Surface Mapping Class Group

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Isotopy and Ambient Isotopy

Definition (Homotopy)

Let X and Y be topological spaces. A homotopy between two continuous maps $\varphi, \psi : X \to Y$ is a continuous map $F : X \times [0,1] \to Y$ such that $F_0 = \varphi$ and $F_1 = \psi$, where $F_t = F(\cdot, t)$

Definition (Smooth Isotopy)

Let M and N be differentiable manifolds. A smooth isotopy between embeddings $\varphi, \psi : M \to N$ is a smooth homotopy F between them, such that every map F_t is an embedding.

Isotopy and Ambient Isotopy

Definition (Ambient Isotopy)

An ambient isotopy on N is an isotopy between id_N and some other diffeomorphism $\varphi : N \to N$, such that every level is a diffeomorphism. Two embeddings $\varphi, \psi : M \to N$ are ambient isotopic if there is an ambient isotopy F of N such that $\psi = F_1 \circ \varphi$.

Theorem

If M is compact, two embeddings $\varphi, \psi : M \to N$ are isomotpic if and only if they are ambiently isotopic.

Isotopy and Ambient Isotopy

A curve on a differentiable manifold M is a smooth map $\gamma : I \to M$ defined on some interval, while a closed curve is a smooth map $\gamma : \mathbb{S}^1 \to M$. A curve is regular if $\gamma'(t) \neq 0$ for all $t \in I$. A curve is if it is injective; a closed curve is simple if and only if it is an embedding. Two simple closed curves are siotopic if and only if they are ambient isotopic.

Geometric Intersection

The algebraic intersection number only detects homology classes, but the geometric intersection number detects homotopy (isotopy) classes.

Definition (Geometric Intersection Number)

Let γ_1 and γ_2 be two simple closed curves in an orientable surface S. The geometric intersection $i(\gamma_1, \gamma_2)$ is the minimum number of intersections of two transverse simple closed curves γ_1', γ_2' homotoptic to γ_1, γ_2.

The geometric intersection number depends only on the homotopy classes of γ_1 and γ_2.

Geometric Intersection

Definition (Minimal Position)

Two simple closed curves γ_1 and γ_2 in an orientable surface S are in minimal position if they intersect transversely in exactly $i(\gamma_1, \gamma_2)$ points.

Theorem (Bigon Criterion)

Two transversely simple closed curves γ_1, γ_2 in S_g are in minimal position if and only if they do not form bigons.

Geometric Intersection

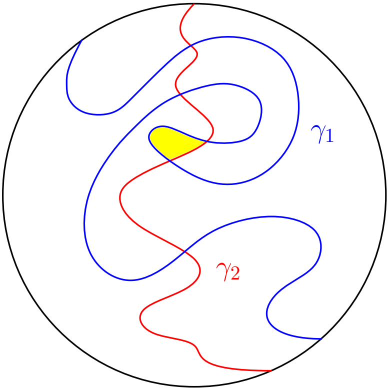


Figure: Bigon criterion.

The bigon criterion leads to an algorithm to calculate the geometric intersection of any pair γ_1 and γ_2 of simple closed curves in S_g: we put them in transver position, and then simplify bigons as much as posibble. After finitely many steps the two curves are in minimal position.

Geometric Intersection

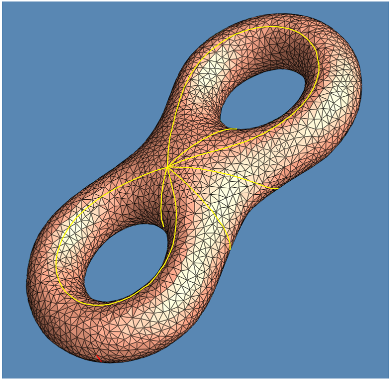


Figure: Hyperbolic geodesics.

Corollary

Let $g \geq 2$ and S_g have a hyperbolic metric. Two simple closed geodesics with distinct supports are always in minimal position.

Corollary

Let $g \geq 2$ and S_g have a hyperbolic metric. Every non-trivial simple closed curve in S_g is isotopic to a simple closed geodesics.

Geometric Intersection

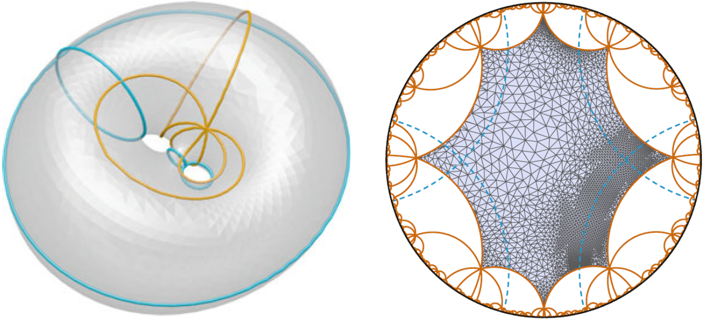


Figure: Hyperbolic geodesics. (By M. Joswig, M. Mehner et al.)

Proposition (Homotopy implies isotopy)

Two non-trivial simple closed curves in S_g are homotopically equivalent if and only if they are isotopic.

Multicurve

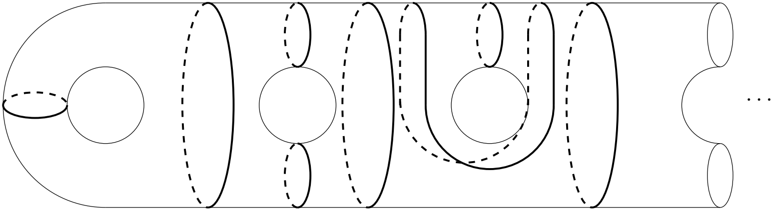


Figure: Pants decomposition

Definition (Multicurve)

A multicurve μ in S_g is a finite set of disjoint non-trivial simple closed curves. μ is essential if there is no parallel components.

Proposition

An essential multicurve μ in S_g with $g \geq 2$ has at most $3g - 3$ components, and it has $3g - 3$ if and only if it is a pants decomposition.

