Gromov-Thurston's Proof for Mostow Rigidity

David Gu

Computer Science Department
Stony Brook University

gu@cs.stonybrook.edu

August 21, 2024

Extension of Homotopies

Extension of Homotopy

Theorem (Extension of Homotopy)

Let $f : M \to N$ be a smooth homotopy equivalence between closed hyperbolic n-manifold. Every lift $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ extends to a continuous map $\tilde{f} : \overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n}$ whose restriction $\partial f : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a homeomorphism.

Quasi-isometry

Definition (Quasi-isometry)

A map $F : X \to Y$ between metric spaces is a quasi-isometry if there are two constants $C_1 > 0$, $C_2 \geq 0$ such that

$$\frac{1}{C_1} d(x_1, x_2) - C_2 \leq d(F(x_1), F(x_2)) \leq C_1 d(x_1, x_2) + C_2$$

for all $x_1, x_2 \in X$ and $d(F(X), y) \leq C_2$ for all $y \in Y$.

One may give every finitely-presented group G a canonical metric using its Cayley graph, and classify the groups using quasi-isometry.

Pseudo-isometry

Definition (Pseudo-isometry)

A map $F : X \to Y$ between metric spaces is a pseudo-isometry if there are two constants $C_1, C_2 > 0$ such that

$$\frac{1}{C_2} d(x_1, x_2) - C_2 \leq d(F(x_1), F(x_2)) \leq C_1 d(x_1, x_2)$$

for all $x_1, x_2 \in X$ and $d(F(X), y) \leq C_2$ for all $y \in Y$.

A pseudo-isometry is C_1-Lipschitz and hence continuous. Let $f : M \to N$ be a smooth map between Riemannian manifolds; the *maximum dilatation* of f at a point $x \in M$ is the maximum $|df_x(v)|/|v|$ where v varies among all the unit tangent vectors. The maximum dilatation of f is the supremum of all maximum dilataions as $x \in M$ varies.

Maximum Dalataion

Definition (Maximum Dalatation)

Let $f : M \to N$ be a smooth map between Riemannian manifolds; the *maximum dilatation* of f at a point $x \in M$ is the maximum $|df_x(v)|/|v|$ where v varies among all the unit tangent vectors. The maximum dilatation of f is the supremum of all maximum dilataions as $x \in M$ varies.

Suppose $f : M \to N$ has the maximum dilataion C, the map f is C-Lipschitz.

pseudo-isometry

Proposition

Let $f : M \to N$ be a smooth homotopy equivalence of closed hyperbolic n-manifolds. The lift $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ is a pseudo-isometry.

Proof.

Since M is compact, the map f has finite maximum dilatation C. Since $\tilde{f}$ is locally like f, it also has maximum dilatation C and is hence C-Lipschitz. The same holds for the homotopy inverse $g : N \to M$. Therefore, there is a $C_1 > 0$ such that

$$d(\tilde{f}(x_1), \tilde{f}(x_2)) \leq C_1 \cdot d(x_1, x_2) \quad \forall x_1, x_2 \in \mathbb{H}^n$$
$$d(\tilde{g}(y_1), \tilde{g}(y_2)) \leq C_1 \cdot d(y_1, y_2) \quad \forall y_1, y_2 \in \mathbb{H}^n$$

□

pseudo-isometry

Proof.

Let $M = \mathbb{H}^n/\Gamma$. The map $\tilde{g} \circ \tilde{f}$ commutes with Γ and has maximum displacement bounded by some $K > 0$, equal to the maximum displacement of the points in a compact Dirichlet domain. Hence

$$d(x_1, x_2) - 2K \leq d(\tilde{g}(\tilde{f}(x_1)), \tilde{g}(\tilde{f}(x_2))) \leq C_1 \cdot d(\tilde{f}(x_1), \tilde{f}(x_2))$$

for all $x_1, x_2 \in \mathbb{H}^n$. Therefore $\tilde{f}$ is a pseudo-isometry with $C_2 = 2K/C_1$. □

Boundary Extension of Pseudo-isometry

Lemma

Let $F : H^n \to \mathbb{H}^n$ *be a pseduo-isometry. There is a* $R > 0$

$$F(\overline{pq}) \subset N_R(\overline{F(p)F(q)})$$

for all distinct points $p, q \in \mathbb{H}^n$*, where* $\overline{pq}$ *represents the segment with end points* p *and* q*,* $N_r(A)$ *the* r*-neighborhood of* A*.*

Lemma

Let $F : \mathbb{H}^n \to \mathbb{H}^n$ *be a pseudo-isometry. There is a* $R > 0$ *such that for all* $p \in \mathbb{H}^n$ *and every half-line* l *starting from* p *there is a unique half-line* l' *starting from* $F(p)$ *such that*

$$F(l) \subset N_R(l').$$

Boundary Extension of Pseudo-isometry

$\partial\mathbb{H}^n$ model

Since $\partial\mathbb{H}^n$ is an equivalence relation of half-lines

$$\{Half\ \ lines\ \ from\ \ O\}/\sim\quad\cong\partial\mathbb{H}^n,$$

where $l_1\sim l_2$ if and only if there is a constant $M>0$, s.t. $d(l_1(t),l_2(t))<M$ for any $t\in(0,\infty)$.

Construction of $\partial F:\partial\mathbb{H}^n\to\partial\mathbb{H}^n$

For each point $x\in\partial\mathbb{H}^n$, choose a half line $l\subset\mathbb{H}^n$, $[l]=x$, find a half-line $l'\subset N_R(F(l))$ to approximate $F(l)$, then define the boundary map

$$\partial F([l])=[l']$$

Boundary Extension of Pseudo-isometry

Lemma

The boundary extension $\partial F : \partial \mathbb{H}^n \to \partial \mathbb{H}^n$ is well-defined and injective.

Proof.

