Hyperbolic Knot Theory

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Trichotomy on Knots

Torus Knot

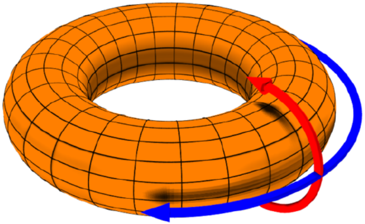


Figure: A torus with a meridian and a longtitude.

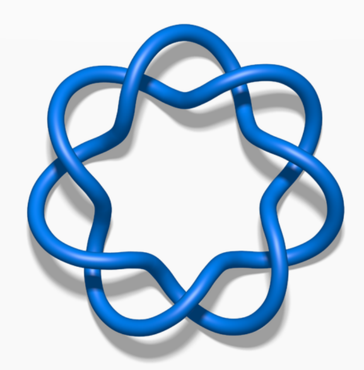
Consider the standardly embedded torus $T \subset S^3$ with oriented meridan m and longtitude l.

Definition (Torus Knot)

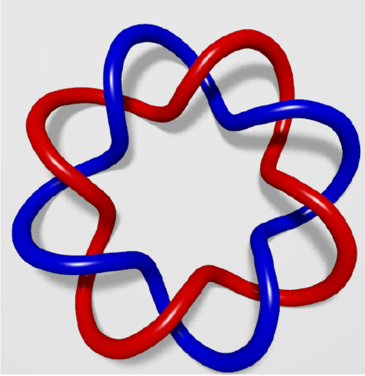
A pair (p, q) of coprime integers determines a simple closed curve in T that is homologically $qm + pl$, and such a curve is called a (p, q)-torus knot.

More generally, a pair of integers (a, b) determines a multicurve in T that is homlogically $bm + al$, called an (a, b)-torus link.

Torus Knot



torus knot $(2, -7)$



torus link $(2, -8)$

Figure: Torus and link knots.

Torus Knot

Proposition

The complement of a (p,q)-torus knot is a Seifert manifold fibering over the orbifold (D,p,q). More precisely, it is

$$(D,(p,r),(q,s))$$

where (r,s) is any pair such that $ps+qr=1$.

Proof.

Let $K\subset T\subset S^3$ be the (p,q)-torus knot. The pair $(r,-s)$ determines another simple closed curve $\alpha\subset T$ that intersects K in one point. The complement of K in a tubular neighbourhood $T\times[-1,1]$ of T is diffeomorphic to $P\times S^1$, where P is a pair of pants. On the tori $T\times\{-1\}$ and $T\times\{+1\}$ the curves $\partial P\times\{pt\}$ and $\{pt\}\times S^1$ are isotopic to α and K. $\square$

Torus Knot

Proof.

The complement of $T \times [-1, 1]$ in S^3 consists of two solid tori, with meridians $(1, 0)$ and $(0, 1)$. Read in basis (α, K) the meridians are (q, s) and (p, r). The complement of K in S^3 is obtained from $P \times S^1$ by filling these curves and hence we get $(D, (p, r), (q, s))$. □

Satellite Knots

Definition (Satellite Knot)

A knot $K \subset S^3$ is a *satellite* if it is complement contains an essential torus.

A knot in a solid torus $D^2 \times S^1$ is *local* if it is contained in a ball, and a *core* if it is isotopic to $\{x\} \times S^1$. An embedding $\varphi : D^2 \times S^1 \hookrightarrow S^3$ is *trivial* (or unknoteed) if the image of a core is a trivial knot.

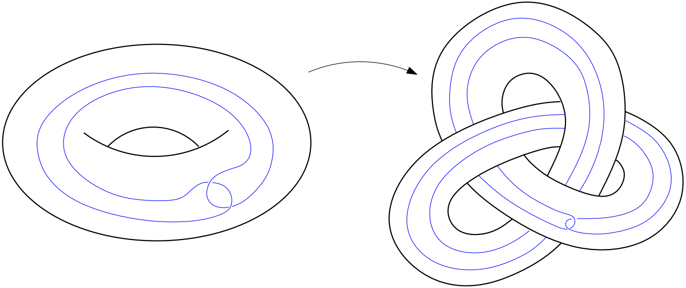


Figure: Satellite knot.

Satellite Knots

Proposition

A knot $K \subset S^3$ is statellite $\iff$ it is the image of a knot $K' \subset D^2 \times S^1$ which is neither local nor a core, along a non-trivial embedding $\varphi : D^2 \times S^1 \hookrightarrow S^3$.

Proof.

