Curvature Flow for Hyperbolic 3-Manifolds with Geodesic Boundaries

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August 27, 2024

Hyperbolic 3-manifold

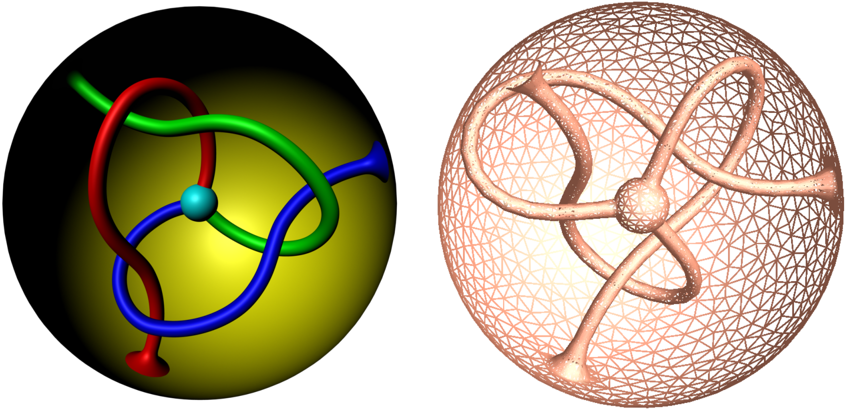


Figure: Thurston Knotted Y-shape.

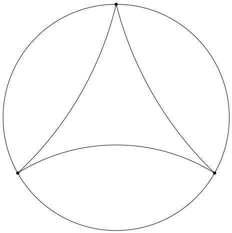
Hyperbolic 3-manifold

Definition (Hyperbolic 3-manifold with Geodesic Boundaries)

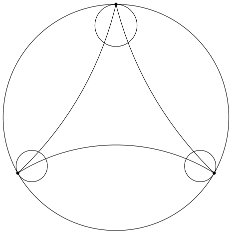
A hyperbolic 3-manifolds with geodesic boundaries have the following topological properties:

1. The genus of boundary surfaces are greater than one.
2. For any closed curve on the boundary surface, if it cannot shrink to a point on the boundary, then it cannot shrink to a point inside the volume.

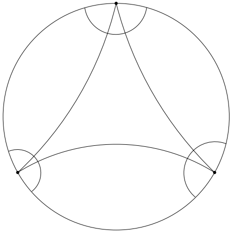
Hyperbolic Ideal Triangles



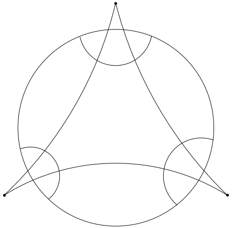
a. ideal triangle



b. decorated ideal triangle



c. hyperideal triangle



d. strictly hyperideal triangle

Figure: Ideal hyperbolic triangles.

Hyperbolic Ideal Tetrahedra

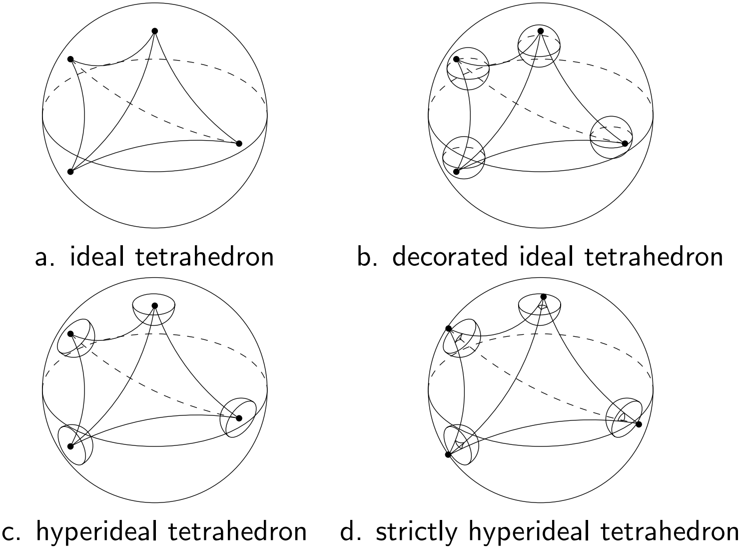


Figure: Ideal hyperbolic tetrahedra.

