Heegard Splitting and Kirby Diagram

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Handles

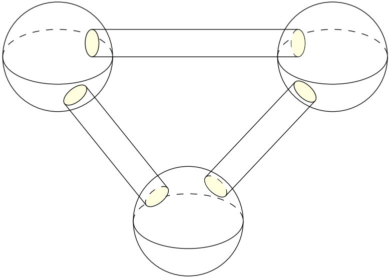

Figure: 0-handles and 1-handles.

Definition (k-handles)

Let M be a (possibly empty or disconnected) n-manifold with boundary and $0 \leq k \leq n$. A *k-handle* is a manifold $D^k \times D^{n-1}$ attached to M along some diffeomorphism $\varphi : \partial D^k \times D^{n-k} \to Y \subset \partial M$, hence producing a new manifold M'.

Handle Decomposition

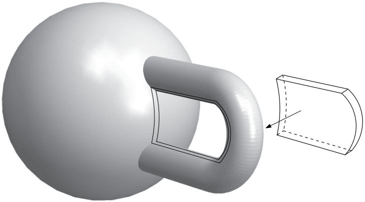

Figure: 2-handle.

Definition (Handle decomposition)

A sequence of handle attachments

$$\emptyset \to M_1 \to M_2 \to \cdots \to M_k = M$$

starting from the empty set and producing a compact manifold M with (possibly empty) boundary is called a *handle decomposition* for M.

Handle Decomposition

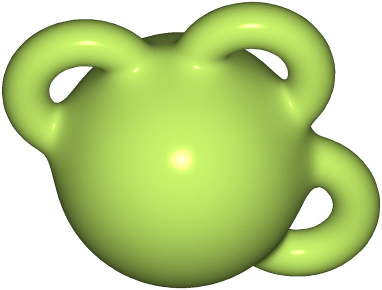

Figure: Handle decomposition. The union of 0-handles and 1-handles is called a handle body.

Proposition

Every compact connected manifold M has a handle decomposition with one 0-handle and at most one n-handle.

If M has a handle decomposition with n_k handles of index k then

$$\chi(M) = \sum_{k=0}^{n} (-1)^k n_k.$$

Heegaard Splitting

Let M_1 and M_2 be compact three-dimensional manifolds with heomeomorphic boundaries, and let $h : \partial M_1 \to \partial M_2$ be any homeomorphism. Gluing the manifold M_1 and M_2 together along this homeomorphisms gives a topological space $M = M_1 \cup_h M_2$.

Definition (Heegaard Splitting)

A splitting of a 3-manifold M into a union of two handlebodies without common interior points is called a *Heegaard splitting*. The genus of the splitting is equal to the genus of the handlebodies.

Heegaard Splitting

Theorem (Heegaard Splitting)

Any closed orientable 3-manifold has a Heegaard splitting.

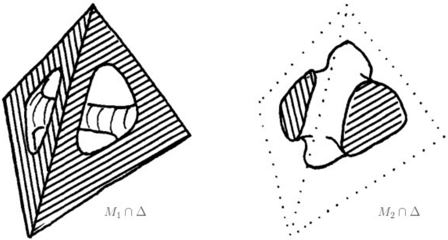

Figure: The intersection between M_k and a tetrahedron Δ.

Heegaard Splitting

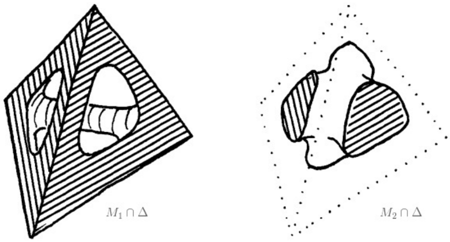

Figure: The intersection between M_k and a tetrahedron Δ.

Proof.

Construct a triangulation T of the 3-manifold M, thicken the triangulation, each vertex is a 0-handle, each edge corresponds to a 1-handle. The union of the 0-handles and 1-handles form a handle body M_1. The complementary $M_2 = M \setminus M_1$ consists of 2-handles and 3-handles, which can be treated as 1-handles and 0-handles, then M_2 is also a handle body. □

Stabilization of Heegaard splitting

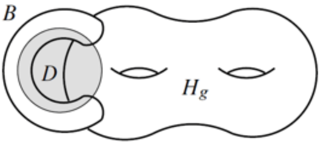

Figure: Stabilization of Heegaard splitting.

