Surface Uniformization

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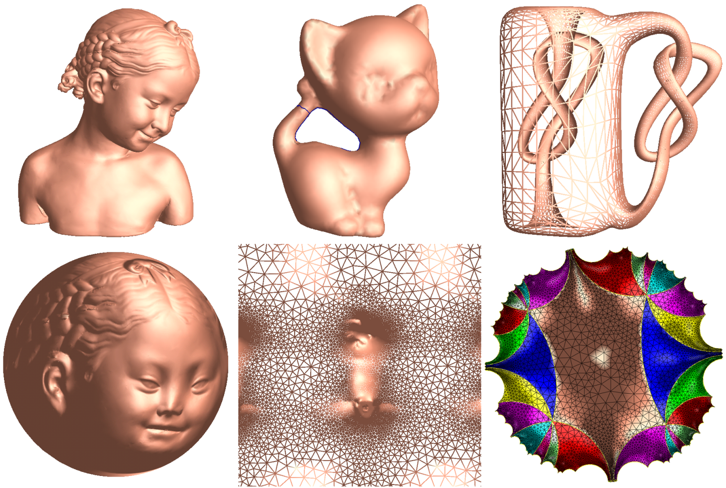


Figure: Closed surface uniformization.

Surface Uniformization

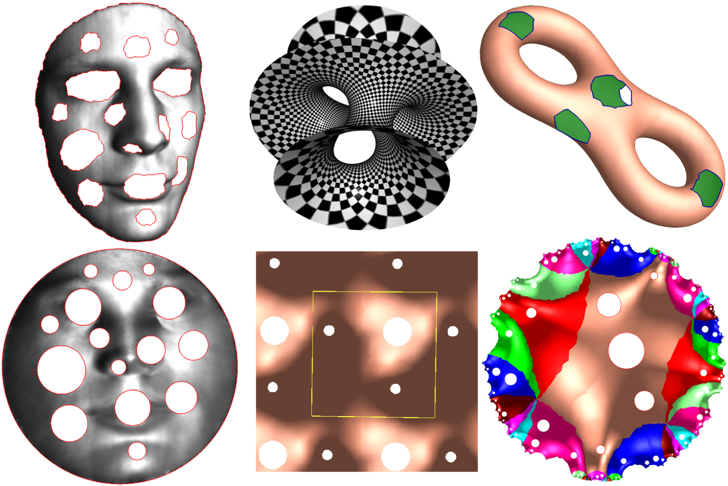


Figure: Open surface uniformization.

Conformal Mapping of Infinite Triangle Mesh

Problem

Suppose we have an infinite triangle mesh, $\tilde{M}$, such as the universal covering space of a closed mesh, fix a point $v_0 \in \tilde{M}$, choose a sequence of neighborhood $E_n \subset \tilde{M}$,

$$v_0 \in E_0 \subset E_1 \subset E_2 \cdots E_n \cdots$$

where each E_k is a topological disk, construct discrete conformal mapping $\varphi_n : E_n \to \mathbb{D}^n$, such that

$$\varphi_n(v_0) = 0, \quad \varphi_n'(v_0) > 0,$$

then what is the is limit of the sequence $\{\varphi_n(v_0)'\}$?

Conformal Mapping of Infinite Triangle Mesh

Answer

1. If $\tilde{M}$ is the universal covering of a torus, then the limit is 0;
2. If $\tilde{M}$ is the universal covering space of a high genus mesh, then the limit is a positive number $\delta > 0$.