Strictly Hyperideal Tetrahedron

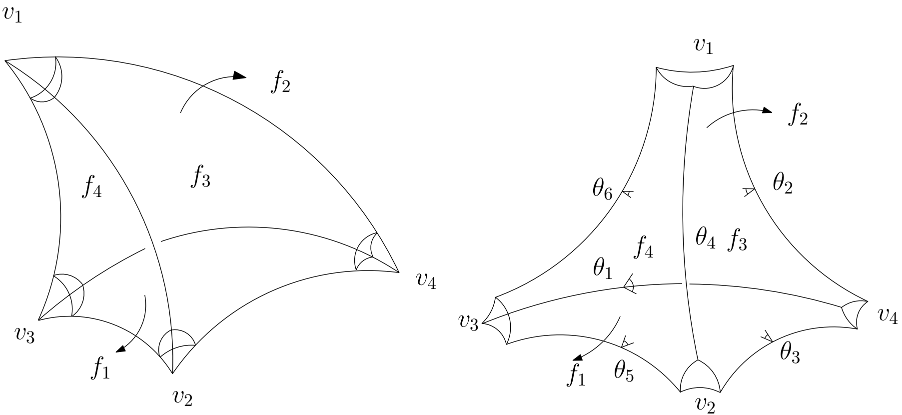


Figure: Strictly hyperideal tetrahedron $[v_1, v_2, v_3, v_4]$, each face f_i is a hyperbolic plane, each edge e_{ij} is a hyperbolic line segment.

Strictly Hyperideal Tetrahedron

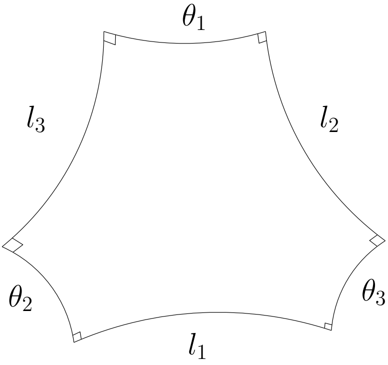


Figure: Hyperbolic hexagon.

The four vertices are truncated by hyperbolic planes,

- the cutting plane at vertex v_i is perpendicular to the edges e_{ij}, e_{ik}, e_{il}.
- Each face is a right-angled hyperbolic hexagon;
- Each cutting section at the vertex is a hyperbolic triangle.

Strictly Hyperideal Tetrahedron

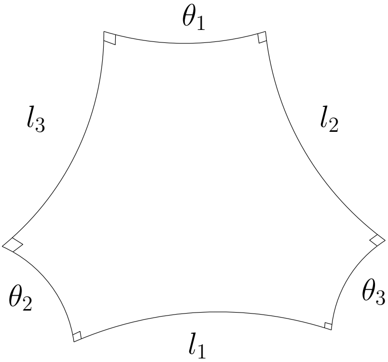


Figure: Hyperbolic hexagon.

Hyperbolic cosine law

$$\cosh l_i = \frac{\cosh \theta_i + \cosh \theta_j \cosh \theta_k}{\sinh \theta_j \sinh \theta_k}$$

$$\cosh \theta_i = \frac{\cosh l_i + \cosh l_j \cosh l_k}{\sinh l_j \sinh l_k}$$

Strictly Hyperideal Triangle

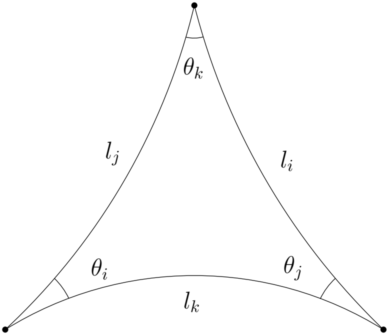


Figure: Hyperbolic triangle.

Hyperbolic cosine law

$$\cosh l_i = \frac{\cos\theta_i + \cos\theta_j \cosh\theta_k}{\sin\theta_i \sin\theta_k}$$

$$\cos\theta_i = \frac{\cosh l_i - \cosh l_j \cosh l_k}{\sinh l_j \sinh l_k}$$

Strictly Hyperideal Tetrahedron

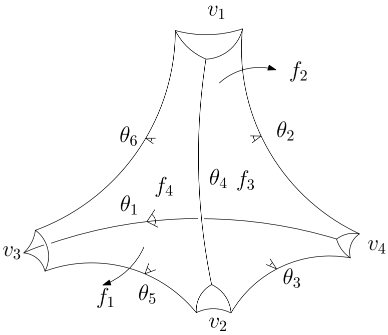


Figure: Strictly hyperbolic tetrahedron.

Theorem (Schlenker)

The volume of a strictly hyperideal tetrahedron is a strictly concave function of its dihedral angles.

$$\left(\frac{\partial^2 V}{\partial \theta_i \partial \theta_j}\right)(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$$

is negative definite.