Given a Heegaard splitting $M = H_g \cup H_g$ of genus g, add an unknotted 1-handle B to H_g to get a handlebody H_{g+1} of genus $g+1$. Here, we call a handle unknotted if there is a 2-disk D in M such that $D \cap H_{g+1} = \partial D$ and the curve ∂D goes along B only once. Next, we thicken the disk D to get $C = D \times I$. Note that $B \cup C$ is homeomorphic to a 3-ball, hence,

$$
\begin{aligned}
M &\cong H_g \cup (B \cup C) \cup H'_g \\
&= (H_g \cup B) \cup (C \cup H'_g)
\end{aligned}
$$

where $H_g \cup B = H_{g+1}$ and $H'_g \cup C = H'_{g+1}$.

Equivalent Heegaard Splitting

Definition (Equivalent Heegaard Splittings)

Two Heegaard splittings of a manifold M are called equivalent if there exists a self-homeomorphism of the manifold M carrying one splitting into the other.

Definition (Stable Equivalent)

Two Heegaard splittings of a manifold M are called stable equivalent if they become equivalent after several stabilizastion operations are applied to each of them.

Theorem

Any two Heegard splittings of an arbitrary three-dimensional manifold are stably equivalent.

Equivalent Heegaard Splitting

Proof.

The central ideas:

- Any Heegard splitting of the manifold M is stable equivalent to the splitting induced by a certain triangulation;
- For any two triangulations T_1 and T_2, there exists a common star subdivision T of them;
- T is a triangulation of M, T_1 is obtained from T by one elementary star subdivision, then the splitting $H(T_1) \cup H'(T_1) = M$ is obtained from $H(T) \cup H'(T) = M$ by means of several stabilization operations.

$\square$

Heegaard Diagram

A Heegaard diagram is a convenient way of presenting closed orientable three-dimensional manifolds.

Definition (Heegaard Diagram)

Let $H \cup H' = M$ be Heegaard diagram splitting of a manifold M, $F = \partial H = \partial H'$ the common boundary of genus g of the two handle bodies, $\mu = \{\mu_1, \mu_2, \dots, \mu_g\}$ the set of meridians of the handle body H and $\nu = \{\nu_1, \nu_2, \dots, \nu_g\}$ the set of meridians of the handle body H'. The triple (F, μ, ν) is called a Heegaard diagram for the manifold M.

Heegaard Diagram

Definition (Heegaard Diagram Homeomorphic)

Two Heegaard diagrams (F, μ, ν) and (F', μ', ν') of manifolds M and M' are called homeomorphic if there exists a homeomorphism $h: F \to F'$ such that $h(\mu) = \mu'$ and $h(\nu) = \nu'$ or $h(\mu) = \nu'$, $h(\nu) = \mu'$ (the order of the meridians is of no importance).

Proposition

Any homeomorphism $h: (F, \mu, \nu) \to (F', \mu', \nu')$ can be extended to a homeomorphism $\hat{h}: M \to M'$.

Let F be a closed orientable surface of genus g and consider two sets of simple closed curves $\mu = (\mu_1, \mu_2, \dots, \mu_g)$ and $\nu = (\nu_1, \nu_2, \dots, \nu_g)$ on it satisfying $\mu_i \cap \mu_j = \emptyset$ and $\nu_i \cap \nu_j = \emptyset$ for $i \neq j$. In order that the triple (F, μ, ν) be the Heegaard diagram of a certain three-dimensional manifold, it is necessary and sufficient that the surface $F - \mu$ and $F - \nu$ be connected.

Equivalent Heegaard Diagrams

Definition (Isotopic Diagrams)

Two Heegaard diagrams (F, μ, ν) and (F, μ', ν') are called isotopic if there exists an isotopy $\varphi_t : F \to F$ such that $\varphi_0 = 1$, $\varphi_1(\mu) = \mu'$ and $\varphi_1(\nu) = \nu'$.

Definition (Semi-isotopic Diagrams)

Two Heegaard diagrams (F, μ, ν) and (F, μ', ν') are called semi-isotopic if there exists isotopies $\varphi_t, \psi_t : F \to F$ such that $\varphi_0 = \psi_0 = 1$, $\varphi_1(\mu) = \mu'$ and $\psi_1(\nu) = \nu'$.

Let (F, μ, ν) be a Heegaard diagram and let β be a simple curve joining the meridians μ_1 and μ_2 of the diagram and having no other common points with the curves of μ. Let C be a closed neighborhood of the union $\mu_1 \cup \mu_2 \cup \beta$ homeomorphic to a disk with two holes and intersecting no other curves of μ. The boundary component of this neighborhood which is not isotopic to the curve μ_1 or μ_2, will be denosted by $\mu_1 \# \mu_2$. The set $(\mu_1 \# \mu_2, \mu_2, \dots, \mu_g)$ will be denoted as $\tilde{\mu}$.

