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Module 2

Lengths and distances (Part 1)

In the last session we defined inner products, now we will use inner product to compute lengths and of vectors and distance between vectors.

Length of vector is defined by inner product using the following equation:

"Length of vector x is defined as square root of the inner product of x with itself"

$$\|x\| = \sqrt{\underbrace{\langle x, x \rangle}_{\geq 0}}$$

Remember the inner product is positive definite, that means this expression is greater or equal to 0

Therefore we can take the square root.

We can now also see that the length of vector depends on the inner product, and depending on the choice of inner product, the length of vector ~~(quite)~~ can be quite different

Similarly, the geometry of vector space, can be very different

The length of x is also called the Norm of x

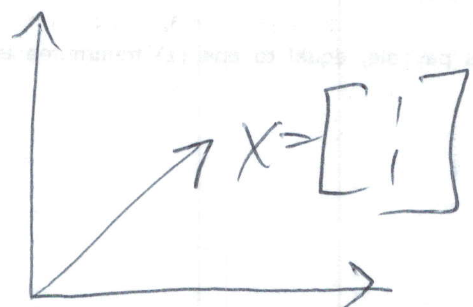
Let's have look at an example:

Assume we interested in computing the length of vector in 2 Dim

"and x is given a vector $\begin{bmatrix}1\\1\end{bmatrix}$"

$$x = \begin{bmatrix}1\\1\end{bmatrix}$$

In diagram:



Now we interested in computing the length of vector.

In order to example compute the length of vector, we need to define an inner product.

So why don't we start with standard dot product.

"We define $\overset{\text{inner product}}{x,y}$ to be x transpose y, then length of X is square root of 2.

$\sqrt{1}+\sqrt{1}=\sqrt{2}$

$$\langle x,y\rangle = x^T y \Rightarrow \|x\| = \sqrt{2} \quad \text{(A)}$$

Lets look at different inner product!

"Lets define x,y to be x transpose times $\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$ times y"

$$\langle x,y\rangle = x^T \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} y \quad \text{(B)}$$

"which we can also write as x_1 times $y_1 - \frac{1}{2}(x_1y_2 + x_2y_1) + x_2y_2$

$$= x_1y_1 - \frac{1}{2}(x_1y_2 + x_2y_1) + x_2y_2$$

Using the definition of this inner product, then

"Length of vector (x) is square root of $\sqrt{x_1^2 - \frac{1}{2}(x_1x_2 + x_2x_1) + x_2^2}$"

$$\therefore \|X\| = \sqrt{x_1^2 - \tfrac{1}{2}(x_1x_2 + x_2x_1) + x_2^2}$$

"and this is identical to square root of $x_1^2 - x_1x_2 + x_2^2$"

$$= \sqrt{x_1^2 - x_1x_2 + x_2^2}$$

we will get smaller values than the dot product definition (A) page 3

of this — expression is positive.

we will now use the definition of this inner product (B)

to compute the length of our vector up here

"the squared norm or inner product of x with itself is

1 plus (−1) is 1."

$$\|x\|^2 = \langle x, x \rangle = \cancel{4(A + 2(-1)) = 1}$$
$$= 1 + 1 - 1 = \underline{1}$$
$$\Rightarrow \|x\| = \underline{1}$$

$\therefore$ 1 is length of vector, using the unusual definition of inner product (B)

whereas the same vector would be longer, have we used the ~~un~~ dot product (A)

The norm we looked at also have some nice properties:

① if we take a vector (x) and ~~(streng)~~ stretch it by scalar λ, then norm of the stretched version is the absolute value of λ times norm of x

$$\therefore \|\lambda x\| = |\lambda| \, \|x\|$$

② Triangle inequality – which says that the norm of x plus y is smaller or equal to the norm of x plus norm of y.

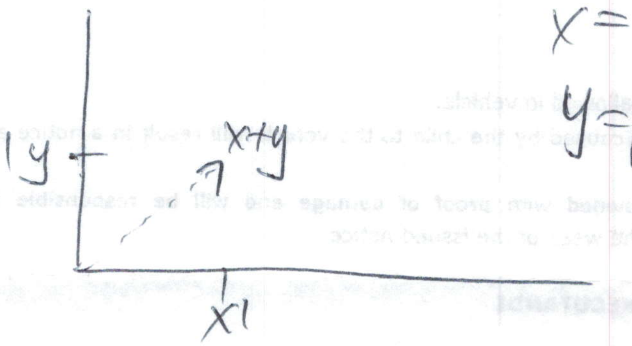
$$\|x+y\| \leq \|x\| + \|y\|$$

Lets have a look at an illustration:

We assume we have a coordinate system in 2 Dim.,

and we use the standard vector $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

then $x+y$ is sitting here.



$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

If we use the dot product as an inner product, then the norm of x is 1, which is same as norm of y.

$$\|x\| = 1 = \|y\|$$

and norm of $x+y$ is square root 2

$$\|x+y\| = \sqrt{2}$$

The triangle inequality says that, x+y norm, is smaller or equal to norm of x, plus norm of y

$$\|x+y\| \leq \|x\| + \|y\|$$

and in our case $\sqrt{2} \leq 2$. Σ

(3) Cauchy-Schwarz inequality

"This says that the absolute value of the inner product of x with y is smaller or equal to product of individuals norms of two vectors"

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|$$

In this session we looked at length of vectors, using the definition of inner products, and now we going to use this to compute distances between vectors in next session.