Discrete Ricci Flow for Polyhedral Euclidean Surfaces

David Gu

Computer Science Department
Stony Brook University

gu@cs.stonybrook.edu

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Discrete Conformal Map

Discrete Conformal Equivalence

Euclidean Surface Triangulation

A *surface* is a connected 2-dimensional manifold, possibly with boundary. A *surface triangulation* is a surface with a CW complex whose faces are triangles which are glued edge-to-edge. The vertices, edges, faces and corner angles are denoted as $V(T), E(T), F(T)$ and $C(T)$ respectively. A *Euclidean surface triangulation* is a surface triangulation equipped with a metric so that $T \setminus V(T)$ is locally isometric to the Euclidean plane, or half-plane if there is boundary, and the edges are geodesic segments.

Discrete Metric

A Euclidean triangulation is uniquely determined by a triangulation T and a function $l : E(T) \to \mathbb{R}_{\geq 0}$ assigning a length on every edge in such a way that the triangle inequalities are satisfied for every triangle in $F(T)$. l is called a *discrete metric* on (S, T), and denote the Euclidean triangulation by (T, l).

Discrete Conformal Equivalence

Definition (Discretely Conformally Equivalent)

Two combinatorially equivalent Euclidean triangulations (T, l) and $(T, \tilde{l})$ are *discretely conformally equivalent* if the discrete metric l and $\tilde{l}$ are related by

$$\boxed{\tilde{l}_{ij} = e^{\frac{u_i + u_j}{2}} l_{ij}}$$

for some $u \in \mathbb{R}^{V(T)}$.

This defines an equivalence relation on the set of discrete metrics on T, of which an equivalence class is called a *discrete conformal class* of discrete metrics, or a *discrete conformal structure* on T.

Discrete Conformal Equivalence

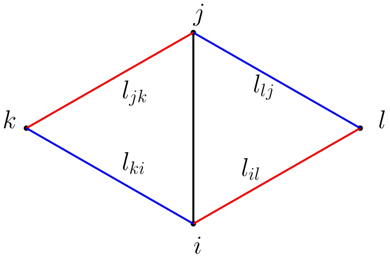

Figure: length cross ratio.

$$\mathrm{lcr}_{ij} = \frac{l_{il} l_{jk}}{l_{lj} l_{ki}}$$

Instead of the edge lengths l, we use the logarithmic lengths

$$\lambda = 2\log l, \quad \tilde{\lambda} = 2\log \tilde{l}$$

discretely conformally equivalent relation becomes linear:

$$\tilde{\lambda}_{ij} = \lambda_{ij} + u_i + u_j.$$

Discrete Conformal Equivalence

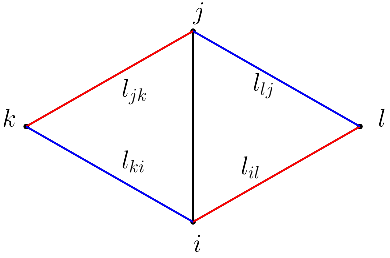

Figure: length cross ratio.

Definition (Length-Cross-Ratio)

For each interior edge ij between triangle ijk and ilj, define the *length-cross-ratio* induced by l to be

$$\mathrm{lcr}_{ij} = \frac{l_{il} l_{jk}}{l_{lj} l_{ki}}. \tag{1}$$

Discrete Conformal Equivalence

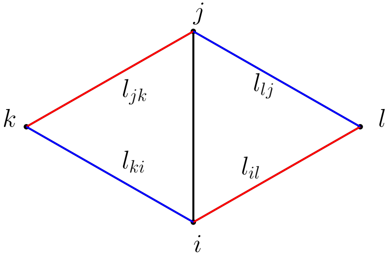

Figure: length cross ratio.

Suppose the quadrilateral *iljk* is embedded in $\mathbb{C}$, the length-cross-ratio lcr_{ij} is the absolute value of the complex cross ratio of the vertex positions z_i, z_l, z_j, z_k,

$$\text{cr}(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

Discrete Conformal Equivalence

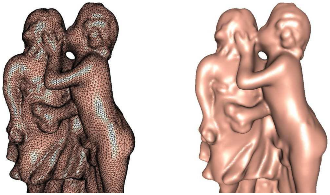

Figure: Euclidean triangulation (T, l).

Two Euclidean triangulations (T, l) and $(T, \tilde{l})$ are discrete conformally equivalent if and only if for each interior edge $ij \in E(T)$, the induced length-cross-ratios are equal:

$$\mathrm{lcr}_{ij} = \widetilde{\mathrm{lcr}}_{ij}.$$

Discrete Conformal Equivalence

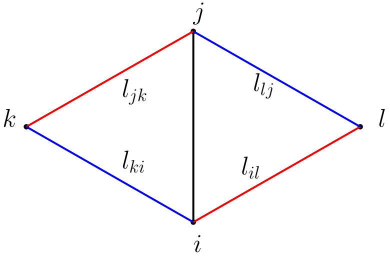

Figure: length cross ratio.

Properties

A necessary condition for length-cross-ratio is as follows: for each interior vertex $v_i \in V_{int}$,

$$\prod_{v_i \sim v_j} \mathrm{lcr}_{ij} = 1. \tag{2}$$

Length-Cross-Ratio

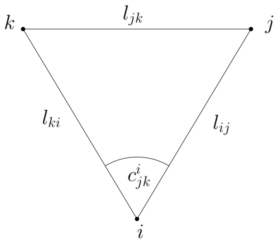

Figure: Corner function.

Lemma

Let $lcr : E_{int} \to \mathbb{R}_{>0}$ be any positive function on the set of interior edges. There exists a positive function $l : E \to \mathbb{R}_{>0}$ on the set satisfying Eqn.(1) for every interior edge ij, if and only if condition Eqn.(2) holds.

Proof.

Given $l \in (\mathbb{R}_{>0})^{E(T)}$, define $c \in (\mathbb{R}_{>0})^{C(T)}$ by

$$c^i_{jk} = \frac{l_{jk}}{l_{ij} l_{ki}}, \tag{3}$$

□

Length-Cross-Ratio

Proof.

in terms of these parameters, the length-cross-ratios induced by l are

$$\mathrm{lcr}_{ij} = \frac{c^i_{jk}}{c^i_{lj}} \tag{4}$$

Given $\mathrm{lcr} \in (\mathbb{R}_{>0})^{E_{int}}$ satisfying condition (2), find a solution $c \in (\mathbb{R}_{>0})^{C(T)}$. One can freely choose one c-value per vertex and successively calculate the c-values on the angles surrounding the same vertex using Eqn. 4. Then the edge length is given by

$$l_{ij} = (c^i_{jk} c^j_{ki})^{-\frac{1}{2}}.$$

□

Discrete Conformal Equivalence

Definition (Möbius Transformation)

The group of Möbius transformations of $\mathbb{R}^n \cup \{\infty\}$ is the group generated by inversions in spheres, or by similarity transformations and the inversion in the unit sphere.