Conformal Mapping of Infinite Triangle Mesh

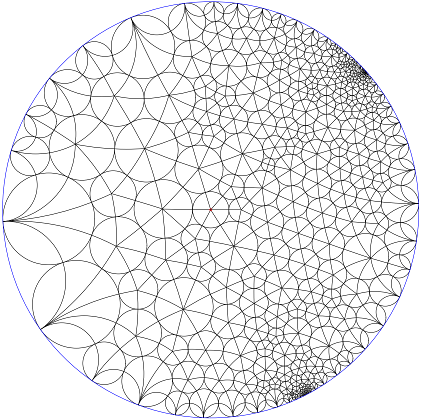
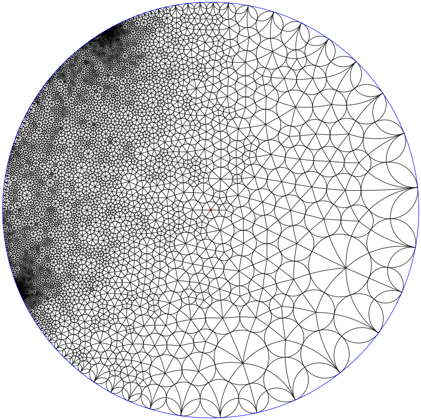


Figure: Discrete Riemann mapping of triangle mesh.

Conformal Mapping of Infinite Triangle Mesh



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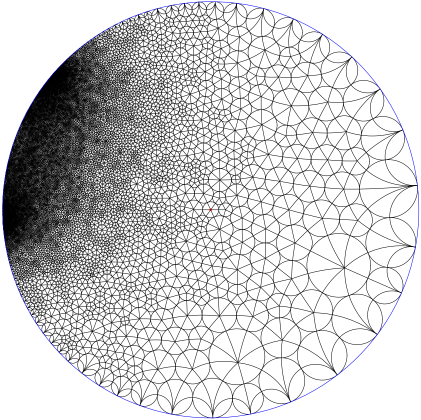

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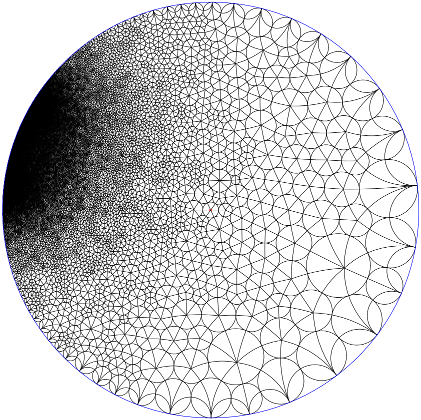
Conformal Mapping of Infinite Triangle Mesh



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Conformal Mapping of Infinite Triangle Mesh

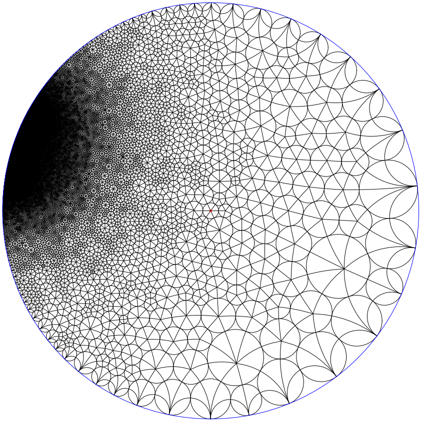

Figure: Discrete Riemann mapping of triangle mesh.

Conformal Mapping of Infinite Triangle Mesh

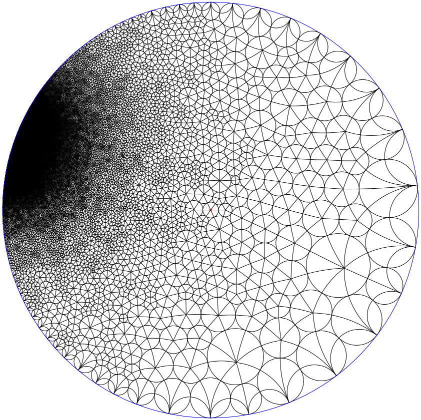

Figure: Discrete Riemann mapping of triangle mesh.

Liuville Theorem

Theorem (Liuville)

Suppose a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is bounded, $|f(z)| < C$, for all $z \in \mathbb{C}$, then $f(z) = const$.

Proof.

