Persistent Homology

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Cech Complex

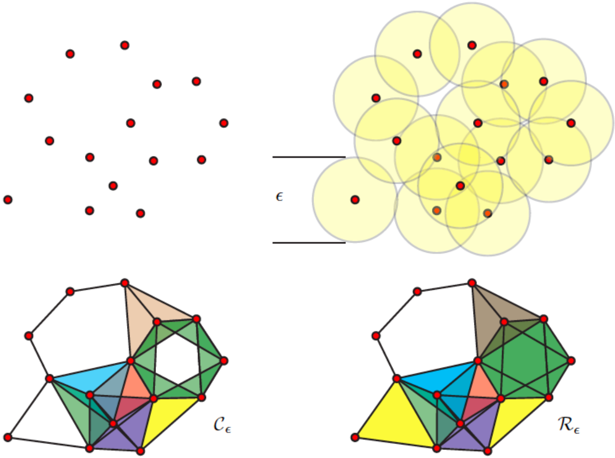


Figure: Cech complex.

Cech Complex

Definition (Cech Complex)

Given a set of points $\{x_\alpha\}$ in Euclidean space $\mathbb{R}^n$, the Cech complex (also known as the nerve), $\mathcal{C}_\varepsilon$, is the abstract simplicial complex where a set of $k+1$ vertices spans a k-simplex whenever the $k+1$ corresponding closed $\varepsilon/2$-ball neighborhoods have nonempty intersection.

Definition (Vietoris-Rips Complex)

Given a set of points $\{x_\alpha\}$ in Euclidean space $\mathbb{R}^n$, the Vietoris-Rips complex, $\mathcal{R}_\varepsilon$, is the abstract simplicial complex where a set S of $k+1$ vertices spans a k-simplex whenever the distance between any pair of points in S is at most ε.

Cech Complex

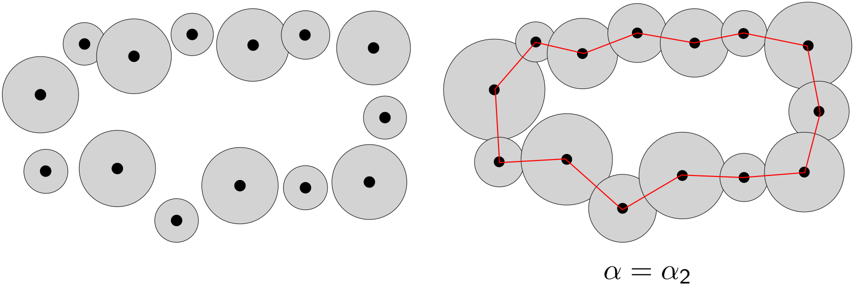


Figure: Cech complex.

Cech Complex

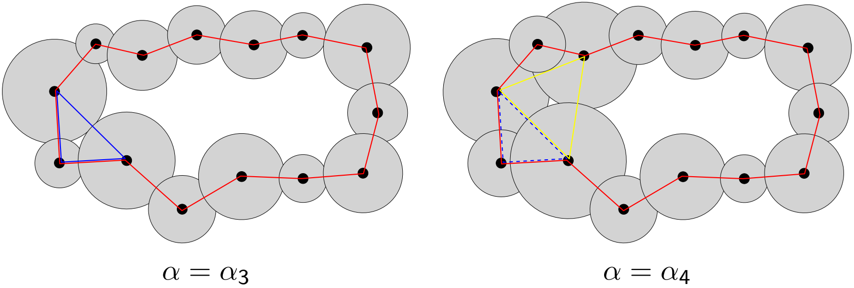
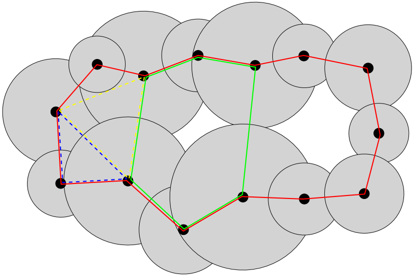
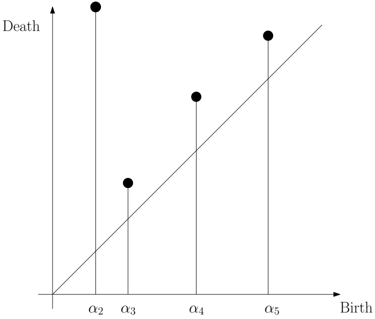


Figure: Cech complex.



$\alpha = \alpha_5$



persistent diagram

Figure: Cech complex.

Filtration

Definition (filtration)

A filtration of a simplicial complex $\mathbb{K}$ is a nested sequence of complexes,

$$\emptyset = \mathbb{K}_{-1} \subset \mathbb{K}_0 \subset \mathbb{K}_1 \subset \cdots \subset \mathbb{K}_n = \mathbb{K}.$$

Example

Suppose $\mathbb{K}$ is a simplicial complex, we sort all the simplices in a sequence

$$\sigma_1^0, \sigma_2^0, \cdots, \sigma_{n_0}^0, \sigma_1^1, \sigma_2^1, \cdots, \sigma_{n_1}^1, \sigma_1^2, \sigma_2^2, \cdots, \sigma_{n_2}^2.$$

where σ_i^k is the i-th k-simplex in $\mathbb{K}$. Then we relabel all the simplices as

$$\sigma^0, \sigma^1, \sigma^2, \cdots,$$

We define $\mathbb{K}_i$ as the union of $\sigma^0, \sigma^1, \ldots, \sigma^i$.

Homology

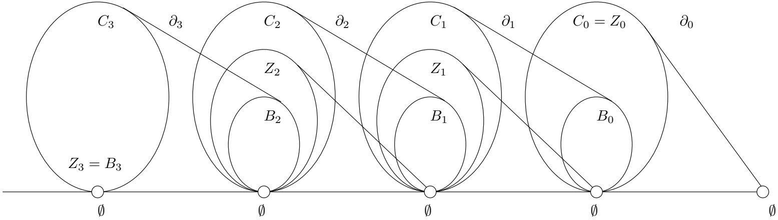


Figure: Chain, cycle, boundary groups and their images under the boundary operators.

$$H_k(\mathbb{K}, \mathbb{Z}_2) = \frac{\text{Ker}\partial_k}{\text{Img}\partial_{k+1}} = \frac{Z_k}{B_k}.$$

Persistent Homology

The inclusion map $f : \mathbb{K}_{i-1} \hookrightarrow \mathbb{K}_i$ defined by $f(x) = x$ induces a homomorphism $f_* : H_p(\mathbb{K}_{i-1}) \to H_p(\mathbb{K}_i)$. The nested sequence of complexes corresponds to a sequence of homology groups connected by the induced maps,

$$0 = H_p(\mathbb{K}_{-1}) \to H_p(\mathbb{K}_0) \to \cdots \to H_p(\mathbb{K}_n) = H_p(\mathbb{K})$$

Persistent homology studies how the homology groups change over the filtration.

Generator and Killer

Definition (positive simplex)

Given a filtration of $\mathbb{K}$, suppose $\mathbb{K}_i - \mathbb{K}_{i-1} = \sigma_i$, where σ_i is a $(k+1)$-simplex. We call σ_i is positive if it belongs to a $(k+1)$-cycle in $\mathbb{K}_i$ and negative otherwise.

A positive simplex is also called a generator, a negative simplex a killer.

Generator and Killer

Definition (Betti Number)

Given a complex K, the i-th Betti number β_i is the rank of $H_i(K)$,

$$\beta_i = \mathrm{Rank} H_i(K, \mathbb{Z}_2)$$

Suppose the number of positive k-simplexes is pos_k, and the number of negative k-simplexes is neg_k, then

$$\boxed{\beta_k = \mathrm{pos}_k - \mathrm{neg}_{k+1}}$$

Definition (Persistent Homology)

Define Z_k^l, B_k^l be the K-th cycle group and k-th boundary group respectively, of the l-complex K^l in a filtration. The p-persistent k-th homology group K^l is

$$H_k^{l,p} := \frac{Z_k^l}{B_k^{l+p} \bigcap Z_k^l}.$$

The p-persistent k-th Betti number $\beta_k^{l,p}$ of K^l is the rank of $H_k^{l,p}$.