Definition

Let T be a triangulation. Suppose $\nu : V(T) \to \mathbb{R}^n$ maps the vertices of each triangle to three affinely independent points. Two maps $\nu, \tilde{\nu} : V(T) \to \mathbb{R}^n$ are related by a Möbius transformation if there is a Möbiustransformation φ such that $\tilde{\nu} = \varphi \circ \nu$.

Möbius Transformation

Lemma

If two maps $V(T) \to \mathbb{R}^n$ are related by a Möbius transformation, then the induced discrete metrics are discretely conformally equivalent.

Proof.

The claim is obvious if the relating Möbius transformation is a similarity transformation. For inversion in the unit sphere, $x \mapsto \frac{1}{\|x\|^2}x$, it follows from the identity

$$\left\| \frac{1}{\|p\|^2}p - \frac{1}{\|q\|^2}q \right\| = \frac{1}{\|p\|\|q\|}\|p-q\|.$$

$\square$

Discrete Conformal Maps

Definition (Circumcircle Preserving Projective Map)

For any two Euclidean triangles, there is a unique projective map that maps one triangle onto the other and the circumcircle of one onto the circumcircle of the other. We call this map the *circumcircle preserving projective map* between the two triangles.

Definition (Discrete Conformal Map)

A *discrete conformal map* from one Euclidean triangulation (T, l) to a combinatorially equivalent Euclidean triangulation $(T, \tilde{l})$ is a homeomorphism whose restriction to every triangle is the circumcircle preserving projective map onto the corresponding image triangle.

Consider two triangulations, (T, l) and $(T, \tilde{l})$. For pairs of triangles, there is a circumcircle preserving projective map. But these maps do not fit together continously across edges in general, unless the triangulations are discretely conformally equivalent.

Discrete Conformal Maps

Theorem

The following two statements are equivalent:

1. (T, l) *and* $(T, \tilde{l})$ *are discretely conformally equivalent.*
2. *There exists a discrete conformal map* $(T, l) \to (T, \tilde{l})$.

Discrete Conformal Maps

Consider two triangles Δ and $\tilde{\Delta}$ in $\mathbb{E}^2$, let (x_i, y_i), $i \in \{1,2,3\}$, be the coordinates of their vertices in Cartesian coordinate system, $v_i = (x_i, y_i, 1)$ and $\tilde{v}_i(\tilde{x}_i, \tilde{y}_i, 1)$ be the homogeneous coordinate vectors. The projective maps $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ that map Δ to $\tilde{\Delta}$ correspond via $f([v]) = [F(v)]$ to the linear maps $F : \mathbb{R}^3 \to \mathbb{R}^3$ of homogeneous coordinates that satisfy

$$F(v_i) = \mu_i \tilde{v}_i$$

for some weights $\mu_i \in \mathbb{R} \setminus \{0\}$.

Lemma

The projective map $f : [v] \to [F(v)]$ maps the circumcircle Δ to the circumcircle of $\tilde{\Delta}$ if and only if

$$(\mu_1, \mu_2, \mu_3) = \mu(e^{-u_1}, e^{-u_2}, e^{-u_3})$$

where u_i's are the conformal factors $\tilde{l}_{ij} = e^{\frac{u_i+u_j}{2}} l_{ij}$ for a single triangle and $\mu \in \mathbb{R} \setminus \{0\}$ is an arbitrary factor.

Hyperbolic Ideal Polyhedra

Decorated Ideal Triangle

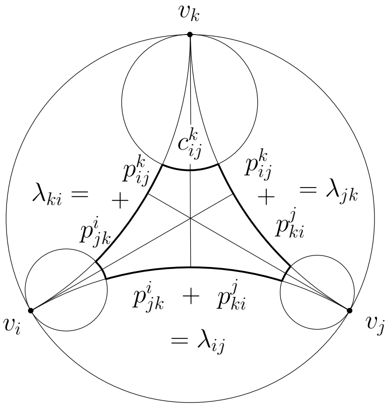

Figure: Decorated ideal triangle.

The signed distances:

$$\lambda_{ij} = p^i_{jk} + p^j_{ki}$$
$$\lambda_{jk} = p^j_{ki} + p^k_{ij}$$
$$\lambda_{ki} = p^k_{ij} + p^i_{jk}$$

$$p^k_{ij} = \frac{1}{2}(-\lambda_{ij} + \lambda_{jk} + \lambda_{ki})$$
$$p^i_{jk} = \frac{1}{2}(\lambda_{ij} - \lambda_{jk} + \lambda_{ki})$$
$$p^j_{ki} = \frac{1}{2}(\lambda_{ij} + \lambda_{jk} - \lambda_{ki})$$

Decorated Ideal Triangle

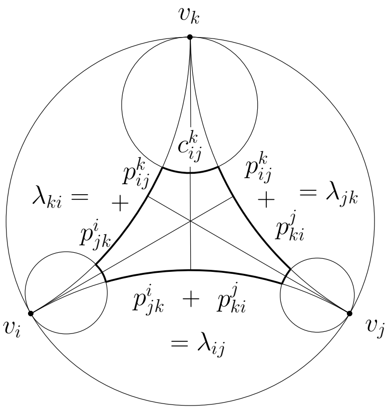

Figure: Decorated ideal triangle.

Lemma (Corner Angle)

The length c_{ij}^k of the arc of the horocycle centered at v_k that is contained in an ideal triangle $[v_i, v_j, v_k]$ is

$$c_{ij}^k = e^{-p_{ij}^k} = e^{\frac{1}{2}(\lambda_{ij} - \lambda_{jk} - \lambda_{ki})}. \tag{5}$$

Decorated Ideal Triangle

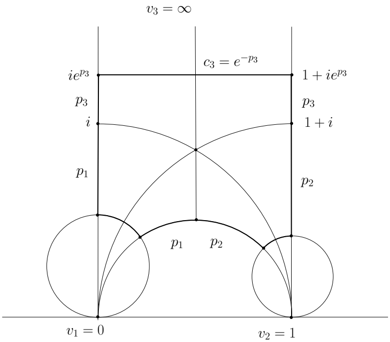

Figure: Decorated ideal triangle.

Proof.

An ideal triangle in the half-plane model with metric

$$ds = \frac{1}{\Im z}|dz|,$$

the length of c_3 is

$$c_3 = \int_0^1 \frac{1}{e^{p_3}} dx = e^{-p_3}.$$

$\square$

Decorated Ideal Tetrahedron

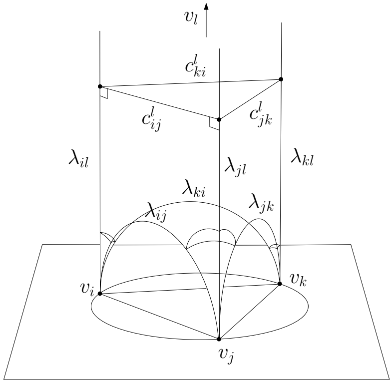

Figure: Decorated ideal tetrahedron.

