# Hyperbolic Three Manifold

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# Discrete Isometry Group

# Hyperbolic Geometry

- $M$ is hyperbolic if it admits a metric with sectional curvature $-1$.
- Equivalently, $M \cong \mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is the hyperbolic space, $\Gamma$ is a discrete group of isometry.

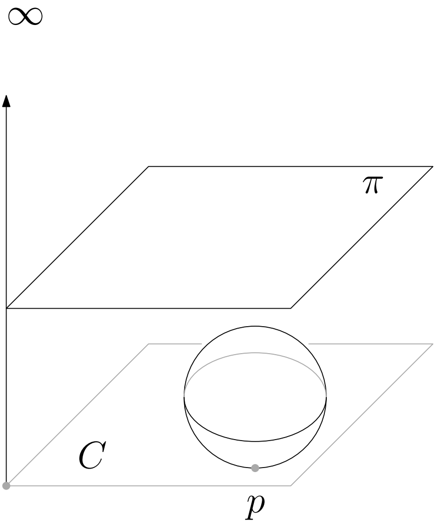
$$\mathbb{H}^3 = \{(x+iy, t) \in \mathbb{C} \times \mathbb{R}^+ | t > 0\}$$

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}$$

# Hyperbolic horosphere

## Definition (Horosphere)

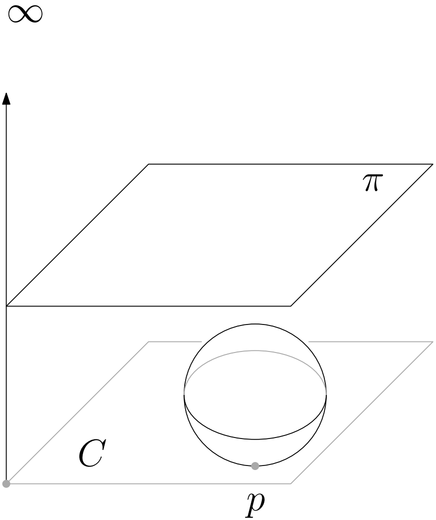
A *horosphere* about $\infty$ in $\partial\mathbb{H}^3$ is a plane parallel to $\mathbb{C}$, consisting of points $\{(x+iy, c) \in \mathbb{C}\times\mathbb{R}\}$ where $c > 0$ is a constant.
When an isometry is applied that takes $\infty$ to some $p \in \mathbb{C}$, a horosphere is taken to a Euclidean sphere tangent to $p$, which is a horosphere about $p$. A *horoball* is the region interior to a horosphere.



# Hyperbolic horosphere

The induced metric on a horosphere is Euclidean:

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2} = \frac{dx^2 + dy^2}{c^2}.$$



# Discrete Isometry Group

Suppose an orientable 3-manifold $M$ has a complete hyperbolic structure. Then the developing map $D : \tilde{M} \to \mathbb{H}^3$ is an isometric covering map, $\mathbb{H}^3$ is the universal cover of $M$. The deck transformations are then the elements of the holonomy group $\rho(\pi_1(M)) = \Gamma \leq PSL(2,\mathbb{C})$. $M$ is homeomorphic to $M \cong \mathbb{H}^3/\Gamma$.

# Discrete Isometry Group

## Definition (Hyperbolic Isometry)

The isometry group of $\mathbb{H}^3$ is $PSL(2,\mathbb{C})$,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}, ad - bc = 1.$$

the matrix is well-defined up to multiplication by $\pm Id$.

## Definition (Conjugacy)

We say $A \in PSL(2,\mathbb{C})$ is *conjugate* to $B \in PSL(2,\mathbb{C})$ if there exists $U \in PSL(2,\mathbb{C})$ such that

$$A = UBU^{-1}.$$

The *trace* of $A$ is the trace of its normalized matrix:

$$\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

# Möbius Transformation

**Theorem**

*Apart from the identity, any element of $PSL(2,\mathbb{C})$ is exactly one of the following:*

1. *elliptic which has two fixed point on $\partial\mathbb{H}^3$ and rotates about the geodesic axis between them in $\mathbb{H}^3$, fixing the axis pointwisely.*
2. *parabollic, which has a single fixed point on $\partial\mathbb{H}^3$.*
3. *loxodromic, which has two fixed points on $\partial\mathbb{H}^3$, which dilates and rotates about the axis between them.*

Parabolics and loxodromics have fixed points on $\partial\mathbb{H}^3$; elliptics have fixed points in the interior of $\mathbb{H}^3$.