A knot $K = \varphi(K')$ constructed in this way is satellite, because the torus $\varphi(S^1 \times S^1)$ is essential in the complement of K: it is incompressible (on one side because φ is non-trivial, and on the other because K' is not local) and not ∂-parallel (because K' is not a core).
Conversely, if K is a satellite knot then its complement contains an essential torus $T \subset S^3$. As every torus in S^3, the torus T bounds a solid torus. Since T is essential, the knot K is contained in this solid torus in a non-local and non-core way. Moreover the solid torus is knotted. Otherwise T would be compressible on the other side. $\square$

Satellite Knots

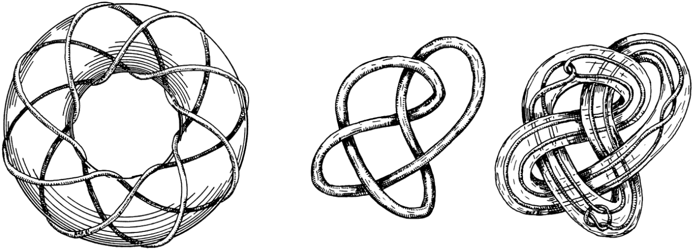


Figure: Torus and Satellite knot.

The non-trival embedding φ sends the core curve of $D^2 \times S^1$ to some non-trivial knot $H \subset S^3$ called the *companion* of K: K can be thought as orbiting as a "satellite" around its companion H.

Essential Surface in Seifert Fibration

Let $M \to S$ be a Seifert fibration. A properly embedded surface $\Sigma \subset S$ is

- *vertical* if it is a union of some regular fibers.
- *horizontal* if it is transverse to all fibers.

Proposition

Let $M \to S$ be a Seifert fibration and M is irreducible. Every essential surface Σ is isotopic to a vertical or horizontal surface.

Essential Surface in Seifert Fibration

Proposition

Let $M \to S$ be a Seifert fibration and M be irreducible. Let $\Sigma \subset M$ be an orientable connected surface. Suppose that

- *Σ is horizontal, or*
- *Σ is vertical and its projection is not as in the following figure.*

Then Σ is essential.

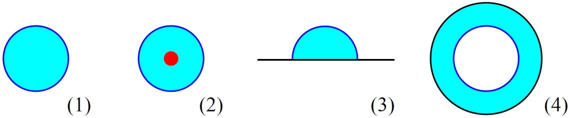


Figure: the boundary of a disk containing zero (1) or one singular cone point (2), an arc parallel to the boundary (3), or a ∂-parallel closed curve (4).

Simple Complement

A compact 3-manifold is simple if it contains no essential sphere, disc, torus, and annulus.

Lemma

Let M be irreducible and ∂-irreducible, with boundary consisting of tori. The manifold M contains no essential tori but contains some essential annuli $\iff$ it is diffeomorphic to one of the following:

$$(D,(p_1,q_1),(p_2,q_2)), \quad (A,(p,q)), \quad P\times S^1$$

with $p_1,p_2 \geq 2$, where P is a pair of pants.

Simple Complement

Proof.

The previous two propositions imply that the Seifert manifolds listed contain vertical essential annuli, but not essential tori.
Conversely, let M contain an essential annulus A. Suppose that A connects two distinct boundary tori T, T' of M. A regular neighborhood of $T \cup T' \cup A$ is diffeomorphic to $P \times S^1$, and its boundary contains a third torus $T" \subset M$. Since $T"$ cannot be essential, it is either boundary parallel or bounds a solid torus in M, and M is diffeomorphic respectively to $P \times S^1$ or a Dehn filling of it. In the latter case, the Dehn filling is not fiber-parallel because M is ∂-reducible, hence we get a Seifert manifold of type $(A, (p, q))$.
If A connects one boundary component to itself we conclude similarly and may also get $M = (D, (p_1, q_1), (p_2, q_2))$. □

Trichotomy on Knots

Proposition

Every knot $K \subset S^3$ is either a torus knot, a satellite knot, or has a simple complement.

Proof.

We only need to prove that if the complement M contains an essential annlus and no essential tori then K is a torus knot. The previous lemma gives $M = (D, (p_1, q_1), (p_2, q_2))$, and to get S^3 back we must have a Dehn filling $S^3 = (S^2, (p_1, q_1), (p_2, q_2), (1, n))$. In particular K is isotopic to a fiber and hence contained in a verticial Heegaard torus for S^3. All Heegaard toris for S^3 are isotopic, hence K is a torus knot. $\square$

Trichotomy on Knots

Theorem (Thurston 1980s)

A compact orientable three-manifold with torus boundary has interior with a hyperbolic structure if and only if there are no embedded essential: spheres, discs, tori or annuli.

