Knots and Hyperbolic 3-Manifolds

David Gu

Computer Science Department
Stony Brook University

gu@cs.stonybrook.edu

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Knot Theory

Definition (Knot)

A knot is an embedding $K : \mathbb{S}^1 \to \mathbb{R}^3$. A knot is a one-dimensional subset of $\mathbb{R}^3$ that is homeomorphic to $\mathbb{S}^1$.

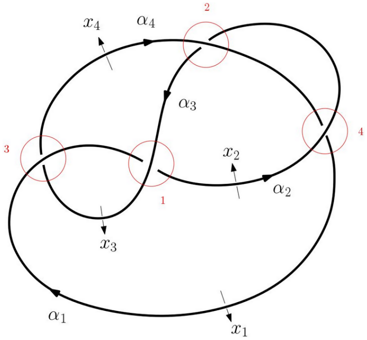


Figure: A planar projection of a knot.

Knot Theory

Definition (Equivalence of Knots)

For K_1, K_2 knots, we say that $K_1 \cong_{isotopic} K_2$ if there exists a (orientation-preserving) homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(K_1) = K_2$.

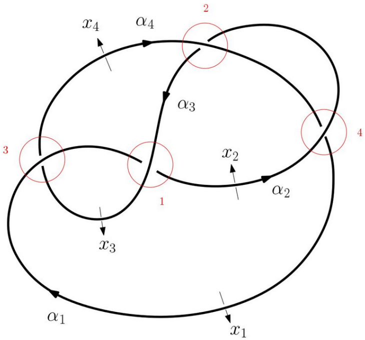


Figure: A planar projection of a knot.

Knot Theory

Definition (Planar Diagram of Knots)

The knot is projected to the plane to obtain a planar graph, such that the valence of each node is four. At each intersection point, the crossing information is recorded to show which strand is above.

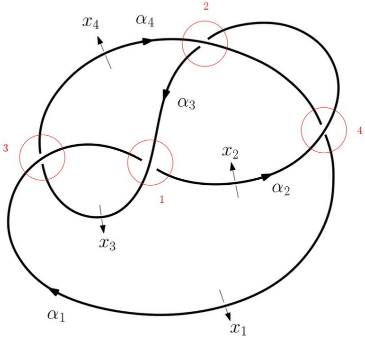


Figure: A planar projection of a knot.

Reidemeister Moves

Definition (Reidemester Moves)

Refer to the following figure. a. straighten wiggly lines; b. undo twists; c. seperate underpasses; d. move a line behind an intersection across the intersection

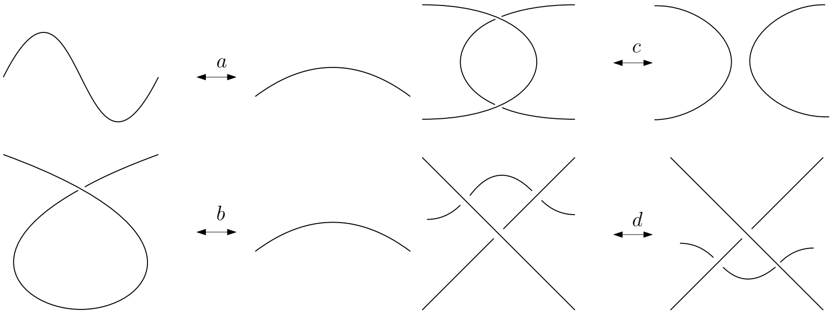


Figure: Reidemeister moves.

Reidemeister Moves

Theorem

$K_1 \cong_{isotopic} K_2$ iff their diagrams can be obatined from each other using the Reidemeister moves.

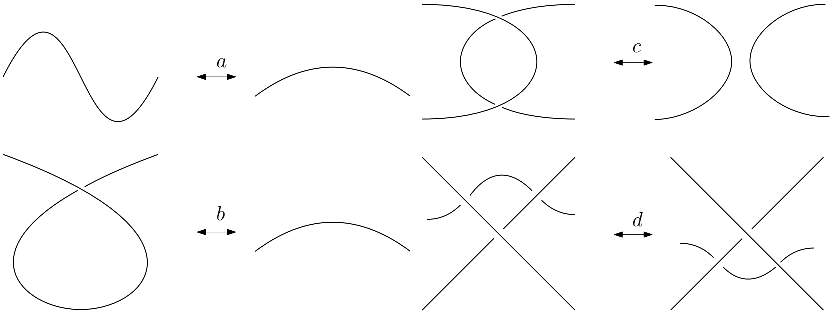


Figure: Reidemeister moves.

