Chapter 4 Convex optimization problems

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Outline

Convex optimization problems

- relevant concepts (for general optimization problems & for convex problems)
- properties of convex problems (local implies global & optimality condition)
- operations preserving convexity (construct new from old)
- many examples of convex problems (LP, QP, QCQP, SOCP, etc.)
- extensions (quasiconvex optimization & geometric programming)
- combination with generalized inequalities (in constraints & in objective functions)

Optimization problem in standard form

$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\
 & h_i(x) = 0, & i = 1, \cdots, p
\end{array}
$$

$x \in \mathbb{R}^n$	optimization variable
$f_0 \colon \mathbb{R}^n \to \mathbb{R}$	objective function (cost function)
$f_i \colon \mathbb{R}^n \to \mathbb{R}$	inequality constraint functions
$h_i \colon \mathbb{R}^n \to \mathbb{R}$	equality constraint functions

Constraints

- **implicit constraints**

$$x \in \mathcal{D} = \left(\bigcap_{i=0}^{m} \mathbf{dom}\, f_i \right) \cap \left(\bigcap_{i=1}^{p} \mathbf{dom}\, h_i \right)$$

- $\mathcal{D}$ is called the **domain** of the problem
- **explicit constraints**

$$f_i(x) \leq 0 \text{ for } 1 \leq i \leq m \quad \text{and} \quad h_i(x) = 0 \text{ for } 1 \leq i \leq p$$

- problem is **unconstrained** if it has no explicit constraints ($m = p = 0$)

Example

$$\text{minimize} \qquad f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints

$$a_i^T x < b_i$$

for each $1 \leq i \leq k$.

Feasibility

- x is **feasible** if $x \in \mathcal{D}$ and x satisfies all constraints
- the set of all feasible points is called the **feasible set** of the problem
- the problem is **infeasible** if the feasible set is empty
- the **feasibility problem** is to determine whether the feasible set is nonempty

$$
\begin{array}{rll}
\text{find} & x & \\
\text{subject to} & f_i(x) \le 0, & i = 1, \cdots, m \\
& h_i(x) = 0, & i = 1, \cdots, p
\end{array}
$$

- it can be rephrased as an optimization problem

$$
\begin{array}{rll}
\text{minimize} & 0 & \\
\text{subject to} & f_i(x) \le 0, & i = 1, \cdots, m \\
& h_i(x) = 0, & i = 1, \cdots, p
\end{array}
$$

Optimality

- The **optimal value** is

$$p^* = \mathbf{inf} \left\{ f_0(x) \;\middle|\; \begin{array}{c} f_i(x) \le 0 \text{ for } 1 \le i \le m \\ h_i(x) = 0 \text{ for } 1 \le i \le p \end{array} \right\} \in \mathbb{R} \cup \{\pm\infty\}$$

- Extreme situations

$$\begin{aligned} p^* &= \infty && \text{if problem is infeasible} \\ p^* &= -\infty && \text{if problem is unbounded below} \end{aligned}$$

- Optimal value may not be achieved.

- x is **optimal** if it is feasible and $f_0(x) = p^*$
- x is **locally optimal** if there exists $R > 0$ such that x is optimal for

$$
\begin{array}{lll}
\text{minimize} & f_0(z) & \\
\text{subject to} & f_i(z) \leq 0, & i = 1, \cdots, m \\
 & h_i(z) = 0, & i = 1, \cdots, p \\
 & \|z - x\|_2 \leq R &
\end{array}
$$

Examples (when $n = 1$, $m = p = 0$)

$f_0(x) = x \log x$	$\mathbf{dom}\, f_0 = \mathbb{R}_{++}$	$p^\star = -1/e$	$x = 1/e$ is optimal
$f_0(x) = -\log x$	$\mathbf{dom}\, f_0 = \mathbb{R}_{++}$	$p^\star = -\infty$	no optimal point
$f_0(x) = 1/x$	$\mathbf{dom}\, f_0 = \mathbb{R}_{++}$	$p^\star = 0$	no optimal point
$f_0(x) = x^3 - 3x$	$\mathbf{dom}\, f_0 = \mathbb{R}$	$p^\star = -\infty$	$x = 1$ is locally optimal

Optimization problems

Convex optimization

Linear optimization

Quadratic optimization

Geometric programming

Generalized inequality constraints

Vector optimization

Convex optimization problem in standard form

$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\
& a_i^T x = b_i, & i = 1, \cdots, p
\end{array}
$$