Multicurve

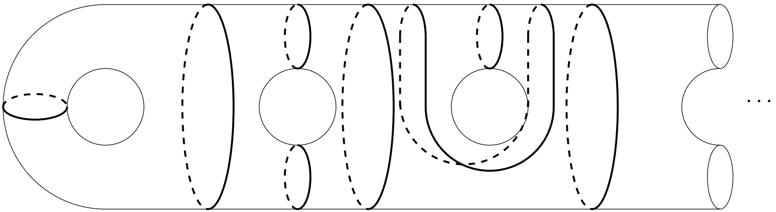


Figure: Pants decomposition

Definition (Geometric Intersection)

The geometric intersection (μ_1, μ_2) of two multicurves μ_1 and μ_2 is the minimum number of intersections of two transverse multicurves μ_1' and μ_2' isotopic to μ_1, μ_2.

Multicurve

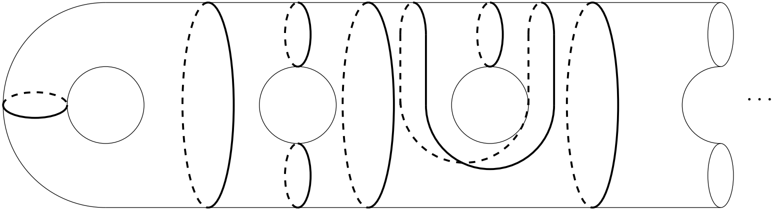


Figure: Pants decomposition

Proposition

Let $\mu_1, \mu_2 \subset S_g$ be two transverse multcurves. Then

$$i(\mu_1, \mu_2) = \sum_{\gamma_1 \subset \mu_1} \sum_{\gamma_2 \subset \mu_2} i(\gamma_1, \gamma_2).$$

Two transverse multicurves μ_1 and μ_2 are in minimal position if they intersect exactly in $i(\mu_1, \mu_2)$ points. μ_1 and μ_2 are in the minimal position if and only if they do not form the bigons.

Multicurve

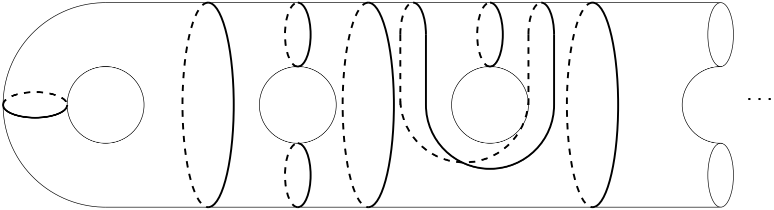


Figure: Pants decomposition

Proposition

Let μ, η be two multicurves in minimal position in S_g. The union $\mu \cap \eta$ of their supports depends up to ambient isotopy only on the isotopy classes of μ and η.

The Alexander Trick

Proposition

Two homeomorphisms $\varphi, \psi : \mathbb{D}^n \to \mathbb{D}^n$ *that coincide on* $\partial\mathbb{D}^n$ *are continuously isotopic, via an isotopy that fixes* $\partial\mathbb{D}^n$ *pointwise.*

Proof.

Consider $f = \varphi \circ \psi^{-1}$ and $\mathrm{id}_{\mathbb{D}^n}$, and define an isotopy that transforms f into $\mathrm{id}_{\mathbb{D}^n}$ fixing $\partial\mathbb{D}^n$,

$$F(x,t) = \begin{cases} x & \|x\| \geq t \\ tf\left(\frac{x}{t}\right) & \|x\| \leq t. \end{cases}$$

□

The Alexander's trick can be enhanced to the smooth setting.

Self-Diffeomorphism

Theorem (Self-Diffeomorphisms of Disks)

Two diffeomorphisms $\varphi, \psi : \mathbb{D}^2 \to \mathbb{D}^2$ *that conincide on* $\partial\mathbb{D}^2$ *are isotopic, via an isotopy that fixes* $\partial\mathbb{D}^2$ *pointwise.*

Theorem (Homotopy to Isotopy of Diffeomorphisms)

Two diffeomorphisms $\varphi, \psi : S_g \to S_g$ *are isotopic if and only if they are homotopic.*

Self-Diffeomorphism

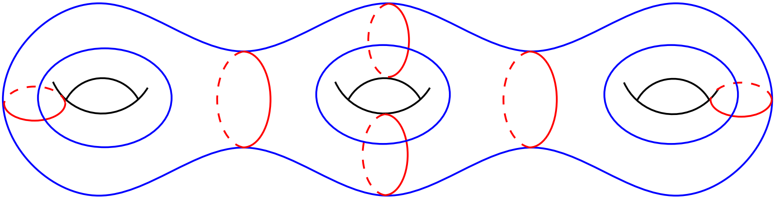


Figure: Two essential multicurves (blue and red) in minimal position, which subdivide the surface into hexagons.

Self-Diffeomorphism

Proof.

By composing ψ^{-1} with φ, the problem is reformulated to the following: the map $\varphi \circ \psi^{-1}$ and *id* are isotopic if and only if they are homotopic (with the boundary fixed).

Fix two essential multicurves μ and η. By hypothesis φ is homotopic to the identity, so the image multicurves $\varphi(\mu)$ and $\varphi(\eta)$ are curve by curve homotopic to μ and η, hence the multicurve $\varphi(\mu)$ is isotopic to μ, and $\varphi(\eta)$ is isotopic to η.

μ and η subdivide the surface into hexagons hence in particular they are in minimal position without bigons. Since φ is a diffeomorphism, $\varphi(\mu)$ and $\varphi(\eta)$ are without bigons, hence in minimal position. By Proposition 11.2, the supports $\mu \cup \eta$ and $\varphi(\mu \cup \eta)$ are ambiently isotopic, so we may suppose that they coincide. $\square$

Self-Diffeomorphism

Proof.

