Alexandrov's Convex Cap Theorem

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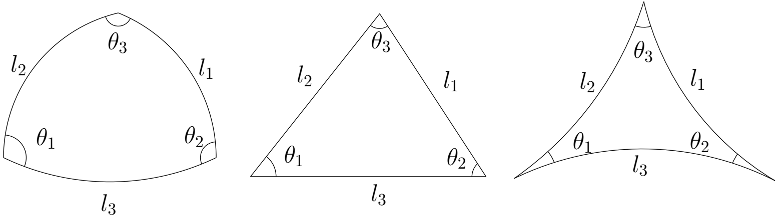
Derivative Cosine Law

Cosine Law



$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k}$$
$$\cos\theta_i = \frac{\cos l_i - \cos l_j \cos l_k}{\sin l_j \sin l_k}$$

$$1 = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k}$$
$$\cos\theta_i = \frac{l_i^2 - l_j^2 - l_k^2}{2 l_j l_k}$$

$$\cosh l_i = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k}$$
$$\cos\theta_i = \frac{\cosh l_i - \cosh l_j \cosh l_k}{\sinh l_j \sinh l_k}$$

Cosine Law

The edge length is a function of the angles: $l_i(\theta_1, \theta_2, \theta_3)$

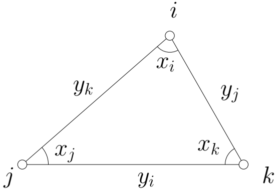
$$\cos(\sqrt{\lambda} l_i) = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k}$$

where $\lambda = +1, 0, -1$ is the curvature of the space $\mathbb{S}^2, \mathbb{H}^2$ or $\mathbb{E}^2$.

$$\boxed{\cos y_i(x_i, x_j, x_k) = \frac{\cos x_i + \cos x_j \cos x_k}{\sin x_j \sin x_k}}$$

where $(x_i, x_j, x_k) \in \mathbb{C}^3$ are complex variables, $(y_i, y_j, y_k) \in \mathbb{C}^3$ are complex functions.

Given a triangle in $\mathbb{H}^2$, $\mathbb{E}^2$ or $\mathbb{S}^2$ of inner angles $\theta_1, \theta_2, \theta_3$ and edge lengths l_1, l_2, l_3, so that θ_i is facing the edge l_i

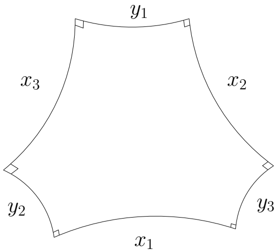


Cosine Law

The edge length is a function of the angles: $y_i(x_1, x_2, x_3)$

$$\cosh(y_i) = \frac{\cosh x_i + \cosh x_j \cosh x_k}{\sinh x_j \sinh x_k}$$

Hyperbolic right-angled hexagon:



Derivative Cosine Law

Theorem (Derivative Cosine Law)

Suppose $\Omega \subset \mathbb{C}^6$ *contains a diagonal point* (a, a, a) *so that* $y(a, a, a) = (b, b, b)$. *Let the indices* $\{i, j, k\}$ *be* $\{1, 2, 3\}$ *and* $A_{ijk} = \sin y_i \sin x_j \sin x_k$, *then*

1. $A_{ijk} = A_{jki}$.
2. $A_{ijk}^2 = 1 - \cos^2 x_i - \cos^2 x_j - \cos^2 x_k - 2\cos x_i \cos x_j \cos x_k$.
3. $\partial y_i / \partial x_i = \sin x_i / A_{ijk}$.
4. $\partial y_i / \partial x_j = \partial y_i / \partial x_i \cos y_k$.
5. $\cos x_i = \frac{\cos y_i - \cos y_j \cos y_k}{\sin y_j \sin y_k}$.

Derivative Cosine Law - Proof of (2)

Let $c_i = \cos x_i$ and $s_i = \sin x_i$,

$$\begin{aligned}
A_{ijk}^2 &= \sin^2 y_i \sin^2 x_j \sin^2 x_k = (1 - \cos y_i^2) \sin^2 x_j \sin^2 x_k \\
&= \left(1 - \left(\frac{c_i + c_j c_k}{s_j s_k}\right)^2\right) s_j^2 s_k^2 \\
&= s_j^2 s_k^2 - (c_i + c_j c_k)^2 \\
&= (1 - c_j^2)(1 - c_k^2) - (c_i + c_j c_k)^2 \\
&= 1 - c_j^2 - c_k^2 + c_j^2 c_k^2 - c_i^2 - 2 c_i c_j c_k - c_j^2 c_k^2 \\
&= 1 - c_i^2 - c_j^2 - c_k^2 - 2 c_i c_j c_k.
\end{aligned}$$

Derivative Cosine Law - Proof of (1)

Consider the analytic function A_{ijk}/A_{jki}, by (2), it takes values 1. By the assumption $y(a,a,a) = (b,b,b)$, which is invariant under the permutations of the indices, we see that value 1 is achieved. Thus $A_{ijk} = A_{jki}$ in the connected set. This shows that (1) holds.