Lemma

Consider the homomorphism $\eta_k^{l,p} : H_k^l \to H_k^{l+p}$*, then*

$$img\ \eta_k^{l,p} \cong H_k^{l,p}$$

Generator

Lemma

For each positive k-simplex σ^i, there exists a non-exact k-cycle c^i, c^i contains σ^i but no other positive k-simplices.

Proof.

Start with an arbitrary a k-cycle that contains σ^i and remove other positive k-simplices by adding their corresponding k-cycles. This method succeeds because each added cycle contains only one positive k-simplex by inductive assumption. $\square$

We use σ^i to represent c^i, and in turn the homologous class $[c^i] = c^i + B_k$.

$$\sigma^i \to c^i \to [c^i] = c^i + B_k. \quad \sigma^i \sim c^i$$

We add $[c^i]$ to the basis of $H_k(\mathbb{K}^i)$.

Generator

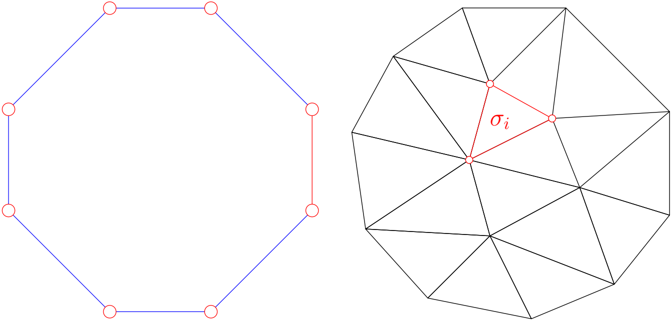


Figure: Generator, positive simplex.

Killer

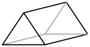
For each negative $(k+1)$-simplex σ^j, its boundary $d = \partial_{k+1}\sigma^j$ is a k-cycle, and can be represented as the linear combination of the basis of $H_k(\mathbb{K}_{j-1})$,

$$[d] = \sum_g [c^g], \{c^g\} \text{ basis } H_k(\mathbb{K}_{j-1}),$$

each $[c^g]$ is represented by a positive k-simplex σ^g, $g < j$, that is not yet paired. The collection of positive non-paired k-simplices is denoted as $\Gamma = \Gamma(d)$,

$$\Gamma(d) := \left\{ \sigma^g : [d] = \sum_g [c^g], \quad \sigma^g \sim c^g \right\}$$

Suppose the youngest positive simplex in $\Gamma(\partial_{k+1}\sigma^j)$ is σ^i, then we form the pair (σ^i, σ^j), and remove $[c^i]$ from $H_k(\mathbb{K}_j)$.
$[c^i]$ is created by σ^i and killed by σ^j, the persistence life of the k-cycle $[c^i]$ is $j - i - 1$.



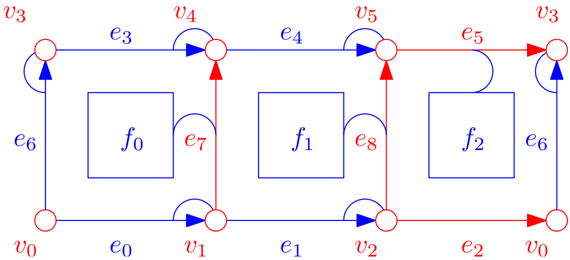


Figure: Generators and killers.

Filtration

$$v_0, v_1, v_2, v_3, v_4, v_5, e_0, e_1, e_2, e_3, e_4, e_5, f_0, f_1, f_2$$

Relabel them as

$$\sigma^0, \sigma^1, \sigma^2, \sigma^3, \sigma^4, \sigma^5, \sigma^6, \sigma^7, \sigma^8, \sigma^9, \sigma^{10}, \sigma^{11}, \sigma^{12}, \sigma^{13}, \sigma^{14}, \sigma^{15}$$

Example Generators

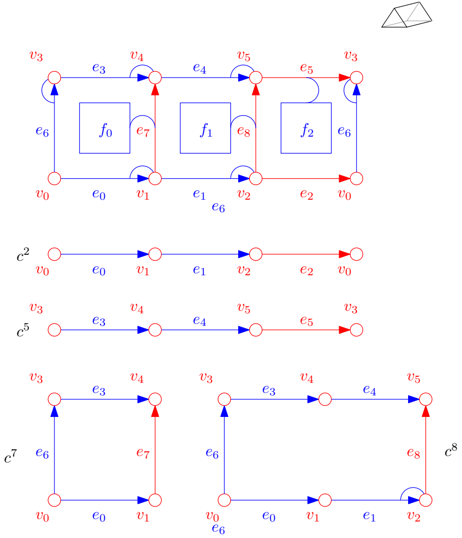


Figure: c_k contains a unique generator e_k.

Example Killers

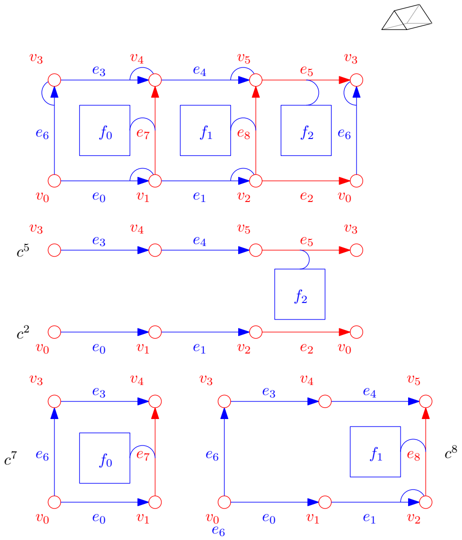
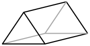


Figure: Killers.

Example Pairing



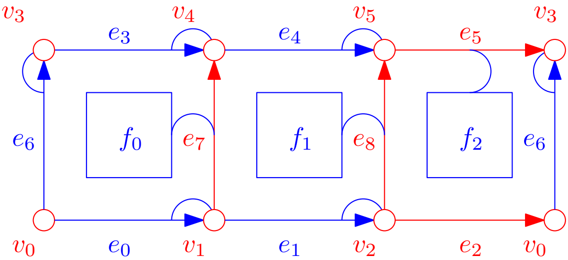


Figure: Generators and killers.

$$
\begin{aligned}
\partial_2 f_2 &= e_2 + e_5 + e_6 + e_8 = (e_5 + 2e_4 + 2e_3) + (e_2 + 2e_1 + 2e_0) + e_6 + e_8 \\
&= (e_5 + e_4 + e_3) + (e_2 + e_1 + e_0) + \partial_2(f_0 + f_1) \\
&= c_5 + c_2 + \partial_2(f_0 + f_1)
\end{aligned}
$$

Key Lemma

Definition (Collision Free Cycle)

A collision free cycle is one where the youngest positive simplex has not been paired (killed).

Lemma (Collision)

Given a filtration, $\mathbb{K}_j - \mathbb{K}_{j-1} = \sigma^j$, σ^i *is the youngest positive simplex in* $\Gamma(\partial_{k+1}\sigma^j)$. *Let* e *be a collision free* k*-cycle in* $\mathbb{K}_{j-1}$ *homologous to* $\partial_{k+1}\sigma^j$. *Suppose the youngest positive simplex in* e *is* σ^g, *then*

$$\sigma^i = \sigma^g.$$

$$\max \Gamma(\partial_{k+1}\sigma^j) = \max(e) \quad \forall e \text{ collision free}, [e] = [\partial\sigma^j].$$

Key Lemma

Proof.

Let f be the sum of the basis cycles, homologous to $d = \partial_{k+1}\sigma^j$. By definition, f's youngest positive simplex is σ^i, namely the youngest simplex in $\Gamma(\partial_{k+1}\sigma^j)$,

$$\sigma^i = \max \Gamma(\partial_{k+1}\sigma^j).$$

This implies that there are no cycles homologous to d in $\mathbb{K}_{i-1}$ or earlier complexes. Let σ^g be the youngest positive simplex in e. $[e] = [d]$, therefore $g \geq i$.

If $g > i$, then $e = f + c$, where c bounds in $\mathbb{K}^{j-1}$. $\sigma^g \notin f$,implies $\sigma^g \in c$, and as σ^g is the youngest in e, it is also the youngest in c. $\square$

Key Lemma

continued.

