Game Theory HW4

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1) a. Compute Pigou for $c_e(x) = a_e x + b_e$ $\quad a,b \geq 0$

$$\alpha(C_e) = \sup_{c \in C} \sup_{x,r \geq 0} \frac{r \cdot c(r)}{x \cdot c(x) + r \cdot c(r) - x \cdot c(r)}$$

$$= \sup_{c \in C} \sup_{x,r \geq 0} \frac{r(a_e r + b_e)}{x(a_e x + b_e) + r(a_e r + b_e) - x(a_e r + b_e)}$$

$$= \sup_{c \in C} \sup_{x,r \geq 0} \frac{r^2 a_e + r b_e}{x^2 a_e + \cancel{x b_e} + r^2 a_e + r b_e - x a_e r - \cancel{x b_e}}$$

$$= \sup_{c \in C} \sup_{x,r \geq 0} \frac{a_e r^2 + b_e r}{a_e x^2 + a_e r^2 + b_e r - a_e x r}$$

Find value of x that minimizes denominator when r, a, and b are fixed.

$$ax^2 - arx + (ar^2 + br)$$

$$\frac{d}{dx} = 2ax - ar = 0$$

$$x = \tfrac{1}{2} r$$

Plug in this value

$$\alpha \geq \frac{ar^2 + br}{\frac{1}{4} ar^2 + ar^2 + br - \frac{1}{2} ar^2}$$

$$\geq \frac{ar^2 + br}{\frac{3}{4} ar^2 + br} \quad \leftarrow \text{dominated by } r^2 \text{ terms}$$

$$\geq \frac{ar^2}{\frac{3}{4} ar^2}$$

$$\alpha = 4/3$$

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1) b. Show $\alpha \leq 4/3$ for nondecreasing, non-negative, concave functions.

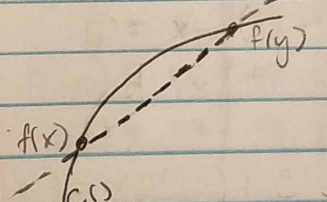
$c(x) \geq \emptyset$ non-negative

$c'(x) = \frac{dc}{dx} \geq \emptyset$ non-decreasing

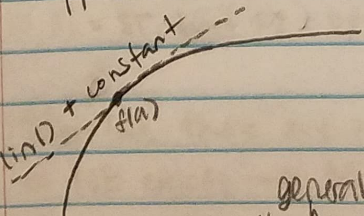
$c''(x) = \frac{d^2c}{dx^2} < \emptyset$ concave

$$\alpha(c) = \sup_{c \in C} \sup_{x,r \geq \emptyset} \frac{r c(r)}{x c(x) + (r-x) c(r)}$$

The definition of a concave function is based on a linear function between 2 points on the concave function:



By translating that function up until there is only 1 intersection point, we see that the linear translated function is the upper bound (α) on C:



If we solve the general form of the linear function for α, (shown above in a), we find an α of $4/3$.

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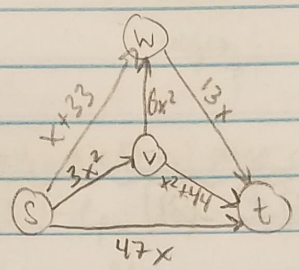
2) Show that NE need not exist in Atomic Congestion games; use the following case b/w s and t.

$r_1 = 1$
$r_2 = 2$



$s \to w \to t$
$s \to v \to w \to t$
$s \to v \to t$
$s \to t$

This case has no stable state (shown below):

	x = 1	x = 2	x = 3
C(s,w) =	34	35	36
C(w,t) =	13	26	39
C(s,v) =	3	12	27
C(v,w) =	6	24	54
C(v,t) =	45	48	53
C(s,t) =	47	94	141

	SWT	SVWT	SVT	ST
SWT	(75, 75)	(73, 75)	(47, 60)	(47, 94)
SVWT	(48, 74)	(120, 120)	(46, 75)	(22, 94)
SVT	(48, 61)	(72, 77)	(80, 80)	(48, 94)
ST	(47, 61)	(47, 62)	(47, 60)	(141, 141)

if P1 = SWT, best P2 = SVT, but then P1 wants to switch to SVWT.
if P1 = SVWT, best P2 = SWT, but then P1 wants to switch to ST.
if P1 = SVT, best P2 = SWT, but then P1 wants to switch to ST.
if P1 = ST, best P2 = SVT, but then P1 wants to switch to SVWT.

Therefore there are no stable cases in this example, and no NE exists → NE need not exist for all atomic congestion games.

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3) Show (G, K, c) admits at least 1 eq flow.

