Module 4
Example

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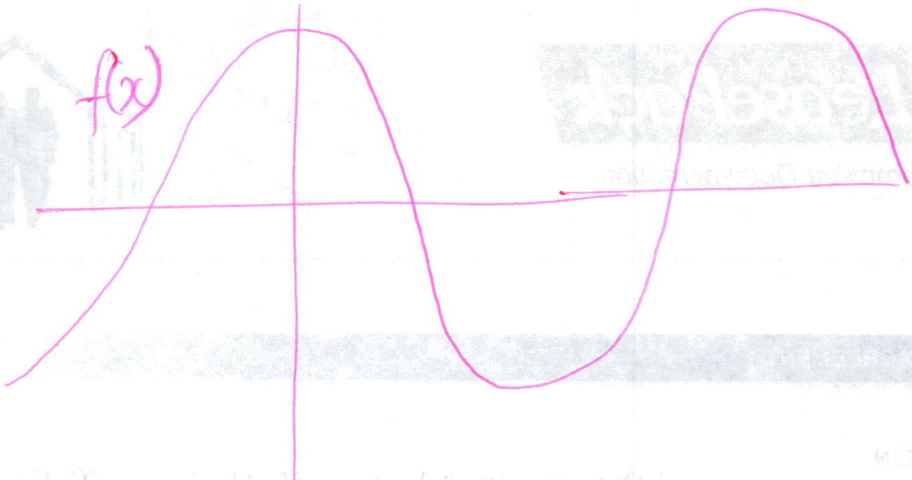
In this session we going to put our understanding of the power series to the test, by looking at two interesting examples

First, Build a Maclaurin Series expansion, the Cos function

Cos is the epitomy (or perfect exemplar representative) of a well behaved function, as it is Certainly Continuous everywhere, as well as been infinitely differentiable

As we building a Maclaurin series, we going want to find at everything about the function at point $x = 0$

So lets Starting doing some differentiation



$$f(x) = \cos(x)$$

$$f'(x) = -\sin(x)$$

$$f''(x) = -\cos(x)$$

$$f^{(3)}(x) = \sin(x)$$

$$f^{(4)}(x) = \cos(x)$$

And what we get if we remind ourselves from earlier session, is the cyclic pattern of Cos and Sin (pos/neg), which takes us Back to the Cos again, after 4 steps. If we know evaluate the derivative, at point $x=0$, we see the Cos terms is either 1 and or -1, and Sin terms are all 0

$f(0) = \cos(0) = 1$

$f'(0) = -\sin(0) = 0$

$f''(0) = -\cos(0) = -1$

$f^{(3)}(0) = \sin(0) = 0$

$f^{(4)}(0) = \cos(0) = 1$

This must mean from a power series perspective, that every other term will have a 0 coefficient

Notice that these 0's will occur whenever we differentiate an odd number of ~~to~~ times

Which means that all the odd ~~Power series~~ powers of x, like x^1, x^3, x^5 will all be absent from the series

The even powers of x, are all what we call, even functions, which means they're symmetrical around vertical axis, just like cos.

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So, we can now bring back our general expression from the Maclaurin series, and just start writing out the terms

The icing on the cake, is that at this point we just notice that we can build a neat summation which fully describes the series, without having to write out all the terms

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots$$

∴ Nice summation notation, eluded above

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Notice, that this expression does not even contain any reference to cos, all all that we need to know is captured by the $(-1)^n$, which just keeps flipping from negative, positive, and negative again.

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~~Hooth~~ Now that we have done all hard work, we can ask computer to plot, the sequence of increasingly accurate Maclaurin serie, which hopefully starting from a horizontal line of $y=1$, will line up with your expectation.

Notice, that outside ~~the~~ of a region of fairly close to point $x=0$, the approximation explodes off and become useless.