# Möbius Transformation

## Lemma

*Apart from the identity, any element of $PSL(2,\mathbb{C})$ is exactly one of the following:*

1. *A is parabolic if and only if $tr(A) = \pm 2$, and if and only if A is conjugate to*
$$z \mapsto z + 1.$$
2. *A is elliptic if and only if $tr(A) \in (-2,2) \subset \mathbb{R} \subset \mathbb{C}$, and if and only if A is conjugate to*
$$z \mapsto e^{2i\theta} z, \quad 2\theta \neq 2n\pi, \forall n \in \mathbb{Z}.$$
3. *A is loxodromic if and only if $tr(A) \in \mathbb{C} - [-2,2]$, and if and only if A is conjugate to*
$$z \mapsto \zeta^2 z \quad |\zeta| > 1.$$

# Kleinian Group

**Definition (Kleinian Group)**

A subgroup of $PSL(2,\mathbb{C})$ is said to be discrete if it contains no sequence of distinct elements converging to the identity element. A discrete subgroup of $PSL(2,\mathbb{C})$ is often called a *Kleinian group*.

# Kleinian Group

## Definition (Properly Discontinuous)

The action of a group $G \leq PSL(2,\mathbb{C})$ on $\mathbb{H}^3$ is *properly discontinuous* if for every closed ball $B \subset \mathbb{H}^3$, the set $\{\gamma \in G | \gamma(B) \cap B \neq \emptyset\}$ is a finite set.

## Definition (Free)

The action of a group $G \leq PSL(2,\mathbb{C})$ on $\mathbb{H}^3$ is *free* if the identity element of $G$ is the only element to have a fixed point in $\mathbb{H}^3$.

The action of $G$ is free if and only if $G$ contains no elliptics.

## Theorem

*The action of a group $G \leq PSL(2,\mathbb{C})$ on $\mathbb{H}^3$ is free and properly discontinuous if and only if $\mathbb{H}^3/G$ is a 3-manifold with a complete hyperbolic structure and with covering projection $\mathbb{H}^3 \to \mathbb{H}^3/G$.*

# Kleinian Group

face-pairing correspond to holonomy isometries:

$$T_E = \frac{i}{\sqrt{\omega}} \begin{pmatrix} 1 & 1 \\ 1 & -\omega^2 \end{pmatrix}$$

$$T_S = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$$

$$T_N = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

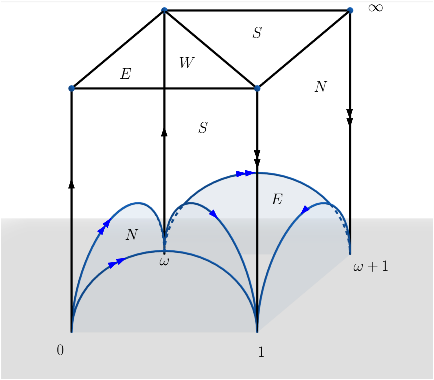



Figure: A fundamental domain of $\mathbb{S}^3 - K$, figure-8 knot $K$.

# Elementary Groups

**Definition (Elementary Groups)**

A subgroup $G \leq PSL(2, \mathbb{C})$ is *elementary* if one of the following holds.

1. The union of all fixed points on $\partial \mathbb{H}^3$ of all non-trivial elements of $G$ is a single point on $\partial \mathbb{H}^3$.
2. The union of all fixed points on $\partial \mathbb{H}^3$ of all non-trivial elements of $G$ consists of exactly two points on $\partial \mathbb{H}^3$.
3. There exists $x \in \mathbb{H}^3$ such that for all $g \in G$, $g(x) = x$.

# Elementary Groups

## Proposition (Discrete Elementary Subgroup)

*Let $G$ be a discrete nontrivial elementary subgroup of $PSL(2,\mathbb{C})$ without elliptics. Then either*

1. *the union of fixed points of nontrivial elements of $G$ is a single point on $\partial\mathbb{H}^3$, $G$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}\times\mathbb{Z}$, and $G$ is generated by parabolics (fixing the same point on $\partial\mathbb{H}^3$), or*
2. *the union of fixed points of nontrivial elements of $G$ consists of two points on $\partial\mathbb{H}^3$, $G$ is isomorphic to $\mathbb{Z}$, and $G$ is generated by a single loxodromic leaving invariant the line between fixed points.*

# Thick and Thin Parts

## Definition (rank-1, rank-2 cusps)

Suppose $G$ is an infinite elementary discrete group in $PSL(2,\mathbb{C})$ fixing a single point on $\partial\mathbb{H}^3$. Let $H$ be the closed horoball of height 1:

$$H = \{(x,y,z)|z \geq 1\}$$

$G$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}\times\mathbb{Z}$.

