PROBABILITY THEORY
MM 5

MM 5: Special probability distributions

Topics: Continuous distributions:

uniform, exponential, normal

What should we learn today?

- A number of r.vs. arises in many diverse, unrelated applications → learn them
- What are the main facts about these distributions:
  - Their cdf and pdf
  - Mean and variance
  - Moment generating function
- In which situations the special distributions arrise and how are they interrelated?

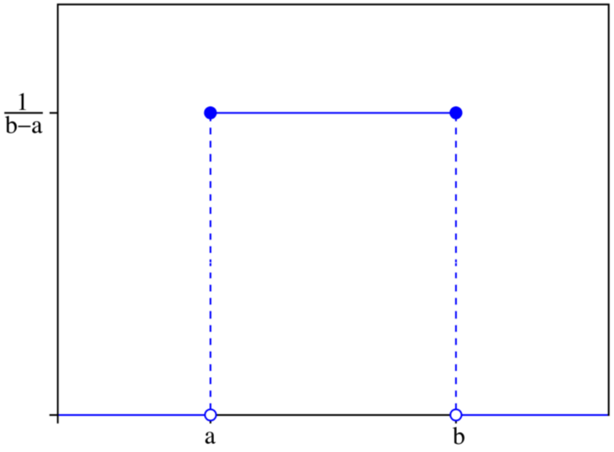
Continuous r.vs.

- We are always limited to measurements of finite precision → every random variable found in practice is in principle a discrete one.
- However in many situations it is convenient and natural to work with an abstraction = continuous r.v.

Uniform r.v.

- The uniform r.v. arises in situations where all values in an interval of the real line are equally likely to occur. Its pdf is given by a constant.

- We can also introduce a uniform distribution over 2-dimentional region: the random vector X, Y is said to have a uniform distribution, if its joint density function is constant for points in this region.



Uniform r.v.

$$f(x) = \begin{cases} c, & \alpha \le X \le \beta \\ 0, & \text{otherwise} \end{cases}$$

$$1 = \int_{-\infty}^{+\infty} f(x)\,dx = c\int_{\alpha}^{\beta} dx = c(\beta - \alpha) \Rightarrow c = \frac{1}{\beta - \alpha}$$

$$P\{a < X < b\} = \frac{1}{\beta - \alpha}\int_{a}^{b} dx = \frac{b-a}{\beta - \alpha}$$

Uniform r.v.: mean and variance

$$E[X] = \int_\alpha^\beta \frac{x}{\beta-\alpha}\,dx = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} = \frac{(\beta+\alpha)(\beta-\alpha)}{2(\beta-\alpha)} = \frac{\beta+\alpha}{2}$$

$$E[X^2] = \int_\alpha^\beta \frac{x^2}{\beta-\alpha}\,dx = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \frac{(\alpha+\beta)^2}{4} = \frac{(\beta-\alpha)^2}{12}$$

Uniform distribution over a 2-dimentional region

$$f(x,y) = \begin{cases} c, & (x,y) \in R \\ 0, & \text{otherwise} \end{cases}$$

$$1 = \int_R f(x,y)\,dx\,dy = c \int_R dx\,dy = c \cdot A,$$

where A is areas of region R $\Rightarrow c = \frac{1}{A}$

Exponential r.v.

- Definition. X is an exponential r.v. with parameter lambda, if its pdf is

$$\boxed{f(x) = \lambda e^{-\lambda x}, \quad x \geq 0}$$

- The moment generating function is

$$\varphi(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(\lambda - t)x} dx = \frac{\lambda}{\lambda - t}; \quad t < \lambda$$

- Memoryless property:

$$\boxed{P(X > s + t | X > t) = P(X > s)}$$

Exponential r.v.

Cdf:

$$F(x) = Pr(X \le x) = \int_{-\infty}^{x} f(y)\,dy = \int_{0}^{x} \lambda e^{-\lambda y}\,dy =$$

$$= \lambda \frac{e^{-\lambda y}}{-\lambda}\Bigg|_{0}^{x} = 1 - e^{-\lambda x}$$

Reliability function $R(x) = Pr(X > x) = 1 - F(x) = e^{-\lambda x}$

Exponential r.v.: mean and variance

$$\varphi(t) = \frac{\lambda}{\lambda - t}$$

$$\varphi'(t) = \frac{\lambda}{(\lambda - t)^2}, \quad \varphi''(t) = \frac{2\lambda}{(\lambda - t)^3}$$

$$E[X] = \varphi'(0) = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{2\lambda}{\lambda^3} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Exponential r.v. : memoryless property

