

Yamabe Equation and Geodesics

David Gu

Computer Science Department
Stony Brook University

gu@cs.stonybrook.edu

July 13, 2024

Computation under Isothermal Coordinates

Isothermal Coordinates

Lemma (Isothermal Coordinates)

Let (S, \mathbf{g}) be a metric surface, use isothermal coordinates

$$\mathbf{g} = e^{2u(x,y)}(dx^2 + dy^2).$$

Then we obtain

$$\omega_1 = e^u dx \quad \omega_2 = e^u dy$$

and the orthonormal frame is

$$\mathbf{e}_1 = e^{-u} \partial_x \quad \mathbf{e}_2 = e^{-u} \partial_y$$

and the connection

$$\omega_{12} = -u_y dx + u_x dy$$

Gaussian Curvature

Proof.

By direct computation, $ds^2 = \omega_1^2 + \omega_2^2$,

$$d\omega_1 = de^u \wedge dx$$

$$= e^u(u_x dx + u_y dy) \wedge dx$$

$$= e^u u_y dy \wedge dx$$

$$d\omega_2 = de^u \wedge dy$$

$$= e^u(u_x dx + u_y dy) \wedge dy$$

$$= e^u u_x dx \wedge dy.$$

therefore

$$\begin{aligned}\omega_{12} &= \frac{d\omega_1}{\omega_1 \wedge \omega_2} \omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2} \omega_2 \\ &= \frac{e^u u_y dy \wedge dx}{e^{2u} dx \wedge dy} e^u dx + \frac{e^u u_x dx \wedge dy}{e^{2u} dx \wedge dy} e^u dy \\ \omega_{12} &= -u_y dx + u_x dy.\end{aligned}$$



Gaussian Curvature

Lemma (Gaussian curvature)

Under the isothermal coordinates, the Gaussian curvature is given by

$$K = -\frac{1}{e^{2u}} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u.$$

Proof.

From

$$\omega_{12} = -u_y dx + u_x dy$$

we get

$$K = -\frac{d\omega_{12}}{\omega_1 \wedge \omega_2} = -\frac{(u_{xx} + u_{yy})dx \wedge dy}{e^{2u}dx \wedge dy} = -\frac{1}{e^{2u}}\Delta u.$$



Gaussian Curvature

Example

The unit disk $|z| < 1$ equipped with the following metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2},$$

the Gaussian curvature is -1 everywhere.

Proof.

$e^{2u} = \frac{4}{(1-x^2-y^2)^2}$, then $u = \log 2 - \log(1 - x^2 - y^2)$.

$$u_x = -\frac{-2x}{1 - x^2 - y^2} = \frac{2x}{1 - x^2 - y^2}.$$



Gaussian Curvature

Proof.

then

$$u_{xx} = \frac{2(1 - x^2 - y^2) - 2x(-2x)}{(1 - x^2 - y^2)^2} = \frac{2 + 2x^2 - 2y^2}{(1 - x^2 - y^2)^2}$$

similarly

$$u_{yy} = \frac{2 + 2y^2 - 2x^2}{(1 - x^2 - y^2)^2}$$

so

$$u_{xx} + u_{yy} = \frac{4}{(1 - x^2 - y^2)^2} = e^{2u}, K = -\frac{1}{e^{2u}}(u_{xx} + u_{yy}) = -1.$$



Yamabe Equation

Lemma (Yamabe Equation)

Conformal metric deformation $\mathbf{g} \rightarrow e^{2\lambda}\mathbf{g} = \tilde{\mathbf{g}}$, then

$$\tilde{K} = \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda)$$

Proof.