- If $G \cong \mathbb{Z}$, the quotient of the horoball $H/G$ is homeomorphic to the space $A\times[1,\infty)$, where $A$ is an annulus, or cylinder. We say $H/G$ is a *rank-1 cusp*.
- If $G \cong \mathbb{Z}\times\mathbb{Z}$, $H/G$ is homeomorphic to $T\times[1,\infty)$, where $T$ is a Euclidean tours. We say $H/G$ is a *rank-2 cusp*.

# Thick and Thin Parts

## Definition (Injective Radius)

Suppose $M$ is a complete hyperbolic 3-manifold and $x \in M$. The *injective radius* of $x$, denoted $\text{injrad}(x)$, is defined to be the supremal radius $r$ such that a metric $r$-ball around $x$ is embedded.

## Definition ($\varepsilon$-thin part and $\varepsilon$-thick part)

Let $M$ be a complete hyperbolic 3-manifold, and let $\varepsilon > 0$. Define the *$\varepsilon$-thin part* of $M$, denoted as $M^{<\varepsilon}$ to be

$$M^{<\varepsilon} = \{x \in M | \text{injrad}(x) < \varepsilon/2\}.$$

Similarly, the $\varepsilon$-thick part, denoted as $M^{>\varepsilon}$ is defined to be

$$M^{>\varepsilon} = \{x \in M | \text{injrad}(x) > \varepsilon/2\}.$$

# Thick and Thin Parts

## Theorem (Structure of Thin Part)

*There exists a universal constant* $\varepsilon_3 > 0$ *such that for* $0 < \varepsilon < \varepsilon_3$*, the* $\varepsilon$*-thin part of any complete, orientable, hyperbolic 3-manifold* $M$ *consists of tubes around short geodesics, rank-1 cusps, and/or rank-2 cusps.*

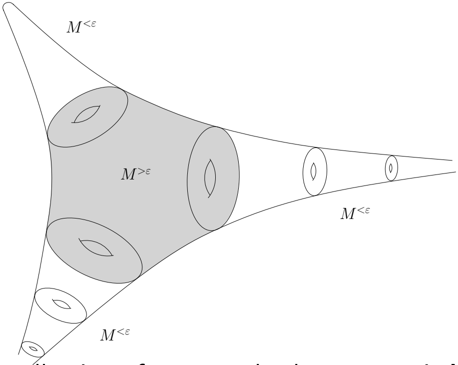



Figure: $M^{<\varepsilon}$ is a collection of cusps and tubes. $\sup \varepsilon_3$ is Margulis constant.

# Finite Volume

**Theorem**

*A hyperbolic 3-manifold $M$ has finite volume if and only if $M$ is closed (compact without boundary), or $M$ is homeomorphic to the interior of a compact manifold $\overline{M}$ with torus boundary components.*

The complement of any knot or link in $\mathbb{S}^3$ with a hyperbolic structure must have finite hyperbolic volume.

# Hyperbolic Ideal Tetrahedron

# Hyperbolic Ideal Tetrahedron

**Definition (Ideal Tetrahedron)**

An *ideal tetrahedron* is a tetrahedron in $\mathbb{H}^3$ with all four vertices on $\partial\mathbb{H}^3$.

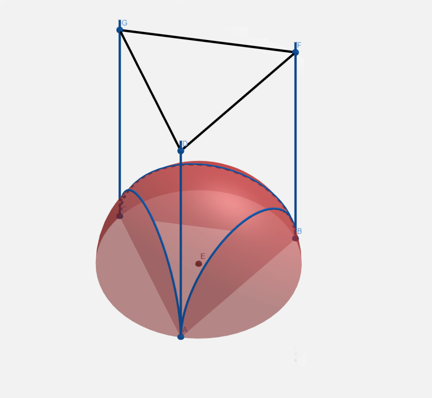



Figure: Hyperbolic ideal tetrahedron.

# Hyperbolic Ideal Tetrahedron

Given an ideal tetrahedron, $e$ is one edge, the two vertices are 0 and $\infty$, the 3rd vertex is 1, then the 4-th vertex is at $z(e)$. $z(e)$ is an invariant of the edge $e$.