Since e is collision free, the cycle created by σ^g, denoted as c^g, is still a non-boundary cycle in $\mathbb{K}_{j-1}$. Hence c^g can't be c, and can't be homologous to c when c becomes a boundary. Namely, when c is killed, σ^g is not paired yet.
It follows that the negative $(k+1)$-simplex that kills c must pair a positive k-simplex in c, which is younger than σ^g, a contradiction.

This lemma shows, when σ^j is added to $\mathbb{K}_{j-1}$, we need to find any collision free cycle e homologous to $\partial_{k+1}\sigma^j$, and pair σ^j with the youngest positive simplex of e.

Pair Algorithm I

Pair(σ)

$c = \partial_p \sigma$

τ is the youngest positive $(p-1)$-simplex in c.

while τ is paired and c is not empty **do**

  find (τ, d), d is the p-simplex paired with τ;

  $c \leftarrow \partial_p d + c$

  Update τ to be the youngest positive $(p-1)$-simplex in c

end while

if c is not empty **then**

  σ is negative p-simplex and paired with τ

else

  σ is a positive p-simplex

endif

Pair Algorithm II

- Suppose the simplices are sorted as $\sigma^0, \sigma_1, \dots, \sigma^{m-1}$, the filtration is constructed as $\mathbb{K}_i - \mathbb{K}_{i-1} = \sigma^i$.
- A linear array $T[0 \dots m-1]$, a pair (σ^i, σ^j) is stored in $T[i]$; together with a list of positive simplices Λ_i defining the cycle created by σ^i and destryoed by σ^j.
- Λ_i are not necessarily the same as the ones in $\Gamma(\partial \sigma^j)$, but the sum of cycles represented by the simplices in Λ_i is homologous to $\partial \sigma^j$.
- Λ_i are generated by the Pair Algorithm II.

Pair Algorithm II

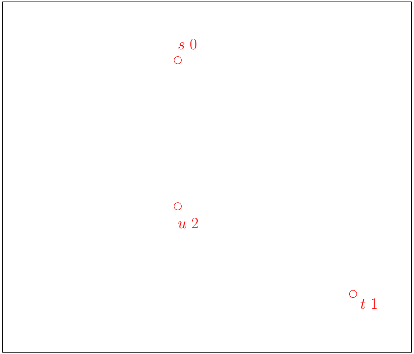
T is initialized as empty;
integer Pair(σ^j)

- $\Lambda \leftarrow \{\sigma \in \partial_{k+1}\sigma^j | \sigma \text{ positive}\}$;
- **while** Λ is not empty **do**
- $i \leftarrow \max(\Lambda)$;
- **if** $T[i]$ is unoccupied **then**
- store j and Λ in $T[i]$;
- return i; //killer
- **end if**
- $\Lambda \leftarrow \Lambda + \Lambda^i$;
- **end while**
- return ∞; //creator

Pair Algorithm II

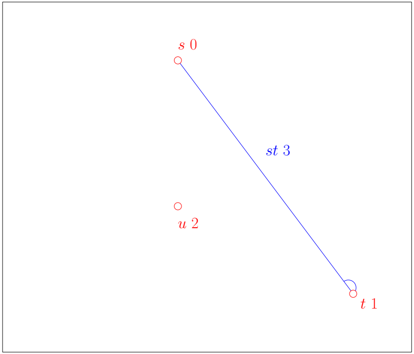
Cycle Search Halts

Observe the array T, we obtain if $\sigma^k \in \Lambda_i$, then $k \leq i$. Therefore in the while loop for cycle search, $\max(\Lambda)$ monotonously decreases, which implies that the search proceeds strictly from right to left in T. It necessarily ends at an unoccupied slot $T[g]$ of the hash table, namely Λ is a collison free cycle homologous to $\partial\sigma^j$, or Λ becomes empty, σ^j is a creator.



s	t	u															
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	∞	∞															

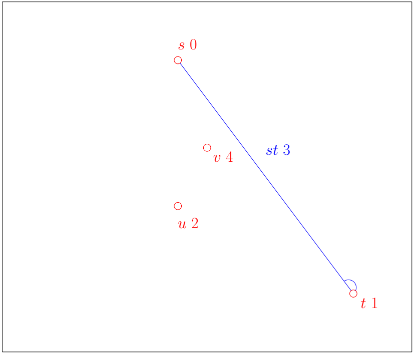
2. s_0, t_1, u_2



s	t	u	st														
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	∞															
	t																
	s																

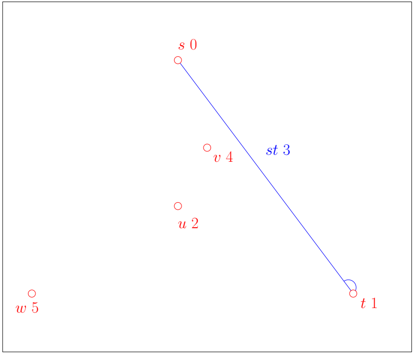
3. $\partial st_3 = s_0 + t_1$, (t_1, st_3), $\Lambda_1 = (t_1, s_0)$

Example



s	t	u	st	v													
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	∞		∞													
	t																
	s																

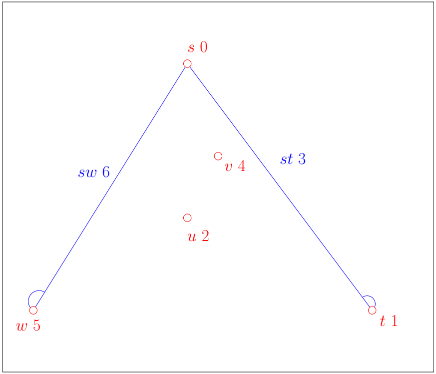
4. v_4



s	t	u	st	v	w												
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	∞		∞	∞												
	t																
	s																

5. w_5

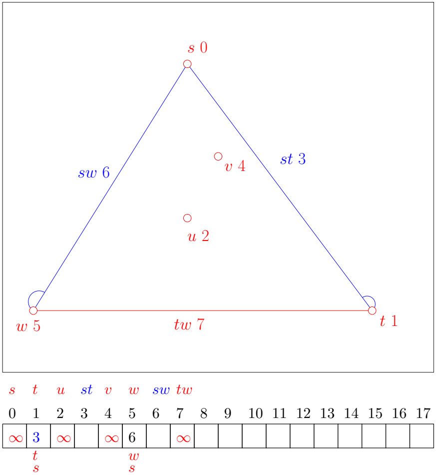
Example



s	t	u	st	v	w	sw											
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	∞		∞	6												
	t s				w s												

6. $\partial sw_6 = s_0 + w_5$, (w_5, sw_6),
$\Lambda_5 = (w_5, s_0)$

Example



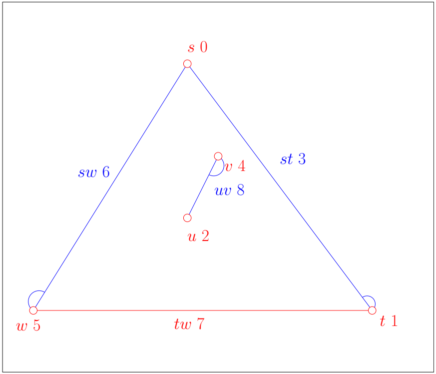
Alg 1.

$$
\begin{aligned}
\partial tw_7 &= w_5 + t_1 = w_5 + t_1 + \partial sw_6 \\
&= w_5 + t_1 + (s_0 + w_5) \\
&= t_1 + s_0 + \partial st_3 \\
&= t_1 + s_0 + (t_1 + s_0) \\
&= 0.
\end{aligned}
$$

Alg 2.

$$
\begin{aligned}
\partial tw_7 &= w_5 + t_1 = w_5 + t_1 + \Lambda_6 \\
&= w_5 + t_1 + (s_0 + w_5) \\
&= t_1 + s_0 + \Lambda_3 \\
&= t_1 + s_0 + (t_1 + s_0) \\
&= 0.
\end{aligned}
$$