According to Cauchy's formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z-a)^{n+1}} dz,$$

here Γ is a circle centered at a with radius r,

$$|f'(a)| = \left| \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z-a)^2} dz \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{C}{r} d\theta = \frac{C}{r},$$

let $r \to \infty$, the derivative goes to 0. Hence the holomorhic function $f(z)$ is constant. □

Liuville Theorem

The unit sphere $\mathbb{S}^2$ is conformal equivalent to the augmented complex plane $\hat{\mathbb{C}}$. Complex plane $\mathbb{C}$ and the unit open disk $\mathbb{D}$ are open sets, therefore they are not homeomorphic to the compact set $\mathbb{S}^2$. Liuville theorem shows $\mathbb{C}$ and $\mathbb{D}$ are not conformally equivalent to each other.

Corollary

The complex plane $\mathbb{C}$ and the unit disk $\mathbb{D}$ are not conformally equivalent.

Proof.

Suppose they are equivalent, there is a biholomorphic function $f : \mathbb{C} \to \mathbb{D}$, according to Liuville, $f(z)$ is constant. Contradiction to biholomorphic function. $\square$

Crescent and Full-Moon Theorem

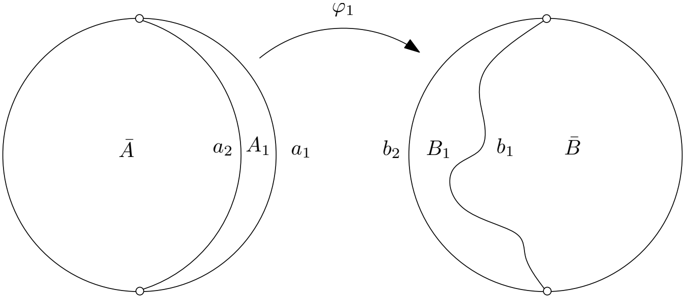

Figure: 9. Initial Map.

Crescent and Full-Moon Theorem

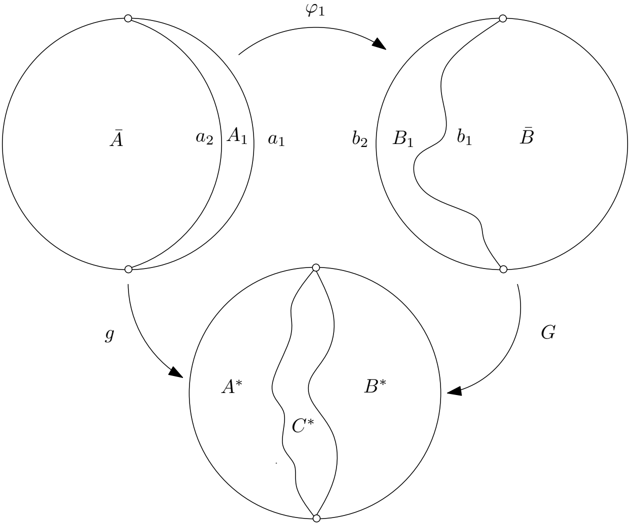

Figure: 10. Analytic extension result.

Crescent and Full Moon

Lemma (Crescent and Full Moon)

As shown in Fig. 9, the boundaries of the crescent domain A_1 are circular arcs a_1 and a_2, they have intersection angle $\pi/2^m$, $m \in \mathbb{Z}^+$. A conformal map $\varphi_1 : A_1 \to B_1$ is defined on the crescent A_1, $\varphi_1(a_k) = b_k$, $k = 1, 2$, b_2 is a circular arc. Then there exist analytic functions, $g, G : \mathbb{D} \to \mathbb{D}$, as shown Fig. 10, satisfying

1. $A^* = g(\bar{A})$, $C^* = g(A_1)$;
2. $B^* = G(\bar{B})$, $C^* = G(B_1)$;
3. $g|_{A_1} = G \circ \varphi_1|_{A_1}$;

and the restriction on a_k's and b_k's, the mappings g and G are homeomorphisms.