Minimize $\Phi(f) = \sum_{e \in E} \left(c_e(f_e) f_e + \sum_{i \in P} c_e(r_i) r_i \right)$ for NE

$e \in E$ ↑ edges

$i \in P$ ↑ players

★ Proof by contradiction: player changes edge

P_h = path

$0 > C_{P_h'}(f') - C_{P_h}(f) = \Phi(f') - \Phi(f)$

new   old

$= \sum_{e \in P_h' - P_h} (c_e(f_e+1)^*(f_e+1) + \sum_{i \in P+1} c_e(r_i)^* r_i) - \sum_{e \in P_h - P_h'} (c_e(f_e)^* f_e + \sum_{i \in P-1} c_e(r_i)^* r_i)$

$i \in P+1$ ↓ one more player

$i \in P-1$ ↓ one less player

$\Phi(f') = \sum_{e \text{ where } f_e = f'_e} (c_e(f_e) f_e + \sum_{i \in P} c_e(r_i) r_i)$ common

$+ \sum_{e \in P_h' - P_h} (c_e(f_e+1)(f_e+1) + \sum_{i \in P+1} c_e(r_i) r_i)$ new only

$+ \sum_{e \in P_h - P_h'} (c_e(f_e-1)(f_e-1) + \sum_{i \in P-1} c_e(r_i) r_i)$ old only

$\Phi(f) = \sum_{e \text{ where } f_e = f'_e} (c_e(f_e) f_e + \sum_{i \in P} c_e(r_i) r_i)$ common

$+ \sum_{e \in P_h' - P_h} (c_e(f_e) f_e + \sum_{i \in P} c_e(r_i) r_i)$ new only

$+ \sum_{e \in P_h - P_h'} (c_e(f_e) f_e + \sum_{i \in P} c_e(r_i) r_i)$ old only

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4) a. local connection game

$cost(u) = \alpha d_u + \sum_v dist(u,v)$ shortest distance

degree of node u

Show that all NE trees for $\alpha > n^2$, $cost(u) = \alpha d_u + \sum_v dist(u,$

Proof by contradiction:

- assume that we have a cycle in our graph G ⇒ Tree is defined as an acyclic graph
- G has m edges
- m must be $\geq n-1$ to be connected

the node (u) that created the cycle to node v incurs a cost of $\alpha + 1 - 2 * dist_{original}(u,v)$ for that edge.

new distance to v

In order for this node to make this choice, we must have $2 * dist_{original}(u,v) > \alpha + 1$.

However, we know this is not possible, since $dist_{original}(u,v) \leq n-1$, and $\alpha > n^2$.

This gives us at the worst case $n-1 > \lceil n^2 \rceil + 1$, which causes a contradiction

4) a. Show PoA bounded by constant.

diameter of graph G → i.e. $\text{dist}(u,v) \leq 2\sqrt{\alpha} \Rightarrow PoA \leq O(\sqrt{\alpha})$ @ NE

$= \text{dist}(u,v) \leq 2\sqrt{n^2}$

Prove by contradiction:

- assume $\text{dist}(u,v) > 2\sqrt{n^2}$
- then a node would reduce its cost by adding edge (u,v), as $2^{\alpha}\text{dist}_{original}(u,v) > \alpha + 1$
- this contradicts our original assumption of Nash Equilibrium

4) b. Show PoS is bounded by 4/3

We showed above that all NE are trees for the given $\alpha > n^2$.

The optimal graph is a star graph, where all nodes are ≤ 2 distance from each other.

For the worst case NE:

$$PoS = \frac{cost(NE)}{cost(Opt)} = \frac{\alpha(n-1) + \sum_{u,v} dist(u,v)}{cost(Opt)}$$

optimal is star graph

$\sum_{u,v} dist(u,v)$: the worst case tree for this component is having a tree where each node has one and only one child (e.g. no shortcuts). In this case, we have $\frac{n!}{2(n-2)!}$ pairs, and each of these pairs would have a max distance of $n-1$.

$$PoS < \frac{n^2(n-1) + \left(\frac{n!}{2(n-2)!}\right)[n-1]}{n^2(n-1) + [\frac{3}{8}(n-1)]}$$

avg distance for star is 3/8 for each other node when we consider all nodes including center

$(n-2)! < n!$ so this term is $< \frac{1}{2}$

$$PoS < \frac{\cancel{n^2(n-1)} + \frac{1}{2}[n-1]}{\cancel{n^2(n-1)} + \frac{3}{8}(n-1)}$$

$$PoS < \frac{\frac{1}{2}[n-1]}{\frac{3}{8}[n-1]}$$

$$PoS < \frac{8}{6} = \frac{4}{3}$$

5) a. Global connection game: show PoA $\leq k$ (# players)

$$PoA = \sup \frac{cost(NE)}{cost(Opt)} = \frac{\lceil k(c_e/n_e) \rceil}{\lfloor k(c_e/n_e) \rfloor}$$

$$\leq \frac{\text{no players share}}{\text{all players share}} = \frac{\cancel{k}c_e}{\frac{\cancel{k}c_e}{k}} = \frac{1}{(1/k)}$$

$$PoA \leq k$$

b. $gain_i(path) = \frac{weight_i}{W}$ $\qquad W = \sum_j w_j \leftarrow$ other players

All players are either indifferent to sharing paths or prefer to share paths (w_i all $\geq \emptyset$). Therefore, any situation where all players with $w_i > \emptyset$ share a path is a Nash Equillibrium. Players with $w_i = \emptyset$ can choose any path they wish.

$\Downarrow$

NE exists

$$\boxed{\Phi(paths) = \sum_e C_e - (n_e/k)(w_i/W)}$$