Use isothermal parameters, $\mathbf{g} = e^{2u}(dx^2 + dy^2)$, $K = -e^{-2u}\Delta u$, similarly $\tilde{\mathbf{g}} = e^{2\tilde{u}}(dx^2 + dy^2)$, $\tilde{K} = -e^{-2\tilde{u}}\Delta\tilde{u}$, $\tilde{u} = u + \lambda$,

$$\begin{aligned}\tilde{K} &= -\frac{1}{e^{2(u+\lambda)}}\Delta(u + \lambda) \\ &= \frac{1}{e^{2\lambda}}\left(-\frac{1}{e^{2u}}\Delta u - \frac{1}{e^{2u}}\Delta\lambda\right) \\ &= \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).\end{aligned}$$

Geodesics

Geodesic Equation

Lemma (Geodesic Equation on a Riemann Surface)

Suppose S is a Riemann surface with a metric, $\rho(z)dzd\bar{z} = e^{2u(z)}dzd\bar{z}$, then a geodesic γ with local representation $z(t)$ satisfies the equation:

$$\ddot{\gamma} + \frac{2\rho_{\gamma}}{\rho}\dot{\gamma}^2 \equiv 0.$$

equivalently,

$$\ddot{\gamma} + 4u_{\gamma}\dot{\gamma}^2 \equiv 0.$$

Geodesic Equation

Proof.

Assume the velocity vector is $\dot{\gamma} = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2$, which is parallel along γ , by parallel transport ODE,

$$\begin{cases} \frac{df_1}{dt} - f_2 \frac{\omega_{12}}{dt} = 0 \\ \frac{df_2}{dt} + f_1 \frac{\omega_{12}}{dt} = 0 \end{cases}$$

Suppose the geodesic has local representation $\gamma(t) = (x(t), y(t))$, then $d\gamma = \dot{x}\partial_x + \dot{y}\partial_y = e^u \dot{x} \mathbf{e}_1 + e^u \dot{y} \mathbf{e}_2$, $\omega_{12}/dt = -u_y \dot{x} + u_x \dot{y}$, $\rho = e^u$,

$$\begin{aligned} \frac{d}{dt}(\rho \dot{x}) - (\rho \dot{y})(-u_y \dot{x} + u_x \dot{y}) &= 0 \\ \frac{d}{dt}(\rho \dot{y}) + (\rho \dot{x})(-u_y \dot{x} + u_x \dot{y}) &= 0 \end{aligned}$$



Geodesic Equation

continued

in turn,

$$\rho\ddot{x} + \dot{\rho}\dot{x} - \dot{y}(-\rho_y\dot{x} + \rho_x\dot{y}) = \rho\ddot{x} + (\rho_x\dot{x} + \rho_y\dot{y})\dot{x} - \dot{y}(-\rho_y\dot{x} + \rho_x\dot{y}) = 0$$

$$\rho\ddot{y} + \dot{\rho}\dot{y} + \dot{x}(-\rho_y\dot{x} + \rho_x\dot{y}) = \rho\ddot{y} + (\rho_x\dot{x} + \rho_y\dot{y})\dot{y} + \dot{x}(-\rho_y\dot{x} + \rho_x\dot{y}) = 0$$

namely

$$\rho\ddot{x} + \rho_x(\dot{x}^2 - \dot{y}^2) + 2\rho_y\dot{x}\dot{y} = 0$$

$$\rho\ddot{y} - \rho_y(\dot{x}^2 - \dot{y}^2) + 2\rho_x\dot{x}\dot{y} = 0$$

The first row plus $\sqrt{-1}$ times the second row,

$$\rho(\ddot{x} + \sqrt{-1}\ddot{y}) + (\rho_x - \sqrt{-1}\rho_y)(\dot{x} + \sqrt{-1}\dot{y})^2 = 0.$$

continued.

Represent $\gamma(t) = z(t)$, where $z = x + \sqrt{-1}y$, $\rho_z = \frac{1}{2}(\rho_x - \sqrt{-1}\rho_y)$, we obtain the equation for geodesic on complex domain,

$$\ddot{\gamma} + \frac{2\rho_\gamma}{\rho}\dot{\gamma}^2 \equiv 0.$$



Geodesic Curvature

Lemma

Given a curve γ on a surface (S, \mathbf{g}) , with isothermal coordinates (x, y) , the angle between ∂_x and $\dot{\gamma}$ is θ , then

$$k_g(s) = \frac{d\theta}{ds} + \frac{\omega_{12}}{ds}$$

Proof.