Definition

We shall say that the diagram $(F, \tilde{\mu}, \tilde{\nu})$ is obtained from the diagram (F, μ, ν) through the operation of adding the curve μ_2 to the curve μ_1 along the curve β.

Equivalent Heegaard Diagrams

Definition (Equivalent Heegaard Diagrams)

The diagrams (F, μ, ν) and (F, μ', ν') are called equivalent if we can pass from one to the other using homeomorphisms, semi-isotopies and operations of adding one meridian to another.

Proposition

Heegaard diagrams are equivalent if and only if the Heegaard splittings corresponding to them are equivalent.

Dehn Surgery

Definition (Link and Knot)

A *link* is a finite set of pairwise disjoint simple closed curves in $\mathbb{R}^3$. A link of one component is called a *knot*.

Definition (Dehn Surgery)

Let K be a knot in a closed orientable three-dimensional manifold M and let $N(K)$ be a regular neighbourhood. When cut along the torus $\partial N(K)$, the manifold M falls into two parts: the complementary space of the knot $M(K) = M - \text{Int}N(K)$ and a solid torus $N(K)$ which is identified with the standard solid torus $D^2 \times S^1$.
Choose a homeomorphism $h : \partial D^2 \times S^1 \to \partial N(K)$ along which we glue the torus $D^2 \times S^1$ back to $M(K)$. The space obtained $Q = M(K) \cup_h N(K)$ is a closed orientable three-dimensional manifold.
We shall say that Q is obtained from the manifold M by surgery along the knot K.

Canonical Longtitudes

Definition (Meridian and Longtitude)

The tubular neighborhood $N(K)$ of a knot $K \subset S^3$ is a solid torus. A *meridian* is a simple closed curve $m \in \partial N(K)$ bounding a disk in $N(K)$ and a *longtitude* is any other simple closed curve l, such that m and l generate $H_1(\partial N(K), \mathbb{Z})$.

The meridian is unique up to sign, bu the longtitude l is not unique ($l + km$ is also a longitude).

Canonical Longtitudes

Proposition

Let $L \subset S^3$ be a link with k components and $M(L)$ its complement. We have $H_1(M, \mathbb{Z}) = \mathbb{Z}^k$, generated by the k meridians.

Proof.

Let $N = N_1 \cup \cdots \cup N_k$ be the solid tori neighborhoods of L and $T_i = \partial N_i$. The Mayer-Vietoris sequence on $S^3 = M(L) \cup N$ gives

$$0 \to H_1(T_1 \cup \cdots \cup T_k) \to H_1(M(L)) \oplus H_1(N_1 \cup \cdots \cup N_k) \to 0,$$

since $H_2(S^3) = H_1(S^3) = 0$. $H_1(T_i) = \mathbb{Z} \times \mathbb{Z}$ and $H_1(N_i) = \mathbb{Z}$ imply that $H_1(M(L)) = \mathbb{Z}^k$. The group $H_1(T_i)$ is generated by (m_i, l_i) and m_i goes to zero in $H_1(N_i)$. Therefore the meridians $m_1, \ldots, m_k$ go to generators of $H_1(M(L))$. □

Canonical Longtitudes

Corollary

Let $K \subset S^3$ be a knot and $M(K)$ be its complement. A unique (up to sign) longtitude $l \subset \partial M$ vanishes in $H_1(M(K), \mathbb{Z})$.

Proof.

In the map $\mathbb{Z} \times \mathbb{Z} = H_1(\partial M(K)) \to H_1(M(K)) = \mathbb{Z}$ the meridian m goes to a generator, hence the kernel is generated by a longtitude l. $\square$

We call l the *canonical longtitude* of K. The torus $T = \partial M(K)$ is hence equipped with a canonical basis (m, l) for $H_1(T, \mathbb{Z})$.

Seifert Surface

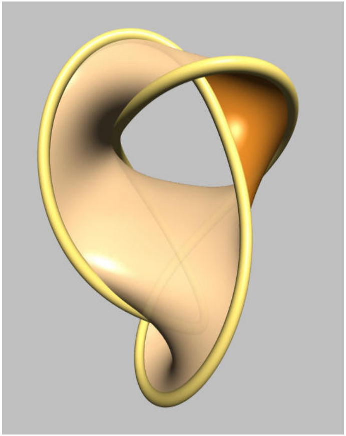

Figure: Seifert surface.

