Prime Decomposition and Haken Hierarchy

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Prime Decomposition

Main Strategy

Definition (Essential Surfaces)

Let M be a compact, oriented three manifold and $S \subset M$ be a properly embedded connected compact surface. We say S is *essential*, if

- $\chi(S) = 2$, S doesn't bound a sphere;
- $\chi(S) = 1$, S is not ∂-parallel;
- $\chi(S) \leq 0$, S is incompressible, ∂-incompressible, and not ∂-parallel.

Main Idea

Cut M along essential surfaces (iteratively) to obtain a decomposition (hierarchy).

- Essential Spheres: Prime decomposition;
- Essential Spheres and Tori: JSJ decomposition;
- Incompressible Surfaces: Haken Hirearchy.

Prime Decomposition

Definition (Connected Sum)

The connected sum $M_1 \# M_2$ of two oriented connected 3-manifolds M_1, M_2 is constructed by removing the interiors of two closed balls from M_1 and M_2, and then gluing the two resulting spheres via any orientation-reversing diffeomorphism.

Definition (∂-Connected Sum)

The ∂-connected sum $M_1 \#_\partial M_2$ of two oriented 3-manifolds with boundary is constructed by gluing two discs $D_1 \subset \partial M_1$ and $D_2 \subset \partial M_2$ via an orientation-reversing diffeomorphism.

Use Van-Kampen, $\pi_1(M_1 \# M_2) = \pi_1(M_1) * \pi_1(M_2)$, and $H_1(M_1 \# M_2, \mathbb{Z}) = H_1(M_1, \mathbb{Z}) \oplus H_1(M_2, \mathbb{Z})$.

Irreducible vs. Prime Manifolds

Definition (Irreducible Manifold)

The manifold M is irreducible if every sphere $S \subset \mathrm{int}(M)$ bounds a ball.

Definition (Prime Manifold)

A connected, oriented 3-manifold M is *prime* if every connected sum $M = M_1 \# M_2$ is trivial, namely either M_1 or M_2 is a sphere.

Proposition

Every oriented 3-manifold $M \neq S^2 \times S^1$ is prime if and only if it is irreducible.

Alexander's Theorem

Theorem (Alexander's Theorem)

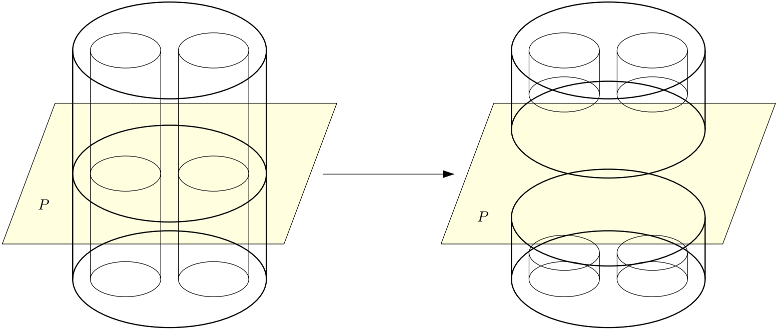
The space $\mathbb{R}^3$ is irreducible.

Proof.

Let $S \subset \mathbb{R}^3$ be a 2-sphere. By small perturbations, the height function $f|_S$ is a Morse function, and the k critical points are with distinct heights $z_1 < z_2 \cdots < z_k$.

$\square$

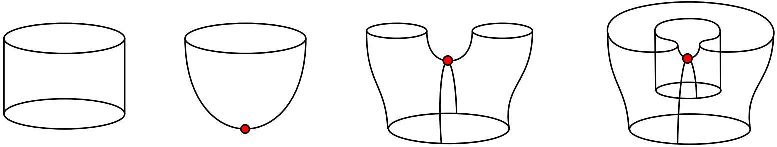
Alexander's Theorem



Proof.

Pick a regular value $u_i \in (z_i, z_{i+1})$ for every $i = 1, \ldots, k-1$. The horizontal plane P at height u_i intersects S transversely into circles. Starting from the innermost ones, we cut S along these circles and cap them by adding pairs of discs. The resulting surface is disjoint from P. $\square$

Alexander's Theorem



Proof.

At every cut a sphere is decomposed into two spheres. If we do this for every $i = 1, 2, \dots, k-1$ we end up with many spheres of the types in the figure that bound balls in $\mathbb{R}^3$.

