Chapter 10 Equality constrained minimization

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equality constrained minimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

- f convex and twice continuously differentiable
- $A \in \mathbb{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- assume optimal value p^* is finite and attained

optimality condition (review)

$$x^* \text{ is optimal} \quad \Longleftrightarrow \quad \begin{array}{c} x^* \in \mathbf{dom}\, f, \quad Ax^* = b, \\ \text{there exists } \nu^* \text{ such that } \nabla f(x^*) + A^T \nu^* = 0 \end{array}$$

equality constrained quadratic minimization (with $P \in \mathbb{S}^n_+$)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Ax = b \end{array}$$

optimality condition

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

- coefficient matrix is called KKT matrix
- KKT matrix is nonsingular if and only if

$$Ax = 0, \quad x \neq 0 \qquad \Longrightarrow \qquad x^T P x > 0$$

- equivalent condition for nonsingularity

$$P + A^T A \succ 0$$

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represent solutions of $\{x \mid Ax = b\}$ as

$$\{x \mid Ax = b\} = \{Fz + \hat{x} \mid z \in \mathbb{R}^{n-p}\}$$

- $\hat{x}$ is any particular solution
- range of $F \in \mathbb{R}^{n \times (n-p)}$ is nullspace of A

reduced or eliminated problem

$$\text{minimize} \qquad f(Fz + \hat{x})$$

- unconstrained problem with variable $z \in \mathbb{R}^{n-p}$
- from solution z^*, obtain x^* and ν^* as

$$x^* = Fz^* + \hat{x}, \qquad \nu^* = -\left(AA^T\right)^{-1} A \nabla f(x^*)$$

example optimal allocation with resource constraint

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + \cdots + f_n(x_n) \\ \text{subject to} & x_1 + \cdots + x_n = b \end{array}$$

eliminate $x_n = b - x_1 - \cdots - x_{n-1}$, namely, choose

$$\hat{x} = be_n, \qquad F = \begin{bmatrix} I \\ -\mathbf{1}^T \end{bmatrix} \in \mathbb{R}^{n \times (n-1)}$$

reduced problem

$$\text{minimize} \qquad f_1(x_1) + \cdots + f_{n-1}(x_{n-1}) + f_n(b - x_1 - \cdots - x_{n-1})$$

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Newton step

Newton step Δx_{nt} of f at feasible x is given by the solution v of

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

interpretations

- Δx_{nt} solves second order approximation (with variable v)

$$\begin{array}{ll} \text{minimize} & \widehat{f}(x+v) = f(x) + \nabla f(x)^T v + (1/2) v^T \nabla^2 f(x) v \\ \text{subject to} & A(x+v) = b \end{array}$$

- Δx_{nt} equations follow from linearizing optimality conditions

$$\nabla f(x+v) + A^T w \approx \nabla f(x) + \nabla^2 f(x) v + A^T w = 0, \qquad A(x+v) = b$$

Newton decrement

$$\lambda(x) = \left(\Delta x_{\rm nt}^T \nabla^2 f(x) \Delta x_{\rm nt}\right)^{1/2} = \left(-\nabla f(x)^T \Delta x_{\rm nt}\right)^{1/2}$$

interpretations

- gives an estimate of $f(x) - p^*$ using quadratic approximation $\widehat{f}$

$$f(x) - \inf_{Ay=b} \widehat{f}(y) = (1/2)\lambda(x)^2$$

- directional derivative in Newton direction

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} f\left(x + t\Delta x_{\rm nt}\right)\right|_{t=0} = -\lambda(x)^2$$

- in general $\lambda(x) \neq \left(\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)\right)^{1/2}$

Newton's method with equality constraints

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given starting point $x \in \mathbf{dom}\, f$ with $Ax = b$, tolerance $\epsilon > 0$

repeat

1. Compute the Newton step and decrement Δx_{nt}, $\lambda(x)$
2. *Stopping criterion.* **quit** if $\lambda^2/2 \leq \epsilon$
3. *Line search.* Choose step size t by backtracking line search
4. *Update.* $x := x + t\Delta x_{\mathrm{nt}}$

- feasible descent method: $x^{(k)}$ feasible and $f\left(x^{(k+1)}\right) < f\left(x^{(k)}\right)$
- affine invariant

Newton's method and elimination

Newton's method for reduced problem

$$\text{minimize } \tilde{f}(z) = f(Fz + \hat{x})$$

- $z \in \mathbb{R}^{n-p}$ are variables, $\hat{x}$ satisfies $A\hat{x} = b$, range of F is the nullspace of A
- Newton's method for $\tilde{f}$ starts at $z^{(0)}$, generates iterates $z^{(k)}$

relation to Newton's method with equality constraints

when starting at $x^{(0)} = Fz^{(0)} + \hat{x}$, iterates are

$$x^{(k)} = Fz^{(k)} + \hat{x}$$

hence no separate convergence analysis is needed

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Newton step at infeasible points

Newton step $\Delta x_{\rm nt}$ of f at infeasible x is given by the solution of

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm nt} \\ w \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ Ax - b \end{bmatrix}$$

interpretation

- $\Delta x_{\rm nt}$ equations follow from linearizing optimality conditions

$$\nabla f(x+v) + A^T w \approx \nabla f(x) + \nabla^2 f(x) v + A^T w = 0, \qquad A(x+v) = b$$

primal-dual interpretation

- write optimality condition as $r(y) = 0$ where

$$y = (x, \nu), \qquad r(y) = \big(\nabla f(x) + A^T\nu, Ax - b\big)$$

- linearizing $r(y) = 0$ gives

$$r(y + \Delta y) \approx r(y) + Dr(y)\Delta y = 0$$

which is equivalent to

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\mathrm{nt}} \\ \Delta \nu_{\mathrm{nt}} \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + A^T\nu \\ Ax - b \end{bmatrix}$$

same as the above equation with $w = \nu + \Delta\nu_{\mathrm{nt}}$

Infeasible start Newton method

given starting point $x \in \mathbf{dom}\, f$, ν, tolerance $\epsilon > 0$, $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$

repeat

1. Compute primal and dual Newton steps Δx_{nt}, $\Delta \nu_{\mathrm{nt}}$
2. *Backtracking line search on* $\|r\|_2$. $t := 1$.
   while $\|r(x + t\Delta x_{\mathrm{nt}}, \nu + t\Delta \nu_{\mathrm{nt}})\|_2 > (1 - \alpha t)\|r(x, \nu)\|_2, \qquad t := \beta t$
3. *Update*. $x := x + t\Delta x_{\mathrm{nt}}, \nu := \nu + t\Delta \nu_{\mathrm{nt}}$

until $Ax = b$ and $\|r(x, \nu)\|_2 \leq \epsilon$

- not a descent method: $f\left(x^{(k+1)}\right) > f\left(x^{(k)}\right)$ is possible
- directional derivative of $\|r(y)\|_2$ in direction $\Delta y = (\Delta x_{\mathrm{nt}}, \Delta \nu_{\mathrm{nt}})$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} \|r(y + t\Delta y)\|_2 \bigg|_{t=0} = -\|r(y)\|_2$$