

de Rham Cohomology, Hodge Decomposition

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Exterior Differential

The homology of a manifold is the difference between the **closed loops** and the **boundary loops**.

The cohomology of a manifold is the difference between the **curl free** vector fields and the **gradient** vector fields.

Consider a planar vector field defined on $\mathbb{C} \setminus \{0\}$,

$$\mathbf{v}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

direct computation $\nabla \times \mathbf{v}(x, y) = 0$.

$$\left| \begin{array}{cc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{array} \right| = 0.$$

But choose the unit circle

$$\oint_{\gamma} \omega = \oint_{\gamma} d \tan^{-1} \frac{y}{x} = 2\pi$$

therefore \mathbf{v} is not a gradient field. Namely, $d\theta$ locally is integrable, globally not.

Smooth Manifold

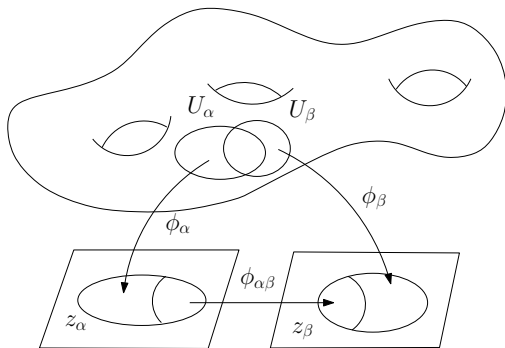


Figure: A manifold.

Definition (Manifold)

A manifold is a topological space M covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ maps U_α to the Euclidean space \mathbb{R}^n . (U_α, ϕ_α) is called a coordinate chart of M . The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of M . Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

If all transition maps $\phi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ are smooth, then the manifold is a differential manifold or a smooth manifold.

Tangent Space

Definition (Tangent Vector)

A tangent vector ξ at the point p is an association to every coordinate chart (x^1, x^2, \dots, x^n) at p an n -tuple $(\xi^1, \xi^2, \dots, \xi^n)$ of real numbers, such that if $(\tilde{\xi}^1, \tilde{\xi}^2, \dots, \tilde{\xi}^n)$ is associated with another coordinate system $(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n)$, then it satisfies the transition rule

$$\tilde{\xi}^i = \sum_{j=1}^n \frac{\partial \tilde{x}^i}{\partial x^j}(p) \xi^j.$$

A smooth vector field ξ assigns a tangent vector for each point of M , it has local representation

$$\xi(x^1, x^2, \dots, x^n) = \sum_{i=1}^n \xi_i(x^1, x^2, \dots, x^n) \frac{\partial}{\partial x_i}.$$

$\{\frac{\partial}{\partial x_i}\}$ represents the vector fields of the velocities of iso-parametric curves on M . They form a basis of all vector fields.

Definition (Push-forward)

Suppose $\phi : M \rightarrow N$ is a differential map from M to N , $\gamma : (-\epsilon, \epsilon) \rightarrow M$ is a curve, $\gamma(0) = p$, $\gamma'(0) = \mathbf{v} \in T_p M$, then $\phi \circ \gamma$ is a curve on N , $\phi \circ \gamma(0) = \phi(p)$, we define the tangent vector

$$\phi_*(\mathbf{v}) = (\phi \circ \gamma)'(0) \in T_{\phi(p)} N,$$

as the push-forward tangent vector of \mathbf{v} induced by ϕ .

Definition (Differential 1-form)

The tangent space $T_p M$ is an n -dimensional vector space, its dual space $T_p^* M$ is called the cotangent space of M at p . Suppose $\omega \in T_p^* M$, then $\omega : T_p M \rightarrow \mathbb{R}$ is a linear function defined on $T_p M$, ω is called a differential 1-form at p .

A differential 1-form field has the local representation

$$\omega(x^1, x^2, \dots, x^n) = \sum_{i=1}^n \omega_i(x^1, x^2, \dots, x^n) dx_i,$$

where $\{dx_i\}$ are the differential forms dual to $\{\frac{\partial}{\partial x_j}\}$, such that

$$\langle dx_i, \frac{\partial}{\partial x_j} \rangle = dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}.$$

High order exterior forms

Definition (Tensor)

A tensor Θ of type (m, n) on a manifold M is a correspondence that associates to each point $p \in M$ a multi-linear map

$$\Theta_p : T_p M \times T_p M \times \cdots \times T_p^* M \cdots \times T_p^* M \rightarrow \mathbb{R},$$

where the tangent space $T_p M$ appears m times and cotangent space $T_p^* M$ appears n times.

