Nielsen-Thurston Classificiation Theorem

David Gu

Computer Science Department
Stony Brook University

gu@cs.stonybrook.edu

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Thurston Simple Curve

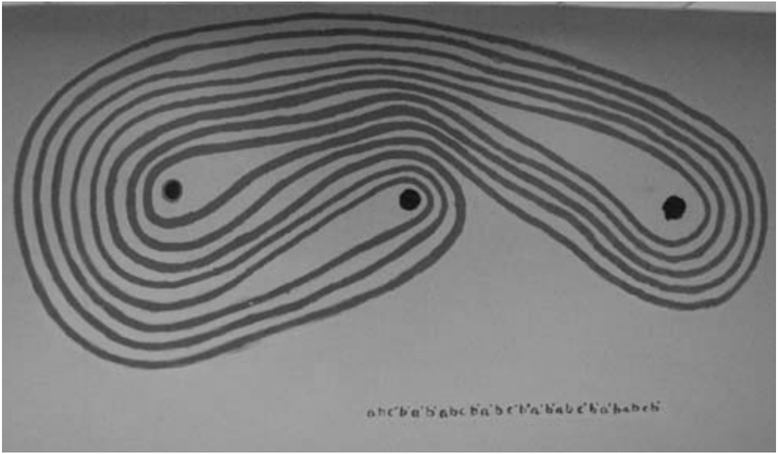


Figure: The painting on the wall of Berkeley mathematics department the iterations of Thurston simple curve by Thurston and Sullivan.

Taffy Pulling

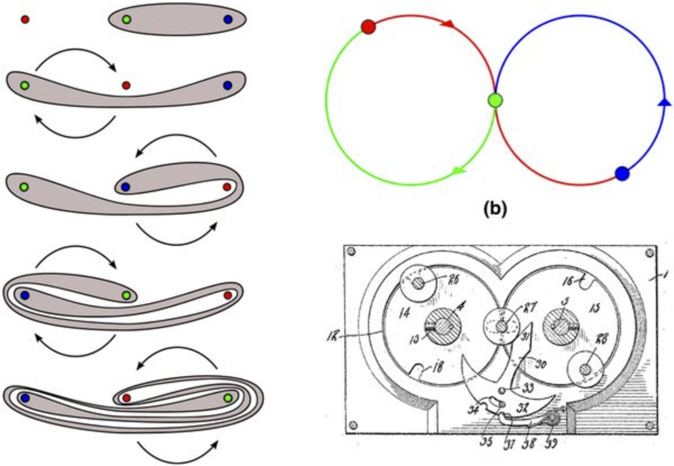
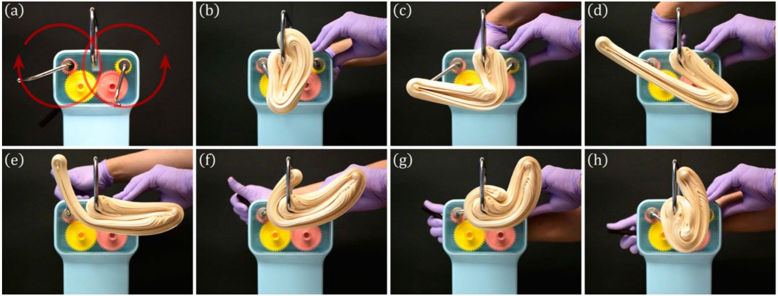


Figure: 3-Rod Taffy pulling, the number of strands increases exponentially fast.

Taffy Pulling



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Half Dehn Twist

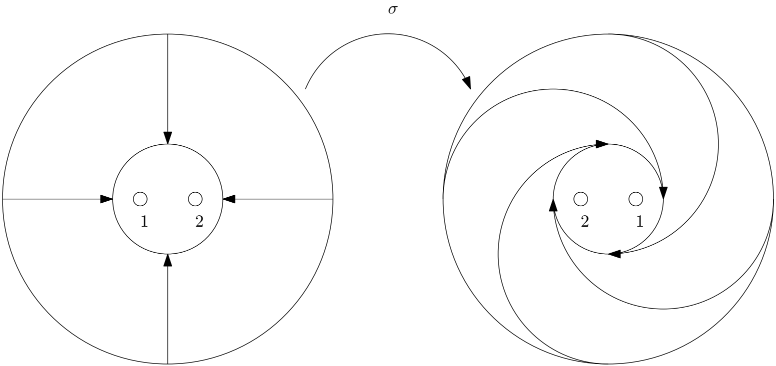


Figure: A planar homeomorphism, half dehn twist, exchanges the punctures 1 and *b*.