Reidemeister Moves

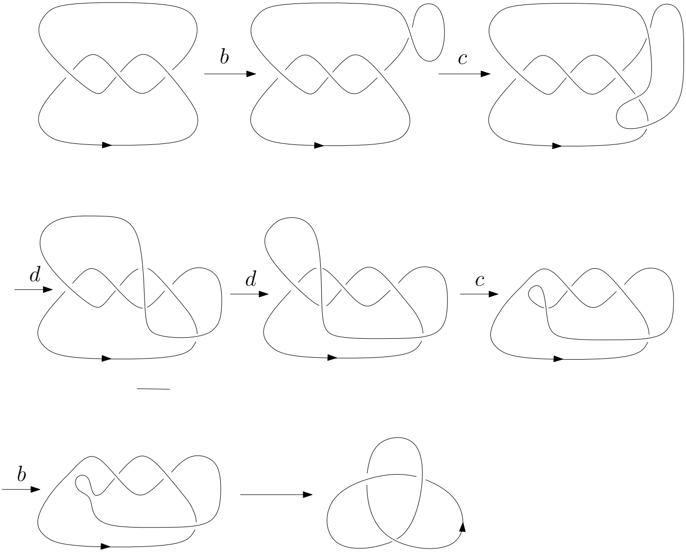


Figure: Reidemeister moves.

Seifert Surface

Definition (Seifert Surface)

Seifert surface of a knot K is an orientable surface Σ with boundary such that $\partial\Sigma = K$.

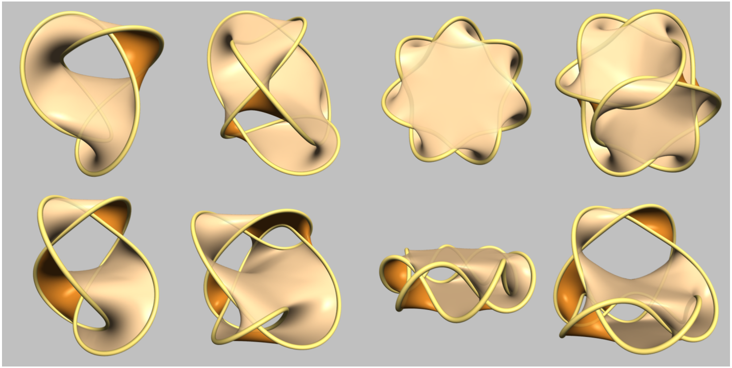


Figure: Seifert surface. (Rendered by Jarke van Wijk and Arjeh Cohen)

Seifert Surface

Definition (Genus of a knot)

Geus of a knot K $g(K)$ is the smallest possible genus of all Seifert surfaces of the knot.

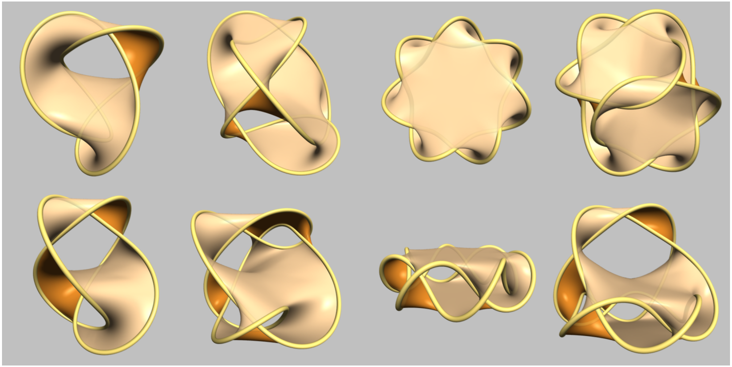


Figure: $g(K) = 0$ iff $K \cong O$.

Seifert Surface Algorithm

1. Choose diagram D and orientation;
2. Smooth every crossing according to orientation; The resulting closed curves are Seifert cycles;
3. Fill each cycle to form a disk, nest them if necessary.
4. Add a twisted strip at each former crossing. The result is Σ_D.

Proposition (Seifert Algorithm)

If D has n crossings and the algorithm produces s Seifert cycles, then

$$g(\Sigma_D) = \frac{1 - s + n}{2}$$

Seifert Surface Algorithm

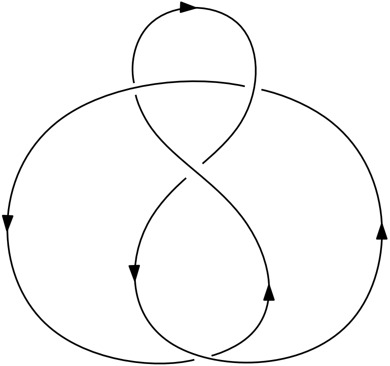


Figure: Step 1. choose an orientation.