The graph $\mu \cup \eta$ is made of vertices and edges. The components of μ and η are pairwise non-homotopic, hence φ sends every component to itself, and it is oreientation-preserving. This implies easily that vertices and edges are sent to themsevles by φ. Hence $\varphi = id$ on vertices and after an isotopy, we may suppose that $\varphi = id$ on edges too.
After an isotopy, we may also suppose that $\varphi = id$ on a regular neighborhood U of $\mu \cup \eta$. The complement of U consists of disks (hexagons). Consider one such disk D, the map φ sends D to itself and is the identity on a collar of ∂D. By theorem 12, there is an isotopy connecting φ to id on every such disk D that fixes pointwise this collar, so we can extend it constantly on the rest of U and get a global isotopy on S_g connecting φ and id. □

Mapping Class Group

Definition (Mapping Class Group)

Let F be a surface (perhaps, with boundary), the mapping class group $H(F)$ of the surface F is defined as a quotient group of the group of homeomorphisms of the surface F onto itself with respect to the subgroup $\mathrm{Iso}(F)$ of homeomrphisms isotopic to the identity.

If $h_t : F \to F$, $h_1 = h$, $h_0 = 1$ is an isotopy of the homeomorphism h to the identity, then fh_tf^{-1} is an isotopy of the conjugate homeomorphism fhf^{-1} to the identity. Hence $\mathrm{Iso}(F)$ is normal.

Each element in the mapping class group $H(F, \partial F)$ is defined by a homeomorphism fixed on the boundary, two homeomorphisms determining the same element if and only if they are isotopic under an isotopyp fixed on the boundary.

Dehn Twist

Definition (Dehn Twist)

Let γ be a non-trivial simple closed curve in the interioir of $S_{g,b,p}$. The Dehn twist along γ is the element $\tau_\gamma \in \mathrm{MCG}(S_{g,b,p})$ defined as follows: pick a tubular neighborhood of γ orientation-preserving diffeomorphic to $\mathbb{S}^1 \times [-1,1]$ where γ lies as $\mathbb{S}^1 \times \{0\}$. Let $f : [-1,1] \to \mathbb{R}$ be a smooth function which is zero in $[-1,-\frac{1}{2}]$ and 2π on $[\frac{1}{2},1]$.
$\tau_\gamma : S_{g,b,p} \to S_{g,b,p}$ be the diffeomorphism that acts on the tubular neighborhood as

$$\tau_\gamma(e^{i\theta}, t) = (e^{i(\theta+f(t))}, t)$$

and on its complementary set in $S_{g,b,p}$ as the identity.

Dehn Twist

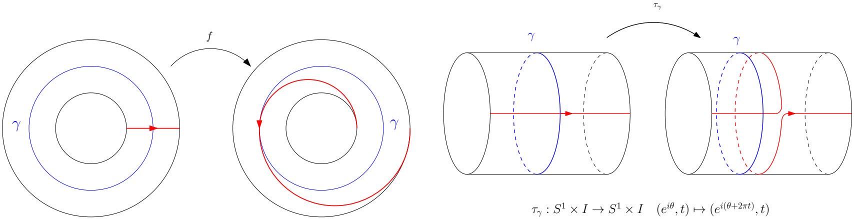


Figure: Denh twist τ_γ.

Dehn Twist

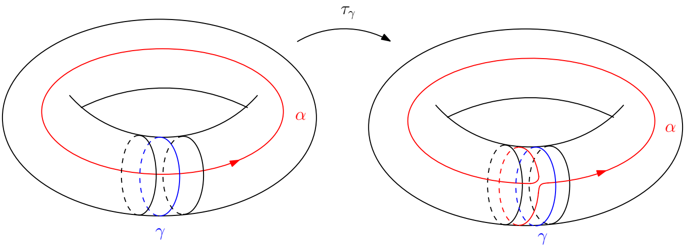


Figure: Denh twist τ_γ.

Braid Group

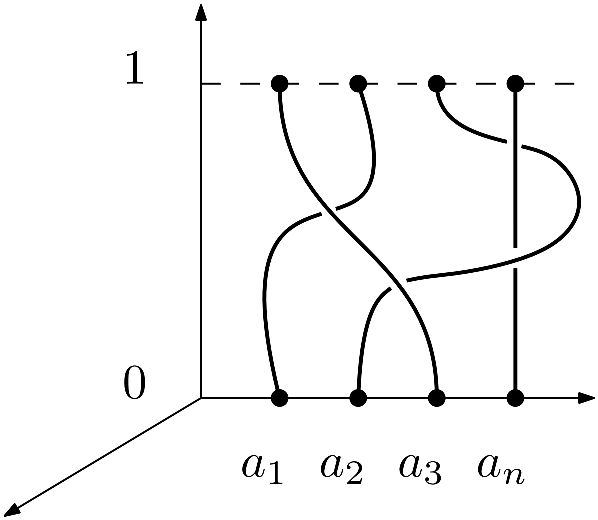


Figure: Braid.

Braid Group

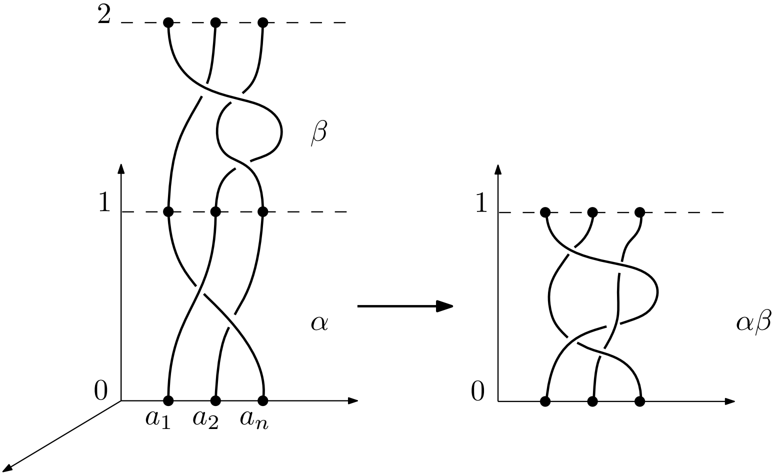


Figure: Product of two braids.