Definition (exterior m -form)

An exterior m -form is a tensor ω of type $(m, 0)$, which is skew symmetric in its arguments, namely

$$\omega_p(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \cdots, \xi_{\sigma(m)}) = (-1)^\sigma \omega_p(\xi_1, \xi_2, \cdots, \xi_m)$$

for any tangent vectors $\xi_1, \xi_2, \cdots, \xi_m \in T_p M$ and any permutation $\sigma \in S_m$, where S_m is the permutation group.

Differential Form

The local representation of ω in (x^1, x^2, \dots, x^m) is

$$\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_m \leq n} \omega_{i_1 i_2 \dots i_m} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_m} = \omega_I dx^I,$$

ω_I is a function of the reference point p , ω is said to be differentiable, if each ω_I is differentiable.

Definition (Wedge product)

A coordinate free representation of wedge product of m_1 -form ω_1 and m_2 -form ω_2 is defined as $(\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \dots, \xi_{m_1+m_2})$ equals

$$\sum_{\sigma \in S_{m_1+m_2}} \frac{(-1)^\sigma}{m_1! m_2!} \omega_1(\xi_{\sigma(1)}, \dots, \xi_{\sigma(m_1)}) \omega_2(\xi_{\sigma(m_1+1)}, \dots, \xi_{\sigma(m_1+m_2)})$$

Wedge product

Give k differential 1-forms, their exterior wedge product is given by:

$$\omega_1 \wedge \omega_2 \cdots \omega_k(v_1, v_2, \dots, v_k) = \begin{vmatrix} \omega_1(v_1) & \omega_1(v_2) & \cdots & \omega_1(v_k) \\ \omega_2(v_1) & \omega_2(v_2) & \cdots & \omega_2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_k(v_1) & \omega_k(v_2) & \cdots & \omega_k(v_k) \end{vmatrix}$$

Exterior is anti-symmetric, suppose $\sigma \in S_k$ is a permutation, then

$$\omega_{\sigma(1)} \wedge \omega_{\sigma(2)} \wedge \cdots \wedge \omega_{\sigma(k)} = (-1)^\sigma \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k.$$

Definition (Pull back)

Suppose $\phi : M \rightarrow N$ is a differentiable map from M to N , ω is an m -form on N , then the pull-back $\phi^*\omega$ is an m -form on M defined by

$$(\phi^*\omega)_p(\xi_1, \dots, \xi_m) = \omega_{\phi(p)}(\phi_*\xi_1, \dots, \phi_*\xi_m), p \in M$$

for $\xi_1, \xi_2, \dots, \xi_m \in T_pM$, where $\phi_*\xi_j \in T_{\phi(p)}N$ is the push forward of $\xi_j \in T_pM$.

Integration in Euclidean space

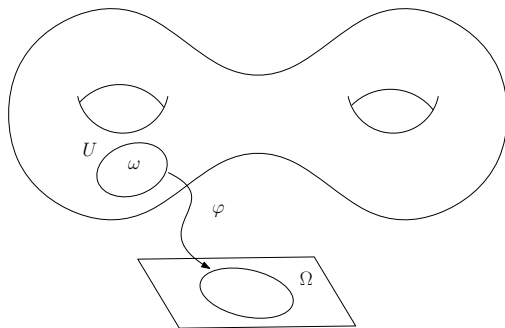
Suppose that $U \subset \mathbb{R}^n$ is an open set,

