Modules 2:43:10

①

Optimization
Constrained optimization method
of Lagrange Multipliers

So we found out about GRAD, and that
lets us to sketch out a neat method for
finding the minima / maxima for
multivariable functions, which we called,

GRADIENT DESCENT.

In this session, we will look at what happens
if we want to find the minima / maxima
subject to some Constraint, of what we
want minima / maxima somewhere on the line or something like that

This is Called (is called the) the method using

LAGRANGE MULTIPLIERS

So what we did in the last ~~video~~ session, is we
looked at function like this one

$$f(x,y) = xe^{-(x^2+y^2)}$$

and there is one of the standard mathlab examples

and plotted below it, ~~is~~ a contour map of values

of I now plot, only the contour map, I can plot

all the vectors of gradf

as we said these vectors are perpendicular to the

contour lines (place here)

and key point are the steepest gradient

Now lets return to the function we looked at

last time $\Rightarrow f(x,y) = x^2 y$

which is the Khan academy standard example for

this problem.

that time when we plotted grad f on both the 3D version and then brought that down to the corresponding Contour map. (place here)

It will be easier if we just if we just work with the Contour map itself. Here is the contour map

What I have (already) done, is use the gradient function in matlab to just plot out the gradient perpendicular to the Contours everywhere in space.

So the low sides are at negative y down here, and high sides are at positive y way up here:
(show pic)

So what happen if I want the maximum value for this function. $f(x,y) = x^2 y$
But constrain it to the circle where the highest point anywhere on that circle (pic)

Say if I have circle with radius a^2, and want to find the minima/maxima on that path as I go around that circle

Now, you can imagine, describing an equation for how x^2y varies as I go around circle, is going to be complete nightmare

But that's not the question we asking actually asked?
What we wanted to know was 1). What's the maximum along that circle, not whats the value everywhere on the circle.

Now, Lagrange, was one of the French mathematicians, that noticed was that when the contours just touched the path, then you have found the maximum/minima point

That is, when f is a little bit smaller, it won't quite touch the path, see here in 3D, but when it's a bit bigger it will cross a couple of times (see contour map)

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But when the Contours just touches, they would have found the minima/maxima of the function f as we go around the circle in this case.

And what Lagrange noticed is that when the Contours touch the path, then the vector perpendicular to Contour is in the same direction, up to to a minus sign, as the vector of the path itself, that perpendicular to path.

So, if we can find grad, we can find the minima/maxima points to solve problem

So if we can find grad perpendicular to Contour on both the path and function, we away/all good.

So we want to maximise the function f of x, y which is equal to $x^2 y$

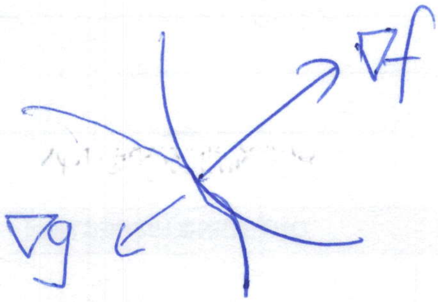
$$f(x,y) = x^2 y$$

Subject to same Constraint, we we will Call the Constraint equation g of (x, y) and that's the $\underset{\wedge}{}$equation (circle) $\underset{\wedge}{}$(circle) x^2+y^2 = some value

$\therefore$ maximise $f(x,y)=x^2y$

Constraint $g(x,y)=x^2+y^2=?^2$

So what we saying is, if ~~we~~ ~~eg~~ we got function doing something like this:



And has it grad ∇f, going that way, and the circle with same path, and has it grad ∇g going that way.

What we doing, is we solving:

$$\nabla f = \lambda \nabla g$$

$\lambda \to$ same number.

Where ~~λ~~ λ is Lagrange's multiplier

That all we need to do

We just need to setup the equation and then solve them. 8

$$\therefore \nabla f = \nabla(x^2 y) = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \lambda \nabla g = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

then differentiate

$\therefore$ we got now two equations with 2 unknowns, and also have a third equation, which is the constraint equation, which being is the actual value of the circle we particularly interested in

So we just go to solve that

$$\therefore \nabla f = \nabla(x^2 y) = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \lambda \nabla g = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

① $2xy = \lambda 2x \Rightarrow \boxed{y = \lambda}$

② $x^2 = \lambda 2y$ (but $y = \lambda$) $= 2y^2 =$

$\boxed{x = \pm\sqrt{2}y}$

③ now take the constraint equation itself

$x^2 + y^2 = a^2 = 3y^2$ $\boxed{y = \pm a/\sqrt{3}}$

These are my solutions

$$y = 1, \quad x = \sqrt{2} \times 1$$

Solutions: $\frac{a}{\sqrt{3}}\begin{pmatrix}\sqrt{2}\\1\end{pmatrix}, \frac{a}{\sqrt{3}}\begin{bmatrix}\sqrt{2}\\-1\end{bmatrix}, \frac{a}{\sqrt{3}}\begin{bmatrix}-\sqrt{2}\\1\end{bmatrix}, \frac{a}{\sqrt{3}}\begin{bmatrix}-\sqrt{2}\\-1\end{bmatrix}$

Now if I find the values of the function

$$f(x,y) = x^2y$$

$$\therefore f(x,y) = \frac{2a^3}{3\sqrt{3}}, \quad -\frac{2a^3}{3\sqrt{3}}, \quad \frac{2a^3}{3\sqrt{3}}, \quad -\frac{2a^3}{3\sqrt{3}}$$

max min max min

So I got 2 positive solutions, and I wanted to find the maximum, those are them (and we got the minima as well for free)

Let's see how it looks like on graph:

(Sage)

So two of our solutions are here ⊗, and

2 here ⊗

When we switch to the 3D view, we can see, that the 2 with positive y are maxima, and 2 with negative y are the minima

So thats really neat

What we've done here, is use our understanding of gradient to find minima / maxima subject to some constraint equation, like straight line, or circle

and very often we would want to make it that some of variables in function we want to plot are fixed in relation to eachother, they have some fixed relationship like they in circle

so thats is handy thing to be able to do.

and very useful.