Chapter 2 Convex sets

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Table of contents

Affine and convex sets

Elementary examples

Operations preserving convexity

three groups of concepts

affine combination	convex combination	conic combination
affine set	convex set	convex cone
affine hull	convex hull	conic hull

Convex combination

convex combination of $x_1, \cdots, x_k \in \mathbb{R}^n$: points of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k, \quad \text{where} \quad \theta_1, \cdots, \theta_k \geq 0 \quad \text{and} \quad \theta_1 + \cdots + \theta_k = 1$$

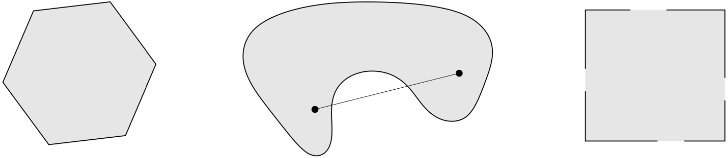
line segment between x_1 and x_2: the set of all convex combinations of x_1 and x_2

$$\{x = \theta x_1 + (1 - \theta) x_2 \mid 0 \leq \theta \leq 1\}$$

Convex set

convex set: $C \subseteq \mathbb{R}^n$ is convex if contains line segment between any pair of points in C

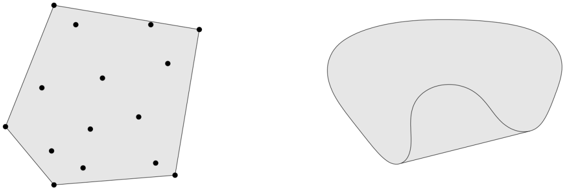
examples (one convex, two nonconvex)



Convex hull

convex hull of $C \subseteq \mathbb{R}^n$: the set of all convex combinations of points in C

$$\mathbf{conv}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C;\ \theta_1, \cdots, \theta_k \geq 0;\ \theta_1 + \cdots + \theta_k = 1\}$$



facts

- the convex hull of C is the smallest convex set containing C
- if C is a convex set, then $\mathbf{conv}\, C = C$

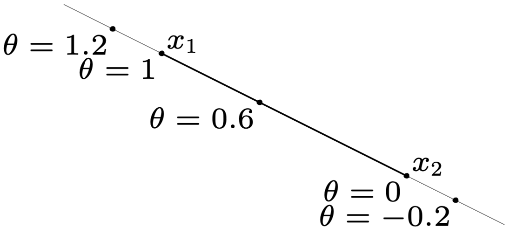
Affine combination

affine combination of $x_1, \cdots, x_k \in \mathbb{R}^n$: points of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k, \quad \text{where} \quad \theta_1 + \cdots + \theta_k = 1$$

line through x_1 and x_2: the set of all affine combinations of x_1 and x_2

$$\{x = \theta x_1 + (1 - \theta) x_2 \mid \theta \in \mathbb{R}\}$$



Affine set

affine set: $C \subseteq \mathbb{R}^n$ is affine if it contains the line through any pair of points in C

example

- the solution set of linear equations $\{x \mid Ax = b\}$ is an affine set
- conversely, every affine set can be expressed as the solution set of a system of linear equations

Affine hull

affine hull of $C \subseteq \mathbb{R}^n$: the set of all affine combinations of points in C

$$\textbf{aff}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C,\ \theta_1 + \cdots + \theta_k = 1\}$$

facts

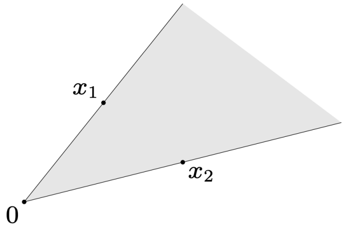
- the affine hull of C is the smallest affine set containing C
- if C is an affine set, then $\textbf{aff}\, C = C$

Conic combination

cone: $C \subseteq \mathbb{R}^n$ is a cone if $\theta x \in C$ for every $x \in C$ and $\theta \geq 0$.

conic combination of $x_1, \cdots, x_k \in \mathbb{R}^n$: points of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k, \quad \text{where} \quad \theta_1, \cdots, \theta_k \geq 0$$



Convex cone

convex cone: $C \subseteq \mathbb{R}^n$ is a convex cone if it is convex and a cone

