# Homology for 3-Manifolds

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# Relative Homology

Let $K$ be a simplicial complex, $K_0$ be a subcomplex of $K$. The relative chain group $C_p(K, K_0)$ is the quotient of the chain groups of $K$ and $K_0$, i.e.

$$C_p(K, K_0) = C_p(K)/C_p(K_0).$$

The general element of $C_p(K, K_0)$ has the form $c + C_p(K_0)$, $c \in C_p(K)$, so it is a coset. So the quotient partitions $C_p(K)$ into cosets, which consists of p-chains in $K$ which are the same on $K \setminus K_0$, but may differ on $K_0$.

# Relative Boundary Operator

$$\begin{array}{ccccc}
c & C_p(K) & \xrightarrow{\partial_p} & C_{p-1}(K) & \partial_p c \\
\downarrow & & & & \downarrow \\
c + C_p(K_0) & C_p(K, K_0) & \xrightarrow{\partial_p} & C_{p-1}(K, K_0) & \partial_p c + C_{p-1}(K_0)
\end{array}$$

Given an element $c + C_p(K_0) \in C_p(K, K_0)$, map $c \to \partial_p c \in C_{p-1}(K)$, consider the coset associated with $\partial_p c$, which is $\partial_p c + C_{p-1}(K_0)$.
Can check $\partial_{p-1}\partial_p = 0$.

# Relative Homology Group

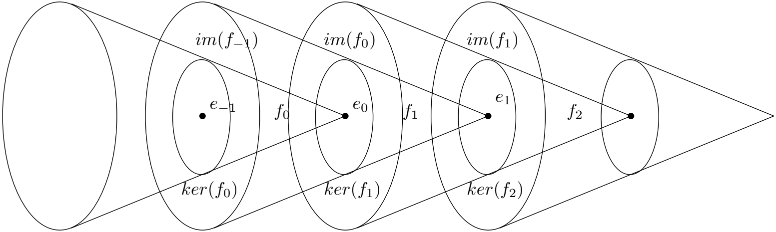
So now, we can have the relative cycles, relative boundaries, relative homology.

- Relative cycles, $Z_p(K, K_0) = \text{Ker } \partial_p$
- Relative boundaries, $B_p(K, K_0) = \text{Img } \partial_{p+1}$
- Relative homology,
$$H_p(K, K_0) = \frac{\text{Ker}\partial_p}{\text{Img}\partial_{p+1}}$$

To distinguish the homology of $K$ fro the pair $(K, K_0)$, we refer to $H_p(K)$ as the absolute homology of $K$, and the element as the absolute classes.

# Relative Homology Group

The continuous maps between spaces induce maps on homology, and this extends to relative homology as well. $f : K \to L$, and subcomplexes $K_0 \subset K$, $L_0 \subset L$, and $f$ maps $K_0$ into $L_0$, then we have an induced map $f_\# : C_p(K, K_0) \to C_p(L, L_0)$, which commutes with the boundary map, induces $f_* : H_p(K, K_0) \to H_p(L, L_0)$. One important case is the induced map from $(K, \emptyset) \to (K, K_0)$.




$$\cdots \longrightarrow G_{-1} \xrightarrow{f_0} G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots$$

Figure: Exact sequence.

# Homological Algebra

Chain maps $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$, short exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d] & A_n \arrow[r, "\partial"] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d] & B_n \arrow[r, "\partial"] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

# Homological Algebra

Yields a long exact sequence

$$\begin{array}{cccccc}
\longrightarrow & H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) \\
\longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(C) \\
\longrightarrow & H_{n-2}(A) & \longrightarrow & H_{n-2}(B) & \longrightarrow & H_{n-2}(C)
\end{array}$$

# Relative Homology Exact Sequence

There exists an exact sequence:

$$\cdots \to H_n(K_0) \xrightarrow{i_*} H_n(K) \xrightarrow{\pi_*} H_n(K, K_0) \xrightarrow{\partial} H_{n-1}(K_0) \to \cdots$$

where

1. $i_* : K_0 \to K$ the inclusion of $K_0$ into $K$
2. $\pi_*$ be induced by the projection $C_*(K) \to C_*(K, K_0)$
3. $\partial : H_*(K, K_0) \to H_{*-1}(K_0)$ be the map that takes a relative cycle to its boundary.