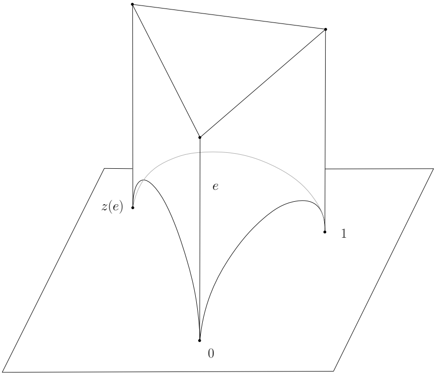



Figure: Hyperbolic ideal tetrahedron.

# Edge Invariance

Suppose edge invariance $z(e_1) = z$, then for $e_2$ edge $(1, \infty)$, apply Möbius transformation

$$w \mapsto \frac{w-1}{z-1}, \quad (0, 1, \infty, z) \mapsto \left(\frac{-1}{z-1}, 0, \infty, 1\right)$$

thus $z(e_2) = \frac{1}{1-z}$.

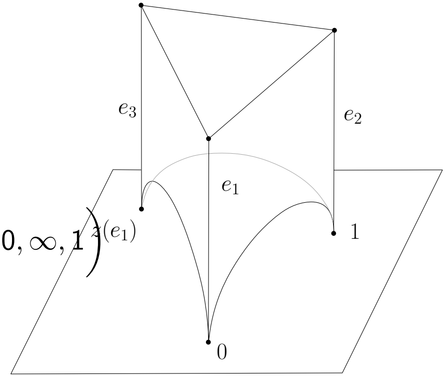



Figure: Hyperbolic ideal tetrahedron.

# Edge Invariance

Suppose edge invariance $z(e_1) = z$, then for $e_3$ edge $(z, \infty)$, apply Möbius transformation

$$w \mapsto \frac{w - z}{-z}, \quad (0, 1, \infty, z) \mapsto \left(1, \frac{1 - z}{-z}, \infty, 0\right)$$

thus $z(e_3) = \frac{z-1}{z}$.

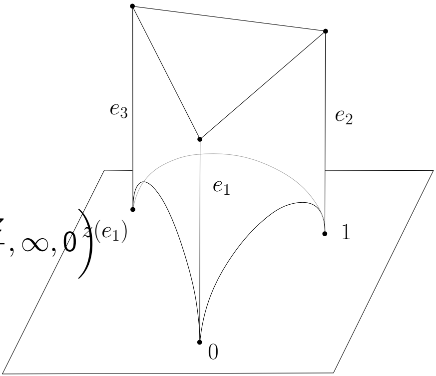


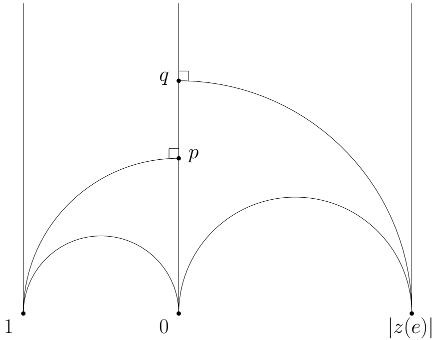

Figure: Hyperbolic ideal tetrahedron.

# Hyperbolic Ideal Tetrahedron

Draw altitudes from 1 and $z(e)$ to $e$, the perpendicular feet are $p$ and $q$ respectively. The signed distance from $p$ to $q$ is

$$\int_p^q \frac{dy}{y} = \ln|z(e)| - \ln 1 = \ln|z(e)|.$$

Hence




$$\ln z(e) = (\text{signed distance between altitudes}) + i(\text{dihedral angle at } e)$$

# Hyperbolic Ideal Tetrahedra

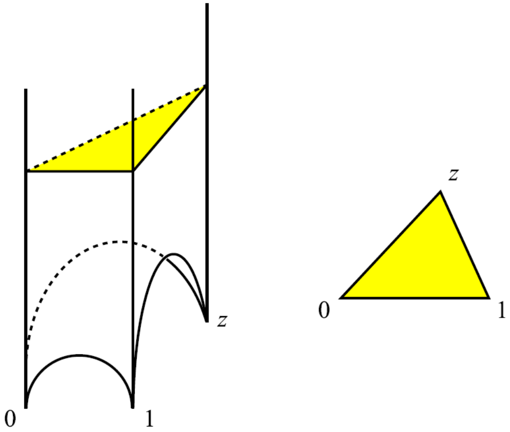



Figure: An ideal tetrahedron with three vertices $0, 1, \infty$ in the half-space model is determined by the position $z \in \mathbb{C} \cup \{\infty\}$ of the fourth vertex. A horosphere centered at the ideal vertex intersects the tetrahedron in a Euclidean triangle uniquely determined up to similarities.