Construct an orthonormal frame $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2\}$ by rotating $\{\mathbf{e}_1, \mathbf{e}_2\}$ by angle θ , hence $\bar{\mathbf{e}}_1$ is the tangent vector of γ .

$$\begin{cases} \bar{\mathbf{e}}_1 &= \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &= -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2 \end{cases}$$

$$\begin{aligned} d\bar{\mathbf{e}}_1 &= -\sin \theta d\theta \mathbf{e}_1 + \cos \theta d\mathbf{e}_1 + \cos \theta d\theta \mathbf{e}_2 + \sin \theta d\mathbf{e}_2 \\ &= (-\sin \theta d\theta - \sin \theta \omega_{12}) \mathbf{e}_1 + (\cos \theta \omega_{12} + \cos \theta d\theta) \mathbf{e}_2 \\ &\quad + (\cos \theta \omega_{13} + \sin \theta \omega_{23}) \mathbf{e}_3 \end{aligned}$$

continued

$$\begin{aligned}\bar{\omega}_{12} &= \langle d\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2 \rangle \\ &= (-\sin\theta d\theta - \sin\theta\omega_{12})(-\sin\theta) + (\cos\theta\omega_{12} + \cos\theta d\theta)\cos\theta \\ &= d\theta + \omega_{12}.\end{aligned}$$

Therefore

$$k_g = \frac{\bar{\omega}_{12}}{ds} = \frac{d\theta}{ds} + \frac{\omega_{12}}{ds}$$



Geodesic Curvature

Lemma (Geodesic Curvature)

Under the isothermal coordinates, the geodesic curvature is given by

$$k_g = e^{-u}(k - \partial_{\mathbf{n}}u)$$

where k is the curvature on the parameter plane, \mathbf{n} is the exterior normal to the curve on the parameter plane.

Proof.

We have $\omega_{12} = -u_y dx + u_x dy$. On the parameter plane, the arc length is dt , then $ds = e^u dt$. The parameterization preserves angle, therefore

$$\begin{aligned} k_g &= \frac{d\theta}{ds} + \frac{-u_y dx + u_x dy}{ds} = \frac{dt}{ds} \left(\frac{d\theta}{dt} + \frac{-u_y dx + u_x dy}{dt} \right) \\ &= e^{-u}(k - \langle \nabla u, \mathbf{n} \rangle) \\ &= e^{-u}(k - \partial_{\mathbf{n}}u) \end{aligned}$$

Geodesic Curvature

Lemma

Given a metric surface (S, \mathbf{g}) , under conformal deformation, $\bar{\mathbf{g}} = e^{2\lambda} \mathbf{g}$, the geodesic curvature satisfies

$$k_{\bar{\mathbf{g}}} = e^{-\lambda}(k_{\mathbf{g}} - \partial_{\mathbf{n}, \mathbf{g}} \lambda)$$

Proof.

$$\begin{aligned} k_{\bar{\mathbf{g}}} &= e^{-(u+\lambda)}(k - \partial_{\mathbf{n}}(u + \lambda)) \\ &= e^{-\lambda}(e^{-u}(k - \partial_{\mathbf{n}} u) - e^{-u} \partial_{\mathbf{n}} \lambda) \\ &= e^{-\lambda}(k_{\mathbf{g}} - \partial_{\mathbf{n}, \mathbf{g}} \lambda) \end{aligned}$$



Definition (geodesic)

Given a metric surface (S, \mathbf{g}) , a curve $\gamma : [0, 1] \rightarrow S$ is a geodesic if $k_{\mathbf{g}}$ is zero everywhere.