We reverse the process and undo all the cuts: at each backward step we have a set of spheres bounding balls. At each backward step we replace two spheres S_1, S_2 bounding balls B_1, B_2 with one sphere S. Isotope S_1 and S_2, so that they intersect in a disc D. If the interiors of B_1 and B_2 are disjoint, then S bounds the ball $B_1 \cup B_2$. If they are not disjoint, then one is contained in the other, say $B_1 \subset B_2$ and S bounds the ball $B_2 \setminus \mathrm{int}(B1)$. $\square$

Prime Manifolds

Proposition

The manifold $S^2 \times S^1$ is prime.

Proof.

Let $S \subset S^2 \times S^1$ is a separating sphere, it separates $S^2 \times S^1$ into two manifolds M and N. Then $\pi_1(S^2 \times S^1) = \pi_1(M) \times \pi_1(N) = \mathbb{Z}$, this implies either $\pi_1(M)$ or $\pi_1(N)$ is trivial, assume $\pi_1(M)$ is trivial. Since M

is trivial, a copy M' of M lifts to the universal cover $S^2 \times \mathbb{R}$ of $S^2 \times S^1$. We identify $S^2 \times \mathbb{R} = \mathbb{R}^3 \setminus \{0\}$. The copy $M' \subset \mathbb{R}^3$, and $\partial M' = S^2$, by Alexander's theorem M' is a ball, so M is a ball. $\square$

Irreducible vs. Prime Manifolds

Proposition (Prime vs. Irreducible)

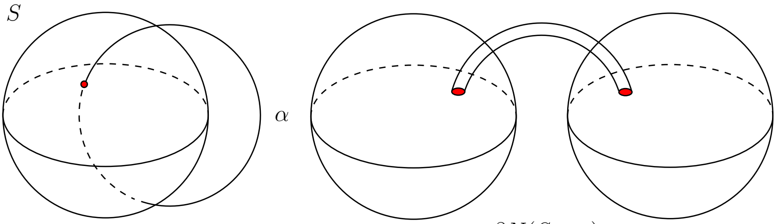
Every oriented 3-manifold $M \neq S^2 \times S^1$ is prime if and only if it is irreducible.

Proof.

The inverse operation of a connected sum $M = M_1 \# M_2$ consists of cutting along a separating sphere $S \subset M$ and then capping off the two resulting manifolds N_1, N_2 with balls. Therefore M is prime if and only if every separating sphere $S \subset M$ bounds a ball.

If M is irreducible, then every separating sphere bounds a ball, then M is prime. □

Irreducible vs. Prime Manifolds

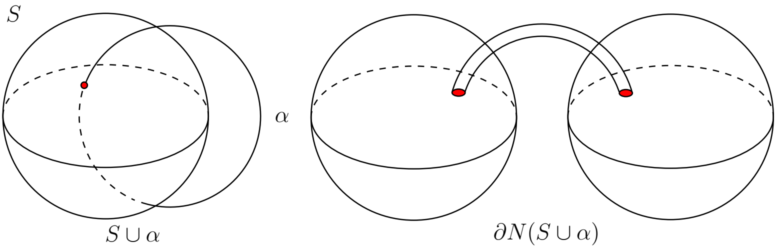


Proof.

Suppose M is prime and not irreducible, then there must be a sphere $S \subset M$ that is non-separating. Otherwise each sphere $S \subset M$ is separating, then S bounds a all, M is irreducible, contradiction. Assume $S \subset M$ is

non-separating. Then there is a simple closed curve $\alpha \subset M$ intersecting S transversely in one point. Pick a neighborhood of $S \cup \alpha$ denoted as $N(S \cup \alpha)$, then the boundary $S' = \partial N(S \cup \alpha)$ is a separating sphere. Since M is prime, then S' bounds a ball B. Therefore $M = N \cup B$. $\square$

Irreducible vs. Prime Manifolds



Proof.

We show $N \cup B = S^2 \times S^1$. $S \cup \alpha$ is embedded in $S^2 \times S^1$, $S = S^2 \times \{y\}$ and $\alpha = \{x\} \times S^1$. Decompose $S^2 = D \cup D'$ and $S^1 = I \cup I'$, then $N = S^2 \times I \cup D \times I'$, its complement $B = D' \times I'$ is a ball. □

Irreducible Manifolds

Proposition

Let $p : M \to N$ be a covering of 3-manifolds. If M is irreducible then N also.