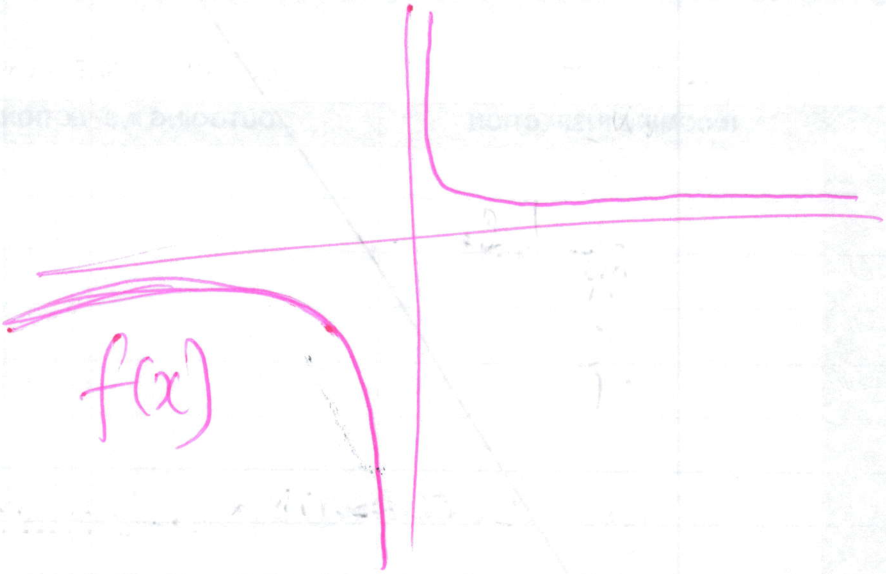
By the time we get our 16th order approximation, we pretty much nailed the region showed in graph.

Although just outside ~~the region~~ of these axis, the function would also be growing hugely positive

So you must be careful when handling series of approximations, that you know the domain in which its acceptable

~~Instead~~ In our second example, we taking the function:

$$f(x) = 1/x$$



$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

It's a nice simple function, But notice the discontinuity at $x=0$

This is NOT a well-behaved function.

In fact, it is so badly behaved, that when we even try and build a zeroth order approximation, we immediately run into problems

As we have to perform the operation $1/0$, But this is undefined

$$f(0) = 1/0 = NaN \quad [\text{Computer: Not a Number}]$$

we need to use a different angle of attack.

Clearly, we aren't going to have much ~~luce~~ luck at point $x=0$

So lets go somewhere else, anywhere else?

$\therefore x=1$

It passes the first test, as we can evaluate the function at this point

$$f(1) = 1/1 = 1$$

But moving away from $x=0$, we then have to use the Taylor Series, instead of Mac Series

So now we need to find a few derivatives of function, and see if we can spot a pattern

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(x-p)^n$$

When we evaluate these functions at point $x=1$, hopefully you'll recognise we get a sequence of factorials emerging

$$f(1) = 1/1 = 1$$

$$f'(1) = -1/1^2 = -1$$

$$f''(1) = 2/1^3 = 2$$

$$f^{(3)}(1) = -6/1^4 = -6$$

$$f^{(4)}(1) = 24/1^5 = 24$$

Just as we did when deriving the power series formula in first place

So if we now substitute these values into our Taylor series formula, the factorial terms will cancel, and all we left with is sum of $(x-1)^n$ with alternative signs which we can simply into the neat summation notation.

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(x-p)^n$$

$$1/x = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$$

$$1/x = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

So once again, lets now pass the formula to the computer and ask it to plot a sequence of increasing approximations starting, as ever, with horizontal line.

But this time, start at height of function $x=1$

There are several interesting features of this example, which tells interesting things of the power series generally

Firstly the approximations ignore the (~~asymptotes~~) asymptote, going straight across it

Furthermore the region of the function where x less than 0 is not described by the approximations.

Secondly, although the function's gradually improving for larger values of x, you can see the tail wildly flailing around, as the sign of each additional term flips from 'pos' to 'neg' and back again

Hope these two examples had made it clear how the Taylor series manages to re-construct well behaved function, like $\cos x$ also why it struggles with something badly behaved $1/x$

Next session, we going to talk briefly on how to ~~lin~~ linearize a function and how this relates to Taylor series analysis that we have covered so far.