Fixed Point, Hopf-Poincarère Index Theorem, Characteristic Class

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Homology and Cohomology Groups

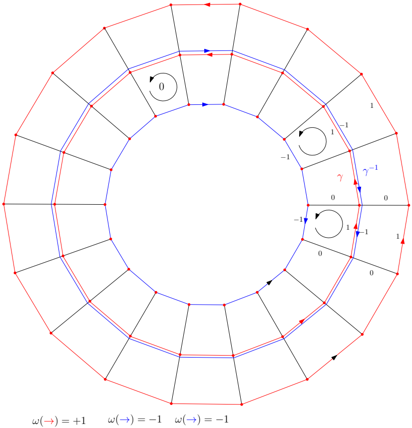


Figure: γ is the generator of $H_1(M,\mathbb{Z})$, ω is the generator of $H^1(M,\mathbb{R})$.

$d\omega = 0$ but $\int_\gamma \omega = 18$, so ω is closed but not exact.

Fixed Point

Brouwer Fixed Point

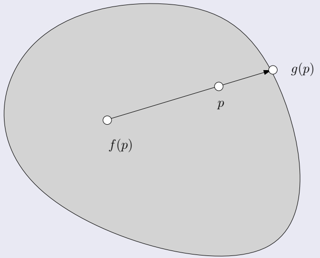


Figure: Brouwer fixed point.

Brouwer Fixed Point

Theorem (Brouwer Fixed Point)

Suppose $\Omega \subset \mathbb{R}^n$ is a compact convex set, $f : \Omega \to \Omega$ is a continous map, then there exists a point $p \in \Omega$, such that $f(p) = p$.

Proof.

Assume $f : \Omega \to \Omega$ has no fixed point, namely $\forall p \in \Omega$, $f(p) \neq p$. We construct $g : \Omega \to \partial\Omega$, a ray starting from $f(p)$ through p and intersect $\partial\Omega$ at $g(p)$, $g|_{\partial\Omega} = id$. i is the inclusion map, $(g \circ i) : \partial\Omega \to \partial\Omega$ is the identity,

$$\partial\Omega \xrightarrow{i} \Omega \xrightarrow{g} \partial\Omega$$

$(g \circ i)_{\#} : H_{n-1}(\partial\Omega, \mathbb{Z}) \to H_{n-1}(\partial\Omega, \mathbb{Z})$ is $z \mapsto z$. But $H_{n-1}(\Omega, \mathbb{Z}) = 0$, then $g_{\#} = 0$. Contradiction. $\square$

Lefschetz Fixed Point

Definition (Index of Fixed Point)

Suppose M is an n-dimensional topological space, p is a fixed point of $f: M \to M$. Choose a neighborhood $p \in U \subset M$, $f_*: H_{n-1}(\partial U, \mathbb{Z}) \to H_{n-1}(\partial U, \mathbb{Z})$,

$$f_*: \mathbb{Z} \to \mathbb{Z}, z \mapsto \lambda z,$$

where λ is an integer, the algebraic index of p, $Ind(f, p) = \lambda$.

Lefschetz Fixed Point

Given a compact topological space M, and a continuous automorophism $f : M \to M$, it induces homomorphisms

$$f_{*k} : H_k(M, \mathbb{Z}) \to H_k(M, \mathbb{Z}),$$

each f_{*k} is represented as a matrix.

Definition (Lefschetz Number)

The Lefschetz number of the automorphism $f : M \to M$ is given by

$$\Lambda(f) := \sum_k (-1)^k Tr(f_{*k} | H_k(M, \mathbb{Z})).$$

Lefschetz Fixed Point

Theorem (Lefschetz Fixed Point)

Given a continuous automorphism of a compact topological space $f : M \to M$, if its Lefschetz number is non-zero, then there is a point $p \in M$, $f(p) = p$.

Proof.

Triangulate M, use a simplicial map to approximate f, then

$$\sum_k (-1)^k Tr(f_k|C_k) = \sum_k (-1)^k Tr(f_k|H_k) = \Lambda(f). \quad (1)$$

If $\Lambda(f) \neq 0$, $\exists \sigma \in C_k$, $f_k(\sigma) \subset \sigma$, from Brouwer fixed point theorem, there is a fixed point $p \in \sigma$. $\square$

Lefschetz Fixed Point

Lemma

$$\sum_k (-1)^k Tr(f_k|C_k) = \sum_k (-1)^k Tr(f_k|H_k) = \Lambda(f).$$

Proof.