Proof.

A sphere $S \subset N$ lifts to many spheres in M, each bounding at least one ball. Pick an inner most such ball B, then $p(B)$ is a ball with boundary S. □

Corollary

Elliptic, flat, hyperbolic 3-manifolds are irreducible.

Proof.

Their universal covering is diffeomorphic to S^3 or R^3. □

Irreducible Manifolds

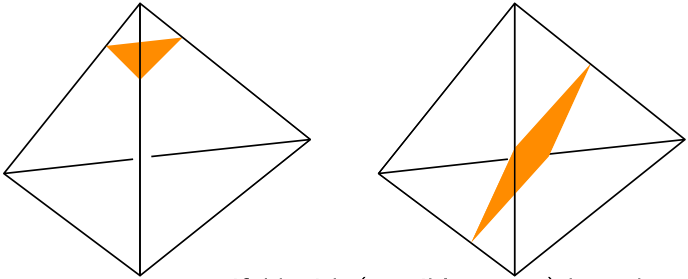
Proposition

If $g \geq 1$ then $S_g \times [0,1]$ is irreducible.

Proof.

Its universal cover is $\mathbb{R}^2 \times [0,1]$ for $g = 1$, or $\mathbb{H}^2 \times [0,1]$ for $g > 1$. It is irreducible, because its interior is diffeomorphic to $\mathbb{R}^3$. $\square$

Normal Surface



Let M be a compact 3-manifold with (possibly empty) boundary with a triangulation T. A properly embedded surface $S \subset M$ is *transverse* to T if it is transverse to all its simplexes.

Definition (Normal Surface)

A *normal surface* is a properly embedded surface S transverse to T that intersects every tetrahedron into triangles or squares.

Normal Surface

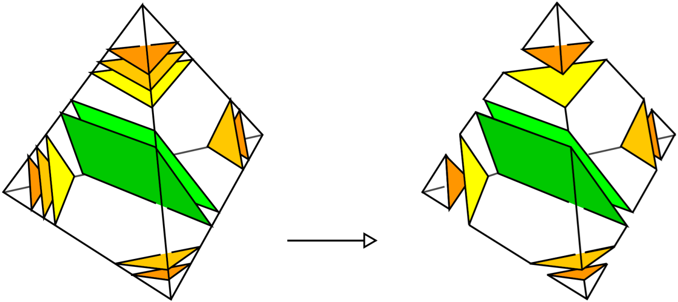


Figure: A tetrahedron Δ is cut along triangles and squares, we get arbitrarily many prisms with triangular or quarilateral basis, and most 6 other pieces.

Normal Surface

Two disjoint connected diffeomorphic surfaces $\Sigma, \Sigma' \subset M$ are *parallel* if they cobound a region diffeomorphic to $\Sigma \times [0,1]$ with $\Sigma \times 0$ and $\Sigma' = \Sigma \times 1$.

Let T be a triangulation of a compact M with boundary. Let t be the number of tetrahedra in T and set $b_2 = \dim H_2(M, \partial M, \mathbb{Z})$.

Lemma

Let S be an orientable normal surface. If S has more than $10t + b_2$ components, then two components Σ, Σ' of S are parallel and cobound a $\Sigma \times [0,1]$ which is disjoint from the other components.

Normal Surface

Proof.

The complement $M \setminus S$ intersects every tetrahedron Δ into polyhedra: there are many prisms lying between parallel triangles or squares, and at most 6 other pieces. These 6 pieces are adjacent to at most 10 triangles and squares. This implies that, except at most $10t$ of them, the

components of S are only adjacent to prims. These prims glue to form I-bundles. Therefore at least $b_2 + 1$ components of S are adjacent to I-bundles on both sides. The twisted I-bundles are at most b_2, and each is adjacent to one surface. Therefore at least one surface is adjacent to a product I-bundle $\Sigma \times [0, 1]$. $\square$

Sphere System

Definition (Sphere System)

A *ball with holes* is a 3-manifold obtained by removing some $k \geq 0$ disjoint open balls from a ball. A *sphere system* for a manifold M is a surface $S \subset M$ consisting of disjoint separating spheres, such that no component in $M \setminus S$ is a ball with holes.