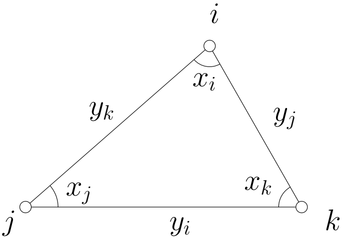
Derivative Cosine Law - Proof of (3)

Given a triangle in $\mathbb{H}^2$, $\mathbb{E}^2$ or $\mathbb{S}^2$ of inner angles x_1, x_2, x_3 and edge lengths y_1, y_2, y_3, so that x_i is facing the edge y_i

$$\frac{\partial}{\partial x_i}\cos y_i = \frac{\partial}{\partial x_i}\frac{\cos x_i + \cos x_j \cos x_k}{\sin x_j \sin x_k}$$

$$-\sin y_i \frac{\partial y_i}{\partial x_i} = -\frac{\sin x_i}{\sin x_j \sin x_k}$$

$$\frac{\partial y_i}{\partial x_i} = \frac{\sin x_i}{\sin y_i \sin x_j \sin x_k}$$



Derivative Cosine Law - Proof of (4)

$$
\begin{aligned}
\frac{\partial}{\partial x_j}\cos y_i &= \frac{\partial}{\partial x_j}\frac{\cos x_i + \cos x_j \cos x_k}{\sin x_j \sin x_k}\\
-\sin y_i \frac{\partial y_i}{\partial x_j} &= \frac{1}{s_k}\frac{(-s_j c_k)s_j - c_j(c_i + c_j c_k)}{s_j^2}\\
&= \frac{1}{s_j^2 s_k}(-s_j^2 c_k - c_i c_j - c_j^2 c_k) = \frac{1}{s_j^2 s_k}(-c_k - c_i c_j)\\
&= -\frac{s_i}{s_j s_k}\frac{c_k + c_i c_j}{s_j s_k} = -\frac{s_i}{s_j s_k}\cos y_k\\
\frac{\partial y_i}{\partial x_j} &= \frac{s_i}{\sin y_i s_j s_k}\cos y_k = \frac{\partial y_i}{\partial x_i}\cos y_k
\end{aligned}
$$

Derivative Cosine Law - Proof of (5)

$$
\begin{aligned}
&\frac{\cos y_i - \cos y_j \cos y_k}{\sin y_j \sin y_k} \\
=&\frac{1}{\sin y_j \sin y_k}\left(\frac{c_i + c_j c_k}{s_j s_k} - \frac{(c_j + c_k c_i)(c_k + c_i c_j)}{s_i s_k \cdot s_i s_j}\right) \\
=&\frac{1}{\sin y_j \sin y_k}\frac{(c_i + c_j c_k)s_i^2 - (c_j + c_k c_i)(c_k + c_i c_j)}{s_i s_k \cdot s_i s_j} \\
=&\frac{(c_i + c_j c_k)(1 - c_i^2) - (c_j + c_k c_i)(c_k + c_i c_j)}{\sin y_j s_i s_k \cdot \sin y_k s_i s_j} \\
=&\frac{1}{A_{ijk} A_{kij}}[(1 - c_i^2)(c_i + c_j c_k) - (c_j c_k + c_i c_j^2 + c_i c_k^2 + c_i^2 c_j c_k)] \\
=&\frac{1}{A_{ijk}^2}(c_i + c_j c_k - c_i^3 - c_j c_k c_i^2 - c_j c_k - c_i c_j^2 - c_i c_k^2 - c_i^2 c_j c_k) \\
=&\frac{c_i}{A_{ijk}^2}(1 - c_i^2 - c_j^2 - c_k^2 - 2 c_i c_j c_k) = c_i \frac{A_{ijk}^2}{A_{ijk}^2} = \cos x_i
\end{aligned}
$$

Derivative Cosine Law

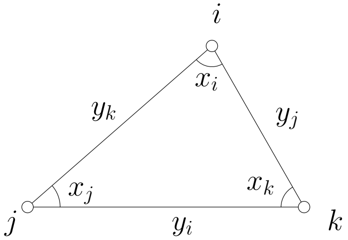
The edge length is a function of the angles: $y_i(x_1, x_2, x_3)$

$$\frac{\partial y_i}{\partial x_i} = \frac{\sin x_i}{A_{ijk}}$$

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_i}{\partial x_i} \cos y_k$$

where $A_{ijk} = \sin y_i \sin x_j \sin x_k$, $\{i, j, k\} = \{1, 2, 3\}$.

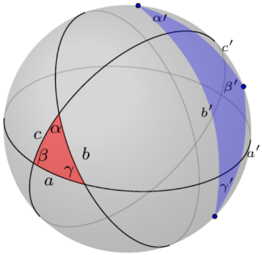
Given a triangle in $\mathbb{H}^2$, $\mathbb{E}^2$ or $\mathbb{S}^2$ of inner angles x_1, x_2, x_3 and edge lengths y_1, y_2, y_3, so that the angle x_i is facing the edge length y_i



Dual Spherical Triangle

$$\alpha + a' = \alpha' + a = \pi$$
$$\beta + b' = \beta' + b = \pi$$
$$\gamma + c' = \gamma' + c = \pi$$

The vertex at α', β' or γ' is the north pole of the equator for the side a, b or c.



Dual Spherical Triangle

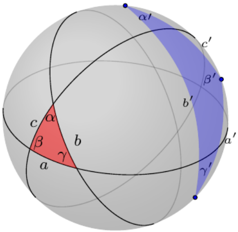
$$\cos(y_i) = \frac{\cos(x_i) + \cos(x_j)\cos(x_k)}{\sin(x_j)\sin(x_k)}$$

$$\cos(\pi - y_i) = \frac{\cos(\pi - x_i) + \cos(\pi - x_j)\cos(\pi - x_k)}{\sin(\pi - x_j)\sin(\pi - x_k)}$$

$$-\cos(x_i') = \frac{-\cos y_i' + \cos y_j' \cos y_k'}{\sin y_j' \sin y_k'}$$

$$\boxed{\cos(x_i') = \frac{\cos y_i' - \cos y_j' \cos y_k'}{\sin y_j' \sin y_k'}}$$

The vertex at α', β' or γ' is the north pole of the equator for the side a, b or c.