Pseudo-Anosov Map

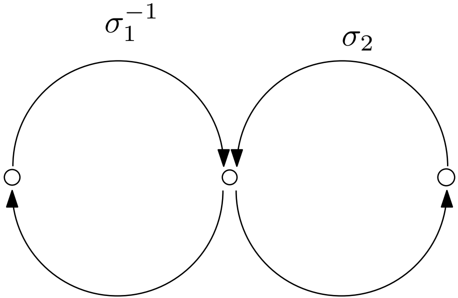


Figure: The mapping class $f = \sigma_1^{-1}\sigma_2$ on $S_{0,4}$.

Pseudo-Anosov Map

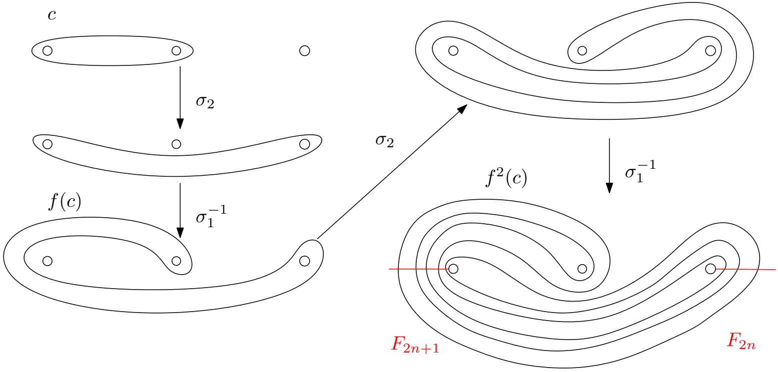


Figure: The first two iterations of c under f.

Fibonacci number $F_0 = 0, F_1 = 1$ and $F_i = F_{i-1} + F_{i-2}$. The number of strands of $f^n(c)$ grows exponentially.

Train Track

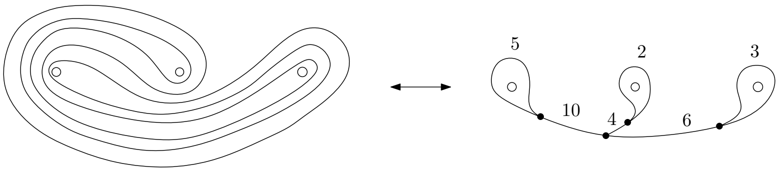


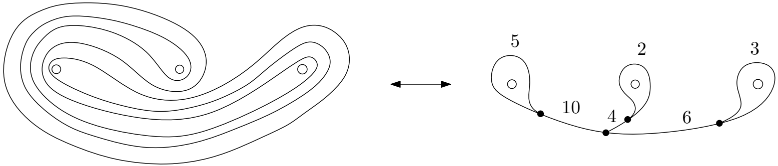
Figure: Converting $f^2(c)$ into a train track. All the weights are determined by the weights $(4, 6)$ by switch conditions.

Definition (Train Track)

A finite graph τ embedded in the surface S, such that

- Each edge of τ is a smooth arc;
- Each vertex of τ has well-defined tangent line, called a switch;
- Each edge is with a nonnegative integer called a *weight* ν (*measure*) satisifying the switch condition: the sums of the weights on each side of the switch are equal to each other.

Train Track



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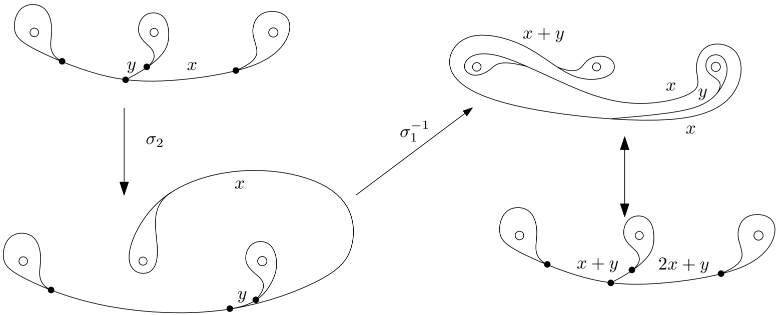


Figure: Applying the map f to the train track (τ, ν), $(0,2),(2,2)$ and $(6,4)$.