Volume Formula

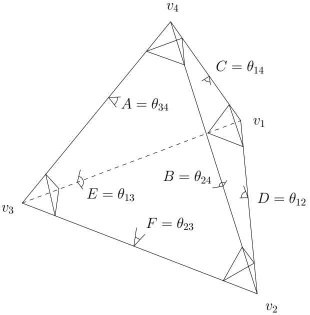


Figure: Volume formula.

Let $T = T(A, B, C, D, E, F)$ be a strictly hyperideal tetrahedron in $\mathbb{H}^3$ whose dihedral angles are A, B, C, D, E, F. Let G be the *Gram matrix* of T:

$$G = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}$$

Volume Formula

Let $a = \exp(\sqrt{-1}A)$,$b = \exp(\sqrt{-1}B)$,..., $f = \exp(\sqrt{-1}F)$, and let $U(z, T)$ be the complex-valued function defined as:

$$U(z,T) := \frac{1}{2}\{Li_2(z) + Li_2(abcdez) + Li_2(acdfz) + Li_2(bcefz) \\ -Li_2(-abcz) - Li_2(-aefz) - Li_2(-bdfz) - Li_2(-cdez)\}$$

where $Li_2(z)$ is the dilogarithm function defined by the analytic continuition of the integral

$$Li_2(z) := -\int_0^z \frac{\log(1-t)}{t}dt = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \quad z \in \mathbb{C}.$$

Volume Formula

We denote by z_1 and z_2 the two complex numbers defined as follows:

$$z_1 := -2\frac{\sin A \sin D + \sin B \sin E + \sin C \sin F - \sqrt{detG}}{ad + be + cf + abf + ace + bcd + def + abcdef},$$

$$z_2 := -2\frac{\sin A \sin D + \sin B \sin E + \sin C \sin F + \sqrt{detG}}{ad + be + cf + abf + ace + bcd + def + abcdef}.$$

Theorem

The volume $V(T)$ of a strictly hyperideal tetrahedron T is given as

$$V(T) = \frac{1}{2}\Im(U(z_1, T) - U(z_2, T)),$$

where $\Im$ means the imaginary part.

Strictly Hyperideal Tetrahedron

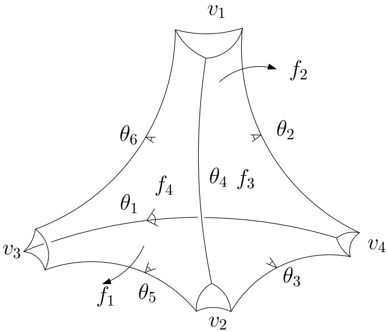


Figure: Strictly hyperbolic tetrahedron.

By Schläfli formula

$$dV = -\frac{1}{2}\sum_i x_i d\theta_i,$$

we obtain

$$\frac{\partial V}{\partial \theta_i} = -\frac{1}{2}x_i.$$

and the Hessian is negative definite:

$$\left(\frac{\partial^2 V}{\partial \theta_i \partial \theta_j}\right) = -\frac{1}{2}\left(\frac{\partial x_i}{\partial \theta_j}\right) < 0.$$

Strictly Hyperideal Tetrahedron

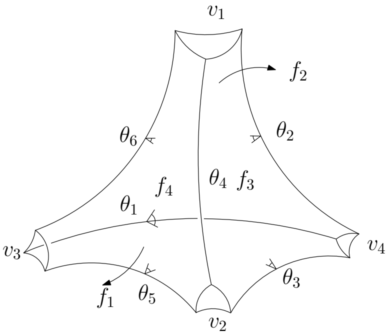


Figure: Strictly hyperbolic tetrahedron.

Lemma

The energy form defined on the strictly hyperbolic tetrahedron Δ

$$H_\Delta(x) = 2V_\Delta(x) + \sum_{i=1}^{6} x_i\theta_i$$

is strictly convex.

Strictly Hyperideal Tetrahedron

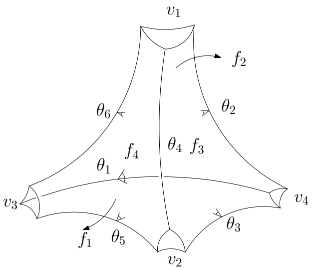


Figure: Strictly hyperbolic tetrahedron.

Proof.