Let l_1, l_2 be two half-lines at bounded distance $d(l_1(t), l_2(t)) < M$ for all t. If $d(l_1'(t), l_2'(t)) \to \infty$, $F(l_k) \subset N_R(l_k')$, we get $d(F(l_1(t)), F(l_2(t))) \to \infty$, contradiction since F is Lipschitz. Therefore l_1', l_2' are at bounded distance and F is well-defined.
Injectivity is proved analogously: if l_1 and l_2 are divergent then l_1' and l_2' also are because F is a pseudo-isometry. $\square$

It remains to prove that the extension $F : \overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n}$ is continuous.

Boundary Extension of Pseudo-isometry

Lemma

Let $F : \mathbb{H}^n \to \mathbb{H}^n$ be a pseudo-isometry. There is a $R > 0$ such that for every line l there is a unique line l' with $F(l) \subset N_R(l')$.

Lemma

Let $F : \mathbb{H}^n \to \mathbb{H}^n$ be a pseudo-isometry. There is a $R > 0$ such that for any line l and hyperplane H orthogonal to l, the image $F(H)$ projects orthogonally to l' onto a bounded segment length smaller than R.

Boundary Extension of Pseudo-isometry

Lemma

Let $F : \mathbb{H}^n \to \mathbb{H}^n$ be a pseudo-isometry. The extension $F : \overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n}$ is continuous.

Proof.

Consider $x \in \partial\mathbb{H}^n$ and its image $F(x) \in \partial\mathbb{H}^n$. Let l be a half line pointing to x: hence l' points to $F(x)$. The half-spaces orthogonal to l' determine a neighborhood system for $F(x)$:consider one such half-space S.
The image $F(l)$ is R-close to l', hence for sufficiently big t the point $F(l(t))$ and its projection into l' lie in S at distance $> R$ from ∂S. The image $F(H(t))$ of the hyperplane $H(t)$ orthogonal to $l(t)$ is also contained in S. Hence the entire half-space bounded by one such $H(t)$ goes inside S through F. This shows that F is continuous at every point $x \in \partial\mathbb{H}^n$. $\square$

Boundary Extension of Pseudo-isometry

previous lemmas show:

Theorem

A pseudo-isometry $F : \mathbb{H}^n \to \mathbb{H}^n$ extends to a continuous map $F : \overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n}$ that injects $\partial\mathbb{H}^n$ to itself.

hence we obtain

Theorem (Extension of Homotopy)

Let $f : M \to N$ be a smooth homotopy equivalence between closed hyperbolic n-manifold. Every lift $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ extends to a continuous map $\tilde{f} : \overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n}$ whose restriction $\partial f : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a homeomorphism.

Boundary Extension of Pseudo-isometry

Proof.

We know that $\tilde{f}$ is a pseudo-isometry and hence extends to a map $\tilde{f} : \overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n}$ that sends injectively $\partial\mathbb{H}^n$ to itself.
Pick a smooth homotopic inverse g for f. The homotopy $id_M \sim g \circ f$ lifts to a homotopy $id_{\mathbb{H}^n} \sim \tilde{g} \circ \tilde{f}$ for some lift $\tilde{g}$. Since M is compact, the latter homotopy moves every point at uniformly bounded distance and hence $\tilde{g} \circ \tilde{f}$ extends continuously to the identity on $\partial\mathbb{H}^n$, and the same holdes for $\tilde{f} \circ \tilde{g}$. Therefore ∂g is the inverse of ∂f on $\partial\mathbb{H}_n$, both of them are homeomorphisms. $\square$

Mostow Rigidity

Lobachevsky Function

The *Lobachevsky function* is defined as

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin t|dt.$$

The function $\log|2\sin t|$ is $-\infty$ on $\pi\mathbb{Z}$ but is integrable, hence Λ is well-defined and continuous on $\mathbb{R}$.

$$\Lambda'(\theta) = -\log|2\sin\theta|, \quad \Lambda''(\theta) = -\cot\theta.$$

Proposition

The function Λ is π-periodic. We have Lambda$(0) = \Lambda(\pi/2) = \Lambda(\pi)$. The function is strictly positive on $(0, \pi/2)$, strictly negative on $(\pi/2, \pi)$, and has absolute maximum and minimum at $\pi/6$ and $5\pi/6$. For all $m \in \mathbb{N}$ the following holds:

$$\Lambda(m\theta) = m\sum_{k=0}^{m-1}\Lambda(\theta + \frac{k\pi}{m}).$$

Ideal Tetrahedron

Definition (Ideal Tetrahedron)

An ideal tetrahedron in $\mathbb{H}^3$ is the convex hull of four non-planar ideal points.

Proposition

For any pair of opposite edges in an ideal tetrahedron Δ there is q unique line γ orthogonal to both and Δ is symmetric with respect to a π-rotation around γ.

Proof.

The opposite edges e and e' are ultraparallel lines in $\mathbb{H}^3$ and hence have a common perpendicular γ. A π-rotation around γ inverts both e and e' but preserve the 4 ideal vertices of Δ, hence Δ itself. $\square$

Hyperbolic Ideal Tetrahedron

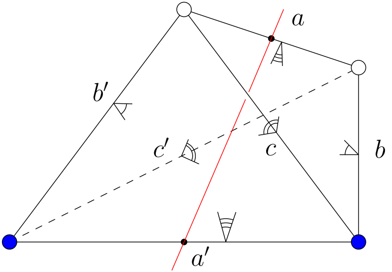

Figure: Symmetry of a hyperbolic ideal tetrahedron.

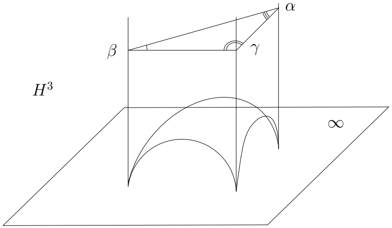

Figure: $\alpha + \beta + \gamma = \pi$.