All ideal tetrahedra are parameterized by the complex cross-ratio of the vertices in the infinite boundary of hyperbolic 3-space $[0, 1, z, \infty]$. A *decorated ideal tetrahedron* is an ideal hyperbolic tetrahedron with a choice of horospheres centered at the vertices. The signed distances between the horospheres are denoted as λ_{ij}.

Decorated Ideal Tetrahedron

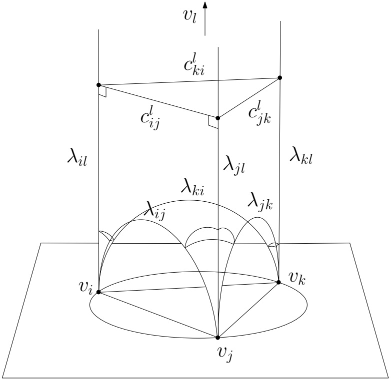

Figure: Decorated ideal tetrahedron.

The intrinsic geometry of a horosphere in hyperbolic space is Euclidean. So the intersection of the tetrahedron with the horosphere centered at v_l is a Euclidean triangle with side lengths $c^l_{ij}, c^l_{jk}, c^l_{ki}$.

Decorated Ideal Tetrahedron

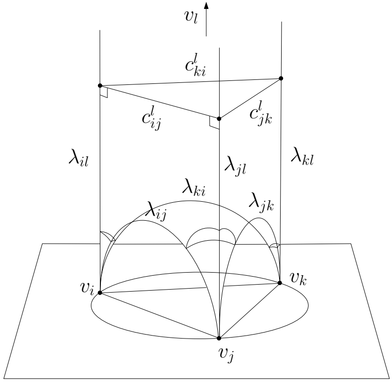

Figure: Decorated ideal tetrahedron.

Lemma

Six real numbers $\lambda_{ij}, \lambda_{jk}, \lambda_{ki}, \lambda_{il}, \lambda_{jl}, \lambda_{kl}$ are the signed distances between horospheres of a decorated ideal tetrahedron if and only if $c_{ij}^l, c_{jk}^l, c_{ki}^l$ determined by Eqn. (5) satisfy the triangle inequalities.

So the six parameters λ determine the congruence class of the ideal tetrahedron (2 real parameters) and the choice of horospheres (4 parameters).

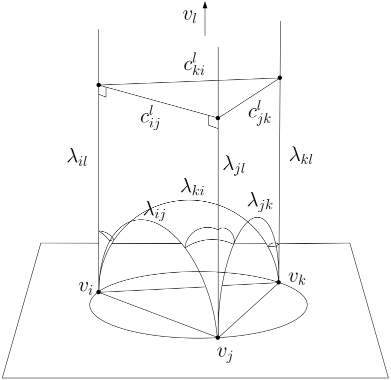

Figure: Decorated ideal tetrahedron.

The space of ideal tetrahedra is parameterized by three dihedral angles

$$
\begin{aligned}
\alpha_{ij} &= \alpha_{kl}, \\
\alpha_{jk} &= \alpha_{il}, \\
\alpha_{ki} &= \alpha_{jl}, \\
\alpha_{il} + \alpha_{jl} + \alpha_{kl} &= \pi.
\end{aligned}
$$

Decorated Ideal Tetrahedron

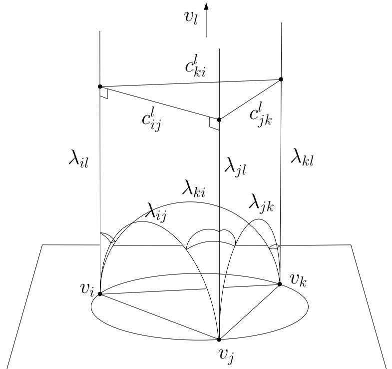

Figure: Decorated ideal tetrahedron.

The volume of an ideal hyperbolic tetrahedron, with dihedral angles α, β, γ of edges adjacent to the same vettexi is

$$V(\alpha, \beta, \gamma) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where the Lobachevsky function is defined as:

$$\Lambda(x) = -\int_0^x \log|2\sin(t)|dt.$$

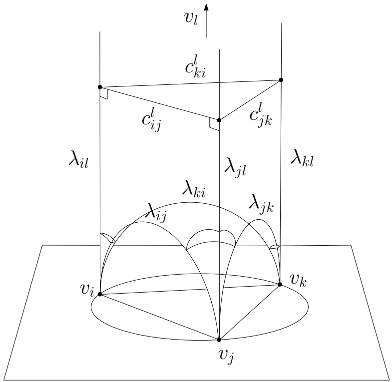

Figure: Decorated ideal tetrahedron.

The Schläfli's differential volume formula say that the volume's derivative is

$$dV = -\frac{1}{2}\sum_{\lambda_{ij}} d\alpha_{ij}$$

where the sum is taken over the six edges ij, λ_{ij} is the signed distance between horospheres centered at the vertices i and j, and α_{ij} is the interior dihedral angle. The choice of horospheres doens't matter because the diheral angle sum at a vertex is π.

Penner Coordinates

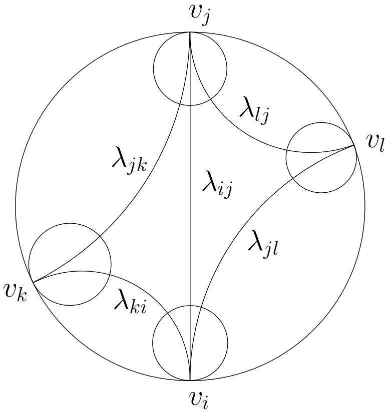

Figure: Penner coordinates.

Since the sides of an ideal hyperbolic triangle are complete geodesics, there is a one-parameter family of ways to glue two sides together. *Penner coordinates* and *shear coordinates* are two ways to describe how ideal triangles are glued together along their edges to form a hyperbolic surface with cusps.

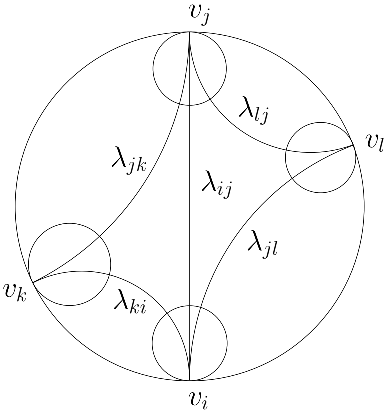

Figure: Penner coordinates.

Suppose T is a triangulated surface and $\lambda \in \mathbb{R}^{E(T)}$. For each triangle $ijk \in T$, take the decorated ideal triangle with horocycle distances $\lambda_{ij}, \lambda_{jk}, \lambda_{ki}$, and glue them so that the horocycles fit together. The result is a hyperbolic surface with cusps at the vertices, together with a particular choice of horocycles centered at the cusps.