Example

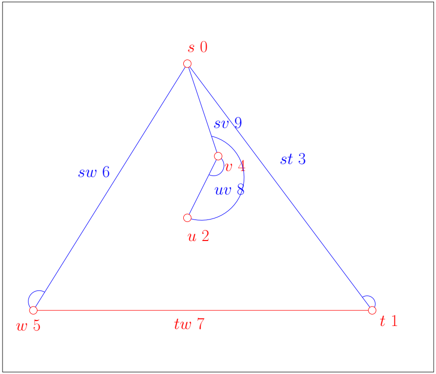


s	t	u	st	v	w	sw	tw	uv									
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	∞		8	6		∞										
	t_1			v_4	w_5												
	s_0			u_2	s_0												

$$8. \partial uv_8 = u_2 + \boxed{v_4}, (v_4, uv_8)$$

$$\Lambda_4 = (v_4, u_2)$$

Example



s	t	u	st	v	w	sw	tw	uv	sv								
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		∞										
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0												

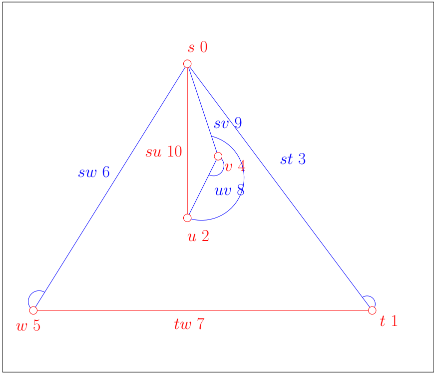
9. (u_2, sv_9) Alg. 1

$$
\begin{aligned}
\partial sv_9 &= s_0 + v_4 \\
&= s_0 + v_4 + \partial uv_8 \\
&= s_0 + v_4 + (u_2 + v_4) \\
&= s_0 + \boxed{u_2} \\
\Lambda_2 &= (s_0 + u_2)
\end{aligned}
$$

Alg. 2

$$
\begin{aligned}
\partial sv_9 &= s_0 + v_4 \\
&= s_0 + v_4 + \Lambda_4 \\
&= s_0 + v_4 + (u_2 + v_4) \\
&= s_0 + \boxed{u_2} \\
\Lambda_2 &= (s_0 + u_2)
\end{aligned}
$$

Example



s	t	u	st	v	w	sw	tw	uv	sv	su							
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		∞			∞							
	t_1	u_2		v_4	w_5												
	s_0	s_0		u_2	s_0												

10. su_{10}

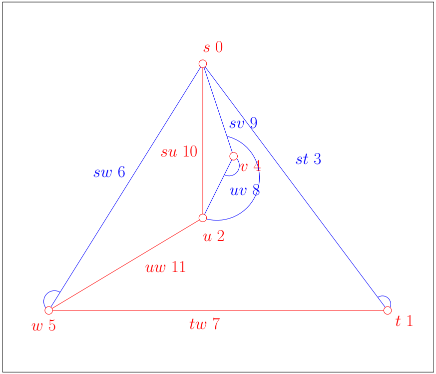
Alg. 1

$$
\begin{aligned}
\partial su_{10} &= s_0 + u_2 = s_0 + u_2 + \partial sv_9 \\
&= s_0 + u_2 + (s_0 + v_4) \\
&= u_2 + v_4 \\
&= u_2 + v_4 + \partial uv_8 \\
&= 0.
\end{aligned}
$$

Alg. 2

$$
\begin{aligned}
\partial su_{10} &= s_0 + u_2 = s_0 + u_2 + \Lambda_2 \\
&= s_0 + u_2 + (s_0 + u_2) \\
&= 0.
\end{aligned}
$$

Example

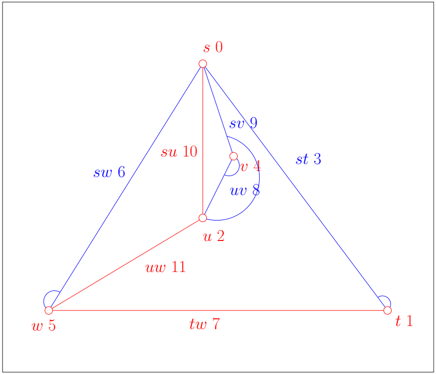


s	t	u	st	v	w	sw	tw	uv	sv	su	uw						
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		∞			∞	∞						
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0												

11. uw_{11} Alg. 1

$$
\begin{aligned}
\partial uw_{11} &= u_2 + w_5 \\
&= u_2 + w_5 + \partial sw_6 \\
&= u_2 + w_5 + (s_0 + w_5) \\
&= s_0 + u_2 + \partial sv_9 \\
&= s_0 + u_2 + (s_0 + v_4) \\
&= u_2 + v_4 + \partial uv_8 \\
&= u_2 + v_4 + (u_2 + v_4) \\
&= 0.
\end{aligned}
$$

Example

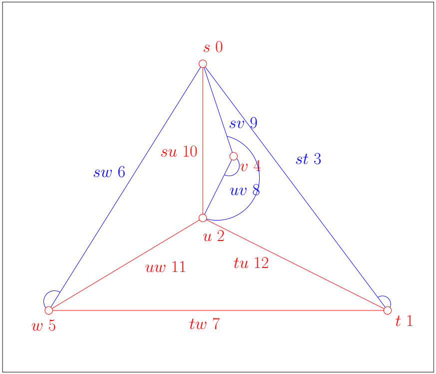


s	t	u	st	v	w	sw	tw	uv	sv	su	uw						
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		∞			∞	∞						
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0												

11. uw_{11} Alg. 2

$$
\begin{aligned}
\partial uw_{11} &= u_2 + w_5 \\
&= u_2 + w_5 + \Lambda_5 \\
&= u_2 + w_5 + (s_0 + w_5) \\
&= s_0 + u_2 + \Lambda_2 \\
&= s_0 + u_2 + (s_0 + u_2) \\
&= 0.
\end{aligned}
$$

Example



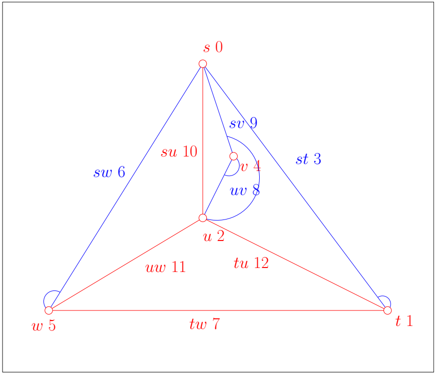
s	t	u	st	v	w	sw	tw	uv	sv	su	uw	tu					
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		∞			∞	∞	∞					
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0												

12. tu_{12}

Alg. 1

$$
\begin{aligned}
\partial tu_{12} &= t_1 + u_2 = t_1 + u_2 + \partial sv_9 \\
&= t_1 + u_2 + (s_0 + v_4) \\
&= t_1 + u_2 + s_0 + v_4 + \partial uv_8 \\
&= t_1 + u_2 + s_0 + v_4 + (u_2 + v_4) \\
&= t_1 + s_0 \\
&= s_0 + t_1 + \partial st_3 \\
&= s_0 + t_1 + (t_1 + s_0) \\
&= 0.
\end{aligned}
$$

Example

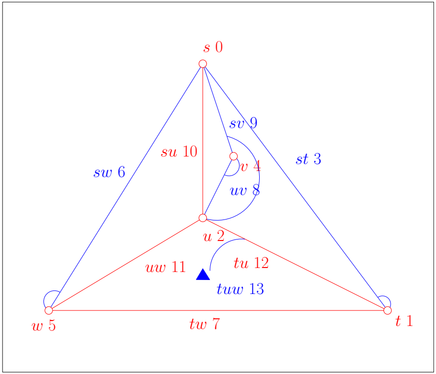


s	t	u	st	v	w	sw	tw	uv	sv	su	uw	tu					
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		∞			∞	∞	∞					
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0												

12. tu_{12}

Alg. 2

$$
\begin{aligned}
\partial tu_{12} &= t_1 + u_2 = t_1 + u_2 + \Lambda_2 \\
&= t_1 + u_2 + (s_0 + u_2) \\
&= t_1 + s_0 + \Lambda_1 \\
&= s_0 + t_1 + (t_1 + s_0) \\
&= 0.
\end{aligned}
$$