Theorem (Thurston 1980s)

A knot $K \subset S^3$ is either a torus knot, a satellite knot, or a hyperbolic knot.

Hyperbolic Knot

Thick and Thin Parts

Definition (rank-1, rank-2 cusps)

Suppose G is an infinite elementary discrete group in $PSL(2,\mathbb{C})$ fixing a single point on $\partial\mathbb{H}^3$. Let H be the closed horoball of height 1:

$$H = \{(x,y,z)|z \geq 1\}$$

G is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}\times\mathbb{Z}$.

- If $G \cong \mathbb{Z}$, the quotient of the horoball H/G is homeomorphic to the space $A\times[1,\infty)$, where A is an annulus, or cylinder. We say H/G is a *rank-1 cusp*.
- If $G \cong \mathbb{Z}\times\mathbb{Z}$, H/G is homeomorphic to $T\times[1,\infty)$, where T is a Euclidean tours. We say H/G is a *rank-2 cusp*.

Thick and Thin Parts

Definition (Injective Radius)

Suppose M is a complete hyperbolic 3-manifold and $x \in M$. The *injective radius* of x, denoted $\text{injrad}(x)$, is defined to be the supremal radius r such that a metric r-ball around x is embedded.

Definition (ε-thin part and ε-thick part)

Let M be a complete hyperbolic 3-manifold, and let $\varepsilon > 0$. Define the *ε-thin part* of M, denoted as $M^{<\varepsilon}$ to be

$$M^{<\varepsilon} = \{x \in M | \text{injrad}(x) < \varepsilon/2\}.$$

Similarly, the ε-thick part, denoted as $M^{>\varepsilon}$ is defined to be

$$M^{>\varepsilon} = \{x \in M | \text{injrad}(x) > \varepsilon/2\}.$$

Thick and Thin Parts

Theorem (Structure of Thin Part)

There exists a universal constant $\varepsilon_3 > 0$ such that for $0 < \varepsilon < \varepsilon_3$, the ε-thin part of any complete, orientable, hyperbolic 3-manifold M consists of tubes around short geodesics, rank-1 cusps, and/or rank-2 cusps.

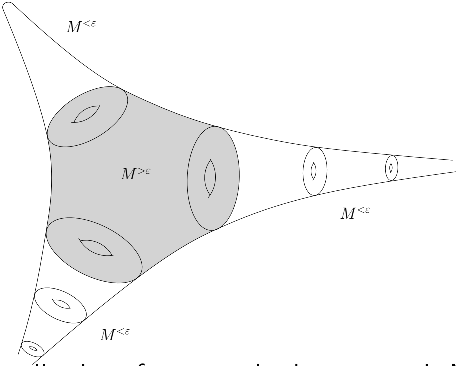


Figure: $M^{<\varepsilon}$ is a collection of cusps and tubes. $\sup \varepsilon_3$ is Margulis constant.

Finite Volume

Theorem

A hyperbolic 3-manifold M has finite volume if and only if M is closed (compact without boundary), or M is homeomorphic to the interior of a compact manifold $\overline{M}$ with torus boundary components.

The complement of any knot or link in $\mathbb{S}^3$ with a hyperbolic structure must have finite hyperbolic volume.

Figure eight Knot

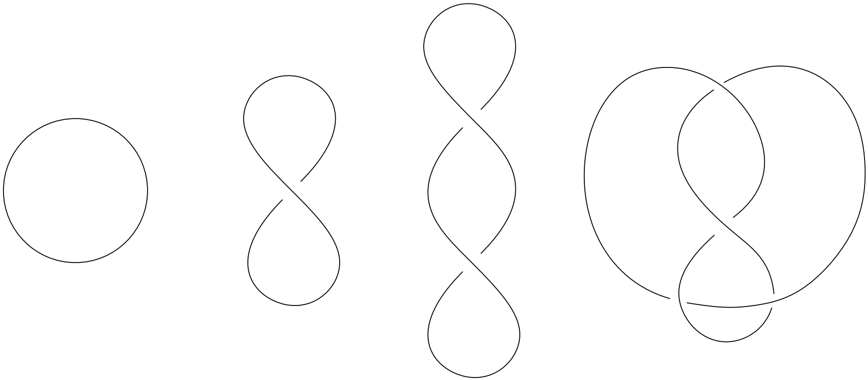


Figure: Figure eight knot.