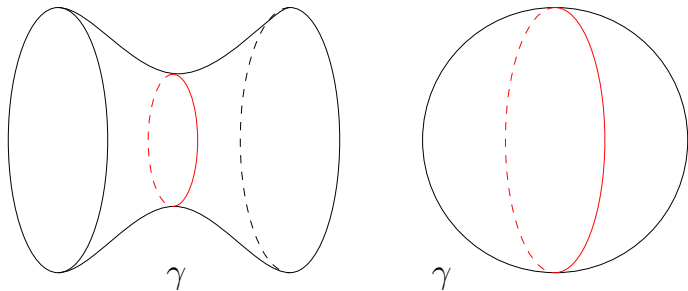


Figure: Stable and unstable geodesics.

Lemma (geodesic)

If γ is the shortest curve connecting p and q , then γ is a geodesic.

Proof.

Consider a family of curves, $\Gamma : (-\varepsilon, \varepsilon) \rightarrow S$, such that $\Gamma(0, t) = \gamma(t)$, and

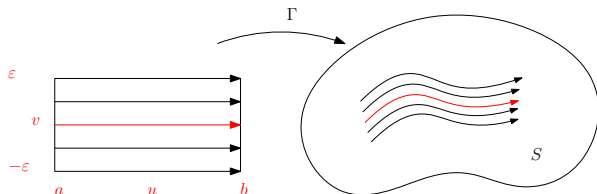
$$\Gamma(s, 0) = p, \Gamma(s, 1) = q, \frac{\partial \Gamma(s, t)}{\partial s} = \varphi(t) \mathbf{e}_2(t),$$

where $\varphi : [0, 1] \rightarrow \mathbb{R}$, $\varphi(0) = \varphi(1) = 0$. Fix parameter s , curve $\gamma_s := \Gamma(s, \cdot)$, $\{\gamma_s\}$ for a variation. Define an energy,

$$L(s) = \int_0^1 \left| \frac{d\gamma_s(t)}{dt} \right| dt, \quad \frac{\partial L(s)}{\partial s} = - \int_0^1 \varphi k_g(\tau) d\tau.$$



First Variation of arc length



Let $\gamma_v : [a, b] \rightarrow M$, where $v \in (-\varepsilon, \varepsilon) \in \mathbb{R}$ be a 1-parameter family of paths. We define the map $\Gamma : [a, b] \times [0, 1] \rightarrow M$ by

$$\Gamma(u, v) := \gamma_v(u).$$

Define the vector fields \mathbf{u} and \mathbf{v} along γ_v by

$$\mathbf{u} := \frac{\partial \Gamma}{\partial u} = \Gamma_*(\partial_u), \quad \text{and} \quad \mathbf{v} := \frac{\partial \Gamma}{\partial v} = \Gamma_*(\partial_v),$$

We call \mathbf{u} the *tangent vector field* and \mathbf{v} the *variation vector field*.

First Variation of arc length

Lemma (First variation of arc length)

If The length of γ_v is given by

$$L(\gamma_v) := \int_a^b |\mathbf{u}(\gamma_v(u))| du.$$

γ_0 is parameterized by arc length, that is, $|\mathbf{u}(\gamma_0(u))| \equiv 1$, then

$$\left. \frac{d}{dv} \right|_{v=0} L(\gamma_v) = - \int_a^b \langle D_{\mathbf{u}} \mathbf{u}, \mathbf{v} \rangle du + \langle \mathbf{u}, \mathbf{v} \rangle \Big|_a^b.$$

If we choose $\mathbf{u} = \mathbf{e}_1$, the tangent vector of γ , $\mathbf{v} = \mathbf{e}_2$ orthogonal to \mathbf{e}_1 , and fix the starting and ending points of paths, then

$$\frac{d}{dv} L(\gamma_v) = - \int_a^b k_g ds.$$

First variation of arc length

Proof.

