Discrete Ricci Flow for Hyperbolic Surfaces

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August 29, 2024

Hyperbolic Surface Ricci Flow

Hyperbolic Delaunay Triangulation

Definition (Hyperbolic Polyhedral Metric)

Denote the set of marked points of $S_{g,n}$ by $\mathcal{B}$. We say that a metric d on $S_{g,n}$ is *hyperbolic polyhedral* if it is locally hyperbolic except points of $\mathcal{B}$ where d can be locally isometric to a hyperbolic cone. For $B_i \in \mathcal{B}$ we define the *curvature* $\kappa_d(B_i)$ to be 2π minus the cone angle of B_i.

Definition (Delaunay Triangulation)

Let T be a geodesic triangulation of $(S_{g,n}, d)$ with vertices at $\mathcal{B}$. We say that T is *Delaunay* if when we develop any two adjacent triangles to $\mathbb{H}^2$, the circumscribed disc of one triangle doesn't contain the fourth vertex in the interior.

Discrete Conformally Euqivalent

Definition (Discrete Conformally Equivalent)

We call two polyhedral hyperbolic metrics d' and d'' *discretely conformally equivalent* if there exists a sequence of pairs $\{(d_t, T_t)\}_{t=1}^m$, where d_t is a polyhedral hyperbolic metric on $S_{g,n}$, T_t is a Delaunay triangulation of $(S_{g,n}, d_t)$, $d_1 = d'$ and $d_m = d''$ and for every t either

1. $d_t = d_{t+1}$ in the sense that $(S_{g,n}, d_t)$ is isometric to $(S_{g,n}, d_{t+1})$ by an isometry isotopic to identity with respect to $\mathcal{B}$, or
2. $T_t = T_{t+1}$ and there exists a function $u : \mathcal{B} \to \mathbb{R}$ such that for every edge e of T_t with vertices B_i and B_j we have

$$\sinh\left(\frac{\mathrm{len}_{d_t}(e)}{2}\right) = \exp(u(B_i) + u(B_j)) \sinh\left(\frac{\mathrm{len}_{d_{t+1}}(e)}{2}\right)$$

Discrete Surface Uniformization

Theorem (Gu-Guo-Luo-Sun-Wu 2018)

Let d be a polyhedral hyperbolic metric on $S_{g,n}$ and $\kappa' : \mathcal{B} \to (-\infty, 2\pi)$ be a function satisfying

$$\sum_{B_i \in \mathcal{B}} \kappa'(B_i) > 2\pi(2-2g).$$

Then there exists a unique metric d' discretely conformally equivalent to d such that $\kappa_{d'}(B_i) = \kappa'(B_i)$ for all $B_i \in \mathcal{B}$.

The condition is necessary by the discrete Gauss-Bonnet theorem:

$$\sum_{B_i \in \mathcal{B}} \kappa'(B_i) = 2\pi(2-2g) + \mathrm{area}(S_{g,n}, d').$$

Discrete Surface Uniformization

Corollary (Discrete Uniformization)

Every polyhedral hyperbolic metric on $S_{g,n}$ is discretely conformally equivalent to a unique hyperbolic metric.

- Xianfeng Gu, Ren Guo, Feng Luo, Jian Sun and Tianqi Wu, **A discrete uniformization theorem for polyhedral surfaces II**, Journal of Differential Geometry (JDG), Volume 109, Number 3, Pages 431-466, 2018.
- Xianfeng Gu, Feng Luo, Jian Sun and Tianqi Wu, **A discrete uniformization theorem for polyhedral surfaces**, Journal of Differential Geometry (JDG), Volume 109, Number 2, Pages 223-256, 2018.

Polyhedral Surface

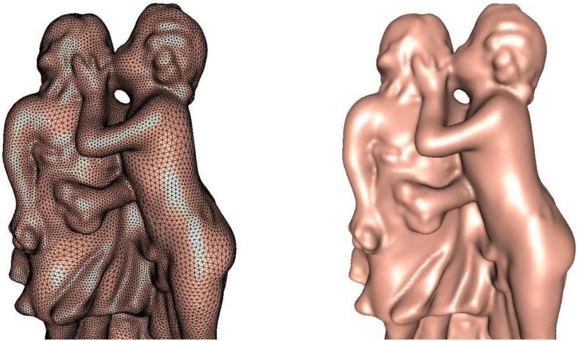


Figure: Polyhedral surface.

Background Geometries

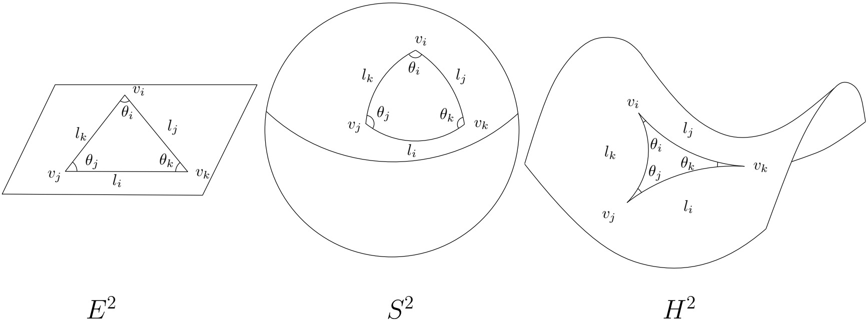


Figure: Constant curvature triangle.

We can glue hyperbolic or spherical triangles isometrically along the common edges to construct the triangle mesh. Then we say the surface is with hyperbolic or spherical background geometry.

Hyperbolic Triangle

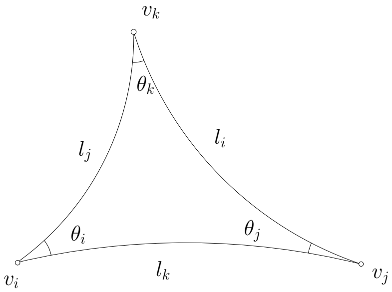


Figure: Hyperbolic triangle.

Cosine law:

$$\cos\theta_i = \frac{\cosh l_j \cosh l_k - \cosh l_i}{\sinh l_j \sinh l_k}$$

Sine law:

$$\frac{\sinh l_i}{\sin\theta_i} = \frac{\sinh l_j}{\sin\theta_j} = \frac{\sinh l_k}{\sin\theta_k}$$

Area

$$A = \frac{1}{2}\sinh l_j \sinh l_k \sin\theta_i$$

Hyperbolic Derivative Cosine Law

Lemma

The hyperbolic derivative cosine law is represented as:

$$\frac{\partial \theta_i}{\partial l_i} = \frac{\sinh l_i}{A}, \frac{\partial \theta_i}{\partial l_j} = -\frac{\sinh l_i}{A} \cos\theta_k$$

Compared with Euclidean cosine law, we replace the edge lengths l_i by $\sinh l_i$.