Definition (Seifert Surface)

A Seifert surface for a knot $K \subset S^3$ is any orientable connected compact surface $S \subset S^3$ with $\partial S = k$.

Every Seifert surface S determines a longtitude l for K: pick a small tubular neighborhood N of K and set $l = S \cap \partial N$. The same knot K has many non-isotopic Seifert surfaces, all induce the same longtitude.

Seifert Surface

Proposition

Every knot K has a Seifert surface S. Every Seifert surface for K induces the same canonical longtitude l.

Proof.

Let M be the complement of K. Let S be a surface representing a generator of $H_2(M, \partial M) = H_1(M) = \mathbb{Z}$. The long exact sequence

$$\cdots \to H_2(M, \partial M) \to H_1(\partial M) \to H_1(M) \to \cdots$$

implies that $[S]$ is mapped to a non-trivial primitive element $\alpha \in H_1(\partial M) = \mathbb{Z} \times \mathbb{Z}$ that is trivial in $H_1(M)$. Therefore $[\partial S] = \alpha = [l]$, we get $\partial S = K$. □

Dehn Filling

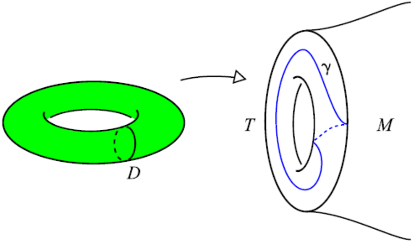

Figure: Dehn filling. The meridian ∂D of the solid torus is attached to $\gamma \subset T$. γ determines the result 3-manifold.

Definition (Dehn Filling)

Let M be a 3-manifold and $T \subset \partial M$ be a boundary torus component. A *Dehn filling* of M along T is the operation of gluing a solid torus $D^2 \times S^1$ to M via a diffeomorphism $\varphi : \partial D^2 \times S^1 \to T$.

Dehn Filling

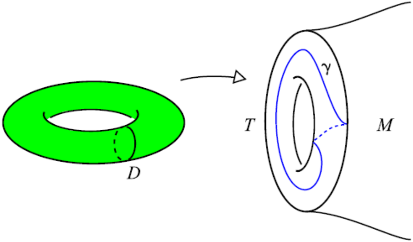

Figure: Dehn filling. The meridian ∂D of the solid torus is attached to $\gamma \subset T$. γ determines the result 3-manifold.

The Dehn filling kills the curve γ. The normalizer of an element $g \in G$ in a group G is the smallest normal subgroup $N(g) \lhd G$ containing g. Suppose the result 3-manifold is M' obtained by Dehn filling, then

$$\pi_1(M') = \pi_1(M)/N(\gamma).$$

Slope on Torus

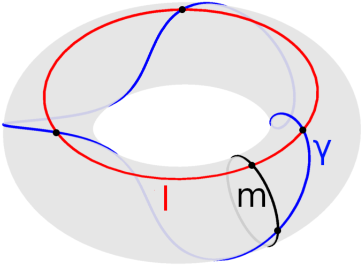

Figure: The slop on a torus $\gamma = \pm(pm + ql)$, denoted as p/q. (by Daniel Tubbenhauer)

Let a *slope* on a torus T be the isotopy class γ of an unoriented homotopically non-trivial simple closed curve. Fix basis (m, l) for $H_1(T, \mathbb{Z})$, every slope may be written as $\gamma = \pm(pm + ql)$ for some coprime pair (p, q). Therefore we get a 1-1 correspondence

$$\mathscr{S} \leftrightarrow \mathbb{Q} \cup \{\infty\}$$

by sending γ to p/q.

Dehn Surgery

Definition (Dehn Surgery)

Let $L \subset M$ be a link with some k components in an orientable 3-manifold M. A *Dehn surgery* on L is a Dehn filling of the complement of L. It is a two-step operation:

1. (drilling) the removal of small open tubular neighborhoods of L, that creates new boundary tori $T_1, \ldots, T_k$;
2. (filling) a Dhen filling of the new boundary tori $T_1, \ldots, T_k$.

The surgered manifold N is determined by the slopes in $T_1, T_2, \ldots, T_k$ that are killed by the Dehn filling. Every torus T_i is equipped with a canonical basis m_i, l_i, th slope is of the form $\pm(p_i m_i + q_i l_i)$.

Dehn Surgery

Definition (Integral Surgery)

A Dehn surgery is integral if the killed slopes are longtitudes of the previously removed solid tori.