Hyperbolic Ideal Tetrahedron

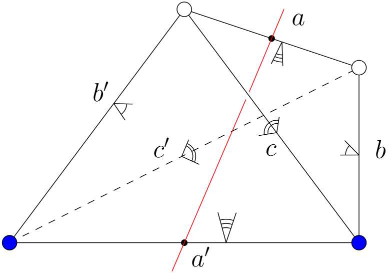

Figure: Symmetry of a hyperbolic ideal tetrahedron.

Theorem

Let Δ be an ideal tetrahedron with dihedral angles α, β and γ. We have

$$Vol(\Delta) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

Corollary

The regular ideal tetrahedron ($\alpha = \beta = \gamma = \pi/3$) is the hyperbolic tetrahedron of maximum volume.

Consider a topological space X and its homology with ring $\mathbb{R}$. We define the *norm* of a cycle $\alpha = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_h\alpha_h$ as follows:

$$|\alpha| = |\lambda_1| + |\lambda_2| + \cdots + |\lambda_h|.$$

Definition (Simplicial Norm)

The *norm* of a class $a \in H_k(X, \mathbb{R})$ is the infimum of the norms of its elements:

$$|a| = \inf\{|\alpha| \mid \alpha \in H_k(X, \mathbb{R}), [\alpha] = a\}.$$

Gromov's Simplicial Volume

Let M be an oriented closed connected manifold: we know that $H_n(M,\mathbb{Z}) \cong \mathbb{Z}$ and the orientation of M determines a fundamental class $[M] \in H_n(M,\mathbb{Z})$ that generates the group. Moreover, $H_n(M,\mathbb{R}) \cong \mathbb{R}$ and there is a natural inclusion

$$\mathbb{Z} \cong H_n(M,\mathbb{Z}) \hookrightarrow H_n(M,\mathbb{R}) \cong \mathbb{R}$$

hence the fundamental class $[M]$ is also an element of $H_n(M,\mathbb{R})$ and has a norm.

Definition (Simplicial Volume)

The *simplicial volume* $\|M\| \in \mathbb{R}_{\geq 0}$ of a closed oriented connected M is the norm of its fundamental class:

$$\|M\| = |[M]|.$$

Gromov's Simplicial Volume

A continuous map $f : M \to N$ between closed oriented n-manifolds induces a homomorphism $f_* : H_n(M, \mathbb{Z}) \to H_n(N, \mathbb{Z})$, and the *degree* of f is the integer $\deg f$ such that

$$f_*([M)] = \deg f \cdot [N].$$

Proposition

Let $f : M \to N$ be a continuous map between closed oriented manifolds. The following inequality holdes:

$$\|M\| \geq |deg f| \cdot \|N\|.$$

Proof.

Every description of $[M]$ as a cycle $\lambda_1\alpha_1 + \cdots + \lambda_h\alpha_h$ induces a description of $f_*([M]) = \deg f[N]$ as a cycle $\lambda_1 f \circ \alpha_1 + \cdots + \lambda_h f \circ \alpha_h$ with the same norm (or less, if there is some cancelation) $\square$

Gromov's Simplicial Volume

Corollary

If M and N are closed orientable and homotopically equivalent n-manifolds then $\|M\| = \|N\|$.

Proof.

A homotopy equivalence consists of two maps $f : M \to N$ and $g : N \to M$, $f \circ g : N \to N$ is homotopic to Id_N and $g \circ f : M \to M$ homotopic to Id_M. Both f and g have degree ± 1. □

Corollary

If M admits a continuous self-map $f : M \to M$ of degree at least 2 then $\|M\| = 0$.

Corollary

Every sphere S^n has norm zero. More generally $\|M \times S^n\| = 0$.

Gromov's Simplicial Volume

The S^2 and T^2 have simplicial volume zero.

Corollary

If $f : M \to N$ is a degree-d covering we have

$$\|M\| = d \cdot \|N\|.$$

Proof.

The cycles can be lifted and projected through the covering. Let $\alpha = \lambda_1\alpha_1 + \cdots + \lambda_h\alpha_h$ represent $[N]$ each α_i is a map $\Delta_n \to N$, which can be lifted to d distinct maps $\alpha_i^1, \ldots, \alpha_i^d : \Delta_n \to N$. The chain $\tilde{\alpha} = \sum_{ij} \lambda_i \alpha_i^j$ is a cycle in M and $f_*(\tilde{\alpha}) = d\alpha$, hence $\|M\| \leq d \cdot \|N\|$. We already know $\|M\| \geq d \cdot \|N\|$. $\square$

Gromov's Simplicial Volume

Proposition

If M is triangulated with k simplicies, then $\|M\| \leq K$.

Proof.

Up to taking a double cover we may suppose that M is oriented. The closed n-manifold M is triangulated into some simplices $\Delta_1, \dots, \Delta_k$, and we fix an orientation-preserving parameterization $s_i : \Delta \to \Delta_i$ of each. We substitute each s_i with

$$\frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} (-1)^{\text{sign}(\sigma)} s_i \circ \sigma.$$

Now $s = s_1 + \cdots + s_k$ is a fundamental cycle and $|s| = k$. $\square$

Gromov's Simplicial Volume for Seifert Manifold

Proposition (Seifert Manifold Simplicial Volume)

If M is a closed Seifert 3-manifold then $\|M\| = 0$.

Proof.

Every Seifert manifold is finitely covered by a product $S \times S^1$ (if $e = 0$) or by a bundle $(S, (1,1))$ with Euler number 1 (if $e \neq 0$). In the first case we have $\|S \times S^1\| = 0$.
In the second, if $S = S^2$ then $(S^2, (1,1)) = S^3$ and $\|S^3\| = 0$. Suppose $\chi(S) \leq 0$, there is a universal $K > 0$ such that $(S, (1,1))$ triangulates with at most $K|\chi(S)| + K$ tetrahedra and therefore
$\|(S, (1,1))\| \leq K|\chi(S)| + K$. For every $e > 0$ there is a degree-e covering $\tilde{S} \to S$ and two degree-e coverings

$$(\tilde{S}, (1,1)) \to (\tilde{S}, (1,e)) \to (S, (1,1))$$

which compose to a degree-e^2 covering $(\tilde{S}, (1,1)) \to (S, (1,1))$. $\square$

Gromov's Simplicial Volume for Seifert Manifold

Proof.