Fixing $u \in [a, b]$, we may consider \mathbf{u} and \mathbf{v} as vector fields along the path $v \mapsto \gamma_v(u)$. Then

$$\begin{aligned}\frac{\partial}{\partial v} |\mathbf{u}(\gamma_v(u))| &= \frac{\partial}{\partial v} \sqrt{|\mathbf{u}(\gamma_v(u))|^2} \\ &= \frac{1}{2|\mathbf{u}(\gamma_v(u))|} \frac{\partial}{\partial v} |\mathbf{u}(\gamma_v(u))|^2 \\ &= \frac{1}{2|\mathbf{u}|} \mathbf{v} |\mathbf{u}|^2 = |\mathbf{u}|^{-1} \langle D_{\mathbf{v}} \mathbf{u}, \mathbf{u} \rangle_{\mathbf{g}} = \langle D_{\mathbf{v}} \mathbf{u}, \mathbf{u} \rangle_{\mathbf{g}}\end{aligned}$$



First variation of arc length

Proof.

$$\frac{d}{dv}L(\gamma_v) = \int_a^b \frac{\partial}{\partial v} |\mathbf{u}(\gamma_v(u))| du = \int_a^b \langle D_v \mathbf{u}, \mathbf{u} \rangle_{\mathbf{g}} du$$

Since $D_v \mathbf{u} - D_u \mathbf{v} = [\mathbf{v}, \mathbf{u}]$, and $[\mathbf{v}, \mathbf{u}] = \Gamma_*([\partial_v, \partial_u]) = 0$,

$$\begin{aligned} \frac{d}{dv}L(\gamma_v) &= \int_a^b \langle D_u \mathbf{v}, \mathbf{u} \rangle_{\mathbf{g}} du \\ &= \int_a^b \left(\frac{d}{du} \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{g}} - \langle \mathbf{v}, D_u \mathbf{u} \rangle_{\mathbf{g}} \right) du \\ &= \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{g}} \Big|_a^b - \int_a^b \langle \mathbf{v}, D_u \mathbf{u} \rangle_{\mathbf{g}} du. \end{aligned}$$



The second derivative of the length variation $L(s)$ depends on the Gaussian curvature of the underlying surface. If $K < 0$, then the second derivative is positive, the geodesic is stable; if $K > 0$, then the secondary derivative is negative, the geodesic is unstable.

Lemma (Uniqueness of geodesics)

Suppose (S, \mathbf{g}) is a closed oriented metric surface, \mathbf{g} induces negative Gaussian curvature everywhere, then each homotopy class has a unique geodesic.

Proof.

The existence can be obtained by variational method. The uniqueness is by Gauss-Bonnet theorem. Assume two geodesics $\gamma_1 \sim \gamma_2$, then they bound a topological annulus Σ , by Gauss-Bonnet,

$$\int_{\Sigma} K dA + \int_{\partial \Sigma} k_g ds = \chi(\Sigma),$$

The first term is negative, the second is along the geodesics, hence 0, $\chi(\Sigma) = 0$. Contradiction. □

Algorithm: Homotopy Detection

Input: A high genus closed mesh M , two loops γ_1 and γ_2 ;

Output: Whether $\gamma_1 \sim \gamma_2$;

- 1 Compute a hyperbolic metric of M , using Ricci flow;
- 2 Homotopically deform γ_k to geodesics, $k = 1, 2$;
- 3 if two geodesics coincide, return true; otherwise, return false;

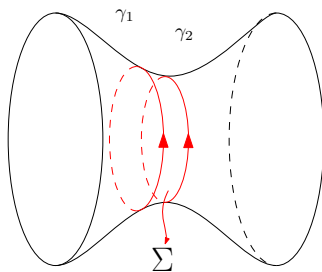


Figure: Geodesics uniqueness.

Algorithm: Shortest Word

Input: A high genus closed mesh M , one loop γ

- 1 Compute a hyperbolic metric of M , using Ricci flow;
- 2 Homotopically deform γ to a geodesic;
- 3 Compute a set of canonical fundamental group basis;
- 4 Embed a finite portion of the universal covering space onto the Poincaré disk;
- 5 Lift γ to the universal covering space $\tilde{\gamma}$. If $\tilde{\gamma}$ crosses b_i^\pm , append a_i^\pm ; crosses a_i^\pm , append b_i^\mp .