Proposition

Let N be obtained by Dehn surgery on a knot $K \subset S^3$ with coefficient $\frac{p}{q}$. The surgered manifold N has $H_1(N) = \mathbb{Z}/p\mathbb{Z}$.

Proof.

Let M be the complement of K. We know the meridian m generates $H_1(M) = \mathbb{Z}$ while the longtitude l is zero there. The Dehn filling kills the element $pm + ql = pm$. $\square$

Homology Sphere

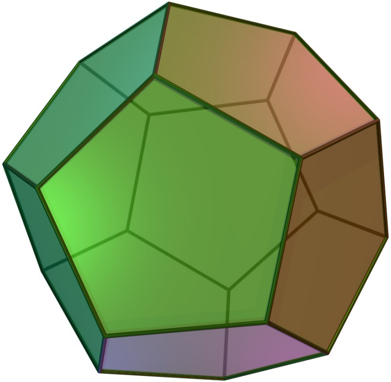

Figure: Poincare's homology sphere.

Each face of the dodecahedron is identified with its opposite face, using the minimal clockwise twist to line up the faces.

Definition (Homology Sphere)

A homology sphere is a closed 3-manifold M having the same integral homology as S^3, that is with trivial $H_1(M,\mathbb{Z})$.

Corollary

If the coefficient is $\frac{1}{q}$ the surgered manifold N is a homology sphere.

Poincaré Homology Sphere

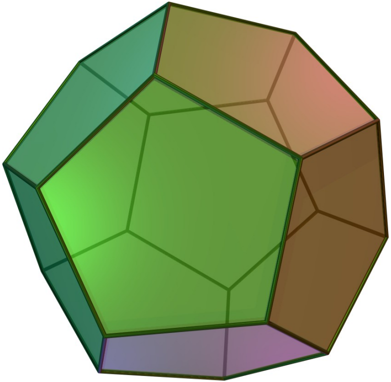

Figure: Poincare's homology sphere.

Equivalently, the Poincare's homology sphere is $SO(3)/I$, where I is the binary icosahedral group.

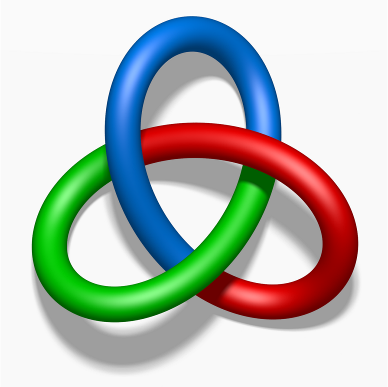

Figure: Trefoil knot with $+1$ surgery.

Kirby Diagram

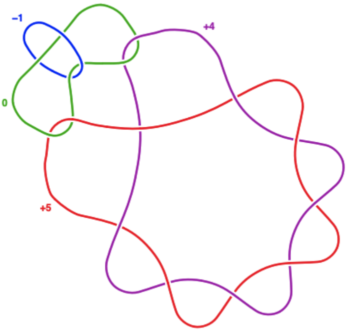

Figure: Kirby diagram.

Definition (Kirby's Diagram)

A Kirby diagram is a link diagram on the plane with a rational number p_i/q_i assigned to each component. Such a diagram defines a Dehn surgery and hence a closed orientable three-manifold.

Heegaard Splitting of S^3

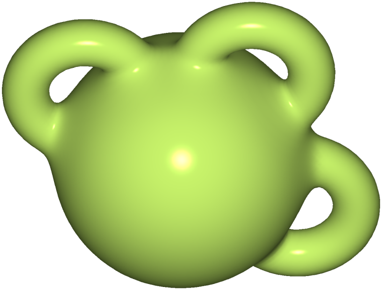

Figure: A genus g handle body embedded in S^3.

The complement of a standard embedded genus g handle body in S^3 is another handle body. They form a genus g Heegaard Splitting of S^3.

Lickorish-Wallace Theorem

Theorem (Lickorish-Wallace)

Every closed orientable three-manifold can be described iva an integral Dehan surgery along a link $L \subset S^3$.

Proof.

Let M be a closed orientable 3-manifold. Pick a Heegaard splitting $M = H_1 \cup_\psi H_2$ where H_1 and H_2 are genus g handle bodies and $\psi : \partial H_1 \to \partial H_2$ is a diffeomorphism. We fix an identification of both H_1 and H_2 with a model handlebody H, so that ψ can be interpreted as an element of the $\mathrm{MCG}(S)$ of the genus g surface $S = \partial H$. □

Lickorish-Wallace Theorem

Proof.

