THE RIESZ-MARKOV-KAKUTANI REPRESENTATION THEOREM

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Abstract. We prove the Riesz-Markov-Kakutani (RMK) Representation Theorem in the setting of a locally compact Hausdorff (LCH) space by using fundamental results from measure theory and point-set topology, presenting much of the necessary background along the way. We also isolate the classical case of a compact interval $[a, b]$ and prove this case independently by using functional analysis. The paper concludes with three major applications of the RMK Theorem.

Contents

1. Introduction

Given a topological space X, we write $C(X)$ for the complex vector space of continuous functions from X into $\mathbb{C}$ under pointwise addition and pointwise scaling. By $C_c(X)$, we denote the collection of functions $f \in C(X)$ whose *support* $\text{supp}(f) := \text{cl}_X\{x \in X : f(x) \neq 0\}$ is compact. Observe that this is a subspace of $C(X)$. Let μ be a locally finite Borel measure on X (see Section 4 in case the definitions are unfamiliar). Since the measure of a compact subset of X under μ is finite, and a continuous complex-valued function on a compact set is bounded, the integral $\int f \, d\mu$ is finite for all $f \in C_c(X)$. Thus, we can define a linear functional T on $C_c(X)$ by $Tf = \int f \, d\mu$ for all $f \in C_c(X)$. As with any integral defined against a measure, T is *positive* in the sense that if $f \in C_c(X)$ takes all of its values in the nonnegative real numbers, then Tf is a nonnegative real number. This expository paper is about the remarkable partial converse to this observation.

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Theorem 1.1 (Riesz-Markov-Kakutani Representation Theorem for C_c)**.** *Let X be an LCH space. If T is a positive linear functional on $C_c(X)$, then there exists a unique Radon measure μ on X such that*

$$Tf = \int f \, d\mu \quad \textit{for all} \quad f \in C_c(X).$$

Hence, the only positive linear functionals on $C_c(X)$ are integrals! In particular, they are integrals with respect to a *Radon measure*, a condition which assures us nice regularity properties generalizing those enjoyed by the Lebesgue measure; cf. Definition 4.3.

The RMK Theorem was first proved for the special case where $X = [0, 1]$ in 1909 by Frigyes Riesz; cf. [12]. In 1938, Andrey Markov published work extending the result to some noncompact spaces; cf. [11]. Shizuo Kakutani proved the result for compact Hausdorff spaces in 1941; cf. [8]. Although LCH spaces are practically always general enough, there does exist an extension of the RMK Theorem to Hausdorff spaces; cf. Chapter V of [10]. The RMK Theorem and its several variants go under a family of names including the Riesz Representation Theorem, Riesz's Theorem, the Riesz-Markov Theorem, and the Riesz-Kakutani Theorem.

The key ingredient in the proof of the RMK Theorem that we present is the Carathéodory Extension Theorem. Urysohn's Lemma is also crucial. However, before giving this proof, we first prove the theorem in the classical case where $X = [a, b]$ without using any measure theory or general topology. Specifically, we use the Hahn-Banach Theorem to prove this case of the theorem in its original formulation in terms of Riemann-Stieltjes integrals. We conclude the paper with a different version of the RMK Theorem and applications. In particular, we use the RMK Theorem to present an alternate approach to measure and integration, explain how the RMK Theorem can be used to prove the existence of a nonzero Haar measure on any locally compact group, and use the RMK Theorem to identify the dual space of ℓ^∞.

The hope is that by giving two very different proofs of the theorem at two very different levels of generality, we will be able to understand why the theorem is true, and how to apply the theorem, better than if we just gave one proof at one level of generality. This approach also allows us to see the RMK Theorem as an application itself of the big theorems from measure theory, topology, and functional analysis mentioned above. Moreover, readers who have studied functional analysis but not measure theory, or measure theory and point-set topology but not functional analysis, may still read much of the paper. Even for sections that assume some background knowledge of one of these areas, we briefly present the relevant special definitions and theorems that the reader may not be acquainted with.

Lastly, we mention that there is another, very different and beautiful, proof of the RMK Theorem for the case of a compact Hausdorff space, which we do not present here. This proof is due to D.J.H. Garling and utilizes Stonean spaces and the Stone-Čech compactification; cf. [7].

2. Preliminaries for the Case $X = [a, b]$

Fix $a, b \in \mathbb{R}$ such that $a < b$. Let $B[a, b]$ denote the complex Banach space of bounded functions from $[a, b]$ into $\mathbb{C}$ under pointwise addition/scaling and the uniform norm. Let $C[a, b]$ denote the Banach subspace of $B[a, b]$ consisting of the collection of functions in $B[a, b]$ that are continuous. By an abuse of notation, we write $\|\cdot\|$ for both the norm on these spaces as well as the operator norm on their dual spaces. Given any subset A of $[a,b]$, we write χ_A for the characteristic function of A on $[a,b]$.

Our main goal at the present is to prove the RMK Theorem in the case that $X=[a,b]$. In this classical case, we simply call the theorem Riesz's Theorem. It should not be confused with the similarly named and probably even more famous Riesz Representation Theorem for Hilbert spaces, which also concerns bounded linear functionals. However, there does exist a similarity between the two theorems that we explain at the end of the next section. To make matters worse, Riesz was also the first to prove the theorem identifying the dual space of L^p for $p\in[1,\infty)$, a result which also sometimes gets called Riesz's Theorem; that result is not discussed here.

We do not follow Riesz's original proof of Riesz's Theorem— functional analysis was still nascent in 1909. See [2] for an excellent modern breakdown of Riesz's original proof, and see [3] for a translation of Riesz's proof into English. Interestingly though, Riesz published alternate proofs of the theorem many times later in his life, the last time being in 1952 with a functional-analytic proof that used a variant of the Hahn-Banach Theorem; cf. [13]. This is pretty much the proof which we present.

Measure theory and Lebesgue integration were also nascent in 1909. Instead of using those notions, Riesz formulated his theorem in terms of Riemann-Stieltjes integrals and functions of bounded variation. We do the same, but afterwards, we also explain how to translate this classical formulation into modern measure-theoretic words. Thus, our exposition begins with Riemann-Stieltjes integrals and functions of bounded variation, but since we are focused on Riesz's Theorem itself, and we don't need a deep understanding of these concepts for that, our presentation of them is very limited. See Chapters 13 and 14 of N.L. Carothers' book [4] for an excellent and highly detailed treatment of these topics (although Carothers restricts his attention to functions which take values just in $\mathbb{R}$ rather than $\mathbb{C}$, everything goes through for the latter class of functions by making the expected changes).