Curvature

Definition (Discrete Curvature)

Given a discrete surface with hyperbolic background geometry $(S, V, \mathcal{T}, l)$, every triangle is a hyperbolic geodesic triangle, the vertex discrete curvature is defined as the angle deficit

$$K(v) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk}, & v \notin \partial S \\ \pi - \sum_{jk} \theta_i^{jk}, & v \in \partial S \end{cases}$$

Theorem (Gauss-Bonnet)

The discrete Gauss-Bonnet theorem is represented as:

$$\sum_{v \notin \partial S} K(v) + \sum_{v \in \partial S} K(v) - Area(S) = 2\pi\chi(S)$$

Discrete Conformal Metric Deformation

Definition (Conformal Deformation)

Given discrete conformal factor function $u : V(\mathcal{T}) \to \mathbb{R}$, hyperbolic vertex scaling is defined as $y := u * l$,

$$\sinh\frac{y_k}{2} = e^{\frac{u_i}{2}}\sinh\frac{l_k}{2}e^{\frac{u_j}{2}}$$

Lemma (Symmetry)

The symmetric relations holds:

$$\frac{\partial\theta_i}{\partial u_j} = \frac{\partial\theta_j}{\partial u_i} = \frac{C_i + C_j - C_k - 1}{A(C_k + 1)}$$

where $S_k = \sinh y_k, C_k = \cosh y_k.$

Discrete Hyperbolic Entropy Energy

Definition (Hyperbolic Entropy Energy)

$$E_f(u_i,u_j,u_k) = \int^{(u_i,u_j,u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k.$$

The Hessian matrix of the entropy energy is:

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A} \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix} \begin{pmatrix} -1 & \cos\theta_3 & \cos\theta_2 \\ \cos\theta_3 & -1 & \cos\theta_1 \\ \cos\theta_2 & \cos\theta_1 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{S_1}{C_1+1} & \frac{S_1}{C_1+1} \\ \frac{S_2}{C_2+1} & 0 & \frac{S_2}{C_2+1} \\ \frac{S_3}{C_3+1} & \frac{S_3}{C_3+1} & 0 \end{pmatrix}$$

which is strictly convex.

Discrete Entropy Energy on a Mesh

Definition (Entropy Energy)

The entropy energy on a triangle mesh with hyperbolic background geometry equals to

$$E(\mathbf{u}) = \int^{\mathbf{u}} \sum_i (\bar{K}_i - K_i) du_i$$

Definition (Hyperbolic Ricci Flow)

Hence the discrete hyperbolic surface Ricci flow is defined as:

$$\frac{du_i(t)}{dt} = \bar{K}_i - K_i(t),$$

which is the gradient flow of the discrete hyperbolic entropy energy. The strict concavity of the discrete entropy ensures the uniqueness of the solution to the flow. The existence is given by Gu-Luo-Sun using Teichmüller theory and hyperbolic geometry.

Uniformizaton of High Genus Surface

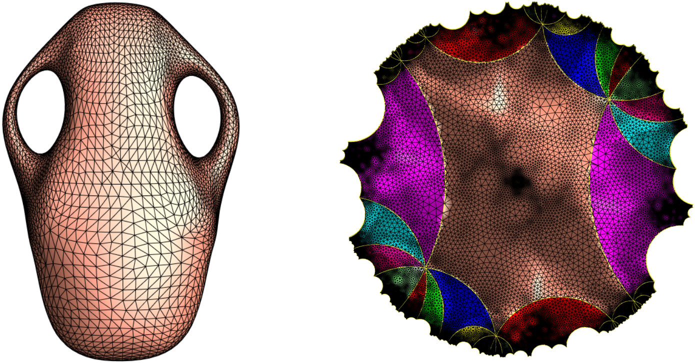


Figure: Uniformization of a genus two surface.

Uniformization

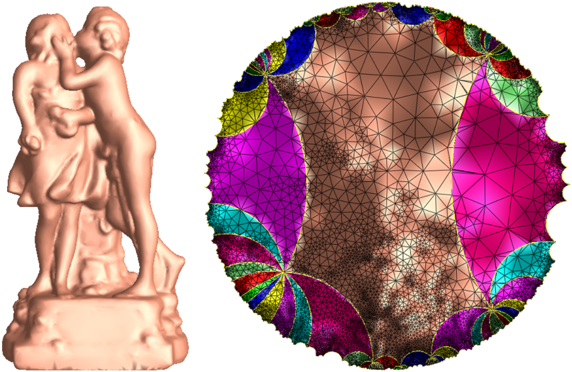


Figure: Uniformization of a genus three surface.

Uniformization

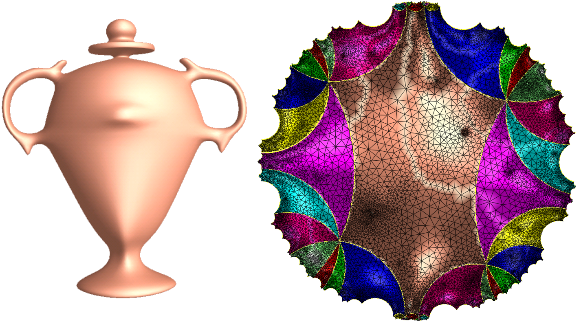


Figure: Uniformization of a genus two surface.

Shortest Word

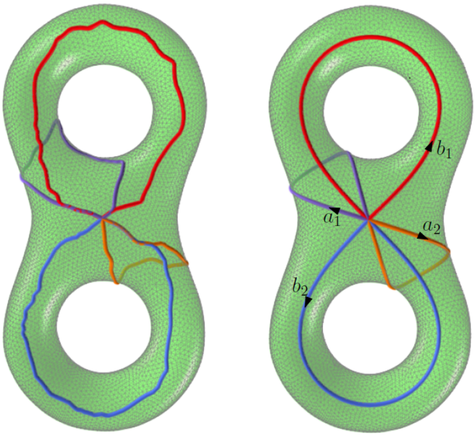
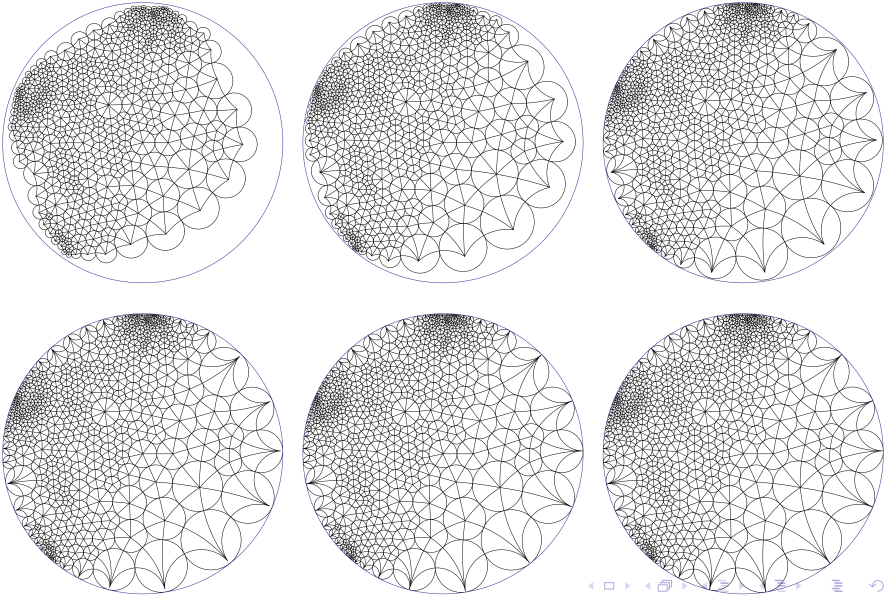


Figure: Shortest word problem.