$C_k = C_k/Z_k \oplus Z_k$, Z_k is the closed chain space; $Z_k = B_k \oplus H_k$, B_k is the exact chain space, H_k is the homology group. $\partial_k : C_k/Z_k \to B_{k-1}$ is isomorphic.

$$\begin{tikzcd} C_k/Z_k \arrow[r, "f_k"] \arrow[d, "\partial_k"'] & C_k/Z_k \arrow[d, "\partial_k"] \\ B_{k-1} \arrow[r, "f_{k-1}"] & B_{k-1} \end{tikzcd}$$

$\square$

Lefschetz Fixed Point

Lemma

$$\sum_k (-1)^k Tr(f_k|C_k) = \sum_k (-1)^k Tr(f_k|H_k) = \Lambda(f).$$

The left hand side depends on the triangulation, the right hand side is independent.

Proof.

$\partial_k \circ f_k \circ \partial_k^{-1} = f_{k-1}$, $Tr(f_k|C_k/Z_k) = Tr(f_{k-1}|B_{k-1})$,

$$\begin{aligned} Tr(f_k|C_k) &= Tr(f_k|C_k/Z_k) + Tr(f_k|Z_k) \\ &= Tr(f_{k-1}|B_{k-1}) + Tr(f_k|B_k) + Tr(f_k|H_k) \end{aligned}$$

□

Euler Number

Lemma

Suppose M is a compact oriented surface with genus g, $f : M \to M$ is a continuous automorphism of M, f is homotopic to the identity map of M, then the Lefschetz number of f equals to the Euler characteristic number of M,

$$\Gamma(f) = \chi(S).$$

Proof.

We construct a triangulation of M and use a simplicial map to approximate the automorphism. Then

$$\Lambda(f) = \Lambda(Id) = |V| + |F| - |E| = \chi(S).$$

□

Poincaré-Hopf Theorem

Isolated Zero Point

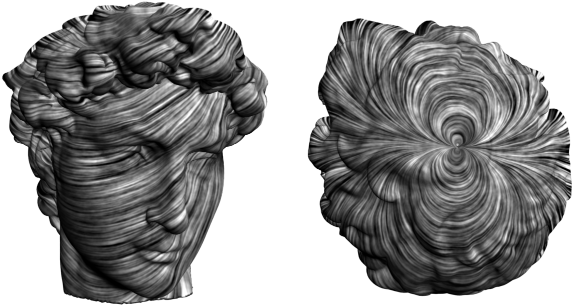
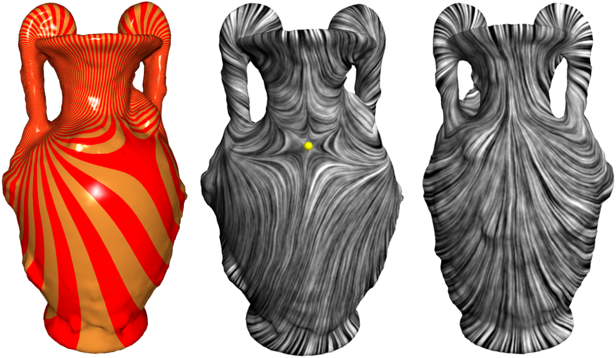


Figure: Islated zero point.

Definition (Isolated Zero)

Given a smooth tangent vector field $\mathbf{v} : S \to TS$ on a smooth surface S, $p \in S$ is called a zero point, if $\mathbf{v}(p) = \mathbf{0}$. If there is a neighborhood $U(p)$, such that p is the unique zero in $U(p)$, then p is an isolated zero point.

Zero Index



Definition (Zero Index)

Given a zero $p \in Z(v)$, choose a small disk $B(p,\varepsilon)$ define a map $\varphi : \partial B(p,\varepsilon) \to \mathbb{S}^1$, $q \mapsto \frac{\mathbf{v}(q)}{|\mathbf{v}(q)|}$. This map induces a homomorphism $\varphi_\# : \pi_1(\partial B) \to \pi_1(\mathbb{S}^1)$, $\varphi_\#(z) = kz$, where the integer k is called the index of the zero.

