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Module 4

Power series Details

We began the last section (earlier) by trying to motivate with fact that we can re-express e to x (e^x) as a power series.

Your mind should be totally blown by shown me that when we differentiate this function term by term, which is not difficult to do, as it's only a polynomial, something rather satisfying happens. Where just as we ~~expect~~ would expect, after the derivative of e^x, this infinitely long series, remains unchanged, which is pretty awesome.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

$$\frac{d}{dx}e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

So now, finally, we going to see the TAYLOR SERIES

Fundamentally we are apply the same logic, as the ~~he~~ Maclaurin Series that we derived previously

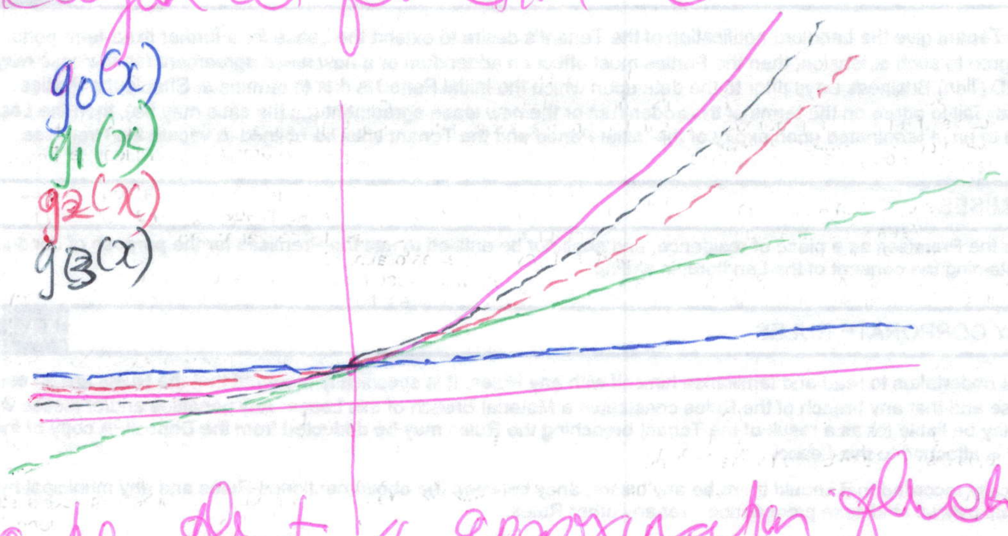
But where as the Maclaurin Series says that if you know everything about a function at point $x=0$, then you can reconstruct everything about it everywhere

Sundy Acknowledges

The Taylor series says that ~~if you know everything about~~ there's nothing special at point $x=0$ and this says if you know everything about the function at any point, you can reconstruct the function anywhere

A small change, but an important one.

Lets look again at function e^x



as well as the first 4 approximation graphs

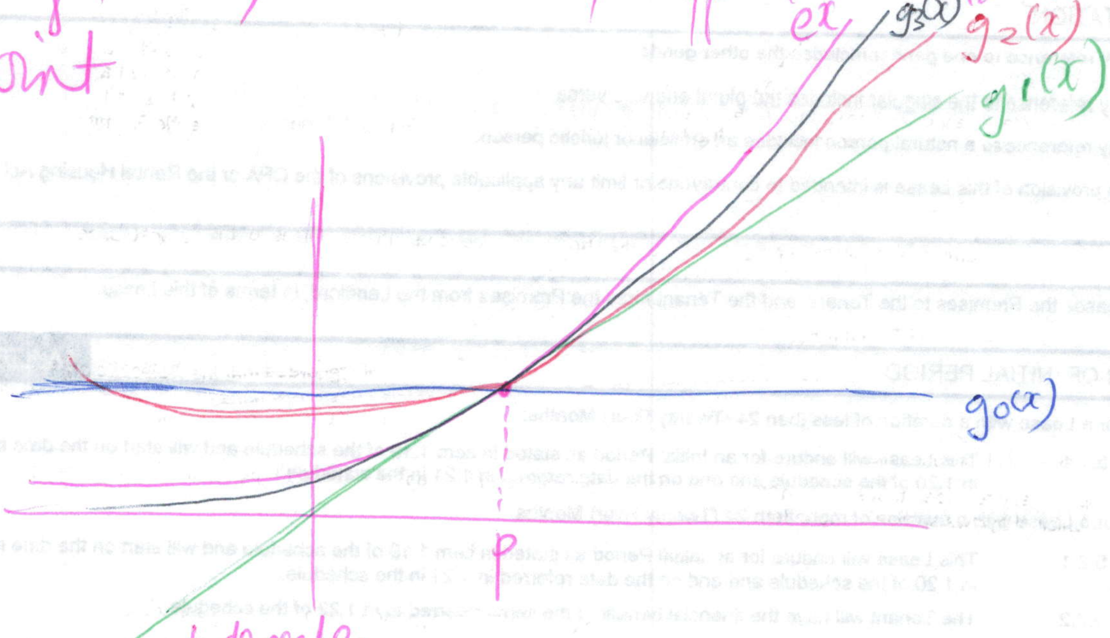
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Using our understanding of the Maclaurin series we can rewrite this in a compact summation notation, such that we could build a polynomial approximation to an arbitrary degree of accuracy.