$$\omega = f(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

then

$$\int_U \omega = \int_U f(x)dx^1 dx^2 \cdots dx^n.$$

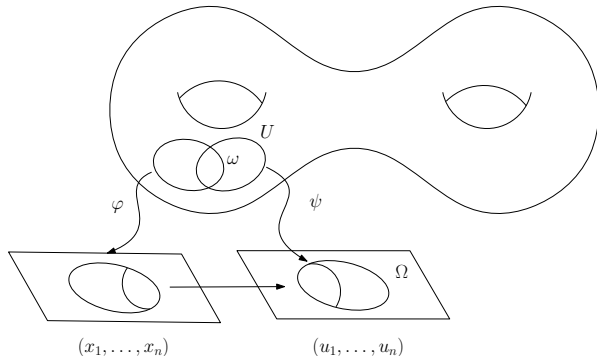
Integration



Suppose $U \subset M$ is an open set of a manifold M , a chart $\phi : U \rightarrow \Omega \subset \mathbb{R}^n$, then

$$\int_U \omega = \int_{\Omega} (\phi^{-1})^* \omega.$$

Integration



Integration is independent of the choice of the charts. Let $\psi : U \rightarrow \psi(U)$ be another chart, with local coordinates (u_1, u_2, \dots, u_n) , then

$$\int_{\phi(U)} f(x) dx^1 dx^2 \cdots dx^n = \int_{\psi(U)} f(x(u)) \det \left(\frac{\partial x^i}{\partial u^j} \right) du^1 du^2 \cdots du^n.$$

Integration on Manifolds

consider a covering of M by coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ and choose a partition of unity $\{f_i\}$, $i \in I$, such that $f_i(p) \geq 0$,

$$\sum_i f_i(p) \equiv 1, \forall p \in M.$$

Then $\omega_i = f_i \omega$ is an n -form on M with compact support in some U_α , we can set the integration as

$$\int_M \omega = \sum_i \int_M \omega_i.$$

Exterior Derivative

Exterior Derivative of a Function

Suppose $f : M \rightarrow \mathbb{R}$ is a differentiable function, then the exterior derivative of f is a 1-form,

$$df = \sum_i \frac{\partial f}{\partial x_i} dx^i.$$

Exterior Derivative of Differential Forms

The exterior derivative of an m -form on M is an $(m+1)$ -form on M defined in local coordinates by

$$d\omega = d(\omega_I dx^I) = (d\omega_I) \wedge dx^I,$$

where $d\omega_I$ is the differential of the function ω_I .

Exterior Derivative

The exterior derivative of a differential 1-form is given by:

$$d\left(\sum \omega_i dx_i\right) = \sum_{i,j} \left(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j}\right) dx_i \wedge dx_j,$$

that of a differential k -form

$$d(\omega_1 \wedge \omega_2 \cdots \wedge \omega_k) = \sum (-1)^{i-1} \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge d\omega_i \wedge \omega_{i+1} \wedge \cdots \wedge \omega_k.$$

Stokes Theorem

Theorem (Stokes)

let M be an n -manifold with boundary ∂M and ω be a differentialble $(n - 1)$ -form with compact support on M , then

$$\int_{\partial M} \omega = \int_M d\omega.$$

Stokes Theorem

Theorem

Suppose Σ is a differential manifold, then we have

$$d^k \circ d^{k-1} = 0.$$

Proof.

Assume ω is a $k - 1$ differential form, D is a $k + 1$ chain, from Stokes theorem, we have

$$\int_D d^k \circ d^{k-1} \omega = \int_{\partial_{k+1} D} d^{k-1} \omega = \int_{\partial_k \circ \partial_{k+1} D} \omega = 0,$$

since $\partial_k \circ \partial_{k+1} = 0$. □

Let $\Omega^k(\Sigma)$ be the sapce of all differential k -forms, $d^k : \Omega^k(\Sigma) \rightarrow \Omega^{k+1}(\Sigma)$ be exterior differential operator.

Definition (Closed form)

k -form $\omega \in \Omega^k(\Sigma)$ is called a closed form, if $d^k \omega = 0$, namely $\omega \in \text{Ker } d^k$.

Definition (Exact Form)

k -differential form $\omega \in \Omega^k(\Sigma)$ is called exact form, if there is a $\tau \in \Omega^{k-1}(\Sigma)$, such that $\omega = d^{k-1} \tau$, namely $\omega \in \text{Img } d^{k-1}$.

Since $d^k \circ d^{k-1} = 0$, exact forms are closed, $\text{Img } d^{k-1} \subset \text{Ker } d^k$.