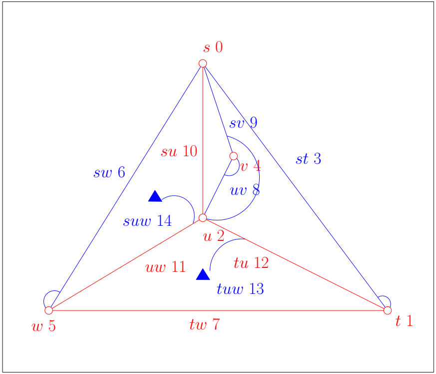
s	t	u	st	v	w	sw	tw	uv	sv	su	uw	tu	tuw				
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		∞			∞	∞	13					
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0							tu_{12} uw_{11} tw_7					

13. tuw_{13},(tu_{12}, tuw_{13})

$$\partial tuw_{13} = tu_{12} + uw_{11} + wt_7$$

$$\Lambda_{12} = (\boxed{tu_{12}}, uw_{11}, wt_7)$$

$$(tu_{12}, tuw_{13})$$



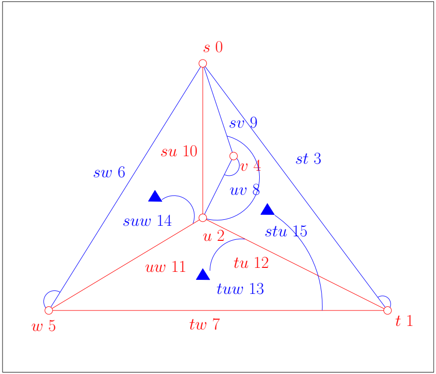
s	t	u	st	v	w	sw	tw	uv	sv	su	uw	tu	tuw	suw			
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		∞			∞	14	13					
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0						uw su	tu uw tw					

14. suw_{14}

$$\partial suw_{14} = uw_{11} + su_{10} + sw_6$$

$$\Lambda_{12} = (\boxed{uw_{11}}, su_{10})$$

$$(uw_{11}, suw_{14})$$

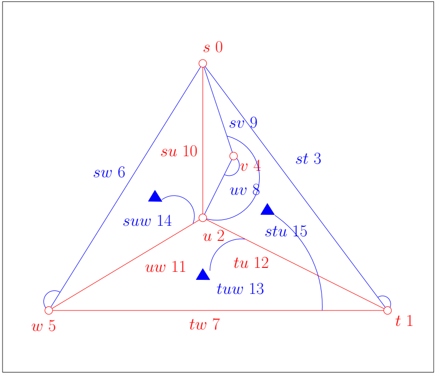


s	t	u	st	v	w	sw	tw	uv	sv	su	uw	tu	tuw	suw	stu		
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		15			∞	14	13					
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0		tw			uw su	tu uw tw						

15. stu,(tw_7, stu_{15}),$\Lambda_7 = (tw_7)$
Alg. 1

$$
\begin{aligned}
&\partial stu_{15}\\
=&su_{10} + tu_{12} + st_3\\
=&su_{10} + st_3 + tu_{12} + \partial tuw_{13}\\
=&su_{10} + st_3 + tu_{12} +\\
&(tu_{12} + uw_{11} + tw_7)\\
=&su_{10} + st + uw_{11} + tw_7\\
=&su_{10} + st_3 + uw_{11} + tw_7 + \partial suw_{14}\\
=&su_{10} + st_3 + uw_{11} + tw_7 +\\
&(sw_6 + su_{10} + uw_{11})\\
=&st_3 + \boxed{tw_7} + sw_6
\end{aligned}
$$

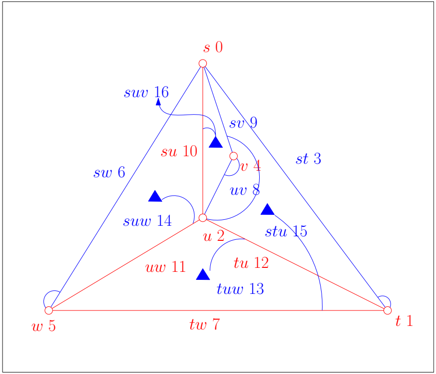
Example



s	t	u	st	v	w	sw	tw	uv	sv	su	uw	tu	tuw	suw	stu		
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		15			∞	14	13					
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0		tw				uw su	tu uw tw					

15. stu,(tw_7, stu_{15}),$\Lambda_7 = (tw_7)$
Alg. 2

$$
\begin{aligned}
&\partial stu_{15} \\
=&su_{10} + tu_{12} + st_3 \\
=&su_{10} + st_3 + tu_{12} + \Lambda_{12} \\
=&su_{10} + st_3 + tu_{12} + \\
&(tu_{12} + uw_{11} + tw_7) \\
=&su_{10} + st + uw_{11} + tw_7 \\
=&su_{10} + st_3 + uw_{11} + tw_7 + \Lambda_{11} \\
=&su_{10} + st_3 + uw_{11} + tw_7 + \\
&(su_{10} + uw_{11}) \\
=&st_3 + \boxed{tw_7}
\end{aligned}
$$



s	t	u	st	v	w	sw	tw	uv	sv	su	uw	tu	tuw	suw	stu	suv	
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		15			16	14	13					
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0		tw			su	uw su	tu uw tw					

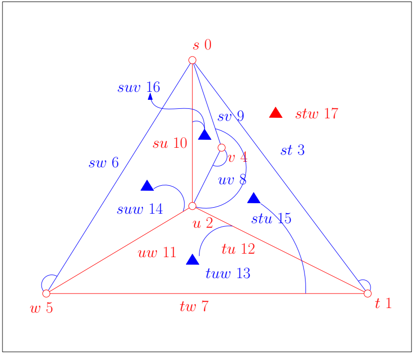
16. suv_{16}

$$\partial suv_{16} = \boxed{su_{10}} + uv_8 + sv_9$$

$$\Lambda_{10} = (su_{10})$$

$$(su_{10}, suv_{16})$$

Example



s	t	u	st	v	w	sw	tw	uv	sv	su	uw	tu	tuw	suw	stu	suv	stw
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		15			16	14	13					∞
	t_1	u_2		v_4	w_5		tw			su	uw	tu					
	s_0	s_0		u_2	s_0						su	uw					
												tw					

17. stw_{17}

Alg. 1

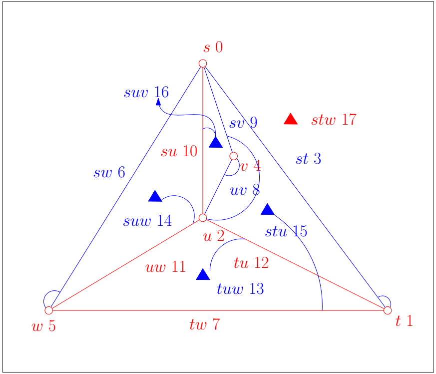
$$
\begin{aligned}
17.\ \partial stw_{17} &= tw_7 + sw_6 + st_3 \\
&= sw_6 + st_3 + tw_7 + \partial stu_{15} \\
&= sw_6 + st_3 + tw_7 + (st_3 + tu_{12} + us_{10}) \\
&= sw_6 + st_3 + tw_7 + (st + tu_{12} + us_{10}) + \partial tuw \\
&= sw_6 + st_3 + tw_7 + (st_3 + tu_{12} + us_{10}) + (tu_{12} + uw_{11} + tw_7) \\
&= sw_6 + us_{10} + uw_{11} \\
&= sw_6 + us_{10} + uw_{11} + \partial suw_{14} \\
&= sw_6 + us_{10} + uw_{11} + (su_{10} + uw_{11} + sw_6) \\
&= 0.
\end{aligned}
$$

Example

Alg. 2

$$
\begin{aligned}
17.\ \partial stw_{17} &= tw_7 + sw_6 + st_3 \\
&= sw_6 + st_3 + tw_7 + \Lambda_7 \\
&= sw_6 + st_3 + tw_7 + (tw_7) \\
&= sw_6 + st_3 \\
&= 0.
\end{aligned}
$$

Example



s	t	u	st	v	w	sw	tw	uv	sv	su	uw	tu	tuw	suw	stu	suv	stw
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
∞	3	9		8	6		15			16	14	13					∞
	t_1 s_0	u_2 s_0		v_4 u_2	w_5 s_0		tw			su	uw su	tu uw tw					

Creater	*Killer*	Λ
t_1	st_3	t_1, s_0
u_2	sv_9	u_2, s_0
v_4	uv_8	v_4, u_2
w_5	sw_6	w_5, s_0
tw_7	stu_{15}	tw_7
su_{10}	suv_{16}	t_1, su_{10}
uw_{11}	suw_{14}	$uw_{11} su_{10}$
tu_{12}	tuw_{13}	tu_{12}, uw_{11}, tw_7

The unpaired creaters are s_0 and stw_{17}.