Therefore

$$\|(S,(1,1))\| = \frac{\|(\tilde{S},(1,1))\|}{e^2} \leq \frac{K|\chi(\tilde{S})| + K}{e^2} \leq \frac{Ke|\chi(S)| + K}{e^2} \to 0.$$

□

Simplicial and Hyperbolic Volume

Theorem (Gromov Generalized Gauss-Bonnet)

Let M be a closed hyperbolic 3-manifold. We have

$$Vol(M) = \nu_3 \|M\|$$

where ν_3 is the volume of the regular ideal tetrahedron in $\mathbb{H}^3$.

Hyperbolic Model

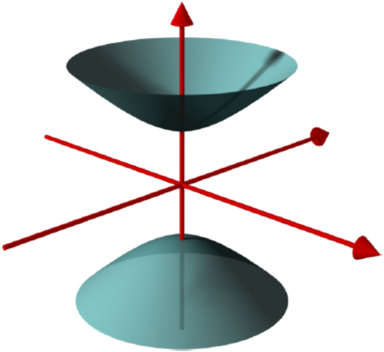

Figure: Hyperboloid Model.

Definition (Lorentzian Scalar Product)

The *Lorentzian scalar product* on $\mathbb{R}^{n+1}$ is:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}.$$

A vector $x \in \mathbb{R}^{n+1}$ is time-like, light-like or space-like if $\langle x, x \rangle$ is negative, null or positive respectively.

Hyperbolic Model

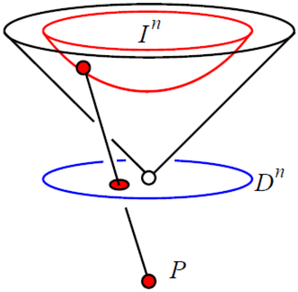

Figure: Poincaré disk Model.

Definition (Poincaré Disk)

The projection towards $P = (0, \dots, 0, -1)$ induces a bijection between the hyperboloid model I^n and the disk model D^n. The emtric tensor g at $x \in D^n$ is:

$$g(x) = \left(\frac{2}{1 - \|x\|^2}\right)^2 (dx_1^2 + dx_2^2 + \cdots + dx_n^2)$$

Hyperbolic Convex Combination

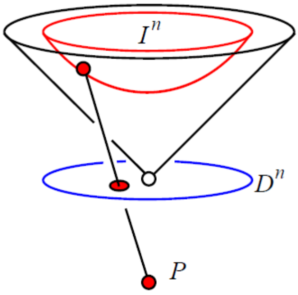

Figure: Poincaré disk Model.

Definition (Convex Combination)

Let $p_1, \dots, p_k$ be k points in I^n and $t_1, \dots, t_n$ be non-negative numbers with $t_1 + \cdots + t_k = 1$. The convex combination

$$p = t_1 p_1 + \cdots + t_k p_k$$

is another point in the space defined as

$$p = \frac{t_1 p_1 + \cdots + t_k p_k}{\| t_1 p_1 + \cdots + t_k p_k \|}$$

where $\|v\| = \sqrt{-\langle v, v \rangle}$ on I^n.

Cycle Straightening

Definition (Straight Singlar k-Simplex)

The straight singular k-simplex with vertices $v_0, \ldots, v_k \in \mathbb{H}^n$ is the map

$$\alpha : \Delta_k \to \mathbb{H}^n$$
$$(t_0, \ldots, t_k) \mapsto t_0 v_0 + \cdots + t_k v_k$$

defined using convex combinations. If the $k+1$ vertices are not contained in a $(k-1)$-plane the singular k-simplex is *non-degenerate* and its image is a hyperbolic k-simplex.

Definition (Straighening)

The *straightening* α^{st} of a singular simplex $\alpha : \Delta_k \to \mathbb{H}^n$ is the straight singular simplex with the same vertices of α. The straightening α^{st} of a singular simplex $\alpha : \Delta_k \to \mathbb{H}^n$ in a hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ is defined by lifting the singular simplex in $\mathbb{H}^n$, straightening it, and projecting it back by M by composition with the covering map.

Cycle Straightening

The straightening extends by linearity to a homomorphism

$$st : C_k(M, \mathbb{R}) \to C_k(M, \mathbb{R}),$$

which commutes with ∂ and hence induces a homomorphism in homology

$$st : H_k(M, \mathbb{R}) \to H_k(M, \mathbb{R})$$

Definition (Abstract Volume)

The abstract volume of a straightened singular n-simplex $\alpha : \Delta_n \to M$ is the volume of its lift in $\mathbb{H}^n$ and can be calculated as

$$\left| \int_\alpha \omega \right|$$

where ω is the volume form on M pulled back along α.

Simplicial Volume Theorem

Proposition

Let M be a closed hyperbolic 3-manifold. We have

$$Vol(M) \leq \nu_3 \|M\|.$$

Proof.