# Excision

**Theorem**

*Let $K_0 \subset K$, $L_0 \subset L$, be pair of simplicial complexes, that $L \subset K$,and $L \setminus L_0 = K \setminus K_0$, then we have isomorphic relative homology*

$$H_p(K, K_0) \cong H_p(L, L_0)$$

*for all dimensions $p$.*

We use the Smith normal form to prove this. We order the simplices in $K$, so that the simplices in $K_0$ appear before those in $K \setminus K_0$. Construct the boundary matrix.

# Excision

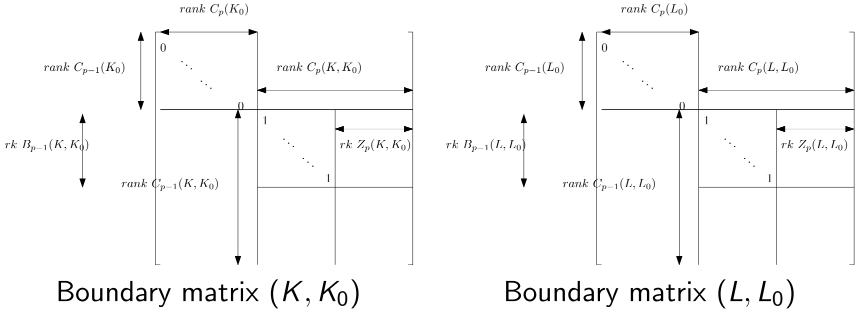



Figure: Proof for excision theorem.

# Homology of 3-Manifolds

## Proposition

*The homology $H_*(M)$ of a closed orientable 3-manifold $M$ is determined by $\pi_1(M)$.*

## Proof.

The group $H_1(M)$ is the Abelianization of $\pi_1(M)$ and $H^1(M) = \mathrm{Hom}(H_1(M), \mathbb{Z})$ which is isomorphic to $H_1(M)$ modulo its torsion. By Poincaré duality $H_2(M) = H^1(M)$ and $H^2(M) = H_1(M)$. Finally $H_3(M) = H_0(M) = \mathbb{Z}$. $\square$

# Euler Characteristic $\chi(M)$

The Euler characteristic $\chi(M)$ of a closed odd dimensional manifold vanishes.

## Proposition

*If $M$ is a compact 3-manifold with boundary, then*

$$\chi(M) = \frac{1}{2}\chi(\partial M).$$

## Proof.

If $M$ is closed and orientable, we have $\chi(M) = \sum_{i=0}^{n}(-1)^i b_i$ and the Betti number $b_i$ and $b_{n-i}$ are equal by Poincaré duality, hence $\chi(M) = 0$. If $M$ is non-orientable then it has an orientable double-cover $N$ and $\chi(N) = 2\chi(M)$, hence $\chi(N) = 0$ implies $\chi(M) = 0$. If $M$ has boundary, then

$$0 = \chi(DM) = 2\chi(M) - \chi(\partial M)$$

where $DM$ is the double of $M$, constructed by taking two copies of $M$ and

# Mayer-Vietoris Sequence

## Definition (Mayer-Vietoris Sequence)

Let $X$ be a topological space and $R$ be a ring, $X = U \cup V$ and $U, V$ are open, the exact Mayer-Vietoris sequence is:

$$\cdots \to H_{n+1}(X) \xrightarrow{\partial_*} H_n(U \cap V) \xrightarrow{\begin{pmatrix} i_* \\ j_* \end{pmatrix}} H_n(U) \oplus H_n(V) \xrightarrow{k_* - l_*} H_n(X) \to \dots$$

where $i : U \cap V \to U, j : U \cap V \to V, k : U \to X$ and $l : V \to X$ are inclusion maps, and $\oplus$ denotes the direct sum of Abelian groups.

# Mayer-Vietoris Sequence

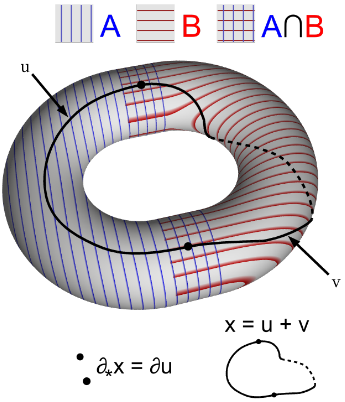



Figure: $\partial_*$ map.