Shear Coordinates

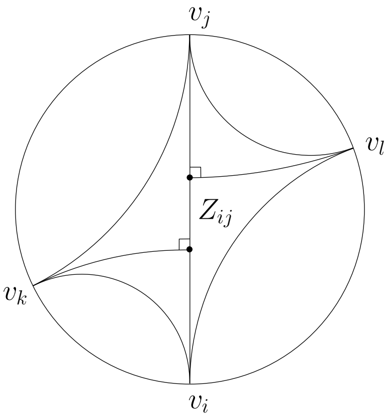

Figure: Shear coordinates.

The shear coordinate Z on an interior edge of an ideal triangulation is the signed distance of the base points of the heights from the opposite vertices.

Shear Coordinates

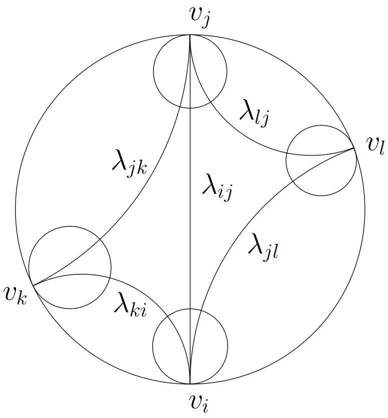

Figure: Penner coordinates.

Lemma

If $\lambda \in \mathbb{R}^{E(T)}$ are the Penner coordinates for an ideal triangulation with a particular choice of horocycles, then the shear coordinates $Z \in \mathbb{R}^{E_{int}}$ are

$$Z_{ij} = \frac{1}{2}(\lambda_{il} - \lambda_{lj} + \lambda_{jk} - \lambda_{ki}),$$

where k and l are the vertices opposite edge ij.

Because $Z_{ij} = p^j_{ki} - p^j_{il}$.

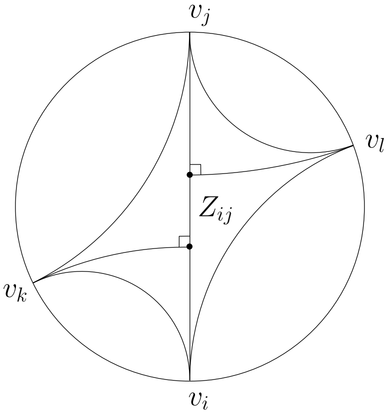

Figure: Shear coordinates.

Lemma

Let (T, l) be a Euclidean triangulation. The shear coordinates $Z \in \mathbb{R}^{E_{int}(T)}$ for the corresponding ideal triangulation are

$$Z_{ij} = \log lcr_{ij}.$$

Thus, for a suitable choice of horocycles, the Penner coordinates $\lambda \in \mathbb{R}^{E(T)}$ are given by $\lambda_{ij} = 2 \log l_{ij}$.

Discrete Conformally Euqivalent

Definition (Discrete Conformally Equivalent)

We call two polyhedral Euclidean metrics d' and d'' *discretely conformally equivalent* if there exists a sequence of pairs $\{(d_t, T_t)\}_{t=1}^{m}$, where d_t is a polyhedral Euclidean metric on $S_{g,n}$, T_t is a Delaunay triangulation of $(S_{g,n}, d_t)$, $d_1 = d'$ and $d_m = d''$ and for every t either

1. $d_t = d_{t+1}$ in the sense that $(S_{g,n}, d_t)$ is isometric to $(S_{g,n}, d_{t+1})$ by an isometry isotopic to identity with respect to $\mathcal{B}$, or
2. $T_t = T_{t+1}$ and there exists a function $u : \mathcal{B} \to \mathbb{R}$ such that for every edge e of T_t with vertices B_i and B_j we have

$$\mathrm{len}_{d_t}(e) = \exp\left(\frac{(u(B_i) + u(B_j))}{2}\right) \mathrm{len}_{d_{t+1}}(e).$$

Convex Prismatic Complex

Prismatic Complexes

Let $(S_{g,n}, d)$ be a hyperbolic cusp-surface with n cusps and T be an ideal geodesic triangulation of $S_{g,n}$ with vertices at cusps. By $E(T)$ and $F(T)$ denote its sets of edges and faces respectively. The set of cusps is denoted by $\mathcal{A} = \{A_1, \ldots, A_n\}$. We fix a horodisk at each A_i, referred as *canonical horodisk*, its boundary as *canonical horocycle*.

Definition (Admissible Weight)

Suppose that a real weight u_i is assigned to every cusp A_i, denote the weight vector by $\mathbf{u} \in \mathbb{R}^n$. A pair $(T, \mathbf{u})$ is called *admissible* if for every decorated ideal triangle $A_iA_jA_h \in F(T)$ there exists a decorated ideal tetrahedron with the lengths of lateral edges A_iB_i, A_jB_j, A_hB_h equal to $-u_i, -u_j$ and $-u_h$.

Prismetic Complex

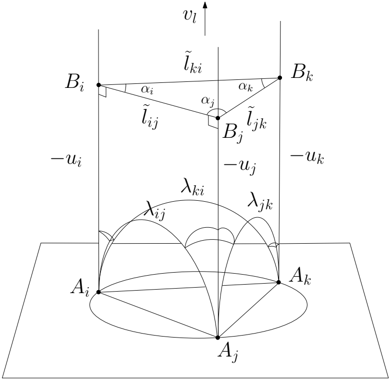

Figure: A decorated ideal tetrahedron.

Let $(T, \mathbf{u})$ be an admissible pair. For each ideal triangle $A_iA_jA_k \in F(T)$ consider a prism (decorated ideal tetrahedron) constructed as previous definition. Since the upper and lateral edge lengths are fixed, it is unique up to isometry. Canonical horocycles coming from $(S_{g,n}, d)$ determine canonical horospheres at each ideal vertex of the prism.

Prismetic Complex

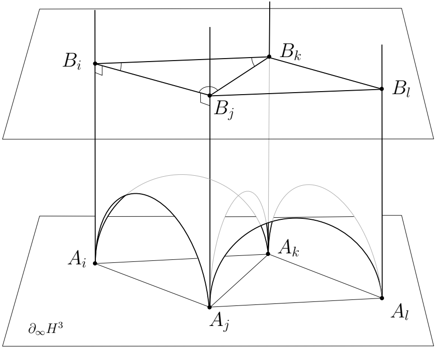

Figure: Glue 2 decorated ideal tetrahedra along the common lateral face.

Definition (Prismetic Complex)

A prismatic complex $K(T, \mathbf{r})$ is a metric space obtained by gluing all these prisms via isometries of lateral faces. We choose gluing isometries in such a way that canonical horspheres at ideal vertices of prims match together.

Prismetic Complex

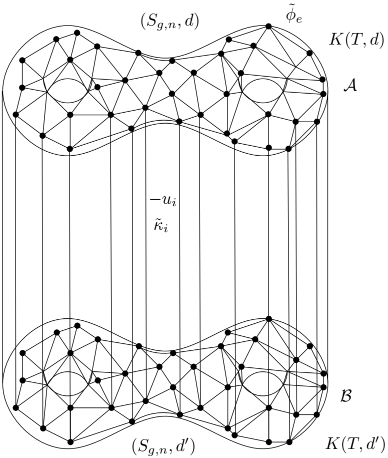

Figure: Convex prismatic complex.