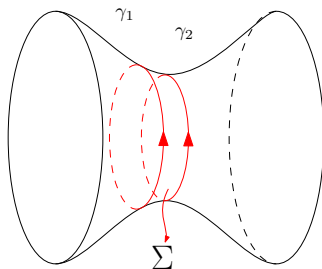


Figure: Geodesic uniqueness

Hyperbolic Geodesics

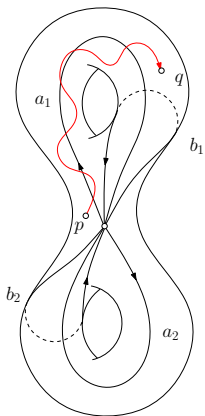
Lemma

Let Σ be a compact hyperbolic Riemann surface, $K \equiv -1$, $p, q \in \Sigma$, then there exists a unique geodesic in each homotopy class, the geodesic depends on p and q continuously.

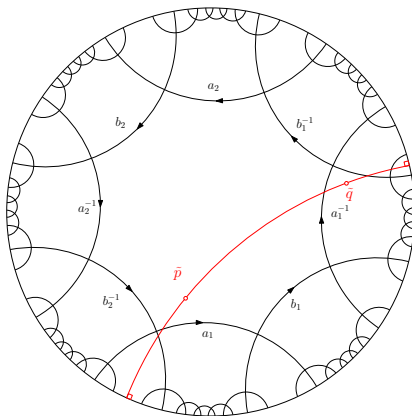
Proof.

Given a path $\gamma : [0, 1] \rightarrow \Sigma$ connecting p and q . Let $\pi : \mathbb{H}^2 \rightarrow \Sigma$ be the universal covering space of Σ . Fix one point $\tilde{p} \in \pi^{-1}(p)$, then there exists a unique lifting of γ , $\tilde{\gamma} : [0, 1] \rightarrow \mathbb{H}^2$, $\tilde{\gamma}(0) = \tilde{p}$ and $\tilde{\gamma}(1) = \tilde{q}$. On the hyperbolic plane, the geodesic between \tilde{p} and \tilde{q} exists and is unique, $\tilde{\gamma}$ depends on \tilde{p} and \tilde{q} continuously. □

Hyperbolic Geodesic

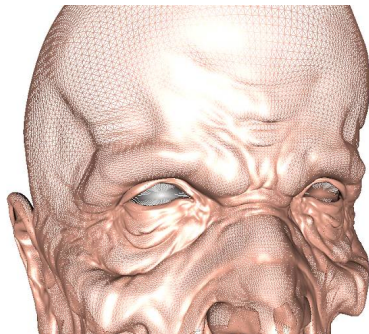


geodesic on surface



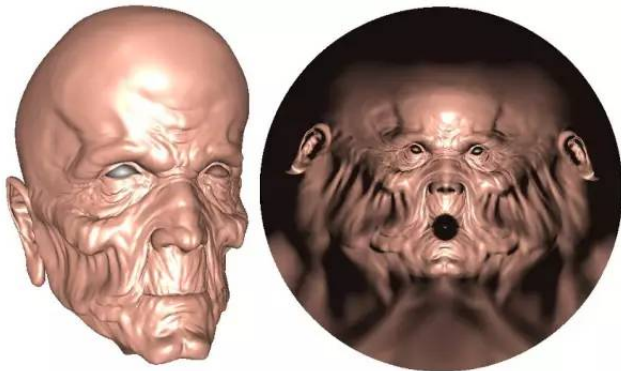
Poincaré's disk model

Finite Element Method



Given a smooth surface (S, \mathbf{g}) , we can construct a sequence of triangle meshes $\varphi_n : S \rightarrow (M_n, \mathbf{d}_n)$, the pull back metric $\{\varphi_n^* \mathbf{d}_n\}$ converge to \mathbf{g} .