Definition 2.1. Let $f:[a,b]\to\mathbb{C}$ and $w:[a,b]\to\mathbb{C}$ be two functions. For any partition $P=(t_0,\dots,t_n)$ of $[a,b]$, and any selection of points $A=(a_1,\dots,a_n)$ chosen so that $a_k\in[t_{k-1},t_k]$ for all $k\in\{1,\dots,n\}$, we define the *Riemann-Stieltjes sum* of f with respect to w, P, and A to be

$$S(f,w,P,A)=\sum_{k=1}^{n} f(a_k)[w(t_k)-w(t_{k-1})].$$

We say that f is *Riemann-Stieltjes integrable* with respect to w if there exists some $I\in\mathbb{C}$ such that for every positive real number ε, there exists a partition P of $[a,b]$ satisfying

$$|S(f,w,Q,A)-I|<\varepsilon$$

for any refinement Q of P and any sampling A of $[a,b]$ chosen with respect to Q as above. Observe that such an I is unique. Consequently, in this case, we define the *Riemann-Stieltjes integral* of f with respect to w to be I and write

$$I=\int_a^b f\ dw.$$

Definition 2.2. Let $w : [a, b] \to \mathbb{C}$ be a function. We define the *total variation* of w to be the extended real number

$$\mathrm{var}(w) = \sup_{(t_0,\dots,t_n)\in P[a,b]} \sum_{k=1}^{n} |w(t_k) - w(t_{k-1})|,$$

where $P[a, b]$ denotes the collection of partitions of $[a, b]$. We say that w is of *bounded variation* if $\mathrm{var}(w)$ is finite.

Let $BV[a, b]$ denote the collection of functions of bounded variation from $[a, b]$ into $\mathbb{C}$.

Proposition 2.3. *Let $f \in C[a, b]$ and $w \in BV[a, b]$.*

(a) The function f is Riemann-Stieltjes integrable with respect to w.
(b) The following estimate holds:

$$\left| \int_a^b f \; dw \right| \le \mathrm{var}(w)\|f\|.$$

(c) If $(P_n = (t_{0,n}, \dots, t_{p_n,n}))$ is a sequence of partitions of $[a, b]$ such that $\mathrm{mesh}(P_n) \to 0$, then

$$S(f, w, P_n, (t_{0,n}, \dots, t_{p_n-1,n})) \to \int_a^b f \; dw \quad \text{as} \quad n \to \infty,$$

where

$$\mathrm{mesh}(P_n) := \max\{t_{1,n} - t_{0,n}, \dots, t_{p_n,n} - t_{p_n-1,n}\}.$$

See Corollary 14.13 and Theorem 14.16 in [4] for proofs of (a) and (b) respectively, and Exercises 49 and 66 in Chapter 14 of the same book for a method of proving (c).

Proposition 2.4.

(a) The set $BV[a, b]$ forms a complex vector space under pointwise addition and pointwise scaling.
(b) Setting $\|w\|_{BV} = |w(a)| + \mathrm{var}(w)$ for all $w \in BV[a, b]$ defines a norm on $BV[a, b]$.
(c) The Riemann-Stieltjes integral $I : C[a, b] \times BV[a, b] \to \mathbb{C}$ given by

$$I(f, w) = \int_a^b f \; dw \quad \text{for all} \quad (f, w) \in C[a, b] \times BV[a, b]$$

is a bilinear form that is bounded in both arguments.

See Lemma 13.3 and Corollary 14.17 in [4] for a proof of this proposition. The reason for the term $|w(a)|$ in (b) is that without it we only get a seminorm. Something else interesting to know, but that we will not use, is that $\|\cdot\|_{BV}$ makes $BV[a, b]$ into a Banach space, and this would not be true if we were to equip $BV[a, b]$ with the uniform norm instead; cf. Theorem 13.4 and the discussion preceding it in [4].

Now, let $BV_0^+[a, b]$ denote the collection of functions $w \in BV[a, b]$ that are right-continuous on (a, b) and satisfy $w(a) = 0$. Unlike the other notation used in this paper, the notation $BV_0^+[a, b]$ is not standard; sometimes $NBV[a, b]$ is used, the N standing for “normalized.” Observe that $BV_0^+[a, b]$ is a subspace of $BV[a, b]$, and that the inherited norm on $BV_0^+[a, b]$ is simply the total variation function.

Lemma 2.5.

(a) If $w \in BV[a,b]$ satisfies $w(a) = 0$, then there exists some $\widetilde{w} \in BV_0^+[a,b]$ such that

$$\operatorname{var}(\widetilde{w}) \leq \operatorname{var}(w) \quad \textit{and} \quad \int_a^b f \, dw = \int_a^b f \, d\widetilde{w} \quad \textit{for all} \quad f \in C[a,b].$$

(b) If $w, u \in BV_0^+[a,b]$ satisfy

$$\int_a^b f \, dw = \int_a^b f \, du \quad \textit{for all} \quad f \in C[a,b],$$

then $w = u$.

See Exercise 52 in Chapter 14 of [4] for a method of proving (a), the key ingredient being the *Jordan decomposition* of a function of bounded variation, as is discussed in that chapter. See Exercise 53 in the same chapter for the statement of (b).

As a final preliminary, recall the following variant of the all-important Hahn-Banach Theorem from functional analysis, which is the key ingredient in our proof of Riesz's Theorem.

Theorem 2.6 (Hahn-Banach Extension Theorem)**.** *Every bounded linear functional T defined on a subspace of a normed space X can be extended to a bounded linear functional on $\widetilde{T}$ defined on the full space X such that $\|\widetilde{T}\| = \|T\|$.*

See Theorem 4.3-2 in [9] for a proof. However, note that this proof of the Hahn-Banach Theorem, which is by far the most common one, crucially uses Zorn's Lemma, a proposition that is equivalent to the Axiom of Choice under Zermelo-Frankel set theory.

3. The Case $X = [a,b]$

Theorem 3.1 (Riesz's Theorem)**.** *For every bounded linear functional T on $C[a,b]$, there exists a unique $w \in BV_0^+[a,b]$ such that $\operatorname{var}(w) = \|T\|$ and*

$$Tf = \int_a^b f \, dw \quad \textit{for all} \quad f \in C[a,b].$$

This landmark theorem gives us an identification of the dual space of $C[a,b]$. Indeed, writing $(C[a,b])'$ for the (continuous) dual space of $C[a,b]$, we can define a map $\varphi : (C[a,b])' \to BV_0^+[a,b]$ by sending each $T \in (C[a,b])'$ to the unique function $w \in BV_0^+[a,b]$ associated to T via Riesz's Theorem. Notice that φ is linear because of the linearity of the Riemann-Stieltjes integral in its integrator (second) argument. Riesz's Theorem also assures us that φ is an isometry because $\operatorname{var}(w) = \|T\|$. In particular, φ is injective. Also, if we start from a $w \in BV_0^+[a,b]$, then setting $Tf = \int_a^b f \, dw$ for all $C[a,b]$ defines a $T \in (C[a,b])'$ because of the linearity and boundedness of the Riemann-Stieltjes integral in its integrand (first) argument. Hence, φ is surjective as well. Altogether, we conclude the following.