A prismatic complex $K(T, \mathbf{u})$ is a complete hyperbolic cone manifold with polyhedral boundary. The boundary consists of two components: the unioin of upper faces form the *upper boundary* coming with a natural isometry to $(S_{g,n}, d)$ with cusp hyperbolic metric; the union of lower faces forms the *lower boundary*, which is isometric to $(S_{g,n}, d')$ for a polyhedral Euclidean metric d' with conical singularities at points B_i.

Prismetic Complex

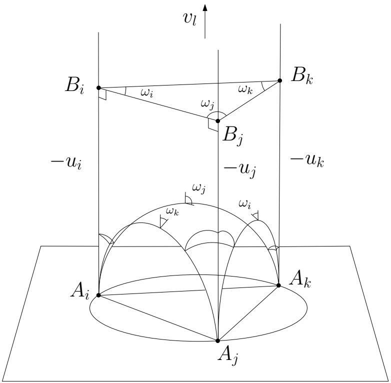

Figure: Dihedral angles of a decorated ideal tetrahedron.

Consider T as a geodesic triangulation of both components. The dihedral angle $\tilde{\phi}_e$ of an edge $e \in E(T)$ is the sum of dihedral angles in both prisms containing e and $\tilde{\theta}_e = \pi - \tilde{\phi}_e$ is its exterior dihedral angle.

The total conical angle $\tilde{\omega}_i$ of an inner edge A_iB_i is the sum of the corresponding dihedral angles of all prisms containing A_iB_i and $\tilde{\kappa}_i = 2\pi - \tilde{\omega}_i$ is the curvature of A_iB_i. The conical angle of the point B_i in the lower boundary is also equal to $\tilde{\omega}_i$.

Convex Prismatic Complex

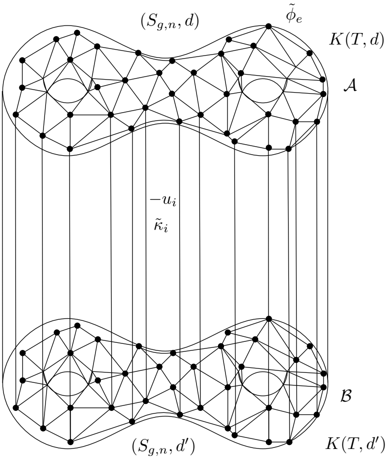

Figure: Convex prismatic complex.

Definition (Convex Complex)

A complex K is called *convex* if for every upper edge its dihedral angle is at most π. If $K = K(T, \mathbf{u})$, then the pair $(K, \mathbf{u})$ is also called *convex*.

Convex Prismetic Complex

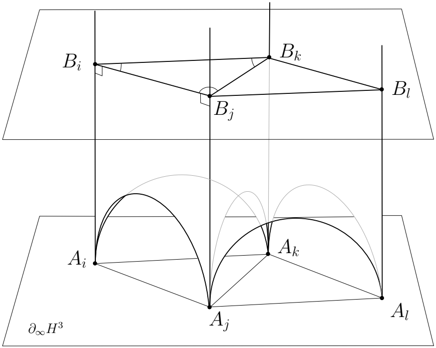

Figure: Glue 2 decorated ideal tetrahedra along the common lateral face.

Let d' be a polyhedral Euclidean metric on $S_{g,n}$ and T be its geodesic triangulation. Denote the set of marked points by $\mathcal{B} = \{B_1, \ldots, B_n\}$. Take a triangle $B_i B_j B_h$, there is a unique up to decorated ideal tetrahedron that have $B_i B_j B_h$ as its lower face. Glue all such tetrahedra together and obtain a complex $K(d', T)$ with the lower boundary isometric to $(S_{g,n}, d')$. Gluing isometries are uniquely defined if we fix the horosphere at each upper vertex passing through the respective lower vertex and match them together.

Convex Prismetic Complex

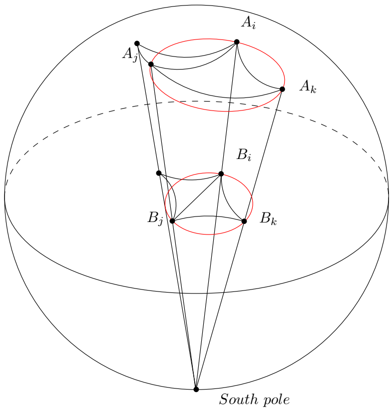

Figure: Lower boundary surface is Delaunay if and only if the upper surface is convex.

Lemma

The semi-ideal prismatic complex $K(d', T)$ is convex if and only if T is a Delaunay triangulation of $(S_{g,n}, d')$, where d' is a polyhedral hyperbolic metric.

Proof.

Given a hyperbolic triangle $B_iB_jB_k$ on the equator plane, construct a semi-ideal prim with roof $A_iA_jA_k$, then the points A_i, B_i and the south pole are colinear. Namely, the projection from $A_iA_jA_k$ to $B_iB_jB_k$ is a stereo-graphic projection from the sphere to the equator plane. The projection maps the circumcircle of $B_iB_jB_k$ to the circumcircle of $A_iA_jA_k$, therefore the projection preserves Delaunay property. □

Convex Prismetic Complex

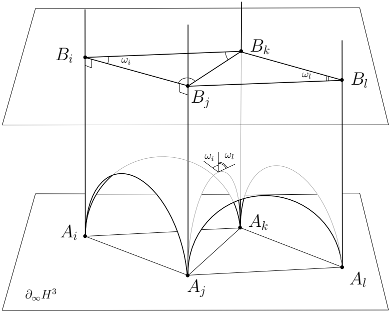

Figure: Lower boundary surface is Delaunay if and only if the upper surface is convex.

Lemma

The complex $K(d', T)$ is convex if and only if T is a Delaunay triangulation of $(S_{g,n}, d')$. Besides, any two convex complexes with isometric lower boundaries are isometric.

Proof.

In an ideal tetrahedron, the dihedral angles on the opposite edges are equal. T is Delaunay, if and only if $\omega_i + \omega_l \leq \pi$, $\tilde{\theta}_{jk} \geq 0$. $\square$

Delaunay Triangulation

Theorem

For each hyperbolic cusp metric d on $S_{g,n}$ there are finitely many Delaunay triangulations of $(S_{g,n}, d)$.

Space of Convex Prismetic Complex $\mathcal{K}$

Space of Convex Complexes

Definition (Space of Convex Complexes)

Denote by $\mathcal{K}$ the set of all convex complexes with the upper boundary isometric to $(S_{g,n}, d)$ considered up to marked isometry (an isometry to itself isotopic to identify with respect to $\mathcal{A}$).

Every $K \in \mathcal{K}$ can be represented as $K(T, \mathbf{r})$.

