Circle Domain Mapping: Koebe's Theorem

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Motivation

Conformal Module for Poly-annulus

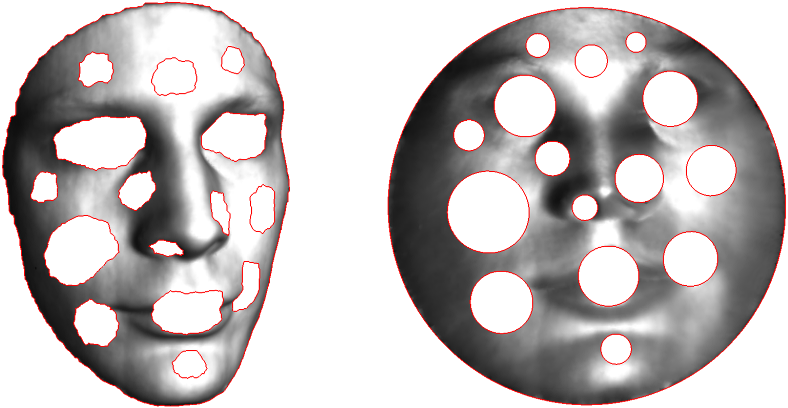


Figure: Conformal mapping from a poly-annulus to a circle domain.

Circle Domain

Definition (Circle Domain)

Suppose $\Omega \subset \hat{\mathbb{C}}$ is a planar domain, if $\partial\Omega$ has finite number of connected components, each of them is either a circle or a point, then Ω is called a circle domain.

Theorem (Koebe)

Suppose S is of genus zero, ∂S has finite number of connected components, then S is conformal equivalent to a circle domain. Furthermore, all such conformal mappings differ by a Möbius transformation.

Schwartz Reflection Principle

Definition (Mirror Reflection)

Given a circle $\Gamma : |z - z_0| = \rho$, the reflection with respect to Γ is defined as:

$$\varphi_\Gamma : re^{i\theta} + z_0 \mapsto \frac{\rho^2}{r} e^{i\theta} + z_0. \tag{1}$$

Two planar domains S and S' are symmetric about Γ, if $\varphi_\Gamma(S) = S'$.

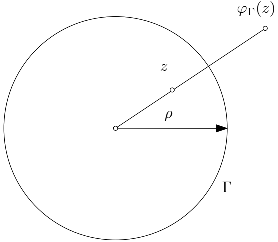


Figure: Reflection about a circle.

Schwartz Reflection Principle

Definition (Reflection)

Suppose Γ is an analytic curve, domain S, S' and Γ are included in a planar domain Ω. There is a conformal map $f : \Omega \to \hat{\mathbb{C}}$, such that $f(\Gamma)$ is a canonical circle, $f(S)$ and $f(S')$ are symmetric about $f(\Gamma)$, then we say S and S' are symmetric about Γ, and denoted as

$$S|S' \quad (\Gamma).$$

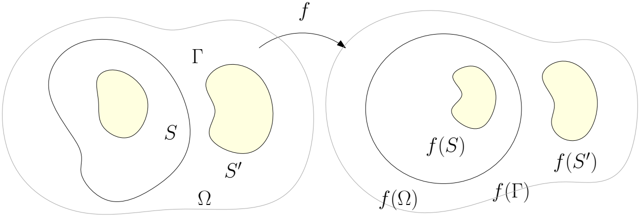


Figure: General symmetry.

Schwartz Reflection Principle

Theorem (Schwartz Reflection Principle)

Assume f is an analytic function, defined on the upper half disk $\{|z| < 1, \Im(z) > 0\}$. If f can be extended to a real continuous function on the real axis, then f can be extended to an analytic function F defined on the whole disk, satisfying

$$F(z) = \begin{cases} f(z), & \Im(z) \geq 0 \\ \overline{f(\bar{z})}, & \Im(z) < 0 \end{cases}$$