- $f_0, f_1, \cdots, f_m$ are convex
- equality constraints are affine, often written as $Ax = b$
- important property: feasible set of a convex problem is convex
- problem is **quasiconvex** if f_0 is quasiconvex (and $f_1, \cdots, f_m$ convex)

Example

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1+x_2^2) \leq 0 \\ & h_1(x) = (x_1+x_2)^2 = 0 \end{array}$$

- f_0 is convex
- not a convex problem: f_1 is not convex, h_1 is not affine
- equivalent (but not identical) to the convex problem

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0 \end{array}$$

Local and global optima

Proposition

Any locally optimal point of a convex optimization problem is globally optimal.

Proof

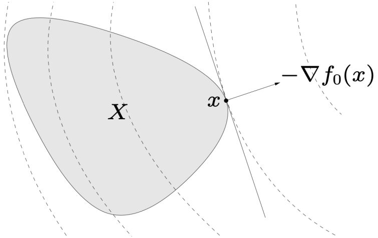
- suppose x is locally optimal, but there exists feasible y with $f_0(y) < f_0(x)$
- there exists $R > 0$ such that $f_0(z) \geq f_0(x)$ for all feasible z with $\|z - x\|_2 < R$
- consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$, then $\|z - x\|_2 = R/2$
- $\|y - x\|_2 > R$ implies $0 < \theta < 1/2$, hence z is feasible by convexity of domain
- by convexity of objective $f_0(z) \leq \theta f_0(y) + (1 - \theta) f_0(x) < f_0(x)$, contradiction

Optimality criterion for differentiable objective

Optimality criterion

Suppose the problem is convex and f_0 is differentiable, then

$$x \text{ is optimal} \quad \Longleftrightarrow \quad \begin{array}{c} x \text{ is feasible and} \\ \nabla f_0(x)^T(y - x) \geq 0 \text{ for all feasible } y \end{array}$$



Geometric interpretation

Either $\nabla f_0(x) = 0$ or $\nabla f_0(x)$ defines a supporting hyperplane to the feasible set at x.

Proof

$(\Longleftarrow)$ For any feasible y, since $y \in \mathbf{dom}\, f_0$, by the convexity of f_0

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y - x).$$

The assumption $\nabla f_0(x)^T (y - x) \geq 0$ implies $f_0(y) \geq f_0(x)$. Hence x is optimal.

$(\Longrightarrow)$ Assume on the contrary that $\nabla f_0(x)^T (y - x) < 0$ for some feasible y, then $z(t) = ty + (1 - t)x$ is feasible for $t \in [0, 1]$ since the feasible set is convex. Then

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} f_0(z(t)) \right|_{t=0} = \nabla f_0(x)^T (y - x) < 0,$$

hence $f_0(z(t)) < f_0(x)$ for $0 < t \ll 1$, which contradicts the optimality of x. $\square$

unconstrained problem

$$\text{minimize} \quad f_0(x)$$

$$x \text{ is optimal} \quad \Longleftrightarrow \quad x \in \mathbf{dom}\, f_0, \quad \nabla f_0(x) = 0$$

equality constrained problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Ax = b \end{array}$$

$$x \text{ is optimal} \quad \Longleftrightarrow \quad \begin{array}{c} x \in \mathbf{dom}\, f_0, \quad Ax = b, \\ \nabla f_0(x) + A^T \nu = 0 \text{ for some vector } \nu \end{array}$$

minimization over nonnegative orthant

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \succeq 0 \end{array}$$

$$x \text{ is optimal} \quad \Longleftrightarrow \quad x \in \mathbf{dom}\, f_0, \quad x \succeq 0, \quad \begin{cases} \nabla f_0(x)_i \geq 0, & \text{if } x_i = 0 \\ \nabla f_0(x)_i = 0, & \text{if } x_i > 0 \end{cases}$$

Sample proof (for unconstrained problems)

- By optimality condition

$$x \text{ is optimal} \iff x \in \mathbf{dom}\, f_0,\ \nabla f_0(x)^T(y-x) \geq 0 \text{ for each } y \in \mathbf{dom}\, f_0$$

- $\nabla f_0(x) = 0$ is clearly sufficient for the above statement.
- Since f_0 is differentiable, $\mathbf{dom}\, f_0$ is open, hence

$$y = x - \varepsilon \nabla f_0(x) \in \mathbf{dom}\, f_0$$

for $0 < \varepsilon \ll 1$. For such y we have

$$\nabla f_0(x)^T(y-x) = -\varepsilon \|\nabla f_0(x)\|_2^2 \leq 0.$$

- Combining above gives

$$\nabla f_0(x) = 0$$

which proves necessity.