Corollary (Sphere System)

Let M be a compact orientable 3-manifold. There is a $K > 0$ such that every sphere system in M contains less than K spheres.

We can transform a sphere system S to a normal surface S', above lemma gives a contradiction on S'.

Prime Decomposition

Theorem (Prime Decomposition)

Every compact oriented 3-manifold M with (possibly empty) boundary decomposes into prime manifolds:

$$M = M_1 \# M_2 \# \dots \# M_k$$

This list of prime factors is unique up to permutations and adding/removing copies of S^3.

Existence by Kneser in 1929, Uniqueness by Milnor in 1962.

Prime Decomposition

Proof.

(Existence) If M contains a non-separating sphere, then the proof of proposition [Prime vs. Irreducible] shows that $M = M' \# (S^2 \times S^1)$. Since $H_1(M) = H_1(M') \oplus \mathbb{Z}$, up to factoring finitely many copies of $S^2 \times S^1$ we may suppose that every sphere in M is separating.

If M is prime we are done. If not, it decomposes as $M = M_1 \# M_2$. We keep decomposing each factor until all factors are prime: this process must end, because a decomposition $M = M_1 \# M_2 \# \ldots \# M_k$ gives rise to a system of $(k-1)$ spheres, and k can not be arbitrarily big by the corollary [Sphere System]. $\square$

Prime Decomposition

Proof.

(Uniqueness) Let two prime decompositions

$$M = M_1 \# \cdots \# M_k \#_h (S^2 \times S^1), \quad M = M_1' \# \cdots \# M_k \#_{h'} (S^2 \times S^1)$$

be with $M_i, M_j' \neq S^2 \times S^1$, so M_i, M_j' are irreducible for all i, j.
We say that a set $S \subset M$ of disjoint spheres is a *reducing set of spheres* for the decomposition $M = M_1 \# \ldots \# M_k \#_h (S^2 \times S^1)$ if $M \setminus S$ consists of precisely one M_i with some holes for each i, and some balls with holes. In general, we may construct S by taking the spheres of the prime decomposition, plus one non-separating sphere inside each $S^2 \times S^1$ summand. Similarly, let S' be reducing set of spheres for the other decomposition. □

Prime Decomposition

Proof.

The observation we make is that if we add to S any sphere Σ disjoint from S, then we still get a reducing set of spheres for the same decomposition. This is because Σ is contained in a holed $N = M_i$ or S^3, and since N is irreducible Σ bounds a ball B there. Therefore, by adding Σ we still get the same holed N, plus a possibly holed (if $B \cap S \neq \emptyset$) ball B. $\square$

Prime Decomposition

Proof.

We assume S and S' intersect transversely in circles and pick an innermost circle in a component of S bounding a disk $D \subset S$. We surger S' along D, thus substituting a component S'_0 of S' with two spheres $S'_1 \cup S'_2$. The result is another sphere system for the same decomposition. We isotope the spheres S'_0, S'_1, S'_2 so that they are disjoint and cobound a ball with two holes B_2: the system $S' \cup S'_1 \cup S'_2$ is still reducing by the observation above. The removal of S'_0 then adds B_2 to the outside of S'_0, and this is equivalent to making one more hole there.

After finitely many suergeries we get $S \cap S' = \emptyset$. By the same observation above $S \cup S'$ is a reducing set of spheres for both decompositions: therefore the piececes M_i and M'_j of the decompositions are pairwise diffeomorphic.

Finally we must have $h = h'$ since $M = N\#_h(S^2 \times S^1) = N\#_{h'}(S^2 \times S^1)$ and $H_1(M) = H_1(N) \oplus \mathbb{Z}^h = H_1(N) \oplus \mathbb{Z}^{h'}$. $\square$

Properly Embedding

Definition (Properly Embedding)

A surface is *properly* embedded in the ambient 3-manifold, namely the boundary of the surface mapped by the embedding to the boundary of the 3-manifold :

$$S \cap \partial M = \partial S,$$

where the intersection is transverse.