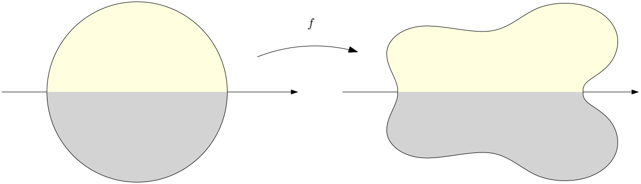
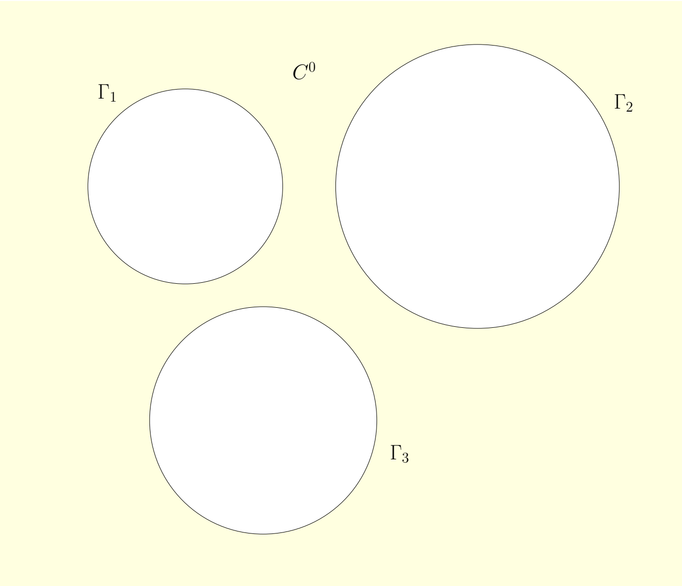
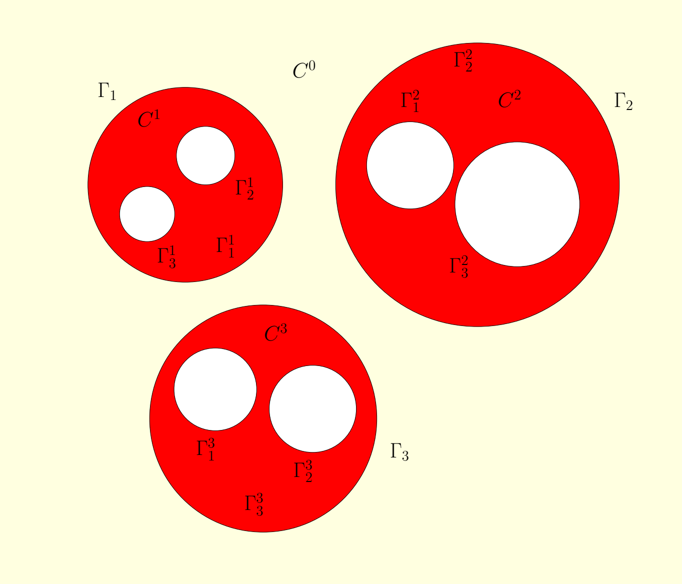


Figure: Schwartz reflection principle.

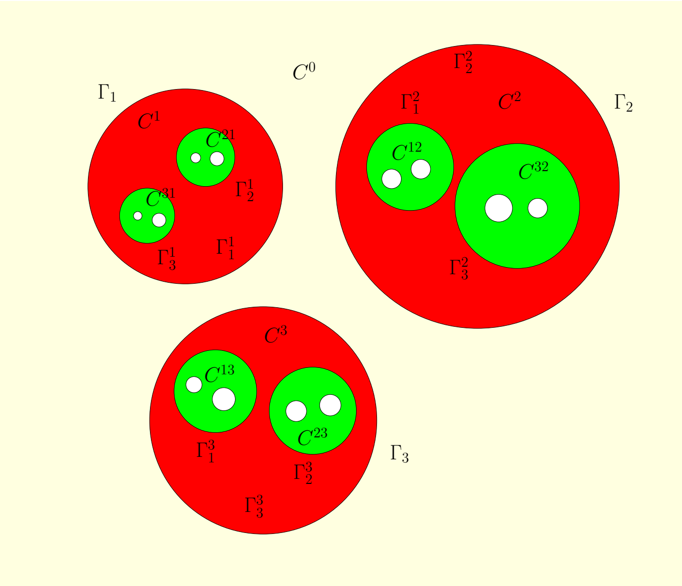
Multiple Reflection



Multiple Reflection



Multiple Reflection



Multiple Reflection

1. Initial circle domain C^0: complex plane remove three disks, its boundary is $\{\Gamma_1, \Gamma_2, \Gamma_3\}$;
2. First level reflection: C^0 is reflected about Γ_{i_1} to C^{i_1}, $i_1 = 1, 2, 3$;