$x \in H_n(X)$ is an n-cycle, $x = u + v$,
$u \subset U$ and $v \subset V$,
$\partial x = \partial u + \partial v = 0$,
$\partial u = -\partial v \in U \cap V$, then

$$\partial_* : H_n(X) \to H_{n-1}(U \cap V),$$
$$\partial_*([x]) = [\partial u]$$

# Mayer-Vietoris Sequence

Analogy with the Seifert-van Kampen theorem: whenever $U \cap V$ is path-connected, the Mayer-Vietoris sequence yields the isomorphism:

$$H_1(X) \cong (H_1(U) \oplus H_1(V))/\mathrm{Ker}(k_* - l_*),$$

where, by exactness

$$\mathrm{Ker}(k_* - l_*) \cong \mathrm{Img}(i_*, j_*).$$

This is the Abelianized statement of the Seifert–van Kampen theorem.

# Homology Group Structure

Let $M$ be a compact oriented connected n-manifold with (possibly empty) boundary. The Abelian group $H_k(M,\mathbb{Z})$ is finitely generated and decomposed as

$$H_k(M,\mathbb{Z}) \cong F_k \oplus T_k$$

where $F_k = \mathbb{Z}^{b_k}$ is free and $T_k$ is finite. The torsion subgroup $T_k$ consists of all finite-order elements in $H_k(M,\mathbb{Z})$. The rank $b_k$ of $F_k$ is the $k$-th Betti number of $M$.

# Torsion

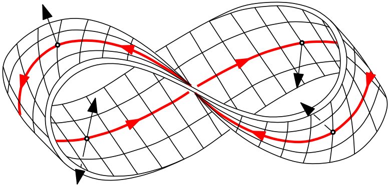


Figure: Möbius band, $\gamma$ is the middle loop.

$$RP^2 = M \cup \mathbb{D}^2,\ M \cap \mathbb{D}^2 = \mathbb{S}^1,$$

$$\pi_1(M) = \langle \gamma \rangle$$
$$\pi_1(\mathbb{D}^2) = \langle e \rangle$$
$$\pi_1(\partial M) = \pi_1(\mathbb{S}^1) = \langle \gamma^2 \rangle$$

By Van Kampan-Seifert,

$$\pi_1(RP^2) = \langle \gamma | \gamma^2 \rangle = \{\gamma, e\}.$$

# Intersection Form

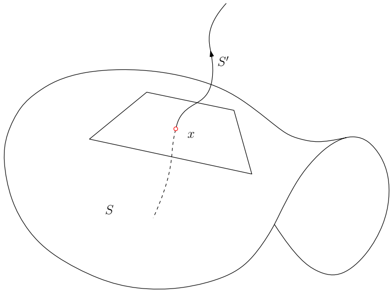



Figure: Intersection form.

Let $G$ and $H$ be finitely generated Abelian groups, a bilinear form

$$\omega : G \times H \to \mathbb{Z}$$

is non-degenerate if for every infinite order element $g \in G$, there is a $h \in H$ such that $\omega(g, h) \neq 0$. If $G = H$, we say that $\omega$ is skew-symmetric if

$$\omega(g_1, g_2) = -\omega(g_2, g_1), \quad \forall g_1, g_2 \in G.$$

A skew-symmetric non-degenerate form is called symplectic.

# Intersection Form

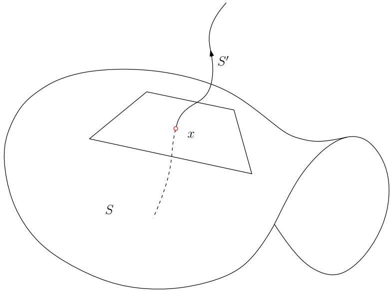



Figure: Intersection form.