# Hyperbolic Ideal Tetrahedra

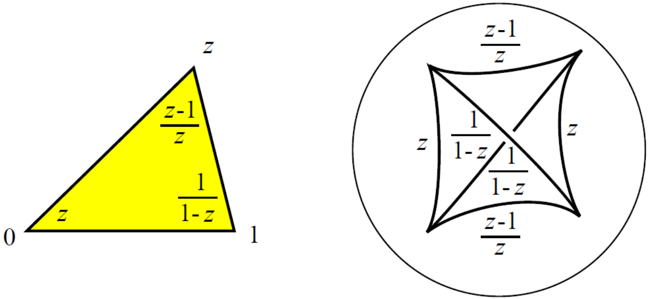
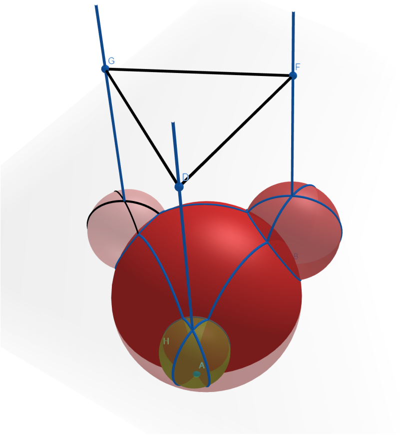



Figure: At each vertex we have a Euclidean triangle: each vertex of the triangle has a complex angle $z$. We can assign the complex angles to the edges through the vertex. The argument is the dihedral angle of the edge.

# Decorated Ideal Tetrahdron

Given an ideal tetrahedron, intersect the horspheres about 0, 1, $\infty$ and $z(e)$ with the ideal tetrahedron to obtain a decorated ideal tetrahedron. Each horosphere is orthogonal to three edges through the horosphere center. The spherical triangle corner angles equal to the corresponding dihedral angles.



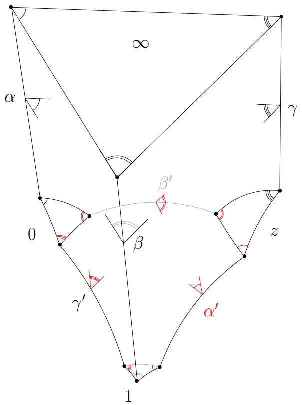
# Decorated Ideal Tetrahedron

Given a decorated ideal tetrahedron, the dihedral angles are

$$\begin{array}{ll} (0,\infty):\alpha & (1,z):\alpha' \\ (1,\infty):\beta & (0,z):\beta' \\ (z,\infty):\gamma & (0,1):\gamma' \end{array}$$

The intersection between the horoball with the ideal tetrahedron is a Euclidean $\Delta$:

$$\begin{array}{rl} \alpha+\beta+\gamma=\pi & \Delta_\infty \\ \alpha+\beta'+\gamma'=\pi & \Delta_0 \\ \beta+\gamma'+\alpha'=\pi & \Delta_1 \\ \gamma+\alpha'+\beta'=\pi & \Delta_z \end{array}$$




$$\alpha'+\beta'+\gamma'=\pi,$$
$$(\alpha,\beta,\gamma)=(\alpha',\beta',\gamma').$$

By the Möbius transformation $w \mapsto 1/(w-1)$, vertex 1 is mapped to $\infty$,

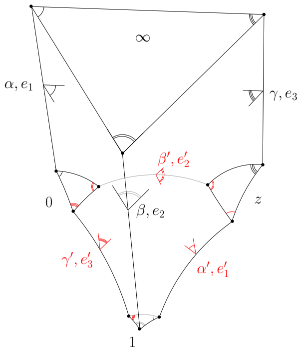
$$(0, 1, \infty, z) \mapsto (-1, \infty, 0, 1/(z-1)),$$

composed with a planar rigid motion to align the original $\Delta_1$ with the original $\Delta_\infty$, $w \mapsto (1-z)w + 1$,

$$(-1, \infty, 0, 1/(z-1)) \mapsto (z, \infty, 1, 0)$$

namely $0 \leftrightarrow z$ and $1 \leftrightarrow \infty$, Then

$$e_1(0,\infty) \leftrightarrow e_1'(z,1)$$
$$e_2(1,\infty) \leftrightarrow e_2(\infty,1)$$
$$e_3(z,\infty) \leftrightarrow e_3'(0,1)$$