Crescent and Full-Moon Theorem

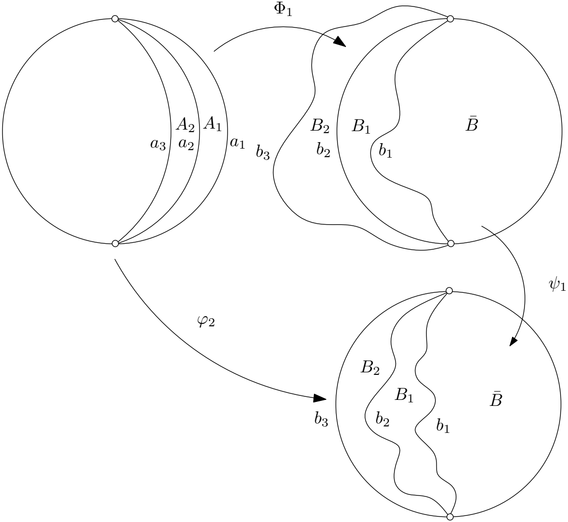

Figure: 11. Analytic extension, step one.

Crescent and Full Moon

Proof.

As shown in Fig. (11), crescents A_1 and A_2 are symmetric about a_2, by the Schwartz reflection principle, analytic function $\varphi_1 : A_1 \to B_1$ can be extended about the circular arc a_2 to

$$\Phi_1 : A_1 + A_2 \to B_1 + B_2,$$

using Riemann mapping

$$\psi_1 : B_1 + B_2 + \bar{B} \to \mathbb{D},$$

which maps the target to the unit disk. For convenience, we relabel $\psi_1(B_1)$, $\psi_1(B_2)$ as B_1 and B_2, then the composition map is:

$$\varphi_2 = \psi_1 \circ \Phi_1 : A_1 + A_2 \to B_1 + B_2.$$

□

Crescent and Full-Moon Theorem

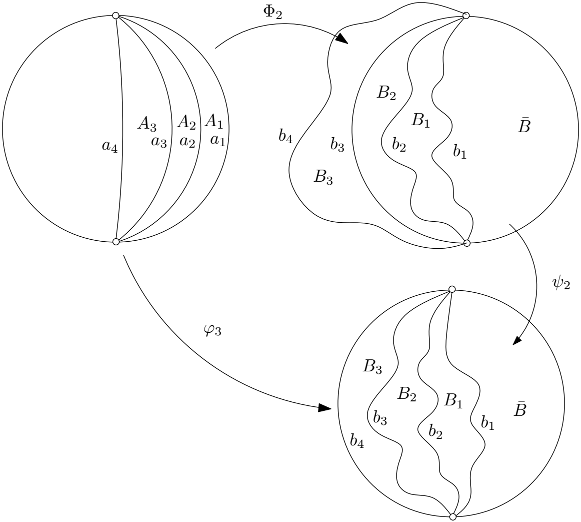

Figure: 12. Analytic extension, step two.

Crescent and Full Moon

continued.

As shown in Fig. (12), we extend $\varphi_2 : A_1 + A_2 \to B_1 + B_2$ again, $A_1 + A_2$ is reflected about a_3 to a crescent A_3, by Schwartz reflection principle,

$$\Phi_2 : (A_1 + A_2) + A_3 \to (B_1 + B_2) + B_3,$$

then composed with the Riemann mapping $\psi_2 : B_1 + B_2 + B_3 + \bar{B} \to \mathbb{D}$, we get the result for the second step extension,

$$\varphi_3 = \psi_2 \circ \Phi_2 : A_1 + A_2 + A_3 \to B_1 + B_2 + B_3.$$

Repeat this procedure, by analytic extension we get conformal mappings:

$$\varphi_k : \sum_{i=1}^{k} A_i \to \sum_{j=1}^{k} B_j,$$

Crescent and Full Moon

continued.

Consider the inner angle of the crescents, the angle of A_k is θ_k, we have recursive relations,

$$\begin{cases} \theta_1 & = & \pi/2^m \\ \theta_2 & = & \pi/2^m \\ \theta_k & = & \sum_{j=1}^{k-1} \theta_j, \ \ k > 2 \end{cases}$$

therefore at the $m+1$ step, all the crescents cover the whole disk. Hence, we obtain analytic function

$$G = \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_2 \circ \psi_1,$$

and

$$g = \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_2 \circ \phi_1.$$

Uniformization

We use a combinatorial representation to define a Riemann surface. Given a Riemann surface M, and a triangulation $\mathcal{T}$. If $\mathcal{T}$ has finite number of faces, then M is a compact surface; if the surface has countable infinite number of faces, then M is an open surface. Van der Waerden proves the existence of a special type of triangulation.