An oriented closed k-submanifold $S \subset M$ defines a class $[S] \in H_k(M)$, if $S$ has boundary and is properly embedded (i.e. $\partial S = \partial M \cap S$), it defines a class $[S] \in H_k(M, \partial M)$. Suppose two oriented submanifolds $S$ and $S'$ have complementary dimensions $k$ and $n-k$ and intersect transversely: every intersection point $x$ is isolated and has a sign $\pm 1$, defined by comparing the orientations of $T_x S \oplus T_x S'$ and $T_x M$, the algebraic intersection $S \cdot S'$ is the sum of these signs. The intersection form is defined as

$$\omega([S], [S']) = S \cdot S'.$$

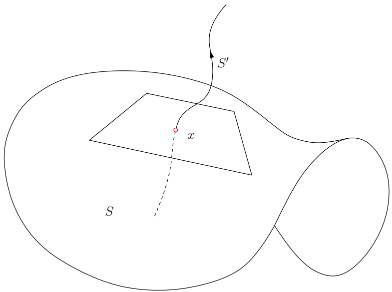



Figure: Intersection form.

Let $M$ be a compact oriented n-manifold with (possibly empty) boundary. Intersection form

$$\omega : H_k(M,\mathbb{Z}) \times H_{n-k}(M,\mathbb{Z}) \to \mathbb{Z}$$

gives an identification

$$\omega(\cdot,\sigma) \in H^k(M,\mathbb{Z}), \sigma \in H_{n-k}(M,\partial M,$$

This is called the Lefschetz duality

$$H^k(M) = H_{n-k}(M,\partial M),$$

for any ring $R$.

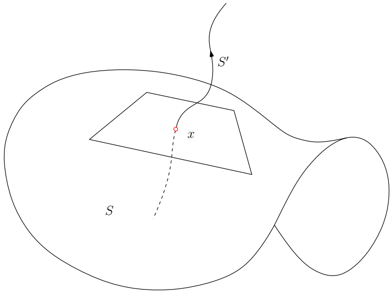



Figure: Intersection form.

If $\partial M = \emptyset$, the this is Poincaré duality,

$$H^k(M) = H_{n-k}(M).$$

In particular

$$H_n(M, \partial M, \mathbb{Z}) = H^0(M, \mathbb{Z}) = \mathbb{Z}$$

and the choice of orientation for $M$ is equivalent to a choice of a generator $[M] \in H_n(M, \partial M, \mathbb{Z})$ called the fundamental class of $M$.

# Lefschetz Duality

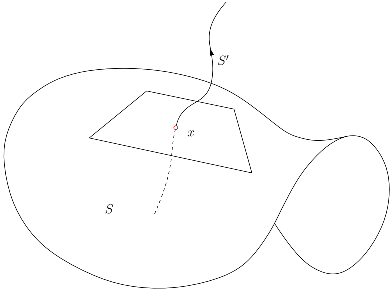



Figure: Intersection form.

The following exact sequence is important:

$$\cdots \to H_n(M) \to H_n(M, \partial M) \to H_{n-1}(\partial M) \to H_{n-1}(M) \to \cdots$$

For non-orientable manifolds, the ring $R$ is $\mathbb{Z}/2\mathbb{Z}$.

# Lefschetz Duality

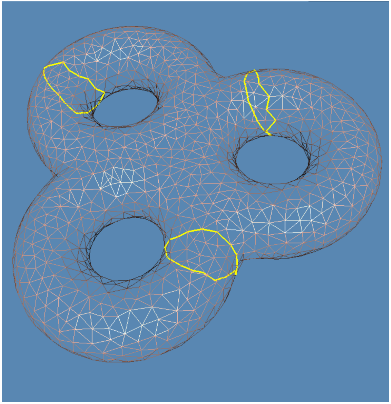


Figure: Closed, orientable surface in $\mathbb{R}^3$.

Suppose $S$ is a closed, orientable surface embedded in $\mathbb{R}^3$, seperating $\mathbb{R}^3$ into two connected components, the interior $I$ and the exterior $O$, $\infty \in O$.

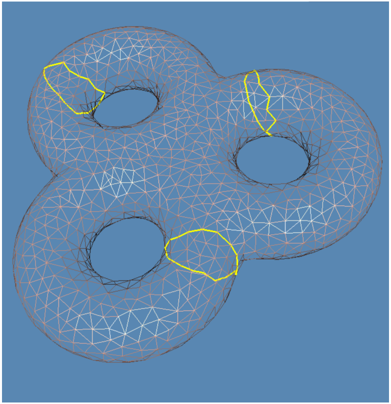


Figure: Closed, orientable surface in $\mathbb{R}^3$.