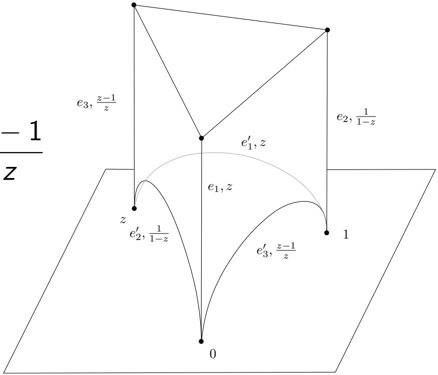

The Euclidean triangles $\Delta_0$, $\Delta_1$, $\Delta_\infty$ and $\Delta_z$ are all similar, one can be mapped to another by Möbius transformation, this implies $z(e_k) = z(e_k')$

# Edge Invariance

$$z(e_1) = z \quad z(e_2) = \frac{1}{1-z} \quad z(e_3) = \frac{z-1}{z} \tag{1}$$

$$z(e_1)z(e_2)z(e_3) = -1$$

$$1 - z(e_1) + z(e_1)z(e_3) = 0$$



# Edge Invariance

$$z(e_1) = z \quad z(e_2) = \frac{1}{1-z} \quad z(e_3) = \frac{z-1}{z} \tag{2}$$

$$z(e_1)z(e_2)z(e_3) = -1$$

$$1 - z(e_1) + z(e_1)z(e_3) = 0$$

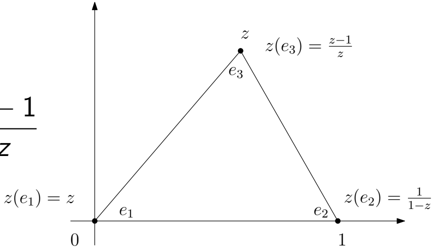



Figure: Edge invariances.

# Consistency & Completeness Equations

# Hyperbolic Ideal Triangulation

## Definition (Triangulation)

Let $\Delta_1, \ldots, \Delta_n$ be identical copies of the standard oriented 3-simplex. A *triangulation* $\mathcal{T}$ is a partition of the $4n$ faces of the tetrahedra into $2n$ pairs, and for each pair a simplicial isometry between two faces. The triangulation is *oriented* if the simplicial isometries are orientation-reversing. If we glue the tretrahedra along the simplicial isometries we get a topological space $X$. Let $M$ be $X$ minus the vertices of the triangulation: we say that $\tau$ is an *ideal triangulation* for $M$.

# Hyperbolic Ideal Triangulation

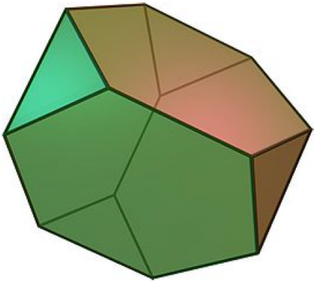


Figure: A truncated tetrahedron.

## Proposition

*If $\mathcal{T}$ is oriented then $M$ is a topological 3-manifold, homeomorphic to the interior of a compact oriented manifold with boundary.*

## Proof.

To prove $M$ is a manifold, we only need to check that a point $x \in e$ has a neighborhood homeomorphic to an open ball. A cycle of tetrahedra is attached to $e$, and since $\mathcal{T}$ is oriented we are certain that a neighborhood of $x$ is a cone over a 2-sphere and not over a projective plane.

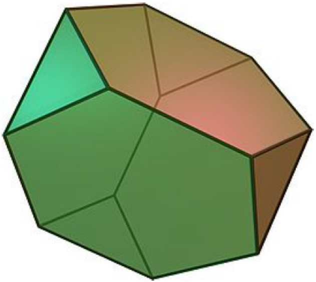


Figure: A truncated tetrahedron.

**Proof.**

If we truncate the tetrahedron before gluing them, we get a compact manifold $N \subset M$ with boundary such that

$$M \setminus N \cong \partial N \times [0, 1).$$

Therefore $M$ is homeomorphic to $\mathrm{int}(N)$. □

# Hyperbolic Structure - Consistency equation

Let $\mathcal{T}$ be an oriented ideal triangulation with tetrahedra $\Delta_1, \dots, \Delta_n$ of a 3-manifold $M$, where each $\Delta_i$ is ideal hyperbolic tetrahedron. The bijection between two ideal triangles is realised by a unique hyperbolic isometry.