Braid Group

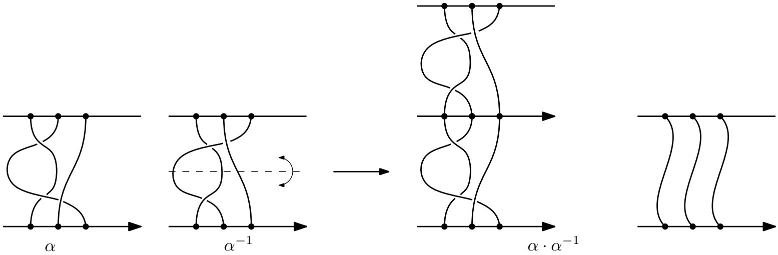


Figure: Inverse element in the braid group.

Braid Group

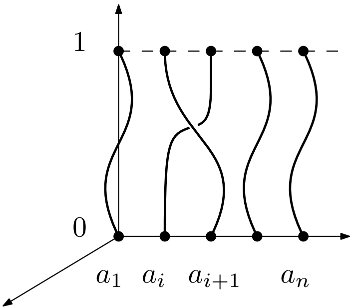


Figure: σ_i, the i-th string goes under the $(i+1)$-th string.

Theorem

The group B_n has the representation

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} : \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| > 1 \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$$

Pure Braid Group

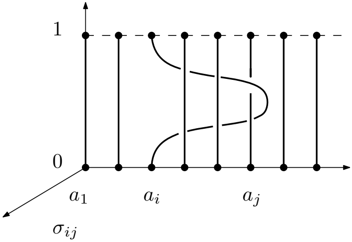


Figure: $\sigma_{ij} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1}^2 \sigma_{j-2}^{-1} \cdots \sigma_i^{-1}$.

Definition (Pure braid group)

There is a natural homomorphism of B_n onto the symmetry group S_n of permuations of n symbols. The kernel K_n is called the pure brain group.

Theorem

The braids σ_{ij}, $1 \leq i < j \leq n$, generate the group K_n.

Mapping Class Group

Let $F = D^2 - \cup_{i=1}^{n} A_i$ be a disk with n holes (disjoint disks $A_1, A_2, \dots, A_n$ centered at the points $a_1, a_2, \dots, a_n$). Each element h of the group $H(F, \partial F)$ can be extended by the identity on each disk A_i, the group $H(F, \partial F)$ is isomorphic to $H(D^2, \cup_i A_i)$.

Theorem (Mapping Class Group for Poly-Annulus)

The group $H(D^2, \cup_i A_i)$ is isomopric to the direct product of the pure braid group K_n by the free Abelian group of rank n.

Mapping Class Group

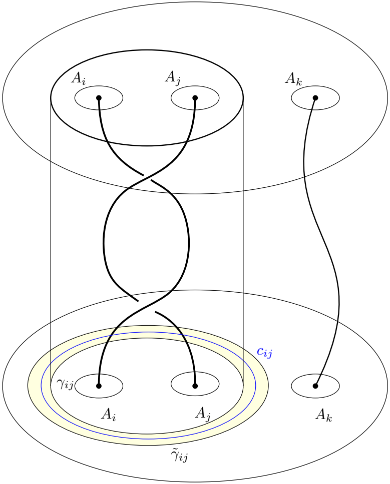


Figure: $\tau_{c_{ij}}$ corresponds to a braid.

Proof.

Consider a simple curve c_{ij} surrounding A_i and A_j, a neighborhood of c_{ij} is an annulus A_{ij} with the inner boundary γ_{ij} and the outer boundary $\tilde{\gamma}_{ij}$. Consider the isotopy of the Dehn twist $\tau_{c_{ij}}$, at each t, the outer boundary is fixed, the inner boundary γ_{ij} is rotated by an angle $2\pi t$, the inner disk bounded by γ_{ij} is also rotated by $2\pi t$. The orbits of the centers of A_i, A_j form a pure braid in K_n. $\square$

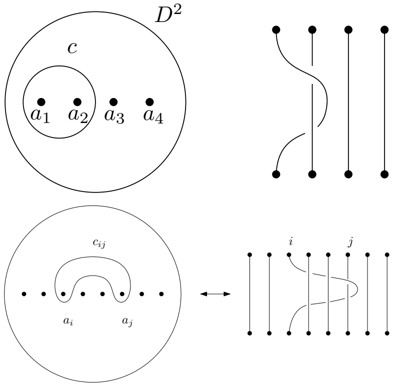


Figure: $\tau_{c_{ij}}$ corresponds to a braid σ_{ij}.

Proof.

Continued.
For any pair of A_i, A_j, c_{ij} is a simple loop surrounding A_i and A_j, the Dehn twist $\tau_{c_{ij}}$ corresponds a pure braid σ_{ij}. The subgroup generated by $\tau_{c_{ij}}$'s is isomorphic to the pure braid group K_n.
The Dehn twists along the boundaries ∂A_i's form a free Abelian group. $\square$

MCG Generators

Definition (c-homeomorphism)

We say that a homoeomorphism h of a surface onto itself is a c-homeomorphism if it is isotopic to a composition of a finite number of Dehn twists.

Theorem (MCG Generators)

Any (fixed on the boundary) homeomorphisms h of a compact orientable surface F onto itself is a c-homeomorphism (if $\partial F = \emptyset$, then h must preserve orientation).

MCG Generators

Definition (c-equivalent)

Two simple non-oriented closed curves a and b in the surface F is c-equivalent (denoted as $a \overset{c}{\sim} b$) if there exists a c-homeomorphism sending one curve into another.

Lemma (c-equivalence)

Any two non-splitting curves on a connected orientable surface F are c-equivalent.

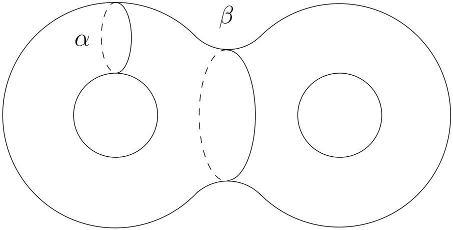
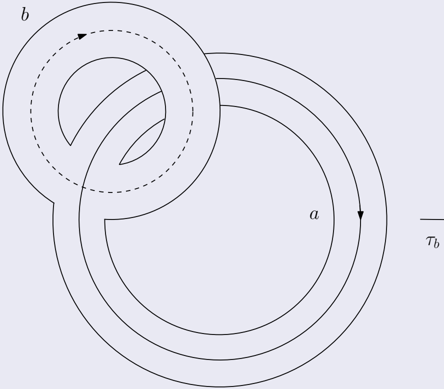


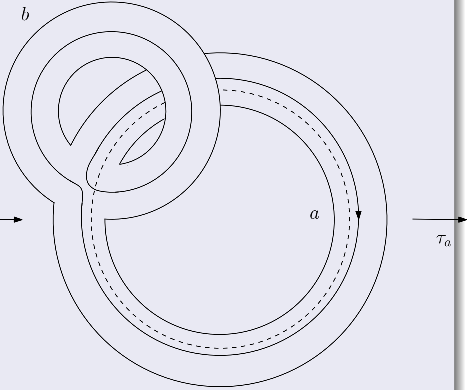
Figure: α is non-splitting, β is splitting, they are not c-equivalent.