Zero Index

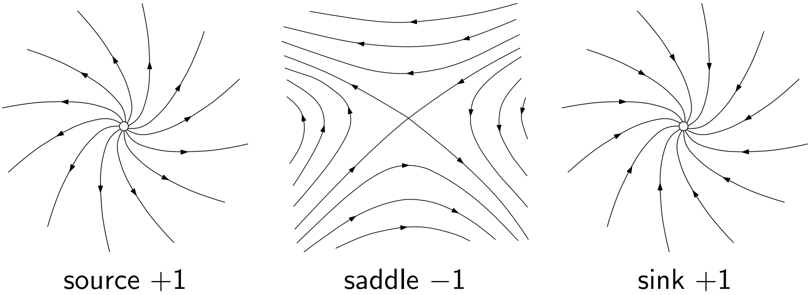


Figure: Indices of zero points.

Poincaré-Hopf

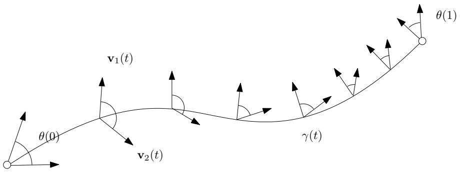
Theorem (Poincaré-Hopf Index)

Assume S is a compact, oriented smooth surface, v is a smooth tangent vector field with isolated zeros. If S has boundaries, then v point along the exterior normal direction, then we have

$$\sum_{p \in Z(v)} Index_p(v) = \chi(S),$$

where $Z(v)$ is the set of all zeros, $\chi(S)$ is the Euler characteristic number of S.

Poincaré-Hopf Theorem



Proof.

Given two vector fields v_1 and v_2 with different isolated zeros. We construct a triangulation T, such that each face contains at most one zero. Define two 2-forms, Ω_1 and Ω_2.

$$\Omega_k(\Delta) = \text{Index}_p(\mathbf{v}_k), \quad p \in \Delta \cap Z(v_k), \quad k = 1, 2.$$

Along $\gamma(t)$, $\theta(t)$ is the angle from $v_1 \circ \gamma(t)$ to $v_2 \circ \gamma(t)$. Define a one form,

$$\omega(\gamma) := \int_\gamma \dot{\theta}(\tau) d\tau.$$

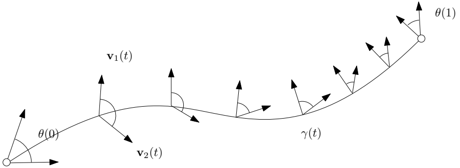
Relation with Fixed Point Theorem

Given a smooth tangent vector field v, we can define a one parameter family of automorphisms, $\varphi(p, t)$,

$$\frac{\partial \varphi(p, t)}{\partial t} = v \circ \varphi(p, t).$$

Then $f_t : p \mapsto \varphi(p, t)$ is an automorophism homotopic to the identity. According to lemma 7, the total index of fixed points of f_t is $\chi(S)$. The fixed points of f_t corresponds to the zeros of v with the sample index.

Poincaré-Hopf Theorem



continued.

Given a triangle Δ, then the relative rotation of v_2 about v_1 is given by

$$\omega(\partial\Delta) = d\omega(\Delta)$$

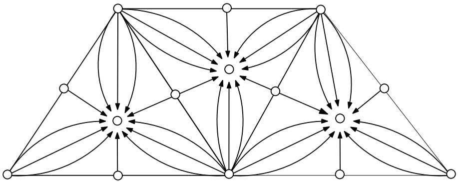
then we get

$$\Omega_2 - \Omega_1 = d\omega.$$

Therefore Ω_1 and Ω_2 are cohomological. The total index of zeros of a vector field

$$\sum_{p \in v_k} \text{Index}_p(v_k) = \int_S \Omega_k$$

Poincaré-Hopf Theorem



continued.

We construct a special vector field, then the total index of all the zeros is

$$\sum_{p \in Z(v)} \mathrm{Index}_p(v) = |V| + |F| - |E| = \chi(S).$$

□

Unit Tangent Bundle of the Sphere

Smooth Manifold

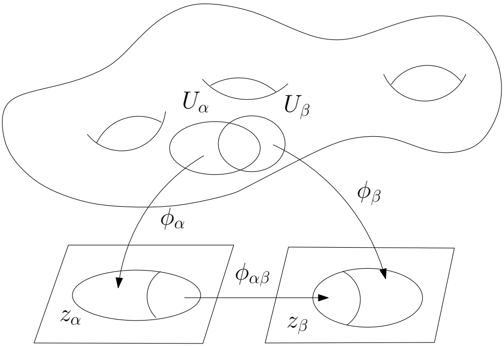


Figure: A manifold.