Equivalent convex problems

Two problems are (informally) **equivalent** if the solution of one is readily obtained from the solution of the other, and vice-versa.

Some common transformations that preserve convexity

- eliminating equality constraints
- introducing equality constraints
- introducing slack variables for linear inequalities
- epigraph form
- minimizing over some variables

eliminating equality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \qquad i = 1, \cdots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{lll} \text{minimize} & f_0(Fz + x_0) & \text{(over } z\text{)} \\ \text{subject to} & f_i(Fz + x_0) \leq 0, & i = 1, \cdots, m \end{array}$$

where F and x_0 are such that

$$Ax = b \quad \Longleftrightarrow \quad x = Fz + x_0 \text{ for some } z$$

introducing equality constraints

$$\begin{array}{lll} \text{minimize} & f_0(A_0x + b_0) & \\ \text{subject to} & f_i(A_ix + b_i) \leq 0, & i = 1, \cdots, m \end{array}$$

is equivalent to

$$\begin{array}{lll} \text{minimize} & f_0(y_0) & (\text{over } x, y_i) \\ \text{subject to} & f_i(y_i) \leq 0, & i = 1, \cdots, m \\ & y_i = A_ix + b_i, & i = 0, 1, \cdots, m \end{array}$$

introducing slack variables for linear inequalities

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \leq b_i, \qquad i = 1, \cdots, m \end{array}$$

is equivalent to

$$\begin{array}{lll} \text{minimize} & f_0(x) & \text{(over } x, s\text{)} \\ \text{subject to} & a_i^T x + s_i = b_i, & i = 1, \cdots, m \\ & s_i \geq 0, & i = 1, \cdots, m \end{array}$$

epigraph form

$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\
 & Ax = b &
\end{array}
$$

is equivalent to

$$
\begin{array}{lll}
\text{minimize} & t & (\text{over } x, t) \\
\text{subject to} & f_0(x) - t \leq 0 & \\
 & f_i(x) \leq 0, & i = 1, \cdots, m \\
 & Ax = b &
\end{array}
$$

partial minimization

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \leq 0, \qquad i = 1, \cdots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \leq 0, \qquad i = 1, \cdots, m \end{array}$$

where

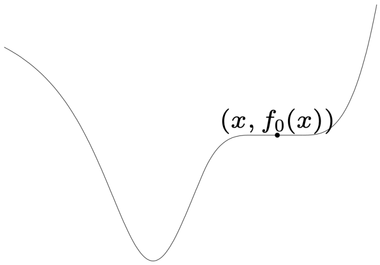
$$\tilde{f}_0(x_1) = \mathbf{inf}_{x_2} f_0(x_1, x_2)$$

Quasiconvex optimization

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\ & Ax = b & \end{array}$$

with $f_0 \colon \mathbb{R}^n \to \mathbb{R}$ quasiconvex, $f_1, \cdots, f_m$ convex.

Remark Locally optimal points may not be globally optimal



Convex representation of sublevel sets of f_0

For quasiconvex f_0 there exists a family of functions ϕ_t such that

- $\phi_t(x)$ is convex in x for each fixed t
- t-sublevel set of f_0 is 0-sublevel set of ϕ_t, i.e. $f_0(x) \le t \iff \phi_t(x) \le 0$
- $\phi_t(x)$ is nonincreasing in t for each fixed x, namely $\phi_s(x) \le \phi_t(x)$ if $s \ge t$

In practice there are usually natural meaningful choices for ϕ_t.

Example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with p convex, q concave, and $p(x) \geq 0$, $q(x) > 0$ on $\mathbf{dom}\, f_0$.