Figure-Eight Knot

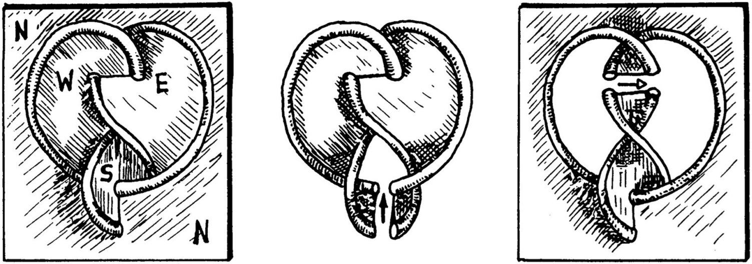


Figure: Georege Francis' description.

Figure-Eight Knot

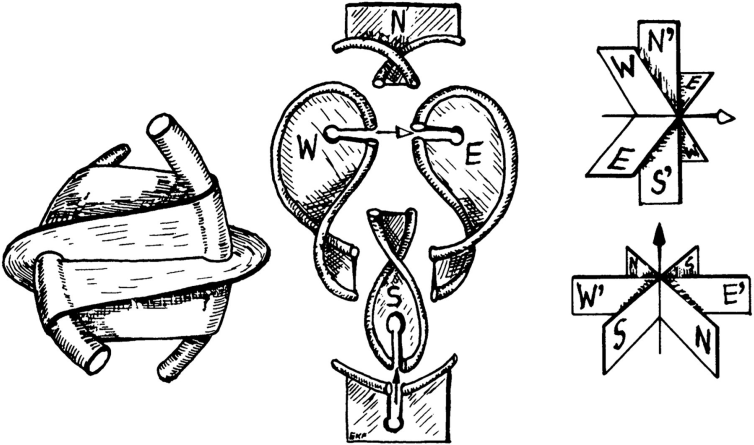


Figure: Georege Francis' description.

Figure-Eight Knot

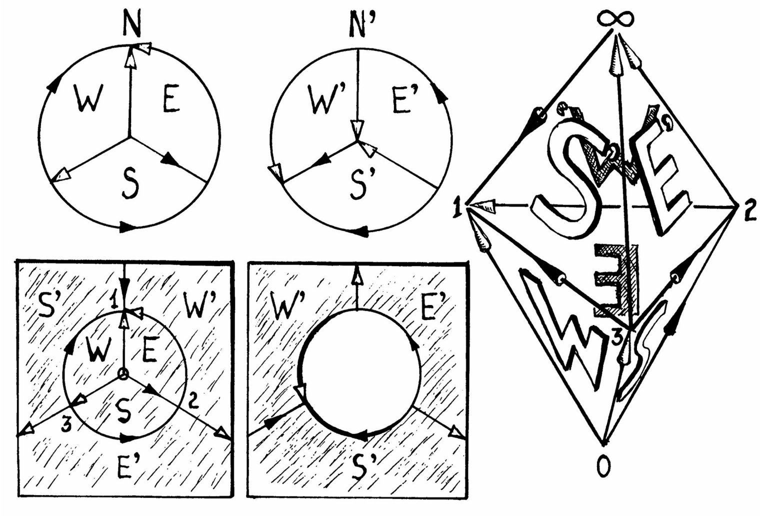
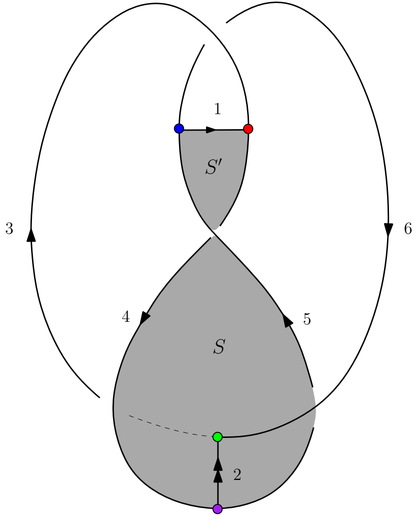


Figure: Georege Francis' description.