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

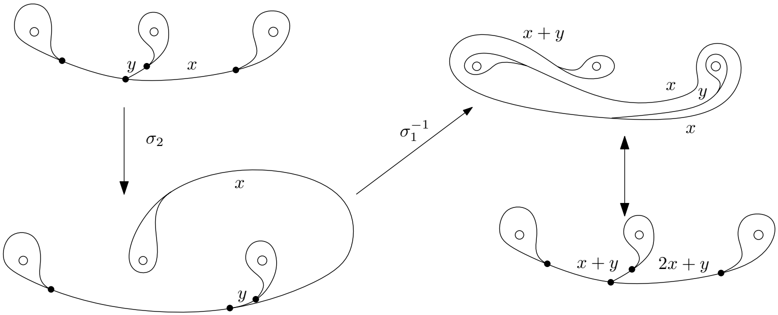


Figure: Applying the map f to the train track (τ, ν), $(0,2), (2,2)$ and $(6,4)$.

$$\lambda = \frac{3+\sqrt{5}}{2}, v_\lambda = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}, \lambda^{-1} = \frac{3-\sqrt{5}}{2}, v_{\lambda^{-1}} = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

Length Functions

Definition (Length Function)

A homotopically nontrivial closed curve γ in S_g defines a *length function*

$$l^\gamma : \mathrm{Teich}(S_g) \to \mathbb{R}_{>0}$$

which assigns to a metric $m \in \mathrm{Teich}(S_g)$ the length $l^\gamma(m)$ of the unique closed geodesic to γ.

Let $\mathscr{S} = \mathscr{S}(S_g)$ be the set of all non-trivial simple closed curves in S_g, considered up to isotopy and orientation reversal. Each element $\gamma \in \mathscr{S}$ induces a length function $l^\gamma : \mathrm{Teich}(S_g) \to \mathbb{R}_{>0}$. Indicate with $\mathbb{R}^{\mathscr{S}}$ the set of all functions $\mathscr{S} \to \mathbb{R}$.

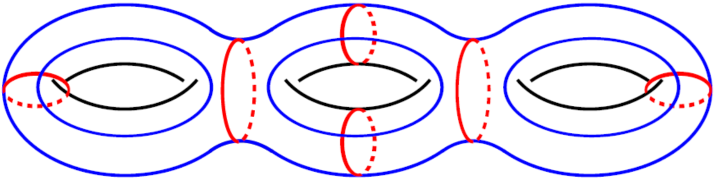


Figure: A frame for Fenchel-Nielsen coordiantes consists of two essential multicurves μ and ν in minimal position, such that : μ is a pants decomposition; ν decomposes every pair of pants in two hexagons.

Definition (Fenchel-Nielse Coordinates)

A frame induces a *Fenchel-Nielsen* map FN : $\text{Teich}(S_g) \to \mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}_{>0}$,

$$m \mapsto (l_1, \ldots, l_{3g-3}, \theta_1, \ldots, \theta_{3g-3}),$$

where $l_i = l^{\gamma_i}(m)$.

$9g - 9$ Theorem

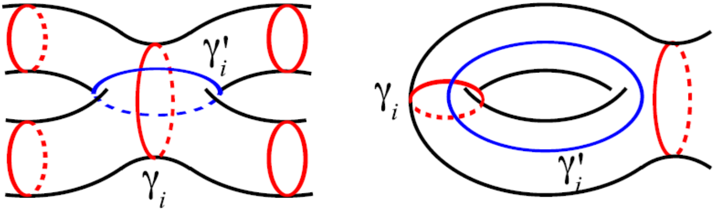


Figure: For each component γ_i of a pants decomposition μ, choose a curve γ_i' that intersects γ_i in one or two points and is disjoint from other components of μ. Two cases, depending on whether the two pants adjacent to γ_i are distinct (left) or not (right).

$9g-9$ Theorem

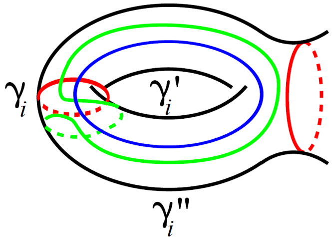


Figure: The curves γ_i (red), γ_i' (blue), and $\gamma_i'' = T_{\gamma_i}(\gamma_i')$ (green) when γ_i is adjacent twice to the same pair of pants.