We can choose

$$\phi_t(x) = p(x) - tq(x)$$

- $\phi_t(x)$ convex in x for $t \geq 0$
- $f_0(x) \leq t \quad \iff \quad \phi_t(x) \leq 0$

Quasiconvex optimization via convex feasibility problems

$$\phi_t(x) \leq 0, \qquad f_i(x) \leq 0, \quad i = 1, \cdots, m, \qquad Ax = b$$

- convex feasibility problem in x for each fixed t
- let p^* be the optimal value for the original quasiconvex problem, then

$$\begin{array}{rcl} \text{above problem feasible} & \Longrightarrow & p^* \leq t \\ \text{above problem infeasible} & \Longrightarrow & p^* \geq t \end{array}$$

Bisection method

---

given $l \leq p^\star$, $u \geq p^\star$, tolerance $\epsilon > 0$

repeat

1. $t := (l + u)/2$
2. solve the above convex feasibility problem
3. **if** feasible, $u := t$; **else** $l := t$

until $u - l \leq \epsilon$

---

requires exactly $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations

Optimization problems

Convex optimization

Linear optimization

Quadratic optimization

Geometric programming

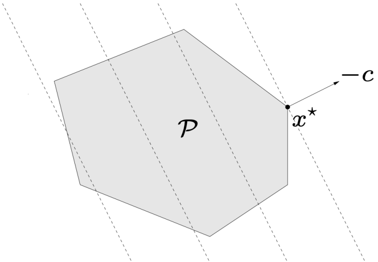
Generalized inequality constraints

Vector optimization

Linear program (LP)

$$
\begin{array}{ll}
\text{minimize} & c^T x + d \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Examples

Diet problem choose quantities $x_1, \cdots, x_n$ of n kinds of food

- one unit of food j costs c_j, contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b \\ & x \succeq 0 \end{array}$$

Piecewise-linear minimization

$$\text{minimize} \quad \mathbf{max}\,\{a_i^T x + b_i \mid i = 1, \cdots, m\}$$

equivalent to the LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \qquad i = 1, \cdots, m \end{array}$$

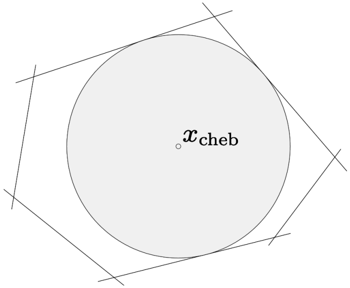
Chebyshev center of a polyhedron

Chebyshev center of

$$\mathcal{P} = \{x \mid a_i^T x \leq b_i,\ i = 1, \cdots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \leq r\}$$



$a_i^T x \leq b_i$ for all $x \in \mathcal{B}$ if and only if

$$\mathbf{sup}\{a_i^T(x_c + u) \mid \|u\|_2 \leq r\} = a_i^T x_c + r\|a_i\|_2 \leq b_i$$

hence x_c and r can be determined by solving the LP

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \leq b_i, \qquad i = 1, \cdots, m \end{array}$$

Linear-fractional program

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

where

$$f_0(x) = \frac{c^T x + d}{e^T x + f}, \qquad \mathbf{dom}\, f_0(x) = \{x \mid e^T x + f > 0\}$$

is a quasiconvex optimization problem; can be solved by bisection method.

If the feasible set is nonempty, then the linear-fractional problem is equivalent to the LP

$$\begin{array}{ll} \text{minimize} & c^T y + dz \\ \text{subject to} & Gy \preceq hz \\ & Ay = bz \\ & e^T y + fz = 1 \\ & z \geq 0 \end{array}$$

Generalized linear-fractional program

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

where

$$
f_0(x) = \mathbf{max}\left\{ \frac{c_i^T x + d_i}{e_i^T x + f_i} \,\middle|\, i = 1, \cdots, r \right\}
$$

$$
\mathbf{dom}\, f_0(x) = \left\{ x \,\middle|\, e_i^T x + f_i > 0,\ i = 1, \cdots, r \right\}
$$

is a quasiconvex optimization problem; can be solved by bisection method.