Take a cycle $\alpha = \lambda_1 \alpha_1 + \cdots + \lambda_k \alpha_k$ that represents $[M]$. Suppose it is straightened. We get

$$\mathrm{Vol}(M) = \int_{[M]} \omega = \int_{\alpha} \omega = \lambda_1 \int_{\alpha_1} \omega + \cdots + \lambda_k \int_{\alpha_k} \omega,$$

since $| \int_{\alpha_i} \omega| < \nu_3$, and

$$\mathrm{Vol}(M) < (|\lambda_1| + \cdots + |\lambda_k|)\nu_3.$$

This holds for all α, hence $\mathrm{Vol}(M) \leq \nu_3 \|M\|$. $\square$

Efficient Cycles

Definition (ε-efficient cycle)

Let $M = \mathbb{H}^3/\Gamma$ be a closed oriented hyperbolic 3-manifold. An *ε-efficient cycle* for M is a straightened cycle

$$\alpha = \lambda_1 \alpha_1 + \cdots + \lambda_k \alpha_k$$

representing $[M]$ where the abstract volume of α_i is bigger than $\nu_3 - \varepsilon$ and the sign of α_i is coherent with the sign of λ_i, for all i.

Lemma

If $\forall \varepsilon > 0$ the manifold M admits an ε-efficient cycle, then we have $Vol(M) \geq \nu_3 \|M\|$.

ε-Efficient Cycles

Proof.

Let $\alpha = \lambda_1\alpha_1 + \cdots + \lambda_k\alpha_k$ be an ε-efficient cycle and ω be the volume form on M. Coherence of signs gives $\lambda_i \int_{\alpha_i} \omega > 0$ for all i. We get

$$\mathrm{Vol}(M) = \int_M \omega = \lambda_1 \int_{\alpha_1} \omega + \cdots + \lambda_k \int_{\alpha_k} \omega \geq (|\lambda_1| + \cdots + |\lambda_k|)(\nu_3 - \varepsilon).$$

Therefore $\mathrm{Vol}(M) \geq \|M\|(\nu_3 - \varepsilon)$ for all $\varepsilon > 0$. $\square$

ε-Efficient Cycles

Definition (Regular Simplex $\Delta(t)$)

For any $t > 0$, let $\Delta(t)$ be a regular tetahedron obtained as follows: pick a point $x \in \mathbb{H}^3$ and a regular tetrahedron in the Euclidean tangent space $T_x\mathbb{H}^3$, centered at the origin with vertices at distance t from it, and project the vertices in $\mathbb{H}^3$ using the exponential map.

Definition (t-simplex)

A t-simplex is a tetrahedron isometric to $\Delta(t)$ equipped with an ordering of its vertices: the ordering allows us to consider it as a straightened singular simplex.

Let $S(t)$ be the set of all t-simplices in $\mathbb{H}^3$.

Haar Measure on t-Simplices

Definition (Haar Measure)

Let G be a Lie group, a non-vanishing left-invariant n-forms ω on G are in 1-1 correspondence with non-trivial n-forms on $T_e G$, since the latter are all proportional, the form ω is unique up to multiplication by a non-zero scalar.
It is a volume form and hence defines a left-invariant locally finite Borel measure on G, called *Haar measure* of G.

The Haar measure on $\mathrm{Isom}(\mathbb{H}^3)$ induces an $\mathrm{Isom}(\mathbb{H}^3)$-invariant measure on $S(t)$.

Construction of ε-Efficient Cycles

Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold and $\pi : \mathbb{H}^3 \to M$ the covering projection. Fix a base point $x_0 \in \mathbb{H}^3$ and consider its orbit $O = \Gamma x_0$. Consider the set

$$\Sigma = \Gamma^4/\Gamma$$

of the 4-tuples (g_0, g_1, g_2, g_3) considered up to the diagonal action of Γ:

$$g \cdot (g_0, g_1, g_2, g_3) = (gg_0, gg_1, gg_2, gg_3).$$

An element $\sigma = (g_0, g_1, g_2, g_3) \in \Sigma$ determines a singular simplex $\tilde{\sigma}$ in $\mathbb{H}^3$ with vertices

$$g_0(x_0), g_1(x_0), g_2(x_0), g_3(x_0) \in O$$

only up to translations by $g \in \Gamma$, hence it gives a singular simplex in M, still denoted as σ.

Construction of ε-Efficient Cycles

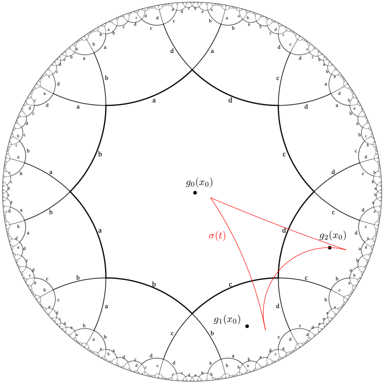

Figure: ε-efficient cycle construction, $\sigma(t)$ is a t-simplex.

Define the chain

$$\alpha(t) = \sum_{\sigma \in \Sigma} \lambda_\sigma(t) \cdot \sigma.$$

the coefficients $\lambda_\sigma(t)$ are defined in the following. For $\sigma \in \Sigma$, $\sigma = (g_0, g_1, g_2, g_3)$, we let $S_\sigma^+(t) \subset S(t)$ be the set of all positive t-simplices whose i-th vertex lies in $D(g_i(x_0))$ for all i. The number $\lambda_\sigma(t)^+$ is the measure of $S_\sigma^+(t)$. $\lambda_\sigma^-(t)$ is defined analogously, and set

$$\lambda_\sigma(t) = \lambda_\sigma^+(t) - \lambda_\sigma^-(t).$$

Construction of ε-Efficient Cycles

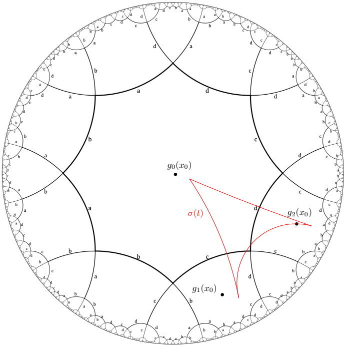

Figure: ε-efficient cycle construction, $\sigma(t)$ is a t-simplex.

Lemma

The chain $\alpha(t)$ has finitely many addenda and is a cycle. If t is sufficiently big the cycle $\alpha(t)$ represents a positive multiple of $[M]$ in the group $H_3(M, \mathbb{R})$.