$S^3 = H_1 \cup_\varphi H_2$ for some $\varphi \in \mathrm{MCG}(S)$. $\psi \circ \varphi^{-1}$ is composition of Dehn twists

$$\psi \circ \varphi^{-1} = \tau_{\gamma_k}^{\pm 1} \circ \cdots \circ \tau_{\gamma_1}^{\pm 1}$$

along some curves $\gamma_i \subset S$. Set $M_i = H_1 \cup_{\psi_i} H_2$ with

$$\psi_i = \tau_{\gamma_i}^{\pm} \circ \cdots \circ \tau_{\gamma_1}^{\pm} \circ \varphi.$$

We have $M_0 = S^3$ and $M_k = M$. We prove that M_i can be described via an integral Dehn surgery along a i-components link in S^3 by induction on i. To obtain that it suffices to check that M_{i+1} can be obtained from M_i via integral Dehn surgery along a knot. We have

$$M_i = H_1 \cup_{\psi_i} H_2, \quad M_{i+1} = H_1 \cup_{\tau_{\gamma_i}^{\pm} \circ \psi_i} H_2$$

This can be accomplished by the next lemma. $\square$

Lickorish-Wallace Theorem

Lemma

Let two homeomorphisms $h_1, h_2 : \partial H_1 \to \partial H_2$ be such $h_1 = h_2 \circ \tau_\gamma$, where τ_γ is a Denh twist along a certain simple closed curve $\gamma \subset \partial H_1$. Then $M_2 = H_1 \cup_{h_2} H_2$ is obtained from $M_1 = H_1 \cup_{h_1} H_2$ by integral surgery along a certain knot $K \subset M_1$ isotopic to the image of the curve γ.

Proof.

Push the curve γ inside the handle body H_1, to obtain a knot $K \subset H_1$. Let $N(K) \subset H_1$ be a regular neighborhood and let $A = S^1 \times I$ be the annulus joining γ and $\partial N(K)$. □

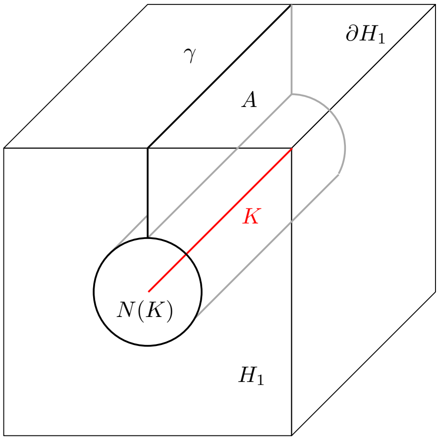

Figure: Lickorish-Wallace theorem.

Proof.

Cut $H_1 - \text{Int}N(K)$ along the annulus A, twist one of boundaries of the cut by $360°$ and glue back, we obtain the homeomorphism

$$\varphi : H_1 - \text{Int}N(K) \to H_1 - \text{Int}N(k)$$

φ restricted on ∂H_1 is the Denh twist τ_γ; restricted on $\partial N(K)$ gives the twist along the longtitude $A \cap N(K)$ of the knot K. □

Lickorish-Wallace Theorem

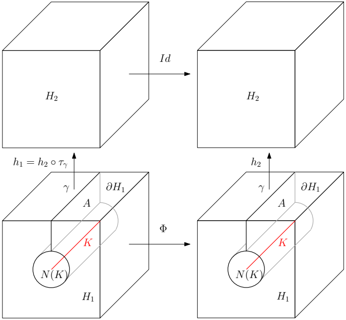

Figure: Lickorish-Wallace theorem.

Proof.

Let $M_i' = H_2 \cup_{h_i} (H_1 - \mathrm{Int}N(K))$, define a $\Phi : M_1' \to M_2'$

$$\Phi(x) := \begin{cases} \varphi(x), & x \in H_1 - \mathrm{Int}N(K) \\ x, & x \in H_2 \end{cases}$$

Since $h_2 = \tau_\gamma \circ h_1$, and $\Phi|_{\partial H_1} = \tau_\gamma$, the consistency shows Φ is a homeomorphism between M_1' and M_2'. Thus, if from each of the manifolds M_1, M_2 we remove a solid torus $N(K)$, we obtain homeomorphic manifolds. This means M_2 is obtained from M_1 through Dehn surgery along K. $\square$

Framed Link

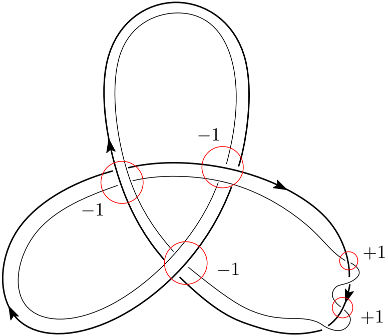

Figure: Framed link.