Discrete Riemann Mapping



Generalized Alexandrov Theorem

Alexandrov Theorem

Theorem (Alexandrov)

For every convex polyhedral Euclidean metric d on $\mathbb{S}^2$ there is a convex polytope $P \subset \mathbb{R}^3$ such that (S^2, d) is isometric to the boundary of P. More over such P is unique up to an isometry of $\mathbb{R}^3$.

Note that P can degenerate to a polygon. In this case, P is doubley covered by the sphere.

Theorem (Rivin)

For every cusp metric d on $S_{0,n}$ there exists a convex ideal polyhedron $P \subset \mathbb{H}^3$ such that $(S_{0,n}, d)$ is isometric to the boundary of P. Moreover, such P is unique up to an isometry of $\mathbb{H}^3$.

Ideal Fuchsian Polyhedra

Define $G := \pi_1(S_g)$, let $\rho : G \to \mathrm{Iso}^+(\mathbb{H}^3)$ be a *Fuchsian representation*: an injective homomorphism such that its image is discrete and there is a geodesic plane invariant under $\rho(G)$. Then $F := \mathbb{H}^3/\rho(G)$ is a complete hyperbolic manifold homeomorphic to $S_g \times \mathbb{R}$. The image of the invariant plane is the so-called *convex core* of F and is homeomorphic to S_g. The manifold F is symmetric with respect to its convex core. The boundary at infinity of F consists of two connected components.

Ideal Fuchsian Polyhedra

A subset of F is called *convex* if it contains every geodesic between any two points. An *ideal Fuchsian polyhedron* P is the closure of convex hull of a finite point set in a connected component of $\partial_\infty F$. It has two boundary components: one is the convex core and the other is isometric to $(S_{g,n}, d)$ for a cusp metric d.

Generalization of Alexandrov Theorem

Generalization of Alexandrov theorem to surfaces of higher genus with cusp metrics:

Theorem

For every cusp metric d on $S_{g,n}$ with $g > 1$, $n > 0$, there exists a Fuchsian manifold F and an ideal Fuchsian polyhedron $P \subset F$ such that $(S_{g,n}, d)$ is isometric to the upper boundary of P. Moreover, F and P are unique up to isometry.

Hyperbolic Trapezoids and Prisms

Trapezoid

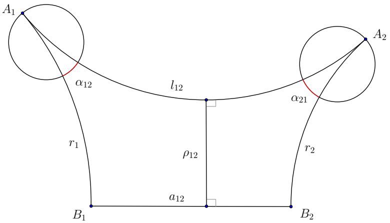


Figure: A semi-ideal ultraparallel trapezoid. Ideal vertices are equipped with horocycles.

Definition (Trapezoid)

A *trapezoid* is the convex hull of a segment $A_1A_2 \subset \overline{\mathbb{H}}^2$ and its orthogonal projection to a line such that the segment A_1A_2 does not intersect this line. It is called *ultra-parallel* if the line A_1A_2 is ultraparallel to the second line. It is called *semi-ideal* if both A_1 and A_2 are ideal. If some vertices are ideal, then they are equipped with *canonical horocycles*.

Trapezoid

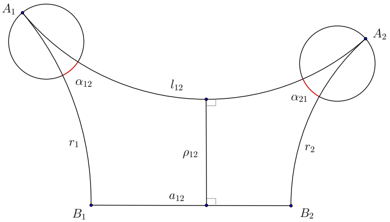


Figure: A semi-ideal ultraparallel trapezoid. Ideal vertices are equipped with horocycles.

Lemma

Let $A_1A_2B_2B_1$ be an ultraparallel trapezoid with $A_1, A_2 \in \mathbb{H}^2$ and $\alpha_{21} = \frac{\pi}{2}$. Then

$$\sinh(r_1) = \sinh(r_2)\cosh(l_{12})$$

$$\tanh(a_{12}) = \frac{\tanh(l_{12})}{\cosh(r_2)}.$$

Trapezoid

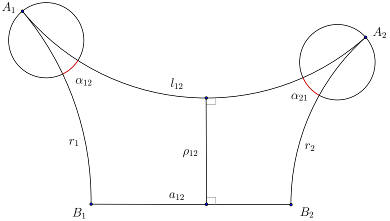


Figure: A semi-ideal ultraparallel trapezoid. Ideal vertices are equipped with horocycles.

Lemma

Let $A_1A_2B_2B_1$ be an ultraparallel trapezoid with $A_1 \in \partial_\infty \mathbb{H}^2$, $A_2 \in \mathbb{H}^2$ and $\alpha_{21} = \frac{\pi}{2}$. Then

$$e^{r_1} = \sinh(r_2) e^{l_{12}}$$

$$\tanh(a_{12}) = \frac{1}{\cosh(r_2)}.$$

Trapezoid

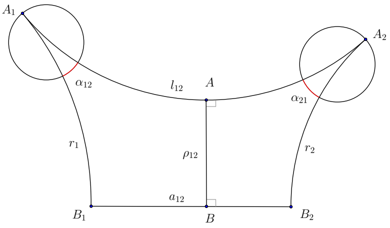


Figure: A semi-ideal ultraparallel trapezoid. Ideal vertices are equipped with horocycles.

Corollary

In an ultraparallel trapezoid $A_1A_2B_2B_1$ the length of the lower edge is uniquely determined by the lengths of the upper edge and the lateral edges.

Proof.

Consider $A \in A_1A_2$ that is the closest point to the line B_1B_2. Let B be its orthogonal projection to B_1B_2. Apply previous lemma to the trapezoids AA_1B_1B and AA_2B_2B. $\square$

Trapezoid

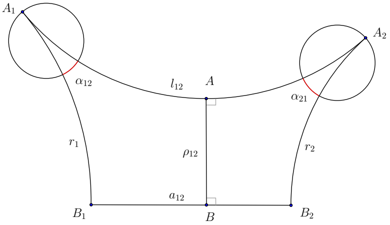


Figure: A semi-ideal ultraparallel trapezoid. Ideal vertices are equipped with horocycles.