Corollary 3.2. *The correspondence φ provided by Riesz's Theorem is an isometric isomorphism between $(C[a,b])'$ and $BV_0^+[a,b]$. In particular, the dual space of $C[a,b]$ can be identified with $BV_0^+[a,b]$.*

Since a dual space is always complete, we get another corollary for free.

Corollary 3.3. *The space $BV_0^+[a,b]$ is Banach.*

Before giving our proof of Riesz's Theorem, which is mostly a reproduction of the proof given for Theorem 4.4-1 in [9], let us first explain the overarching ideas. Continuous functions can be uniformly approximated by step functions, which are by definition finite linear combinations of characteristic functions of intervals, so pretend for a moment that T is also defined on these characteristic functions to see how T should act on them. At the very least, we want to find a function $w : [a, b] \to \mathbb{C}$ such that

$$T\chi_{[a,x]} = \int_a^b \chi_{[a,x]} \; dw$$

for all $x \in (a, b]$. Observe that if w is right-continuous on (a, b), then for each $x \in (a, b]$, the characteristic function $\chi_{[a,x]}$ is Riemann-Stieltjes integrable with respect to w and

$$\int_a^b \chi_{[a,x]} \; dw = w(x) - w(a).$$

Also note that this is not true if we replace right-continuity with left-continuity. For example, consider $[a, b] = [0, 1]$, $x = 1/2$, and the left-continuous function $w = \chi_{[0,1/2]}$ (as both the integrand and integrator). In this case, $\chi_{[0,1/2]}$ is not even Riemann-Stieltjes integrable with respect to w.

If we also require that $w(a) = 0$, then our formula simplifies to $w(x) = T\chi_{[a,x]}$ for all $x \in (a, b]$. Working backwards like this tells us how to define w in terms of T. However, we still have the issue that T is not actually defined on characteristic functions. We use the Hahn-Banach Theorem to bypass this issue by extending T to a bounded linear functional of the same norm defined on $B[a, b]$, a space that does include characteristic functions. The formal proof is just a matter of checking that if w is defined in this manner, then everything really works as promised. Here it is.

Proof. Since $C[a, b]$ is a subspace of $B[a, b]$, we can apply the Hahn-Banach Theorem to extend T to a bounded linear functional $\widetilde{T} : B[a, b] \to \mathbb{C}$ such that $\|\widetilde{T}\| = \|T\|$. Using this extension, we can define a map $w : [a, b] \to \mathbb{C}$ by

$$w(x) = \begin{cases} 0 & x = a \\ \widetilde{T}\chi_{[a,x]} & x \in (a, b]. \end{cases}$$

We start by showing that $\operatorname{var}(w) \leq \|T\|$. Suppose that $(t_0, \ldots, t_n)$ is a partition of $[a, b]$. Let

$$z_k = \overline{e^{i \arg(w(t_k) - w(t_{k-1}))}} \quad \text{for all} \quad k \in \{1, \ldots, n\},$$

where we make the convention of setting $\arg(0) = 0$ and we choose any convention for the argument function on $\mathbb{C} \setminus \{0\}$. With this convention, for any $z \in \mathbb{C}$, we have that $z = |z|e^{i \arg(z)}$, and so, $|z| = z/e^{i \arg(z)} = z\overline{e^{i \arg(z)}}$. Hence, by the linearity and boundedness of $\widetilde{T}$,

$$\begin{aligned}
\sum_{k=1}^{n} |w(t_k) - w(t_{k-1})| &= \sum_{k=1}^{n} z_k [w(t_k) - w(t_{k-1})] \\
&= z_1 \widetilde{T}\chi_{[a,t_1]} + \sum_{k=2}^{n} z_k [\widetilde{T}\chi_{[a,t_k]} - \widetilde{T}\chi_{[a,t_{k-1}]}] \\
&= \widetilde{T}\left(z_1 \chi_{[a,t_1]} + \sum_{k=2}^{n} z_k (\chi_{[a,t_k]} - \chi_{[a,t_{k-1}]}) \right) \\
&\leq \|\widetilde{T}\| \left\| z_1 \chi_{[a,t_1]} + \sum_{k=2}^{n} z_k (\chi_{[a,t_k]} - \chi_{[a,t_{k-1}]}) \right\| \\
&\leq \|\widetilde{T}\| = \|T\|,
\end{aligned}$$

where the final inequality is due to the fact that the function inside the norm is bounded by 1. Since our partition was arbitrary, we may therefore conclude that $\mathrm{var}(w) \leq \|T\|$. In particular, w is of bounded variation.

Next, we prove that T is given by integration against w. Suppose that $f \in C[a,b]$. Choose a sequence of partitions $(P_n = (t_{0,n}, \ldots, t_{p_n,n}))$ of $[a,b]$ such that $\mathrm{mesh}(P_n) \to 0$ as $n \to \infty$, e.g. let P_n be the partition that divides $[a,b]$ into n equal parts. For each $n \in \mathbb{N}$, define the step function

$$s_{P_n} = f(t_{0,n})\chi_{[a,t_{1,n}]} + \sum_{k=2}^{p_n} f(t_{k-1,n})[\chi_{[a,t_{k,n}]} - \chi_{[a,t_{k-1,n}]}].$$

Observe that each of these maps is in $B[a,b]$. Also notice that for each $n \in \mathbb{N}$, we have that $s_{P_n}(a) = f(t_{0,n}) = f(a)$ and that if $k \in \{1, \ldots, p_n\}$ and $x \in (t_{k-1,n}, t_{k,n}]$, then $s_{P_n}(x) = f(t_{k-1,n})$. Since f is uniformly continuous, we therefore see that

$$\|s_{P_n} - f\| = \sup_{x \in [a,b]} |s_{P_n}(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

That is, $s_{P_n} \to f$ in $B[a,b]$ as $n \to \infty$, which implies that

$$\widetilde{T}s_{P_n} \to \widetilde{T}f = Tf \quad \text{as} \quad n \to \infty$$

because $\widetilde{T}$ a bounded extension of T. As such, once we show that $\widetilde{T}s_{P_n} \to \int_a^b f \, dw$ as $n \to \infty$ too, we will have established the fact that T is given by integration against w. Indeed, after applying linearity and our definition of w, the terms of the sequence reveal themselves to simply be Riemann-Stieltjes sums:

$$\begin{aligned}
\widetilde{T}s_{P_n} &= \widetilde{T}\left(f(t_{0,n})\chi_{[a,t_{1,n}]} + \sum_{k=2}^{p_n} f(t_{k-1})[\chi_{[a,t_{k,n}]} - \chi_{[a,t_{k-1,n}]}]\right) \\
&= f(t_{0,n})\widetilde{T}\chi_{[a,t_{1,n}]} + \sum_{k=2}^{p_n} f(t_{k-1,n})[\widetilde{T}\chi_{[a,t_{k,n}]} - \widetilde{T}\chi_{[a,t_{k-1,n}]}] \\
&= f(t_{0,n})w(t_{1,n}) + \sum_{k=2}^{p_n} f(t_{k-1,n})[(w(t_{k,n}) - w(t_{k-1,n})] \\
&= \sum_{k=1}^{p_n} f(t_{k-1,n})[(w(t_{k,n}) - w(t_{k-1,n})] \\
&= S(f, w, P_n, (t_{0,n}, \ldots, t_{p_{n-1},n})) \\
&\to \int_a^b f\ dw \quad \text{as} \quad n \to \infty,
\end{aligned}$$