Lemma

Let $(T', \mathbf{u})$ *and* $K'' = (T'', \mathbf{u})$ *be two convex pairs. Then the complexes* $K' = K(T', \mathbf{r})$ *and* $K'' = (T'', \mathbf{u})$ *are marked isometric.*

Corollary

The map $\mathbf{u} : \mathcal{K} \to \mathbb{R}^n$ *is injective.*

Hence $\mathcal{K}$ can be parameterizeed by $\mathbf{u}$.

Space of Convex Complexes

Proof.

The top surface for both K' and K'' is $(S_{g,n}, d)$, the bottom surfaces are $(S_{g,n}, d')$ and $(S_{g,n}, d'')$. d' is determined by d and $\mathbf{u}'$, d'' is determined by d and $\mathbf{u}''$ by the formula

$$c_{ij}^k = e^{-p_{ij}^k} = e^{\frac{1}{2}(\lambda_{ij} - \lambda_{jk} - \lambda_{ki})}.$$

namely

$$\tilde{l}_{ij} = e^{\frac{1}{2}(\lambda_{ij} + u_i + u_j)}$$

where $\lambda_{ij} = 2 \log l_{ij}$. we obtain d' is isometric to d'', therefore T' equals to T''. Hence K' and K'' are marked isometric. $\square$

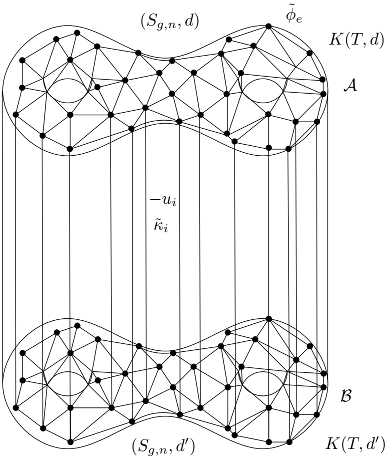

Figure: Convex prismetic complex.

Theorem

The lower boundary metric d' on $S_{g,n}$ is discretely conformally equivalent to d'' if and only if the upper boundaries of $K(d')$ and $K(d'')$ are isometric.

Convex Prismetic Complex

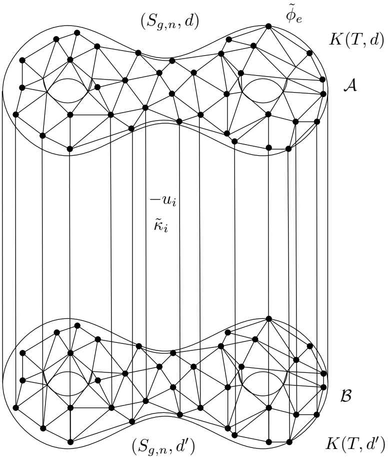

Figure: Convex prismetic complex.

Proof.

Assume that the upper boundaries of $K(d')$ and $K(d'')$ are both isometric to $(S_{g,n}, d)$ for a cusp metric d. Let $\mathcal{K}$ be the set of convex complexes realizing $(S_{g,n}, d)$. Choose a decoration on $(S_{g,n}, d)$. □

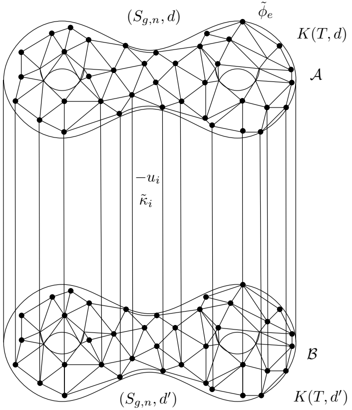

Figure: Convex prismetic complex.

Proof.

First, assume that convex complexes $K(d'), K(d'') \in \mathcal{K}(T)$ for a triangulation T, T is Delaunay for both d' and d''. Take $e \in E(T)$ and denote its lengths in d' and d'' by l' and l'' respectively. By u'_i and u'_j denote the weights of its endpoints in $K(d')$, and u''_i and u''_j in $K(d'')$. Then

$$
\begin{aligned}
l'_{12} &= e^{\frac{1}{2}(\lambda_{12}+u'_1+u'_2)} \\
l''_{12} &= e^{\frac{1}{2}(\lambda_{12}+u''_1+u''_2)} \\
l''_{12} &= l'_{12} e^{\frac{1}{2}[(u''_1-u'_1)+(u''_2-u'_1)]}
\end{aligned}
$$

Thus, d' is discrete conformally equivalent to d''. □

Convex Prismetic Complex

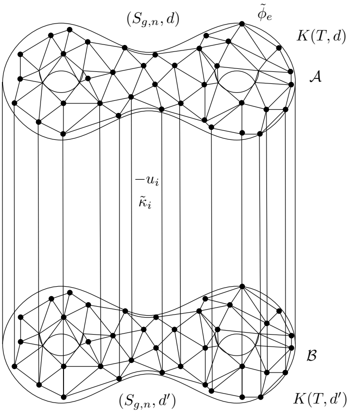

Figure: Convex prismetic complex.

Assume that d' and d'' are in the different cells $\mathcal{K}(T')$ and $\mathcal{K}(T'')$. The decomposition $\mathcal{K} = \bigcup \mathcal{K}(T)$ is finite and the boundaries of cells $\mathcal{K}(T)$ are piecewise analytic as subsets of $\mathbb{R}^n$. Then $K(d')$ and $K(d'')$ can be connected by a path in $\mathcal{K}$ transversal to the boundaries of all cells and intersecting them m times. All intersection points correspond to distinct convex complexes, denote their lower boundary metrics by $d_1, \ldots, d_m$, $d_0 = d'$, $d_{m+1} = d''$. A segment between d_i and d_{i+1} of the path belongs to $\mathcal{K}(T_i)$ for some triangulation T_i, T_i is Delaunay for both d_i and d_{i+1}. Hence d_i is discretely conformally equivalent. Then so are d_0 and d_{m+1}.

Convex Prismetic Complex

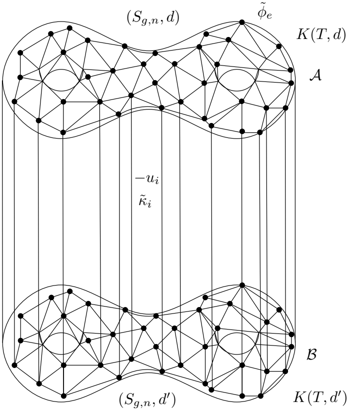

Figure: Convex prismetic complex.

In the opposite direction, assume that d' and d'' are discretely conformally equivalent and have a common Delaunay triangulation T. Then there exists a function $u : \mathcal{B} \to \mathbb{R}$ such that for each edge $e \in E(T)$ with endpoints B_i and B_j we have

$$\text{len}_{d'}(e) = e^{\frac{1}{2}(u(B_i)+u(B_j))}\text{len}_{d''}(e)$$

Consider $K(d')$ and $K(d'')$, then T is a face triangulation of both these complexes. Choose an horosection at each vertex of the upper boundaries in both $K(d')$ and $K(d'')$. Let $-u_i'$ and $-u_i''$ be the distance from the horosections at $A_i \in \mathcal{A}$ to B_i in $K(d')$ and $K(d'')$ respectively.