Figure 8 Knot



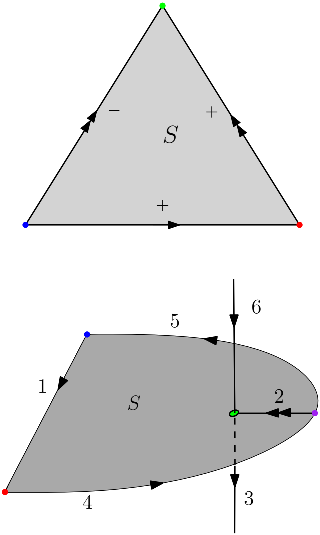


Figure: $\partial S = 1 \cdot 4 \cdot 2 \cdot 2^{-1} \cdot 5$

Figure 8 Knot

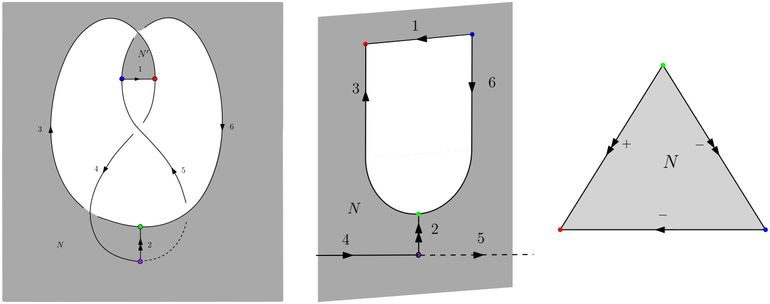


Figure: $N = 3 \cdot 1^{-1} \cdot 6 \cdot 2^{-1} \cdot 2$

Figure 8 Knot

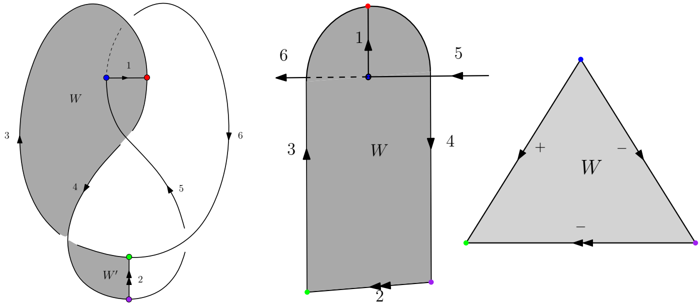


Figure: $\partial W = (3 \cdot 1^{-1} \cdot 1 \cdot 4 \cdot 2)^{-1}$

Figure 8 Knot

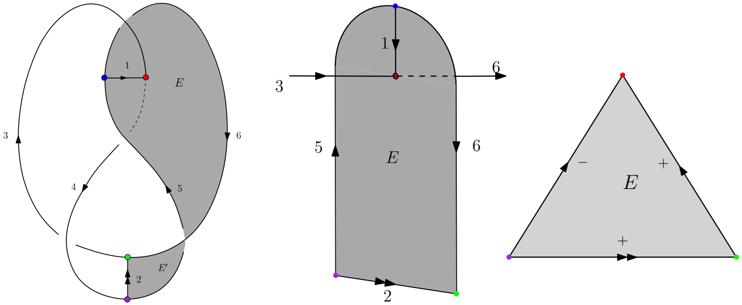


Figure: $\partial E = (6 \cdot 2^{-1} \cdot 5 \cdot 1 \cdot 1^{-1})^{-1}$

Figure 8 Knot

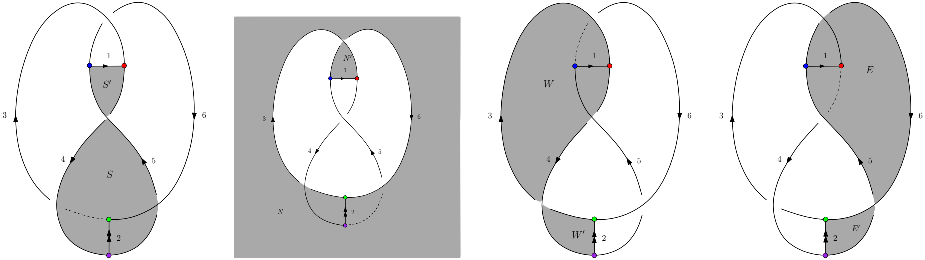


Figure 8 Knot

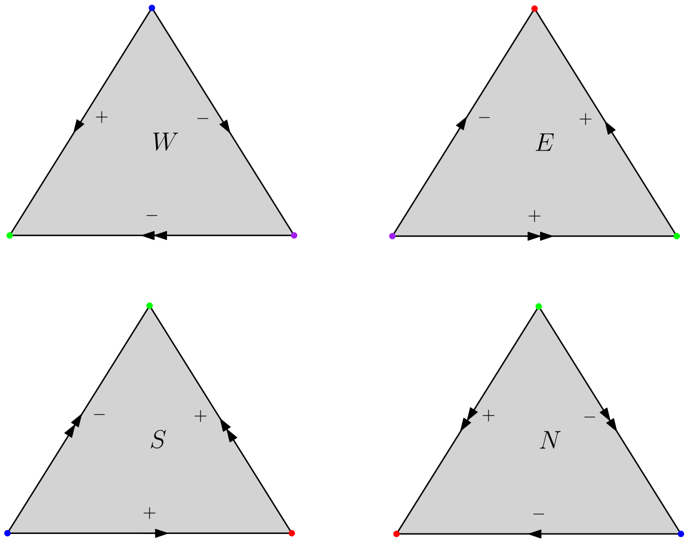
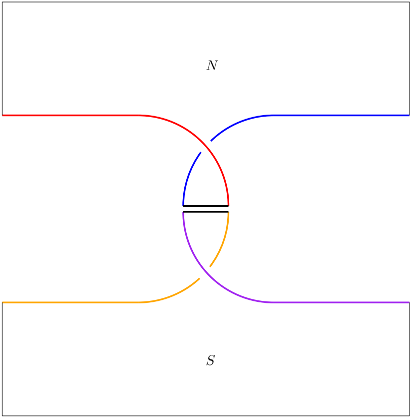