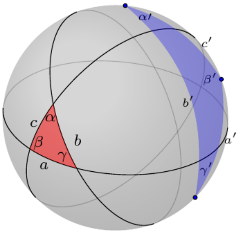
Derivative Cosine Law

$$\frac{\partial y_i}{\partial x_i} = \frac{\sin x_i}{\sin x_i \sin y_j \sin y_k} = \frac{\sin x_i}{A_{ijk}}$$

$$\frac{\partial(\pi - y_i)}{\partial(\pi - x_i)} = \frac{\sin(\pi - x_i)}{\sin(\pi - x_i)\sin(\pi - y_j)\sin(\pi - y_k)}$$

$$\boxed{\frac{\partial x_i'}{\partial y_i'} = \frac{\sin y_i'}{\sin y_i' \sin x_j' \sin x_k'} = \frac{\sin y_i'}{A_{ijk}'}}$$

The vertex at α', β' or γ' is the north pole of the equator for the side a, b or c.



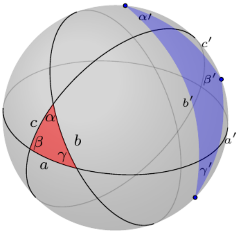
Derivative Cosine Law

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_i}{\partial x_i}\cos y_k$$

$$\frac{\partial(\pi - y_i)}{\partial(\pi - x_j)} = -\frac{\partial(\pi - y_i)}{\partial(\pi - x_i)}\cos(\pi - y_k)$$

$$\boxed{\frac{\partial x'_i}{\partial y'_j} = -\frac{\partial x'_i}{\partial y'_i}\cos x'_k}$$

The vertex at α', β' or γ' is the north pole of the equator for the side a, b or c.



Derivative Cosine Law

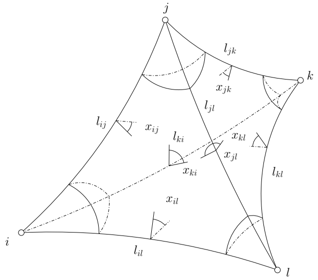
Theorem (Schlaefli Formula)

Let T be a 3-simplex with $\mathbb{S}^3$ or $\mathbb{H}^3$ metric, the sectional curvature is $\lambda = 1$ or -1 respectively. The dihedral angles are $(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})$. The volume of T is denoted as $V(x)$, Schlaefli formula is

$$dV(x_{ij}) = \frac{\lambda}{2} \sum_{ij \in T} l_{ij} dx_{ij}. \tag{1}$$

If T is with $\mathbb{E}^3$ metric,

$$0 = \sum_{ij \in T} l_{ij} dx_{ij}. \tag{2}$$

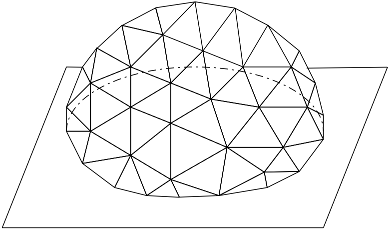


Alexandrov Convex Cap Theorem

Definition (Convex Cap)

A convex cap is a convex polytope C in $\mathbb{R}^3$ with the following properties:

1. C is contained in the upper half-space $\mathbb{R}^3_+$, and $C \cap \mathbb{R}^3_+ \neq \emptyset$. The face $C \cap \mathbb{R}^3_+$ of C is called the base of the cap C;
2. the orthogonal projection $\mathbb{R}^3 \to \mathbb{R}^2 = \partial\mathbb{R}^3_+$ maps C to its base;



Convex Cap

Definition (Euclidean polyhedral metric)

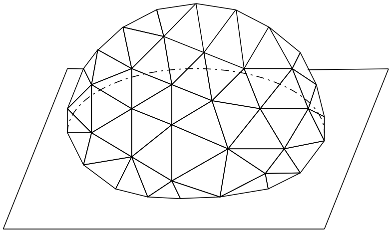
Let M be a surface, possibly with boundary. A metric structure on M is called a Euclidean polyhedral metric if there is a finite set $\Sigma \subset M$ of points called singularity points, such that

1. any regular interior point x has a neighborhood isometric to an open subset of the Euclidean plane; any regular boundary point x has a neighborhood isometric to an open subset of the half-plane;
2. any singular interior point x has a neighborhood isometric to an open subset of a cone with x at the apex of the cone; any singular boundary point x has a neighborhood isometric to an open subset of an angular region, with x at the angle's vertex.

A Euclidean polyhedral metric is called convex, if all of the angles at the interior singularities are less than 2π, and all of the angles at the boundary singularities are less than π.