Mayer-Vietoris sequence is

$$\begin{aligned}\cdots \to & H_2(\mathbb{R}^3) \to H_1(S) \\ \to & H_1(I) \oplus H_1(O) \to H_1(X)\end{aligned}$$

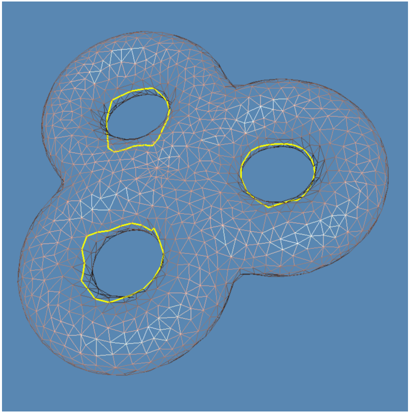
$\mathbb{R}^3$ can shrink to a point, $H_2(\mathbb{R}^3) = 0$, $H_1(\mathbb{R}^3) = 0$,

$$0 \to H_1(S) \to H_1(I) \oplus H_1(O) \to 0,$$

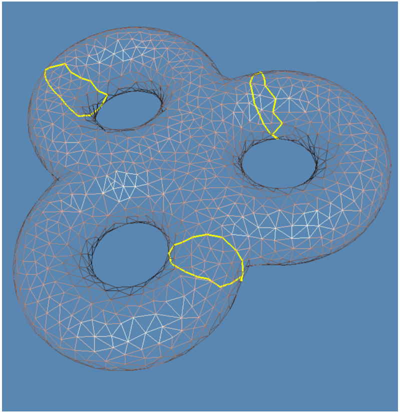
Therefore

$$H_1(S) \cong H_1(I) \oplus H_1(O).$$

# Handle Loops and Tunnel Loops



generators of $H_1(I)$ Tunnel loops



generators of $H_1(O)$ Handle loops

Figure: $H_1(S) \cong H_1(I) \oplus H_1(O)$.

# Handle Loops and Tunnel Loops

Let $M$ be oriented, the boundary $\partial M$ may be disconnected and inherits an orientation. $H_1(\partial M) \cong \mathbb{Z}^{2n}$ for some $n$, $H_1(\partial M)$ is with the intersection form $\omega$. A subgroup $L < H_1(\partial M)$ is Lagrangian if $\omega|_L \equiv 0$. When $\text{rank}(L) = n$, we say that $L$ has maximal rank.

## Lemma

*Let $M$ be an oriented compact 3-manifold with boundary. The image of the map*

$$\partial : H_2(M, \partial M, \mathbb{Z}) \to H_1(\partial M, \mathbb{Z})$$

*is a Lagrangian subgroup of $H_1(\partial M, \mathbb{Z})$ of maximal rank.*

**Proof.**

We have two parings

$$
\begin{aligned}
\omega : & H_1(\partial M) \times H_1(\partial M) \to \mathbb{Z} \\
\eta : & H_2(M, \partial M) \times H_1(M) \to \mathbb{Z}
\end{aligned}
$$

The latter is provided by Lefschetz duality and is non-degenerate after quotienting the torsion subgroup. We have

$$
\omega(\partial \alpha, \beta) = \eta(\alpha, i_* \beta),
$$

for any $\alpha \in H_2(M, \partial M)$, and $\beta \in H_1(M)$, $\square$

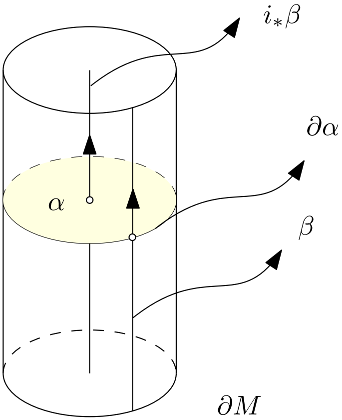



Figure: $\omega(\partial\alpha, \beta) = \eta(\alpha, i_*\beta)$.

Consider the long exact sequence

$$H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \xrightarrow{i_*} H_1(M)$$

Now if $\beta = \partial\alpha'$

$$\begin{aligned} \omega(\partial\alpha, \beta) &= \eta(\alpha, i_*\beta), \\ \omega(\partial\alpha, \partial\alpha') &= \eta(\alpha, i_*\partial\alpha') \\ &= \eta(\alpha, 0) = 0, \end{aligned}$$

hence $\mathrm{Img}\partial$ is Lagrangian.