Proof.

Case 1. the geometric intersection point of a and b is one, the $\tau_b \tau_a$ sends a to b.

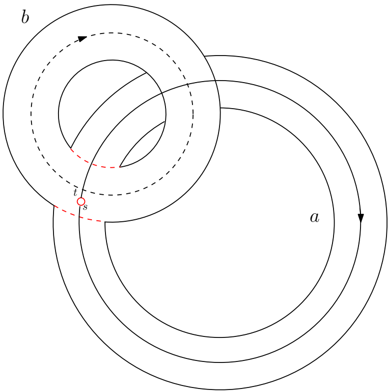


a. initial setting

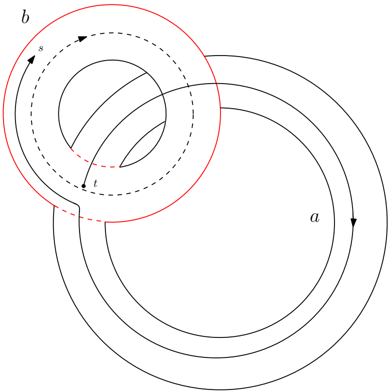


b. Dehn twist τ_b.

MCG Generators



cut a near outer rim



rotate outer rim, fix inner rim

Figure: The outer rim of the annulus rotates 2π angle, the inner rim is fixed. The end vertex s is near the outer rim, rotates with the rim. The end s is near the inner rim, so it is fixed.

MCG Generators

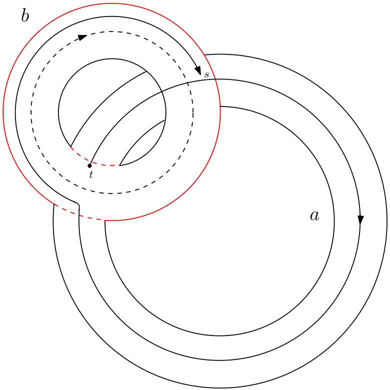
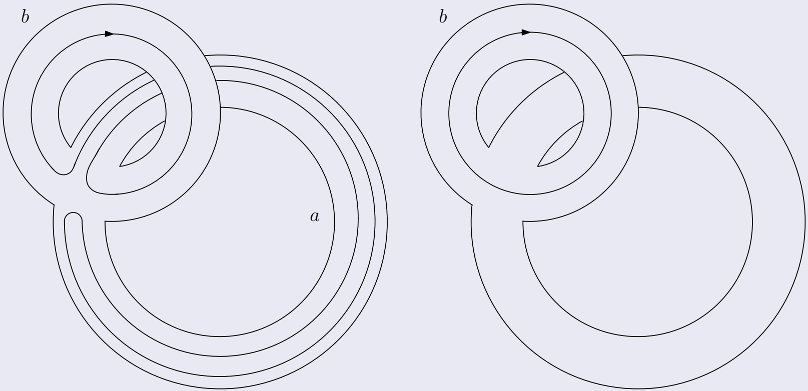


Figure: s goes around one cycle, reconnects with t.

MCG Generators

Proof.



c. Dehn twist τ_a

d. isotopy

Figure: Case 1. $i(a,b) = 1$.

MCG Generators

Proof.

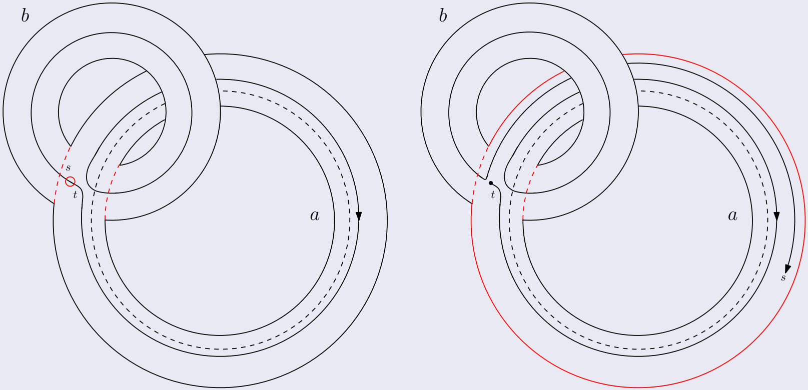


Figure: rotate the outer rim, fix the inner rim.

□

MCG Generators

Proof.

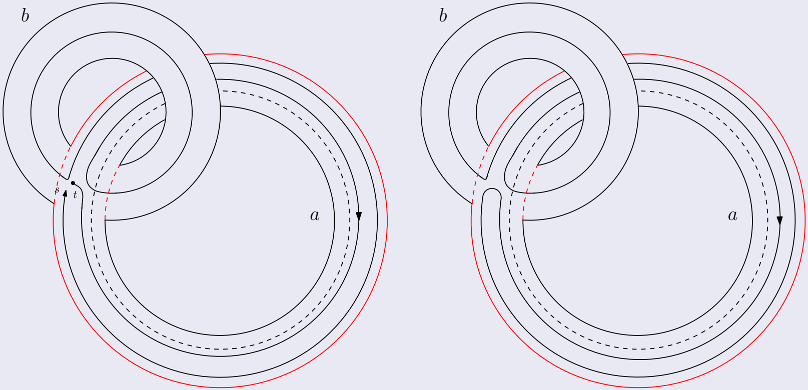


Figure: rotate the outer rim, fix the inner rim.