where the limit at the end is due to our hypothesis that $\operatorname{mesh}(P_n) \to 0$, see Proposition 2.3(c). Hence, we have shown that

$$Tf = \int_a^b f\ dw \quad \text{for all} \quad f \in C[a,b].$$

Now, since $w \in BV[a,b]$ and $w(a) = 0$, according to Lemma 2.5(a), there exists some $\widetilde{w} \in BV_0^+[a,b]$ such that

$$\operatorname{var}(\widetilde{w}) \le \operatorname{var}(w) \le \|T\| \quad \text{and} \quad Tf = \int_a^b f\ dw = \int_a^b f\ d\widetilde{w} \quad \text{for all} \quad f \in C[a,b].$$

By an abuse of notation, we denote this modified function by w as well.

The estimate given in Proposition 2.3(b) yields that

$$|Tf| = \left|\int_a^b f\ dw\right| \le \operatorname{var}(w)\|f\| \quad \text{for all} \quad f \in C[a,b].$$

Thus, $\|T\| \le \operatorname{var}(w)$ as well, implying that $\operatorname{var}(w) = \|T\|$. Finally, the uniqueness of $w \in BV_0^+[a,b]$ follows from Lemma 2.5(b). □

Another proof of Riesz's Theorem that is based on Helly's Selection Theorem rather than on functional analysis is given in Chapter 14 of [4].

The way that we stated Riesz's Theorem does not look quite the same as setting $X = [a,b]$ in the RMK Theorem. However, it really is a special case, but of a different version of the RMK Theorem, Theorem 5.3. This can be realized via a correspondence between $BV_0^+[a,b]$ and $M[a,b]$, the space of complex Radon measures on $[a,b]$ (Section 5 has the relevant definitions for this space). This correspondence works through notion of *Borel-Stieltjes measures*, the details are in Section 3.5 of [6]. Furthermore, see Theorem 9.9 in [1] for a direct proof of the measure-theoretic formulation of Riesz's Theorem that is based on the Hahn Extension Theorem (a variant of the Carathéodory Extension Theorem).

Lastly, we explain the similarity between Riesz's Theorem and the Riesz Representation Theorem for Hilbert spaces.

Theorem 3.4 (Riesz Representation Theorem)**.** *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. If T is a bounded linear functional on H, then there exists a unique vector $v \in H$ such that $\|T\| = \|v\|$ and $Tx = \langle x, v \rangle$ for all $x \in H$.*

Given any Hilbert space H, there exists a measure space (X, Σ, μ) such that H is inner product isomorphic to $L^2(\mu)$, where the codomain of the functions in $L^2(\mu)$ is defined to be the scalar field of H (either $\mathbb{R}$ or $\mathbb{C}$). Hence, by an abuse of notation, the Riesz Representation Theorem is equivalent to the statement that for any bounded linear functional T on H, there exists some unique $\varphi \in L^2(\mu)$ such that $\|T\| = \|\varphi\|_2$ and

$$Tf = \int f\overline{\varphi}\, d\mu \quad \text{for all} \quad f \in L^2(\mu).$$

In this way, the Riesz Representation Theorem is also a result saying that the only bounded linear functionals on a space are integrals.

4. Preliminaries for the General Case

Before getting to our proof of the general case of the RMK Theorem, we establish the necessary measure-theoretic and topological framework. As with our treatment of Riemann-Stieltjes integration and functions of bounded variation in Section 2, we are very brief in order to get to the proof at hand.

The key result from measure theory that we need is the following variant of the Carathéodory Extension Theorem.

Theorem 4.1 (Carathéodory's Theorem for Outer Measures)**.** *If μ^* is an outer measure on a set X, then the collection of μ^*-measurable subsets of X is a σ-algebra of X, and the restriction of μ^* to this σ-algebra is a measure on X.*

See Theorem 1.11 in [6] for a proof, and see the discussion preceding the theorem for the relevant definitions.

Definition 4.2. Let X be a topological space. The *Borel algebra* of X is the σ-algebra of X generated by the collection of open subsets of X. Members of the Borel algebra of X are called *Borel subsets* of X. A *Borel measure* on X is an (unsigned extended-valued σ-additive) measure whose domain is precisely the Borel algebra of X.

The Borel algebra is the natural place to do measure theory in a general topological space, and thus, Borel measures are the natural measures to consider. See Chapter 1 of [6] for a detailed treatment of Borel measures. As with any σ-algebra, the Borel algebra induces a class of measurable functions. Namely, a function $f : X \to \mathbb{C}$ is defined to be *Borel measurable* if the inverse image of any open subset of $\mathbb{C}$ under f is a Borel subset of X. Observe that continuous functions $f : X \to \mathbb{C}$ are Borel measurable, and so are characteristic functions $\chi_E : X \to \mathbb{C}$, where E is a Borel subset of X. Therefore, if f is compactly supported, then it makes sense to take its integral with respect to a Borel measure on X, and it always makes sense to do so for χ_E.

From now on, fix an LCH space X.

Definition 4.3. A Borel measure μ on X is called a *Radon measure* if

(a) μ is *locally finite:* for any $x \in X$, there exists a neighborhood U of x in X such that $\mu(U) < \infty$.

(b) μ is *outer regular:* for any Borel subset E of X,

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ is open in } X\}.$$

(c) μ is *weakly inner regular:* for any open subset U of X,

$$\mu(U) = \sup\{\mu(K) : K \subseteq U,\ K \text{ is compact}\}.$$

To be clear, all suprema and infima above, and in what follows, are taken in the extended real line. Notice that (d) makes sense because compact sets always belong to the Borel algebra, and also note that local finiteness implies that the measure of any compact set is finite. As we mentioned in the introduction, the point of this definition is to generalize the nice topological regularity of the Lebesgue measure. See Chapter 7 of [6] for a detailed treatment of Radon measures. Be warned, however, that there are several different definitions of Radon measures in the literature. Some definitions are equivalent to ours, while others are not. Moreover, many people call what we defined to be a Radon measure a *regular Borel measure* instead, while some people have a slightly different definition for the latter. There are also notions of *outer Radon measures* and *inner Radon measures.* Finally, there are *Baire measures*, which are related to all of the above, but certainly different. It's all very confusing.

The key topological ingredient for our proof of the RMK Theorem is the following variant of Urysohn's Lemma.