Figure 8 Knot



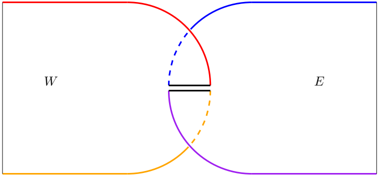


Figure: Crossing.

Figure 8 Knot - Combinatorial Operations front-back sides

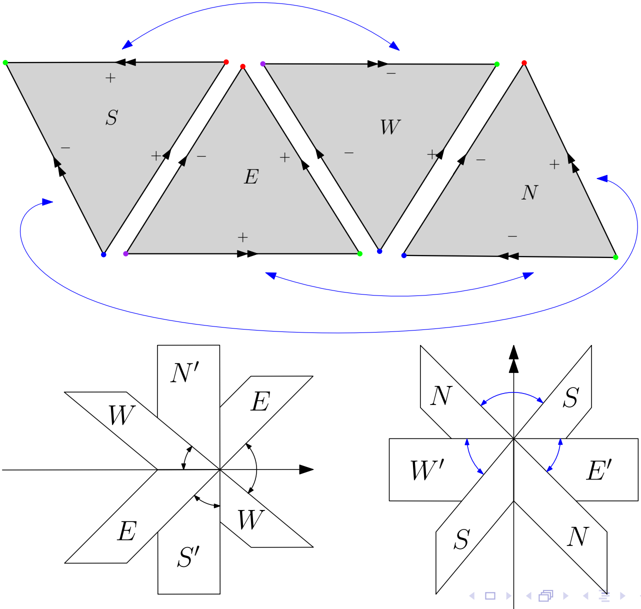


Figure 8 Knot - Combinatorial Operations front-back sides

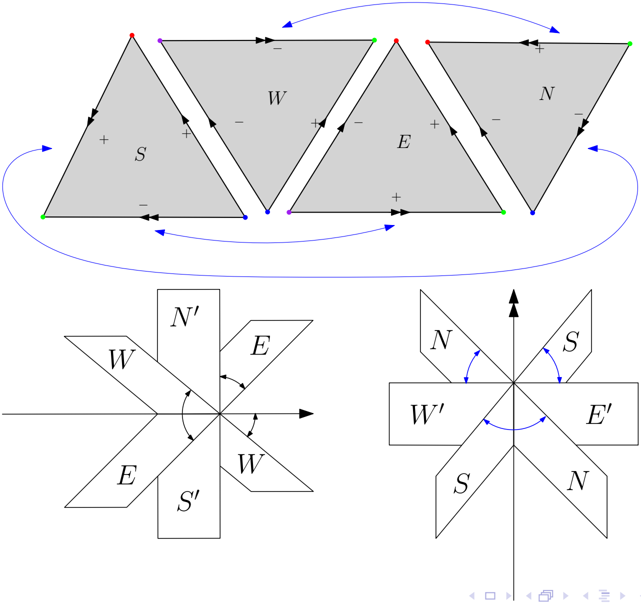


Figure 8 Knot

Delete the figure-8 know from this space, remove 3,4,5 and 6 1-cells.

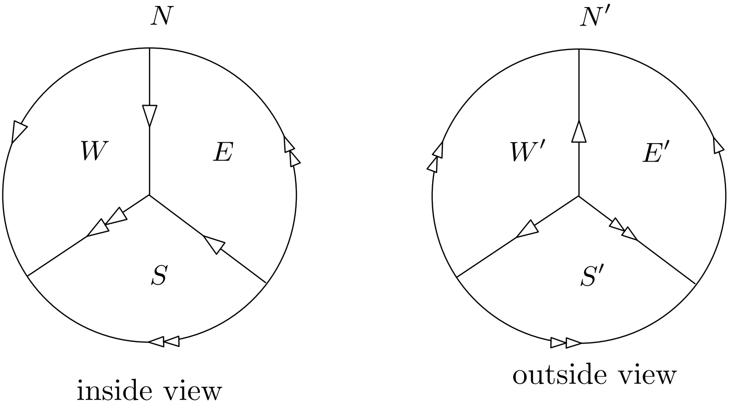


Figure: Simplified double tetrahedra. Left frame, the upper half space is the interior of one tetrahedron, viewing from top means viewing from interior. Right frame, viewing from bottom and flipping over means viewing from outside of the tetrahedron.