Seifert Surface Algorithm

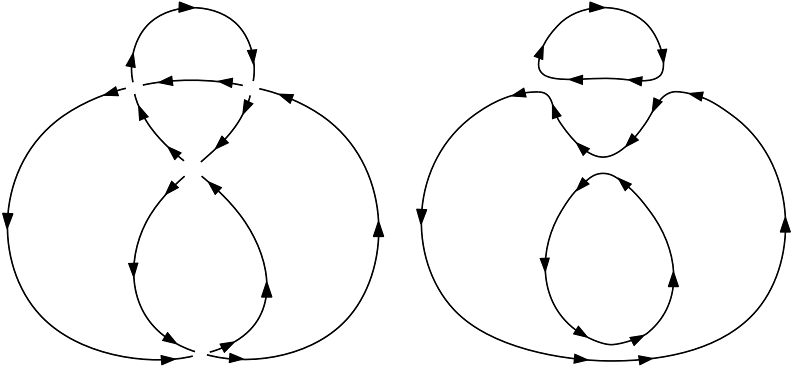


Figure: Step 2. Seifert cycles.

Seifert Surface Algorithm

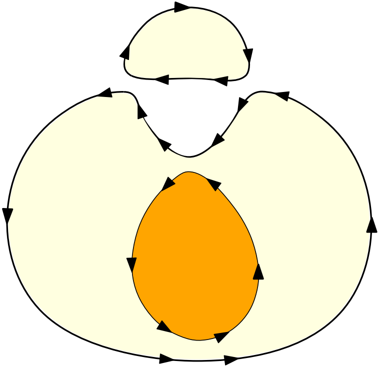


Figure: Step 3. Fill cycles. Step 4. Attach twisted strips.

Connect Sum of Knots

Definition (Connect Sum of Knots)

The connect sum of knots K_1 and K_2 (written as $K_1 \# K_2$) is given by cutting knots K_1 and K_2 at some point and pasting them together.

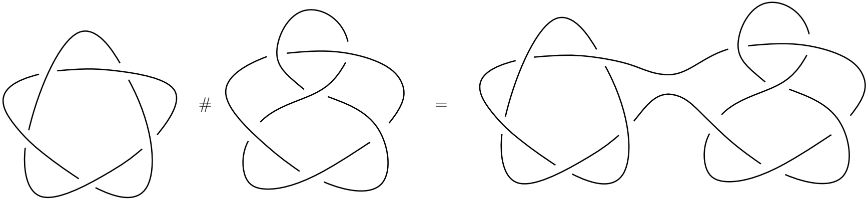


Figure: $K_1 \# K_2$

Connect Sum of Knots

Theorem (Connect Sum of Knots)

$g(K_1 \# K_2) = g(K_1) + g(K_2)$.

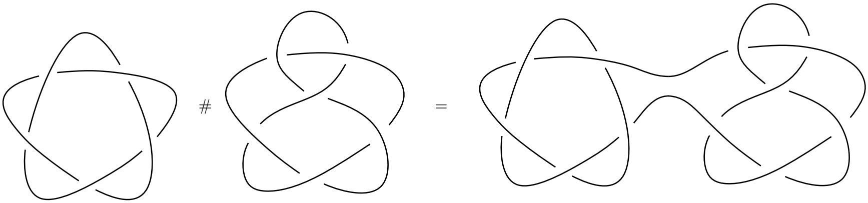


Figure: $K_1 \# K_2$

Definition (Prime Knot)

A knot cannot be decomposed as a connect sum is called a prime knot.

Knot Theory

Theorem (Gordon-Luecke)

If the complement spaces of two knots are homeomorphic, then the knots are isotopic. $\mathbb{R}^3 \setminus K_1 \sim \mathbb{R}^3 \setminus K_2$ *implies* $K_1 \cong K_2$.

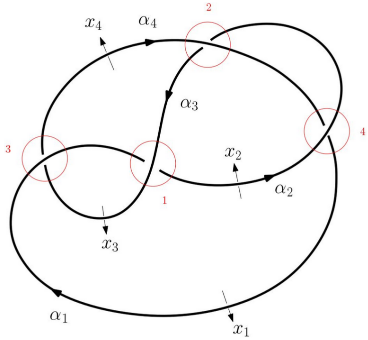


Figure: A planar projection of a knot.