Theorem ($9g-9$)

The following map is injective:

$$L : Teich(S_g) \to \mathbb{R}^{9g-9}_{>0}$$
$$m \mapsto (l^{\gamma_i}(m), l^{\gamma_i'}(m), l^{\gamma_i''}(m))$$

Teich(S_g) Embeds in $\mathbb{P}(\mathbb{R}^{\mathscr{S}})$

Consider the infinite-dimensional projective space $\mathbb{P}(\mathbb{R}^{\mathscr{S}})$, let the projection map is

$$\pi : \mathbb{R}^{\mathscr{S}} \to \mathbb{P}(\mathbb{R}^{\mathscr{S}}).$$

Proposition

The composition

$$\pi \circ i : Teich(S_g) \to \mathbb{P}(\mathbb{R}^{\mathscr{S}})$$

is injective.

$\mathscr{S}$ Embeds in $\mathbb{P}(\mathbb{R}^{\mathscr{S}})$

Definition (Geometric Intersection Number)

A simple closed curve $\gamma \in \mathscr{S}$ defines a functional $i(\gamma) \in \mathbb{R}^{\mathscr{S}}$ by setting:

$$i(\gamma)(\eta) := i(\gamma, \eta).$$

Proposition

The composition

$$\pi \circ i : \mathscr{S} \to \mathbb{P}(\mathbb{R}^{\mathscr{S}})$$

is injective.

Proof.

Let $\gamma_1, \gamma_2 \in \mathscr{S}$ be distinct. There is always a curve $\eta \in \mathscr{S}$ with $i(\gamma_1, \eta) \neq 0$ and $i(\gamma_2, \eta) = 0$. $\square$

$\mathscr{S}$ Embeds in $\mathbb{P}(\mathbb{R}^{\mathscr{S}})$

Proposition

Consider $Teich(S_g)$ *and* $\mathscr{S}$ *as subsets of* $\mathbb{R}^{\mathscr{S}}$*, then they are disjoint in* $\mathbb{P}(\mathbb{R}^{\mathscr{S}})$*.*

Proof.

For each $\gamma \in \mathscr{S}$ we have $i(\gamma, \gamma) = 0$, where every curve has positive length on any hyperbolic metric. $\square$

Thurston's Compactification Theorem

Theorem (Thurston's Compactification)

The closure $\overline{Teich(S_g)}$ of $Teich(S_g)$ in $\mathbb{P}(\mathbb{R}^{\mathscr{S}})$ is homeomorphic to the closed disk D^{6g-6}. Its interior is $Teich(S_g)$ and its boundary sphere contains $\mathscr{S}$ as a dense subset.

Hyperboloid Model for $\mathbb{H}^2$

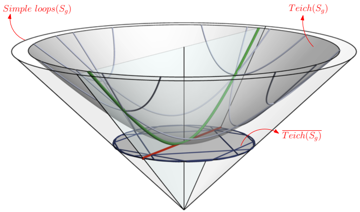


Figure: The hyperboloid model for $\mathbb{H}^2$. Teich(S_g) corresponds to the hyperboloid; $\mathscr{S}$ corresponds to the light cone; $\mathbb{R}^{2,1}$ corresponds to the space of $\mathbb{R}^{\mathscr{S}}$ (furthermore the space of geodesic currents); $\langle v_1, v_2\rangle$ corresponds to $i(\alpha,\beta)$.

- Lorentzian form in $\mathbb{R}^{2,1}$ $\langle v_1, v_2\rangle = x_1x_2 + y_1y_2 - z_1z_2$;
- The hyperboloid $\langle v, v\rangle = -1$, $\mathbb{H}^2$;
- The Light cone $\langle v, v\rangle = 0$, $\partial\mathbb{H}^2$;
- Klein model of $\mathbb{H}^2$, compactification.

Geodesics

- Let M be a complete hyperbolic manifold, $\mathscr{G}(M)$ indicates the set of all supports of all complete non-trivial geodesics $\mathbb{R} \to M$. We call an element of $\mathscr{G}(M)$ a *geodesic*, we say it is *simple* if it has a simple geodesic parameterization.
- Consider the set $\mathscr{G} = \mathscr{G}(\mathbb{H}^2)$ of lines in $\mathbb{H}^2$. A line is determined by its end points
$$\mathscr{G} \leftrightarrow (\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \setminus \Delta)/\sim,$$
where $\Delta = \{(a,a)|a \in \partial\mathbb{H}^2\}$ is the diagonal and $(a,b) \sim (b,a)$. $\mathscr{G}$ is homeomorphic to an open Möbius strip.