Convex Prismetic Complex

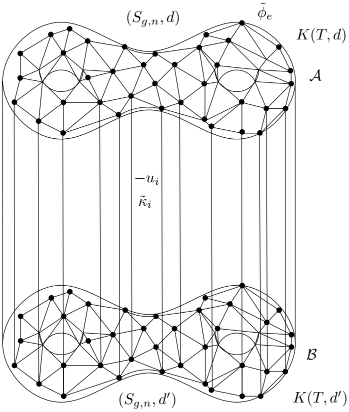

Figure: Convex prismetic complex.

We can choose the horosections such that for every i, $\frac{u_i''-u_i'}{2} = u(B_i)$. Then

$$\tilde{l}_{ij} = e^{\frac{1}{2}(\lambda_{ij}+u_i+u_j)}$$

shows for each $e \in E(T)$ its length in the upper boundary $K(d')$ is the same as in the upper boundary of $K(d'')$ (with respect to the chosen horosection). Therefore, the upper boundary metrics of $K(d')$ and $K(d'')$ together with the chosen decorations have the same Penner coordinates, hence they are isometric. When d' and d'' are discretely conformally equivalent and do not have a common Delaunay triangulation T, is inductively reduced to the last case.

Discrete Hilbert-Einstein Action

Volume

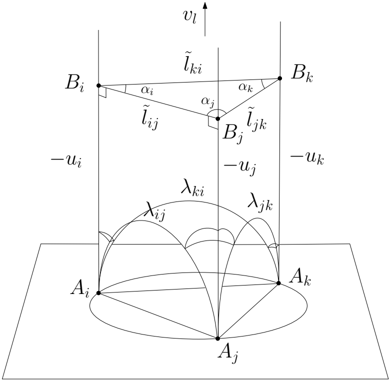

Figure: A decorated ideal tetrahedron.

By Schläfli differential volume formula:

$$dV = -\frac{1}{2}\sum \lambda_{ij} d\alpha_{ij}$$

λ_{ij} is the signed distance between horospheres and α_{ij} is the interior dihedral angle. Now define

$$\hat{V}(\lambda_{12}, \lambda_{23}, \lambda_{31}, \lambda_{14}, \lambda_{24}, \lambda_{34})$$
$$= \frac{1}{2}\sum_{ij} \alpha_{ij}\lambda_{ij} + V(\alpha_{14}, \alpha_{24}, \alpha_{34}),$$

where dihedral angles $\alpha_{12} = \alpha_{34}$, $\alpha_{23} = \alpha_{14}$, $\alpha_{31} = \alpha_{24}$, which are the functions of the λ_{ij}.

Volume

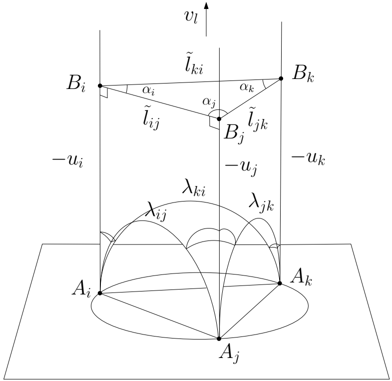

Figure: A decorated ideal tetrahedron.

The ideal tetrahedron volume
$V(\alpha,\beta,\gamma)=\Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma)$.

$$
\begin{aligned}
d\hat{V} &= \frac{1}{2}\sum_{ij}(d\alpha_{ij}\lambda_{ij}+\alpha_{ij}d\lambda_{ij})+dV\\
&= \frac{1}{2}\sum_{ij}(d\alpha_{ij}\lambda_{ij}+\alpha_{ij}d\lambda_{ij})\\
&- \frac{1}{2}\sum\lambda_{ij}d\alpha_{ij}
\end{aligned}
$$

We obtain

$$
\boxed{d\hat{V}=\frac{1}{2}\sum\alpha_{ij}d\lambda_{ij}}.
$$

Volume

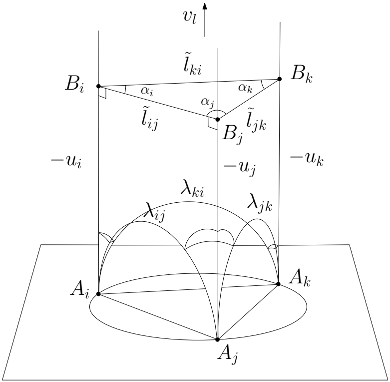

Figure: A decorated ideal tetrahedron.

Define energy $E_{T,\Theta,\lambda}(\mathbf{u})$ as

$$\sum_{ijk\in T} 2\hat{V}(\lambda_{ij}, \lambda_{jk}, \lambda_{ki}, -u_i, -u_j, -u_k) - \sum_{ij\in E(T)} \Phi_{ij}\lambda_{ij} + \sum_{i\in V(T)} \Theta_i u_i$$

where

$$\Phi_{ij} = \begin{cases} \pi, & ij \text{ interior edge} \\ \frac{\pi}{2}, & ij \text{ boundary edge} \end{cases}$$

Θ_i is the target cone angle at the vertex i, and the term in blue is constant.

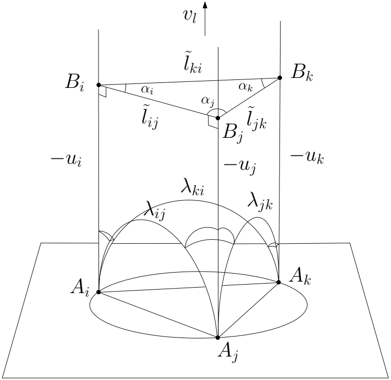

Figure: A decorated ideal tetrahedron.

$$\frac{\partial}{\partial u_i} E_{T,\Theta,\lambda} = \Theta_i - \sum_{jk:ijk\in T} \alpha^i_{jk}$$

$$=(\Theta_i - 2\pi) + \left(2\pi - \sum_{jk} \alpha^i_{jk}\right)$$

$$=\tilde{\kappa}_i - \kappa'_i.$$

Discrete Hilbert-Einstein Functional

Definition (Hilbert-Einstein Functional)

For $\mathbf{r} \in \mathbb{R}^n$ let T be any face triangulation of the convex complex $K(\mathbf{r})$. We introduce *the discrete Hilbert-Einstein functional* over the space of convex complexes $\mathcal{K}$ identified with $\mathbb{R}^n$:

$$S(\mathbf{u}) := 2\text{vol}(K(\mathbf{u})) + \sum_{1 \leq i \leq n} u_i \tilde{\kappa}_i - \sum_{e \in E(T)} l_e \tilde{\theta}_e.$$

Consider a function $\kappa' : \mathcal{A} \to (-\infty, 2\pi)$. Define the *modified discrete Hilbert-Eistein functional*:

$$S_{\kappa'}(\mathbf{u}) := S(\mathbf{u}) - \sum_{1 \leq i \leq n} u_i \kappa'_i.$$

The value $S(\mathbf{u})$ doesn't depend on the choice of T, because two face trangulations of $K(\mathbf{u})$ are different only in flat edges, for which $\tilde{\theta}_e = 0$.