Corollary

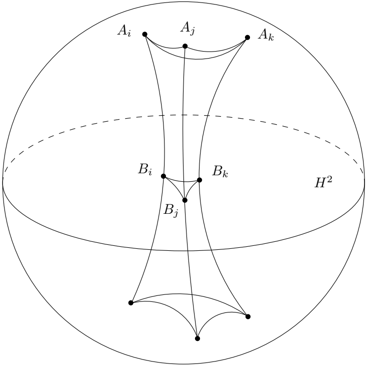
For a semi-ideal ultraparallel trapezoid we have

$$\cosh(a_{12}) = 1 + \frac{2}{\sinh^2(\rho_{12})}$$

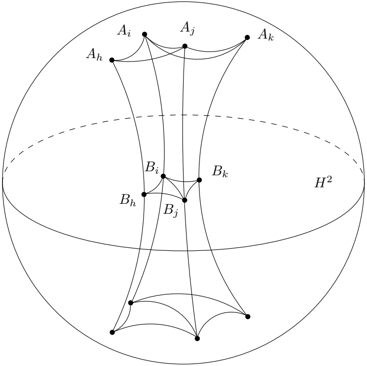
$$\cosh(a_{12}) = 1 + 2e^{l_{12}-r_1-r_2}$$

$$\alpha_{12}^2 = e^{r_2-r_1-l_{12}} + e^{-2r_1}$$

Prism



a semi-deal prism



gluing of two prisms

Figure: A hyperbolic triangle $[B_i, B_j, B_k]$ on $\mathbb{H}^2$. Draw perpendicular lines through B_i, B_j and B_k intersecting $\partial_\infty \mathbb{H}^3$ at A_i, A_j and A_k. The convex hull of A_i, A_j, A_k and B_i, B_j, B_k is a semi-ideal prism.

Prism

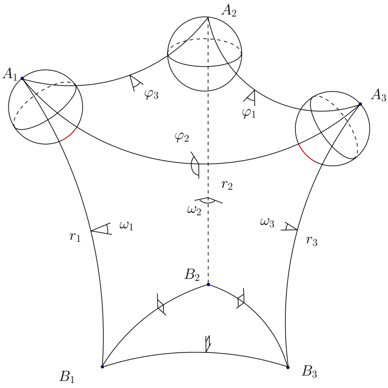


Figure: A semi-ideal prism. Ideal vertices are equipped with horospheres.

Definition (Prism)

A *prism* is the convex hull of a triangle $A_1A_2A_3 \subset \overline{\mathbb{H}}^3$ and its orthogonal projection to a plane such that the triangle $A_1A_2A_3$ does not intersect this plane. It is called *semi-ideal* if all A_1, A_2 and A_3 are ideal. If some vertices are ideal, then they are equipped with *canonical horospheres*.

Prism

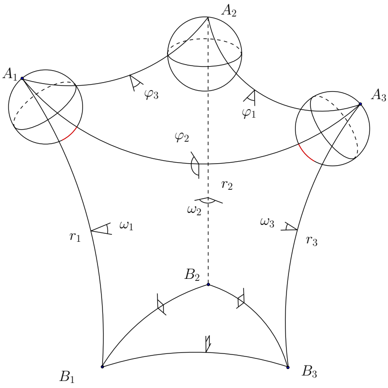


Figure: A semi-ideal prism. Ideal vertices are equipped with horospheres.

Corollary

An ultraparallel trapezoid or an ultraparallel prism is determined up to isometry (mapping canonical horocycles/horospheres, if any, to canoical horocycles/horospheres) by the lengths of the upper edges and the lateral edges.

Convex Prismatic Complex

Prismatic Complexes

Let $(S_{g,n}, d)$ be a hyperbolic cusp-surface with n cusps and T be an ideal geodesic triangulation of $S_{g,n}$ with vertices at cusps. By $E(T)$ and $F(T)$ denote its sets of edges and faces respectively. The set of cusps is denoted by $\mathcal{A} = \{A_1, \ldots, A_n\}$. We fix a horodisk at each A_i, referred as *canonical horodisk*, its boundary as *canonical horocycle*.

Definition (Admissible Weight)

Suppose that a real weight r_i is assigned to every cusp A_i, denote the weight vector by $\mathbf{r} \in \mathbb{R}^n$. A pair $(T, \mathbf{r})$ is called *admissible* if for every decorated ideal triangle $A_iA_jA_h \in F(T)$ there exists a semi-ideal prism with the lengths of lateral edges A_iB_i, A_jB_j, A_hB_h equal to r_i, r_j and r_h.

Prismetic Complex

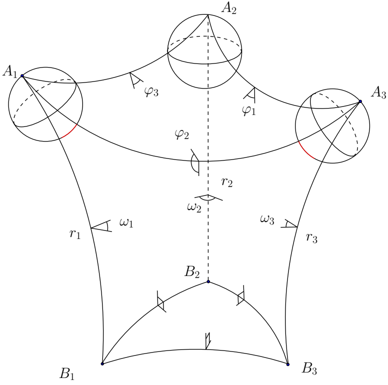


Figure: A semi-ideal prism. Ideal vertices are equipped with horospheres.

Let $(T, \mathbf{r})$ be an admissible pair. For each ideal triangle $A_iA_jA_h \in F(T)$ consider a prism constructed as previous definition. Since the upper and lateral edge lengths are fixed, it is unique up to isometry. Canonical horocycles coming from $(S_{g,n}, d)$ determine canonical horospheres at each ideal vertex of the prism.

Prismetic Complex

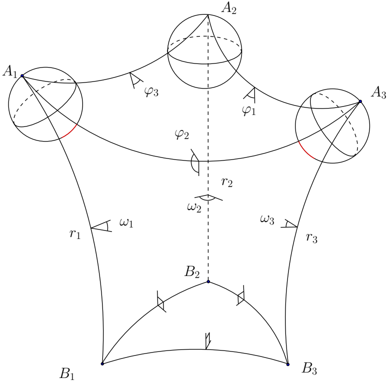


Figure: A semi-ideal prism. Ideal vertices are equipped with horospheres.

Definition (Prismetic Complex)

A prismatic complex $K(T, \mathbf{r})$ is a metric space obtained by gluing all these prisms via isometries of lateral faces. We choose gluing isometries in such a way that canonical horspheres at ideal vertices of prims match together.

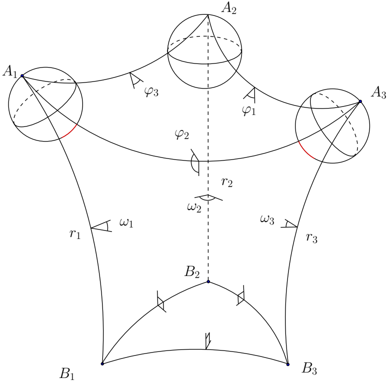


Figure: A semi-ideal prism. Ideal vertices are equipped with horospheres.

A prismatic complex $K(T, \mathbf{r})$ is a complete hyperbolic cone manifold with polyhedral boundary. The boundary consists of two components: the unioin of upper faces form the *upper boundary* coming with a natural isometry to $(S_{g,n}, d)$; the union of lower faces forms the *lower boundary*, which is isometric to $(S_{g,n}, d')$ for a polyhedral hyperbolic metric d' with conical singularities at points B_i.

