Algorithmic Strategies 2024/25

Week 3 – Backtracking

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UNIVERSIDADE DE COIMBRA

Outline

Backtracking

Reading about problem solving with backtracking

- J. Erickson, Algorithms, Chapter 2
- J. Edmonds, How to think about algorithms, Chapter 17
- S.S. Skiena, M.G. Revilla, Programming Challenges, Chapter 8

Backtracking

- Mostly used for optimization and constraint satisfaction problems.
- Backtracking uses recursion but stops it once an invalid partial solution is found, that is, extending this partial solution will always lead to an invalid/worse solution.
- Although it has the same time complexity of brute-force enumeration, it should be faster in practice
- We must carefully choose the **representation of the solution**.

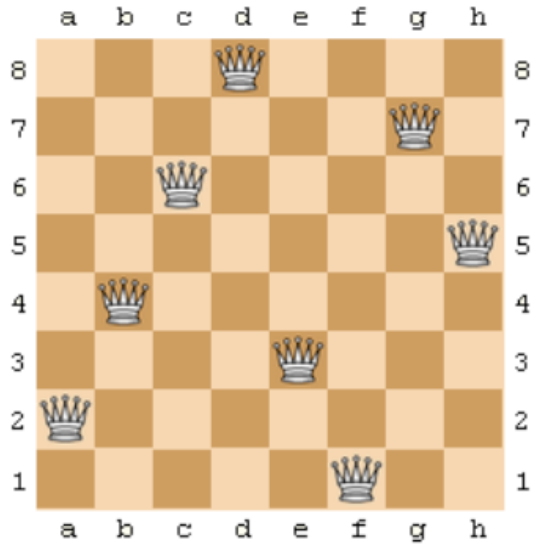
Backtracking template (for a decision problem)

Function $BT(s)$

```
if reject(s) = true then                          {rejection test}
    return false
if accept(s) = true then                          {base case}
    output(s)
    return true
while condition(s) = true do
    s' = update(s)                                {new candidate solution}
    if BT(s') = true then                         {recursive step}
        return true
return false
```

Examples

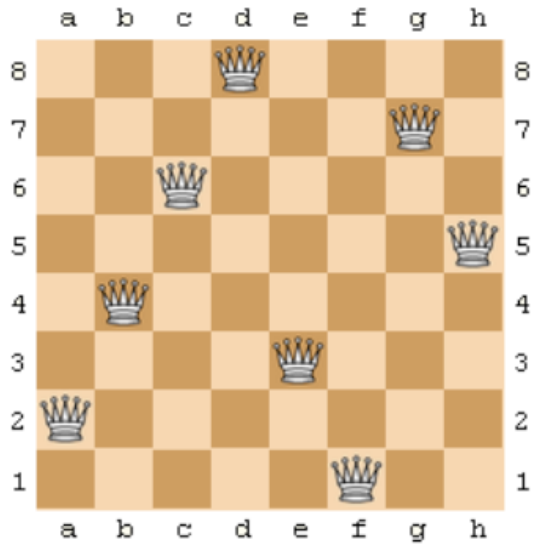
Problem: 8 Queens Problem



Find the position for 8 queens in a 8x8 chessboard such that no queen is able to capture another queen by using queen's move.

Examples

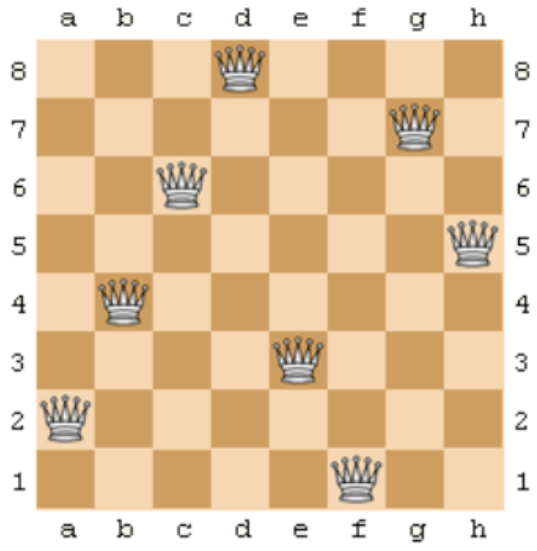
Problem: 8 Queens Problem



Solution representation: Number all squares from 1 to 64.
The solution is a boolean list of 64 positions.