Theorem 4.4 (Urysohn's Lemma for LCH Spaces)**.** *If $K \subseteq U \subseteq X$, where U is open in X, and K is compact, then there exists some $f \in C_c(X)$ such that $f(X) \subseteq [0,1]$, $f = 1$ on K, and* $\operatorname{supp}(f) \subseteq U$.

See Theorem 4.32 in [6] for a proof of this result. In our proof of Riesz's Theorem, we used the Hahn-Banach Theorem to subvert the issue that characteristic functions are not continuous by extending our functional to a space that does include characteristic functions. This issue persists when trying to prove the general RMK Theorem. However, it's no longer clear how to use Hahn-Banach to work-around it. In this case, we tackle the issue with Urysohn's Lemma, which gives us continuous approximations of characteristic functions.

We also need the following topological lemma.

Lemma 4.5. *If K is a compact subset of X and $\{U_1, \dots, U_n\}$ is an open cover of K, then there exists a partition of unity on K subordinate to $\{U_1, \dots, U_n\}$ consisting of compactly supported functions.*

See Proposition 4.41 in [6] for a proof (whose essential ingredient is Urysohn's Lemma), and see the discussion above the proposition for the definition of such a partition of unity.

Given $f, g \in C_c(X)$, we write $g \leq f$ if $(f - g)(x) \in [0, \infty)$ for all $x \in X$. We can give $C_c(X)$ the uniform norm $\|\cdot\|$ in order to make it into a normed space. The positivity condition in the RMK Theorem is therefore motivated by the following.

Proposition 4.6. *Let T be a positive linear functional on $C_c(X)$. For each compact subset K of X, there exists some $C \in \mathbb{R}$ such that $|Tf| \leq C\|f\|$ for all $f \in C_c(X)$ satisfying* $\operatorname{supp}(f) \subseteq K$.

Proof. Suppose that K is a compact subset of X. By Urysohn's Lemma, there exists some $g \in C_c(X)$ such that $g(X) \subseteq [0,1]$ and $g = 1$ on K. So, for all $f \in C_c(X)$

such that $\operatorname{supp}(f) \subseteq K$, the function $\|f\| \cdot g$ is also in $C_c(X)$, and $|f| \leq \|f\| \cdot g$. By the linearity and positivity of T, it follows that

$$T|f| \leq T(\|f\| \cdot g) = (Tg)\|f\|$$

for all such f. Now, if $f \in C_c(X)$ is real-valued, then $-|f| \leq f \leq |f|$, so $-T|f| \leq Tf \leq T|f|$, which is equivalent to the inequality $|Tf| \leq T|f|$. Thus, we deduce the desired estimate $|Tf| \leq C\|f\|$ for all real-valued $f \in C_c(X)$ such that $\operatorname{supp}(f) \subseteq K$, where $C = Tg \in \mathbb{R}$.

This in turn implies the general complex-valued case. Indeed, suppose that $f \in C_c(X)$ satisfies $\operatorname{supp}(f) \subseteq K$. Since $\operatorname{supp}(\operatorname{Re} f)$ and $\operatorname{supp}(\operatorname{Im} f)$ are closed subsets of the compact set $\operatorname{supp}(f)$, we have that $\operatorname{Re} f, \operatorname{Im} f \in C_c(X)$, where $\operatorname{supp}(\operatorname{Re} f)$ and $\operatorname{supp}(\operatorname{Im} f)$ are subsets of K as well, so there exist $C_1, C_2 \in \mathbb{R}$ such that

$$\begin{aligned} |Tf| = |T(\operatorname{Re} f + i \operatorname{Im} f)| \leq |T(\operatorname{Re} f)| + |T(\operatorname{Im} f)| &\leq C_1\|\operatorname{Re} f\| + C_2\|\operatorname{Im} f\| \\ &\leq (C_1 + C_2)|\|f\|. \end{aligned}$$

□

Corollary 4.7. *If X is compact, then any positive linear functional on $C_c(X)$ is bounded.*

(Remember that we already supposed that X is Hausdorff.) The reverse inclusion does not hold. For example, consider the compact Hausdorff space $[0, 1]$, the function of bounded variation $w : [0, 1] \to \mathbb{C}$ defined by $w(x) = -x$ for all $x \in [0, 1]$, and the linear functional T on $C_c[0, 1] = C[0, 1]$ defined by $Tf = \int_0^1 f \, dw$ for all $f \in C[0, 1]$. This functional is bounded, and $\chi_{[0,1]} \geq 0$, but $T\chi_{[0,1]} = -1 < 0$.

5. The General Case

We now come to our proof of the general case of the RMK Theorem, as stated in Theorem 1.1. This proof is mostly a reproduction of the proof given for Theorem 7.2 in [6].

Proof. Throughout this proof, given an open subset U of X and a function $f \in C_c(X)$, we write $f \prec U$ if $f(X) \subseteq [0, 1]$ and $\operatorname{supp}(f) \subseteq U$. Also, given any subset A of X, we write χ_A for the characteristic function of A on X.

We begin by proving uniqueness. Suppose that λ is a Radon measure on X such that $Tf = \int f \, d\lambda$ for all $f \in C_c(X)$. Let U be an open subset of X. If $f \in C_c(X)$ satisfies $f \prec U$, then $f \leq \chi_U$, so

$$Tf = \int f \, d\lambda \leq \int \chi_U \, d\lambda = \lambda(U) = \sup\{\lambda(K) : K \subseteq U, \ K \text{ compact}\},$$

where the final equality is due to the weak inner regularity of λ. On the other hand, if K is a compact subset of U, Urysohn's Lemma tells us that there exists an $f \in C_c(X)$ such that $f \prec U$ and $f = 1$ on K. Thus,

$$\lambda(K) = \int_K f \, d\lambda \leq \int f \, d\lambda = Tf.$$

So,

$$\sup\{Tf : f \in C_c(X), \ f \prec U\} = \sup\{\lambda(K) : K \subseteq U, \ K \text{ compact}\} = \lambda(U).$$

As such, we have shown that the action of λ on the open subsets of X is determined by T. By the outer regularity of λ, it follows that λ is entirely determined by T.

Like what we saw in the $[a,b]$ case, this uniqueness argument also suggests how to define such a Radon measure. First, set

$$\mu(U) = \sup\{Tf : f \prec U,\ f \in C_c(X)\}$$

for all open subsets U of X (note that all of the Tf's there are real due to the positivity of T). Then, define $\mu^* : \mathcal{P}(X) \to [0,\infty]$ by

$$\mu^*(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ open in } X\}$$

for all $E \in \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the power set of X. The outline of our proof is now as follows.