Prismetic Complex

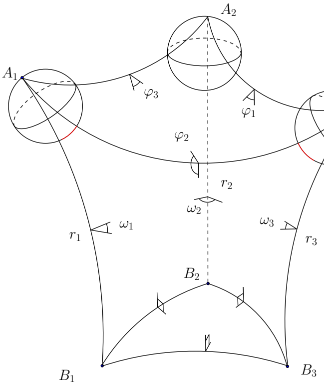


Figure: A semi-ideal prism. Ideal vertices are equipped with horospheres.

Consider T as a geodesic triangulation of both components. The dihedral angle $\tilde{\phi}_e$ of an edge $e \in E(T)$ is the sum of dihedral angles in both prisms containing e and $\tilde{\theta}_e = \pi - \tilde{\phi}_e$ is its exterior dihedral angle.

The total conical angle $\tilde{\omega}_i$ of an inner edge A_iB_i is the sum of the corresponding dihedral angles of all prisms containing A_iB_i and $\tilde{\kappa}_i = 2\pi - \tilde{\omega}_i$ is the curvature of A_iB_i. The conical angle of the point B_i in the lower boundary is also equal to $\tilde{\omega}_i$.

Convex Prismetic Complex

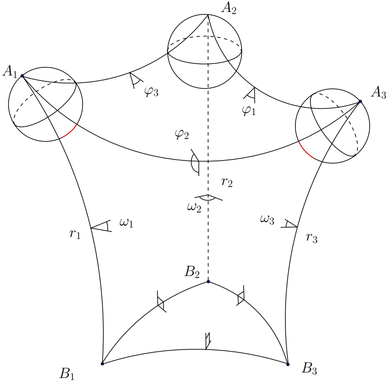


Figure: A semi-ideal prism. Ideal vertices are equipped with horospheres.

Definition (Convex Complex)

A complex K is called *convex* if for every upper edge its dihedral angle is at most π. If $K = K(T, \mathbf{r})$, then the pair $(K, \mathbf{r})$ is also called *convex*.

Lemma

Let K be a convex complex. Then each prism of K is ultraparallel.

Convex Prismetic Complex

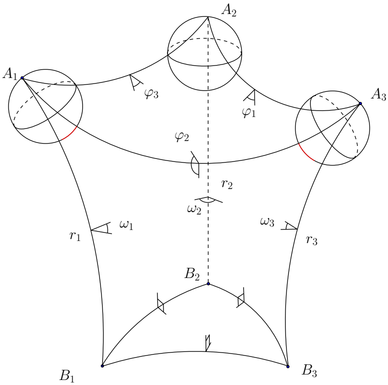


Figure: A semi-ideal prism. Ideal vertices are equipped with horospheres.

Let d' be a polyhedral hyperbolic metric on $S_{g,n}$ and T be its geodesic triangulation. Denote the set of marked points by $\mathcal{B} = \{B_1, \ldots, B_n\}$. Take a triangle $B_iB_jB_h$, there is a unique up to isometry semi-ideal prism that have $B_iB_jB_h$ as its lower face. Glue all such prisms together and obtain a complex $K(d', T)$ with the lower boundary isometric to $(S_{g,n}, d')$. Gluing isometries are uniquely defined if we fix the horosphere at each upper vertex passing through the respective lower vertex and match them together.

Convex Prismetic Complex

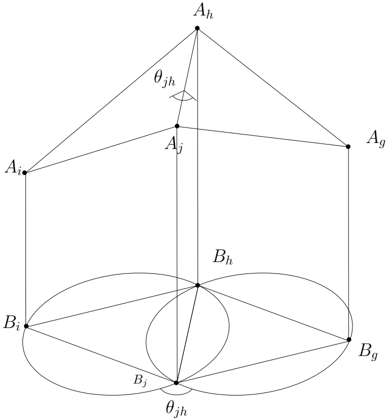


Figure: Lower boundary surface is Delaunay if and only if the upper surface is convex.

Lemma

The complex $K(d', T)$ is convex if and only if T is a Delaunay triangulation of $(S_{g,n}, d')$. Besides, any two convex complexes with isometric lower boundaries are isometric.

Proof.

Geometric observation (Leibon): the intersection angle between circumscribed circles of two adjacent triangles $B_iB_jB_h$ and $B_jB_hB_g$ is equal to the dihedral angle of the upper edge A_jA_h. $\square$

Delaunay Triangulation

Theorem

For each hyperbolic cusp metric d on $S_{g,n}$ there are finitely many Epstein-Penner triangulations of $(S_{g,n}, d)$.

Space of Convex Prismetic Complex $\mathcal{K}$

Space of Convex Complexes

Definition (Space of Convex Complexes)

Denote by $\mathcal{K}$ the set of all convex complexes with the upper boundary isometric to $(S_{g,n}, d)$ considered up to marked isometry (an isometry to itself isotopic to identify with respect to $\mathcal{A}$).

Every $K \in \mathcal{K}$ can be represented as $K(T, \mathbf{r})$.

Lemma

Let $(T', \mathbf{r})$ and $K'' = (T'', \mathbf{r})$ be two convex pairs. Then the complexes $K' = K(T', \mathbf{r})$ and $K'' = (T'', \mathbf{r})$ are marked isometric.

Corollary

The map $\mathbf{r} : \mathcal{K} \to \mathbb{R}^n$ is injective.

Hence $\mathcal{K}$ can be parameterizeed by $\mathbf{r}$.

Space of Convex Complexes

Proof.

The top surface for both K' and K'' is $(S_{g,n}, d)$, the bottom surfaces are $(S_{g,n}, d')$ and $(S_{g,n}, d'')$. d' is determined by d and $\mathbf{r}'$, d'' is determined by d and $\mathbf{r}''$ by the formula

$$\cosh(a_{12}) = 1 + 2e^{l_{12} - r1 - r2}$$

we obtain d' is isometric to d'', therefore T' equals to T''. Hence K' and K'' are marked isometric. $\square$

Convex Prismetic Complex

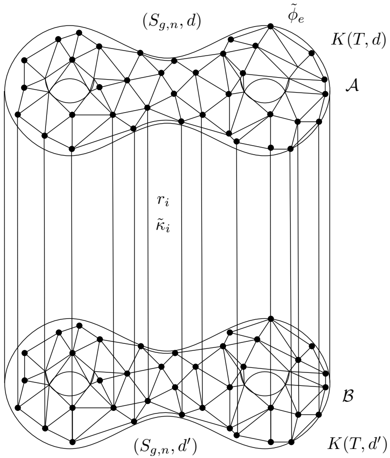


Figure: Convex prismetic complex.

Theorem

The lower boundary metric d' on $S_{g,n}$ is discretely conformally equivalent to d'' if and only if the upper boundaries of $K(d')$ and $K(d'')$ are isometric.