Uniformization

Lemma (Van der Waerden)

Assume $\tilde{M}$ is an open surface, then its triangulation can be sorted,

$$\mathcal{T} = \{\Delta_1, \Delta_2, \Delta_3, \cdots, \Delta_n, \cdots\}$$

such that for any $n = 1, 2, \cdots$,

$$\mathcal{T}_n := \bigcup_{k=1}^{n} \Delta_k$$

and Δ_{n+1} has only one intersection edge (and the third non-intersecting vertex), or two edges, namely $\mathcal{T}_n$ is a topological disk.

Uniformization

Let $\tilde{M}$ be the universal covering space of a Riemann surface, then $\tilde{M}$ is a simply connected Riemann surface, its triangulation $\mathcal{T}$ is sorted in Van der Waerden pattern. All the edges of $\mathcal{T}$ are analytic arcs, and every face Δ_k is covered by a conformal local chart.

Lemma

For any $n > 0$, the interior of

$$E_n = \Delta_1 + \Delta_2 + \cdots + \Delta_n,$$

is conformally mapped onto the open unit disk, $\varphi_n : E_n \to R_n$, R_n is an open unit disk, and the restriction on the boundary,

$$\varphi_n|_{\partial E_n} : \partial E_n \to \partial R_n$$

is topological homeomorphic.

Uniformization

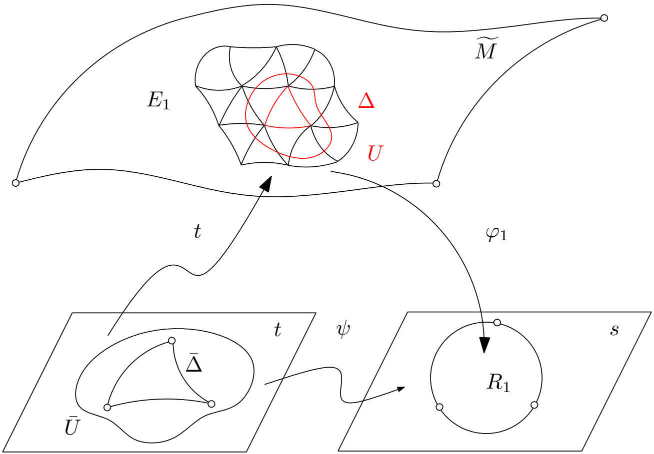

Figure: 13. Initial induction step.

Uniformization

Proof.

Step one: when $n = 1$, as shown in Fig. (13), E_1 (bounded by the red loop) only includes one triangle Δ_1, denote $\Delta = \Delta_1$. Δ is covered by a conformal coordinate system (U, t), $\Delta \subset U$. Let $\bar{\Delta}$, $\bar{U}$ are the pre-images of Δ, U on the t-plane,

$$t(\bar{\Delta}) = \Delta, \quad t(\bar{U}) = U.$$

$\bar{\Delta}$ is a simply connected domain, its boundary is piecewise analytic curves. According to Riemann mapping theorem, there is a holomorphic map $\psi : \bar{\Delta} \to R_1$, from $\bar{\Delta}$ to the unit disk R_1 on s-plane, and the restriction on the boundary is topological homeomorphic,

$$\psi|_{\partial\bar{\Delta}} : \partial\bar{\Delta} \to \partial R_1,$$

then construct a holomorphic map $\varphi_1 = \psi \circ t^{-1} : E_1 \to R_1$, its restriction on the boundary is a homeomorphism. □