∂-irreducible

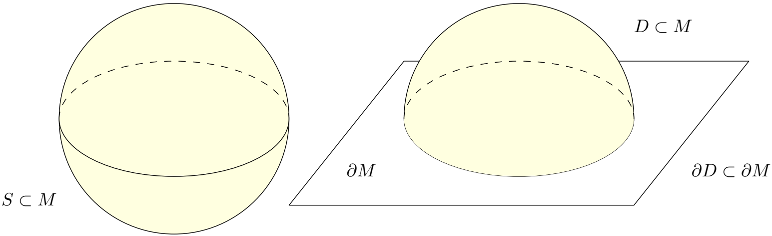


Figure: Non-essential sphere $S \subset M$ and non-essential hemi-sphere (disk) $D \subset M$.

Definition (∂-parallel)

Let M be a compact 3-manifold with (possibly empty) boundary. A properly embedded surface $S \subset M$ is *∂-parallel* if it is obtained by pushing inside M the interior of a compact surface $S' \subset \partial M$, possibly with boundary.

∂-irreducible

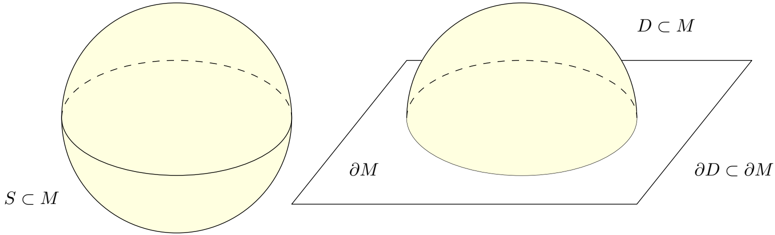


Figure: Non-essential sphere $S \subset M$ and non-essential hemi-sphere (disk) $D \subset M$.

Definition (Essential Sphere/Disk)

Properly embedded sphere $S \subset M$ is *essential*, if it doesn't bound a ball. A disk $D \subset M$ is *essential*, if it is not ∂-parallel.

Definition (∂-irreducible manifold)

The manifold M is *irreducible* (∂-irreducible) if it dosen't contain essential spheres (disks).

Decomposition Along Discs

Definition (Disc System)

A *disc system* in M is a set of pairwise disjoint, non-parallel essential disks.

Proposition

There is a $K > 0$ such that every disc system in M cannot contain more than K discs.

The oppositie operation of cutting a manifold along a properly embedded disc is a 1-hanle addition.

Theorem (∂-prime decomposition)

Every compact oriented irreducible 3-manifold M is obtained by adding 1-handles to a finite set

$$M_1, M_2, \ldots, M_k$$

of connected irreducible and ∂-irreducible 3-manifolds. The list is unique up to permutations and adding/removing balls.

Haken Manifold

Incomressible Surface

Definition (Compressing Disc)

M is a compact orientable 3-manifold with (possibly empty) boundary. Let $S \subset M$ be a properly embedded orientable surface. A *compressing disc* for S is a disc $D \subset M$ with $\partial D = D \cap S$, such that ∂D doesn't bound a disc in S.

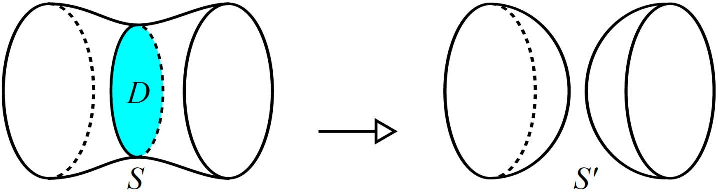


Figure: Compression operator: A surface S is surgered along a compressing disk D. The operation consists of removing an annular tubular neighborhood of ∂D in S and adding two parallel copies of D. The result is a new surface S'.

Definition (∂-Compressing Disc)

M is a compact orientable 3-manifold with boundary. Let $S \subset M$ be a properly embedded orientable surface. A *-compressing disc* for S is a disc $D \subset M$ with $\partial D = D \cap S$, such that ∂D doesn't bound a disc in S, D touches ∂M in a segment.

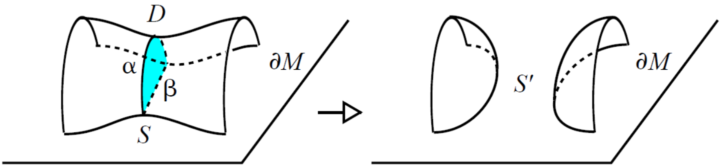


Figure: ∂-Compression operator: A surface S is surgered along a ∂-compressing disk D, which touces the boundary ∂M in a segment.