Wirtinger presentation

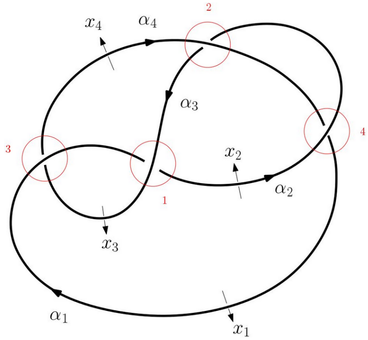


Figure: A planar projection of a knot.

Orient the knot diagram; the intersection divides the knot into segments $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$; draw an arrow x_k under α_k, such that $\{\alpha_k, x_k\}$ form a right hand frame.

Wirtinger presentation

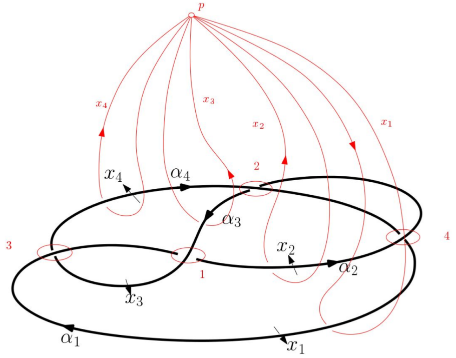


Figure: Wirtinger presentation.

Choose the base point p at the infinity of positive z-axis; connect p to the starting point of x_k, the target point of x_k to p, to form a loop x_k. The loops $\{x_1, x_2, x_3, x_4\}$ form the generators of $\pi_1(\mathbb{R}^3 \setminus K, p)$.

Wirtinger presentation

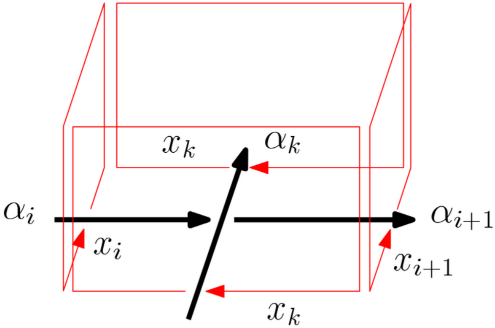


Figure: Wirtinger presentation.

Each crossing gives one relation

$$x_k x_i x_k^{-1} = x_{i+1}$$

Wirtinger presentation

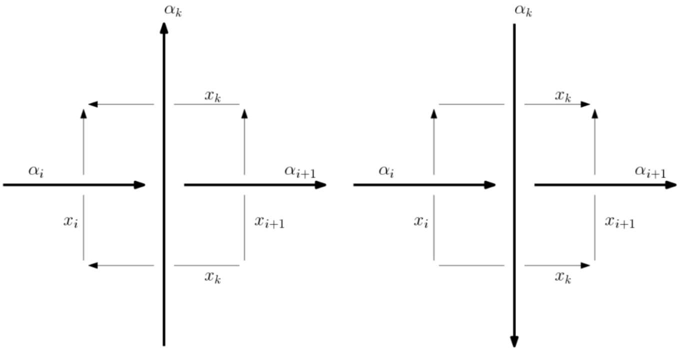
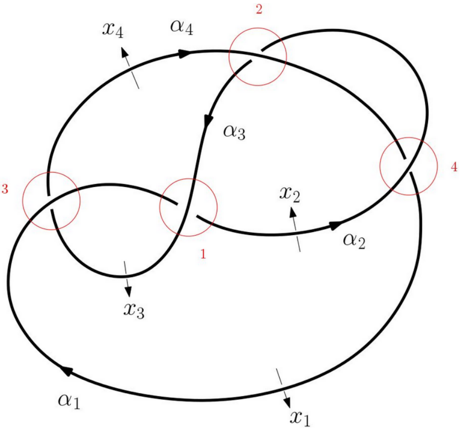


Figure: Wirtinger presentation.