# Lagrangian subgroup

Claim: if $\text{Img}\partial < H_1(\partial M) = \mathbb{Z}^{2n}$ is Lagrangian, then $\text{rank}(\text{Img}\partial) \leq n$.
The symplectic intersection form $\omega$ of $H_1(\partial M)$ can be represented as

$$\omega = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{bmatrix}_{2n\times 2n}$$

Let $e_k$ is the $k$-th base vector, then $\omega(e_k, e_k) = 0$, but $\omega(e_{2k}, e_{2k+1}) = 1$.
Therefore $\text{Img}\partial$ intersects $\text{Span}\{e_{2k}, e_{2k+1}\}$ must be one dimension.
Therefore $\text{rank}(\text{Img}\partial)$ is at most $n$.

$$\text{rank}(\text{Img}\partial) \leq \frac{1}{2} b_1(\partial M). \quad (1)$$

# Maximal Rank

By exact sequence

$$\cdots \to H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \xrightarrow{i_*} H_1(M) \to \cdots,$$

we obtain

$$\begin{aligned} H_2(M, \partial M) &= \text{Ker } \partial + \text{Img } \partial && (2) \\ H_1(\partial M) &= \text{Ker } i_* + \text{Img } i_* && (3) \\ \text{ker } i_* &= \text{Img } \partial && (4) \end{aligned}$$

By Lefschetz duality

$$H_2(M, \partial M) = H^1(M) \tag{5}$$

Furthermore, assume $\alpha \in \text{Ker}\partial$, then

$$\eta(\alpha, i_*\beta) = \omega(\partial\alpha, \beta) = \omega(0, \beta) = 0,$$

therefore

$$\text{Ker } \partial \perp_\eta \text{Img } i_* \tag{6}$$

# Maximal Rank

$$
\begin{aligned}
H_2(M,\partial M) &= \text{Ker } \partial + \text{Img } \partial \\
H_1(\partial M) &= \text{Ker } i_* + \text{Img } i_* \\
\text{ker } i_* &= \text{Img } \partial \\
H_2(M,\partial M) &= H^1(M) \\
\text{Ker } \partial &\perp_\eta \text{Img } i_*
\end{aligned}
$$

$$
\begin{aligned}
b_1(M) &= \text{rk Ker } \partial + \text{rk Img } \partial \\
b_1(\partial M) &= \text{rk Ker } i_* + \text{rk Img } i_* \\
\text{rk ker } i_* &= \text{rk Img } \partial \\
\text{rk } H_2(M,\partial M) &= \text{rk } H^1(M) = b_1(M) \\
\text{rk Ker } \partial + \text{rk Img } i_* &\leq b_1(M)
\end{aligned}
$$

# Maximal Rank

$$
\begin{aligned}
b_1(M) &= \text{rk Ker } \partial + \text{rk Img } \partial \\
b_1(\partial M) &= \text{rk Ker } i_* + \text{rk Img } i_* \\
\text{rk ker } i_* &= \text{rk Img } \partial \\
\text{rk } H_2(M, \partial M) &= \text{rk } H^1(M) = b_1(M) \\
\text{rk Ker } \partial + \text{rk Img } i_* &\leq b_1(M)
\end{aligned}
$$

The summation of the top two equations

$$
\begin{aligned}
b_1(M) + b_1(\partial M) &= 2\text{rk Img}\partial \\
&+ \text{rk Ker } \partial + \text{rk Img } i_* \\
&\leq 2\text{rk Img}\partial + b_1(M)
\end{aligned}
$$