Incomressible Surface

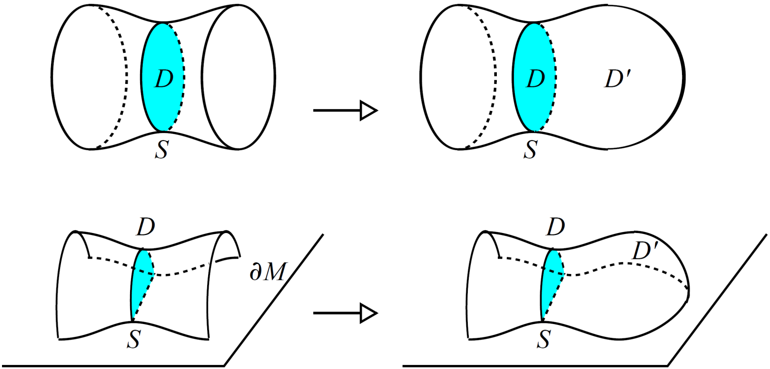


Figure: A surface S is incompressible (∂-incompressible) if the existence of disk D implies the existence of $D' \subset S$, D and D' form sphere and bounds a ball.

Incomressible Surface

Definition (Incompressible Surface)

A properly embedded connected orientable surface $S \subset M$ with $\chi(S) \leq 0$ is *compressible* (∂-compressible) if it has a compressing (∂-compressing) disk, and *incompressible* (∂-incompressible) otherwise.

Definition (∂-Incompressible Surface)

A properly embedded connected orientable surface $S \subset M$ with $\chi(S) \leq 0$ is ∂-*compressible* if it has a ∂-compressing disk, and ∂-*incompressible* otherwise.

Incomressible Surface

Proposition

The surface S' is obtained by compressing (∂-compressing) S, S' may have one or two components S'_i, and $\chi(S'_i) > \chi(S)$ for each component.

Proof.

$\chi(S') = \chi(S) + 2$. If S' has one component we are done, so suppose $S' = S'_1 \cup S'_2$. Since ∂D did not bound a disk in S, no S'_i is a sphere, hence $\chi(S'_i) \leq 1$ implies $\chi(S'_i) > \chi(S)$. $\square$

Corollary

Let $S \subset M$ be any properly embedded surface. After compressing (∂-compressing) it a finite number of times it transforms into a disjoint union of spheres, disks, and incompressible (∂-incompressing) surfaces.

Incomressible Surface

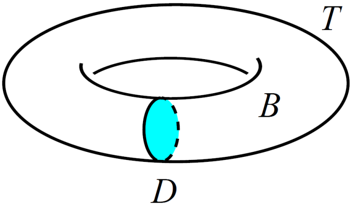
Proposition

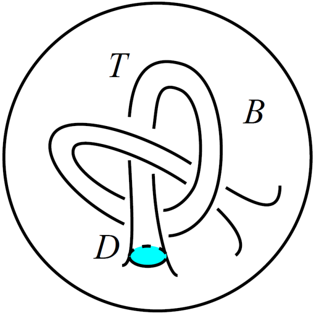
Let $S \subset M$ be an orientable, connected, properly embedded surface with $\chi(S) \leq 0$, the inclusion map is $i : S \to M$, S is incompressible if and only if $i_\# : \pi_1(S) \to \pi_1(M)$ is injective.

Proof.

If $i_\# : \pi_1(S) \to \pi_1(M)$ is injective, then S is incompressible. Otherwise there is a disk D compresses S, ∂D is trivial in $\pi_1(M)$ but non-trivial in $\pi_1(S)$, contradiction. $\square$

Incomressible Surface





Proposition

Let $T \subset M$ be a torus in an irreducible 3-manifold, one of the following holds:

1. *T is incompressible,*
2. *T bounds a solid torus,*
3. *T is contained in a ball.*

Incomressible Surface

Proof.