Step (i): show that μ^* is an outer measure on X. **Step (ii)**: show that every open subset of X is μ^*-measurable. Assuming those two results, the Carathéodory Extension Theorem implies that every Borel subset of X is μ^*-measurable and that the restriction of μ^* to the Borel algebra of X is a measure on X. We can denote this new restricted measure by μ because μ^* agrees with our previously defined μ on all open subsets of X. Notice that μ is an outer regular Borel measure by definition. **Step (iii)**: show that

$$\mu(K) = \inf\{Tf : f \geq \chi_K,\ f \in C_c(X)\}$$

for all compact subsets K of X. By Urysohn's Lemma, for each compact subset K of X, there exists some $f \in C_c(X)$ such that $f(X) \subseteq [0,1]$ and $f = 1$ on K. Hence, $f \geq \chi_K$. As T maps into $\mathbb{C}$ by definition, this implies that $\mu(K) < \infty$. Therefore, by the local compactness of X, we have that μ is locally finite. **Step (iv)**: use the identity derived in the previous step to show that μ is weakly inner regular. **Step (v)**: show that

$$Tf = \int f\ d\mu \quad \text{for all} \quad f \in C_c(X).$$

Once we complete each of these steps, the proof is done.

Step (i). As $T(0) = 0$ (one zero being the function and one being the number), we have that $\mu(\emptyset) = 0$. If we show that $\mu(\bigcup_{j=1}^\infty U_j) \leq \sum_{j=1}^\infty \mu(U_j)$ for any sequence (U_j) of open subsets of X, then we get that

$$\begin{aligned}
\mu^*(E) &= \inf\{\mu(U) : E \subseteq U,\ U \text{ open in } X\} \\
&= \inf\left\{\mu\left(\bigcup_{j=1}^\infty U_j\right) : (U_j) \text{ is an open cover of } E \text{ in X}\right\} \\
&\leq \inf\left\{\sum_{j=1}^\infty \mu(U_j) : (U_j) \text{ is an open cover of } E \text{ in X}\right\}
\end{aligned}$$

for all subsets E of X. However, the inequality above is actually an equality because we can just take the trivial open cover $\{X\}$ of E to contradict the possibility of strict inequality. Thus, if we show that $\mu(\bigcup_{j=1}^\infty U_j) \leq \sum_{j=1}^\infty \mu(U_j)$ for any sequence (U_j) of open subsets of X, then we get that

$$\mu^*(E) = \inf\left\{\sum_{j=1}^\infty \mu(U_j) : (U_j) \text{ an open cover of } E \text{ in X}\right\},$$

which implies that μ^* is an outer measure on X; cf. Proposition 1.10 in [6]— this blueprint for the construction of an outer measure from an elementary family works for arbitrary sets. So, suppose that (U_j) is a sequence of open subsets of X. Assume that $f \in C_c(X)$ satisfies $f \prec \bigcup_{j=1}^{\infty} U_j$. Since $\operatorname{supp}(f)$ is compact, there exists some $n \in \mathbb{N}$ such that $\{U_1, \ldots, U_n\}$ still covers $\operatorname{supp}(f)$. By Lemma 4.5, it follows that there exist $g_1, \ldots, g_n \in C_c(X)$ such that $\sum_{j=1}^{n} g_j = 1$ on $\operatorname{supp}(f)$ and $g_j \prec U_j$ for all $j \in \{1, \ldots, n\}$. Hence, $f \leq \sum_{j=1}^{n} g_j$, so due to the positivity and linearity of T, we have

$$Tf \leq T\left(\sum_{j=1}^{n} g_j\right) = \sum_{j=1}^{n} Tg_j \leq \sum_{j=1}^{n} \mu(U_j) \leq \sum_{j=1}^{\infty} \mu(U_j).$$

As such, we may conclude that $\mu(\bigcup_{j=1}^{\infty} U_j) \leq \sum_{j=1}^{\infty} \mu(U_j)$.

Step (ii). It suffices to show that if U is open in X, and E is any subset of X such that $\mu^*(E) < \infty$, then

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U);$$

this is because the reverse inequality holds for any outer measure and any two sets, and the case where $\mu^*(E) = \infty$ is trivial. In order to establish this estimate, we start by just considering the case where E is also open in X. It follows that $E \cap U$ is open in X. Suppose that ε is a positive real number. By the definition of $\mu(E \cap U)$, there exists some $f \in C_c(X)$ such that $f \prec E \cap U$ and $Tf > \mu(E \cap U) - \varepsilon$. As X is Hausdorff, the set $E \setminus \operatorname{supp}(f)$ is also open in X, so likewise, there exists some $g \in C_c(X)$ such that $g \prec E \setminus \operatorname{supp}(f)$ and $Tg > \mu(E \setminus \operatorname{supp}(f)) - \varepsilon$. Thus, $f + g \in C_c(X)$ with $f + g \prec E$, which then implies that

$$\begin{aligned} \mu^*(E) = \mu(E) \geq T(f+g) = Tf + Tg &\geq \mu(E \cap U) + \mu(E \setminus \operatorname{supp}(f)) - 2\varepsilon \\ &\geq \mu^*(E \cap U) + \mu^*(E \setminus \operatorname{supp}(f)) - 2\varepsilon \\ &\geq \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon. \end{aligned}$$

Sending $\varepsilon \to 0$, we obtain the desired inequality. Now, for the general case, since $\mu^*(E) < \infty$, there exists an open subset V of X such that $E \subseteq V$ and $\mu(V) < \mu^*(E) + \varepsilon$. Thus, by what we showed for the open case,

$$\mu^*(E) + \varepsilon > \mu(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U) \geq \mu^*(E \cap U) + \mu^*(E \setminus U).$$

Again, as ε is arbitrary, this establishes the desired general inequality.

Step (iii). Let K be a compact subset of X. First, suppose that $f \in C_c(X)$ satisfies $f \geq \chi_K$ Let ε be a positive real number such that $\varepsilon < 1$. Furthermore, let $U = \{x \in X : f(x) > 1-\varepsilon\}$. Note that U_ε is open in X. If $g \in C_c(X)$ satisfies $g \prec U$, then $g \leq (1-\varepsilon)^{-1} f$, so $Tg \leq (1-\varepsilon)^{-1} Tf$. Hence, $\mu(K) \leq \mu(U) \leq (1-\varepsilon)^{-1} Tf$. Sending $\varepsilon \to 0$, this yields that $\mu(K) \leq Tf$. Consequently,

$$\mu(K) \leq \inf\{Tf : f \geq \chi_K,\ f \in C_c(X)\}.$$

For the reverse inequality, note that by the definition of $\mu(K)$, there exists some open subset U of X such that $K \subseteq U$ and $\mu(K) + \varepsilon \geq \mu(U)$. By Urysohn's Lemma, there then exists some $f \in C_c(X)$ such that $f \geq \chi_K$ and $f \prec U$. Hence, $\mu(U) \geq Tf$, which then gives us that

$$\mu(K) + \varepsilon \geq \inf\{Tf : f \geq \chi_K,\ f \in C_c(X)\}.$$

Sending $\varepsilon \to 0$ again, we get the reverse inequality.