We obtain

$$
\text{rank}(\text{Img } \partial) \geq \frac{1}{2} b_1(\partial M)
$$

Compare with the inequality (1), we obtain

$$
\text{rank}(\text{Img } \partial) = \frac{1}{2} b_1(\partial M).
$$

# Maximal Rank

## Corollary

*Let $M$ be an oriented compact 3-manifold. We have*

$$b_1(M) \geq \frac{b_1(\partial M)}{2}$$

## Proof.

$$H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M)$$

$$H_2(M, \partial M) = \mathrm{Ker}\partial + \mathrm{Img}\ \partial$$

$$\mathrm{rank}(H_2(M, \partial M)) \geq \mathrm{rank}(\mathrm{Img}\ \partial)$$

$$\mathrm{rank}(H^1(M)) \geq \mathrm{rank}(\mathrm{Img}\ \partial)$$

$$b_1(M) \geq \frac{1}{2} b_1(\partial M).$$

□

# Maximal Rank

## Corollary

*The boundary of a simply connected compact 3-manifold consists of spheres.*

## Proof.

$$b_1(M) \geq \frac{1}{2} b_1(\partial M).$$

implies $b_1(\partial M) = 0$, hence $\partial M$ consists of spheres. $\square$

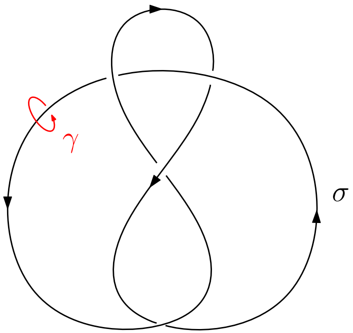



Figure: The homology of the figure-8 knot.

## Corollary

*Suppose $K$ is a knot, the complementary space $M = \mathbb{S}^3 \setminus K$, then*

$$H_1(M) = H_2(M, \partial M) = \mathbb{Z}$$

## Proof.

Consider a tubular neighborhood of $K$, denoted as $I$, then $I \cap M = T^2$,

$$H_1(M) \oplus H_1(I) = H_1(T^2).$$

By Lefschetz duality,

$$H_1(M, \mathbb{Z}) = H_2(M, \partial M, \mathbb{Z}).$$

# Non-orientable Surfaces

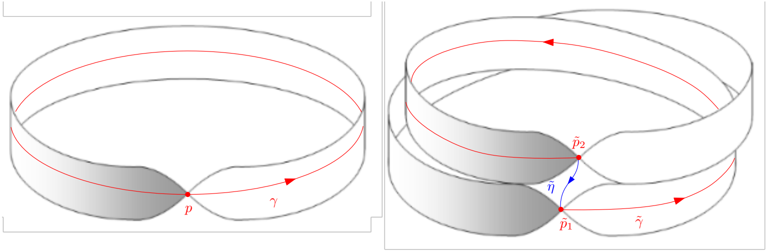



Figure: The double cover of a Möbius band.

A non-orientable properly embedded surface $S \subset M$ defines a non-trivial class $[S] \in H_2(M, \partial M, \mathbb{Z}_2)$, as opposite to orientable surfaces, this class $[S]$ is always non-trivial.

# Non-orientable Surfaces

## Proposition

*Let $M$ be orientable. Every non-orientable properly embedded surface $S \subset M$ defines a non-trivial class $[S] \in H_2(M, \partial M, \mathbb{Z}_2)$. The manifold $M$ cannot contain more than dim $H_2(M, \partial M, \mathbb{Z}_2)$ disjoint non-orientable surfaces.*

## Proof.

Suppose $S \subset M$ is an non-orientable surface. A tubular neighborhood of $S$ is diffeomorphic to the orientable interval bundle $S \times I$, whose boundary is the orientable double cover of $S$, denoted as $\tilde{S}$. A close loop $\gamma \subset S$ through the base point $p \in S$ is lifted to a path $\tilde{\gamma} \subset \tilde{S}$, connecting $\tilde{p}_1$ and $\tilde{p}_2$. Draw an arc $\tilde{\sigma}$ connecting $\tilde{p}_2$ and $\tilde{p}_1$. The loop $\alpha := \tilde{\gamma}\tilde{\eta}$ is in the tubular neighborhood of $S$ and intersects $S$ transversely in one point. $\square$

# Non-orientable Surfaces

**Proof.**

The class $[S] \in H_2(M, \partial M; \mathbb{Z}_2) = H^1(M; \mathbb{Z}_2)$ sends $\alpha$ to $1 \in \mathbb{Z}_2$ and is hence non-trivial.
If $S = S_1 \sqcup \cdots \sqcup S_k$ are all non-orientable, each $S_i$ has its own $\alpha_i$ and therefore the elements $[S_1], \ldots, [S_k] \in H_2(M, \partial M; \mathbb{Z}_2)$ are linearly independent. □

**Corollary**

*A simply-connected three-manifold does not contain any closed non-orientable surface.*

**Proof.**

Because $H_2(M, \partial M) = H^1(M)$, $\dim H_2(M, \partial M, \mathbb{Z}_2) = 0$. □