Convex Cap

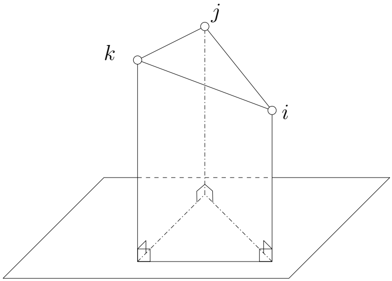
Let D be a convex polyhedral disk such that $\sigma \cap \partial D \neq \emptyset$, where Σ is the singular set of D. A geodesic triangulation T of D is a decomposition of D into triangles by geodesics with end points in Σ. By $\mathcal{E}(T)$ and $\mathcal{F}(T)$ we denote the sets of edges and triangles of T. An edge ij of T is called a boundary edge, if it is contained in the boundary of D; otherwise it is called an interior edge.



Convex Cap

Definition (Prism)

A prism is a convex polytope isometric to the lower hull of three non-collinear points in $\mathbb{R}^3_+$.



Convex Cap

Definition (Generalized Convex Cap)

A generalized convex cap C with the upper boundary D is a polyhedron glued from prisms, whose upper base are the triangles of a geodesic triangulation T of D. The identification pattern of the prims corresponds to the combinatorics of the triangulation. Besides, the following properties should hold:

1. the heights of the boundary vertices are 0, i.e. in a prim that contains a boundary edge ij this edge is shared by the upper and the lower base;
2. for every interior edge $ij \in \mathcal{E}(T)$, the dihedral angle θ_{ij} is either not defined or doesn't exceed π. Here θ_{ij} is the sum of the two dihedral angles of the prims at the edge ij.

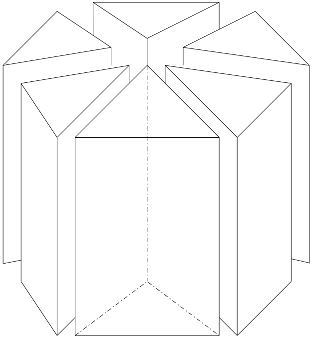


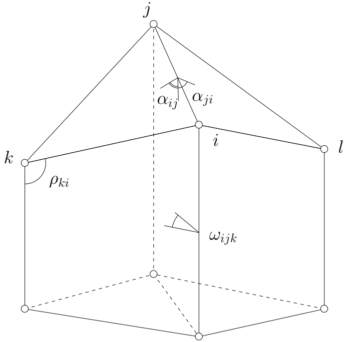
Figure: Generalized convex cap.

Definition (Curvature)

Let (T, h) be a generalized convex cap. For any interior singularity $i \in \Sigma \setminus \partial D$ denoted by ω_i the sum of the dihedral angles of the prism at the edge under the vertex i. The angle defect

$$k_i = 2\pi - \omega_i$$

is called the curvature at the i-th height.



$$\begin{aligned}\theta_i &= \alpha_{ij} + \alpha_{ji} \\ \omega_i &= \sum_{ijk \in \mathcal{F}(T)} \omega_{ijk} \\ k_i &= 2\pi - \omega_i\end{aligned} \tag{3}$$

Space of Generalized Convex Caps

Theorem

The space $\mathcal{C}(D) := \{$*generalized convex caps with the upper boundary* $D\}$ *is a non-empty bounded convex polyhedron in* $\mathbb{R}^{\Sigma}$*. Namely, it is the set of points that satisfy conditions:*

$$\begin{aligned} h_i &= 0 \quad \forall i \in \partial D & (4) \\ h_i &\leq d_i \quad \forall i, & (5) \end{aligned}$$

where d_i *is the distance in* D *from* i *to* ∂D*,*

$$\begin{aligned} h_i &\geq ext_{jkl}(i) & (6) \\ h_i &\geq h_j & (7) \end{aligned}$$

Eqn. 7 for each Euclidean quadrilateral $ikjl$ *with the angle at* i *greater or equal* π*, and Eqn. 7 for each Euclidean triangle* jij*.*

Space of Generalized Convex Caps

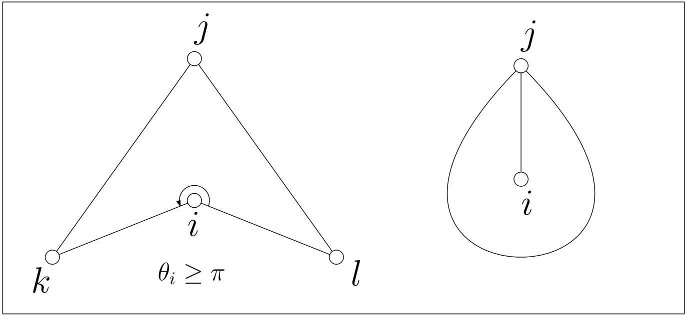


Figure: left frame $h_i \geq \text{ext}_{jkl}(i)$, where ext_{jkl} is the linear extension of the function defined on jkl; right frame $h_i \geq h_j$.

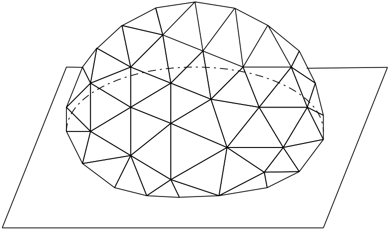
Space of Generalized Convex Caps

Lemma

There is a natural decomposition

$$\mathcal{C}(D) = \bigcup_T \mathcal{C}^T(D),$$

where $\mathcal{C}^T(D)$ consists of the caps that have a representative of the form (T, h). For every geodesic triangulation T, the space $\mathcal{C}^T(D)$ is a bounded convex polyhedron in $\mathbb{R}^\Sigma$.



For each interior edge ij, shared by triangles ijk and jil

$$\text{ext}_{ijk}(l) \geq h_l.$$

For each face iji, $h_i \geq h_j$.