Geodesics

Proposition

If $S = \mathbb{H}^2/\Gamma$ *is a complete hyperbolic surface there is a natural bijection*

$$\mathscr{G}(S) \leftrightarrow \mathscr{G}/\Gamma.$$

Proof.

Every geodesic in S lifts to a Γ-orbit of lines in $\mathbb{H}^2$. $\square$

Geodesics

Proposition

The Γ-oribit of a line $l \in \mathscr{G}$ is discrete if and only if l projects to a closed geodesic in S_g.

Proof.

Let $\pi(l) \subset S_g$ be the projection of l in S_g. If $\pi(l)$ is not a closed geodesic, since S_g is compact, then there is a small disk $D \subset S_g$ intersecting $\pi(l)$ into infinitely many distinct segments, hence there is a small disk $D \subset \mathbb{H}^2$ intersecting infinitely many lines of the Γ-orbit of l, therefore the Γ-orbit is not discrete. $\square$

Geodesics Current

Definition (Geodesic Current)

Let $S_g = \mathbb{H}^2/\Gamma$ be a hyperbolic surface. A *geodesic current* on S_g is a locally finite Γ-invariant Borel measure on $\mathscr{G} = \mathscr{G}(\mathbb{H}^2)$.

Let $\mathscr{M}(\mathscr{G})$ be the space of all locally finite Borel measures on $\mathscr{G}$, $\mathscr{C} = \mathscr{C}(S_g)$ be the set of all geodesic currents in S_g.

Geodesics Current for a Closed Geodesic

A closed geodesic γ on $S_g = \mathbb{H}^2/\Gamma$ lifts to a discrete Γ-orbit of lines in $\mathbb{H}^2$. The Diract measure on this discrete set is locally finite and Γ-invariant, hence it is a geodesic current.

Proposition

If $l \in \mathscr{G}$ is an atomic point for a geodesic current μ, that is if $\mu(\{l\}) > 0$, then l projects to a closed geodesic in S_g.

We can interpret every closed geodesic in S_g as a particular geodesic current with discrete support. The closed geodesics form a discrete subset in $\mathscr{C}$. We get an embedding

$$\mathscr{S} \hookrightarrow \mathscr{C}.$$

Liouville Measure

Definition

Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be a parameterized geodesic and $U_\gamma \subset \mathscr{G}$ be the open set consisting of all lines intersecting γ, except γ itself. We can parametrize U_γ via the homeomorphism $\mathbb{R} \times (0, \pi) \to U_\gamma$, that sends (t, θ) to the line that intersects γ at the point $\gamma(t)$ with angle θ. Define a volume 2-form on U_γ,

$$L_\gamma = \frac{1}{2} \sin \theta dt \wedge d\theta.$$

Proposition

The charts U_γ form a differentiable atlas for $\mathscr{G}$. The 2-form L_γ match up to sign and hence define a measure L on $\mathscr{G}$.

Liouville Current

Definition (Liouville Current)

Let $S_g = \mathbb{H}^2/\Gamma$ be equipped with a hyperbolic metric. The Liouville measure on $\mathscr{G}$ is $\mathrm{Isom}(\mathbb{H}^2)$-invariant: in particular it is Γ-invariant and hence defines a current $L \in \mathscr{C}(S_g)$, called the Liouville current.

Every metric $m \in \mathrm{Teich}(S_g)$ induces a Liouville current $L_m \in \mathscr{C}$, and in this way we get a Liouville map

$$\mathrm{Teich}(S_g) \to \mathscr{C}$$

that sends m to L_m.

Projective Frame Bundle

We denote by $\mathscr{I} \subset \mathscr{G} \times \mathscr{G}$ the open subset consisting of all pairs of incident distinct lines in $\mathbb{H}^2$, which can be interpreted as the set of triples (p, l_1, l_2) with $p \in \mathbb{H}^2$ and l_1, l_2 two distinct vectors in the tangent plane $T_p\mathbb{H}^2$.