Finite Element Method



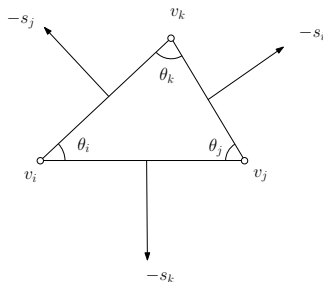
For each M_n , construct a harmonic map $f_n : M_n \rightarrow \mathbb{D}^2$. Then $\{f_n\}$ converge to the smooth harmonic map $f : S \rightarrow \mathbb{D}^2$.

Lemma (Discrete Harmonic Energy)

Given a piecewise linear function $f : M \rightarrow \mathbb{R}$, then the harmonic energy of f is given by

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} (f(v_i) - f(v_j))^2.$$

$$w_{ij} = \cot \theta_{ij}^k + \cot \theta_{ji}^l.$$



Definition (Barry-centric Coordinates)

Given a Euclidean triangle with vertices $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$ the bary-centric coordinates of a planar point $\mathbf{p} \in \mathbb{R}^2$ with respect to the triangle are $(\lambda_i, \lambda_j, \lambda_k)$, $\mathbf{p} = \lambda_i \mathbf{v}_i + \lambda_j \mathbf{v}_j + \lambda_k \mathbf{v}_k$, where

$$\lambda_i = \frac{(\mathbf{v}_j - \mathbf{p}) \times (\mathbf{v}_k - \mathbf{p}) \cdot \mathbf{n}}{(\mathbf{v}_j - \mathbf{v}_i) \times (\mathbf{v}_k - \mathbf{v}_i) \cdot \mathbf{n}}$$

the ratio between the area of the triangle $\mathbf{p}, \mathbf{v}_j, \mathbf{v}_k$ and the area of $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$. λ_j and λ_k are defined similarly.

By direct computation, the sum of the bary-centric coordinates is 1

$$\lambda_i + \lambda_j + \lambda_k = 1.$$

If \mathbf{p} is the interior point of the triangle, then all components of the bary-centric coordinates are positive.

Lemma

Suppose $f : \Delta \rightarrow \mathbb{R}$ is a linear function,

$$f(p) = \lambda_i f(v_i) + \lambda_j f(v_j) + \lambda_k f(v_k),$$

then the gradient of the function is

$$\nabla f(p) = \frac{1}{2A}(s_i f(v_i) + s_j f(v_j) + s_k f(v_k)).$$

its harmonic energy is

$$\int_{\Delta} \langle \nabla f, \nabla f \rangle dA = \frac{\cot \theta_i}{2} (f_j - f_k)^2 + \frac{\cot \theta_j}{2} (f_k - f_i)^2 + \frac{\cot \theta_k}{2} (f_i - f_j)^2. \quad (1)$$

Proof.

We have

$$\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k = \mathbf{n} \times \{(\mathbf{v}_k - \mathbf{v}_j) + (\mathbf{v}_i - \mathbf{v}_k) + (\mathbf{v}_j - \mathbf{v}_i)\} = \mathbf{0}$$

therefore

$$\langle \mathbf{s}_i, \mathbf{s}_i \rangle = \langle \mathbf{s}_i, -\mathbf{s}_j - \mathbf{s}_k \rangle = -\langle \mathbf{s}_i, \mathbf{s}_j \rangle - \langle \mathbf{s}_i, \mathbf{s}_k \rangle.$$

pick a point $\mathbf{p} = \lambda_i \mathbf{v}_i + \lambda_j \mathbf{v}_j + \lambda_k \mathbf{v}_k$, bary-centric coordinates

$$\lambda_i = \frac{1}{2A} \langle \mathbf{v}_k - \mathbf{v}_j, \mathbf{p} - \mathbf{v}_j, \mathbf{n} \rangle = \frac{1}{2A} \langle \mathbf{n} \times (\mathbf{v}_k - \mathbf{v}_j), \mathbf{p} - \mathbf{v}_j \rangle$$

hence

$$\lambda_i = \frac{1}{2A} \langle \mathbf{p} - \mathbf{v}_j, \mathbf{s}_i \rangle, \lambda_j = \frac{1}{2A} \langle \mathbf{p} - \mathbf{v}_k, \mathbf{s}_j \rangle, \lambda_k = \frac{1}{2A} \langle \mathbf{p} - \mathbf{v}_i, \mathbf{s}_k \rangle,$$

where A is the triangle area.