Example: von Neumann model of a growing economy

$$\begin{array}{lll} \text{maximize} & \min\left\{x_i^+/x_i \,\middle|\, i=1,\cdots,n\right\} & \text{(over } x, x^+\text{)} \\ \text{subject to} & x^+ \succeq 0 & \\ & Bx^+ \preceq Ax & \end{array}$$

with domain $\{(x, x^+) \mid x \succ 0\}$

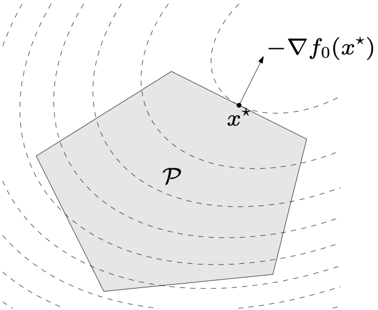
- $x, x^+ \in \mathbb{R}^n$: activity levels of n sectors, in current and next period
- $(Ax)_i$, $(Bx^+)_i$: produced resp. consumed amounts of good i
- x_i^+/x_i: growth rate of sector i

allocate activity to maximize growth rate of slowest growing sector

Quadratic program (QP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P x + q^T x + r \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- $P \in \mathbb{S}^n_+$ thus objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Example

Least-squares

$$\text{minimize} \qquad \|Ax - b\|_2^2$$

- analytical solution $x^* = A^\dagger b$ (where $A^\dagger$ is pseudo-inverse)
- can add linear constraints such as $l \preceq x \preceq u$

Linear program with random cost

Consider the linear program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- Assume c is random vector with mean $\bar{c}$ and covariance Σ
- Then $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$

$$\begin{aligned} \mathbf{E}\left(c^T x\right) &= \mathbf{E}\left(c\right)^T x = \bar{c}^T x \\ \mathbf{var}\left(c^T x\right) &= \mathbf{E}\left(c^T x - \bar{c}^T x\right)^2 = x^T \mathbf{E}\left((c-\bar{c})(c-\bar{c})^T\right) x = x^T \Sigma x \end{aligned}$$

We modify the above LP to the following QP

$$
\begin{array}{ll}
\text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- To keep both the expected cost and the cost variance (risk) under control, choose a linear combination of both as the new objective, called **risk-sensitive cost**.
- $\gamma > 0$ is the **risk-aversion parameter**, which controls the trade-off between expected cost and variance.
- Coefficient vector $(1, \gamma)$ lies in the interior of the dual cone of the nonnegative quadrant.

Quadratically constrained quadratic program (QCQP)

$$
\begin{array}{lll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 & \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, & i = 1, \cdots, m \\
 & Ax = b &
\end{array}
$$

- $P_i \in \mathbb{S}^n_+$ thus objective and constraints are convex quadratic
- feasible region is intersection of m ellipsoids and an affine set if $P_1, \cdots, P_m \in \mathbb{S}^n_{++}$

Second-order cone program (SOCP)

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \qquad i = 1, \cdots, m \\ & Fx = G \end{array}$$

with $A_i \in \mathbb{R}^{n_i \times n}$ and $F \in \mathbb{R}^{p \times n}$

- inequalities are called second-order cone constraints since

$$(A_i x + b_i, c_i^T x + d_i) \in \text{ second-order cone in } \mathbb{R}^{n_i + 1}$$

- if $n_i = 0$, reduces to LP
- if $c_i = 0$, reduces to QCQP (with linear objective)

Robust linear program

Parameters in optimization problems are often uncertain. Consider the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \qquad i = 1, \cdots, m \end{array}$$

- There can be uncertainty in c, a_i, b_i (in a_i for example)
- There are two common approaches to handle uncertainty
  - deterministic model
  - stochastic model

- deterministic model: constraints must hold for all $a_i \in \mathcal{E}_i$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \qquad i = 1, \cdots, m \end{array}$$

- stochastic model: a_i is random variable; constraints must hold with probability η

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \qquad i = 1, \cdots, m \end{array}$$

deterministic approach via SOCP

- choose ellipsoid as $\mathcal{E}_i$ with $\bar{a}_i \in \mathbb{R}^n$ and $P_i \in \mathbb{R}^{n \times n}$

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$$

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \qquad i = 1, \cdots, m \end{array}$$

- equivalent SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \qquad i = 1, \cdots, m \end{array}$$

which follows from

$$\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$$

stochastic approach via SOCP

- assume $a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$ is Gaussian, then $a_i^T x \sim \mathcal{N}(\bar{a}_i^T x, x^T \Sigma_i x)$ is also Gaussian

$$\mathbf{prob}(a_i^T x \le b_i) = \Phi\left(\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2}\right)$$

  with $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t$ cumulative distribution function of $\mathcal{N}(0,1)$
- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \le b_i) \ge \eta, \qquad i = 1, \cdots, m \end{array}$$