Convex Prismetic Complex

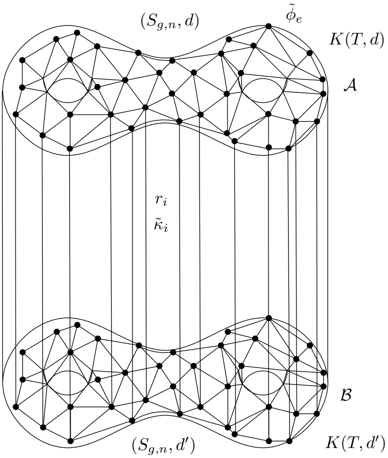


Figure: Convex prismetic complex.

Proof.

Assume that the upper boundaries of $K(d')$ and $K(d'')$ are both isometric to $(S_{g,n}, d)$ for a cusp metric d. Let $\mathcal{K}$ be the set of convex complexes realizing $(S_{g,n}, d)$. Choose a decoration on $(S_{g,n}, d)$. □

Convex Prismetic Complex

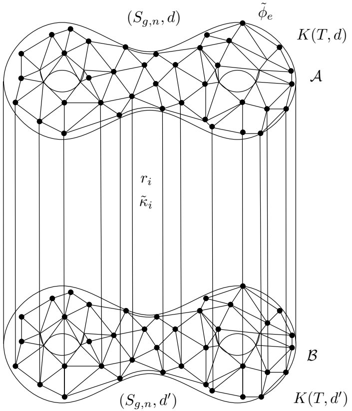


Figure: Convex prismetic complex.

Proof.

First, assume that convex complexes $K(d'), K(d'') \in \mathcal{K}(T)$ for a triangulation T, T is Delaunay for both d' and d''. Take $e \in E(T)$ and denote its lengths in d' and d'' by a' and a'' respectively. By r_i' and r_j' denote the weights of its endpoints in $K(d')$, and r_i'' and r_j'' in $K(d'')$. Then

$$
\begin{aligned}
\cosh(a') =& 1 + 2\exp(l_{12} - r_1' - r_2') \\
\cosh(a'') =& 1 + 2\exp(l_{12} - r_1'' - r_2'') \\
\sinh(\frac{a'}{2}) =& \sinh(\frac{a''}{2}) \\
& \exp\left(\frac{r_i'' - r_i'}{2} + \frac{r_j'' - r_j'}{2}\right).
\end{aligned}
$$

Thus, d' is discrete conformally equivalent to d''. $\square$

Convex Prismetic Complex

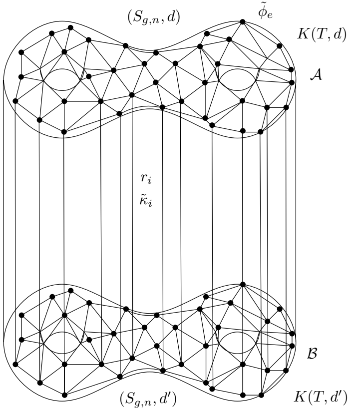


Figure: Convex prismetic complex.

Assume that d' and d'' are in the different cells $\mathcal{K}(T')$ and $\mathcal{K}(T'')$. The decomposition $\mathcal{K} = \bigcup \mathcal{K}(T)$ is finite and the boundaries of cells $\mathcal{K}(T)$ are piecewise analytic as subsets of $\mathbb{R}^n$. Then $K(d')$ and $K(d'')$ can be connected by a path in $\mathcal{K}$ transversal to the boundaries of all cells and intersecting them m times. All intersection points correspond to distinct convex complexes, denote their lower boundary metrics by $d_1, \ldots, d_m$, $d_0 = d'$, $d_{m+1} = d''$. A segment between d_i and d_{i+1} of the path belongs to $\mathcal{K}(T_i)$ for some triangulation T_i, T_i is Delaunay for both d_i and d_{i+1}. Hence d_i is discretely conformally equivalent. Then so are d_0 and d_{m+1}.

Convex Prismetic Complex

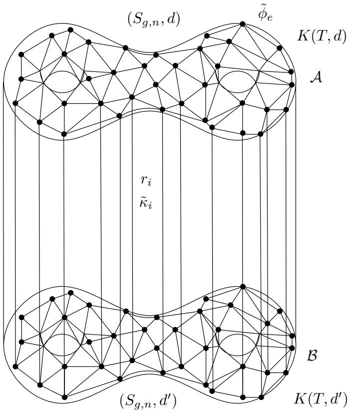


Figure: Convex prismetic complex.

In the opposite direction, assume that d' and d'' are discretely conformally equivalent and have a common Delaunay triangulation T. Then there exists a function $u : \mathcal{B} \to \mathbb{R}$ such that for each edge $e \in E(T)$ with endpoints B_i and B_j we have

$$\sinh\left(\frac{\text{len}_{d'}(e)}{2}\right) = e^{u(B_i)+u(B_j)} \sinh\left(\frac{\text{len}_{d''}(e)}{2}\right)$$

Consider $K(d')$ and $K(d'')$, then T is a face triangulation of both these complexes. Choose an horosection at each vertex of the upper boundaries in both $K(d')$ and $K(d'')$. Let r_i' and r_i'' be the distance from the horosections at $A_i \in \mathcal{A}$ to B_i in $K(d')$ and $K(d'')$ respectively.

Convex Prismetic Complex

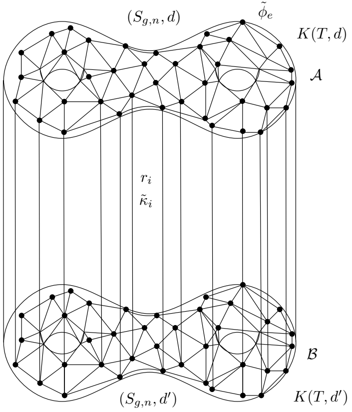


Figure: Convex prismetic complex.

We can choose the horosections such that for every i, $\frac{r_i'' - r_i'}{2} = u(B_i)$. Then

$$\cosh(a_{12} = 1 + 2\exp(l_{12} - r_1 - r_2)$$

shows for each $e \in E(T)$ its length in the upper boundary $K(d')$ is the same as in the upper boundary of $K(d'')$ (with respect to the chose horosection). Therefore, the upper boundary metrics of $K(d')$ and $K(d'')$ together with the chosen decorations have the same Penner coordinates, hence they are isometric. When d' and d'' are discretely conformally equivalent and do not have a commone Delaunay triangulation T, is inductively reduced to the last case.

Discrete Hilbert-Einstein Action

Discrete Hilbert-Einstein Functional

Definition (Hilbert-Einstein Functional)

For $\mathbf{r} \in \mathbb{R}^n$ let T be any face triangulation of the convex complex $K(\mathbf{r})$. We introduce *the discrete Hilbert-Einstein functional* over the space of convex complexes $\mathcal{K}$ identified with $\mathbb{R}^n$:

$$S(\mathbf{r}) := -2\text{vol}(K(\mathbf{r})) + \sum_{1\leq i\leq n} r_i\tilde{\kappa}_i + \sum_{e\in E(T)} l_e\tilde{\theta}_e.$$

Consider a function $\kappa' : \mathcal{A} \to (-\infty, 2\pi)$. Define the *modified discrete Hilbert-Eistein functional*:

$$S_{\kappa'}(\mathbf{r}) := S(\mathbf{r}) - \sum_{1\leq i\leq n} r_i\kappa'_i.$$

The value $S(\mathbf{r})$ doesn't depend on the choice of T, because two face trangulations of $K(\mathbf{r})$ are different only in flat edges, for which $\tilde{\theta}_e = 0$.