If T is not incompressible, it compresses along a compressing disk D. The result of the compression is a sphere $S \subset M$ which bounds a ball $B \subset M$ since M is irreducible. If B is disjoint from T, then T bounds a solid torus; If B contains T, then case 3 holds. $\square$

Compression Annuli

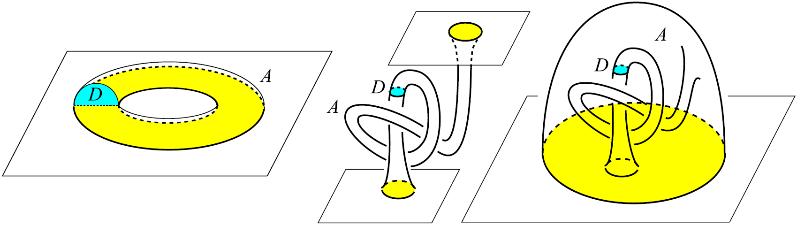


Figure: An annulus A in an irreducible and ∂-irreducible 3-manifold is either incompressible, or is parallel to annulus in ∂M (left), or bounds a tube (center), or is contained in a ball intersecting ∂M in a disc (right).

Compression Annuli

Proposition

Let $A \subset M$ be a properly embedded annulus in an irreducible and ∂-irreducible 3-manifold. One of the following holds:

1. *A is incompressible and ∂-incompressible,*
2. *A bounds a tube,*
3. *A is parallel to an annulus in ∂M,*
4. *A is contained in a ball B intersecting ∂M in a disc.*

Corollary

Let $A \subset M$ be a properly embedded annulus in an irreducible and ∂-irreducible 3-manifold. If the components of ∂A are non-trivial and non-parallel in ∂M, the annulus A is compressible and ∂-incompressible.

Compression Annuli

Proof.

If A compresses along a disc D, it transforms into two discs that are parallel to two discs $D_1, D_2 \subset \partial M$ since M is ∂-irreducible. If $D_1 \cap D_2 = \emptyset$ then A bounds a tube as in middle frame; if $D_1 \subset D_2$ then A is contained in a ball B intersecting ∂M in D_2 as in right frame.

If A ∂-compresses along a disc D, it transforms into a disc which is again ∂-parallel and hence A is as in left frame. $\square$

Haken Manifolds

Definition (Haken Manifold)

A *Haken manifold* is a compact, oriented 3-manifold M with (possibly empty) boundary, which is irreducible, ∂-irreducible, and contains an incompressible and ∂-incompressible surface.

Haken Manifolds

Proposition

Every boundary component X of a Haken manifold M has $\chi(X) \leq 0$ and is incompressible.

Proof.

No component X of ∂M is a sphere. Otherwise, suppose X is a sphere, since M is irreducible, any sphere in it bounds a ball B, then $M = B$. A ball doesn't contain any incompressible surface, hence M is not Haken. Contradiction. Hence $\chi(X) \leq 0$.
Since M is ∂-irreducible, for any disk D, $\partial D \subset X$ implies ∂D bounds a disk $D' \subset X$, hence D is not compressing. Hence X is incompressible. $\square$

Haken Manifolds

Proposition

Let M be an oriented, compact, irreducible, and ∂-irreducible 3-manifold with (possibly empty) boundary. Every non-trivial homology class $\alpha \in H_2(M, \partial M; \mathbb{Z})$ is represented by a disjoint union of incompressible and ∂-incompressible oriented surfaces.

Proof.

Every class α is represented by a properly embedded oriented surface S. A compression or ∂-compression doesn't alter the homology class of the surface: we have $S' - S = \partial B$ where $B = D \times [-1, 1]$ is a tubular neighborhood of the compressing disk D. Hence $[S'] = [S] = \alpha$.
We compress S until its connected comoents are either incompressible and ∂-incompressible surfaces, discs, or spheres. Since M is irreducible and ∂-irreducible, dics and spheres bound balls and hence homologically triviale. So they can be removed. □

Haken Manifolds

Corollary

Let M be oriented, compact, irreducible, and ∂-irreducible. If $H_2(M, \partial M; \mathbb{Z}) \neq \{e\}$ then M is Haken.

Corollary

Let M be oriented, compact, irreducible, and ∂-irreducible. If $\partial M \neq \emptyset$ and $M \neq B$, then M is Haken.

Proof.

If ∂M contains a sphere, it bounds a ball B and hence $M = B$. Otherwise $H_1(\partial M)$ has positive rank, and hence $H_2(M, \partial M) = H^1(M)$ also has positive rank since $b_1(M) \geq \frac{b_1(\partial M)}{2}$. □

Haken Manifolds

Proposition

Let M be compact and irreducible, and $S \subset M$ be either an essential disc or an incompressible surface. Let M' be obtained by cutting M along S. The following holds:

- *the manifold M' is irreducible;*
- *a closed $\Sigma \subset M'$ is incompressible in M' if and only if it is so in M.*

Corollary

If we cut a Haken 3-manifold along a closed incompressible surface, we get a disjoint union of Haken 3-manifolds.