Incidence Matrix

Assuming an ordering of the $(p-1)$ simplices and of the p-simplices, the boundary of a p-chian can be obtained by multiplication of the corresponding vector with the incidence matrix,

$$\partial(c_p) = D_p c_p.$$

The incidence matrix is defined as

$$D_p[i,j] = \begin{cases} 1 & \sigma_i^{p-1} \in \sigma_j^p \\ 0 & \sigma_i^{p-1} \notin \sigma_j^p \end{cases}$$

Incidence Matrix and Betti Number

A classic algorithm computes the Betti numbers of K by reducing its incidence matrices to Smith normal form. It uses row and column operations to zero out all entries except along an initial portion of the diagonal.

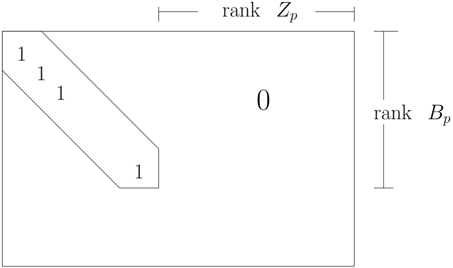


Figure: Smith norm of incidence matrix in $\mathbb{Z}_2$.

The Betti number

$$\boxed{\beta_p = \text{rank} Z_p - \text{rank} B_p}.$$

Pairing Algorithm

Definition (Monotonous Filtering)

A filtering is monotonous, if in the ordering of K, any simplex σ is proceeded by its faces.

An algorithm computes the persistence diagrams by pairing the simplices, which uses column operator to reduce D to another $0-1$ matrix R. Let $\text{low}_R(j)$ be the row index of the last 1 in column j of R, and (undefined if the column is zero).

Definition (Reduced Matrix and Pairing)

We call R reduced and low_R a pairing function, if

$$\text{low}_R(j) \neq \text{low}_R(j'),$$

whenever $j \neq j'$ specify two non-zero columns.

Pairing Algorithm

Algorithm: Incidence matrix reduction

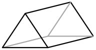
1. $R \leftarrow D$
2. **for** $j = 1$ **to** n **do**
3. **while** $\exists j' < j$ **with** $\text{low}_R(j') = \text{low}_R(j)$ **do**
4. add column j' to column j
5. **endwhile**
6. **endfor**.

The pairing is given by

$$\boxed{(\sigma_i, \sigma_j) \iff i = \text{low}_R(j)}.$$

σ_i is positive, it generates a homology class; σ_j is negative, it kills a homolog class.

Killer



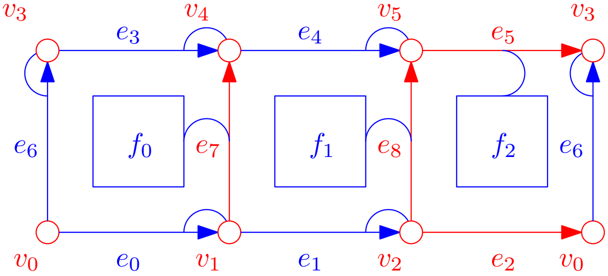


Figure: Generators and killers.

Pairing by matrix induction

Boundary operator ∂_1, incidence matrix D_1,

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_0	1	0	1	0	0	0	1	0	0
v_1	1	1	0	0	0	0	0	1	0
v_2	0	1	1	0	0	0	0	0	1
v_3	0	0	0	1	0	1	1	0	0
v_4	0	0	0	1	1	0	0	1	0
v_5	0	0	0	0	1	1	0	0	1

Pairing by matrix induction

$1+2$, $4+5$, $3+7$, $4+8$

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_0	1	0	1	0	0	0	1	0	0
v_1	1	1	1	0	0	0	0	1	0
v_2	0	1	0	0	0	0	0	0	1
v_3	0	0	0	1	0	1	1	1	0
v_4	0	0	0	1	1	1	0	0	1
v_5	0	0	0	0	1	0	0	0	0

Pairing by matrix induction

$1 + 2, 3 + 5, 3 + 8$

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_0	1	0	0	0	0	0	1	0	0
v_1	1	1	0	0	0	0	0	1	0
v_2	0	1	0	0	0	0	0	0	1
v_3	0	0	0	1	0	0	1	1	1
v_4	0	0	0	1	1	0	0	0	0
v_5	0	0	0	0	1	0	0	0	0

Pairing by matrix induction

$6+7$, $6+8$

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_0	1	0	0	0	0	0	1	1	1
v_1	1	1	0	0	0	0	0	1	0
v_2	0	1	0	0	0	0	0	0	1
v_3	0	0	0	1	0	0	1	0	0
v_4	0	0	0	1	1	0	0	0	0
v_5	0	0	0	0	1	0	0	0	0

Pairing by matrix induction

$0+7, 1+8, 0+8$

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_0	1	0	0	0	0	0	1	0	0
v_1	1	1	0	0	0	0	0	0	0
v_2	0	1	0	0	0	0	0	0	0
v_3	0	0	0	1	0	0	1	0	0
v_4	0	0	0	1	1	0	0	0	0
v_5	0	0	0	0	1	0	0	0	0

Generators e_2, e_5, e_7, e_8, corresponding to 0 columns. Killers corresponds to non-zero columns. Pairing

$$(e_0, v_1), (e_1, v_2), (e_3, v_4), (e_4, v_5), (e_6, v_3)$$

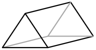
Pairing by matrix induction

$$\begin{array}{c|ccc} & f_0 & f_1 & f_2 \\ \hline e_0 & 1 & 0 & 0 \\ e_1 & 0 & 1 & 0 \\ e_2 & 0 & 0 & 1 \\ e_3 & 1 & 0 & 0 \\ e_4 & 0 & 1 & 0 \\ e_5 & 0 & 0 & 1 \\ e_6 & 1 & 0 & 1 \\ e_7 & 1 & 1 & 0 \\ e_8 & 0 & 1 & 1 \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow^{2+3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow^{1+3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The pairing is