Definition (de Rham Cohomology)

Assume Σ is a differential manifold, then de Rham complex is

$$\Omega^0(\Sigma, \mathbb{R}) \xrightarrow{d^0} \Omega^1(\Sigma, \mathbb{R}) \xrightarrow{d^1} \Omega^2(\Sigma, \mathbb{R}) \xrightarrow{d^2} \Omega^3(\Sigma, \mathbb{R}) \xrightarrow{d^3} \dots$$

$$H_{dR}^k(\Sigma, \mathbb{R}) := \frac{\text{Ker } d^k}{\text{Im } d^{k-1}}$$

Theorem

The de Rham cohomology group $H_{dR}^m(M)$ is isomorphic to the cohomology group $H^m(M, \mathbb{R})$

$$H_{dR}^m(M) \cong H^m(M, \mathbb{R}).$$

Hodge Operator

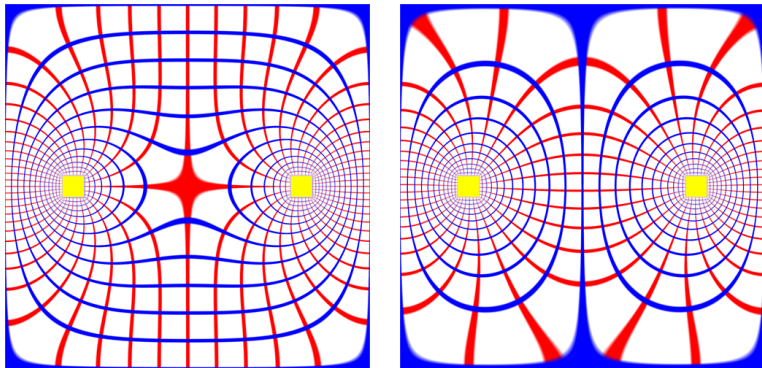


Figure: Hodge star operator.

Hodge Star Operator - First Definition

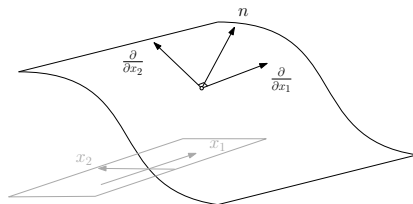
Suppose M is a Riemannian manifold, we can locally find oriented orthonormal basis of vector fields, and choose parameterization, such that

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right\}$$

form an oriented orthonormal basis. let

$$\{dx_1, dx_2, \dots, dx_n\}$$

be the dual 1-form basis.



Hodge star operator

Definition (Hodge Star Operator)

The Hodge star operator $*$: $\Omega^k(M) \rightarrow \Omega^{n-k}(M)$ is a linear operator

$$*(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k) = dx_{k+1} \wedge dx_{k+2} \wedge \cdots \wedge dx_n.$$

Hodge Star Operator

Let $\sigma = (i_1, i_2, \dots, i_n)$ be a permutation, then the Hodge star operator

$$*(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = (-1)^\sigma dx_{i_{k+1}} \wedge dx_{i_{k+2}} \wedge \cdots \wedge dx_{i_n}.$$

Definition

Let $\eta, \zeta \in \Omega^k(M)$ are two k -forms on M , then the norm is defined as

$$(\eta, \zeta) = \int_M \eta \wedge {}^*\zeta.$$

$\Omega^k(M)$ is a Hilbert space.

Hodge Star Operator - Second Equivalent Definition

Given a Riemannian manifold (M, \mathbf{g}) , $\mathbf{g} = (g_{ij})$, which gives the inner product in the tangent space $T_p(M)$,

$$g_{ij} = \langle \partial_i, \partial_j \rangle_{\mathbf{g}}.$$

its inverse matrix is (g^{ij}) , satisfies

$$\sum_{j=1}^n g_{ij} g^{jk} = \delta_i^k.$$

Definition (Dual Inner Product)

Given a n dimensional Riemannian manifold (M, \mathbf{g}) , the dual inner product $\langle \cdot, \cdot \rangle_{\mathbf{g}} : T_p^*(M) \times T_p^*(M) \rightarrow \mathbb{R}$, $\forall \omega, \eta \in T_p^*(M)$, $\omega = \sum_{i=1}^n \omega_i dx^i$, $\eta = \sum_{i=1}^n \eta_i dx^i$, then

$$\langle \omega, \eta \rangle_{\mathbf{g}} = \sum_{i,j=1}^n g^{ij} \omega_i \eta_j.$$

