Module 2

Inner Products (definition) Part 2

In this session we will be looking at generalization of dot product in order to compute ~~ang~~ angles, lengths and ~~to~~ distances

Sometimes it is necessary to ~~compute~~ use unconventional way to ~~compute~~ measure these geometric properties

And the inner product allows us to do exactly this kind of thing.

The inner product is a generalization of dot product, but with the same idea in mind we want to express geometric properties, such as lengths and angles between vectors.

Lets define what an inner product actually is

Defenition: We looking at two vectors, for any vectors x and y, in vector space V.

~~(The inner product)~~

$$x, y \in V$$

The inner product is defined as a symmetric, positive definite bilinear mapping.

What does that mean?...

~~(we take~~ 0)

We take a mapping that takes two inputs, out of this vector space, mapping from V time V to real numbers $\mathbb{R}$

$$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$$

We say this function is ~~symme~~

- symmetric
- positive definite
- bilinear

Lets unpack it.

Lets start with Bilinearity...

Bilear means that for vectors x, y and z in vector space V, and real numbers λ (lamda), we get that λ times x plus z, and y, can be written as λ times inner product between x and y plus inner product between z and y.

Bilinear:

$$x, y, z \in V, \ \lambda \in \mathbb{R}$$

$$\langle \underbrace{\lambda x + z}_{\text{linearity}}, y \rangle = \lambda \langle x, y \rangle + \langle z, y \rangle$$

So this is linearity only in the first ~~part~~ argument of the function — and we require linearity in second argument of this function.

Similarly, we will then require that inner product between x and λy plus z is λ times inner product between x and y, plus inner product between x and z.

$$\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$$

lm

This means linearity also in second argument

Bilinear $\Rightarrow$ linearity in both argument. of the function.

Positive Definite means...
That inner product of x with itself is greater or equal to 0 and equality holds if and only if x is zero vector.

$$\langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0 \iff x = 0$$

And last component that we need is symmetry:

Symmetric:
mean that the inner product of x and y is the same as inner product of y and x

$$\langle x, y \rangle = \langle y, x \rangle$$

the order does not matter.

Lets look at an example in $\mathbb{R}^2$

If we define our inner product to be X transpose times identity matrix times y, then we get exactly the dot product that we flamiliar with

$$\langle x, y \rangle = X^T I y \Leftarrow \text{dot product}$$

Now lets have a look at different example, where we define our inner product to be X transpose times A times y, where A is matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Then we can also write the inner product to be

2 times $x_1 y_1 + x_2 y_1 + x_1 y_2 + 2 x_2 y_2$

and this inner product is different from dot product

$$\langle x, y \rangle = X^T A y$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\to 2x_1y_1 + x_2y_1 + x_1y_2 + 2x_2y_2$$

Any symmetric, positive definite matrix in the equation, defines a valid inner product.

In this session we introduced the concept of inner product, which we will use in next session to discuss geometric properties of vectors such as length and angles.