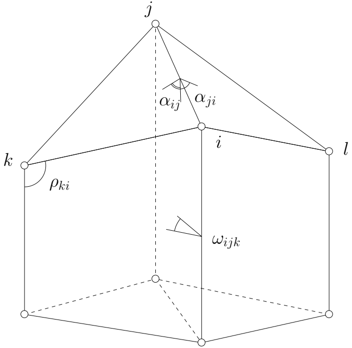
Total Scalar Curvature

Definition (Total Scalar Curvature)

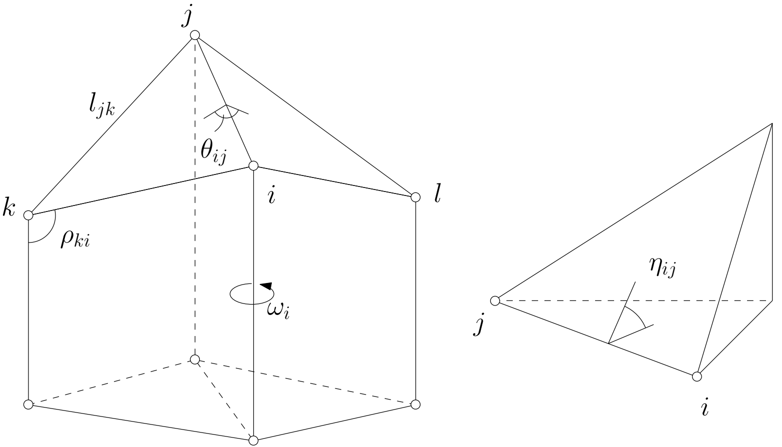
Let C be a generalized convex cap represented by (T, h). The total scalar curvature of C is defined as

$$S(C) = \sum_{\Sigma \setminus \partial D} h_i k_i + \sum_{\text{int } D} l_{ij}(\pi - \theta_{ij}) + \sum_{\partial D} l_{ij}\left(\frac{\pi}{2} - \eta_{ij}\right).$$

Here k_i is the curvature at the i-th height, l_{ij} the length of edge $ij \in \mathcal{E}(T)$, θ_{ij} the total dihearl angle at an interior edge ij, η_{ij} the dihedral angle at a boundary edge $ij \in \partial D$.



Total Scalar Curvature



$$S(C) = \sum_{\Sigma \setminus \partial D} h_i k_i + \sum_{\text{int } D} l_{ij}(\pi - \theta_{ij}) + \sum_{\partial D} l_{ij} \left(\frac{\pi}{2} - \eta_{ij} \right)$$

Total Scalar Curvature

Definition (Edge Weight)

Let C be a generalized convex cap represented by (T, h). For any $i \neq j \in \Sigma$ put

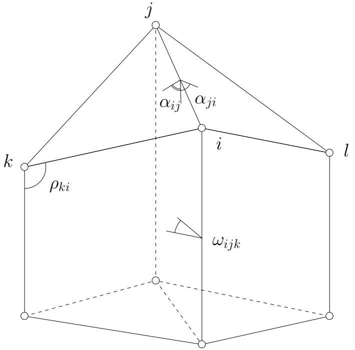
$$a_{ij} = \begin{cases} \frac{\cot \alpha_{ij} + \cot \alpha_{ji}}{l_{ij} \sin^2 \rho_{ij}} & ij \text{ interior edge of } T \\ 0 & \text{otherwise.} \end{cases}$$

Here α_{ij} and α_{ji} are the dihedral angles of the prisms at the edge ij, thus $\alpha_{ij} + \alpha_{ji} = \theta_{ij}$; ρ_{ij} is the angle between the edge ij and the i-th height. If $h_i = 0$, then the angle ρ_{ij} is the angle between the edge ij and the vector $(0, 0, -1)$ at the vertex i.
If there are several interior edgges in T that joint i and j, then a_{ij} is the sum of the corresponding expressions over all such edges.

Edge Weight

Properties of edge weights:

- If $\theta_{ij} = \pi$, then $\cot\alpha_{ij} + \cot\alpha_{ji} = 0$. Therefore a_{ij} doesn't depend on the choice of a triangulation T.
- $\rho_{ij} + \rho_{ji} = \pi$, therefore $a_{ij} = a_{ji}$.



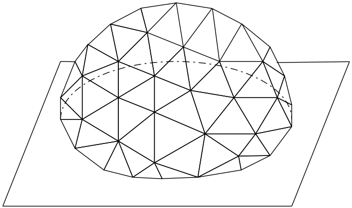
C^2 function on the Space of Generalized Convex Caps

Definition (C^2 function)

Let $f : X \to \mathbb{R}$ be a continuous function on a polyhedron $X \subset \mathbb{R}^n$. We say f is of class C^1 on X, if there exists continuous functions $f_i : X \to \mathbb{R}$ for $i = 1, 2, \dots, n$ such that for any $x \in X$ and any $\xi \in \mathbb{R}^n$ such that $x + \varepsilon\xi \in X$ for all sufficiently small positive ε, we have

$$\frac{\partial f}{\partial \xi}(x) = \sum_{i=1}^{n} f_i \xi_i.$$

We say that the function f is of class C^2 iff $f \in C^1(X)$ and $f_i \in C^1(X)$ for all i.



$S : \mathcal{C}(D) \to \mathbb{R}$ is a C^2 function.