MCG Generators

Proof.

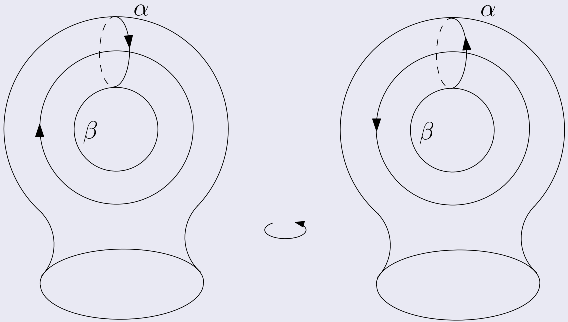


Figure: Handle twisting.

Furthermore, $\tau_b \tau_a \tau_b$ sends the curve a into the curve b, and the curve b into the curve a^{-1}. $(\tau_b \tau_a \tau_b)^2$ reverse the orientation of both curves. Then the homeomorphism $(\tau_b \tau_a \tau_b)^2$ is isotopic to twisting the handle 180 degree along th base.

Proof.

Case 2. $i(a,b)=0$,

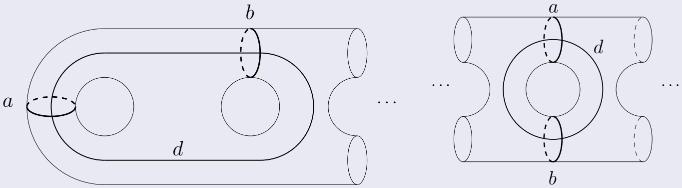


Figure: Case 2. $i(a,b)=0$.

$a\cup b$ is non-splitting (left) or splitting (right), there is a simple closed curve d which intersects each of them at exactly one point and does not split the surface. Then $a \overset{c}{\sim} d$ and $b \overset{c}{\sim} d$, it follows that $a \overset{c}{\sim} b$. □

Proof.

Case 3. $i(a,b) = k > 1$,

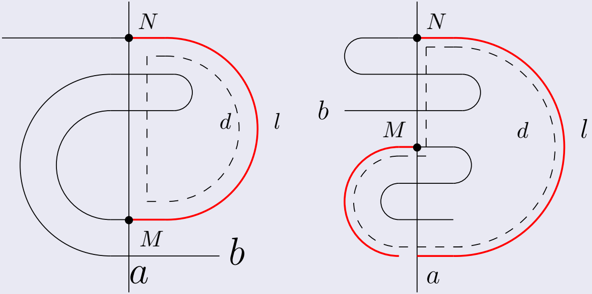


Figure: Case 3. $i(a,b) = k > 1$.

By induction on k, assume that $i(a,b) < k$ implies $a \overset{c}{\sim} b$. Chose

MCG Generators

Proof.

From among the intersection points, choose M and N neighboring on b. Let $l \subset b$ be the arc joining these two points. M and N split a into m_1 and m_2. One of the two loops $l \cup m_1$ and $l \cup m_2$ is non-splitting, denoted as d, otherwise $(l \cup m_1) \cup (l \cup m_2 = a \cup l)$ is splitting, and l is removable, so a is splitting, contradiction.
Shrink d slightly, then $i(a,d) < i(a,b) = k$, by induction assumption $a \overset{c}{\sim} d$, $b \overset{c}{\sim} d$, hence the $a \overset{c}{\sim} b$. $\square$

MCG Generators

Theorem (MCG Generators)

Any (fixed on the boundary) homeomorphisms h of a compact orientable surface F onto itself is a c-homeomorphism (if $\partial F = \emptyset$, then h must preserve orientation).

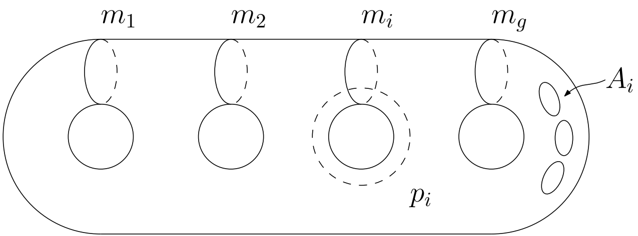


Figure: A genus g orientable surface F with boundaries $A_1, A_2, \ldots, A_m$.

MCG Generators

Proof.

We choose a family $m_1, m_2, \ldots, m_g$ of disjoint simple closed curves which cut the surface to a disk with $2g + m - 1$ holes. A homeomorphism h sends the curves m_i to some other curves $h(m_i)$. Since the curves m_1 and $h(m_1)$ are non-splitting, by lemma 23, there exists a c-homeomorphism s_1 such that $s_1 h(m_1) = m_1$.
similarly, there exists a c-homeomorphism s_2 such that $s_2 s_1 h(m_2) = m_2$. The homeomorphism s_2 can be thought of as fixed on m_1 and then $s_2 s_1 h(m_1) = m_1$. To this end, we cut the surface F along the curve m_1, then apply the lemma 23 and glue the edges of the cut again. In a similar way we return all the remaining curves m_i to their places. Let $s = s_g \ldots s_1$ be the thus obtained homeomorphism such that $sh(m_i) = m_i$, $1 \leq i \leq g$.

□

MCG Generators

Proof.

The homeomorphism sh can send the curve m_i into itself with orientation reversal. This shortcoming can be eliminated by choosing a curve p_i, which intersects m_i at exactly one point and does not intersect the other curves m_j, $j \neq i$, and multiplying the homeomorphism sh by $(\tau_{p_i}\tau_{m_i}\tau_{p_i})^2$. Thus, we can assume the homeomorphism sh to be fixed on the curves m_i. Therefore it determines a homeomorphism fixed on the boundary of the surface F cut along the curves m_i, i.e. the homeomorphism fixed on the boundary of a disk with $2g + m - 1$ holes. By theorem 19, this homeomorphism can be decomposed into the product of Dehn twists. Hence sh, as well as h, decomposes into a product of twists. □

MCG Generators

Let $L = \{l_1, l_2, \ldots, l_m\}$ be an arbitrary family of pairwise disjoint simple closed curves in F.