$$\partial C^{i_1} = \Gamma_{i_1}^{i_1} - \sum_{j \neq i_1} \Gamma_j^{i_1},$$

   where $\Gamma_{i_1}^{i_1} = \Gamma_{i_1}$.
3. Second level reflection: C^{i_1} is reflected about Γ_{i_2} to $C^{i_1 i_2}$, $i_1 \neq i_2$; the boundary of $C^{i_1 i_2}$ are $\Gamma_j^{i_1 i_2}$, when $j \neq i_1$, $\Gamma_j^{i_1 i_2}$ is an interior boundary; when $j = i_1$, $\Gamma_j^{i_1 i_2}$ is the exterior boundary, $\Gamma_{i_1}^{i_1 i_2} = \Gamma_{i_1}^{i_2}$.

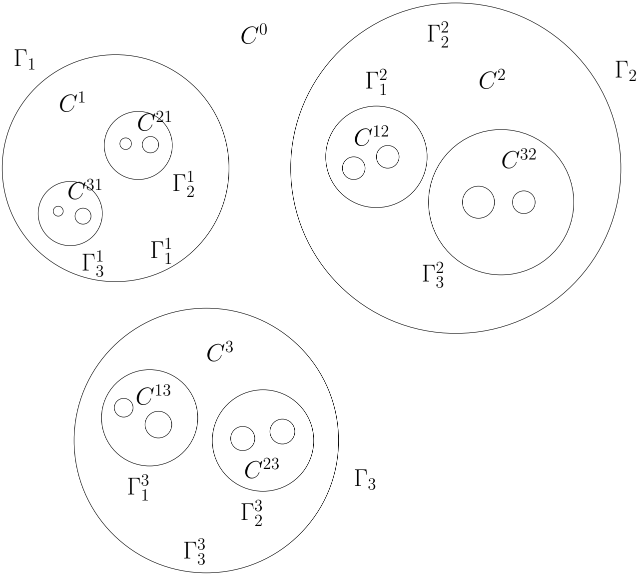
$$\partial C^{i_1 i_2} = \Gamma_{i_1}^{i_2} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2}$$

   when $j = i_1$, $\Gamma_{i_1}^{i_1 i_2} = \Gamma_{i_1}^{i_2}$;

Multiple Reflection

4. Third level reflection: $C^{i_1 i_2}$ is reflected about Γ_{i_3} to $C^{i_1 i_2 i_3}$, $i_1 \neq i_2$, $i_2 \neq i_3$; the boundary of $C^{i_1 i_2 i_3}$ are $\Gamma_j^{i_1 i_2 i_3}$, when $j \neq i_1$, $\Gamma_j^{i_1 i_2 i_3}$ is an interior boundary; when $j = i_1$, $\Gamma_j^{i_1 i_2 i_3}$ is the exterior boundary, $\Gamma_{i_1}^{i_1 i_2 i_3} = \Gamma_{i_1}^{i_2 i_3}$.
$$\partial C^{i_1 i_2 i_3} = \Gamma_{i_1}^{i_2 i_3} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2 i_3}.$$
5. The m-level reflection: $C^{i_1 i_2 \dots i_{m-1}}$ is reflected about Γ_{i_m} to $C^{i_1 i_2 \dots i_{m-1} i_m}$, $i_k \neq i_{k+1}$; the boundary of $C^{i_1 i_2 \dots i_{m-1} i_m}$, $i_k \neq i_{k+1}$ are $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$, when $j \neq i_1$, $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$ is an interior boundary; when $j = i_1$, $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$ is the exterior boundary, $\Gamma_{i_1}^{i_1 i_2 \dots i_{m-1} i_m} = \Gamma_{i_1}^{i_2 \dots i_{m-1} i_m}$ is an interior boundary,
$$\partial C^{i_1 i_2 \dots i_m} = \Gamma_{i_1}^{i_2 i_3 \dots i_m} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2 \dots i_m}.$$

Multiple Reflection



Multiple Reflection

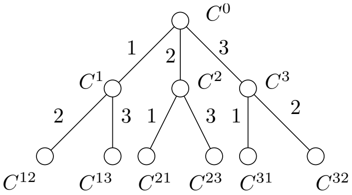


Figure: Reflection tree.