Examples

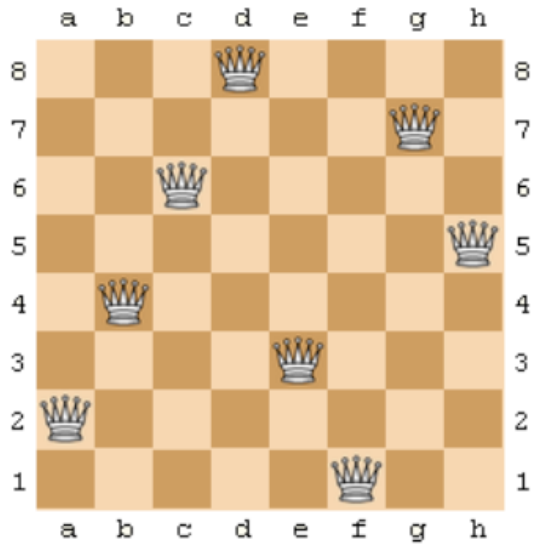
Problem: 8 Queens Problem



Solution representation: Number all squares from 1 to 64.
The solution is a boolean list of 64 positions.
This gives $2^{64} \approx 1.84 \times 10^{19}$ solutions!

Examples

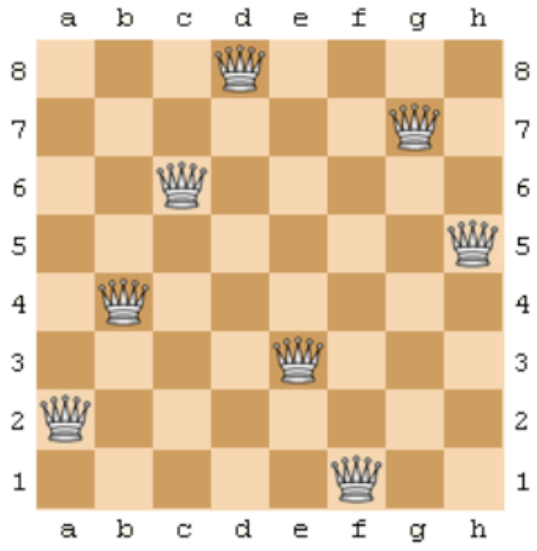
Problem: 8 Queens Problem



Solution representation: List of 8 elements, one for each queen. Assign a cell number to queen i at position i in the list.

Examples

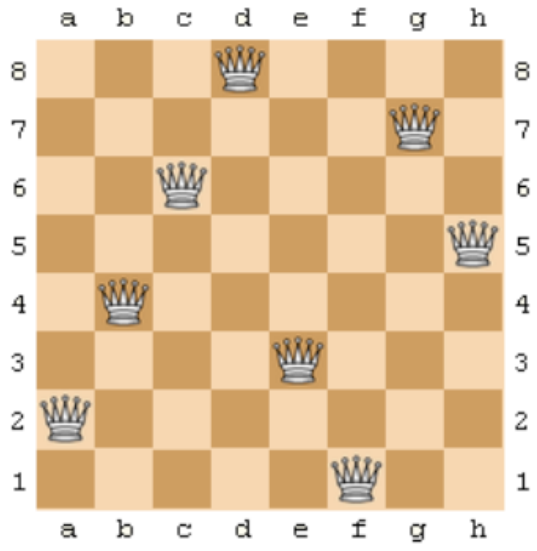
Problem: 8 Queens Problem



Solution representation: List of 8 elements, one for each queen.
Assign a cell number to queen i at position i in the list.
This gives $64^8 \approx 2.81 \times 10^{14}$ solutions!

Examples

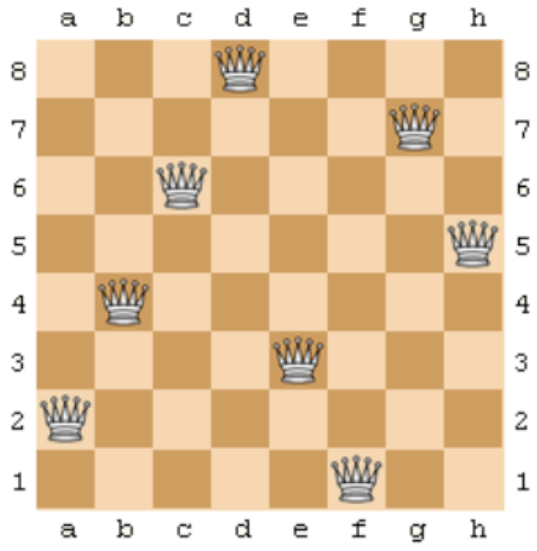
Problem: 8 Queens Problem



Solution representation: List of 8 elements, one for each queen. Two queens cannot be in the same column. Assign row number.