Uniformization

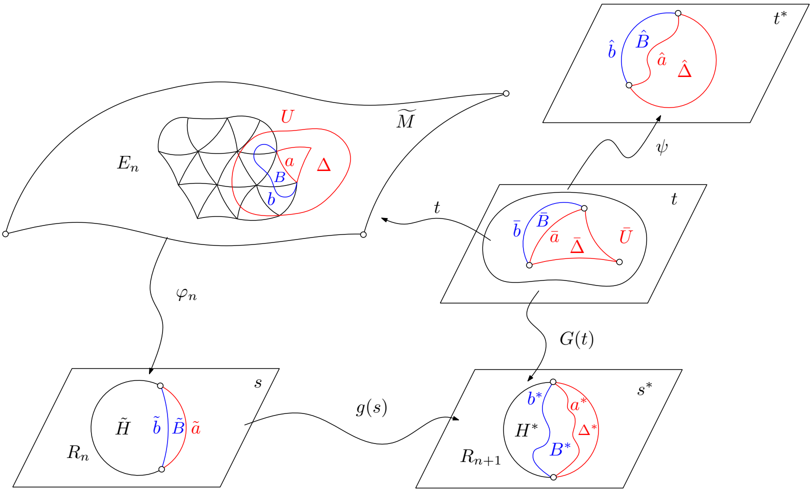

Figure: 14. Induction step.

continued.

Step two: As shown in Fig. (14), when $n > 1$, assume at the n-th step, E_n is conformally mapped onto the unit disk R_n on s-plane, $\varphi_n : E_n \to R_n$, the restriction on the boundary $\varphi_n|_{\partial E_n} : \partial E_n \to \partial R_n$ is homeomorphic. We consider $E_{n+1} = E_n + \Delta_{n+1}$. Let $\Delta = \Delta_{n+1}$, covered by a local conformal coordinates (U, t), the preimages of U and Δ are $\bar{U}$ and $\bar{\Delta}$ respectively in the local coordinate system,

$$t\left(\bar{U}\right) = U, \quad t\left(\bar{\Delta}\right) = \Delta.$$

E_n and Δ intersect at an analytic arc a, $\Delta \cap E_n = a$. The image of a under φ_n is $\tilde{a}$, $\varphi_n(a) = \tilde{a}$. The conformal local parametric representation of a is $\bar{a}$, $t(\bar{a}) = a$.

Uniformization

continued.

In the unit disk R_n on the s-plane, draw a circular arc $\tilde{b}$, two circular arcs $\tilde{a}$ and $\tilde{b}$ have the same ending points, and the intersection angles at the ending points equal to $\pi/2^k$, where k is a big positive integer. The circlar arcs bound a crescent $\tilde{B}$, the pre-image of $\tilde{B}$ on $\tilde{M}$ is B; the image of B on the t-image is $\bar{B}$, $\varphi_n(B) = \tilde{B}, \quad t(\bar{B}) = B$. We want to show the existence of holomorphic maps $s^* = g(s)$ and $s^* = G(t)$, satisfying:

1. $g(\tilde{B}) = B^*$, $g(\tilde{H}) = H^*$, where $\tilde{H} = R_n - \tilde{B}$;
2. $G(\bar{B}) = B^*$, $G(\bar{\Delta}) = \Delta^*$;
3. on domain $\bar{B}$, $G(t) = g \circ \varphi_n \circ t$;
4. $R_{n+1} = B^* + H^* + \Delta^*$

The combination of $g(s)$ and $G(t)$ gives the conformal mapping from E_{n+1} to R_{n+1}.

Uniformization

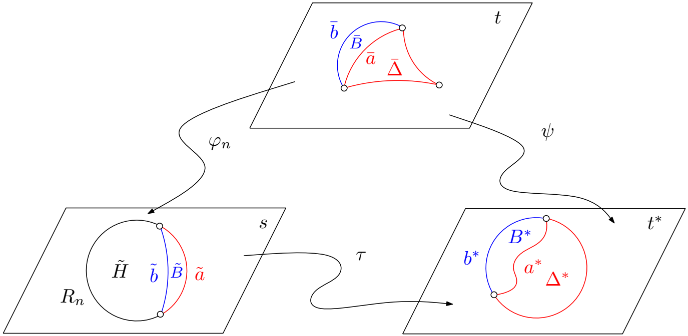

Figure: 15. Combination of conformal mappings.

Uniformization

continued.