- Each node represents a domain $C^{i_1 i_2 \cdots i_m}$;
- Each edge represents a circle Γ_k, $k = 1, \ldots, n$;
- Father and Son share an edge i_1

$$\Gamma_{i_1}^{i_1 i_2 \cdots i_m} = \Gamma_{i_1}^{i_2 \cdots i_m}.$$

- Each node $C^{(i)}$, $(i) = i_1 i_2 \ldots i_m$ is the path from the root to $C^{(i)}$,

$$C^{(i)} = \varphi_{\Gamma_{i_m}} \circ \varphi_{\Gamma_{i_{m-1}}} \cdots \varphi_{\Gamma_{i_1}}(C^0).$$

Multiple Reflection

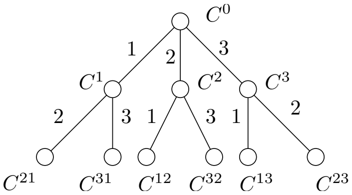


Figure: Embedding tree.

- Father node $C^{i_2\cdots i_m}$ and child node $C^{i_1 i_2\cdots i_m}$ is connected by edge i_1, the exterior boundary of child equals to an interior boundary of the father

$$\Gamma_{i_1}^{i_1 i_2\cdots i_m} = \Gamma_{i_1}^{i_2\cdots i_m}.$$

- From the root C^0 to $C^{i_1\cdots i_m}$, the path is inverse to the index

$$(i)^{-1} = i_m i_{m-1}\cdots i_2 i_1,$$

starting from C^0 crosses Γ^{i_m} to C^{i_m}, crosses $\Gamma^{i_m}_{i_{m-1}}$ to $C^{i_{m-1}i_m}$; when arrives at $C^{i_{k-1}\cdots i_1}$, crosses $\Gamma^{i_{k-1}\cdots i_1}_{i_k}$ to $C^{i_k i_{k-1}\cdots i_1}$; and eventually reach $C^{(i)}$.

Hole Area Estimation

Lemma

Suppose $C^{(i)}$ is an interior node in the reflection tree,

$$(i) = i_1 i_2 \cdots i_m,$$

its exterior boundary is $\Gamma_{i_1}^{(i)}$, interior boundaries are $\Gamma_j^{(i)}$, $j \neq i_1$, we have the estimate:

$$\sum_{j \neq i_1} \alpha(\Gamma_j^{(i)}) \leq \mu^4 \alpha(\Gamma_{i_1}^{(i)}).$$

Hole Area Estimation

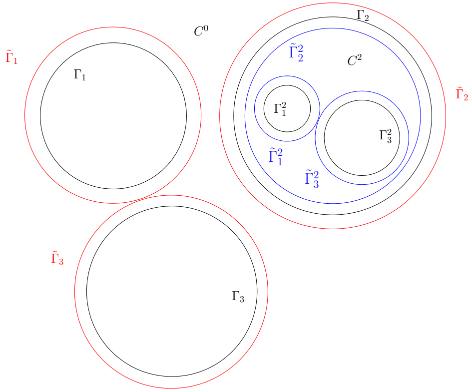


Figure: Hole area estimation.

Enlarge all Γ_k's by factor μ^{-1} to $\tilde{\Gamma}_k$, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_3$ touch each other; reflect C^0 about Γ_2

- $\Gamma_k | \Gamma_k^2 \quad (\Gamma_2)$.
- $\tilde{\Gamma}_k | \tilde{\Gamma}_k^2 \quad (\Gamma_2)$.

$$\alpha(\tilde{\Gamma}_1^2) = \mu^{-2}\alpha(\Gamma_1^2)$$

$$\alpha(\tilde{\Gamma}_3^2) = \mu^{-2}\alpha(\Gamma_3^2)$$

$$\alpha(\tilde{\Gamma}_2^2) = \mu^{2}\alpha(\Gamma_2)$$

$$\alpha(\Gamma_1^2)+\alpha(\Gamma_3^2) = \mu^2(\alpha(\tilde{\Gamma}_1^2)+\alpha(\tilde{\Gamma}_3^2)) \leq \mu^2\alpha(\tilde{\Gamma}_2^2) = \mu^4\alpha(\Gamma^2).$$

Hole Area Estimation

Lemma

Suppose the boundaries of the initial circle domain C^0 *are* $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$*, consider the reflection tree with* m *layers, then the total area of the holes bounded by the interior boundaries of leaf nodes is no greater than* μ^{4m} *times the area bounded by* Γ_k*'s,*

$$\sum_{(i)=i_1 i_2 \dots i_m} \sum_{k \neq i_1} \alpha(\Gamma_k^{(i)}) \leq \mu^{4m} \sum_{i=1}^{n} \alpha(\Gamma_i). \quad (2)$$

Proof.