Examples

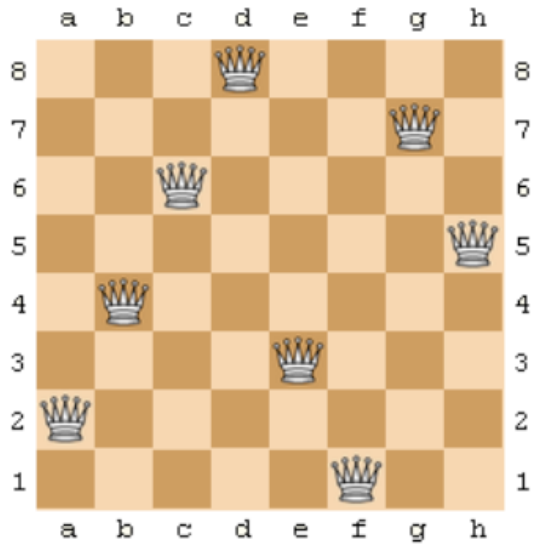
Problem: 8 Queens Problem



Solution representation: List of 8 elements, one for each queen. Two queens cannot be in the same column. Assign row number. This gives $8^8 \approx 1.67 \times 10^7$ solutions!

Examples

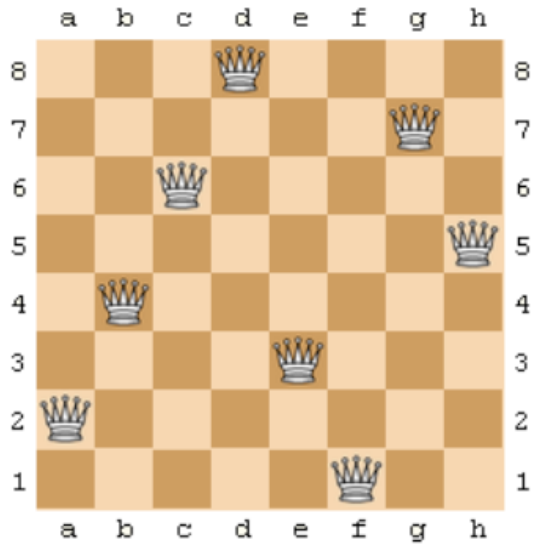
Problem: 8 Queens Problem



Solution representation: List of 8 elements, one for each queen. Two queens cannot be in the same row and column: a permutation.

Examples

Problem: 8 Queens Problem



Solution representation: List of 8 elements, one for each queen. Two queens cannot be in the same row and column: a permutation.This gives $8! \approx 40320$ solutions!

Examples

```
Function nQueens(col)
  if attack(col) = true then                          {rejection test}
    return false
  if col = N then                                     {base case}
    return true
  for i = 1 to N do                                   {for all unvisited rows}
    if row[i] = false then
      row[i] = true
      Q[col + 1] = i                                  {assignment}
      if nQueens(col + 1) = true then                 {recursive step}
        return true
      row[i] = false
  return false
```

Examples

- Solution representation is given by list Q, which assigns a row number to a queen in a given column.
- At each recursive step, the queen $col + 1$ is tested in all empty rows.
- Function $attack(col)$ checks if queen at column col is attacking any queen at column $j < col$.
- First call is $nQueens(0)$

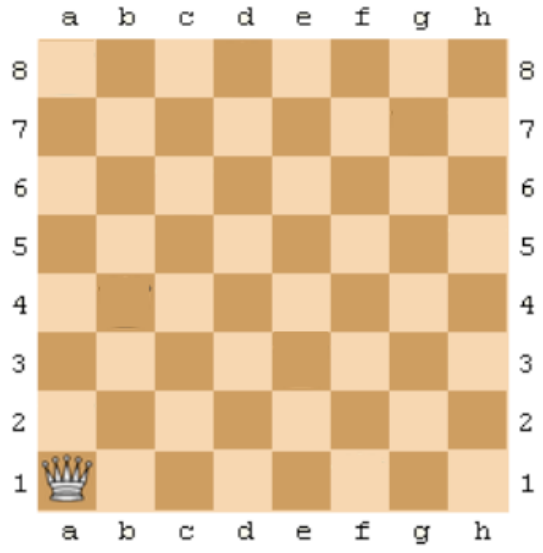
Examples

```
Function nQueens(col)
  if col = N + 1 then                                   {base case}
    return true
  for i = 1 to N do                                     {for all unvisited rows}
    if row[i] = false then
      row[i] = true
      Q[col] = i                                        {assignment}
      if attack(col) = false then
        if nQueens(col + 1) = true then                 {recursive step}
          return true
      row[i] = false
  return false
```