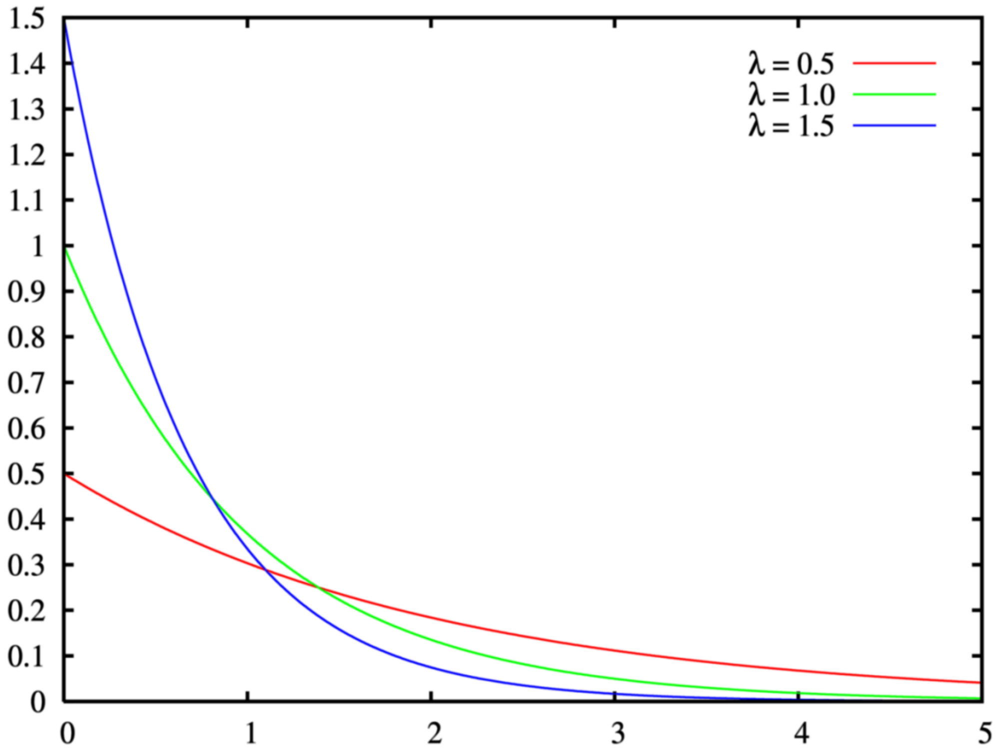
$$P(X > s + t | X > t) = P(X > s)$$

- → Old item is as good as new. There is no need to remember the age of the functioning item.

$$P\{X > s+t \mid X > t\} = \frac{P\{X > s+t, X > t\}}{P\{X > t\}} =$$

$$= \frac{P\{X > s+t\}}{P\{X > t\}} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P\{X > s\}$$

Pdf of exponential r.v.



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Poisson process

- Suppose that events are occurring at random moments in time. N(t) is the number of events in a time interval [0,t].
- The events are said to follow a Poisson process if

$$P\{N(t)=k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k=0,1,2,\ldots$$

Poisson process

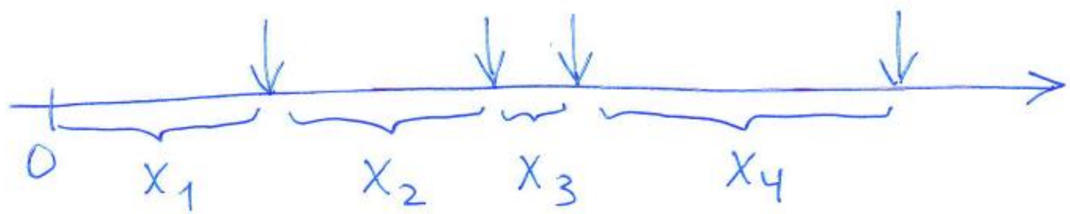
- The events are said to constitute a Poisson process with rate lambda if

1. *N(0)=0*
2. The number of events occuring in disjoint time intervals are independent
3. The distribution depends only on the length of the interval
4. $$\lim_{h \to 0} \frac{P(N(h)=1)}{h} = \lambda \qquad \lim_{h \to 0} \frac{P(N(h) \geq 2)}{h} = 0$$

- Under these assumptions, the number of events in any interval t has a Poisson distribution with mean λt.

Poisson process: very important property

The interarrival times between the events of a Poisson process are independent exponential random variables each with rate λ.