Let $S_g = \mathbb{H}^2/\Gamma$, the diagonal action of Γ on $\mathscr{I}$ is properly discontinuous, the map

$$\mathscr{I} \to \mathscr{I}/\Gamma$$

is a topological covering. (p, l_1, l_2) form a frame of T_pS_g, $\mathscr{I}/\Gamma$ is treated as some projective quotient of the frame bundle on S_g.

Geometric Properties of Intersection Form

Proposition (Simple Closed Geodesics)

If $\alpha, \beta \in \mathscr{S}$, the value of the form $i(\alpha, \beta)$ is the geometric intersection of the simple closed curves α and β.

In particular $i(\alpha, \alpha) = 0$, namely $\alpha \in \mathscr{C}$ is on the light cone.

Proposition (Simple Closed Geodesic and Hyperbolic Metric)

If $\alpha \in \mathscr{S}$ and $m \in Teich(S_g)$, the value of the form $i(\alpha, L_m) = l^\alpha(m)$ is the length of the geodesic representative of α in the metric m.

The intersection form i on $\mathscr{C}$ generalizes both the geometric intersection of curves and the length functions on Teichmüller space. The Liouville map $\text{Teich}(S_g) \to \mathcal{C}$ is injective.

Geometric Properties of Intersection Form

Proposition

Let $s \subset \mathbb{H}^2$ be a geodesic segment of length l. The lines in $\mathbb{H}^2$ intersecting s in $\mathscr{G}$ form a set of Liouville measure l.

Proof.

The set has the measure

$$\int_0^\pi \int_0^l \frac{1}{2} \sin\theta dt d\theta = l \int_0^\pi \frac{1}{2} \sin\theta d\theta = l.$$

$\square$

Proposition (Hyperbolic Metrics)

If $m \in Teich(S_g)$, the value of the form

$$i(L_m, L_m) = -\pi^2 \chi(S_g).$$

Analogy to Hyperboloid Model

- For each simple closed curve $\alpha \in \mathscr{C}(S_g)$, $i(\alpha,\alpha) = 0$;
- For each hyperbolic metric $m \in \text{Teich}(S_g)$, $i(m,m) = -\pi^2\chi(S_g)$;
- $\mathscr{S} \hookrightarrow \mathscr{C}$ is injective;
- $\text{Teich}(S_g) \hookrightarrow \mathscr{C}$ is injective;
- $\mathscr{S}$ is dense in the boundary of $\overline{\text{Teich}(S_g)}$

Filling geodesic currents

Definition

We say that a geodesic current $\alpha \in \mathscr{C}$ fills the surface $S_g = \mathbb{H}^2/\Gamma$, if every line in $\mathbb{H}^2$ intersects transversely at least one line in the support of α

A Liouville measure fills S_g since its support is the whole of $\mathscr{G}$.

Definition

We say that k closed geodesics $\gamma_1, \gamma_2, \ldots, \gamma_k$ fill S_g if the geodesic current

$$\gamma_1 + \gamma_2 + \cdots + \gamma_k$$

does.

Proposition

Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be closed geodesics. If $S_g \setminus (\gamma_1 \cup \cdots \cup \gamma_k)$ consists of polygons, the curves fill S_g.

Compactness Criterion

Proposition (Compactness criterion)

If $\alpha \in \mathscr{C}$ fills S_g, the set of all $\beta \in \mathscr{C}$ with $i(\alpha, \beta) \leq M$ is compact, for all $M > 0$.

Corollary

The Liouville map $Teich(S_g) \hookrightarrow \mathscr{C}$ is proper and a homeomorphism onto its image.

Corollary

Let $\gamma_1, \ldots, \gamma_k$ be some closed geodesics that fill S_g. The metrics $m \in Teich(S_g)$ with $l^{\gamma_i}(m) \leq M$ form a compact subset of $Teich(S_g)$ for all $M > 0$.

Projective Currents

Definition

Projectived Current Space The current space $\mathscr{C}$ is equipped with a multiplication by positive scalars, hence we can define its projectivisation

$$\pi : \mathscr{C} \setminus 0 \to \mathbb{P}\mathscr{C}$$

where $\mathbb{P}\mathscr{C} = (\mathscr{C} \setminus 0)/\sim$ with $\alpha \sim \lambda\alpha$ for all $\lambda > 0$.

Proposition

The space $\mathbb{P}\mathscr{C}$ is compact.

Proof.