Figure 8 Knot

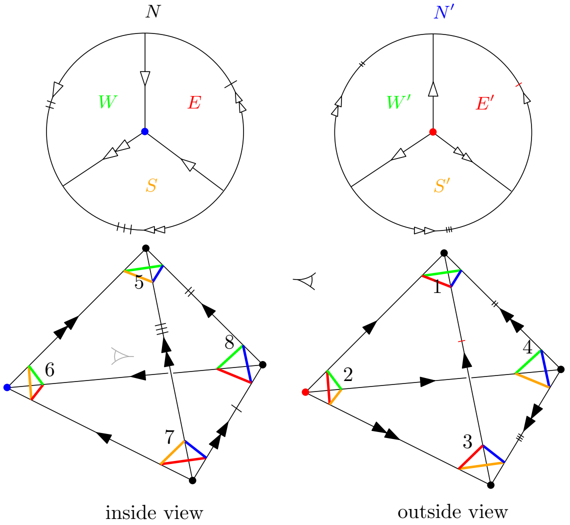
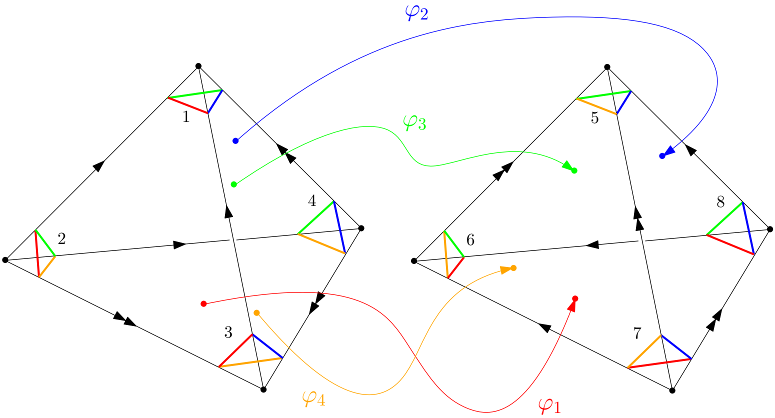


Figure: Double tetrahedra matching.

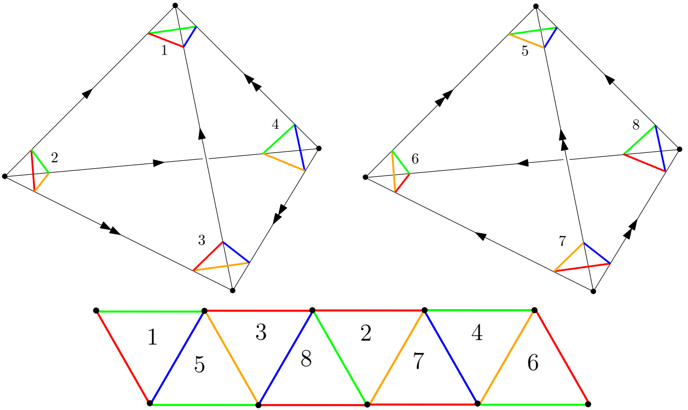
Gluing Equation



$\color{red}{\varphi 1}$	$\color{blue}{\varphi 2}$	$\color{green}{\varphi 3}$	$\color{orange}{\varphi 4}$
$\{1,2,3\}$	$\{1,3,4\}$	$\{1,4,2\}$	$\{3,2,4\}$
$\{6,7,8\}$	$\{5,8,7\}$	$\{5,6,8\}$	$\{5,7,6\}$

$$1 \xrightarrow{\varphi 1} 6 \xrightarrow{\varphi 3} 4 \xrightarrow{\varphi 2} 7 \xrightarrow{\varphi 4} 2 \xrightarrow{\varphi 3} 8 \xrightarrow{\varphi 1} 3 \xrightarrow{\varphi 4} 5 \xrightarrow{\varphi 2} 1$$

