Chapter 1 Introduction

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Mathematical optimization

(mathematical) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \qquad i = 1, \cdots, m \end{array}$$

- optimization (decision) variables $\quad x = (x_1, \cdots, x_n)$
- objective function $\quad f_0 \colon \mathbb{R}^n \to \mathbb{R}$
- constraint functions $\quad f_i \colon \mathbb{R}^n \to \mathbb{R}, \qquad i = 1, \cdots, m$

optimal solution x^*

the vector x that gives the smallest objective value among all vectors satisfying the constraints

Examples

portfolio optimization

- variables: amounts invested in different assets
- constraints: budget, max/min investment per asset, minimum return
- objective: overall risk or return variance

device sizing in electronic circuits

- variables: device widths and lengths
- constraints: manufacturing limits, timing requirements, maximum area
- objective: power consumption

data fitting

- variables: model parameters
- constraints: prior information, parameter limits
- objective: measure of misfit or prediction error

Solving optimization problems

general optimization problems

- very difficult to solve
- methods involve some compromises (e.g. very long computation time, or not always finding the solution)

exceptions: certain problem classes can be solved efficiently and reliably

- least-square problems
- linear programming problems
- convex optimization problems

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Least-squares

$$\text{minimize} \qquad \|Ax - b\|_2^2$$

solving least-squares

- analytic solution: $x^* = (A^TA)^{-1}A^Tb$
- reliable and efficient algorithms and software
- computation time proportional to n^2k (when $A \in \mathbb{R}^{k\times n}$); less if structured
- a mature technology

using least-squares

- least-squares problems are easy to recognize
- a few standard techniques increase flexibility (e.g. including weights, adding regularization terms)

Linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \qquad i = 1, \cdots, m \end{array}$$

solving linear programs

- no analytical formula for solution
- reliable and efficient algorithms and software
- computation time proportional to $n^2 m$ if $m \geq n$; less if structured
- a mature technology

using linear programming

- no as easy to recognize as least-squares problems
- a few standard tricks used to convert problems into linear programs (e.g. problems involving ℓ_1- or ℓ_∞-norms, piecewise linear functions)

Convex optimization problems

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \qquad i = 1, \cdots, m \end{array}$$

- objective and constraint functions are convex

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

  if $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$
- includes least-squares problems and linear programs as special cases

solving convex optimization problems

- no analytical solution
- reliable and efficient algorithms
- computation time (roughly) proportional to $\mathbf{max}\{n^3, n^2m, F\}$ where F is the cost of evaluating f_i's and their first and second derivatives
- almost a technology

using convex optimization

- often difficult to recognize
- many tricks for transforming problems into convex form
- surprisingly many problems can be solved via convex optimization

Nonlinear (nonconvex) optimization

traditional techniques for general nonconvex problems involve compromises

local optimization methods

- find a point that minimizes the objective function among feasible points near it
- fast, can handle large problems
- require initial guess
- provide no information about distance to global optimum

global optimization methods

- find the global solution
- worst-case complexity grows exponentially with problem size

the above algorithms are often based on solving convex subproblems

roles of convex optimization in nonconvex problems

- initialization for local optimization
- convex heuristics for nonconvex optimization
- bounds for global optimization

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theory

- basic convex analysis
- recognize and formulate problems as convex optimization problems
- Lagrangian duality, characterize optimal solutions

algorithms

- problem types: unconstrained, equality constrained, inequality constrained
- algorithms: Newton's algorithm, interior-point methods

applications

- data fitting, probability and statistics, computational geometry

Brief history of convex optimization

theory (convex analysis): ca 1900-1970

algorithms

- 1947: simplex algorithm for linear programming (Dantzig)
- 1960s: early interior-point methods (Fiacco & McCormick, Dikin, ...)
- 1970s: ellipsoid method and other subgradient methods
- 1984: polynomial-time interior-point methods for linear programming (Karmarkar)
- 1994: polynomial-time interior-point methods for nonlinear convex optimization (Nesterov & Nemirovski)

applications

- before 1990: mostly in operations research; few in engineering
- since 1990: many new applications in engineering (control, signal processing, communications, circuit design, ...); new problem classes (semidefinite and second-order cone programming, robust optimization, ...)