Definition (L-admissible)

We refer to a simple closed curve $c \subset F$ as L-admissible if it intersects each curve of L at no more two points, and if at two points then in opposite direction (i.e. the intersection indices at these two points must have opposite signs.)

Lemma

The group $H(F, \partial F)$ is generated by twists along L-admissible curves.

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Proof.

The subgroup of $H(F, \partial F)$ generated by twists along L-admissible curves will be denoted by G_L. Since by theorem 24 the group $H(F, \partial F)$ is generated by twists, it follows that to prove $H(F, \partial F) = G_L$ it suffices to prove that $\tau_p \in G_L$ for any simple closed curve $p \subset F$. If the curve p is L-admissible, the inclusion $\tau_p \in G_L$ follows from the definition of G_L. If p is not L-admissible, then two cases may occur. $\square$

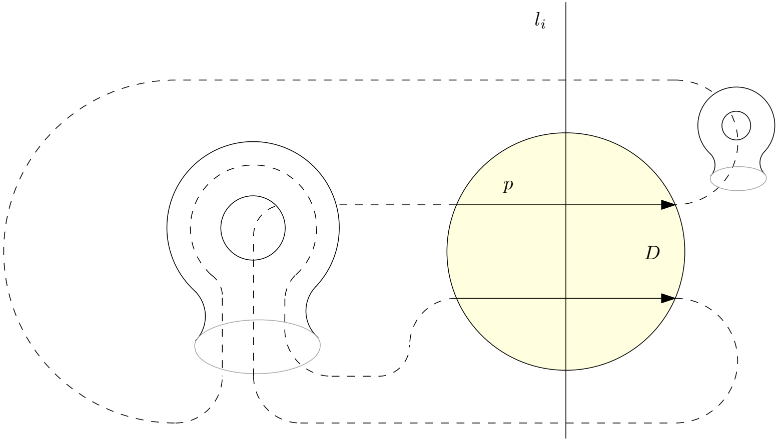

Figure: Case 1. The curve p intersects any curve l_i from L at two neighboring points in one direction.

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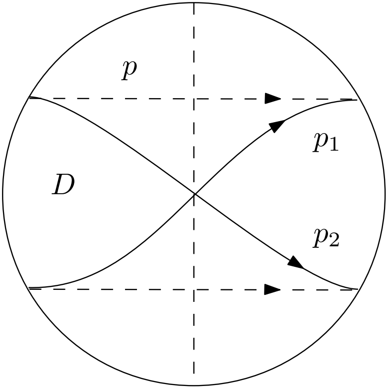

Figure: Case 1. The curve p intersects any curve l_i from L at two neighboring points in one direction.

$$\tau_p = \tau_{p_1}\tau_{p_2}\tau_{p_1}^{-1}$$

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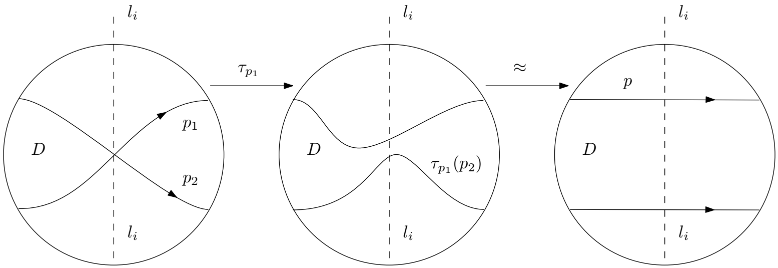

Figure: Case 1. $\tau_{p_1}(p_2)$ is isotopic to p.

Then the twist along p is isotopic to the composition of the twist $\tau_{p_1}^{-1}$, which sends p into the curve p_2, the twist τ_{p_2} and the twist τ_{p_1}, which returns p_2 to p. Namely $\tau_p = \tau_{p_1}\tau_{p_2}\tau_{p_1}^{-1}$. p_1 and p_2 intersect L with smaller number of points each. By induction, τ_{p_1}, τ_{p_2} are in G_L, so $\tau_p \in G_L$.

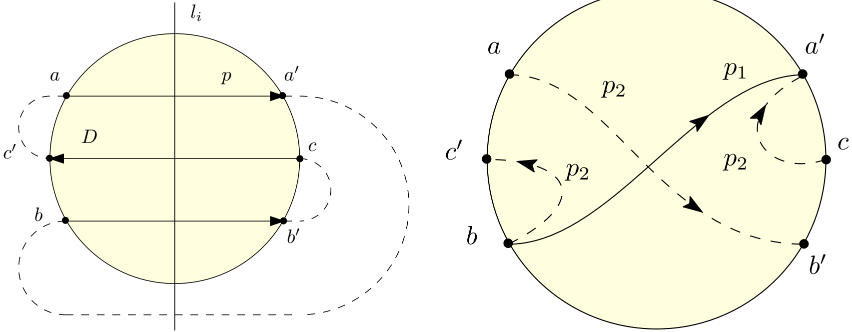

Figure: Case 2. The curve p intersects any curve l_i in L at three neighboring points in alternating directions (left frame).

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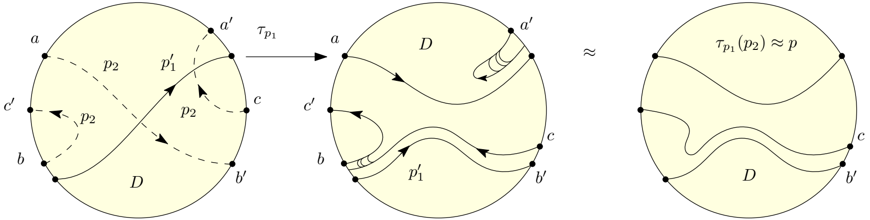

Figure: Case 2. $\tau_{p_1}(p_2)$ is isotopic to p.

Shift p_1 slightly to p_1' such that p_1' intersects p_2 transversely. Similarly to case 1, $\tau_p = \tau_{p_1}\tau_{p_2}\tau_{p_1}^{-1}$. p_1, p_2 intersect the family L at a smaller number of points. By induction, $\tau_{p_1}, \tau_{p_2} \in G_L$, so does τ_p.