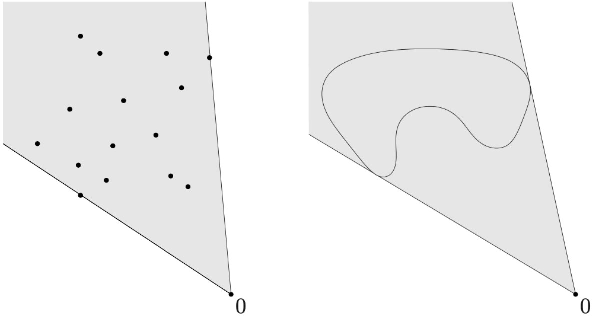
fact

C is a convex cone $\iff$ C contains all conic combinations of points in itself

Conic hull

conic hull of $C \subseteq \mathbb{R}^n$: the set of all conic combinations of points in C

$$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C;\ \theta_1, \cdots, \theta_k \geq 0\}$$



facts

- the conic hull of C is the smallest convex cone containing C
- if C is a convex cone, then its conic hull is itself

Exercises

- Study the following concepts from text:

$$\textbf{affine dimension}, \qquad \textbf{relative interior}.$$

- Suppose that $C \subseteq \mathbb{R}^n$ is convex, then $\mathbf{int}\, C$ and $\mathbf{cl}\, C$ are also convex.
- Suppose that $C \subseteq \mathbb{R}^n$ is convex, then

$$\mathbf{int}\, C = \varnothing \qquad \Longleftrightarrow \qquad C \text{ is contained in a hyperplane.}$$

- Suppose that $C \subseteq \mathbb{R}^n$ is convex, and $\mathbf{int}\, C \neq \varnothing$, then

$$\mathbf{cl}(\mathbf{int}\, C) = \mathbf{cl}\, C.$$

Affine and convex sets

Elementary examples

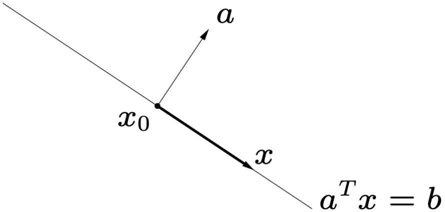
Operations preserving convexity

four types of elementary examples

- LP type: hyperplanes, halfspaces, polyhedra
- ball type: Euclidean balls, ellipsoids, norm balls
- cone type: second-order cone, norm cones
- matrix type: positive semidefinite cone

Hyperplanes

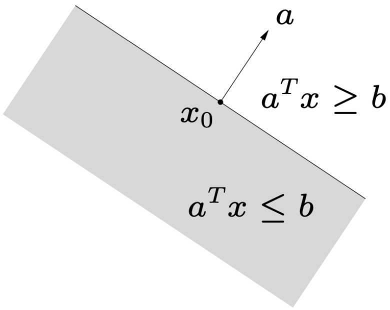
hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R})$



fact: hyperplanes are affine and convex

Halfspaces

halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R})$



fact: halfspaces are convex

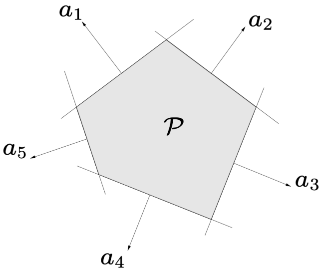
Polyhedra

polyhedron: solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \quad Cx = d$$

where $A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{p\times n}$, $\preceq$ is componentwise inequality

i.e. polyhedra are intersections of finite number of halfspaces and hyperplanes;



fact: polyhedra are convex

Euclidean balls

Euclidean ball with center x_c and radius r: two equivalent representations

- set of the form
$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\}$$
- set of the form
$$B(x_c, r) = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

fact: Euclidean balls are convex

Ellipsoids

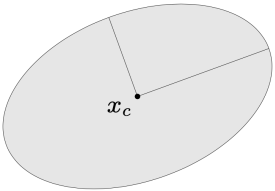
ellipsoid: two equivalent representations

- set of the form

$$\{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\} \quad \text{with} \quad P \in \mathbb{S}^n_{++}$$