Smooth Manifold

Definition (Manifold)

A manifold is a topological space M covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a coordinate chart of M. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of M. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

If all transition maps $\phi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ are smooth, then the manifold is a differential manifold or a smooth manifold.

Tangent Space

Definition (Tangent Vector)

A tangent vector ξ at the point p is an association to every coordinate chart $(x^1, x^2, \cdots, x^n)$ at p an n-tuple $(\xi^1, \xi^2, \cdots, \xi^n)$ of real numbers, such that if $(\tilde{\xi}^1, \tilde{\xi}^2, \cdots, \tilde{\xi}^n)$ is associated with another coordinate system $(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n)$, then it satisfies the transition rule

$$\tilde{\xi}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}(p)\xi^j.$$

A smooth vector field ξ assigns a tangent vector for each point of M, it has local representation

$$\xi(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \xi_i(x^1, x^2, \cdots, x^n)\frac{\partial}{\partial x_i}.$$

$\{\frac{\partial}{\partial x_i}\}$ represents the vector fields of the velocities of iso-parametric curves on M. They form a basis of all vector fields.

Push forward

Definition (Push-forward)

Suppose $\phi : M \to N$ is a differential map from M to N, $\gamma : (-\epsilon, \epsilon) \to M$ is a curve, $\gamma(0) = p$, $\gamma'(0) = \mathbf{v} \in T_p M$, then $\phi \circ \gamma$ is a curve on N, $\phi \circ \gamma(0) = \phi(p)$, we define the tangent vector

$$\phi_*(\mathbf{v}) = (\phi \circ \gamma)'(0) \in T_{\phi(p)} N,$$

as the push-forward tangent vector of $\mathbf{v}$ induced by ϕ.

Unit Tangent Bundle

Definition (UTM)

The unit tangent bundle of the unit sphere is the manifold

$$UTM(S) := \{(p, v) | p \in S, v \in T_p(S), |v|_{\mathbf{g}} = 1\}.$$

The unit tangent bundle of a surface is a 3-dimensional manifold. We want to compute its triangulation and its fundamental group.

Sphere

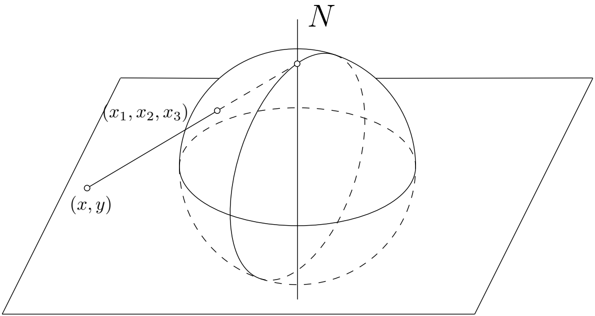


Figure: Stereo-graphic projection

$$(x, y) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right)$$

$$\mathbf{r}(x, y) = (x_1, x_2, x_3) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2}\right)$$

Sphere

$$\mathbf{r}_x = \partial_x = \frac{2}{(1+x^2+y^2)^2}\left(1-x^2+y^2, -2xy, 2x\right)$$
$$\mathbf{r}_y = \partial_y = \frac{2}{(1+x^2+y^2)^2}\left(-2xy, 1+x^2-y^2, 2y\right)$$

$$\langle \partial_x, \partial_x \rangle = \frac{4}{(1+x^2+y^2)^2}$$
$$\langle \partial_y, \partial_y \rangle = \frac{4}{(1+x^2+y^2)^2}$$
$$\langle \partial_x, \partial_y \rangle = 0$$

Unit Tangent Bundle of the Sphere

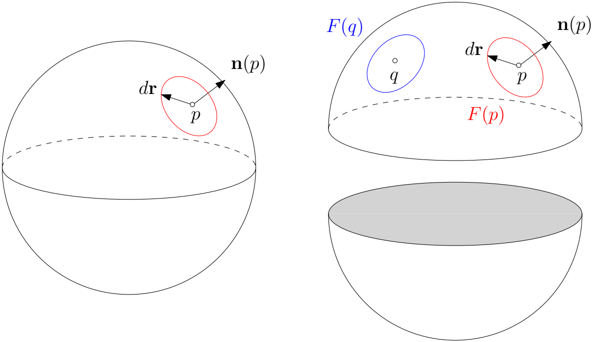


Figure: Unit tangent bundle.