- equivalent SOCP when $\eta > 1/2$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta)\|\Sigma_i^{1/2} x\|_2 \le b_i, \qquad i = 1, \cdots, m \end{array}$$

Monomials and posynomials

- **monomial function**

$$f(x) = c x_1^{a_1} \cdots x_n^{a_n}, \qquad \mathbf{dom}\, f = \mathbb{R}^n_{++}$$

with $c > 0$ and $a_i \in \mathbb{R}$

- **posynomial function**

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} \cdots x_n^{a_{nk}}, \qquad \mathbf{dom}\, f = \mathbb{R}^n_{++}$$

sum of monomials

change variables to $y_i = \log x_i$ and take logarithm

- monomial $f(x) = cx_1^{a_1} \cdots x_n^{a_n}$ transforms to

$$\log f(e^{y_1}, \cdots, e^{y_n}) = a^T y + b, \qquad (b = \log c)$$

- posynomial $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1}, \cdots, e^{y_n}) = \log \left(\sum_{k=1}^{K} e^{a_k^T y + b_k} \right), \qquad (b_k = \log c_k)$$

Geometric program (GP)

geometric program in standard form

$$
\begin{array}{rll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 1, & i = 1, \cdots, m \\
& h_i(x) = 1, & i = 1, \cdots, p
\end{array}
$$

with f_i posynomial, h_i monomial

geometric program in convex form

change variables to $y_i = \log x_i$ and take logarithm of objective and constraints

$$
\begin{array}{ll}
\text{minimize} & \log\left(\sum_{k=1}^{K} e^{a_{0k}^T y + b_{0k}}\right) \\
\text{subject to} & \log\left(\sum_{k=1}^{K} e^{a_{ik}^T y + b_{ik}}\right) \le 0, \qquad i = 1, \cdots, m \\
& Gy + d = 0
\end{array}
$$

Example

Frobenius norm diagonal scaling

- Assume $M \in \mathbb{R}^{n\times n}$ defines a linear transformation. After scaling the coordinates by $D = \mathbf{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n\times n}$, the resulting matrix becomes DMD^{-1}.
- How to choose D such that DMD^{-1} is small under the Frobenius norm?

$$\|DMD^{-1}\|_F^2 = \sum_{i,j=1}^{n} (DMD^{-1})_{ij}^2 = \sum_{i,j=1}^{n} M_{ij}^2 d_i^2 / d_j^2.$$

- It is an unconstrained geometric program

$$\text{minimize} \qquad \sum_{i,j=1}^{n} M_{ij}^2 d_i^2 / d_j^2$$

with variable $d = (d_1, \ldots, d_n)$.

Convex problem with generalized inequality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \qquad i = 1, \cdots, m \\ & Ax = b \end{array}$$

- $f_0 \colon \mathbb{R}^n \to \mathbb{R}$ is convex
- $f_i \colon \mathbb{R}^n \to \mathbb{R}^{k_i}$ is K_i-convex, where K_i is a proper cone
- same properties as standard convex problem
  (convex feasible set, local optimum is global, etc)

Conic form problem (cone program)

special case of above with affine objective and constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b \end{array}$$

extends linear programming ($K = \mathbb{R}^m_+$) to nonpolyhedral cones

Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

with $F_i, G \in \mathbb{S}^k$

- inequality constraint is called **linear matrix inequality** (LMI)
- includes problems with multiple LMI constrains:

$$x_1 F_1' + \cdots + x_n F_n' + G' \preceq 0 \quad \text{and} \quad x_1 F_1'' + \cdots + x_n F_n'' + G'' \preceq 0$$

  is equivalent to single LMI

$$x_1 \begin{bmatrix} F_1' & 0 \\ 0 & F_1'' \end{bmatrix} + \cdots + x_n \begin{bmatrix} F_n' & 0 \\ 0 & F_n'' \end{bmatrix} + \begin{bmatrix} G' & 0 \\ 0 & G'' \end{bmatrix} \preceq 0$$

LP as equivalent SDP

LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

equivalent SDP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{diag}(Ax - b) \preceq 0 \end{array}$$

note different interpretation of generalized inequality

SOCP as equivalent SDP

SOCP

$$
\begin{array}{ll}
\text{minimize} & f^T x \\
\text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \qquad i = 1, \cdots m
\end{array}
$$

equivalent SDP

$$
\begin{array}{ll}
\text{minimize} & f^T x \\
\text{subject to} & \begin{bmatrix} (c_i^T x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \qquad i = 1, \cdots m
\end{array}
$$