Orthonormal Basis

Let $\{\theta_1, \theta_2, \dots, \theta_n\}$ is a set of orthonormal basis

$$\langle \theta_i, \theta_j \rangle_{\mathbf{g}} = \delta_i^j.$$

Basis of $\Omega^k(M)$

We use $\{\theta_i\}$ to construct the basis of $\Omega^k(M)$,

$$\Omega^k(M) := \text{Span}\{\theta_{i_1} \wedge \theta_{i_2} \wedge \dots \wedge \theta_{i_k} \mid i_1 < i_2 < \dots < i_k\}.$$

Dual Inner Product

We define dual inner product $\langle \cdot, \cdot \rangle_{\mathbf{g}} : \Omega^k(M) \times \Omega^k(M)$ as follows:

$$\langle \theta_{i_1} \wedge \dots \wedge \theta_{i_k}, \theta_{j_1} \wedge \dots \wedge \theta_{j_k} \rangle = \delta_{i_1 \dots i_k}^{j_1 \dots j_k}.$$

Riemannian Volume Element

Riemannian volume Element

Let $G = \det(g_{ij})$, then in the local coordinates, the Riemannian volume element is defined as

$$\omega_{\mathbf{g}} = \sqrt{G} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Definition (Hodge Star Operator)

$$* : \Omega^k(M) \rightarrow \Omega^{n-k}(M),$$

$$\omega \wedge *\eta = \langle \omega, \eta \rangle_{\mathbf{g}} \omega_{\mathbf{g}}.$$

Therefore

$$*(1) = \omega_{\mathbf{g}}, \quad *\omega_{\mathbf{g}} = 1.$$

Definition (Inner Product)

Let (M, \mathbf{g}) be a n dimensional Riemannian manifold, ζ and η are differential k -forms, $0 \leq k \leq n$, then ζ and η inner product is defined as

$$(\zeta, \eta) := \int_M \zeta \wedge {}^*\eta = \int_M \langle \zeta, \eta \rangle_{\mathbf{g}} \omega_{\mathbf{g}}$$

Hodge Star Operator on Surface - Type I

Suppose (S, \mathbf{g}) is a surface with a Riemannian metric, with isothermal coordinates (u, v) , the metric is

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2),$$

Then

$$\frac{\partial}{\partial x_1} = e^{-\lambda} \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial x_2} = e^{-\lambda} \frac{\partial}{\partial v},$$

And

$$dx_1 = e^{\lambda} du, \quad dx_2 = e^{\lambda} dv.$$

Hodge Star

$$*dx_1 = dx_2, \quad *du = dv$$

$$*dx_2 = -dx_1, \quad *dv = -du$$

$$*(1) = dx_1 \wedge dx_2 = e^{2\lambda} du \wedge dv, \quad *(dx_1 \wedge dx_2) = 1.$$

Hodge Star Operator on Surface - Type II

Suppose (S, \mathbf{g}) is a surface with a Riemannian metric, with isothermal coordinates (u, v) , the metric is

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2),$$

surface area element is

$$\omega_{\mathbf{g}} = e^{2\lambda(u,v)} du \wedge dv.$$

Given 1-forms $\omega = \omega_1 du + \omega_2 dv$ and $\tau = \tau_1 du + \tau_2 dv$, its wedge product is

$$\omega \wedge \tau = (\omega_1 \tau_2 - \omega_2 \tau_1) du \wedge dv.$$

Inner product is

$$\langle \omega, \tau \rangle_{\mathbf{g}} = e^{-2\lambda(u,v)} (\omega_1 \tau_1 + \omega_2 \tau_2).$$

Hodge Star Operator on Surface

$$(\omega_1 du + \omega_2 dv) \wedge *du = \langle \omega, du \rangle_{\mathbf{g}} \omega_{\mathbf{g}} = e^{-2\lambda} \omega_1 e^{2\lambda} du \wedge dv,$$

This shows $*du = dv$, similarly $*dv = -du$.

$$*(\omega_1 du + \omega_2 dv) = \omega_1 dv - \omega_2 du.$$

Hence $**\omega = -\omega$.