continued

The linear function is

$$\begin{aligned} f(\mathbf{p}) &= \lambda_i f_i + \lambda_j f_j + \lambda_k f_k \\ &= \frac{1}{2A} \langle \mathbf{p} - \mathbf{v}_j, f_i \mathbf{s}_i \rangle + \frac{1}{2A} \langle \mathbf{p} - \mathbf{v}_k, f_j \mathbf{s}_j \rangle + \frac{1}{2A} \langle \mathbf{p} - \mathbf{v}_i, f_k \mathbf{s}_k \rangle \\ &= \langle \mathbf{p}, \frac{1}{2A} (f_i \mathbf{s}_i + f_j \mathbf{s}_j + f_k \mathbf{s}_k) \rangle - \frac{1}{2A} (\langle \mathbf{v}_j, f_i \mathbf{s}_i \rangle + \langle \mathbf{v}_k, f_j \mathbf{s}_j \rangle + \langle \mathbf{v}_i, f_k \mathbf{s}_k \rangle). \end{aligned}$$

Hence we obtain the gradient

$$\nabla f = \frac{1}{2A} (f_i \mathbf{s}_i + f_j \mathbf{s}_j + f_k \mathbf{s}_k).$$

continued

we compute the harmonic energy

$$\begin{aligned}& \int_{\Delta} \langle \nabla f, \nabla f \rangle dA \\&= \frac{1}{4A} \langle f_i \mathbf{s}_i + f_j \mathbf{s}_j + f_k \mathbf{s}_k, f_i \mathbf{s}_i + f_j \mathbf{s}_j + f_k \mathbf{s}_k \rangle \\&= \frac{1}{4A} \left(\sum_i \langle \mathbf{s}_i, \mathbf{s}_i \rangle f_i^2 + 2 \sum_{i < j} \langle \mathbf{s}_i, \mathbf{s}_j \rangle f_i f_j \right) \\&= \frac{1}{4A} \left(- \sum_i \langle \mathbf{s}_i, \mathbf{s}_j + \mathbf{s}_k \rangle f_i^2 + 2 \sum_{i < j} \langle \mathbf{s}_i, \mathbf{s}_j \rangle f_i f_j \right) \\&= - \frac{1}{4A} (\langle \mathbf{s}_i, \mathbf{s}_j \rangle (f_i - f_j)^2 + \langle \mathbf{s}_j, \mathbf{s}_k \rangle (f_j - f_k)^2 + \langle \mathbf{s}_k, \mathbf{s}_i \rangle (f_k - f_i)^2)\end{aligned}$$

continued

Since

$$\frac{\langle \mathbf{s}_i, \mathbf{s}_j \rangle}{2A} = -\cot \theta_k, \quad \frac{\langle \mathbf{s}_j, \mathbf{s}_k \rangle}{2A} = -\cot \theta_i, \quad \frac{\langle \mathbf{s}_k, \mathbf{s}_i \rangle}{2A} = -\cot \theta_j.$$

Hence the harmonic energy is

$$\int_{\Delta} \langle \nabla f, \nabla f \rangle dA = \frac{\cot \theta_i}{2} (f_j - f_k)^2 + \frac{\cot \theta_j}{2} (f_k - f_i)^2 + \frac{\cot \theta_k}{2} (f_i - f_j)^2.$$

□.

Lemma (Discrete Harmonic Energy)

Given a piecewise linear function $f : M \rightarrow \mathbb{R}$, then the harmonic energy of f is given by

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} (f(v_i) - f(v_j))^2.$$

$$w_{ij} = \cot \theta_{ij}^k + \cot \theta_{ji}^l.$$

Proof.

We add the harmonic energies on all faces together, and merge the items associated with the same edge, then each edge contributes

$$\frac{1}{2} w_{ij} (f_j - f_i)^2.$$