As shown in Fig. (15), by Riemann mapping, there is a mapping $t^* = \psi(t)$, mapping $\bar{\Delta} + \bar{B}$ to $\Delta^* + B^*$, the center of the disk is inside Δ^*. Then the composition

$$\tau = \psi \circ \varphi_n^{-1}, \quad t^* = \tau(s)$$

maps the crescent $\tilde{B}$ to B^*. Note that $\tau : \tilde{B} \to B^*$ is defined on crescent $\tilde{B}$, not defined on $\tilde{H}$. By crescent-full moon lemma, there exist holomorphic functions g and G, this proves the existence of $\varphi_{n+1} : E_{n+1} \to R_{n+1}$. By induction, the lemma holds.

Uniformization

Theorem (Open Riemann Surface Uniformization)

Simply connected open Riemann surface is conformal equivalent to the whole complex plane $\mathbb{C}$ or the unit open disk $\mathbb{D}$.

Proof.

As shown in Figure 16, construct a sequence of holomorphic functions

$$\varphi_{1,n}(s) = \varphi_n \circ \varphi_1^{-1},$$

univalent on R_1, and normalized at $s = 0$, $\varphi_{1,n}(0) = 0$, $\varphi'_{1,n}(0) = 1$. Then $\{\varphi_{1,n}\}$ is a normal family. We choose subsequence $\Gamma_1 \subset \{\varphi_{1,n}\}$, which converges to univalent function in the interior of R_1, denoted as

$$\Gamma_1 : \varphi_1^1(p), \varphi_2^1(p), \varphi_3^1(p), \cdots$$

converges to a univalent function $\varphi_0(p)$ in E_1. $\square$

Construction of Normal Family

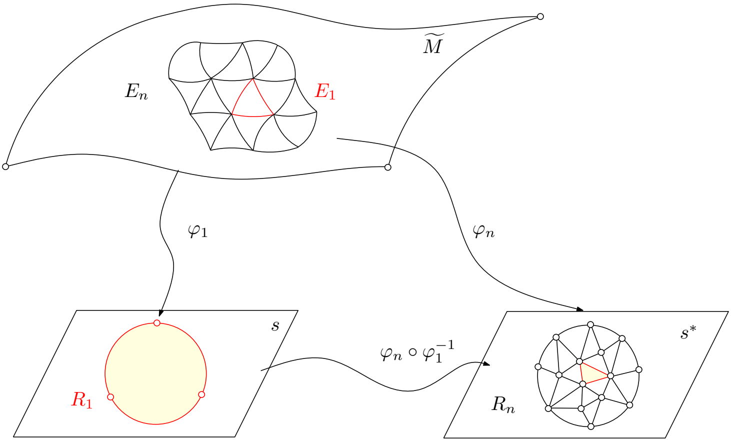

Figure: 16. Construction of normal family $\{\varphi_n \circ \varphi_1^{-1}\}$.

Uniformization

continued.

Construct a sequence of holomorphic functions

$$\varphi_{2,n}(s) = \varphi_n^1 \circ \varphi_2^{-1},\ \ \varphi_n^1 \in \Gamma_1,$$

from $\{\varphi_{2,n}\}$ choose subsequence

$$\Gamma_2 : \varphi_1^2(p), \varphi_2^2(p), \cdots$$

converges to a univalent holomorphic function on E_2, and the restriction on E_1 equals to $\varphi_0(p)$, we still denote it as $\varphi_0(p)$.

continued.

Furthermore, construct a sequence of functions

$$\varphi_{3,n}(s) = \varphi_n^2 \circ \varphi_3^{-1},\ \ \varphi_n^1 \in \Gamma_2,$$

from $\{\varphi_{3,n}\}$ choose subsequence

$$\Gamma_3 : \varphi_1^3(p), \varphi_2^3(p), \cdots$$

converges to a univalent holomorphic function on E_3, and the restriction on E_2 equals to $\varphi_0(p)$, we still denote it as $\varphi_0(p)$. Repeat this step, apply diagonal principle, we obtain a function sequence

$$\varphi_1^1(p), \varphi_2^2(p), \varphi_3^3(p), \ldots$$

where $\varphi_k^k(p)$ are well defined on E_n $(k \geq n)$, and converge to $\varphi_0(p)$ on E_n.