Discrete Hilbert-Einstein Functional

Lemma

For every $\mathbf{r} \in \mathbb{R}^n$, $S(\mathbf{r})$ *is twice continuously differentiable and*

$$\frac{\partial S}{\partial r_i} = \tilde{\kappa}_i.$$

Proof.

By generalized Schläffli's formula, for a prism $P = A_i A_j A_h B_h B_j B_i \subset K$, we have

$$-2d\mathrm{vol}(P) = r_i d\omega_i + r_j d\omega_j + r_h d\omega_h + l_{jh} d\phi_i + l_{ih} d\phi_j + l_{ij} d\phi_h.$$

Summing these equalities over all prims we obtain

$$-2d\mathrm{vol}(K(\mathbf{r})) = -\sum_{1 \le i \le n} r_i d\tilde{\kappa}_i - \sum_{e \in E(T)} l_e d\tilde{\theta}_e.$$

Discrete Hilbert-Einstein Functional

Proof.

Thus,

$$dS(\mathbf{r}) = \sum_{1\le i\le n} \tilde{\kappa}_i dr_i + \sum_{e\in E(T)} \tilde{\theta}_e dl_e = \sum_{1\le i\le n} \tilde{\kappa}_i dr_i.$$

since l_e's are always fixed, $dl_e = 0$. $\square$

Corollary

For every $\mathbf{r} \in \mathbb{R}^n$, $S_{\kappa'}(\mathbf{r})$ *is twice continuously differentiable and*

$$\frac{\partial S_{\kappa'}}{\partial r_i} = \tilde{\kappa}_i - \kappa'_i.$$

If $\mathbf{r}$ is a critical point of $S_{\kappa'}$, then for all i, $\tilde{\kappa}_i = \kappa'_i$.

Concavity of Hilbert-Einstein Functional

Lemma

Define

$$X_{ij} := \frac{\partial^2 S}{\partial r_i \partial r_j} = \frac{\partial k_i}{\partial r_j}.$$

Then for every $1 \leq i \leq n$*:*

1. $X_{ii} < 0$,
2. *for* $i \neq j$, $X_{ij} > 0$,
3. *for every* $1 \leq i \leq n$, $\sum_{1 \leq j \leq n} X_{ij} < 0$,
4. *the second derivatives are continuous at every point* $\mathbf{r} \in \mathbb{R}^n$*. In particular, this implies that* $X_{ij} = X_{ji}$*.*

Namely the matrix is diagonally dominated.

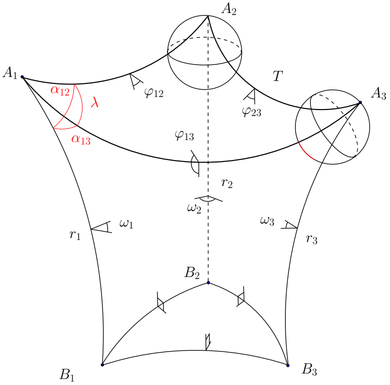


Figure: Hessian matrix.

The solid angle at the vertex A_1 cuts a Euclidean tringle from the canonical horosphere at A_1 with side lengths equal to α_{12}, α_{13} and λ; its respective angles are ϕ_{13}, ϕ_{12} and ω_1. By consine law we have

$$\cos(\omega_1) = \frac{\alpha_{12}^2 + \alpha_{13}^2 - \lambda^2}{2\alpha_{12}\alpha_{13}}.$$

We obtain the derivatives of ω_1

$$\frac{\partial \omega_1}{\partial \alpha_{12}} = -\frac{\cot(\phi_{12})}{\alpha_{12}}, \quad \frac{\partial \omega_1}{\partial \alpha_{13}} = -\frac{\cot(\phi_{13})}{\alpha_{13}}$$

Concavity of Hibert-Einstein Functional

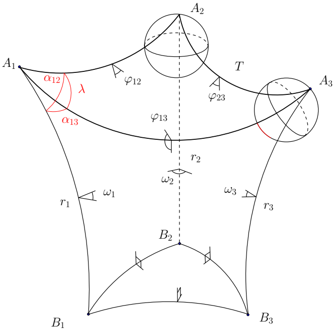


Figure: Hessian matrix.

By $\alpha_{12}^2 = e^{r_2 - r_1 - l_{12}} + e^{-2r_1}$, we obtain

$$\frac{\partial \alpha_{12}}{\partial r_1} = \frac{-\alpha_{12}^2 - e^{-2r_1}}{2\alpha_{12}}, \quad \frac{\partial \alpha_{12}}{\partial r_2} = \frac{\alpha_{12}^2 - e^{-2r_1}}{2\alpha_{12}},$$

Consider a deformation of this prism fixing the upper face. Then

$$\begin{aligned}
\frac{\partial \omega_1}{\partial r_1} &= \frac{\partial \omega_1}{\partial \alpha_{12}} \frac{\partial \alpha_{12}}{\partial r_1} + \frac{\partial \omega_1}{\partial \alpha_{13}} \frac{\partial \alpha_{13}}{\partial r_1} \\
&= \frac{\cot \phi_{12}}{2\alpha_{12}^2}(\alpha_{12}^2 + e^{-2r_1}) + \frac{\cot \phi_{13}}{2\alpha_{13}^2}(\alpha_{13}^2 + e^{-2r_1}) \\
\frac{\partial \omega_1}{\partial r_2} &= \frac{\partial \omega_1}{\partial \alpha_{12}} \frac{\partial \alpha_{12}}{\partial r_2} = \frac{\cot \phi_{12}}{2\alpha_{12}^2}(-\alpha_{12}^2 + e^{-2r_1}) \\
\frac{\partial \omega_1}{\partial r_3} &= \frac{\partial \omega_1}{\partial \alpha_{13}} \frac{\partial \alpha_{13}}{\partial r_3} = \frac{\cot \phi_{13}}{2\alpha_{13}^2}(-\alpha_{13}^2 + e^{-2r_1})
\end{aligned}$$

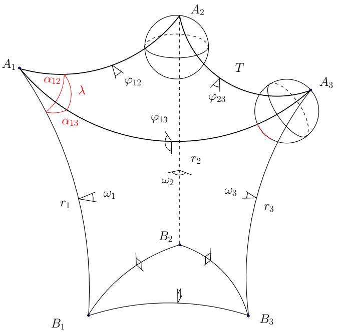


Figure: Hessian matrix.