Lets determine distribution of X_n:

$$P\{X_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

no events occurs in the interval [0, t]

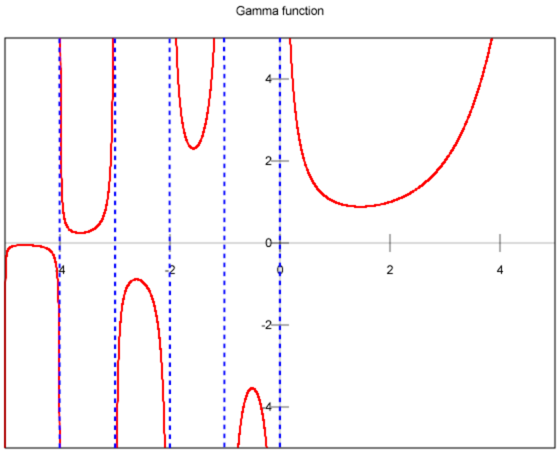
$$P\{X_2 > t \mid X_1 = s\} = P\{\text{no events in } (s, s+t]\} = e^{-\lambda t}$$

Gamma distribution

- Definition. A r.v. is said to have a gamma distribution with paramaters $(\alpha, \lambda), \quad \alpha > 0, \ \lambda > 0$ if its pdf is given by

$$f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)}, \quad x \geq 0$$

$$\Gamma(\alpha) = \int_0^{\infty} \lambda e^{-\lambda x} (\lambda x)^{\alpha - 1} dx$$



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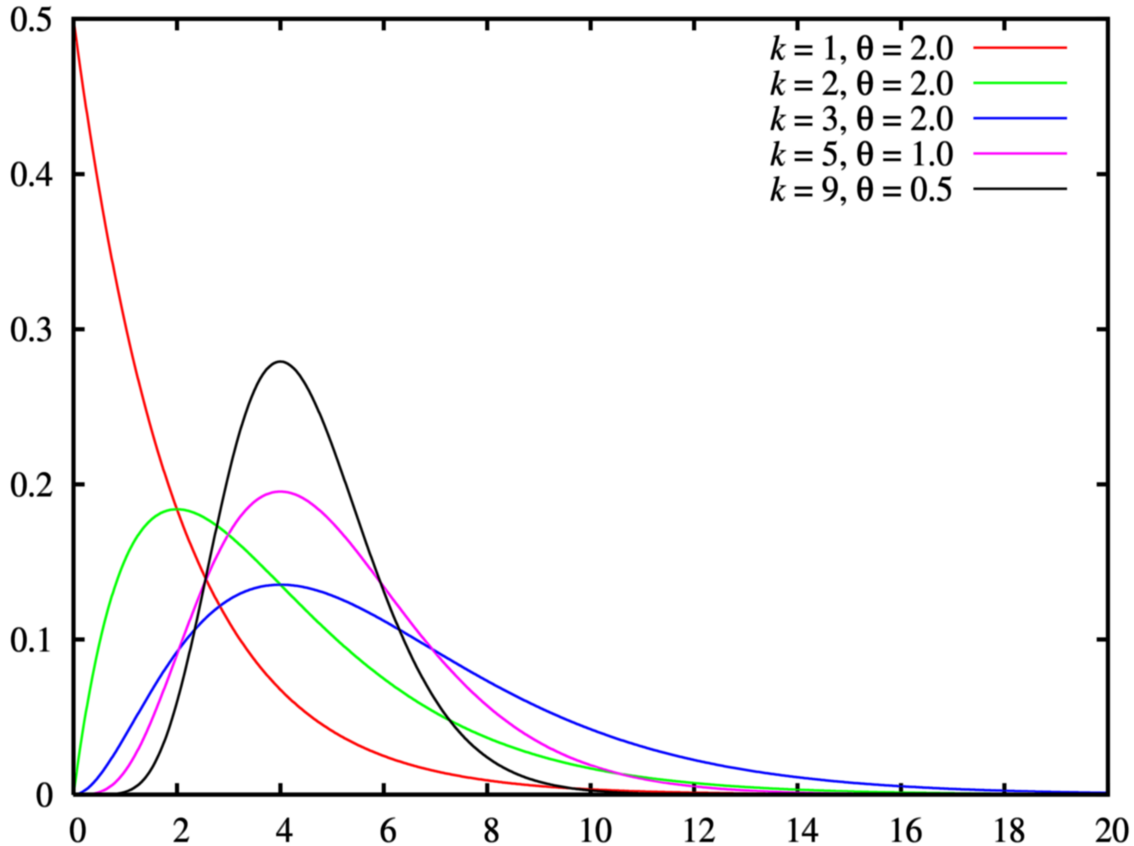
Sum of exponential r.v.