By Schläfli

$$
\begin{aligned}
dH_\Delta &= 2dV_\Delta + \sum_i (dx_i\theta_i + x_i d\theta_i) \\
&= -\sum_i x_i d\theta_i + \sum_i dx_i\theta_i + \sum_i x_i d\theta_i \\
&= \sum_i \theta_i dx_i.
\end{aligned}
$$

$$
\left(\frac{\partial^2 H}{\partial x_i \partial x_j}\right) = \left(\frac{\partial x_i}{\partial \theta_j}\right) = \left(\frac{\partial \theta_i}{\partial x_j}\right)^{-1} > 0.
$$

□

Initial Triangulation

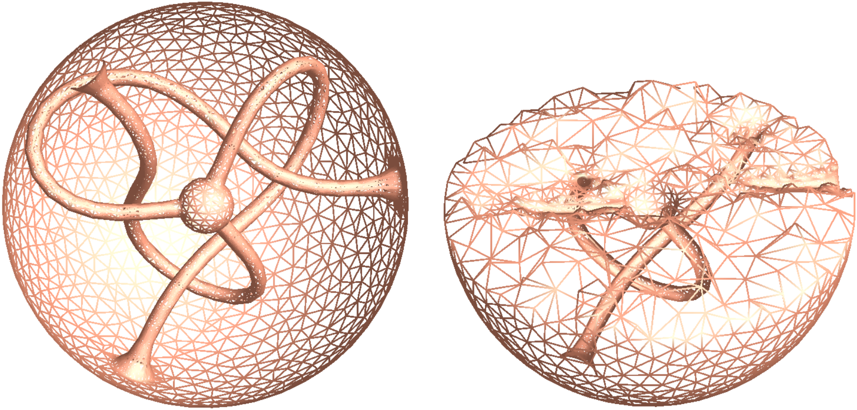


Figure: Given the boundary surface triangulation, we use the volumetric Delaunay refinement algorithm to tessellate the interior volume with tetrahedra.

Augmented Tetrahedral Mesh

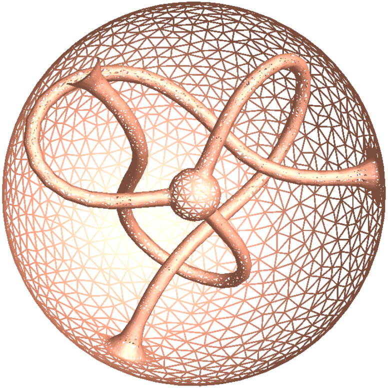
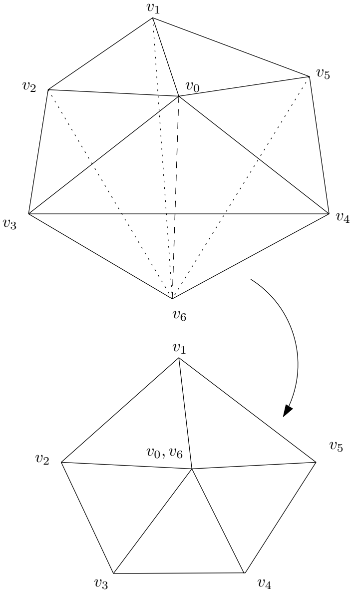


Figure: Thurston's knotty Y-shape.

Suppose the boundary components are $\partial M = \bigcup_{i=1}^{n} S_i$. Create a cone vertex v_i for each boundary component S_i; for each triangle face $f_j \in S_i$, construct a tetrahedron $T_j^i = (f_j, v_i)$, whose vertex set consists of v_i and those of f_j. M is augumented with a set of cone vertices and a set of newly constructed tetrahedra.

Volumetric Mesh Simplification



Take the *edge collapsing* operation iteratively, until all the vertices are removed except for those cone vertices $\{v_1, v_2, \ldots, v_n\}$ created in the previous step.

Volumetric Mesh Simplification

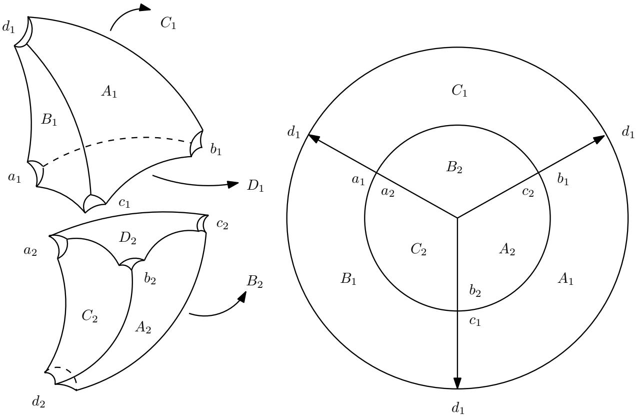


Figure: Simplified triangulation.

For each tetrahedron in the simplified triangulation, trim off its vertices by the original boundary surface, and make it a truncated tetrahedron (strictly hyperideal tetrahedron).

