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Orthogonal Matrices:

It will be really useful if we can make a transformation matrix, whose column vectors make a new basis

→ and whose component vectors are perpendicular, or orthogonal to each other.

Here:
- we will look at how to do this and WHY its useful.

Lets define a new operation on a matrix, called a transpose.

we interchange all of the elements of rows and columns of matrix.

$$A_{ij}^{T} = A_{ji}$$

Have matrix A with element i,j, and transpose of it is where I <u>interchange</u> i,j.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^{T} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$\Rightarrow$ diagonal stays same $(1, 4)$, But $2,3$ flips ☺ ✓

another way of looking at it

Eg have:

$$A^{n \times n}$$

$$\begin{array}{c} \begin{bmatrix} \begin{pmatrix} \hat{a}_1 \end{pmatrix} & \begin{pmatrix} \hat{a}_2 \end{pmatrix} & \cdots & \begin{pmatrix} \hat{a}_n \end{pmatrix} \end{bmatrix} \\ \begin{matrix} a_{i1} & \quad a_{i2} & \qquad a_{in} \end{matrix} \end{array}$$

A $n \times n$ matrix, with series of column vectors, which will be the basis for new transformed vector space.

what can we say about vectors $a_1 \ldots a_n$

- they are of unit length $\therefore \hat{a}$.
- and they are orthogonal to each other.

$$\therefore a_1 \cdot a_2 = 0$$

$$\therefore a_i \cdot a_j = 0 \text{ if } i \neq j$$

$$\text{and } = 1 \text{ if } i = j$$

Now lets pre-multiply A, with its Transpose

$$\overset{A^T}{\begin{pmatrix} (& \hat{a}_1 &) \\ (& \hat{a}_2 &) \\ (& \vdots &) \\ & \hat{a}_n & \end{pmatrix}} \overset{A^{nn}}{\begin{bmatrix} \Big(& \Big(& \Big(\\ a_{i1} & a_{i2} & a_n \end{bmatrix}}$$

A^T = series of row vectors, which is $\hat{a}$, but now rows

what happens when we multiply $A^T A$?

$$A^TA = I$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \\ 0 & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix} \Rightarrow I$$

above 1 = $a_i \cdot a_j = 1$ if $i=j$ [how we get 1] } ortho-normal.
above 0 = $a_j \cdot a_p = 0$ if $i \neq j$ [how we get 0]

∴ found ⇒ ~~where~~ in A, where we have vectors that are normal to each other in unit length. i.e. when they orthonormal, then A^T will be the identity.

But $A^T = A^{-1}$ (is really inverse of A)

∴ orthonormal vector set, then transpose is inverse.

, that means don't have get inverse long way, can just do transpose and get the inverse.

A is thus called an orthogonal matrix

Another property of orthogonal matrix is, since all the basis vectors are of unit length, it will scale space by factor of 1.

∴ determinant of orthogonal matrix will either be 1 or -1

$$|A| = \pm 1$$

(minus 1 comes about if they flip new basis set flips space around, i.e. make it left handed from right hand it was originally)

Notice: if $A^T = A^{-1}$, the following must be also be true

$$AA^T = \cancel{A} I$$

(previously $A^T A = I$)

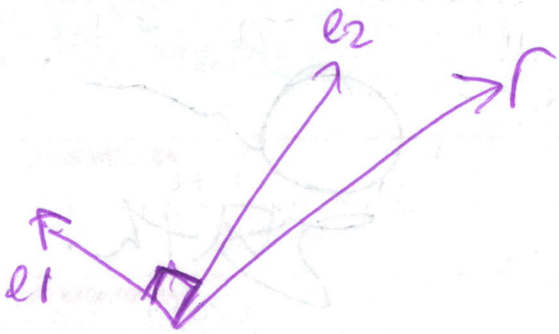
∴ pre or post multiply to get I

That then also mean, that the rows of the orthogonal matrix are also orthonormal to each other.

(∴ A^T is also orthonormal Basis set.)

Previously we said that transforming a vector onto a new coordinate system is just taking the projection or dot product of that vector onto each of the new basis vectors, as long as they are orthogonal to each other.

Eg if have vector, and project it into new set axis:



∴ take dot product:

① $r \cdot e2$

② $r \cdot e1$

and we'll have component in new set axis.

In Data Science, what we saying ~~is, that~~ is wherever possible we want to use an orthonormal basis vector set when we transform our data

That means A (our transformation matrix), will be an orthogonal matrix, therefore the transpose (A^T), or inverse A^{-1} will be really easy to compute

- means transformation can be reversible since space don't get collapsed.
- means projection is just the dot product. (all things, nice and lovely)
- and if we arrange the basis vector in correct order, then the determinant will be 1.

Recap

- we looked at transpose
- which allowed us to find the most convenient basis vector set of all, i.e. orthonormal basis vector set.
- which together gives us an orthogonal matrix
- whose inverse ~~are~~ is its transpose