- Proposition. If $X1, ..., Xn$ are independent exponential r.v. with parameters lambda, then their sum is a gamma r.v. with parameters *(n, lambda)* .

Gamma distribution



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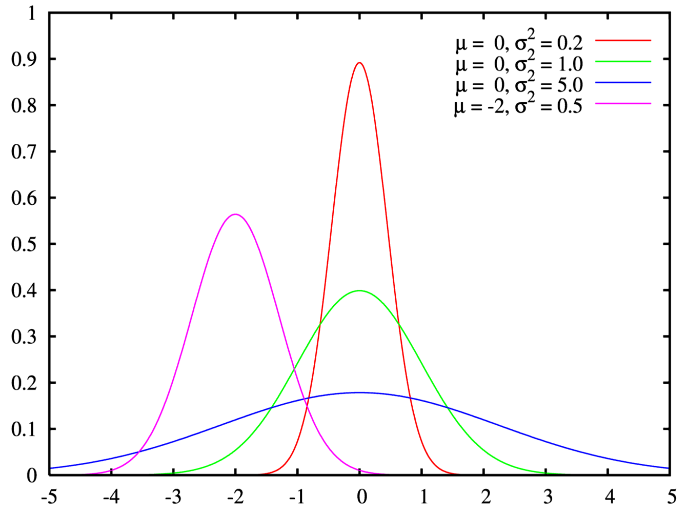
Normal r.v. (Gaussian r.v.)

- Definition. A r.v. is said to be normally distributed if its pdf is

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$



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How to calculate normal distribution

- In practise our goal is to find different probabilities. They can be calculated by finding a corresponding integral.
- However, this integral does not have a closed-form expression → need numerically evaluate it every time
- Nowadays:
- E.g. Matlab `P = normcdf(X,mu,sigma)`
- Old days:
- Traditionally integrals have been evaluated by looking up tables → it is unrealistic to provide tables for all values of μ and σ → use tables for unit (standard) normal distribution $\mu=0$ and $\sigma=1$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-y^2/2} dy$$

How to calculate normal distribution

$$X \sim N(\mu, \sigma^2) \qquad Z = \frac{X-\mu}{\sigma}$$

$\Rightarrow$ Z has a standard normal distribution

$$P\{X < b\} = P\left\{\frac{X-\mu}{\sigma^{\cancel{2}}} < \frac{b-\mu}{\sigma^{\cancel{2}}}\right\} = P\left\{Z < \frac{b-\mu}{\sigma^{\cancel{2}}}\right\} =$$

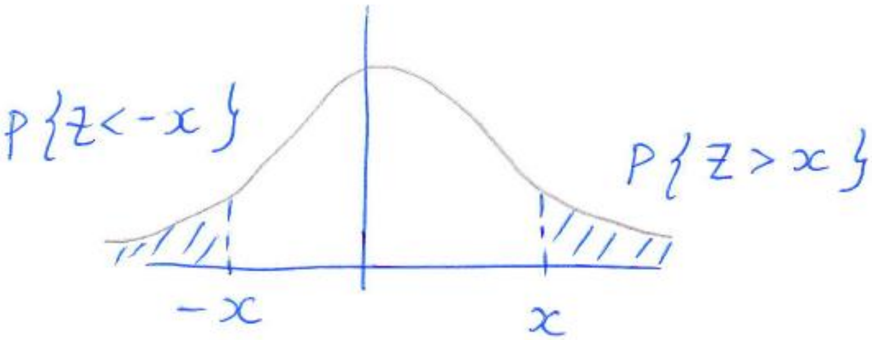
$$= \Phi\left(\frac{b-\mu}{\sigma}\right)$$

$$P\{a < X < b\} = P\left\{\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right\} = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

How to calculate normal distribution

Table gives values for $\Phi(x)$ only for nonnegative x.

$$\Phi(-x) = P\{Z < -x\} = P\{Z > x\} = 1 - \Phi(x)$$



Example

- A communication system accepts a positive voltage V as input, and outputs a voltage Y=aV+N, N is noise.