Volumetric Mesh Simplification

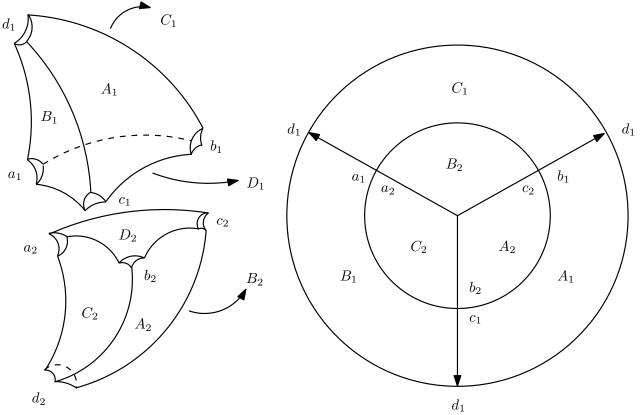


Figure: Simplified triangulation.

The edge collapsing operation doesn't change the fundamental group of the 3-manifold, by Mostow rigidity, the hyperbolic structure is unchanged.

Volumetric Mesh Simplification

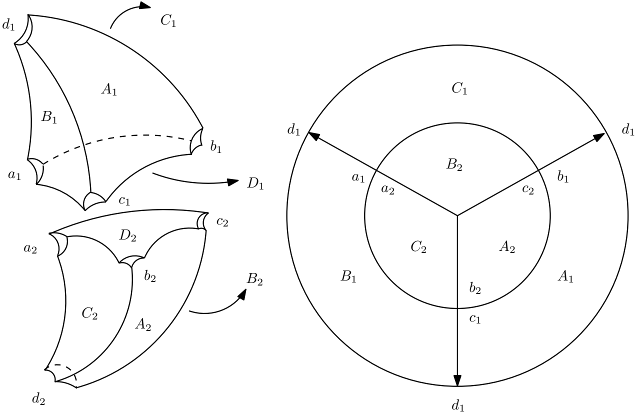


Figure: Simplified triangulation.

The edge collapsing operation doesn't change the fundamental group of the 3-manifold, by Mostow rigidity, the hyperbolic structure is unchanged.

Gluing Pattern

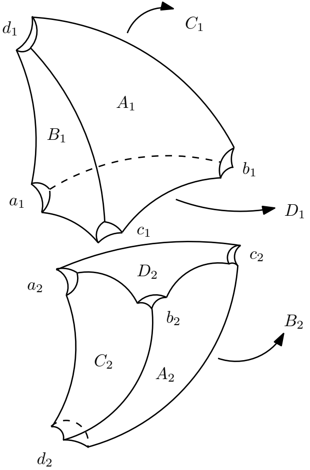


Figure: Strictly hyperideal triangulation.

$$A_1 \leftrightarrow B_2 \; \{b_1 \leftrightarrow c_2, d_1 \leftrightarrow a_2, c_1 \leftrightarrow d_2\}$$
$$B_1 \leftrightarrow A_2 \; \{c_1 \leftrightarrow b_2, d_1 \leftrightarrow c_2, a_1 \leftrightarrow d_2\}$$
$$C_1 \leftrightarrow C_2 \; \{a_1 \leftrightarrow a_2, d_1 \leftrightarrow b_2, b_1 \leftrightarrow d_2\}$$
$$D_1 \leftrightarrow D_2 \; \{a_1 \leftrightarrow a_2, b_1 \leftrightarrow c_2, c_1 \leftrightarrow b_2\}$$

Hyperbolic Cone Metric

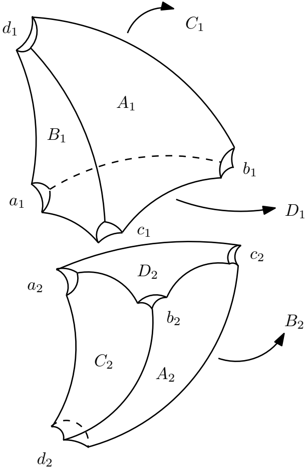


Figure: Strictly hyperideal triangulation.

Definition (Hyperbolic Cone Metric)

Given an ideal triangulated 3-manifold (M, T), let $\mathcal{E}(T)$ be the set of edges in T and let n be the number of edges in $\mathcal{E}(T)$. An assignment $x : \mathcal{E}(T) \to \mathbb{R}_{>0}$ is called a *hyperbolic cone metric associated to the triangulation* T if for each tetrahedron $\Delta \in T$ with edges $e_1, e_2, \ldots, e_6$, the numbers $x_i = x(e_i)$, are the edge lengths of a hyperideal tetrahedron in $\mathbb{H}^3$.