A tangent vector at $\mathbf{r}(x, y)$ is given by: $d\mathbf{r}(x, y) = \mathbf{r}_x(x, y)dx + \mathbf{r}_y(x, y)dy$. On the equator

$$((x, y), (dx, dy)) = ((\cos\theta, \sin\theta), (\cos\tau, \sin\tau)).$$

Unit Tangent Bundle of the Sphere

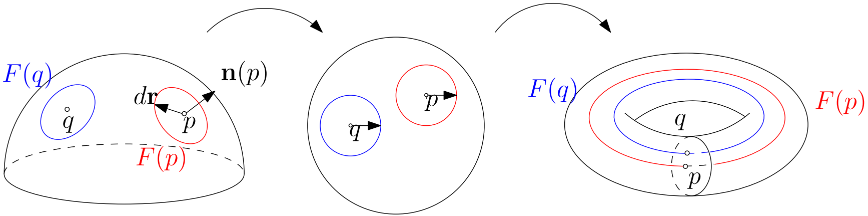


Figure: Unit tangent bundle.

The unit tangent bundle of a hemisphere is a direct product $\mathbb{S}^1 \times \mathbb{D}^2$, where $\mathbb{S}^1$ is the fiber of each point, $\mathbb{D}^2$ is the hemisphere. The boundary of the UTM of the hemisphere is a torus $\mathbb{S}^1 \times \partial\mathbb{D}^2$.

Sphere

$$(u, v) = \left(\frac{x_1}{1+x_3}, \frac{-x_2}{1+x_3} \right)$$

$$\mathbf{r}(u, v) = (x_1, x_2, x_3) = \left(\frac{2u}{1+u^2+v^2}, \frac{-2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2} \right)$$

$$\mathbf{r}_u = \partial_u = \frac{2}{(1+u^2+v^2)^2}(1-u^2+v^2, 2uv, -2u)$$

$$\mathbf{r}_u = \partial_v = \frac{2}{(1+u^2+v^2)^2}(-2uv, -1-u^2+v^2, -2v)$$

$$\langle \partial_u, \partial_u \rangle = \frac{4}{(1+u^2+v^2)^2}$$

$$\langle \partial_v, \partial_v \rangle = \frac{4}{(1+u^2+v^2)^2}$$

$$\langle \partial_u, \partial_v \rangle = 0$$

Chart transition

Let $z = x + iy$ and $w = u + iv$, Then

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{x_1 - ix_2}{1 - x_3} : \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{x_1 - ix_2}{1 + x_3} = w.$$

Therefore $dw = -\frac{1}{z^2} dz$,

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

this gives the Jacobi matrix,

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \frac{1}{(x^2 + y^2)^2} \begin{bmatrix} y^2 - x^2 & -2xy \\ 2xy & y^2 - x^2 \end{bmatrix}$$

Gluing Map

Construct the unit tangent bundle of the sphere. The unit tangent bundle of the upper hemisphere is a solid torus, the unit tangent bundle of the lower hemisphere is also a solid torus. The unit tangent bundle of the equator is a torus, $\varphi : (z, dz) \mapsto (w, dw)$, $z = e^{i\theta}$, $dz = e^{i\tau}$,

$$\varphi : (z, dz) \mapsto \left(\frac{1}{z}, -\frac{1}{z^2} dz \right), (\theta, \tau) \mapsto (-\theta, \pi - 2\theta + \tau)$$

Automorphism of the Torus

$$\varphi : (\tau, \theta) \mapsto (\tau - 2\theta + \pi, -\theta)$$

φ	(τ, θ)	(τ', θ')
A	$(0, 0)$	$(\pi, 0)$
B	$(2\pi, 0)$	$(3\pi, 0)$
C	$(2\pi, 2\pi)$	$(-\pi, -2\pi)$
D	$(0, 2\pi)$	$(-3\pi, -2\pi)$

Table: Corresponding corner points.