Variational Principle

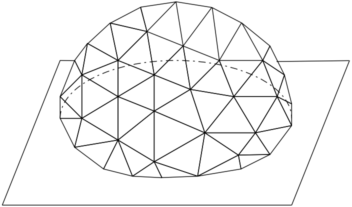
Theorem

The function S is of class C^2 on $\mathcal{C}(D)$. Its partial derivatives are:

$$\frac{\partial S}{\partial h_i} = k_i \quad (8)$$

$$\frac{\partial^2 S}{\partial h_i \partial h_j} = a_{ij} \quad (9)$$

$$\frac{\partial^2 S}{\partial h_i^2} = -\sum_{j \in \Sigma} a_{ij} \quad (10)$$



$S : \mathcal{C}(D) \to \mathbb{R}$ is a C^2 function.

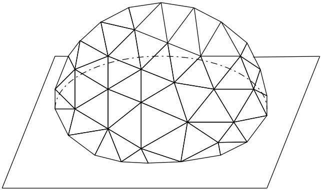
Variational Principle

$$dS = d\left(\sum_{\Sigma\setminus\partial D} h_i k_i + \sum_{\mathrm{int}D} l_{ij}(\pi - \theta_{ij}) + \sum_{\partial D} l_{ij}(\pi/2 - \eta_{ij})\right)$$

$$= \sum_{\Sigma\setminus\partial D} (dh_i k_i + h_i dk_i) + \sum_{\mathrm{int}D}(dl_{ij}(\pi - \theta_{ij}) - l_{ij} d\theta_{ij})$$

$$+ \sum_{\partial D}(dl_{ij}(\pi/2 - \eta_{ij}) - l_{ij} d\eta_{ij}) \quad \text{by Schlafli and constant}$$

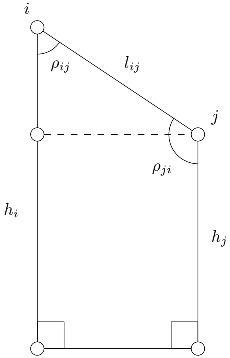
$$= \sum_{\Sigma\setminus\partial D} k_i dh_i \implies \frac{\partial S}{\partial h_i} = k_i.$$



$\partial\rho_{ij}/\partial h_i$

$$
\begin{aligned}
\cos\rho_{ij} &= \frac{h_i - h_j}{l_{ij}} \\
\frac{\partial}{\partial h_i}\cos\rho_{ij} &= \frac{\partial}{\partial h_i}\frac{h_i - h_j}{l_{ij}} \\
-\sin\rho_{ij}\frac{\partial\rho_{ij}}{\partial h_i} &= \frac{1}{l_{ij}} \\
\frac{\partial\rho_{ij}}{\partial h_i} &= -\frac{1}{l_{ij}\sin\rho_{ij}}
\end{aligned}
\tag{11}
$$

$$
\boxed{\frac{\partial\rho_{ij}}{\partial h_i} = -\frac{1}{l_{ij}\sin\rho_{ij}}}
$$



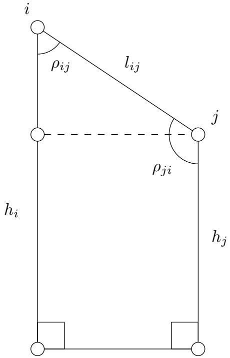
$\partial \rho_{ij} / \partial h_i$

$$\rho_{ij} + \rho_{ji} = \pi$$

$$\frac{\partial \rho_{ji}}{\partial h_j} = -\frac{1}{l_{ij} \sin \rho_{ji}} = -\frac{1}{l_{ij} \sin \rho_{ij}} = \frac{\partial \rho_{ij}}{\partial h_i}$$

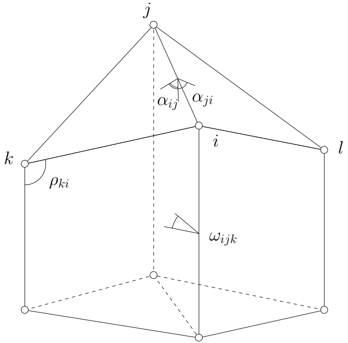
$$\frac{\partial \rho_{ij}}{\partial h_j} = -\frac{\partial \rho_{ji}}{\partial h_j} = -\frac{\partial \rho_{ij}}{\partial h_i} \tag{12}$$

$$\boxed{\frac{\partial \rho_{ij}}{\partial h_i} = -\frac{1}{l_{ij} \sin \rho_{ij}} = -\frac{\partial \rho_{ij}}{\partial h_j}}$$



ω_i can be viewed as the function of the angles ρ_{ij}, $ij \in \mathcal{E}(T)$, as long as $h \in \mathcal{C}^T(D)$,

$$
\begin{aligned}
\theta_i &= \alpha_{ij} + \alpha_{ji} \\
\omega_i &= \sum_{ijk \in \mathcal{F}(T)} \omega_{ijk} \\
d\omega_i &= \sum_{ij \in \mathcal{E}(T)} \frac{\partial \omega_i}{\partial \rho_{ij}} d\rho_{ij} \\
\frac{\partial \omega_i}{\partial \rho_{ij}} &= \frac{\partial \omega_{ijk}}{\partial \rho_{ij}} + \frac{\partial \omega_{ijl}}{\partial \rho_{ij}}
\end{aligned}
\tag{13}
$$



Spherical Derivative Cosine Law

$x_i \sim \omega_{ijk}$, $y_j \sim \rho_{ik}$, $y_k \sim \rho_{ij}$,
$x_j \sim \alpha_{ij}$, $x_k \sim \alpha_{ik}$