Vertex and Edge Curvatur

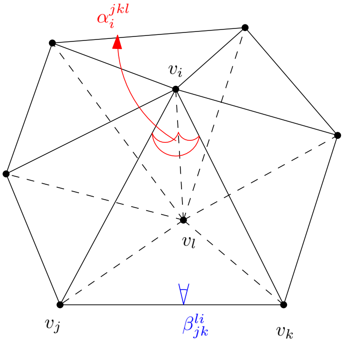


Figure: Discrete Curvatures.

Definition (Curvatures)

Given a 3-manifold M represented as a tetrahedron mesh T, for a tetrahedron $[v_i, v_j, v_k, v_l]$, let $\{\alpha_i^{jkl}, \alpha_j^{kli}, \alpha_k^{lij}, \alpha_l^{ijk}\}$ denote the solid angles at the vertices, $\{\beta_{ij}^{kl}, \beta_{kl}^{ij}, \beta_{jk}^{li}, \beta_{li}^{jk}, \beta_{ki}^{lj}, \beta_{lj}^{ki}\}$, edge curvature is defined as

$$K(e_{ij}) = \begin{cases} 2\pi - \sum_{kl} \beta_{ij}^{kl} & e_{ij} \notin \partial M \\ \pi - \sum_{kl} \beta_{ij}^{kl} & e_{ij} \in \partial M \end{cases}$$

vertex curvature

$$K(v_i) = \begin{cases} 4\pi - \sum_{jkl} \alpha_i^{jkl} & v_i \notin \partial M \\ 2\pi - \sum_{jkl} \alpha_i^{jkl} & v_i \in \partial M \end{cases}$$

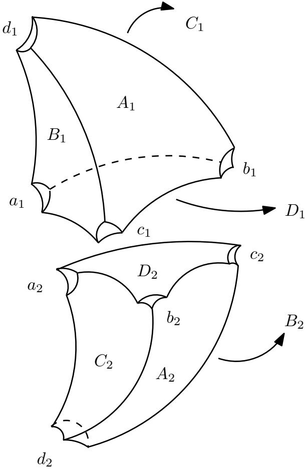


Figure: Strictly hyperideal triangulation.

Definition (Hilbert-Eistein Action)

Given an ideal triangulated 3-manifold (M, T), the set of all hyperbolic cone metrics associated to T is denoted by $\Omega(M, T)$. The Hilber-Einstein action on hte metric space is defined as $H : \Omega(M, T) \to \mathbb{R}$:

$$H(x) := 2V_M(x) - \sum_{\{i,j\}\in\mathcal{E}(T)} x_{ij}K_{ij}$$

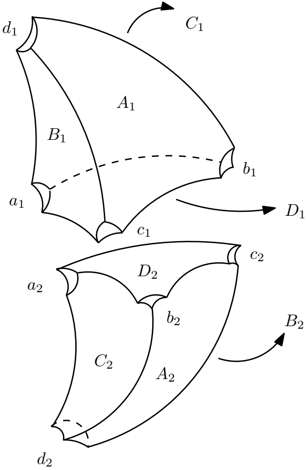


Figure: Strictly hyperideal triangulation.

Lemma

The Hilbert-Eistein action $H : \Omega(M, T) \to \mathbb{R}$ is strictly convex.

Proof.

Because the action equals to the summation of the convex energies defined on all tetrahedra plus a linear term,

$$H(x) = \sum_{\Delta \in T} H_\Delta(x) + 2\pi \sum_{ij \in \mathcal{E}(T)} x_{ij}$$

then it is strictly convex. $\square$

Curvature Flow

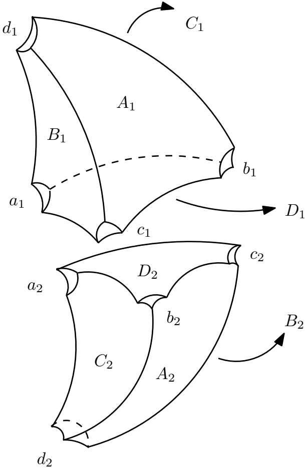


Figure: Strictly hyperideal triangulation.

Definition (Discrete Curvature Flow)

Given a 3-manifold M with a hyperideal triangulation T, the *discrete curvature flow* is defined as

$$\frac{dx_{ij}}{dt} = K_{ij}(x).$$

Curvature Flow

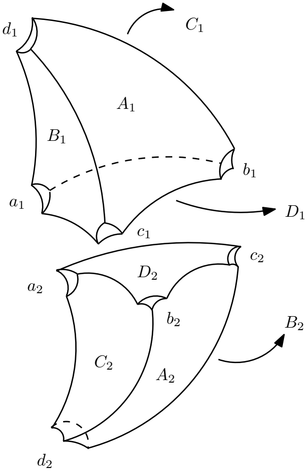


Figure: Strictly hyperideal triangulation.