Hodge Decomposition

Codifferential operator

Definition

The codifferential operator $\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ is defined as

$$\delta = (-1)^{kn+n+1*} d^*,$$

where d is the exterior derivative.

Lemma

The codifferential is the adjoint of the exterior derivative, in that

$$(\delta\zeta, \eta) = (\zeta, d\eta).$$

Laplace Operator

Definition (Laplace Operator)

The Laplace operator $\Delta : \Omega^k(M) \rightarrow \Omega^k(M)$,

$$\Delta = d\delta + \delta d.$$

Lemma

The Laplace operator is symmetric

$$(\Delta\zeta, \eta) = (\zeta, \Delta\eta)$$

and non-negative

$$(\Delta\eta, \eta) \geq 0.$$

Proof.

$$(\Delta\zeta, \eta) = (d\zeta, d\eta) + (\delta\zeta, \delta\eta).$$



Harmonic Forms

Definition (Harmonic forms)

Suppose $\omega \in \Omega^k(M)$, then ω is called a k -harmonic form, if

$$\Delta\omega = 0.$$

Lemma

ω is a harmonic form, if and only if

$$d\omega = 0, \delta\omega = 0.$$

Proof.

$$0 = (\Delta\omega, \omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega).$$



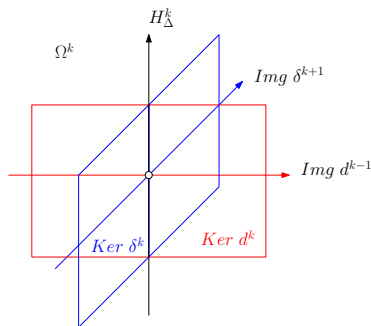
Hodge Decomposition

Definition (Harmonic form group)

All harmonic k -forms form a group, denoted as $H_{\Delta}^k(M)$.

Theorem (Hodge Decomposition)

$$\Omega_k = \text{img } d^{k-1} \oplus \text{img } \delta^{k+1} \oplus H_{\Delta}^k(M).$$



Hodge Decomposition

Proof.

$(\operatorname{img} d^{k-1})^\perp = \{\omega \in \Omega^k(M) \mid (\omega, d\eta) = 0, \forall \eta \in \Omega^{k-1}(M)\}$, because $(\omega, d\eta) = (\delta\omega, \eta)$, so $(\operatorname{img} d^{k-1})^\perp = \ker \delta^k$. similarly, $(\operatorname{img} \delta^{k+1})^\perp = \ker d^k$. Because $\operatorname{img} d^{k-1} \subset \ker d^k$, $\operatorname{img} \delta^{k+1} \subset \ker \delta^k$, therefore $\operatorname{img} d^{k-1} \perp \operatorname{img} \delta^{k+1}$,

$$\Omega^k = \operatorname{img} d^{k-1} \oplus \operatorname{img} \delta^{k+1} \oplus (\operatorname{img} d^{k-1} \oplus \operatorname{img} \delta^{k+1})^\perp$$

$$(\operatorname{img} d^{k-1} \oplus \operatorname{img} \delta^{k+1})^\perp = (\operatorname{img} d^{k-1})^\perp \cap (\operatorname{img} \delta^{k+1})^\perp = \ker \delta^k \cap \ker d^k = H_\Delta^k.$$



Hodge Decomposition

suppose $\omega \in \ker d^k$, then $\omega \perp \operatorname{img} d^{k+1}$, then $\omega = \alpha + \beta$, $\alpha \in \operatorname{img} d^{k-1}$, $\beta \in H_{\Delta}^k(M)$, define project $h : \ker d^k \rightarrow H_{\Delta}^k(M)$,

Theorem

Suppose ω is a closed form, its harmonic component is $h(\omega)$, then the map:

$$h : H_{dR}^k(M) \rightarrow H_{\Delta}^k(M).$$

is isomorphic.

Each cohomologous class has a unique harmonic form.