Eigenvalue minimization

$$\text{minimize} \quad \lambda_{\mathbf{max}}(A(x))$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ with given $A_i \in \mathbb{S}^k$

equivalent SDP with variables $(x, t) \in \mathbb{R}^{n+1}$

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

follows from

$$\lambda_{\mathbf{max}}(A) \leq t \quad \Longleftrightarrow \quad A \preceq tI$$

Matrix norm minimization

$$\text{minimize} \quad \|A(x)\|_2 = \left(\lambda_{\mathbf{max}}\left(A(x)^T A(x)\right)\right)^{1/2}$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ with given $A_i \in \mathbb{R}^{p \times q}$

equivalent SDP with variables $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0 \end{array}$$

follows from

$$\begin{aligned} \|A\|_2 \leq t \quad &\Longleftrightarrow \quad A^T A \preceq t^2 I, \quad t \geq 0 \\ &\Longleftrightarrow \quad \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0 \end{aligned}$$

Vector optimization

general vector optimization problem

$$\begin{array}{lll} \text{minimize (with respect to } K) & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\ & h_i(x) = 0, & i = 1, \cdots, p \end{array}$$

vector objective $f_0 \colon \mathbb{R}^n \to \mathbb{R}^q$ minimized with respect to proper cone $K \subseteq \mathbb{R}^q$

convex vector optimization problem

$$\begin{array}{lll} \text{minimize (with respect to } K) & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\ & Ax = b & \end{array}$$

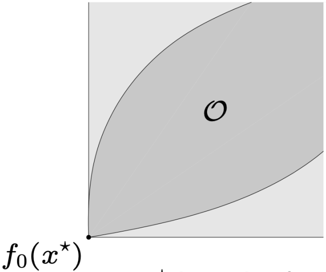
where f_0 is K-convex and $f_1, \cdots, f_m$ are convex

Optimal and Pareto optimal points

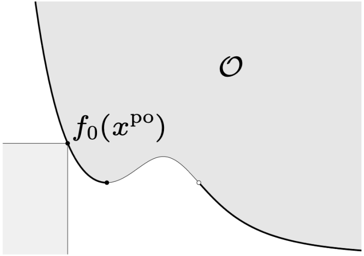
set of achievable objective values

$$\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$$

- feasible x^* is optimal if $f_0(x^*)$ is the minimum value of $\mathcal{O}$ (optimal value)
- feasible x^{po} is Pareto optimal if $f_0(x^{\mathrm{po}})$ is a minimal value of $\mathcal{O}$ (Pareto optimal value)



$x^\star$ is optimal



x^{po} is Pareto optimal

Multicriterion optimization

vector optimization problem with $K = \mathbb{R}^q_+$

$$f_0(x) = (F_1(x), \cdots, F_q(x))$$

- q different objectives F_i, we want all of them to be small
- feasible x^* is optimal if

$$y \text{ feasible} \quad \Longrightarrow \quad f_0(x^*) \preceq f_0(y)$$

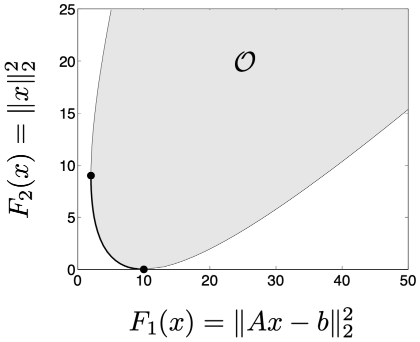
  if an optimal point exists, the objectives are noncompeting
- feasible x^{po} is Pareto optimal if

$$y \text{ feasible}, \quad f_0(y) \preceq f_0(x^{\text{po}}) \quad \Longrightarrow \quad f_0(x^{\text{po}}) = f_0(y)$$

  if multiple Pareto optimal values exist, there is a trade-off between the objectives

Regularized least-squares

$$\text{minimize (with respect to } \mathbb{R}^2_+) \qquad \left(\|Ax - b\|_2^2, \|x\|_2^2\right)$$