Proof.

We prove that the sum is finite. Let d, T be the diameters of $D(x_0)$ and of a t-simplex. We write $\sigma = (id, g_1, g_2, g_3)$ for all $\sigma \in \Sigma = \Gamma^4/\Gamma$, i.e. all simplices have their first vertex at x_0. If $\lambda_\sigma(t) \neq 0$ then $d(g_i(x_0), x_0) < 2d + T$ for all i: therefore $\alpha(t)$ has finitely many addenda (because O is discrete). $\square$

Construction of ε-Efficient Cycles

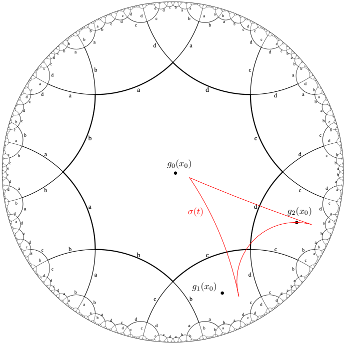

Figure: ε-efficient cycle construction, $\sigma(t)$ is a t-simplex.

Proof.

We prove that $\alpha(t)$ is a cycle. The boundary $\partial\alpha(t)$ is a linear combination of straight 2-simplices with vertices in (g_0x_0, g_1x_0, g_2x_0) as g_0, g_1 and g_2 vary. The coefficient of one such 2-simplex is

$$\sum_{g\in\Gamma}\Big(-\lambda_{(g,g_0,g_1,g_2)}(t) + \lambda_{(g_0,g,g_1,g_2)}(t) - \lambda_{(g_0,g_1,g,g_2)}(t) + \lambda_{(g_0,g_1,g_2,g)}(t)\Big)$$

$\square$

Construction of ε-Efficient Cycles

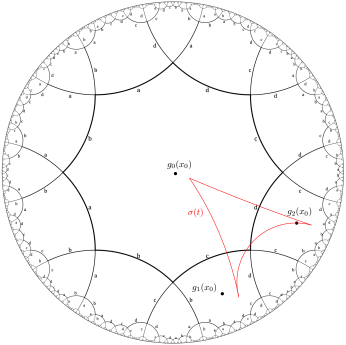

Figure: ε-efficient cycle construction, $\sigma(t)$ is a t-simplex.

Prove that each addendum summed along $g \in \Gamma$ is zero; only consider the last item

$$\sum_{g\in\Gamma} \lambda_{(g_0,g_1,g_2,g)}(t) = \sum_{g\in\Gamma} \lambda_{(g_0,g_1,g_2,g)}(t)^+ - \sum_{g\in\Gamma} \lambda_{(g_0,g_1,g_2,g)}(t)^-.$$

The 1st (2nd) addendum measures the positive (negative) t-simplices whose first 3 vertices lie in $D(g_0(x_0)), \ldots, D(g_3(x_0))$. These subsets have the same volume in $S(t)$ since they are related by the involution $r : S(t) \to S(t)$ that mirrors a simplex with respect to its first facet.

Construction of ε-Efficient Cycles

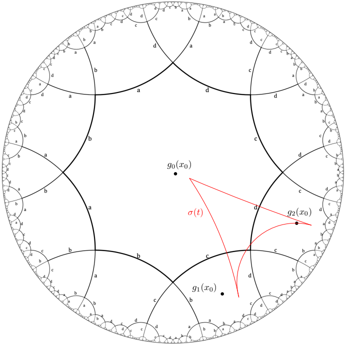

Figure: ε-efficient cycle construction, $\sigma(t)$ is a t-simplex.

Show that for sufficiently large t the cycle is a positive multiple of $[M]$. Let t be sufficient big so that two vertices in a t-simplex have distance bigger than $2d$. This implies that if there is a positive t-simplex with vertices in $D(g_0(x_0)), \dots, D(g_3(x_0))$, then any straight simplex with vertices in $D(g(x_0)), \dots, D(g_3(x_0))$ is positive. Therefore in the expression

$$\alpha(t) = \sum_{\sigma \in \Sigma} \lambda_\sigma(t) \cdot \sigma$$

the signs of $\lambda_\sigma(t)$ and σ are coherent and

$$\int_{\alpha(t)} \omega = \sum_{\sigma \in \Sigma} \lambda_\sigma(t) \cdot \int_\sigma \omega > 0.$$

Therefore $\alpha(t)$ is a positive multiple of $[M]$.

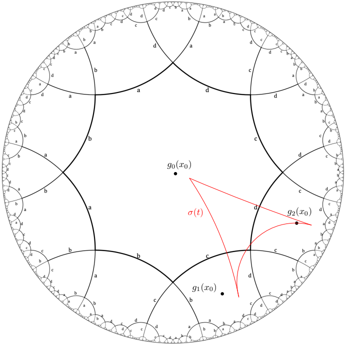

Figure: For sufficiently big t we have $\alpha(t) = k_t[M]$ in homology for some $k_t > 0$. The rescale $\bar{\alpha}(t) = \alpha(t)/k_t$ hence represents $[M]$, an ε-efficient cycle.

Lemma

For any $\varepsilon > 0$ there is a $t_0 > 0$ such that $\bar{\alpha}(t)$ is ε-efficient for all $t > t_0$.

Proof.