We glue $h$ tetrahedra around each edge $e$. Let $z_1, z_2, \dots, z_h$ be the complex moduli associated to the edges of the $h$ tetrahedra incident to $e$. The sum of dihedra angles equals to $2\pi$, and the product of the complex moduli equals to 1.

# Hyperbolic Structure - Consistency Condition

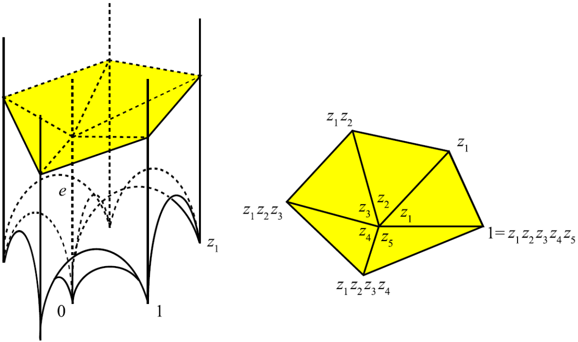



Figure: The consistency equation.

The consistency equations are (assuming $\mathfrak{Im}(z_i) > 0$ for all $i$),

$$z_1 z_2 \ldots z_h = 1$$

$$\arg(z_1) + \arg(z_2) + \cdots + \arg(z_h) = 2\pi$$

# Consistency Equation

Gluing of ideal tetrahedra. Fix an edge $e$ of the gluing, and let $T_1$ be a tetrahedron with vertices $0, 1, \infty, z(e_1)$, where $e_1$ running from 0 to $\infty$ is glued to $e$; then glue $T_2$ with $0, 1, \infty, z(e_2)$ to $T_1$ $e_2$ running from 0 to $\infty$ is glued to $e$, the fourth vertex is at $z(e_1)z(e_2)$. Repeat this procedure. The fourth vertex of the final tetrahedron will be at

$$z(e_1)z(e_2)\cdots z(e_n).$$

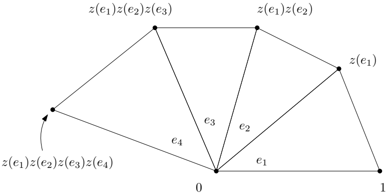



Figure: Gluing equation.

# Consistency Equation

## Theorem (Consistency Equation)

*Let $M^3$ admit a topological ideal triangulation such that each ideal tetrahedron has a hyperbolic structure. The hyperbolic structures on the ideal tetrahedra induce a hyperbolic structure on the gluing, $M$, if and only if for each edge $e$,*

$$\prod z(e_i) = 1 \quad \sum \arg(z(e_i)) = 2\pi,$$

*where the product and sum are over edges that glue to $e$.*

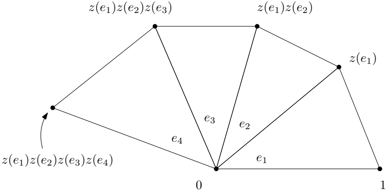



Figure: Gluing equation.

# Hyperbolic Structure - Completeness Equation

Our goal is to construct a *complete finite-volume* hyperbolic metric on $M$. If $M$ has such a metric the link of very ideal vertex of $\mathcal{T}$ is a triangulated torus and identifies a cusp of $M$. Namely, $M$ is the interior of a compact 3-manifold $N$ bounded by some tori.

Every boundary torus $T \subset \partial N$ is triangulated by $\mathcal{T}$. Every triangle in $T$ is the truncation triangle of some $\Delta_i$ and hence inherits the complex moduli of the three adjacent edges of $\Delta_i$, thus it has a Euclidean structure well-defined up to similarities. The triangle Euclidean structure determines a Euclidean structure of the torus $T$.