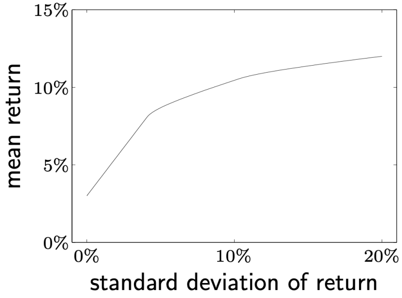


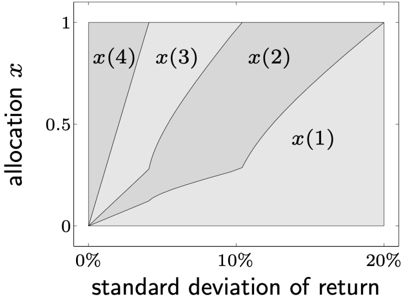
the optimal trade-off curve, shown darker, is formed by Pareto optimal points

Risk-return trade-off in portfolio optimization

$$\begin{array}{ll} \text{minimize (with respect to } \mathbb{R}^2_+) & (-\bar{p}^T x, x^T \Sigma x) \\ \text{subject to} & \mathbf{1}^T x = 1 \\ & x \succeq 0 \end{array}$$

- $x \in \mathbb{R}^n$ investment portfolio; x_i fraction invested in asset i
- $p \in \mathbb{R}^n$ (relative) asset price, random variable with mean $\bar{p}$ and covariance Σ
- $r = p^T x$ (relative) return, random variable with mean $\bar{p}^T x$ and variance $x^T \Sigma x$





Scalarization

To find Pareto optimal points, choose $\lambda \succ_{K^*} 0$ and solve scalar problem

$$\begin{array}{lll} \text{minimize} & \lambda^T f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\ & h_i(x) = 0, & i = 1, \cdots, p \end{array}$$

- if x is optimal for scalar problem, then it is Pareto optimal for vector optimization problem
- for convex vector optimization problem, can find (almost) all Pareto optimal points by varying $\lambda \succ_{K^*} 0$

Scalarization for multicriterion problems

In this more concrete situation

$$K = K^* = \mathbb{R}^q_+.$$

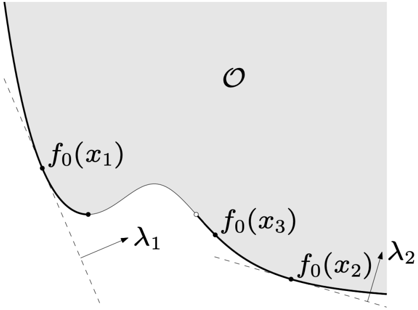
To find Pareto optimal points, write

$$\lambda = \begin{bmatrix} a_1 \\ \vdots \\ a_q \end{bmatrix} \in \mathbb{R}^q_{++} \qquad \text{and} \qquad f_0(x) = \begin{bmatrix} F_1(x) \\ \vdots \\ F_q(x) \end{bmatrix},$$

then minimize the positive weighted sum

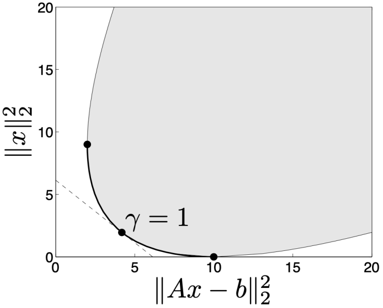
$$\lambda^T f_0(x) = a_1 F_1(x) + \cdots + a_q F_q(x)$$

Geometric interpretation



- $\mathcal{O}$ is the set of achievable objective values
- Pareto optimal values $f_0(x_1)$ and $f_0(x_2)$ can both be obtained by scalarization: $f_0(x_1)$ minimizes $\lambda_1^T u$ and $f_0(x_2)$ minimizes $\lambda_2^T u$ over all $u \in \mathcal{O}$
- $f_0(x_3)$ is Pareto optimal, but cannot be found by scalarization

Regularized least-square problem



Take $\lambda = (1, \gamma)$ with $\gamma > 0$

$$\text{minimize} \qquad \|Ax - b\|_2^2 + \gamma \|x\|_2^2$$

least-square problem for fixed $\gamma > 0$

Risk-return trade-off problem

Take $\lambda = (1, \gamma)$ with $\gamma > 0$

$$\begin{array}{ll} \text{minimize} & -\bar{p}^T x + \gamma x^T \Sigma x \\ \text{subject to} & \mathbf{1}^T x = 1 \\ & x \succeq 0 \end{array}$$

quadratic program for each fixed $\gamma > 0$