Uniformization

continued.

Since $\{E_n\}$ exhausts the whole open Riemann surface $\tilde{M}$, $\varphi_0(p)$ is univalent, and maps $\tilde{M}$ to a simply connected domain R on s-plane. Since $\tilde{M}$ is open, R can't be the augmented complex plane $\mathbb{C} \cup \{\infty\}$. Hence, R is either the whole complex plane $\mathbb{C}$, or a domain on the complex plane. In the second situation, by Riemann mapping theorem, R can be conformally mapped to the unit disk $\mathbb{D}$. $\square$

Compact Surface Uniformization

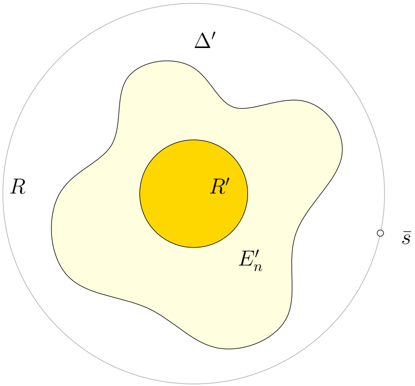

Figure: Compact surface case.

Uniformization

Theorem (Compact Riemann Surface Uniformization)

Compact simply connected Riemann surface is conformal equivalent to the unit sphere.

Proof.

Suppose $\tilde{M}$ has a triangulation $\mathcal{T}$, which includes a finite number of faces,

$$\mathcal{T}_n = \Delta_1 + \Delta_2 + \cdots + \Delta_n,$$

the last triangle Δ_n has three common eges with T_{n-1}. Choose an interior point $q \in \Delta_n$, remove this point, we obtain an open Riemann surface,

$$\tilde{M}_0 = \tilde{M} \setminus \{q\},$$

according to open Riemann surface uniformization theorem, there is a conformal mapping, $\varphi : \tilde{M}_0 \to \mathbb{C}$, $s = \varphi(p)$, which maps the open Riemann surface either to a unit disk or the whole complex plane. □

continued.

on s-complex plane, let $\varphi(\Delta_n \setminus \{q\}) = \Delta'$, $\varphi(E_{n-1}) = E'_n$, point $o \in E_{n-1}$, $\varphi(o) = 0$. Let $R' \subset E'_n$ be a disk centered at the origin, then Δ' is outside R'.

Function $w = 1/s$ maps Δ' to a bounded domain on w-plane. Consider the function $w = 1/\varphi(p)$ defined on $\tilde{M} \setminus \{q\}$, w is bounded in a neighborhood of q, hence q is a removable singularity of function w. Let the image of q in w-plane is $w(q)$.

Assume $R = \varphi(\tilde{M} \setminus \{q\})$ is not the whole complex plane, but the unit disk, $R = \mathbb{D}$. Choose a point sequence $s_1, s_2, \cdots$, its accumulation point is on the unit circle. The corresponding point sequence on the surface is $p_1, p_2, \cdots$. Since $\tilde{M}$ is compact, the accumulation point of the point sequence is on the surface. For any point $p \in \tilde{M} \setminus \{q\}$, $\varphi(p)$ is an interior point of R, $\varphi(p) \notin \partial R$. Since $\lim_{n\to\infty} s_n \in \partial R$, therefore $\lim_{n\to\infty} p_n \neq p$. Hence

$$\lim_{n\to\infty} p_n = q.$$

continued.

For any point on the unit circle, $\bar{s} \in \partial R$, there is a point sequence converging to $\bar{s}$, hence

$$1/\bar{s} = w(q),$$

but $\bar{s}$ has infinite many values, hence $w(q)$ has infinite many values, contradiction. Hence the assumption is incorrect, $R = \varphi(\tilde{M} \setminus \{q\})$ is the whole complex plane, $\tilde{M}$ is conformal equivalent to the augmented complex plane $\mathbb{C} \cup \{\infty\}$. $\square$