$a = 10^{-2}$; N is normal r.v. with $\mu = 0$, $\sigma = 2$

Find value of V that gives $P\{Y < 0\} = 10^{-3}$

$P\{Y<0\} = P\{aV + N < 0\} = P\{N < -aV\} =$

$= P\{Z < -\frac{aV}{\sigma}\} = P\{Z < -\frac{10^{-2}V}{2}\} =$

$= 1 - \varphi\left(\frac{10^{-2}V}{2}\right) = 10^{-3}$

$\varphi\left(\frac{10^{-2}V}{2}\right) = 0.999$ $\quad \frac{10^{-2}V}{2} \approx 3.1$ $\quad V \approx 620$

Table

TABLE A1 *Standard Normal Distribution Function:* $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

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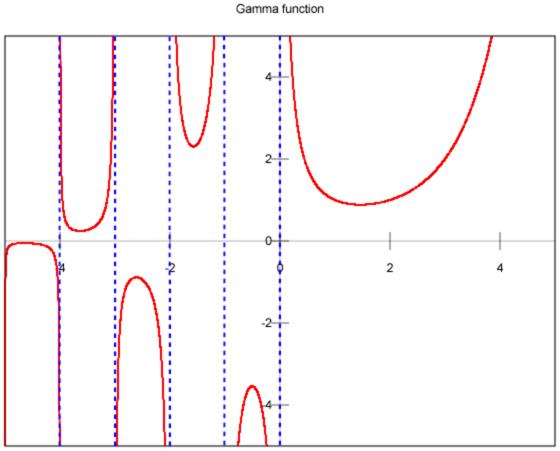

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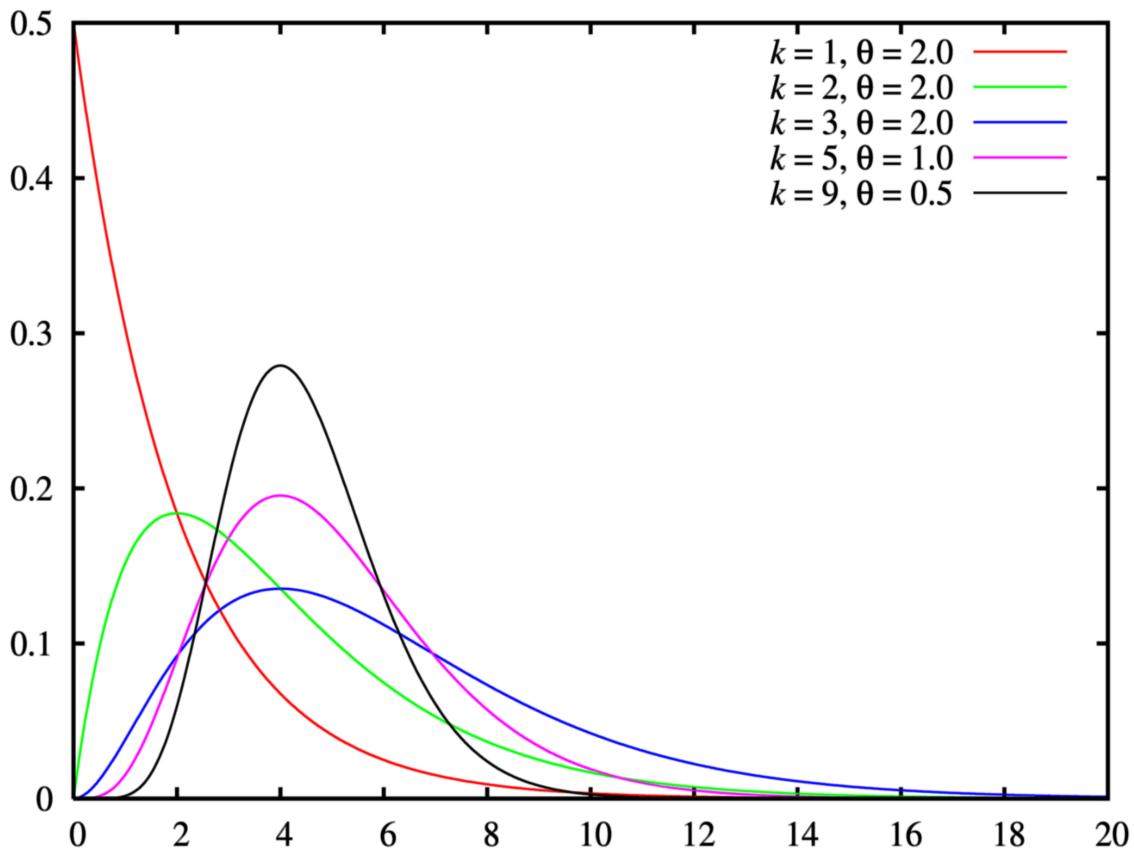

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