By induction on m. The area bounded by the exterior boundaries of the nodes in the $k+1$-layer is no greater than μ^4 times that of the k-layer. The total area of the interior boundaries of leaf nodes is no greater than the area bounded by the exterior boundaries of leaf nodes. □

Uniqueness

Theorem (Uniqueness)

Given two circle domains $C_1, C_2 \subset \hat{\mathbb{C}}$, $f : C_1 \to C_2$ is a univalent holomorphic function, then f is a linear rational, namely a Möbius transformation.

Proof.

Assume both C_1 and C_2 include ∞, and $f(\infty) = \infty$. Since f is holomorhic, it maps the boundary circles of C_1 to those of C_2. By Schwartz reflection principle, f can be extended to the multiple reflected domains. By the area estimation of the holes Eqn. 2, the multiple reflected domains cover the whole $\hat{\mathbb{C}}$, hence f can be extended to the whole $\hat{\mathbb{C}}$, since $f(\infty) = \infty$, f is a linear function. If $f(\infty) \neq \infty$, we can use a Möbius map to transform $f(\infty)$ to ∞. $\square$

Existence

Definition (Kernel)

Suppose $\{B_n\}$ is a family of domains on the complex plane, $\infty \in B_k$ for all k. Suppose B is the maximal set: $\infty \in B$, and for any closed set $K \subset B$, there is an N, such that for any $n > N$, $K \subset B_n$. Then B is called the kernel of $\{B_n\}$.

Definition (Domain Convergence)

We say a sequence $\{B_n\}$ converges to its kernel B, if any sub-sequence $\{B_{n_k}\}$ of $\{B_n\}$ has the same kernel B. We denote $B_n \to B$.

Goluzin Theorem

Theorem (Goluzin)

Let $\{A_n\}$ be a sequence of domains on the complex domain. Any domain A_n includes ∞, $n = 1, 2, \cdots,$. Assume $\{A_n\}$ converges to its kernel A. Let $\{f_n(z)\}$ be a family of analytic function, for all n, $f_n(z)$ maps A_n to B_n surjectively, such that $f_n(\infty) = \infty$, $f_n'(\infty) = 1$. Then $\{f_n(z)\}$ uniformly converges to a univalent analytic function $f(z)$ in the interior of A, if and only if $\{B_n\}$ converges to its kernel B, then the univalent analytic function $f(z)$ maps A to B surjectively.

Existence

Theorem (Existence)

On the z-plane, every n-connected domain Ω can be mapped to a circle domain on the ζ-plane by a univalent holomorphic function. Choose a point $a \in \Omega$, there is a unique map which maps a to $\zeta = \infty$, and in a neighborhood of $z = a$, the map has the power series

$$\frac{1}{z-a} + a_1(z-a) + \cdots \text{ if } a \neq \infty$$

$$z + \frac{a_1}{z} + \cdots \text{ if } a = \infty$$

Existence

Proof.

According to Hilbert theorem, all n-connected domains are conformally equivalent to slit domains. We can assume Ω is a slit domain. We use $\mathcal{S}$ represent all the n-connected slit domains with horizontal slits, and $\mathcal{C}$ the n-connected circle domains. We label all the boundaries of the domains, $\partial\Omega = \bigcup_{k=1}^{n} \gamma_k$. For each slit γ_k, we represent it by the starting point p_k and the length l_k, then we get the coordinates of the slit domain Ω

$$(p_1, l_1, p_2, l_2, \cdots, p_n, l_n).$$

Hence $\mathcal{S}$ is a connected open set in $\mathbb{R}^{3n}$. Similarly, consider a circle domain $\mathcal{D} \in \mathcal{C}$, we use the center and the radius to represent each circle (q_k, r_k), and the coordinates of $\mathcal{D}$ are given by,

$$(q_1, r_1, q_2, r_2, \cdots, q_n, r_n).$$

$\mathcal{C}$ is also a connected open set in $\mathbb{R}^{3n}$. □

Existence

continued

Consider a normalized univalent holomorphic function $f : \Omega \to \mathcal{D}$, $\Omega \in \mathcal{S}$ and $\mathcal{D} \in \mathcal{C}$, f maps the k-th boundary curve γ_k to the k-th circular boundary of $\mathcal{D}$. By the existence of slit mapping and the uniqueness of circle domain mapping, we have