Proof.

Let $\Sigma \subset M'$ be a sphere. Since M is irreducible, the sphere Σ bounds a ball $B \subset M$. The ball can not contain S because all surfaces in a ball are compressible. Therefore $B \subset M'$ and M' is irreducible. Hence M' is irreducible.

To prove the second assertion, we show that Σ has a compressing disk D in M if and only if it has one in M'. If D lies in M' then it lies also in M as well. Conversely, suppose D lines in M. Put D in transverse position with respect to S and pick an innermost intersection circle in D, bounding a disc $D' \subset D$. Since D' can not compress S, and since M is irreducible, the disc D' can be isotoped away from S. This simplifies $D \cap S$ and after finitely many steps we get $D \cap S = \emptyset$ and hence $D \subset M'$. □

Haken Manifolds - Normal Surfaces

Proposition

Let M be Haken and T be a triangulation for M. Every compact surface $S \subset M$ whose components are all incompressible and ∂-incompressible is isotopic to a normal surface.

Corollary

Let M be a Haken manifold. There is a $K > 0$ such that every set S of pairwise disjoint and non-parallel incompressible and ∂-incompressible surfaces in M consists of at most K elements.

Haken Manifolds - Hierarchies

Definition (Hierarchy)

A *hierarchy* for a Haken 3-manifold M is a sequence of 3-manifolds

$$M = M_0 \xrightarrow{S_0} M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \cdots \xrightarrow{S_{h-1}} M_h,$$

where each M_{i+1} is obtained cutting M_i along a properly embedded (possibly disconnected) surface $S_i \subset M_i$, such that the following holds:

- every component of S_i is an incompressible and ∂-incompressible surface or an essential disc, for all i;
- the final manifold M_h consists of balls.

The number h is the *height* of the hierarchy.

Theorem

Every Haken manifold has a hierarchy of height 3.

Proof.

Let S_0 be a maximal family of pairwise disjoint and non-parallel closed incompressible surfaces in M. We cut $M_0 = M$ along S_0 and get M_1. □

Line Bundles

Proposition (Line Bundles)

Fix $g \geq 1$. The product $M = S_g \times [-1,1]$ is irreducible and ∂-irreducible. The incompressible and ∂-incompressible surfaces in M are precisely the following (up to isotopy)

- *the horizontal surface $S_g \times 0$,*
- *a vertical annulus $\gamma \times [-1,1]$ for each non-trivial simple closed curve $\gamma \subset S_g$.*

Proof.

The universal cover of $S_g \times [-1,1]$ is $\mathbb{H}^2 \times [-1,1]$, which is irreducible, so $S_g \times [-1,1]$ is irreducible. The inclusion map induces the homomorphism $i_\# : \pi_1(S_g \times 0) \to \pi_1(S_g \times [-1,1])$, which is injective, so $S_g \times 0$ is incompressible.

The annulus boundary $\partial(\gamma \times [-1,1]) = \gamma \times \{-1\} \cup \gamma \times \{1\}$, which are non-trivial and non-parallel in $\partial(S_g \times [-1,1])$. Hence the annulus is incompressible.

Haken Manifolds - Hierarchies

Theorem

Every Haken manifold has a hierarchy of height 3.

Lemma

Every Haken manifold M contains an oriented surface S ("spanning surface"), whose components are incompressible and ∂-incompressible, such that $[\partial S \cap X] \in H_1(X, \mathbb{Z})$ is non-trivial for every boundary component X of M.

Proof.

Let S_0 be a maximal family of pairwise disjoint and non-parallel closed incompressible surfaces in M. We cut $M_0 = M$ along S_0 and get M_1. Every connected component M_1^i of M_1 is Haken, and there is a "spanning surface" $S_1^i \subset M_1^i$, cut along the spanning $S_1 = \cup_i S_1^i$ to obtain M_2. Then every M_2 contains no closed incompressible surface, and is a handlebody. Cut it along a set S_2 of essential discs to get balls. $\square$