Definition (Framed Link)

A framed knot is a knot K with the longtitude on $\partial N(K)$, which bound a band. A framed link is formed by framed knots.

The manifold obtained from M by a surgery along a framed link $L \subset S^3$ will be denoted by $\chi(M; L)$.

Framed Link

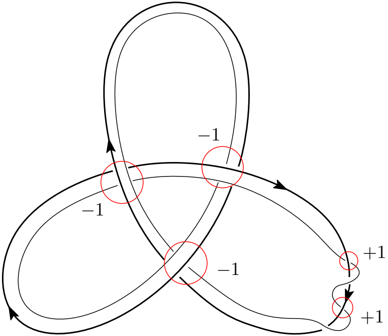

Figure: Framed link.

Definition (Linking Coefficient)

Two non-intersecting curves γ_1 and γ_2 in S^3, the linking coefficient $\text{lk}(\gamma_1, \gamma_2)$ is calculated from the projections of the curves onto a plane: count how many times γ_1 goes under γ_2 from the right to the left and how many times it goes under from the left to the right, the linking coefficient is equal to the difference of these numbers.

Cobordism

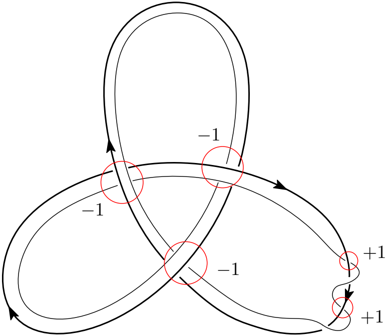

Figure: Framed link.

Definition (Coboardism)

An orientable four-dimensional manifold W is called a cobordism between two closed orientable 3-manifolds M_1 and M_2 if its boundary consists of two components one of which is homeomorphic to M_1 and the second to the manifold M_2.

Cobordism

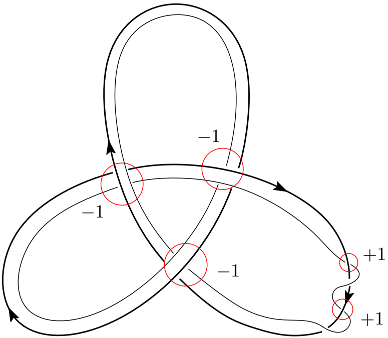

Figure: Framed link.

Let K be a framed knot with longtitude l. The center of disk D^2 is O, and an arbitrary point on its boundary is a. There is a homeomorphism $h : S^1 \times D^2 \to N(K)$ such that $h(S^1 \times \{0\} = K$ and $h(S^1 \times \{a\} = l$. Glue a handle $D^2 \times D^2$ of index 2 to the manifold $M \times I$ along the embedding

$$h : S^1 \times D^2 = \partial D^2 \times D^2$$
$$\to N(K) \subset M = M \times \{1\}$$

gives a four dimensional manifold $W = M \times I \cup_h D^2 \times D^2$

Cobordism

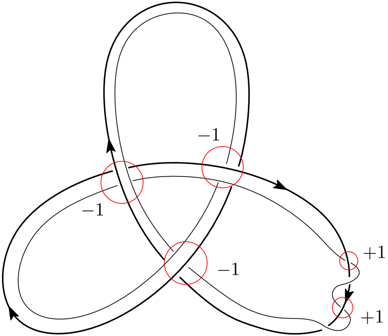

Figure: Framed link.

Theorem

The manifold W is a cobordism between the manifold M and $\chi(M; K)$, that is $\partial W = \partial_- W \cup \partial_+ W$, where $\partial_- W = M$ and $\partial_+ W = M$.

Kirby Move I

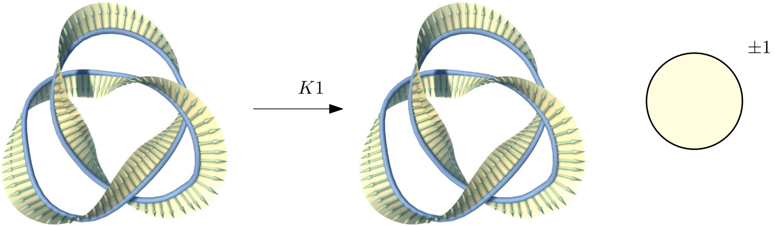

Figure: Kirby transformation $K1$.