Torus Automorphism on UCS

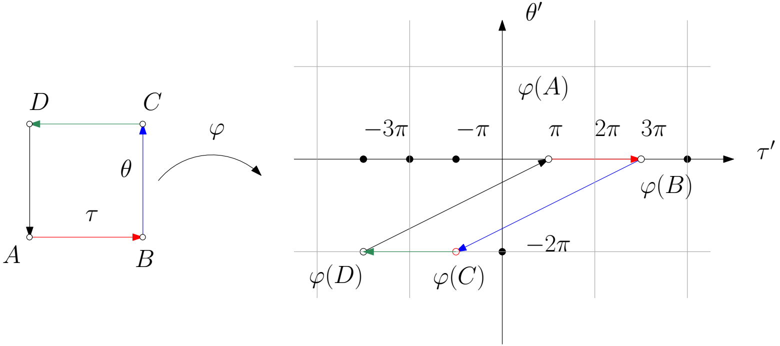


Figure: Torus automorophism.

This induces an automorphism of the fundamental group of the torus, $\varphi_\# : \pi_1(T^2) \to \pi_1(T^2)$,

$$\varphi_\# : a \mapsto a, \quad b \mapsto a^{-2}b^{-1}.$$

Torus Automorphism on UCS

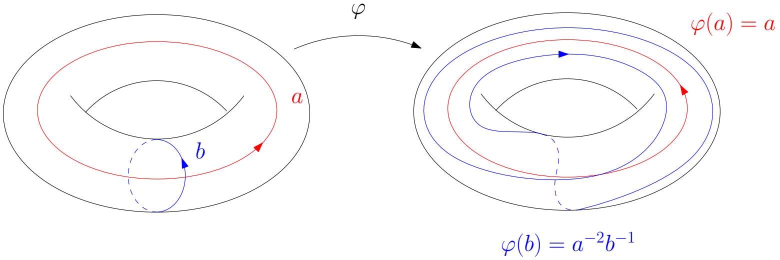


Figure: Torus automorophism.

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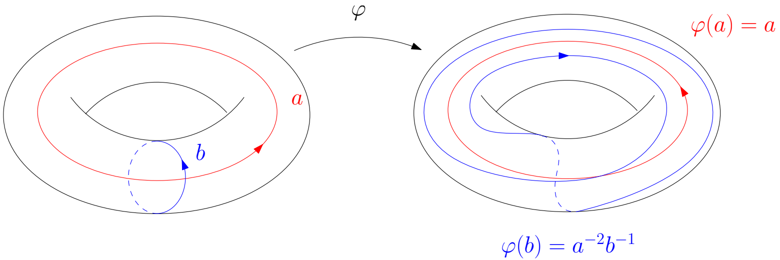
Torus Automorphism on UCS



$\pi_1(M_1) = \langle a_1 \rangle$, $\pi_1(M_2) = \langle a_2 \rangle$, $M_1 \cap M_2 = T^2$, $\pi_1(T^2) = \langle a, b | [a, b] \rangle$, then the π_1 of the unit tangent bundle is

$$\pi_1(M_1 \cup M_2) = \langle a_1, a_2 | a_1 a_2, a_2^{-2} b_2^{-1} \rangle = \mathbb{Z}_2.$$

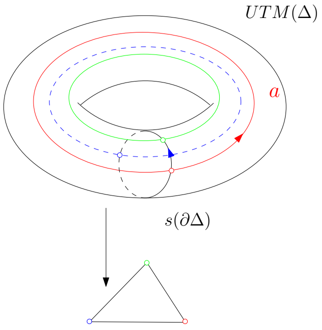


Figure: Local obstruction.