Pick an $\alpha \in \mathscr{C}$ that fills S_g, the set $C = \{\beta \in \mathscr{C} | i(\alpha, \beta) = 1\}$ is compact. Since α fills S_g, for all $\beta \in \mathcal{C}$, $i(\alpha, \beta) > 0$, hence $\lambda\beta \in C$ for some $\lambda > 0$. $\pi(C) = \mathbb{P}\mathscr{C}$. □

Projective Currents

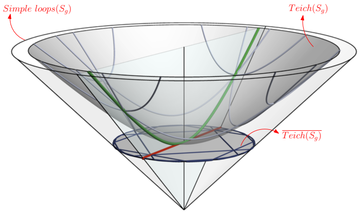


Figure: The hyperboloid model for $\mathbb{H}^2$. Teich(S_g) corresponds to the hyperboloid; $\mathscr{S}$ corresponds to the light cone; $\mathbb{R}^{2,1}$ corresponds to the space of $\mathbb{R}^{\mathscr{S}}$ (furthermore the space of geodesic currents); $\langle v_1, v_2 \rangle$ corresponds to $i(\alpha, \beta)$.

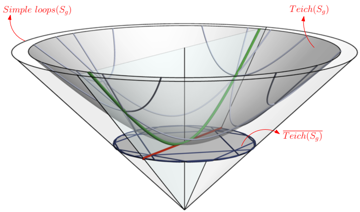
Proposition

The composition $\mathscr{S} \to \mathscr{C} \setminus 0 \to \mathbb{P}\mathscr{C}$ is injective.

Proposition

The composition $Teich(S_g) \to \mathscr{C} \setminus 0 \to \mathbb{P}\mathscr{C}$ is injective and a homemorphism onto its image.

Both $\mathscr{S}$ and Teich(S_g) are embedded in the projective current space $\mathbb{P}\mathscr{C}$.



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Definition (Compactification)

The closure $\overline{\text{Teich}(S_g)}$ of $\text{Teich}(S_g)$ inside $\mathbb{P}\mathscr{C}$ is called the Thurston compactification of Teichmüller space.

The boundary is

$$\partial\overline{\text{Teich}(S_g)} = \overline{\text{Teich}(S_g)} \setminus \text{Teich}(S_g).$$

Proposition

The Thurston boundary consists of all projective currents $[\alpha]$ with $i(\alpha,\alpha) = 0$.

Definition (Geodesic Lamination)

A geodesic lamination λ is a set of disjoint simple complete geodesics in S, whose union is a closed subset of S.Each geodesic may be closed or open and is called a *leaf* ; their union is the support of λ.

The following examples of geodesic laminations are fundamental:

- a finite set of disjoint closed geodesics in S,
- a set of disjoint lines in H2 whose union is closed.

Definition

A geodesic lamination $\lambda \subset S_g$ is minimal if it contains no proper non-empty sublamination.

A geodesic lamination $\lambda \subset S_g$ is minimal if and only if every leaf in λ is dense.

Transverse Measure

Let $\lambda \subset S$ be a geodesic lambination in a hyperbolic surface S. A *transverse arc* to λ is the support of a simple regular curve $\alpha : [a, b] \to S$ transverse to each leaf of λ, whose end points $\alpha(a)$ and $\alpha(b)$ are not contained in λ.

Definition (Transverse Measure)

A transverse measure for a lamination $\lambda \subset S$ is a locally Borel measure L_α on each transverse arc α such that:

1. If $\alpha' \subset \alpha$ is a sub-arc of α, the measure $L_{\alpha'}$ is the restriction of L_α;
2. the support of L_α is $\alpha \cap \lambda$;
3. the measure is invariant through isotopies of transverse arcs.

Definition (Measured Geodesic Lamination)

A measured geodesic lamination is a geodesic lamination equipped with a transverse measure.

A lamination λ formed by a finite set of disjoint closed geodesics $\gamma_1, \ldots, \gamma_k$ has a natural transverse measure: for any transverse arc α, the measure L_α on α is just the Dirac measure supported in $\alpha \cap \lambda$. More generally, we may assign a positive weight $a_i > 0$ at each γ_i and define a measured geodesic lamination by giving the weight a_i at each intersection $\alpha \cap \gamma_i$. By varying weights we get distinct measured laminations with the same support.

$$\mathscr{S} \subset \mathscr{M} \subset \mathscr{ML} \subset \mathscr{C}$$

Simple closed curves $\mathscr{S}$, multicurves $\mathscr{M}$, measured geodesic laminations $\mathscr{ML}$, current space $\mathscr{C}$.