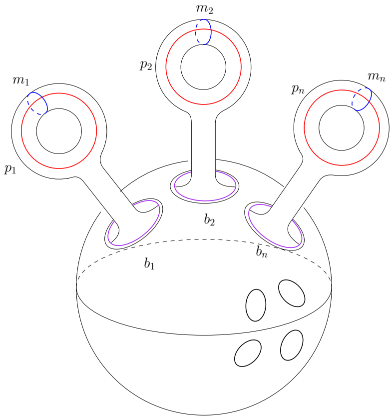

Figure: Compact orientable surface F.

Theorem

On an arbitrary compact orientable surface F, there exists a finite set of simple closed curves $c_1, c_2, \ldots, c_N$, such that the group $H(F, \partial F)$ is generated by the twists $\tau_{c_1}, \tau_{c_2}, \ldots, \tau_{c_N}$.

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Proof.

The surface F is represented as a sphere with n handles and k holes. The meridians of the handles are denoted by $m_1, m_2, \ldots, m_n$, their longtitudes by $p_1, p_2, \ldots, p_n$, and their bases by $b_1, b_2, \ldots, b_n$. Let the subgroup $H_{m,p} \subset H(F, \partial F)$ be generated by twists along all the meridians m_i and all the longtitudes p_i. Let the subgroup $\text{Fix}(m) \subset H(F, \partial F)$ be generated by the homeomorphisms which are fixed on all the meridians m_i and on the boundary ∂F. We claim that the group $H(F, \partial F)$ is generated by these two subgroups.

This is surfficient for the proof of the theorem 27 since cutting the surface F along the meridians yields a disk with $2n + k - 1$ holes, whose mapping class group is finitely generated according to theorem 19. $\square$

Proof.

We set L to be the set of $m_1, m_2, \ldots, m_n$, $b_1, b_2, \ldots, b_n$ consisting of the meridans and bases of handles. By lemma 26, twists along L-admissible curves generate the group $H(F, \partial F)$.

Let p be an arbitrary L-admissible curve. Consider one handle, p intersects m_i and b_i no more than two points, and if at exactly two, then in opposite directions. $\square$

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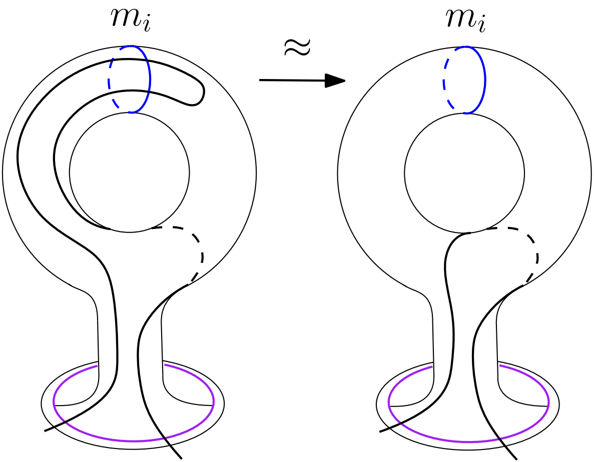

Proof.

If p intersects the meridian m_i at two points, this intersection can be eliminated by an isotopy. □

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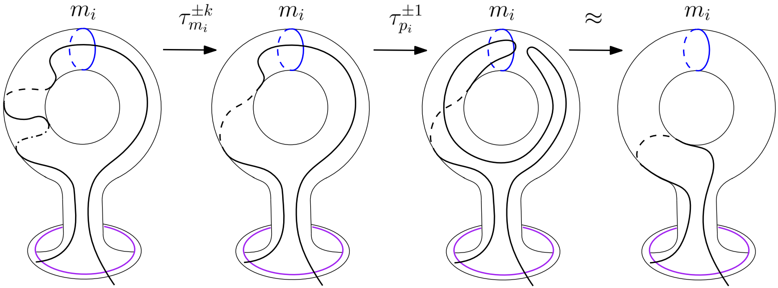

Proof.

If p intersects the meridian m_i at one point, several twists along the meridian m_i, one twist along the longtitude p_i and an isotopy will eliminate the intersection with m_i. Similarly we can eliminate the intersection of p with other meridians. □

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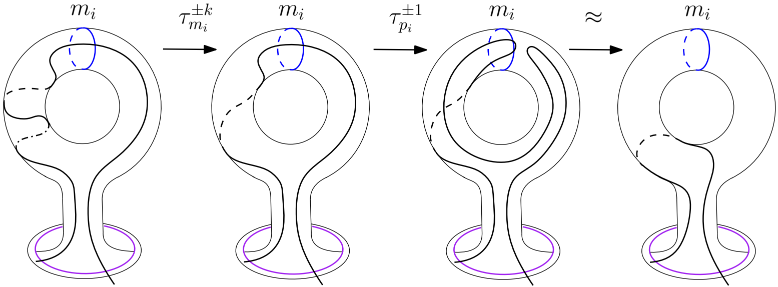

Proof.

So any L-admissible curve p can be sent into a curve p' using a composition $h \in H_{m,p}$, which doesn't intersect the meridians, and then $\tau_p = h^{-1}\tau_{p'}h$, where $\tau_{p'} \in \mathrm{Fix}(m)$. This implies the theorem. $\square$

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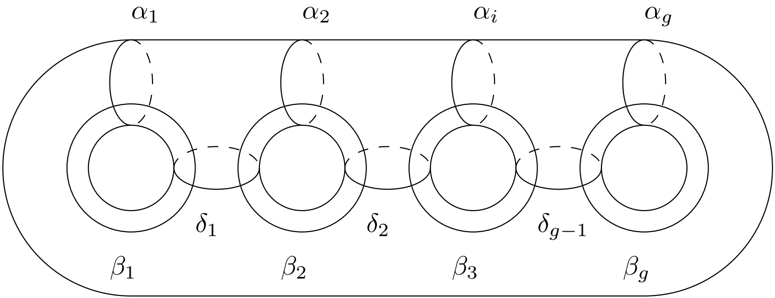

Mapping class gorup generators by Lickorish. In fact, only $\{\alpha_k, \beta_k\}$ and δ_1 are enough for $S_{g,0}$ by Humphrises.