda I) = 0$$

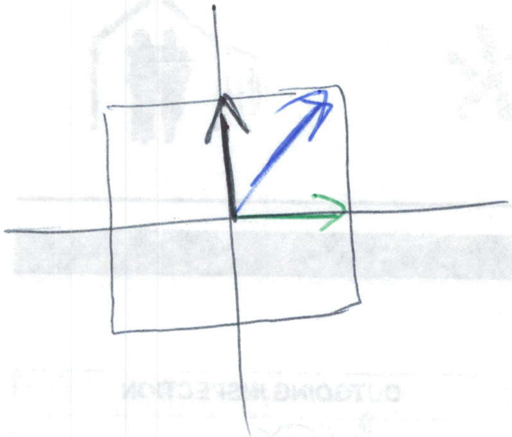
$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

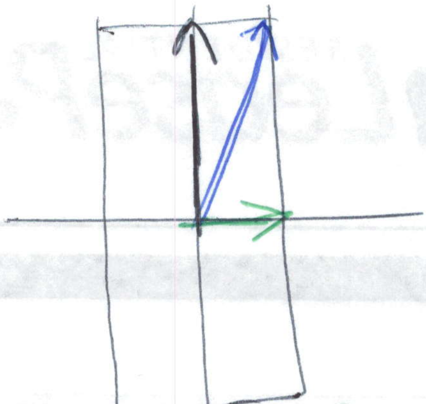
Evaluating the determinant gives us, Characteristic polynomial.

$$\lambda^2 - \left(\genfrac{}{}{0pt}{}{a+b}{\cancel{ad-bc}}\right)\lambda + ad - bc = 0$$

The Eigenvalues are simply the solution to this equation. and plug E'values back into the original expression to calculate our E'vectors.

Rather than continuing with our generalized form, Lets apply this to simple transformation that we know the eigen solution.





⇒ Vertical scaling by factor of 2 and represented by transformation matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

– we then apply the method just described, and take the determinant of $(A - \lambda I)$ and set 0 and solve

$$\det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$$

$$= \Big\{ (1-\lambda)(2-\lambda) = 0$$

This means our equation must have solution where $\lambda = 1$ and $\lambda = 2$.

$$\therefore (A - \lambda I)x = 0$$

⇒ Now we substitute these values back in.

$$\therefore \lambda = 1: \begin{bmatrix} 1-1 & 0 \\ 0 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = 0$$

$$\lambda = 2: \begin{bmatrix} 1-2 & 0 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ 0 \end{bmatrix} = 0$$

So when $\lambda = 1$, $x_2 = 0$ (A)
" " $\lambda = 2$, $-x_1 = 0$

(A) But, in case where $\lambda = 1$, we have eigenvector, where x_2 must be zero, but we do not know anything about the x_1 term.

This means, any vector that points along the horizontal axis could be an Eigen vector of system.

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So write that by saying:

where $\lambda = 1$, x $\overset{(t)}{\text{can}}$ equal anything along the horizontal axis, as long its 0 (x_2) along the vertical direction.

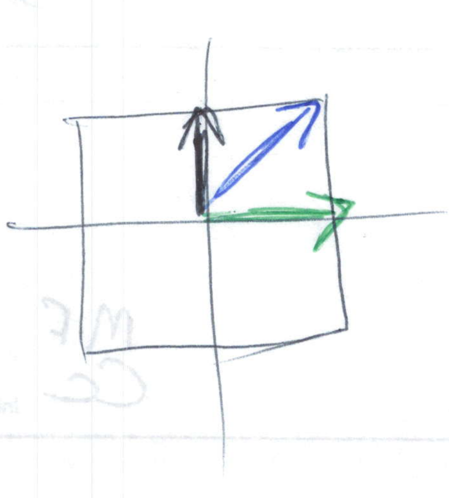
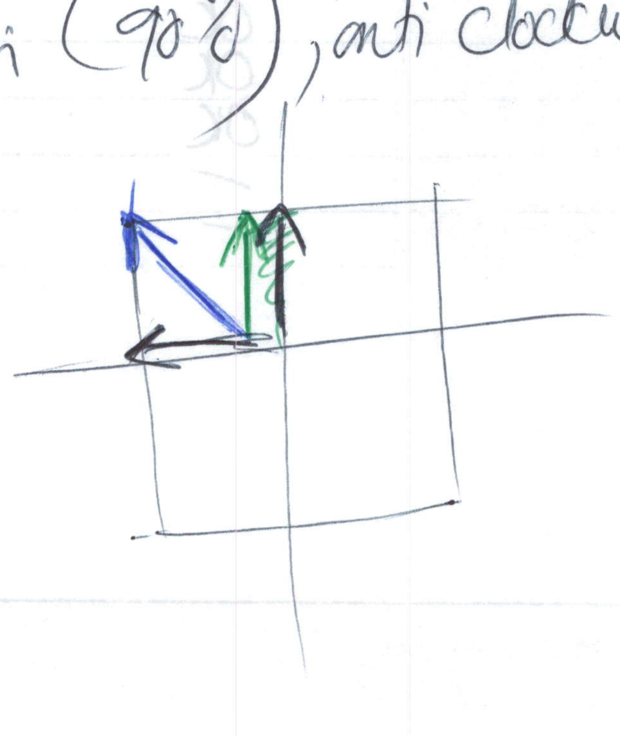
$\therefore \lambda = 1, X = \begin{bmatrix} t \\ 0 \end{bmatrix}$ ← Eigenvector

$\therefore$ arbitrary parameter t

Similarly for $\lambda = 2$, $X = \begin{bmatrix} 0 \\ t \end{bmatrix}$ ← Eigenvector.

$\therefore$ two Eigenvalues, with two corresponding Eigenvectors

Lets look at rotation $(90°)$, anti clockwise.

⇒ Remember, we stated previously we got no eigen vectors at all.

∴ transformation matrix corresponding to 90° rotation:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$ [no real number solution]

∴ no real eigenvectors.

Truth: we will never have to perform this → 100 dimensions
calculation by hand ∴ computer to rescue (iterative numerical methods)

⇒ Good to develop a strong conceptual understanding of eigen problems is very useful than (been really good at calculating it by hand)

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Recap:

- Translating our geometrical understanding into robust mathematical expression.
- Validated it on few test cases.
- Convinced underlying concept is better approach than doing calculations manually.

⟹ next lecture: how to use eigenvectors as basis when changing a basis.