Let d be the diameter of the Dirichlet domain $D(x_0)$. Let a quasi t-simplex be a simplex whose vertices are at distance $< d$ from those of a t-simplex. By construction $\bar{\alpha}(t)$ is a linear combination of quasi t-simplices.
We now show that for any $\varepsilon > 0$ there is a $t_0 > 0$ such that for all $t > t_0$ every quasi t-simplex has volume bigger than $\nu_3 - \varepsilon$. By contradiction, let Δ^t be a sequence of quasi t-simplices of volume smaller than $\nu_3 - \varepsilon$ with $t \to \infty$. $\square$

Construction of ε-Efficient Cycles

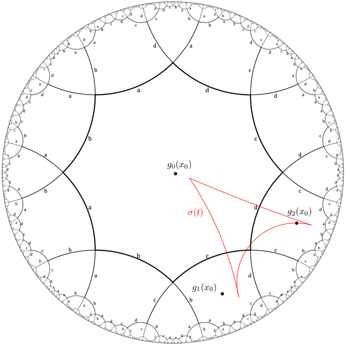

Figure: For sufficiently big t we have $\alpha(t) = k_t[M]$ in homology for some $k_t > 0$. The rescale $\bar{\alpha}(t) = \alpha(t)/k_t$ hence represents $[M]$, an ε-efficient cycle.

Proof.

The vertices of Δ^t are d-close to a t-simplex Δ^t_*, and we move the pair Δ^t, Δ^t_* isometrically so that the t-simplices Δ^t_* have the same barycenter. Now both the vertices of Δ^t and Δ^t_* tend to the vertices of an ideal regular tetrahedron in $\partial\mathbb{H}^3$, their volumes tend to ν_3. Contradiction. □

Corollary

Let M be a closed hyperbolic 3-manifold. We have

$$Vol(M) \geq \nu_3 \|M\|.$$

Simplicial Volume

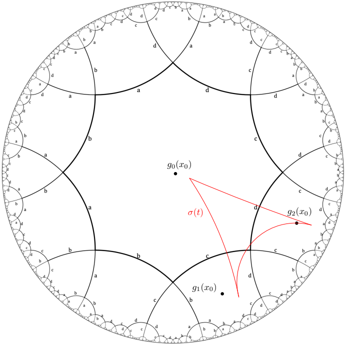

Figure: For sufficiently big t we have $\alpha(t) = k_t[M]$ in homology for some $k_t > 0$. The rescale $\bar{\alpha}(t) = \alpha(t)/k_t$ hence represents $[M]$, an ε-efficient cycle.

Theorem

Let M be a closed hyperbolic 3-manifold. We have

$$Vol(M) = \nu_3 \|M\|.$$

Corollary

Two homotopically equivalent and closed hyperbolic 3-manifolds have the same volume.

Mostow Rigidity

Theorem (Mostow Rigidity)

Let M and N be two closed connected orientable hyperbolic 3-manifolds. Every isomorphism $\pi_1(M) \xrightarrow{\sim} \pi_1(N)$ between fundamental groups is induced by a unique isometry $M \xrightarrow{\sim} N$.

Corollary

Two closed orientable hyperbolic 3-manifolds with isomorphic fundamental groups are isometric.

ε-efficient cycle

Lemma

For the ε-efficient cycle $\alpha(t) = \sum_{\sigma\in\Sigma} \lambda_\sigma(t) \cdot \sigma$ the norm

$$|\alpha(t)| = \sum_{\sigma\in\Sigma} |\lambda_\sigma(t)|$$

is independent of t for sufficiently big t.

Proof.

Let $S_0 \subset S(t)$ be the set of all t-simplices having the first vertex in the Dirichlet domain $D(x_0)$ of the fixed base point $x_0 \in \mathbb{H}^3$. It follows from the definitions that $|\alpha(t)|$ equals the measure of S_0 for surfficiently big t. Moreover the set S_0 is in natural correspondence with the set of all isometries that send x_0 to some point in $D(x_0)$: its volume doesn't depend on t. $\square$

Mostow Rigidity

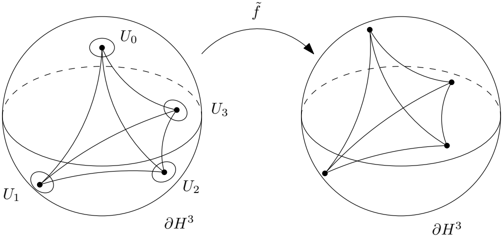

Figure: Let $f : M \to N$ be a smooth homotopy equivalence. f lifts to a map $\tilde{f} : \mathbb{H}^3 \to \mathbb{H}^3$ which extends continuously to a homeomorphism $\partial\tilde{f} : \partial\mathbb{H}^3 \to \mathbb{H}^3$.

Lemma

The extension $\partial\tilde{f} : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ sends the vertices of every regular ideal simplex to the vertices of some regular ideal simplex.

Mostow Rigidity

Proof.

Let $w_0, \ldots, w_3$ be vertices of a regular ideal simplex and suppose by contradiction that their images $\partial\tilde{f}(w_0), \ldots, \partial\tilde{f}(w_3)$ span a non-regular ideal simplex, which has volume smaller than $\nu_3 - 2\delta$ form some $\delta > 0$. There are neighbourhoods U_i of w_i in $\overline{\mathbb{H}^3}$ for $i = 0, \ldots, 3$ such that the volume of the simplex with vertices $\partial\tilde{f}(u_0), \ldots, \partial\tilde{f}(u_3)$ is smaller than $\nu_3 - \delta$ for any choice $u_i \in U_i$.

We say that a singular simplex $\sigma \in \Sigma$ is *bad* if its i-th vertex is contained in U_i for all i. Let $\Sigma^{\text{bad}} \subset \Sigma$ be the subset of all bad singular simplices and define

$$\alpha(t)^{\text{bad}} = \sum_{\sigma \in \Sigma^{\text{bad}}} \lambda_\sigma(t) \cdot \sigma.$$

We fix $g_0 \in \Gamma$ so that $D(g_0 x_0) \subset U_0$. Let $S^{\text{bad}} \subset S(t)$ be the set of all bad t-simplices with first vertex in $D(g_0 x_0)$. If t is sufficiently big, the volume of S^{bad} is bigger than a constant independent of t. $\square$

Proof.