$$(f_0, e_7), (f_1, e_8), (f_2, e_5)$$

Topological Annulus



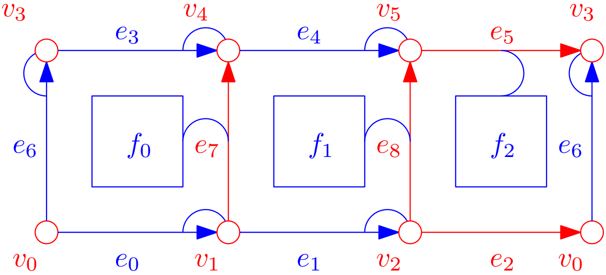
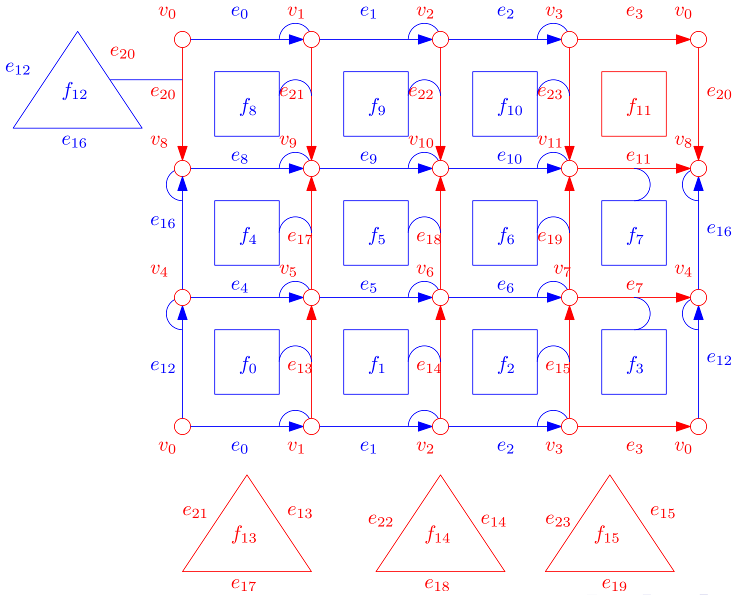
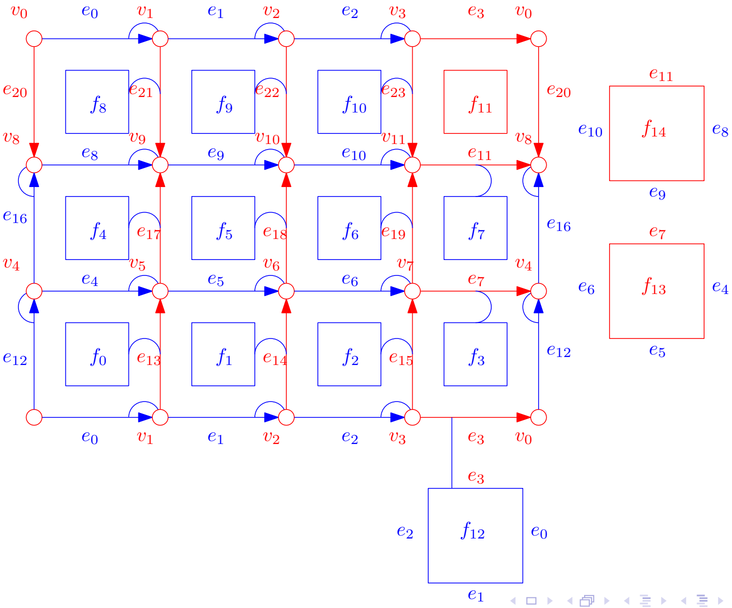


Figure: Topological Annulus. The unpaired creator is v_0 and e_2.

Topological Torus



Topological Torus



Topological Torus

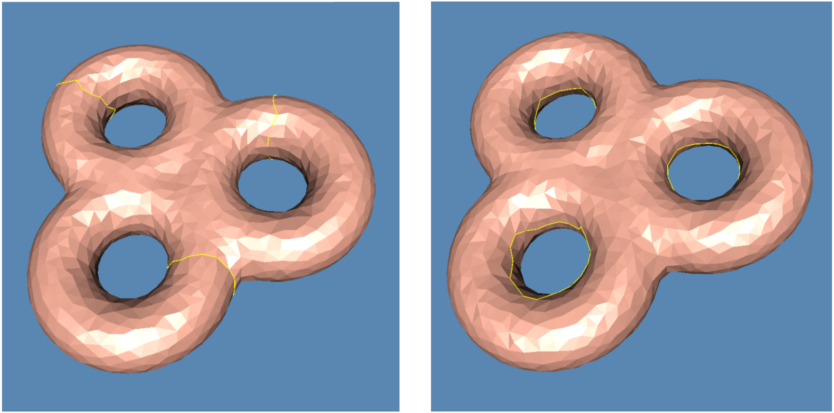


Figure: Handle and tunnel loops.

Handle Loop and Tunnel Loop

1. The simplices on the surface M are added into the filtration in any arbitrary order. Since $H_1(M)$ is of rank $2g$, the algorithm Pair generates $2g$ number of unpaired positive edges.
2. The simplices up to dimension 2 in I are added into the filtration. Since $H_1(I)$ of rank g, half of $2g$ positive edges generated in step 1 get paired with the negative triangles in I. Each pair correponds to a killed loop, these g loops are handle loops.
3. Or the simplices up to dimension 2 in O are added into the filtration. Since $H_1(O)$ of rank g, half of $2g$ positive edges genrated in step 2 get paried with the negative triangles in O. Each pair corresponds to a killed loop, these g loops are tunnel loops.

Handle Loops, Tunnel Loops

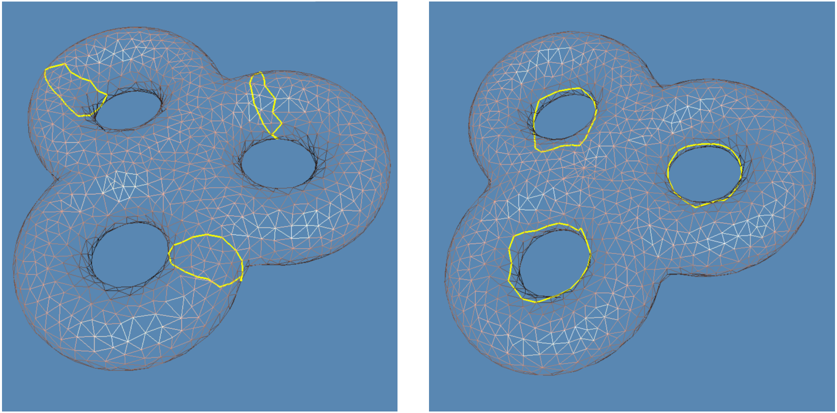


Figure: Handle and tunnel loops.

Topological Torus

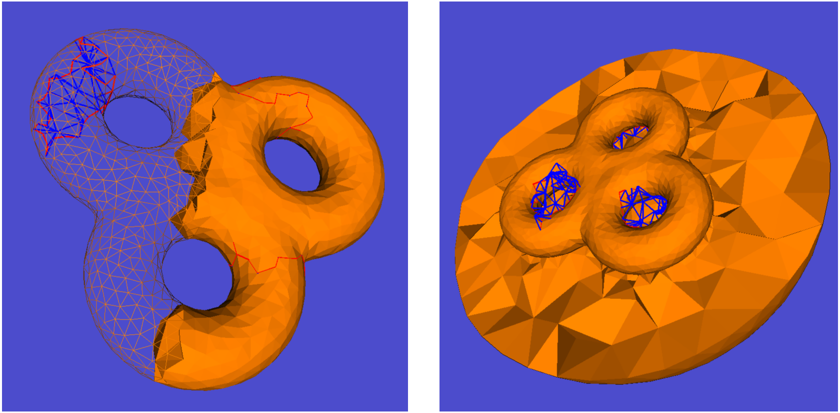


Figure: Interior and exterior volumes.

Topological Quadrilaterals

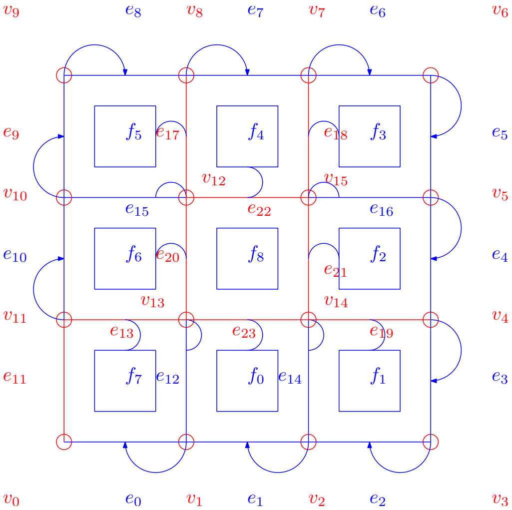
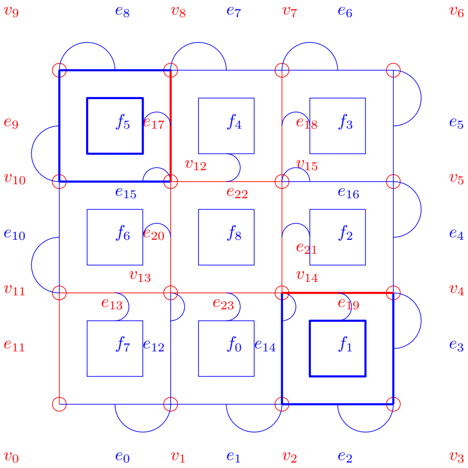


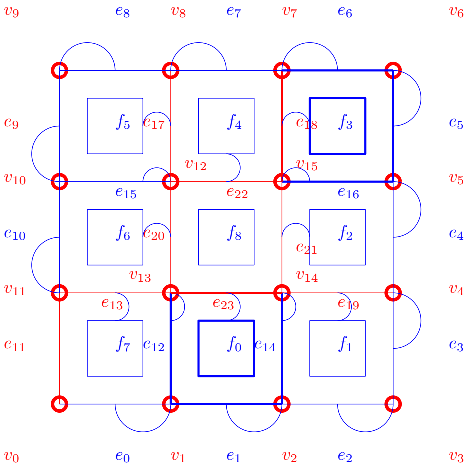
Figure: Quadrilateral example.