# Hyperbolic Structure - Completeness Equation

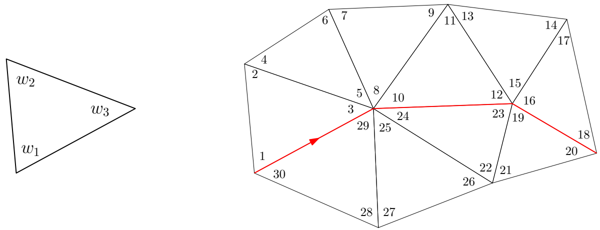



Figure: Completeness equation. $\mu(\gamma) = -w_{30}w_{29}w_{25}w_{24}w_{23}w_{19}w_{20}$

## Definition ($\mu$ homomorphism)

Pick $\gamma \in \pi_1(T)$, define $\mu(\gamma) \in \mathbb{C}^*$ to be $(-1)^{|\gamma|}$ times the product of all the complex moduli that $\gamma$ encounters at its right side with $|\gamma|$ being the number of edges of $\gamma$.

# Hyperbolic Structure - Completeness Equation

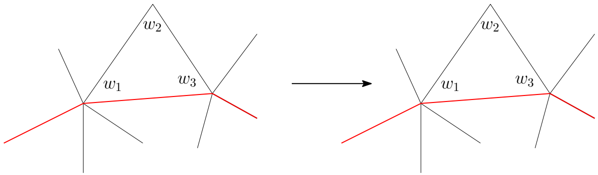



Figure: Completeness equation $\mu(\gamma)$.

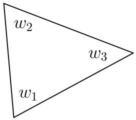
## Proposition

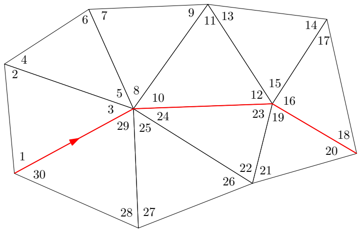
*The element $\mu(\gamma)$ is well-defined and $\mu : \pi_1(T) \to \mathbb{C}^*$ is a homomorphism.*

## Proof.

Two different paths for $\gamma$ are related by moves in the figure. This move does not affect $\mu(\gamma)$ since $w_1 w_2 w_3 = -1$ and the product of the moduli around a vertex is $+1$. The map $\mu$ is clearly a homomorphism. $\square$

# Hyperbolic Structure - Completeness Equation



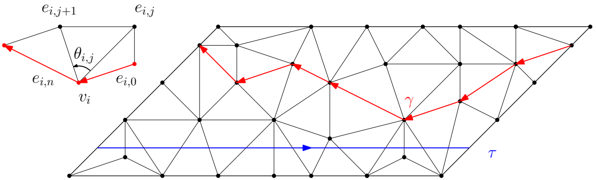

Let $C(T) \subset M$ be a closed collar of the torus $T$ in $N$, intersected with $M$, which is diffeomorphic to $T \times [0, +\infty)$.

## Proposition

*The following facts are equivalent:*

1. *the homomorphism $\mu$ is trivial,*
2. *there is a Euclidean structure on $T$ that induces all the moduli,*
3. *the manifold $C(T)$ is complete and contains a truncated cusp.*

# Hyperbolic Structure - Completeness Equation




Let $T$ be a flat torus, $\gamma$ is a path. $v_i \in \gamma$, the incident edges on the right side are $e_{i,0}, \dots, e_{i,n}$, the complex angle between $e_{i,j}$ and $e_{i,j+1}$ is $w_{i,j}$

$$w_{i,j} = \frac{|e_{i,j+1}|}{|e_{i,j}|} e^{\sqrt{-1}\theta_{i,j}}$$

therefore

$$\prod_{i,j} w_{i,j} = \prod_{i,j} \frac{|e_{i,j+1}|}{|e_{i,j}|} e^{\sqrt{-1}\theta_{i,j}} = e^{\sqrt{-1}\sum_{i,j}\theta_{i,j}} = e^{\sqrt{-1}(\sum_i \pi - k_i)} = (-1)^{|\gamma|}$$

Let $\tau$ be a straight line on $T$ homotopic to $\gamma$, $\gamma$ and $\tau$ bound a flat annulus, by Gauss-Bonnet, we obtain $\sum_i k_i = 0$.

# Hyperbolic Structure - Completeness Equation

## Corollary

*The hyperbolic manifold $M$ is complete if and only if $\mu$ is trivial for every torus $T \subset \partial N$.*

## Proposition

*Let $\mathcal{T}$ be an ideal triangulation of $M = int(N)$ with $n$ tetrahedra and $\partial N$ consisting of $c$ tori. If a point $z = (z_1, \ldots, z_n)$ with $Im(z_i) > 0$ satisfies the $n$ consistency equations and the $2c$ completeness equations, then $M$ admits a finite-volume complete hyperbolic metric.*