Theorem (Discrete Curvature Flow)

For any ideal triangulated 3-manifold, under the discrete curvature flow, the discrete edge curvature $K_{ij}(t)$ evolves according to a discrete heat equation,

$$\frac{dK_{ij}(t)}{dt} = \sum_{\{\alpha,\beta\}\in\mathcal{E}(T)} h_{ij}^{\alpha\beta} K_{\alpha\beta}(t).$$

where the matrix $[h_{ij}^{\alpha\beta}]_{n\times n}$ is symmetric negative definite. The total curvature

$$\sum_{\{i,j\}\in\mathcal{E}(T)} K_{ij}(t)^2$$

is strictly decreasing along the flow unless $K_{ij}(t) = 0$ for all $ij \in \mathcal{E}(T)$.

Curvature Flow

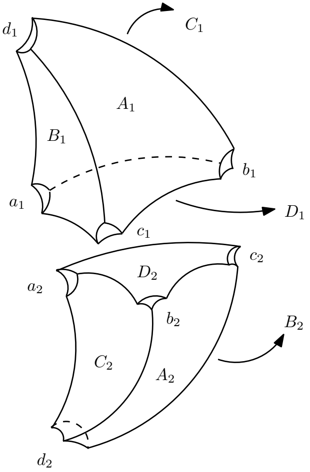


Figure: Strictly hyperideal triangulation.

Proof.

The gradient of the Hilber-Eistein action is $\nabla H(x) = -K(x)$, its Hessian $D^2 H(x) = -(\partial K/\partial x) > 0$ is positive definite.
The curvature flow is $dx/dt = K(x,t)$, hence

$$\begin{aligned} dK(x,t)/dt &= (\partial K/\partial x) dx/dt \\ &= (-D^2 H) K(x,t) \end{aligned}$$

where $-D^2 H < 0$. Hence

$$\begin{aligned} \frac{d \sum K_{ij}^2}{dt} = \frac{d\langle K, K\rangle}{dt} &= 2K^T \frac{dK}{dt} \\ &= 2K^T(-D^2 H)K < 0. \end{aligned}$$

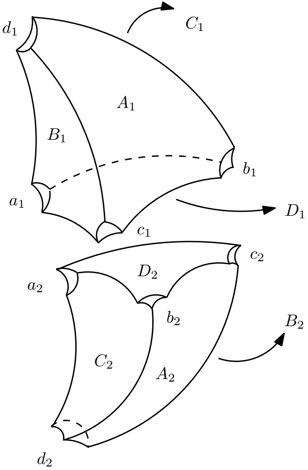


Figure: Strictly hyperideal triangulation.

Theorem (Equilibrium of Curvature Flow)

For any ideal triangulated 3-manifold (M, T), the equilibrium points of the discrete curvature flow are the complete hyperbolic metric with totally geodesic boundary. Furthermore, each equilibrium point is a local attractor of the flow.

Proof.

Since $\nabla H(x) = K(x)$, at the equilibrium point x^* the curvature are zeros $K(x^*) = 0$. □

Curvature Flow

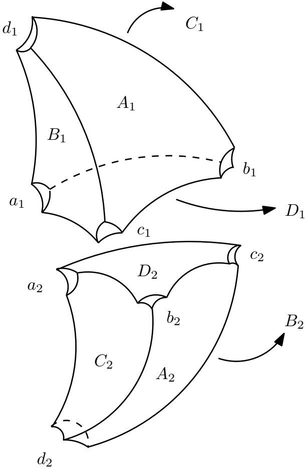


Figure: Strictly hyperideal triangulation.

Theorem (Curvature Map)

For any ideal triangulated 3-manifold (M, T), the curvature map $\Pi : \Omega(M, L) \to \mathbb{R}^n$, sending a metric x to its curvature $K(x)$ is a local diffeomorphism. In particular, a hyperbolic cone metric associated to an ideal triangulation is locally determined by its cone angles.

Proof.