Step (iv). Suppose that U is open in X. Assume that $f \in C_c(X)$ and $f \prec U$. Let $A = \operatorname{supp}(f)$. Observe that $f \leq g$ for all $g \in C_c(X)$ such that $g \geq \chi_A$. As

T is positive and linear, it follows that $Tf \leq Tg$ for all such g. Thus, since A is compact, we can use the identity derived in the previous step to yield that

$$Tf \leq \inf\{Tg : g \geq \chi_A,\ C_c(X)\} = \mu(A) \leq \sup\{\mu(K) : K \subseteq U,\ K \text{ compact}\}.$$

Therefore,

$$\mu(U) = \sup\{Tf : f \prec U,\ f \in C_c(X)\} \leq \sup\{\mu(K) : K \subseteq U,\ K \text{ compact}\}.$$

On the other hand, suppose that K is a compact subset of U. By Urysohn's Lemma, there exists some $g \in C_c(X)$ such that $g \geq \chi_K$ and $g \prec U$. Hence, reusing the identity derived in the previous step,

$$\mu(K) = \inf\{Tf : f \in C_c(X),\ f \geq \chi_K\} \leq Tg \leq \sup\{Tf : f \in C_c(X),\ f \prec U\} = \mu(U).$$

Therefore,

$$\sup\{\mu(K) : K \subseteq U,\ K \text{ compact}\} \leq \mu(U)$$

as well. Combining our estimates, we deduce that μ is weakly inner regular.

Step (v). Let $C_c(X \to [0,1])$ denote the subset of functions in $C_c(X)$ whose image is contained in $[0,1]$. The linear span of $C_c(X \to [0,1])$ is $C_c(X)$. Thus, it suffices to show that $Tf = \int f\ d\mu$ for all $f \in C_c(X \to [0,1])$. To that end, suppose that $f \in C_c(X \to [0,1])$. Let $N \in \mathbb{N}$. Set $K_0 = \operatorname{supp}(f)$ and $K_j = \{x \in X : f(x) \geq j/N\}$ for all $j \in \{1, \ldots, N\}$. It is given that K_0 is compact, and moreover, for each $j \in \{1, \ldots, N\}$, the set K_j is a closed subset of K_0, and so, K_j is likewise compact. Hence, we can define a function $f_j \in C_c(X)$ by

$$f_j(x) = \begin{cases} 0 & x \notin K_{j-1} \\ f(x) - \frac{j-1}{N} & x \in K_{j-1} \setminus K_j \\ \frac{1}{N} & x \in K_j \end{cases}$$

for all $x \in X$ (notice that $\operatorname{supp}(f_j) \subseteq K_{j-1}$). As $\frac{\chi_{K_j}}{N} \leq f_j \leq \frac{\chi_{K_{j-1}}}{N}$, we have that

$$\frac{\mu(K_j)}{N} \leq \int f_j\ d\mu \leq \frac{\mu(K_{j-1})}{N}.$$

Also, as

$$\mu(K_j) = \inf\{Tg : g \in C_c(X),\ g \geq \chi_{K_j}\},$$

we have that $\mu(K_j) \leq N \cdot Tf_j$. And furthermore, if U is an open subset of X containing K_{j-1}, then $Nf_j \prec U$, so $Tf_j \leq \mu(U)/N$. Thus,

$$\frac{\mu(K_j)}{N} \leq Tf_j \leq \frac{\mu(K_{j-1})}{N}$$

as well. Lastly, note that $f_1 + \cdots + f_N = f$, so

$$\frac{1}{N}\sum_{j=1}^{N} \mu(K_j) \leq \int f\, d\mu \leq \frac{1}{N}\sum_{j=0}^{N-1} \mu(K_j) \quad \text{and} \quad \frac{1}{N}\sum_{j=1}^{N} \mu(K_j) \leq Tf \leq \frac{1}{N}\sum_{j=0}^{N-1} \mu(K_j).$$

This implies that

$$\left|Tf - \int f\ d\mu\right| \leq \frac{\mu(K_0) - \mu(K_N)}{N} \leq \frac{\mu(K_0)}{N}.$$

Since μ is locally finite, $\mu(K_0) < \infty$. Therefore, taking $N \to \infty$, we may conclude that

$$Tf = \int f\ d\mu.$$

□

We conclude this section with a useful variant of the RMK Theorem. Let $C_0(X)$ denote the collection of functions in $C(X)$ that *vanish at infinity,* i.e. the $f \in C(X)$ such that for every positive real number ε, there exists a compact subset K of X such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$. Observe that this is a subspace of $C(X)$. In fact, equipping $C(X)$ with the uniform norm, we have that $C_0(X)$ is the closure of $C_c(X)$ in $C(X)$; cf. Proposition 4.35 in [6]. For this last bit of the section, we assume a working knowledge of complex Borel measures, as in Section 3.3 of [6].

Definition 5.1. A *signed Radon measure* on X is a signed Borel measure on X whose positive and negative variations are Radon. A *complex Radon measure* on X is a complex Borel measure on X whose real and imaginary parts are signed Radon measures.

Let $M(X)$ denote the collection of complex Radon measures on X.

Proposition 5.2. *The set $M(X)$ is a complex vector space under pointwise addition/scaling and the norm given by $\|\mu\|_M = |\mu|(X)$ for all $\mu \in M(X)$, where $|\mu|$ denotes the total variation of μ.*

See Proposition 7.6 in [6] for a proof.

Theorem 5.3 (Riesz-Markov-Kakutani Representation Theorem for C_0)**.** *If T is a bounded linear functional on $C_0(X)$, then there exists a unique complex Radon measure μ on X such that $\|T\| = \|\mu\|_M$ and*

$$Tf = \int f \; d\mu \quad \text{for all} \quad f \in C_0(X).$$

See Theorem 7.17 in [6] for a proof, which is quite short having already proved the RMK Theorem for C_c. As with functions of bounded variation, the key extra fact needed here is a *Jordan decomposition*: functionals on $C_0(X)$ can be decomposed into a sum of positive ones.

Since $C_0(X) = C(X)$ when X is compact, this yields the following corollary. (Again, remember that we already supposed that X is Hausdorff.)

Corollary 5.4. *The correspondence between $[C_0(X)]'$ and $M(X)$ provided by the RMK Theorem is an isometric isomorphism. In particular, if X is compact, then the dual space of $C(X)$ can be identified with $M(X)$.*

What a beautiful result.