Left and right frames show:

$$x_k x_i x_k^{-1} = x_{i+1} \quad x_k x_{i+1} x_k^{-1} = x_i.$$

Wirtinger presentation



$R_1 : x_1x_3 = x_3x_2$

$R_2 : x_4x_2 = x_3x_4$

$R_3 : x_3x_1 = x_1x_4$

$R_4 : x_2x_4 = x_1x_2$

R_1, R_2, R_3 imply R_4;

$$x_2 = x_3^{-1}x_1x_3$$

$$x_4 = x_1^{-1}x_3x_1$$

Figure:
$\pi_1(\mathbb{R}^3 \setminus K, p) = \langle x_1, x_3 | x_1^{-1}x_3x_1x_3^{-1}x_1x_3 = x_3x_1^{-1}x_3x_1 \rangle$

Trichotomy: (p, q)-torus knots

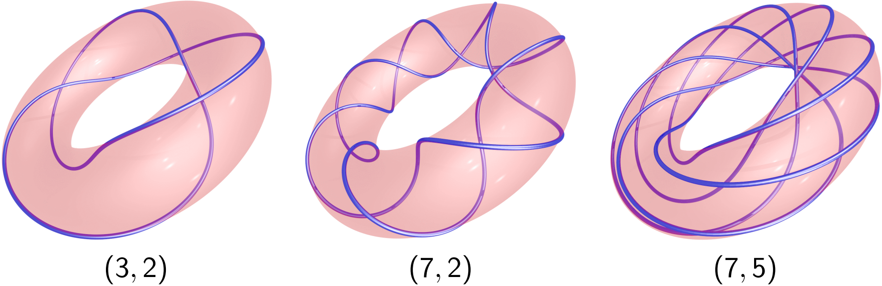


Figure: Torus knot: $pa + qb$, $gcd(p, q) = 1$. (Rendered by Mitch Richling)

A torus knot is a knot which can be embedded on the torus as a simple closed curve.

Trichotomy: Satellite Knots

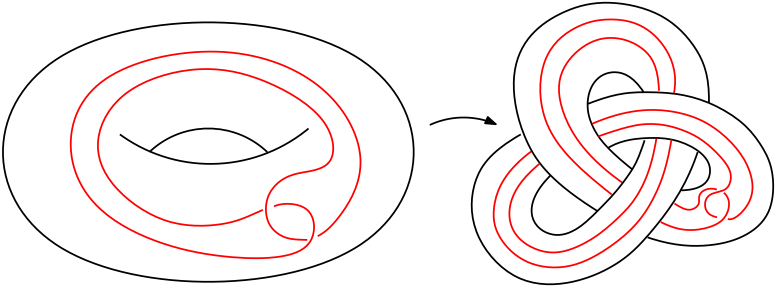


Figure: Satellite knot.

A knot K is embedded in a solid torus, which is another non-trivial knot J, call it $Sat_J(K)$. Namely, a satellite knot is a knot that contains an incompressible, non-boundary parallel torus in its complement.

Hyperbolic 3-manifolds

- The Poincare half-space model of hyperbolic 3-space $\mathbb{H}^3 = \{(x, y, t) | t > 0\}$ with metric $ds^2 = \frac{dx^2+dy^2+dt^2}{t^2}$. The boundary of $\mathbb{H}^3$ is $\mathbb{C} \cup \infty$ called the sphere at infinity.
- Geodesic planes $\mathbb{H}^2$ are vertical planes or upper hemispheres of spheres orthogonal to the xy-plane (with centers on the xy-plane).
- Geodesics are lines or half circle orthogonal to the xy-plane.
- $\text{Isom}^+(\mathbb{H}^3) = \text{PSL}(2, \mathbb{C})$ which acts as Mobius transforms on $\mathbb{C} \cup \infty$ extending this action by isometries.
- The horizontal planes $t =$ are scaled Euclidean planes called horospheres.

Hyperbolic 3-manifolds

A 3-manifold M is said to be hyperbolic if it has a complete, finite volume hyperbolic metric.

- $\pi_1(M) = \Gamma$ acts by covering translations as isometries and hence has a discrete faithful representation in $PSL(2, \mathbb{C})$.
- (Margulis thick-thin theorem, 1978) If M is orientable and noncompact then $M = \circ M'$ where $\partial M' = \cup T^2$. Each end is of the form $T^2 \times [0, \infty)$ with each section is scaled Euclidean metric, called a cusp.
- (Mostow-Prasad Rigidity theorem, 1968) Hyperbolic structure on a 3-manifold is unique. This implies geometric invariants are topological invariants.