Since $D^2H(x) = -(\partial K/\partial x) > 0$, it is the Jacobian matrix of the curvature map. $\square$

Realization of a hyperideal tetrahedron

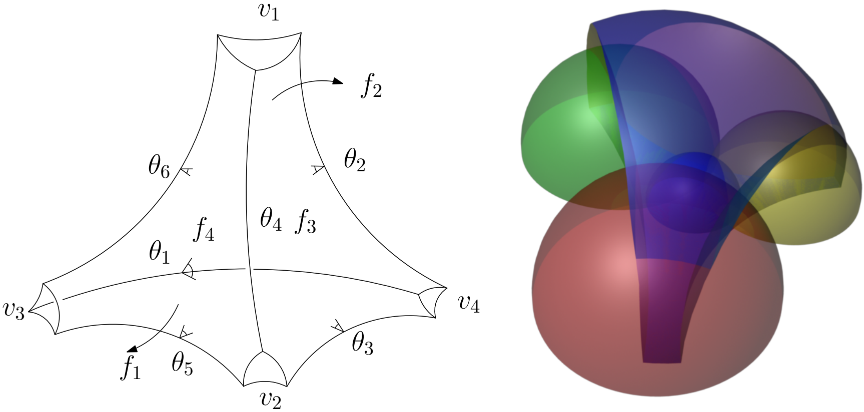


Figure: Realization of a hyperideal tetraheron, 4 faces are hyperbolic hexagons, 4 vertex sections are hyperbolic triangles.

Realization of a hyperideal tetrahedron

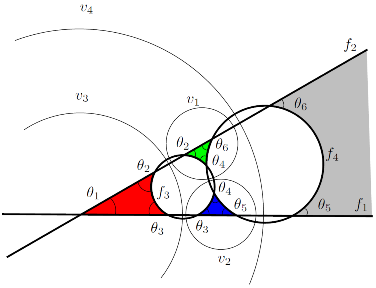


Figure: Realization of a hyperideal tetraheron by circle pattern

Each hyperbolic plane intersect the ∞-plane at a circle.

1. Set the circle f_1 to be the real axis;
2. Set the circle f_2 to be $y = \tan\theta_1 x$;
3. Set the circle f_3 to be the unit circle with center $((\cos\theta_2 + \cos\theta_3)\cot\theta_1, \cos\theta_3)$,
4. circle v_i is orthogonal to circles f_j, f_k and f_l.

Realization of a hyperideal tetrahedron

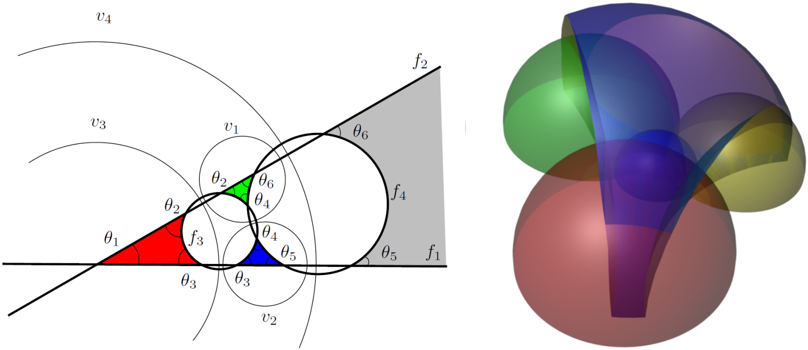


Figure: Replace each planar circle by a hemisphere, using CSG to build the hyperideal tetrahedron.

Realization of a fundamental domain

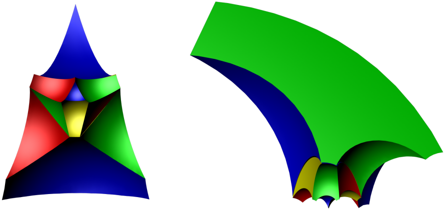


Figure: Glue $f_0 \in T_0$ with $f_1 \in T_0$ by a Mobius transformation, such that the face circles and the vertex circles coincide.

Realization of UCS

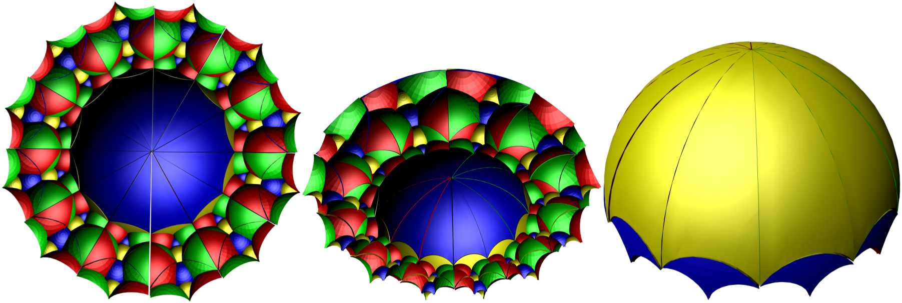


Figure: Glue the pair of hexagonal faces of different copies of the fundamental domains to construct a finite portion of the universal covering space.