What, if now, instead of expanding around the point $x=0$, we want to start from point $x=p$.

Ok, to begin with, lets look at 4 approximations at function e^x this point



the expression to describe of these approximations will still require the value of function as well as all of its derivatives at this point

But how do we now adjust the general equation to allow for an arbitrary expansion point $x=p$.

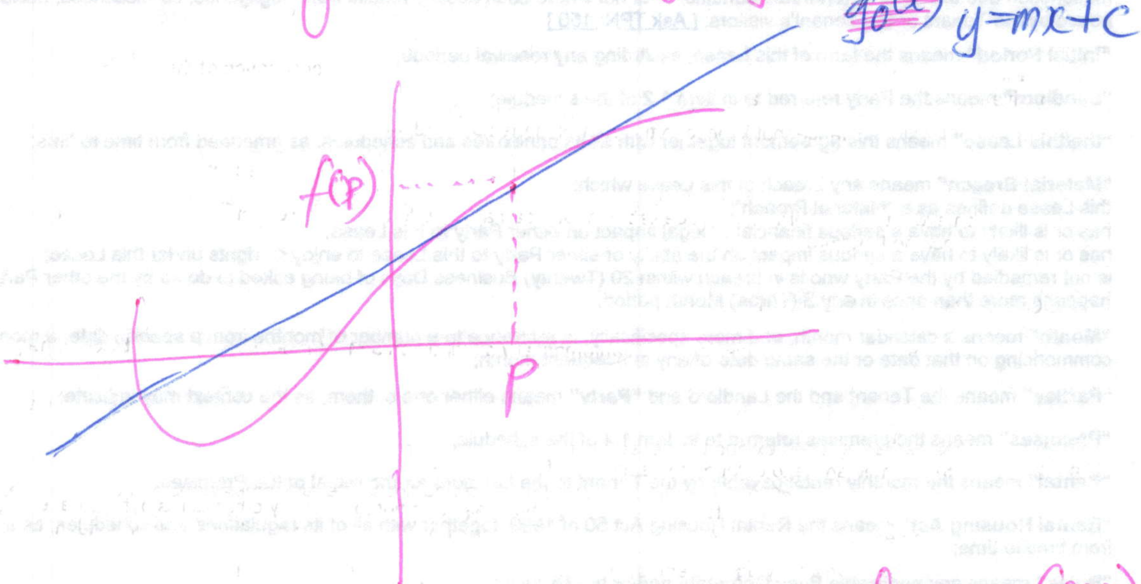
Clearly, zeroth order term, this is just a straightforward, its a horizontal line, that uses the point $f(p)$ everywhere

If we take closer look at 1st order approximation $g_1(x)$, this should give us enough insight that we will then be able to tackle all the higher order terms.

So we essentially looking to build a TANGENT to Curve at point p.

All straight lines are of form $y = mx + c$ so lets work through this by putting in place all the information available to us.

There's some nice function we going to deal with.



So we looking at point p, which is of f of p ($f(p)$) height.

$\therefore$ Coordinates at this point $= (p, f(p))$

So we want to build an approximation function that its first order, its 1st y-intercept and gradient and look like

that blue line will have the equation $y = mx + c$.

we know immediately that m is the gradient of this line, and we know the gradient of our function here: just $f'(p)$

So we can say

$$y = f'(p)x + c$$

all is left is for us to find c

and to do this, we going to need to know x, y

But we do know a point on our line, is the point p

$\Rightarrow$ so we just sub this in

$$f(p) = f'(p)p + c \xrightarrow{\text{rearrange}} c = f(p) - f'(p)p$$

then subst back to $y = mx + c$

$$y = f'(p)x + f(p) - f'(p)p$$

Last thing, we going to do, is ~~that~~ take this and factorize this $f'(p)$ out.

$$y = f'(p)(x-p) + f(p)$$

What this shows ~~up~~ us, is that building our approximation around point p, when we use our gradient term $f'(p)$, rather than applying it directly to x, we instead now apply it to $x-p$, which we can think of as how far you away from p.

Now we can write down our first two approximation functions to $f(x)$ at point p.

$$g_0(x) = f(p)$$
$$g_1(x) = f(p) + f'(p)(x-p)$$

Thinking back at Maclaurin Series, all we need to do to convert to Taylor Series, is use the 2nd derivative at p, rather than at 0.

All also replace x, with $x-p$.

But notice that factor of $\frac{1}{2}$ still remains

$$g_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$$

Maclaurin Series

So we can now look at the compact summation notation of the Maclaurin series, and upgrade the whole thing to the more general Taylor Series Form

Noticing, of course, if we decide to set p to 0, then the expression would actually become identical.

Maclaurin Series

$$g(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^{n}$$

Taylor Series

$$\boxed{g(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(p)}{n!}(x-p)^{n}}$$

Now we have our one-dimensional Taylor Series in all its glory. Which will allow us to conveniently re-express functions into a polynomial series.

Remainder module, we ~~will~~ will play around with this result, as well as extend it to higher higher dimensions