Margulis Thick-Thin Theorem

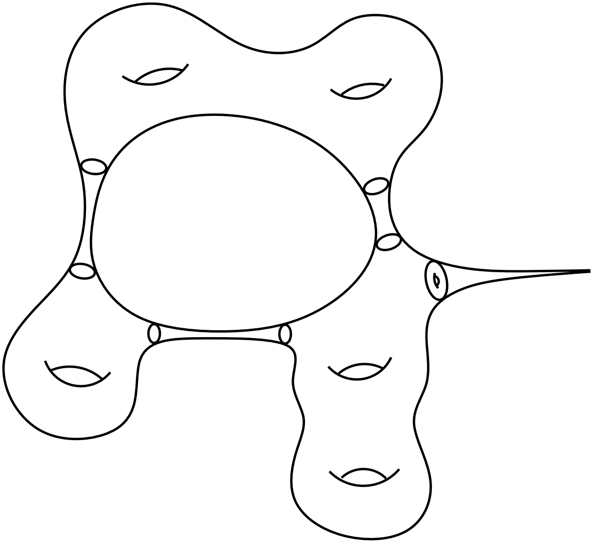


Figure: Margulis thick-thin theorem.

Knots and 3-manifolds

Definition (Dehn Surgery)

Since the boundary of $M = S^3 - \circ N(k)$ is a torus, we can attach a solid torus by choosing a curve (p, q) on ∂M which becomes medridian. The resulting closed 3-manifold is called (p, q)-Dehn filling of M. In general the process of drilling a simple closed curve and filling it with a solid torus is called Dehn surgery.

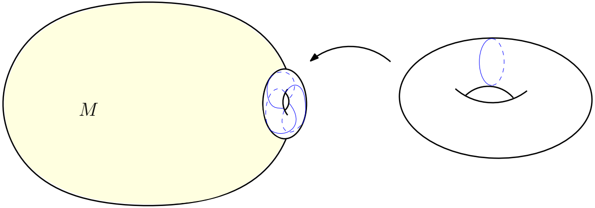


Figure: Dehn filling.

Lickorish-Wallace Theorem

Definition (Link)

Given a collection of knots $\{K_i\}$, define a link $L = \bigcup_i K_i$ is the disjoint union of the knots.

Theorem (Lickorish-Wallace, 1960)

Any closed, orientable, connected 3-manifold can be obtained by Dehn surgery on a link in S^3.

Different Dehn surgeries on different links may give the same 3-manifold. The process of identifying which links/surgeries give the same 3-manifold is Kirby Calculus.

Thurston's Theorems

Theorem (Geometrization of Haken manifolds)

If M is a compact irreducible atoroidal Haken manifold with torus boundary, then the interior of M is hyperbolic.

Theorem (Geometrization of knot complements)

Every knot in S^3 is either a torus knot, a satellite knot or a hyperbolic knot.

Theorem (Dehn Surgery Theorem)

Let $M = S^3 \setminus K$, where K is a hyperbolic knot. Then (p, q)-Dehn filling on M is hyperbolic for all but finitely many (p, q).

Hyperbolic 3-Manifolds

- Hyperbolic structures on some link complements can be described using circle packing and dual packings via the Koebe-Andreev-Thurston circle packing theorem which relates circle packings to triangulations of S^2.
- Hyperbolic structures on fibered 3-manifolds i.e. surface bundles can be obtained by using pseudo-Anosov monodromy.
- Hyperbolic 3-manifolds also arise as quotients of arithmetic lattices in $PSL(2,\mathbb{C})$ e.g. finite index subgroups of $PSL(2,\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of some number field.

Hyperbolic 3-Manifolds

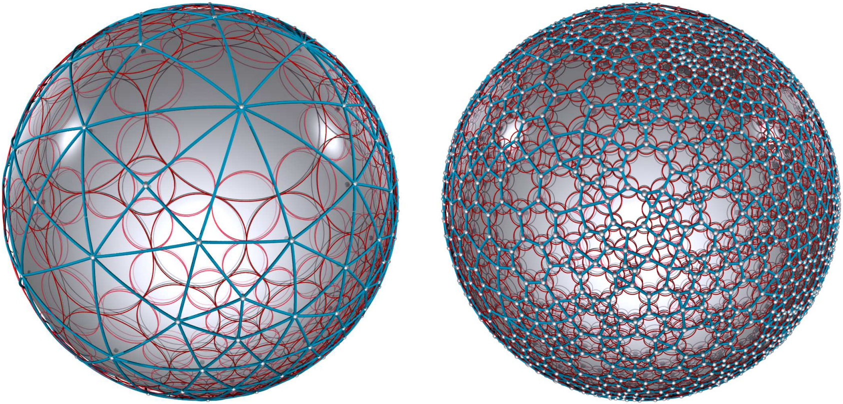


Figure: Koebe-Andreev-Thusrton circle packing theorem.