Corollary 5.5. *The space $M(X)$ is Banach.*

6. Applications

6.1. Alternate Approach to Measure Theory.

The RMK Theorem gives an alternate abstract construction of *the* Borel measure on $\mathbb{R}$ (i.e. the restriction of the Lebesgue measure on $\mathbb{R}$ to its Borel algebra). Indeed, let $T : C_c(\mathbb{R}) \to \mathbb{R}$ be the Riemann integral over $\mathbb{R}$, given by $Tf = \int_{-\infty}^{\infty} f(x)\; dx$ for all $f \in C_c(\mathbb{R})$. Since $\mathbb{R}$ is an LCH space and T is a positive linear functional, the RMK Theorem tells us that there exists a Radon measure μ on $\mathbb{R}$ such that $\int f \; d\mu = \int_{-\infty}^{\infty} f(x)\; dx$ for all $f \in C_c(\mathbb{R})$. Now, let $a, b \in \mathbb{R}$ be such that $a \leq b$, and consider the characteristic function $\chi_{[a,b]} : \mathbb{R} \to \mathbb{R}$. Choose a sequence of functions (f_k) in $C_c(\mathbb{R})$ such that $f_k \to \chi_{[a,b]}$ pointwise as $k \to \infty$, e.g. "tent-shaped" functions that use increasingly steep lines to connect 0 and 1 near a and b. By the Dominated Convergence Theorem,

$$\begin{aligned}\mu[a,b] = \int \chi_{[a,b]}\ d\mu = \lim_{k\to\infty} \int f_k\ d\mu = \lim_{k\to\infty} \int_{-\infty}^{\infty} f_k(x)\ dx &= \int_{-\infty}^{\infty} \chi_{[a,b]}(x)\ dx \\ &= b-a.\end{aligned}$$

Next, weak inner regularity comes in to give us that μ agrees with the Borel measure on all open subsets of $\mathbb{R}$. Furthermore, outer regularity therefore implies that μ agrees with the Borel measure on all Borel subsets of $\mathbb{R}$. Therefore, μ must be the Borel measure on $\mathbb{R}$. This is good enough because in the words of the great contemporary analyst Barry Simon, "passing from Borel to Lebesgue measurable functions is the work of the devil. Don't even consider it!" [14]. An analogous approach works for any $\mathbb{R}^n$.

Additionally, by the RMK Theorem, we could equivalently define a Radon measure on an LCH space X to be a positive linear functional on $C_c(X)$. This functional-analytic approach to measure theory bypasses the usual mound of definitions and step-by-step constructions, immediately getting to the heart of the matter. Although this approach does not work for general measure spaces, the class of LCH spaces is so broad and ubiquitous that it is essentially good enough. On the other hand, whether it is pedagogically sound to do this is another issue.

6.2. **Haar Measures.**

Definition 6.1. Let G be a topological group, whose group operation we denote with multiplicative notation. We say that a Radon measure μ on G is a *left Haar measure* on G if it is left G-invariant, i.e. for all $x \in G$ and all Borel subsets E of G, we have that $\mu(xE) = \mu(E)$. A *right Haar measure* on G is defined analogously.

To give just the archetypal example, the Borel measure on $\mathbb{R}^n$ is a left and right Haar measure on the abelian topological group $(\mathbb{R}^n, +)$.

The following seminal result about Haar measures can be most easily proved by using the RMK Theorem.

Theorem 6.2. *If G is a locally compact group, then there exists a nonzero left Haar measure on G that is unique up to multiplication by a positive real constant.*

To be clear, by *nonzero* here we mean that $\mu(E) \neq 0$ for some Borel subset E of G. Also, a *locally compact group* actually refers to a topological group whose topology is locally compact and Hausdorff— there is a good justification for this seemingly perverse naming convention, but we don't need to get into that here. Theorem 6.2 is also true if we replace "left" with "right," choosing the former is just a convention. Let us sketch how to use the RMK Theorem to prove the existence part of Theorem 6.2, following the presentation in [5] (uniqueness is an independent matter for which the RMK Theorem is not used).

The rough idea of why such Haar measure should exist is as follows. Given an open subset U of G, we can measure the size of any subset A of G relative to U by the "minimal number" $(A : U)$ of translates xU needed to cover A, where $x \in G$. Notice that this relative measure is left G-invariant: $(gA : U) = (A : U)$ for all $g \in G$. Also, if A is compact, then $(A : U)$ is finite. As U shrinks to a point, the quantity $(A : U)$ should converge to a real number, and it makes intuitive sense to define this number to be the measure of A. However, what we do instead is fix a compact subset K of X, and define the measure of A to be the limit of the ratio $(A : U)/(K : U)$ as U shrinks to a point. This normalizes the measure so that the measure of K is 1. Note how this idea generalizes the definition of the Lebesgue outer measure on $\mathbb{R}^n$ and even the general construction of an outer measure from an elementary family in an arbitrary set.

However, it is quite hard to verify that this purported limit does exist and defines a measure. Instead, we convert the idea to the setting of positive linear functionals, which turn out to be easier to work with. Namely, by the RMK Theorem, Theorem 6.2 would follow if we found a nonzero positive linear functional $T : C_c(G) \to \mathbb{C}$ that is left-invariant in the sense that $T(L_x f) = Tf$ for all $x \in G$ and $f \in C_c(G)$, where the left translation $L_x f : G \to \mathbb{C}$ is defined by $(L_x f)(y) = f(x^{-1}y)$ for all $y \in G$. The approach described above translates to this setting by defining

$$(f : g) = \inf \left\{ \sum_{j=1}^{n} c_j : c_1, \ldots, c_n \in (0, \infty),\ s_1, \ldots, s_n \in G,\ f \leq \sum_{j=1}^{n} c_j L_{s_j} g \right\}$$

for all $f, g \in C_c(G)$ such that $f, g \geq 0$ and $g \neq 0$. We then fix a nonzero $f_0 \in C_c(G)$ such that $f_0 \geq 0$ and consider the limit of $(f : \varphi)/(f_0 : \varphi)$ as $\operatorname{supp}(\varphi)$ shrinks to $\{1\}$. In some sense, this limit converges to our desired Tf. It turns out that this is much easier to verify, the details are given in [5].

One major application of Theorem 6.2 is that it allows us to generalize the definition of the Fourier transform to general locally compact abelian groups by defining it via the guaranteed nonzero left Haar measure on the group. This is the beginning of the profound and modern subject of abstract harmonic analysis. The existence of Haar measures on locally compact groups is also crucial in geometry for the theory of Lie groups. For example, John von Neumann's partial solution of Hilbert's Fifth Problem was based on a weaker version of Theorem 6.2.

6.3. The Dual of ℓ^∞.

By ℓ^∞, we mean $\ell^\infty(\mathbb{N} \to \mathbb{C})$, the Banach space of bounded complex sequences under the uniform norm. The Stone-Čech compactification of $\mathbb{N}$, denoted by $\beta\mathbb{N}$, is a compact Hausdorff space "containing" $\mathbb{N}$ as dense subspace. In particular, $C_c(\beta\mathbb{N}) = C(\beta\mathbb{N})$. It is straightforward to show that ℓ^∞ is isometrically isomorphic to $C(\beta\mathbb{N})$. Hence, the dual space of ℓ^∞ is isometrically isomorphic to the dual of $C(\beta\mathbb{N})$. And by the RMK Theorem, the latter is isometrically isomorphic to $M(\beta\mathbb{N})$. Therefore the dual space of ℓ^∞ can be identified with $M(\beta\mathbb{N})$. Similar things can be deduced about the dual of any L^∞ space.

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