Measured Geodesic Lamination

The measured geodesic laminations can be represented as colorings (weights) on traintracks.

Theorem

The following homemorphism holds

$$\partial Teich(S_g) = \mathbb{P}\mathscr{M}\mathscr{L} \cong \mathbb{S}^{6g-7}.$$

The set $\mathbb{P}\mathscr{S}$ *is dense in PML.*

Trichotomy

Let S_g have genus $g \geq 2$. The mapping class group $MCG(S_g)$ acts natually on the whole space $\mathscr{C}$ of currents in particular on the compactification $\overline{\mathrm{Teich}(S_g)} \cong D^{6g-6}$ of the Teichmüller space.

Definition (Trichotomy)

Let $\varphi \in \mathrm{MCG}(S_g)$ be a non-trivial element. By Brouwer's fixed point theorem, φ fixes at least one point in $\overline{\mathrm{Teich}(S_g)}$. We say that φ is:

1. *finite order* if it fixes a hyperbolic mertric $m \in \mathrm{Teich}(S_g)$;
2. *reducible* if it fixes a multicurve $\mu \in \mathscr{M}$;
3. pseudo-Anosov in all the other cases.

Finite order elements

Proposition

A non-trivial element $\varphi \in MCG(S_g)$ is finite order if and only if it has indeed finite order in $MCG(S_g)$.

Suppose that φ preserves the isotopy class $[m] \in \text{Teich}(Sg)$ of a hyperbolic metric m in S_g. We can choose a representative φ that fixes m. Since the isometry group of a closed hyperbolic manifold is finite, then $\varphi^n = \text{id}$.

Reducible elements

If φ fixes a multicurve μ, one can cut S_g along μ and look at the restriction of φ to the resulting pieces: after extending all the theory to surfaces with boundary, we can hence study inductively each piece.
The cases (1) and (2) are not exclusive: there are isometries of hyperbolic surfaces that preserve some multicurve. Both cases are necessary, however: there are finite order elements that are not reducible and reducible maps that are not of finite order.

Pseudo-Anosov Elements

Theorem

Let $\varphi \in MCG(S_g)$ be a pseudo-Anosov element. There are two measured geodesic laminations $\mu_s, \mu_u \in \mathscr{M}\mathscr{L}$ and a real number $\lambda > 1$ such that

$$\varphi(\mu_s) = \lambda\mu_s, \quad \varphi(\mu_u) = \frac{1}{\lambda}\mu_u.$$

The laminations μ_s and μ_u are full and minimal, and they altogether fill S_g.

By Brouwer's fixed point theorem, a pseudo-Anosov element φ has a fixed point in $\overline{\text{Teich}(Sg)}$ which is (by definition) neither a metric nor a multicurve. Therefore φ fixes a measured projective lamination $[\mu] \in \mathbb{P}\mathscr{M}\mathscr{L}$ which is minimal and full.

Pseudo-Anosov Elements

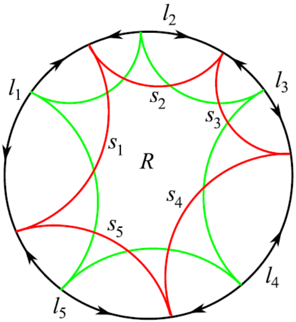

Figure: The appropriate lift of φ to $\mathbb{H}^2$ acts on $\partial\mathbb{H}^2$ with $2k$ fixed points that are alternatively attractive and repelling. By joining the repelling points we find another lamination μ_u fixed by φ.

Consider the preimage $\tilde{\mu}_s \subset \mathbb{H}^2$ of μ_s, and after replacing φ with a finite power we may choose a lift $\tilde{\varphi}$ of φ that fixes a complementary pologonal region R of $\tilde{\mu}_s$ and its sides, hence in particular its vertices of R. The k vertices of R divide $\partial\mathbb{H}^2$ into $\tilde{\varphi}$-invariant arcs $l_1, \dots, l_k$, corresponding to the sides $s_1, \dots, s_k$. Since the endpoints p and q of l_i are attractors, $\tilde{\varphi}$ fixes at least one point in the interior of l_i, which must be repulsive. The closure of the projection of the k green lines is another invariant geodesic lamination μ_u.