Kirby Move II

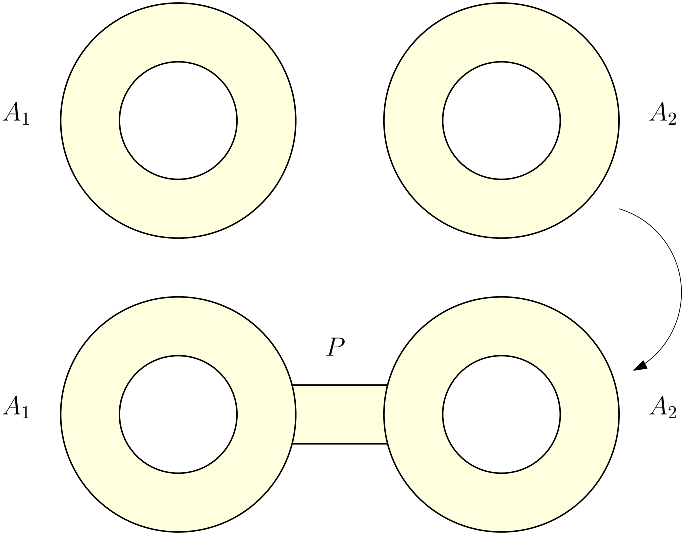

Figure: Kirby transformation $K2$.

Kirby Move II

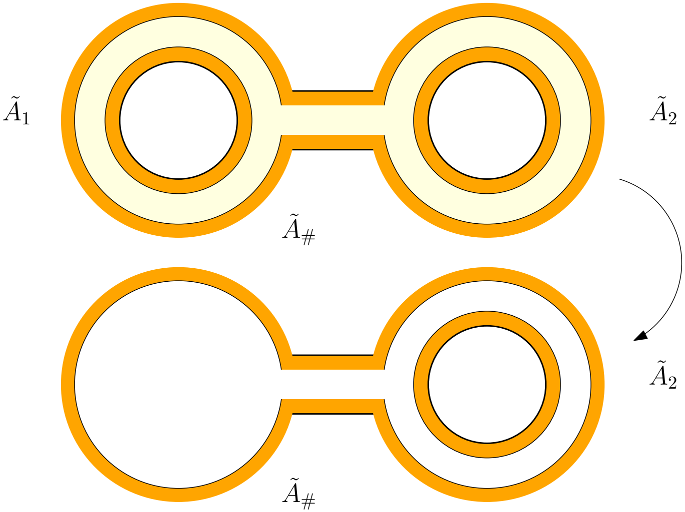

Figure: Kirby transformation $K2$. The frame of $\tilde{A}_\#$ equals to $fr(A_1) + fr(A_2) + lk(A_1, A_2)$.

Kirby Calculus

Theorem (Kirby Calculus)

A framed link $L1 \subset S^3$ can be joined to a framed link $L_2 \subset S^3$ by a chain of transformations K_1, K_1^{-1}, K_2 if and only if the manifolds $\chi(S^3; L_1)$ and $\chi(S^3; L_2)$ are homeomorphic.

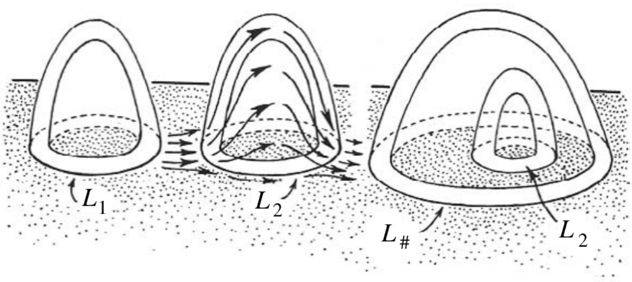

Figure: Handle slides

Kirby Calculus

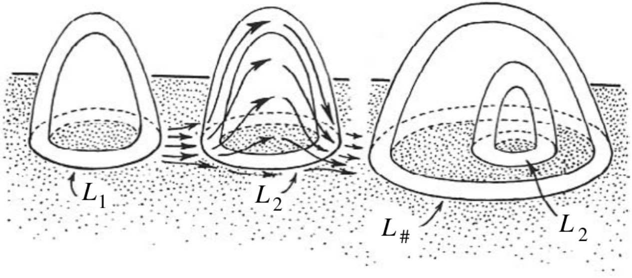

Figure: Handle slides

Proof.

All ± 1 surgeries along unknots give back S^3 so the first Kirby move holds; the second Kirby move is a handle slide. □