Consider the complex $K(T, \mathbf{r})$. Let $E^{or}(T)$ be the set of oriented edges of T, $E_i^{orp}(T) \subset E_i^{or}(T)$ denote the set of oriented edges starting from A_i but ending not in A_i; $E_i^{orl}(T) \subset E_i^{or}(T)$ the set of oriented loops from A_i to A_i. For an oriented edge $\vec{e} \in E_i^{or}(T)$ denote by $\alpha_{\vec{e}}$ the length of the arc of horosphere at A_i between A_iB_i and $\vec{e}$.

Consider $\tilde{\omega}_i$ as the sum of angles in all prisms incident to A_i,

$$
\begin{aligned}
\frac{\partial \tilde{\omega}_i}{\partial r_i} &= -X_{ii} \\
&= \sum_{\vec{e} \in E_i^{orp}(T)} \frac{\alpha_{\vec{e}}^2 + e^{-2r_i}}{2\alpha_{\vec{e}}^2} (\cot \phi_{\vec{e}+} + \cot \phi_{\vec{e}-}) + \\
&\quad + \sum_{\vec{e} \in E_i^{orl}(T)} \frac{e^{-2r_i}}{\alpha_{\vec{e}}^2} (\cot \phi_{\vec{e}+} + \cot \phi_{\vec{e}-}),
\end{aligned}
$$

where $\phi_{\vec{e}+}$ and $\phi_{\vec{e}-}$ are the dihedral angles at $\vec{e}$ in two prisms containing $\vec{e}$.

Concavity of Hibert-Einstein Functional

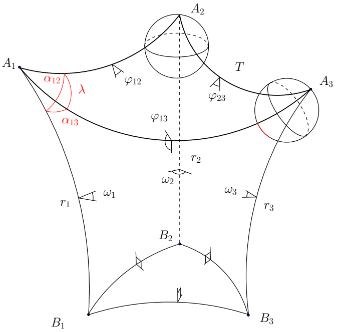


Figure: Hessian matrix.

For every $e \in E(T)$ we have

$$\phi_{\vec{e}+} + \phi_{\vec{e}-} \leq \pi,$$

hence $(\cot\phi_{\vec{e}+} + \cot\phi_{\vec{e}-}) \geq 0$ and $\frac{\partial\tilde{\omega}_i}{\partial r_i} = -X_{ii} \geq 0$.
Similarly, for $i \neq j$ denote by $E_{ij}^{orp}(T) \subset E_i^{orp}(T)$ the set of all oriented edges starting from A_i and ending at A_j, then

$$\begin{aligned}
\frac{\partial\tilde{\omega}_i}{\partial r_j} &= -X_{ii} \\
&= \sum_{\vec{e}\in E_{ij}^{orp}(T)} \frac{e^{-2r_i} - \alpha_{\vec{e}}^2}{2\alpha_{\vec{e}}^2}(\cot\phi_{\vec{e}+} + \cot\phi_{\vec{e}-}) \\
&= -\sum_{\vec{e}\in E_{ij}^{orp}(T)} \frac{e^{r_j - r_i - l_e}}{2\alpha_{\vec{e}}^2}(\cot\phi_{\vec{e}+} + \cot\phi_{\vec{e}-}) < 0
\end{aligned}$$

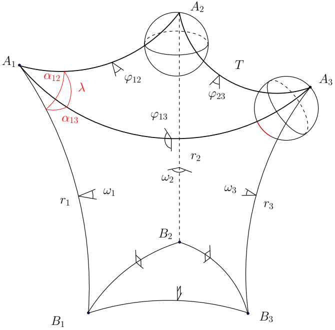


Figure: Hessian matrix.

For this we obtain for every i,

$$\sum_{1\le j\le n}\frac{\partial\tilde{\omega}_i}{\partial r_j}=\sum_{1\le j\le n}-X_{ij}$$

$$=\sum_{\vec{e}\in E_i^{or}(T)}\frac{e^{-2r_i}}{\alpha_{\vec{e}}^2}(\cot\phi_{\vec{e}+}+\cot\phi_{\vec{e}-})>0.$$

Concavity of Hibert-Einstein Functional

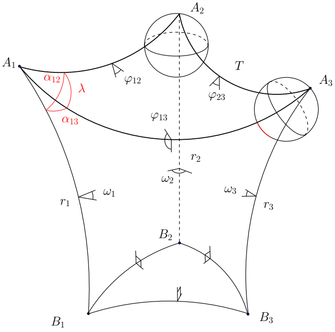


Figure: The volume of a prism is concave wrt its dihedral angles by applying Legendre transformation.

Corollary

The functions S and $S_{\kappa'}$ are strictly concave over $\mathbb{R}^n$.

Proof.

The Hessian X of S is negatively definite over $\mathbb{R}^n$:

$$\mathbf{r}^T X \mathbf{r} = \sum_{ii} X_{ii} r_i^2 + \sum_{1 \le i < j \le n} 2 X_{ij} r_i r_j$$

$$= - \sum_{1 \le i < j \le n} X_{ij} (r_i - r_j)^2 + \sum_{i=1}^{n} r_i^2 \sum_{j=1}^{n} X_{ij} < 0.$$

$\square$

Convex Prismetic Complex

Theorem (Main)

For every cusp metric d on $S_{g,n}$, $g > 1$, $n > 0$ with the set of cusps $\mathcal{A}$ and a function $\kappa' : \mathcal{A} \to (-\infty, 2\pi)$ satisfying

$$\sum_{A_i \in \mathcal{A}} \kappa'(A_i) > 2\pi(2 - 2g)$$

there exists a unique up to isometry convex complex with the upper boundary isometric to $(S_{g,n}, d)$ and the curvature $\tilde{\kappa}_i$ of each edge $A_i B_i$ equal to $\kappa'(A_i)$.

Main Theorem

Lemma

Consider a cube Q in $\mathbb{R}^n$, $Q := \{\mathbf{r} \in \mathbb{R}^n : \max |r_i| \leq q\}$. If

$$\sum_{1 \leq i \leq n} \kappa'_i > 2\pi(2 - 2g),$$

then for sufficiently large q, the maximum of $S_{\kappa'}(\mathbf{r})$ over Q is attained in the interior of Q.

This can be proved by analyzing the behavior of $S_{\kappa'}$ near infinity.

Main Theorem

Proof.

For the main theorem, fix a sufficiently large Q, $\bar{Q}$ is compact, $S_{\kappa'}(\mathbf{r})$ has maximum points. Since $S_{\kappa'}(\mathbf{r})$ is strictly concave, the maximum point is unique. The maximum of $S_{\kappa'}(\mathbf{r})$ is an interior point $\mathbf{r}^*$ of Q, at the maximum point $\nabla S_{\kappa'}(\mathbf{r}^*) = 0$, $\tilde{\kappa}_i = \kappa'_i$. $\square$

The curvature flow is the gradient flow of the Hilbert-Eistein energy:

$$\frac{dr_i}{dt} = \kappa'_i - \tilde{\kappa}_i.$$