1. Every circle domain $\mathcal{D} \in \mathcal{C}$ corresponds to a unique slit domain $\Omega \in \mathcal{S}$;
2. Every slit domain $\Omega \in \mathcal{S}$ corresponds to at most one circle domain $\mathcal{D} \in \mathcal{C}$.

Then we establish a mapping from circle domains to slit domains $\varphi : \mathcal{C} \to \mathcal{S}$.

Existence

continued

Assume $\{\mathcal{D}_n\}$ is a family of circle domains, converge to the kernel $\mathcal{D}^*$. The domain convergence definition is consistent with the convergence of coordinates, namely, the boundary circles of $\mathcal{D}_n$ converge to the corresponding boundary circles of $\mathcal{D}^*$, denoted as $\lim_{n\to\infty} \mathcal{D}_n = \mathcal{D}^*$. The convergence of slit domains can be similarly defined. By Goluzin's theorem, we obtain the mapping $\varphi : \mathcal{C} \to \mathcal{S}$ is continuous:

$$\varphi(\lim_{n\to\infty} \mathcal{D}_n) = \lim_{n\to\infty} \varphi(\mathcal{D}_n).$$

By the uniqueness of circle domain mapping, we obtain φ is injective. We will prove the mapping φ is surjective.

Existence

continued

$\mathcal{C}$ is an open set in Euclidean space $\varphi : \mathcal{C} \to \mathcal{S}$ is injective continuous map. According to invariance of domain theorem, $\varphi(\mathcal{C})$ is an open set, $\varphi : \mathcal{C} \to \varphi(\mathcal{C})$ is a homeomorphism.
Choose a circle domain $\mathcal{D}_0 \in \mathcal{C}$, its corresponding slit domain is $\varphi(\mathcal{D}_0) = \Omega_0 \in \mathcal{S}$, then $\Omega_0 \in \varphi(\mathcal{C})$. Choose another slit map $\Omega_1 \in \mathcal{S}$, we don't know if Ω_1 is in $\varphi(\mathcal{C})$ or not. We draw a path $\Gamma : [0,1] \to \mathcal{S}$, $\Gamma(0) = \Omega_0$ and $\Gamma(1) = \Omega_1$. Let

$$t^* = \sup\{t \in [0,1] | \forall 0 \leq \tau \leq t, \Gamma(\tau) \in \varphi(\mathcal{C})\},$$

namely Γ from starting point to t^* belongs to $\varphi(\mathcal{C})$.

Existence

continued

By the definition of domain convergence,

$$\lim_{n\to\infty} \Gamma(t_n) \to \Gamma(t^*).$$

By $\{\Gamma(t_n)\} \subset \varphi(\mathcal{C})$, there is a family of circle domains $\{\mathcal{D}_n\} \subset \mathcal{C}$, $\varphi(\mathcal{D}_n) = \Gamma(t_n)$. Let $\lim_{n\to\infty} \mathcal{D}_n = \mathcal{D}^*$, by domain limit theorem, we have

$$\varphi(\mathcal{D}^*) = \varphi(\lim_{n\to\infty} \mathcal{D}_n) = \lim_{n\to\infty} \varphi(\mathcal{D}_n) = \lim_{n\to\infty} \Gamma(t_n) = \Gamma(t^*),$$

namely $\varphi(\mathcal{D}^*) = \Gamma(t^*)$, hence $\Gamma(t^*) \in \varphi(\mathcal{C})$. But $\varphi(\mathcal{C})$ is an open set, hence if $t^* < 1$, t^* can be further extended. This contradict to the choice of t^*, hence $t^* = 1$. Therefore $\Omega_1 \in \varphi(\mathcal{C})$. Since Ω_1 is arbitrarily chosen, hence $\varphi : \mathcal{C} \to \mathcal{S}$ is surjective. This proves the existence of the circle domain mapping.