Gluing Equation



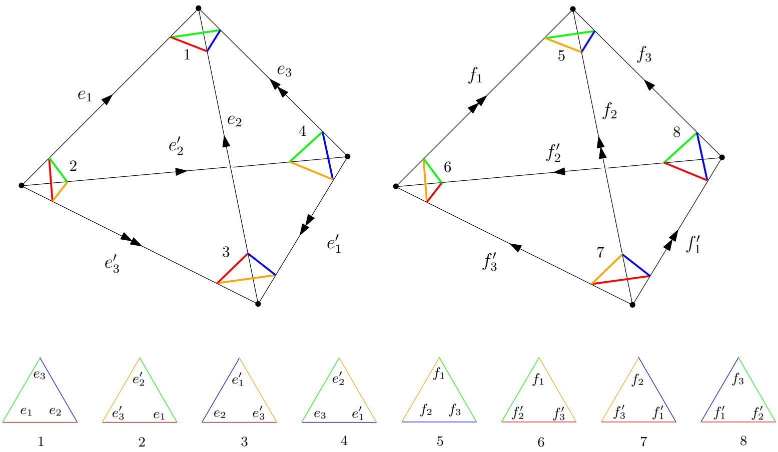


Figure: Edge invariances on vertex sections.

product of edge invariances is

$$z_1 w_3 z_2 w_2 z_2 w_3 = z_1 z_2^2 w_2 w_3^2 = 1$$

$$z \frac{1}{(1-z)^2} \frac{1}{1-w} \frac{(w-1)^2}{w^2} = 1$$

$$z' = \frac{1}{1-z}, w' = \frac{w-1}{w}$$

$$z'^2 \frac{z'-1}{z'} w'^2 \frac{w'-1}{w'} = 1$$

$$z'(z'-1)w'(w'-1) = 1$$

We obtain

$$z' = \frac{1 \pm \sqrt{1 + 4/(w'(w'-1))}}{2}$$

The imaginary parts of z' and w' to be strictively greater than 0.

The edge sequence surrounding the center is $e_1, f_3, e_2, f_2', e_2', f_3'$,

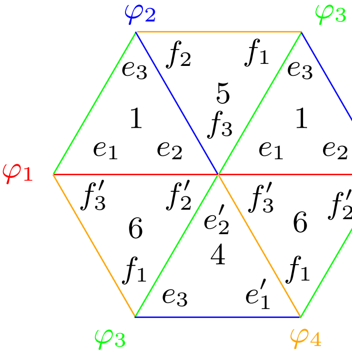


Figure: Gluing equation.

Gluing Equation

$$z_2 w_2 z_2 w_3 z_1 w_3 = z_1 z_2^2 w_2 w_3^2 = 1$$

$$\frac{z}{(1-z)^2}\frac{1}{1-w}\frac{(w-1)^2}{w^2} = 1$$

The solution exists provided that the discriminant $1 + 4/(w'(w'-1))$ is not positive real.

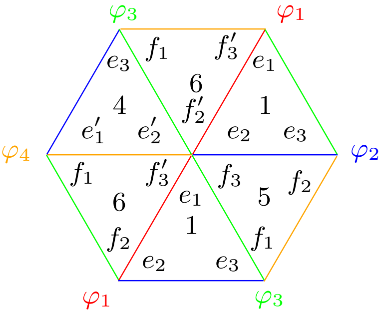


Figure: Gluing equation.

Gluing Equation

The

$$1 = z_2^{-1} w_2 z_1^{-1} w_3 z_2^{-1} w_2 z_1^{-1} w_3$$
$$= z_1^{-2} z_2^{-2} w_2^2 w_3^2$$
$$= \left(\frac{z_1 z_2}{w_2 w_3}\right)^2$$

$$1 = \frac{z_3}{w_3} = \frac{1}{1-z_2}\frac{1}{w_3} = \frac{1}{(1-z')w'}$$

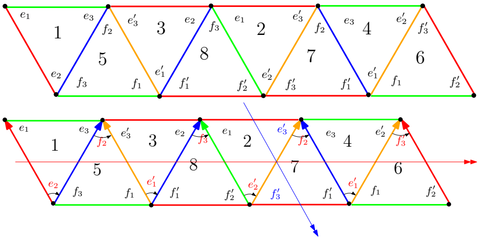


Figure: Completeness equation.

$$\begin{cases} (1-z')w' & = & 1 & \text{Completeness} \\ z'(z'-1)w'(w'-1) & = & 1 & \text{Curvature} \end{cases}$$

We obtain

$$z'^2 - z' + 1 = 0, \quad z' = \frac{1 \pm i\sqrt{3}}{2}$$

The imaginary part of z' is positive

$$z' = z_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}, w' = \frac{1}{1-z'} = \frac{1}{2} + i\frac{\sqrt{3}}{2} = w_3.$$

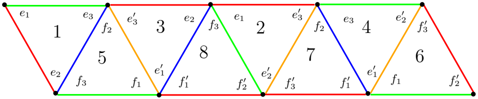


Figure: Completeness equation.