Harmonic 1-forms

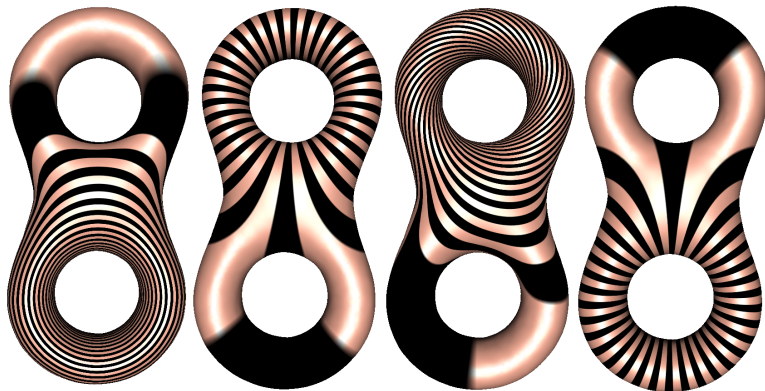
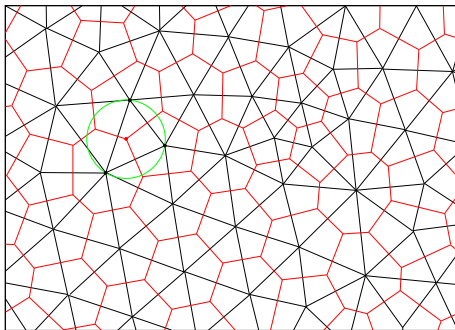


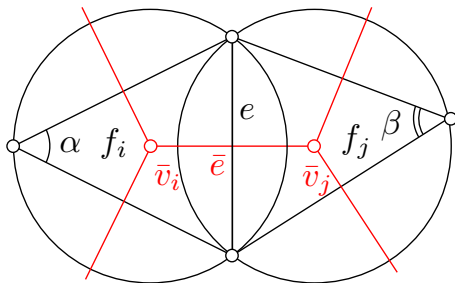
Figure: Harmonic 1-form group basis.

Dual Mesh



Delaunay triangulation T and Voronoi diagram D , every prime edge e corresponds to an dual edge \bar{e} . ω is a one-form on T . $^*\omega$ is a one-form on D .

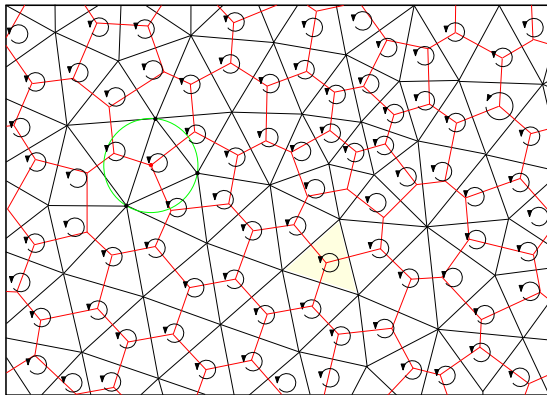
Discrete Hodge Operator



Discrete Hodge star operator,

$$\frac{\omega(e)}{e} = \frac{*\omega(\bar{e})}{|\bar{e}|}, \quad *\omega(\bar{e}) = \frac{|\bar{e}|}{|e|}\omega(e) = \frac{1}{2}(\cot \alpha + \cot \beta)\omega(e).$$

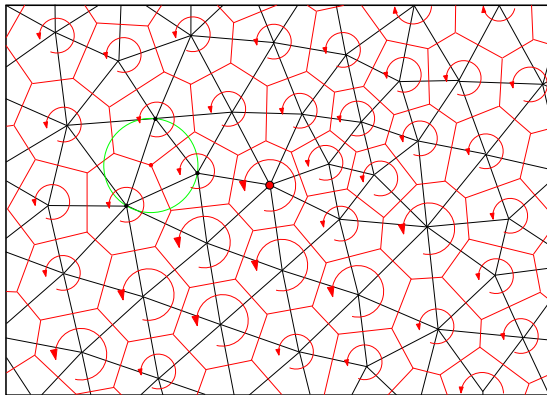
Discrete Harmonic One-form



ω is closed,

$$d\omega = 0.$$

Discrete Harmonic One-form



ω is coclosed,

$$\delta\omega = {}^*d^*\omega = 0.$$