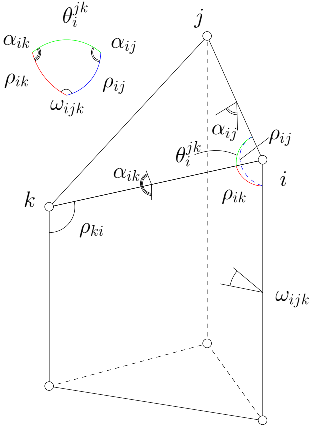
$$\frac{\partial x_i}{\partial y_k} = -\frac{\partial x_i}{\partial y_i}\cos x_j = -\frac{\sin y_i}{A_{ijk}}\cos x_j$$

$$A_{ijk} = \sin x_j \sin y_k \sin y_i$$

$$\frac{\partial x_i}{\partial y_k} = -\frac{\sin y_i}{\sin x_j \sin y_k \sin y_i}\cos x_j$$

$$= -\frac{\cot x_j}{\sin y_k} = -\frac{\cot \alpha_{ij}}{\sin \rho_{ij}}$$

$$\boxed{\frac{\partial \omega_{ijk}}{\partial \rho_{ij}} = -\frac{\cot \alpha_{ij}}{\sin \rho_{ij}}}$$



$$d\omega_i = \sum_{ij \in \mathcal{E}(T)} \frac{\partial \omega_i}{\partial \rho_{ij}} d\rho_{ij}$$

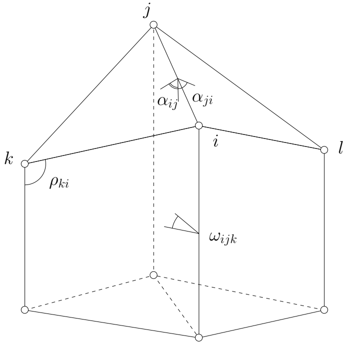
$$d\rho_{ij} = \frac{\partial \rho_{ij}}{\partial h_i} dh_i + \frac{\partial \rho_{ji}}{\partial h_j} dh_j$$

$$\frac{\partial \rho_{ij}}{\partial h_i} = -\frac{1}{l_{ij} \sin \rho_{ij}} = -\frac{\partial \rho_{ij}}{\partial h_j}$$

$$\frac{\partial \omega_{ijk}}{\partial \rho_{ij}} = -\frac{\cot \alpha_{ij}}{\sin \rho_{ij}}$$

$$\frac{\partial \omega_{jil}}{\partial \rho_{ij}} = -\frac{\cot \alpha_{ji}}{\sin \rho_{ji}}$$

ω_i can be viewed as the function of the angles ρ_{ij}, $ij \in \mathcal{E}(T)$, as long as $h \in \mathcal{C}^T(D)$,



$d\omega_i$

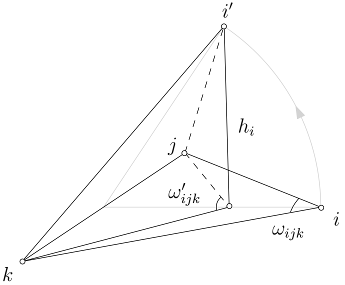
$$
\begin{aligned}
d\omega_i &= \sum_{ij\in\mathcal{E}(T)} \frac{\partial\omega_i}{\partial\rho_{ij}} d\rho_{ij} = \sum_{ij\in\mathcal{E}(T)} \frac{\partial\omega_i}{\partial\rho_{ij}} \left(\frac{\partial\rho_{ij}}{\partial h_i} dh_i + \frac{\partial\rho_{ij}}{\partial h_j} dh_j \right) \\
&= \sum_{ij\in\mathcal{E}(T)} \frac{\partial\omega_i}{\partial\rho_{ij}} \left(-\frac{1}{l_{ij}\sin\rho_{ij}} \right) (dh_i - dh_j) \\
&= \sum_{ij\in\mathcal{E}(T)} \left(-\frac{\cot\alpha_{ij} + \cot\alpha_{ji}}{\sin\rho_{ij}} \right) \left(-\frac{1}{l_{ij}\sin\rho_{ij}} \right) (dh_i - dh_j) \\
&= \sum_{ij\in\mathcal{E}(T)} \frac{\cot\alpha_{ij} + \cot\alpha_{ji}}{l_{ij}\sin^2\rho_{ij}} (dh_i - dh_j) \\
&= \sum_{ij\in\mathcal{E}(T)} a_{ij}(dh_i - dh_j)
\end{aligned}
$$

$$
\frac{\partial^2 S}{\partial h_i^2} = \frac{\partial k_i}{\partial h_i} = -\frac{\partial\omega_i}{\partial h_i} = -\sum_j a_{ij}, \quad \frac{\partial^2 S}{\partial h_i \partial h_j} = \frac{\partial k_i}{\partial h_j} = -\frac{\partial\omega_i}{\partial h_j} = a_{ij}
$$

ω_i can be viewed as the function of the heights $(h_1, h_2, \ldots, h_n)$, as long as $h \in \mathcal{C}^T(D)$,

$$d\omega_i = \sum_{ij \in \mathcal{E}(T)} \frac{\cot \alpha_{ij} + \cot \alpha_{ji}}{l_{ij} \sin^2 \rho_{ij}} (dh_i - dh_j)$$

$h_i \uparrow$ then $\omega_i \uparrow$, $\omega_j \downarrow$.