Topological Quadrilaterals



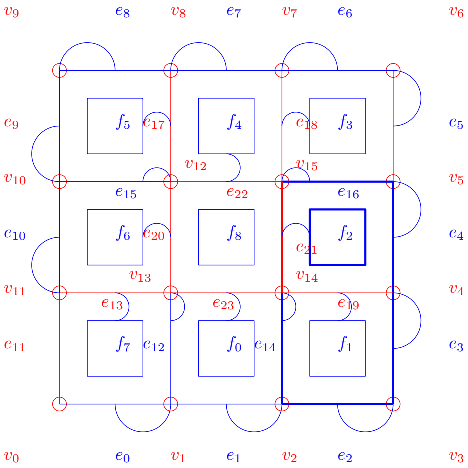
$$\partial f_5 = c_{17} = e_{15} + e_9 + e_8 + e_{17}, \ \partial f_1 = c_{19} = e_3 + e_2 + e_{14} + e_{19}.$$

Topological Quadrilaterals



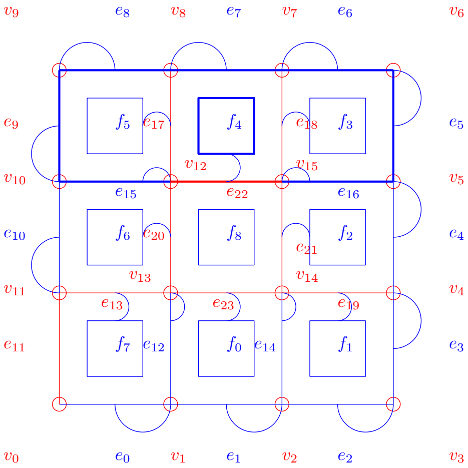
$$\partial f_3 = c_{18} = e_{16} + e_5 + e_6 + e_{18},\ \partial f_0 = c_{23} = e_{12} + e_{14} + e_1 + e_{23}.$$

Topological Quadrilaterals



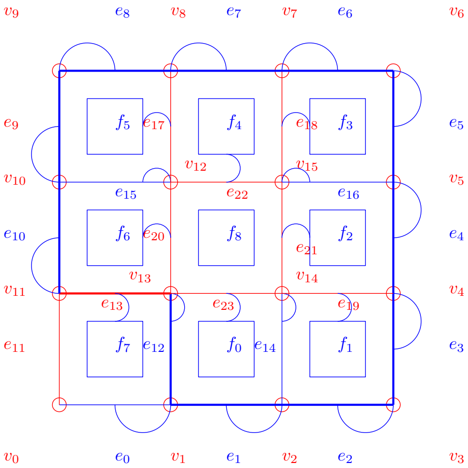
$$c_{21} = e_{16} + e_4 + e_3 + e_2 + e_{14} + e_{21},\ \partial f_2 = c_{19} + c_{21}$$

Topological Quadrilaterals



$$c_{22} = e_{15} + e_4 + e_{16} + e_9 + e_5 + e_8 + e_7 + e_6 + e_{22}, \ \partial f_4 = c_{22} + c_{18} + c_{17}$$

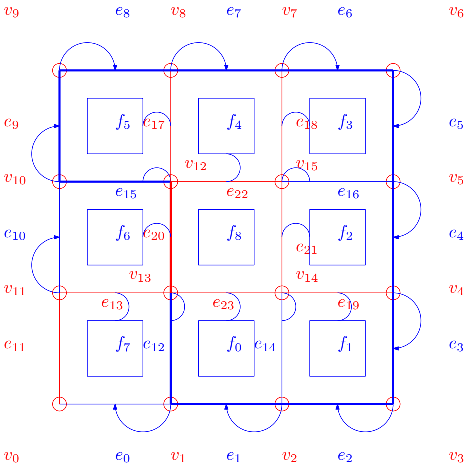
Topological Quadrilaterals



$c_{13} = e_{12} + e_{10} + e_9 + e_8 + e_7 + e_6 + e_5 + e_4 + e_3 + e_2 + e_1 + e_{13}, \partial f_7 = c_{11} + c_{13}$

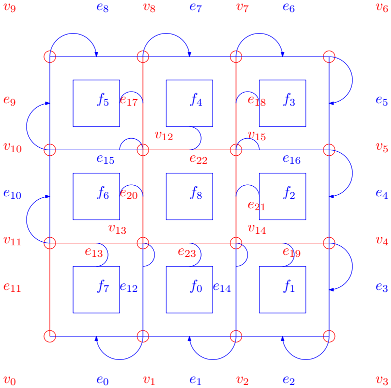
$c_{11} = e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 + e_9 + e_{10} + e_{11}$

Topological Quadrilaterals



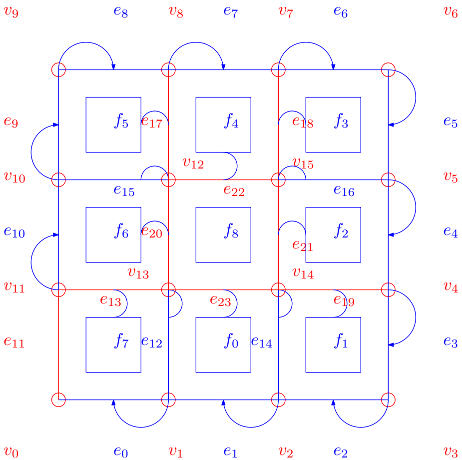
$$c_{20} = e_{15} + e_9 + e_8 + e_7 + e_6 + e_5 + e_4 + e_3 + e_2 + e_2 + e_{12} + e_{20}, \partial f_6 = c_{20} + c_{13}.$$

Topological Quadrilaterals



$$\partial f_0 = c_{23}, (e_{23}, f_0),$$
$$\partial f_1 = c_{19}, (e_{19}, f_1),$$
$$\partial f_2 = c_{21} + \partial f_1, (e_{21}, f_2),$$
$$\partial f_3 = c_{18}, (e_{18}, f_3),$$
$$\partial f_5 = c_{17}, (e_{17}, f_5),$$
$$\partial f_4 =$$
$$c_{17} + c_{18} + c_{22}, (e_{22}, f_4),$$
$$\partial f_6 = c_{13} + c_{20}, (e_{20}, f_6),$$
$$\partial f_7 = c_{13} + c_{11}, (e_{13}, f_7),$$

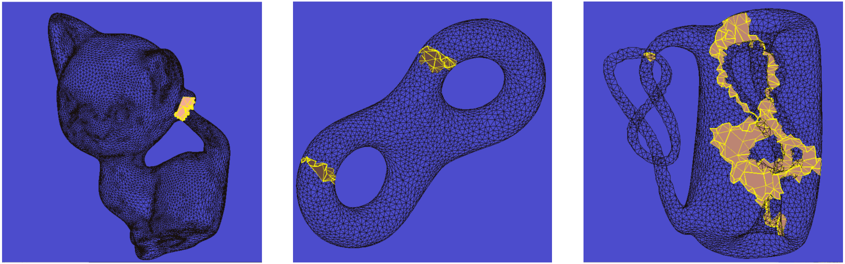
Topological Quadrilaterals



$$\partial f_8 \to \partial f_0, \partial f_4, \partial f_2, \partial f_6, \partial f_1, \partial f_3, \partial f_5, \partial f_7$$

$$\to e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 + e_9 + e_{10} + e_{11}$$

Relative Homology



Compute $H_2(M, \partial M)$ to obtain meridians. From $H_2(M, \partial M)$ to compute $H_1(M)$.