We have proved that $|\alpha(t)^{\text{bad}}|/|\alpha(t)| > C > 0$ independent of t. We may assume that $\alpha(t)$ represents $[M]$ up to renormalization. The map $f : M \to N$ has degree one and hence sends $\alpha(t)$ to a class

$$f_*(\alpha(t)) = \sum_{\sigma\in\Sigma} \lambda_\sigma(t)\cdot (f\circ\sigma)^{\text{st}}$$

representing $[N]$. Since a C-portion of $\alpha(t)$ is bad, a C-portion of simplices in $f_*(\alpha(t))$ has volume smaller than $\nu_3 - \delta$ and hence

$$\text{Vol}(N) = \int_{f_*(\alpha(t))} \omega < |\alpha(t)|((1-C)\nu_3 + C(\nu_3-\delta)) = |\alpha(t)|(\nu_3 - \delta C).$$

Since this holds for all t and $|\alpha(t)| \to \|M\|$ we get

$$\text{Vol}(N) < \|M\|(\nu_3 - \delta C) = \text{Vol}(M) - \delta C\cdot\|M\|.$$

Contradiction to $\text{Vol}(M) = \text{Vol}(N)$. $\square$

Symmetry of Regular Ideal Tetrahedra

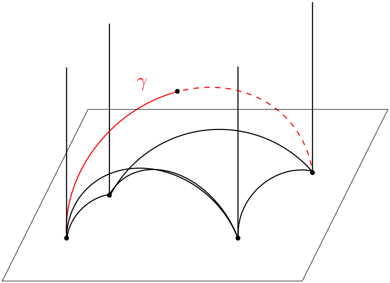

Figure: Symmetric regular ideal hyperbolic tetrahedra.

Proposition

Every ideal triangle in $\mathbb{H}^3$ is the face of precisely two regular ideal tetrahedra.

Proof.

Pick the line γ orthogonal to the barycenter of the triangle: the vertex of a regular ideal tetrahedron must be an endpoint of γ. $\square$

Mostow Rigidity

Proposition

Let $f : M \to N$ be a smooth homotopic equivalence between closed hyperbolic orientable 3-manifolds. The boundary map $\partial \tilde{f} : \partial \mathbb{H}^3 \to \partial \mathbb{H}^3$ is the trace of an isometry $\psi : \mathbb{H}^3 \to \mathbb{H}^3$.

Let $v_0, \ldots, v_3 \in \partial \mathbb{H}^3$ be vertices of a regular ideal tetrahedron Δ. The lift $\tilde{f}$ sends them to the vertices of some regular ideal tetrahedron, and let ψ be the unique isometry of $\mathbb{H}^3$ such that $\psi(v_i) = \partial \tilde{f}(v_i)$ for all i.
There is a unique point $v_4 \neq v_3$ such that v_0, v_1, v_2, v_4 are the vertices of an ideal regular tetrahedron, and $\psi(v_4)$ is the unique point other than $\psi(v_3)$ such that $\psi(v_0), \psi(v_1), \psi(v_2), \psi(v_4)$ are the vertices of an ideal regular tetrahedron. We also have $\partial \tilde{f}(v_4) = \psi(v_4)$.
If we mirror Δ along its faces iteratively we get a tessellation of $\mathbb{H}^3$ via regular ideal tetrahedra, whose ideal vertices form a dense subset of $\partial \mathbb{H}^3$.
By iterating this argument in all directions the functions ψ and $\tilde{f}$ coincide on this dense subset and hence on the whole $\partial \mathbb{H}^3$.

Mostow Rigidity

Theorem (Mostow Rigidity)

Let $f : M \to N$ be a homotopic equivalence between closed orientable hyperbolic 3-manifolds. The map f is homotopically equivalent to an isometry.

Proof.

Set $M = \mathbb{H}^3/\Gamma$ and $N = \mathbb{H}^3/\Gamma'$, and pick a lift $\tilde{f}$. We have for an isomorphism $f_* : \Gamma \to \Gamma'$, the following diagram commutes,

$$\begin{array}{ccc} \mathbb{H}^3 & \xrightarrow{\tilde{f}} & \mathbb{H}^3 \\ g \downarrow & & \downarrow f_*(g) \\ \mathbb{H}^3 & \xrightarrow{\tilde{f}} & \mathbb{H}^3 \end{array}$$

$$\tilde{f} \circ g = f_*(g) \circ \tilde{f} \quad \forall g \in \Gamma$$

Proof.

The boundary extension of $\tilde{f}$ is the trace of an isometry $\psi : \mathbb{H}^3 \to \mathbb{H}^3$ and hence

$$\begin{array}{ccc} \partial\mathbb{H}^3 & \xrightarrow{\psi} & \partial\mathbb{H}^3 \\ g\downarrow & & \downarrow f_*(g) \\ \partial\mathbb{H}^3 & \xrightarrow{\psi} & \partial\mathbb{H}^3 \end{array}$$

the following relation holds at $\partial\mathbb{H}^3$:

$$\psi \circ g = f_*(g) \circ \psi \quad \forall g \in \Gamma$$

both terms are isometries, and isometries are determined by their boundary traces: hence the relation holds for all points in $\mathbb{H}^3$. Therefore ψ descends to an isometry $\psi : M \to N$. A homotopy between f and ψ may be constructed from a convex combination of $\tilde{f}$ and ψ in $\mathbb{H}^n$, which is also Γ-equivariant and hence descends. □

Mostow Rigidity

Mostow rigidity shows that the entire geometry of a closed hyperbolic 3-manifold is a topological invariant: the volume of the manifold, the geodesic spectrum etc.

Corollary

Let M be a closed orientable hyperbolic 3-manifold. The natural map

$$Isom(M) \to Out(\pi_1(M))$$

is an isomorphism.

Proof.

The map is injective. Every automorphism of $\pi_1(M)$ is represented by a homotopy equivalence, which is in turn homotopic to an isometry by Mostow's rigidity. Hence the map is surjective. □

For a surface, $\text{Isom}(S)$ is finite and $\text{Out}(\pi_1(S))$ is infinite.