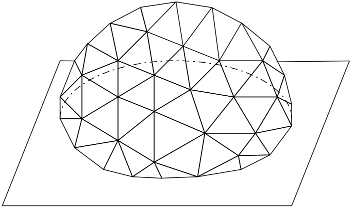
$h'_i > h_i$, $\omega'_{ijk} > \omega_{ijk}$

Variational Principle

For any generalized convex cap $C \in \mathcal{C}(D)$, since C is convex, $\theta_{ij} < \pi$, $a_{ij} \geq 0$. For any $\mathbf{h}$ vector,

$$\mathbf{h}^T D^2 S \mathbf{h} = -\sum_{ij \in \mathcal{E}(T)} a_{ij}(h_i - h_j)^2 \leq 0.$$

since $h_i = 0$ on ∂D, the above is zero if and only if $\mathbf{h} = \mathbf{0}$. Therefore, D^S is negative definite, S is strictly concave.

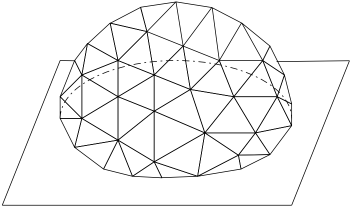


$S : \mathcal{C}(D) \to \mathbb{R}$ is a concave C^2 function.

Global Rigidity

Let $n = |\Sigma \setminus \partial D|$, the curvature map $K_D : \mathcal{C}(D) \to \mathbb{R}^n$,

$$(h_1, h_2, \dots, h_n) \mapsto (k_1, k_2, \dots, k_n).$$



$S : \mathcal{C}(D) \to \mathbb{R}$ is a concave C^2 function.

Definition (Global Rigidity)

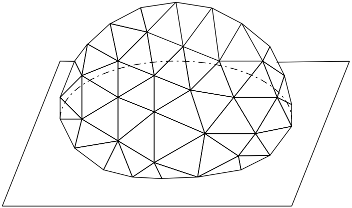
One says that generalized convex caps with the upper boundary D are globally rigid iff the map K_D is injective.

Global Rigidity

Theorem (Global Rigidity)

For any convex polyhedral disk D, generalized convex caps with the upper boundary D are globally rigid.

Proof: The function S is strictly concave, $K_D = \nabla S$, its domain $\mathcal{C}(D)$ is compact and convex, therefore the map K_D is a homeomorphism onto its image. $\square$



$S : \mathcal{C}(D) \to \mathbb{R}$ is a concave C^2 function.

Global Rigidity

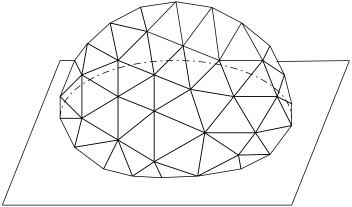
Lemma

Let $f \in C^1(X)$ be a strictly convex or strictly concave function on a compact convex set $X \subset \mathbb{R}^n$. Then the map $\nabla f : X \to \mathbb{R}^n$ is a homeomorphism onto the image.

Proof: Let x and y be two different points in X, $f(\lambda x + (1-\lambda)y)$ is a convex C^1 function,

$$\frac{\partial f}{\partial \xi}(\lambda x+(1-\lambda)y), \quad \xi = (y-x)/\|y-x\|$$

is a monotonous. $\frac{\partial f}{\partial \xi}(x) \neq \frac{\partial f}{\partial \xi}(y)$, since $\frac{\partial f}{\partial \xi} = \langle \nabla f, \xi \rangle$, it follows that $\nabla f(x) \neq \nabla f(y)$. $\square$



$S : \mathcal{C}(D) \to \mathbb{R}$ is a concave C^2 function.

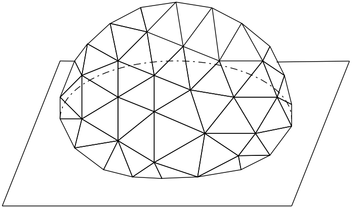
Infinitesimal Rigidity

Definition

Let $C \in \mathcal{C}(D)$ be a generalized convex cap. One says that C is infinitesimally rigid iff the Jacobian of the map K_D at C has full rank.

Theorem

Classical convex caps that have dimension 3 without vertical faces are infinitesimally rigid.



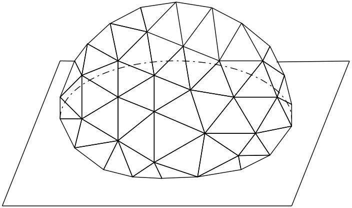
$S : \mathcal{C}(D) \to \mathbb{R}$ is a concave C^2 function.

Alexandrov Convex Cap Theorem

Theorem (Alexandrov Convex Cap)

Let D be a disk with a convex Euclidean polyhedral metric. Then there exists a convex cap $C \subset \mathbb{R}^3$ with the upper boundary isometric to D. Besides, C is unique up to a rigid motion.

Proof: Let $C \in \mathcal{C}(D)$ be a maximum point of the functional S. If C lies in the interior of $\mathcal{C}(D)$, then we have $k(C) = \nabla S = 0$, C is a classical convex cap with no vertical faces. The functional S is strictly concave, therefore it has only one local maximum on the convex space $\mathcal{C}(D)$, and the uniqueness follows. □



The local maximum C lies in the interior of $\mathcal{C}(D)$ can be proved by more detailed analysis.