Discrete Hilbert-Einstein Functional

Lemma

For every $\mathbf{u} \in \mathbb{R}^n$, $S(\mathbf{u})$ *is twice continuously differentiable and*

$$\frac{\partial S}{\partial u_i} = \tilde{\kappa}_i.$$

Proof.

By generalized Schläffli's formula, for a prism $P = A_iA_jA_hB_hB_jB_i \subset K$, we have

$$2d\mathrm{vol}(P) = -\sum \lambda_{ij} d\alpha_{ij} = u_i d\omega_i + u_j d\omega_j + u_h d\omega_h - l_{jh} d\phi_i - l_{ih} d\phi_j - l_{ij} d\phi_h.$$

Summing these equalities over all prims we obtain

$$2d\mathrm{vol}(K(\mathbf{u})) = -\sum_{1 \le i \le n} u_i d\tilde{\kappa}_i + \sum_{e \in E(T)} l_e d\tilde{\theta}_e.$$

Discrete Hilbert-Einstein Functional

Proof.

Thus,

$$dS(\mathbf{u}) = 2d\mathrm{vol}(K(\mathbf{u})) + \sum_{1\leq i\leq n} (du_i\tilde{\kappa}_i + u_i d\tilde{\kappa}_i) - \sum_{e\in E(T)} (dl_e\tilde{\theta}_e + l_e d\tilde{\Theta}_e).$$

$$= \sum_{1\leq i\leq n} \tilde{\kappa}_i du_i - \sum_{e\in E(T)} \tilde{\theta}_e dl_e = \sum_{1\leq i\leq n} \tilde{\kappa}_i du_i.$$

since l_e's are always fixed, $dl_e = 0$. □

Corollary

For every $\mathbf{u} \in \mathbb{R}^n$, $S_{\kappa'}(\mathbf{u})$ is twice continuously differentiable and

$$\frac{\partial S_{\kappa'}}{\partial u_i} = \tilde{\kappa}_i - \kappa'_i.$$

If $\mathbf{u}$ is a critical point of $S_{\kappa'}$, then for all i, $\tilde{\kappa}_i = \kappa'_i$.

Concavity of Hilbert-Einstein Functional

Lemma

Define

$$X_{ij} := \frac{\partial^2 S}{\partial u_i \partial u_j} = \frac{\partial k_i}{\partial u_j}.$$

Then for every $1 \leq i \leq n$:

1. $X_{ii} > 0$,
2. *for $i \neq j$, $X_{ij} < 0$,*
3. *for every $1 \leq i \leq n$, $\sum_{1 \leq j \leq n} X_{ij} = 0$,*
4. *the second derivatives are continuous at every point $\mathbf{u} \in \mathbb{R}^n$. In particular, this implies that $X_{ij} = X_{ji}$.*
5. *the Hessian matrix has one dimensional null space, spaced by $(1, 1, \ldots, 1)^T$.*

Namely the matrix is diagonally dominated.

Concavity of Hibert-Einstein Functional

Proof.

Suppose ij is shared by faces ijk and jil in the Delaunay triangulation T, then

$$X_{ij} = \frac{\partial k_i}{\partial u_j} = -\frac{1}{2}(\cot\alpha_{ij}^k + \cot\alpha_{ji}^l) < 0,$$

For the diagonal element $\alpha_i + \alpha_j + \alpha_k = \pi$, hence

$$\frac{\partial \alpha_i}{\partial u_i} = -\frac{\partial \alpha_j}{\partial u_i} - \frac{\partial \alpha_k}{\partial u_i} = -\frac{\partial \alpha_i}{\partial u_j} - \frac{\partial \alpha_i}{\partial u_k} = -\frac{1}{2}(\cot\alpha_{ij}^k + \cot\alpha_{ki}^j)$$

Hence

$$X_{ii} = \frac{\partial k_i}{\partial u_i} = \sum_{i\sim j}\frac{1}{2}(\cot\alpha_{ij}^k + \cot\alpha_{ji}^l) = -\sum_{i\sim j} X_{ij} > 0.$$

For any $\mathbf{u}$, we have

$$\mathbf{u}^T X \mathbf{u} = \sum_{i\sim j}\frac{1}{2}(\cot\alpha_{ij}^k + \cot\alpha_{ji}^l)(u_i - u_j)^2$$

the Hessian is semi-positive-definite with one dimensional null space space by $(1,\ldots,1)^T$. □

Concavity of Hibert-Einstein Functional

Corollary

The functions S and $S_{\kappa'}$ are strictly convex over $\{\mathbf{u} \in \mathbb{R}^n : \sum_{i=1}^n u_i = 0\}$.

Convex Prismatic Complex

Theorem (Main)

For every cusp metric d on $S_{g,n}$, $g \geq 1$, $n > 0$ with the set of cusps $\mathcal{A}$ and a function $\kappa' : \mathcal{A} \to (-\infty, 2\pi)$ satisfying

$$\sum_{A_i \in \mathcal{A}} \kappa'(A_i) = 2\pi(2 - 2g)$$

there exists a unique up to isometry convex complex $K(\mathbf{u})$, where $\sum_{i=1}^{n} u_i = 0$, with the upper boundary isometric to $(S_{g,n}, d)$ and the curvature $\tilde{\kappa}_i$ of each edge $A_i B_i$ equal to $\kappa'(A_i)$.

Main Theorem

Lemma

Consider a cube Q in $\mathbb{R}^n$, $Q := \{\mathbf{u} \in \mathbb{R}^n : \max |r_i| \le q\}$. If

$$\sum_{1 \le i \le n} \kappa'_i = 2\pi(2 - 2g),$$

then for sufficiently large q, the minimum of $S_{\kappa'}(\mathbf{u})$ over Q is attained in the interior of Q.

This can be proved by analyzing the behavior of $S_{\kappa'}$ near infinity.

Main Theorem

Proof.

For the main theorem, fix a sufficiently large Q, $\bar{Q}$ is compact, $S_{\kappa'}(\mathbf{u})$ has minimum points. Since $S_{\kappa'}(\mathbf{u})$ is strictly convex on the hyper-plane $\sum_i u_i = 0$, the minimum point is unique. The minimum of $S_{\kappa'}(\mathbf{u})$ is an interior point $\mathbf{u}^*$ of Q, at the minimum point $\nabla S_{\kappa'}(\mathbf{u}^*) = 0$, $\tilde{\kappa}_i = \kappa'_i$. $\square$

The curvature flow is the negative gradient flow of the Hilbert-Einstein energy:

$$\frac{du_i}{dt} = \kappa'_i - \tilde{\kappa}_i.$$