Obstruction Class

The topological obstruction for the existence of global section $\varphi : \mathbb{S}^2 \to UTM(\mathbb{S}^2)$ is constructed as follows:

1. Construct a triangulation $\mathcal{T}$, which is refined enough such that the fiber bundle of each face is trivial (direct product).
2. For each vertex v_i, choose a point on its fiber, $\varphi(v_i) \in F(v_i)$
3. For each edge $[v_i, v_j]$, choose a curve connecting $\varphi(v_i)$ and $\varphi(v_j)$ in the restiction of the UTM on $[v_i, v_j]$, which is annulus;
4. For each face Δ, $\varphi(\partial\Delta)$ is a loop in the fiber bundle of Δ, $[\varphi(\partial\Delta)]$ is an integer, an element in $\pi_1(UTM(\Delta))$, this gives a 2-form Ω on the original surface M,
$$\Omega(\Delta) = [\varphi(\partial\Delta)].$$
5. If Ω is zero, then global section exists. Otherwise doesn't exists.
6. Different constructions get different Ω's, but all of them are cohomological. Therefore $[\Omega] \in H^2(M, \mathbb{R})$ is the characteristic class of fiber bundle.

Obstruction Class

Lemma

Given two sections $\varphi, \bar{\varphi} : \mathbb{S} \to UTM(S)$, they incudes two 2-forms Ω_2, $\bar{\Omega}_2$. Then there exists a 1-form h, such that

$$\forall \sigma^2, \quad \delta h(\sigma^2) = \Omega^2(\sigma^2) - \bar{\Omega}^2(\sigma^2).$$

Proof.

$\forall \sigma_a^0 \in B^{(0)}$, construct a path in the fiber $p_a : [0,1] \to F$, such that

$$p_a(0) = \bar{\varphi}(\sigma_a^0), \quad p_a(1) = \varphi(\sigma_a^0)$$

Given a 1-simplex σ_a^1, with boundary $\partial \sigma_a^1 = \sigma_j^0 - \sigma_i^0$, construct a loop

$$l_a = p_i \varphi(\sigma_a^1) p_j^{-1} \bar{\varphi}(\sigma_a^1)^{-1}.$$

□

Obstruction Class

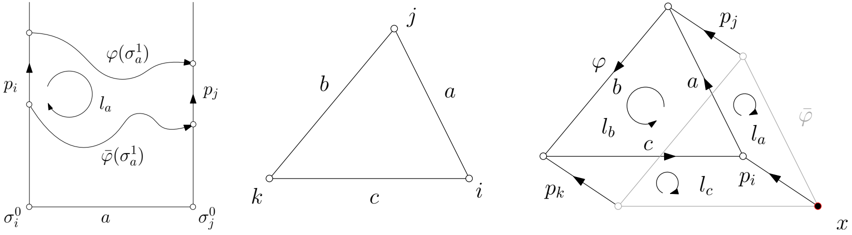


Figure: Denote $a = \varphi(\sigma_a^1)$, $b = \varphi(\sigma_b^1)$ and $c = \varphi(\sigma_c^1)$.

$$l_a := p_i \varphi(\sigma_a^1) p_j^{-1} \bar{\varphi}(\sigma_a^1)^{-1} = p_i a p_j^{-1} \bar{a}^{-1}$$
$$l_b := p_j b p_k^{-1} \bar{b}^{-1} \sim \bar{a} p_j b p_k^{-1} \bar{b}^{-1} \bar{a}^{-1}$$
$$l_c := p_k c p_i^{-1} \bar{c}^{-1} \sim \bar{a} \bar{b} p_k c p_i^{-1} \bar{c}^{-1} \bar{b}^{-1} \bar{a}^{-1}$$

continued

$$
\begin{aligned}
[l_a][l_b][l_c] &= (iaj^{-1}\bar{a}^{-1})(\bar{a}jbk^{-1}\bar{b}^{-1}\bar{a}^{-1})(\bar{a}\bar{b}kci^{-1}\bar{c}^{-1}\bar{b}^{-1}\bar{a}^{-1}) \\
&= iaj^{-1}jbk^{-1}kci^{-1}\bar{c}^{-1}\bar{b}^{-1}\bar{z}^{-1} \\
&= (iabci^{-1})(\bar{c}^{-1}\bar{b}^{-1}\bar{a}^{-1})
\end{aligned}
$$

Let the 1-form $h(\sigma_a^1) := [l_a]$, then

$$
\begin{aligned}
\delta h(\sigma^2) &= [l_a][l_b][l_c] \\
&= [iabci^{-1}][\bar{c}^{-1}\bar{b}^{-1}\bar{a}^{-1}] \\
&= [abc][(\bar{a}\bar{b}\bar{c})]^{-1} \\
&= C_2(\sigma^2)(\bar{C}(\sigma^2))^{-1}
\end{aligned}
$$

□