- set of the form

$$\{x_c + Au \mid \|u\|_2 \leq 1\} \qquad \text{with } A \text{ square and nonsingular}$$



fact: ellipsoids are convex

Norm balls

norm ball with center x_c and radius r: set of the form

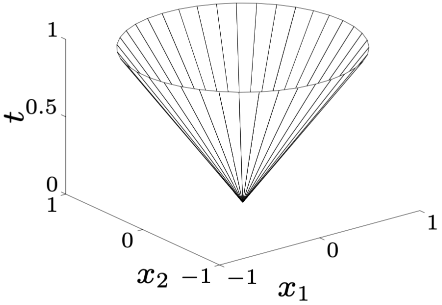
$$\{x \mid \|x - x_c\| \leq r\}$$

fact: norm balls are convex

Norm cones

second-order cone:

$$\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_2 \le t\}$$



norm cone: for any norm $\|\cdot\|$ on $\mathbb{R}^n$

$$\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| \le t\}$$

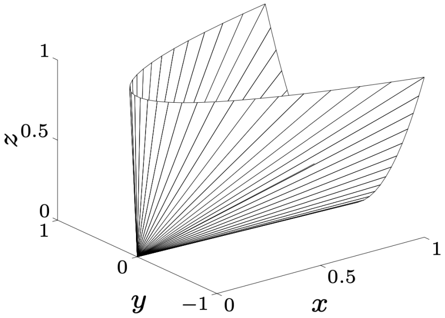
fact: norm cones are convex cones

Positive semidefinite cone

positive semidefinite cone

$$\mathbb{S}^n_+ = \{X \in \mathbb{S}^n \mid X \succeq 0\}$$

fact: positive semidefinite cone $\mathbb{S}^n_+$ is a convex cone



Affine and convex sets

Elementary examples

Operations preserving convexity

Establishing convexity

Practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, 0 \leq \theta \leq 1 \qquad \Longrightarrow \qquad \theta x_1 + (1-\theta)x_2 \in C$$

2. reconstruct C from known convex sets by operations preserving convexity:
   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection

an arbitrary intersection of convex sets is convex

example

the positive semidefinite cone $\mathbb{S}^n_+$ is convex

Affine function

affine function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is of the form

$$f(x) = Ax + b \qquad \text{with } A \in \mathbb{R}^{m\times n} \text{ and } b \in \mathbb{R}^m$$

- the image of a convex set under f is convex

$$S \subseteq \mathbb{R}^n \text{ convex} \qquad \Longrightarrow \qquad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbb{R}^m \text{ convex} \qquad \Longrightarrow \qquad f^{-1}(C) = \{x \in \mathbb{R}^n \mid f(x) \in C\} \text{ convex}$$

examples

- scaling and translation: if $S \subseteq \mathbb{R}^n$ is convex, $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}^n$, then

$$\alpha S = \{\alpha x \mid x \in S\} \qquad \text{and} \qquad S + a = \{x + a \mid x \in S\}$$

  are convex

- projection: if $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then

$$T = \{x_1 \in \mathbb{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbb{R}^n\}$$

  is convex

- sum: if $S_1, S_2 \subseteq \mathbb{R}^n$ are both convex, then

$$S_1 + S_2 = \{x + y \mid x \in S_1,\ y \in S_2\}$$

  is convex

- solution set of linear matrix inequality

$$\{x \in \mathbb{R}^n \mid x_1 A_1 + \cdots + x_n A_n \preceq B\}$$

where $A_i, B \in \mathbb{S}^m$, is convex

proof
inverse image of the positive semidefinite cone under the affine function

$$f\colon \mathbb{R}^n \to \mathbb{S}^m, \qquad f(x) = B - (x_1 A_1 + \cdots + x_n A_n)$$

- hyperbolic cone

$$\left\{x \in \mathbb{R}^n \;\middle|\; x^T P x \le \left(c^T x\right)^2, c^T x \ge 0\right\}$$

where $P \in \mathbb{S}^n_+$ and $c \in \mathbb{R}^n$, is convex

proof

inverse image of the second-order cone

$$\{(z, t) \mid z^T z \le t^2, t \ge 0\}$$

under the affine function $f\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $f(x) = (P^{1/2}x, c^T x)$

Perspective function

perspective function $P\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ given by

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \mathbb{R}^n \times \mathbb{R}_{++} = \{(x,t) \mid t > 0\}$$

- images of convex sets under perspective function are convex
- inverse images of convex sets under perspective function are convex

Linear-fractional functions

linear-fractional function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ given by

$$f(x) = \frac{Ax+b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

it is the composition of an affine function g and the perspective function P, where

$$g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}$$

- images of convex sets under linear-fractional functions are convex
- inverse images of convex sets under linear-fractional functions are convex

example

$$f(x) = \frac{x}{x_1 + x_2 + 1}, \qquad \mathbf{dom}\, f = \{(x_1, x_2) \mid x_1 + x_2 + 1 > 0\}$$