- Only feasible partial solutions are called recursively.
- First call is $nQueens(1)$

Examples

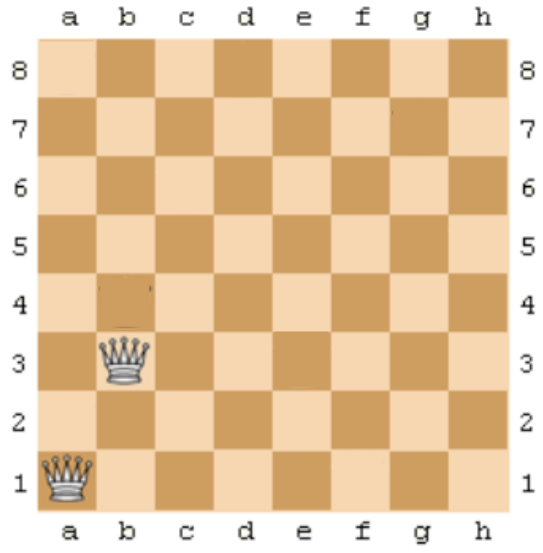
Problem: 8 Queens Problem



$$Q[a] = 1$$

Examples

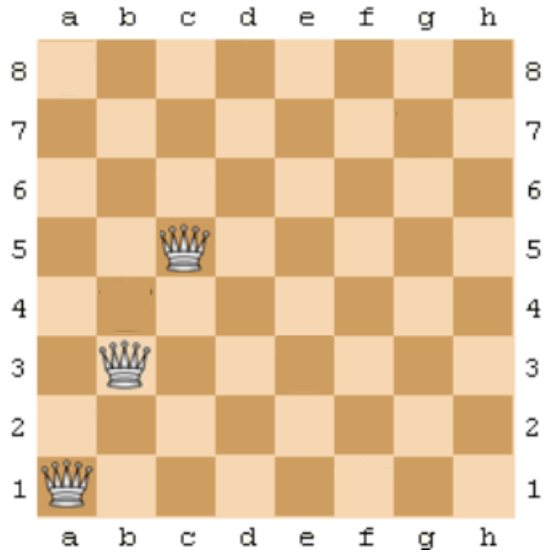
Problem: 8 Queens Problem



$Q[b] = 3$

Examples

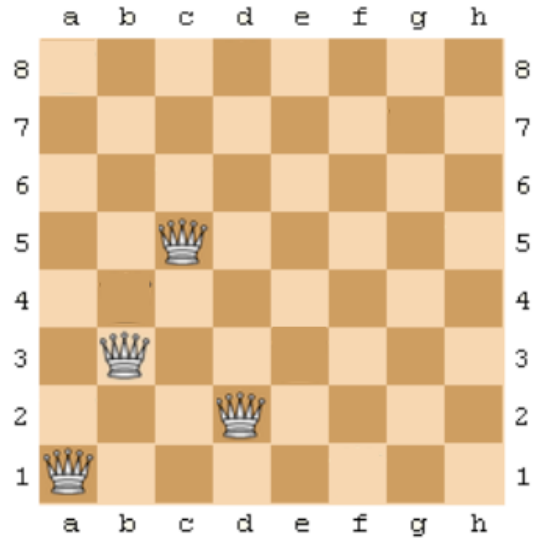
Problem: 8 Queens Problem



$$Q[c] = 5$$

Examples

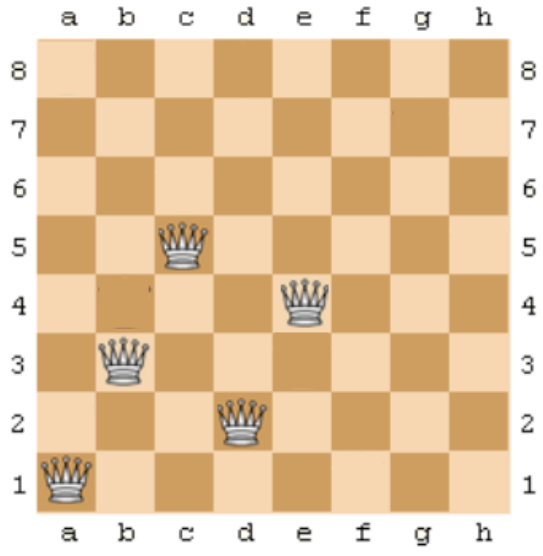
Problem: 8 Queens Problem



$$Q[d] = 2$$

Examples

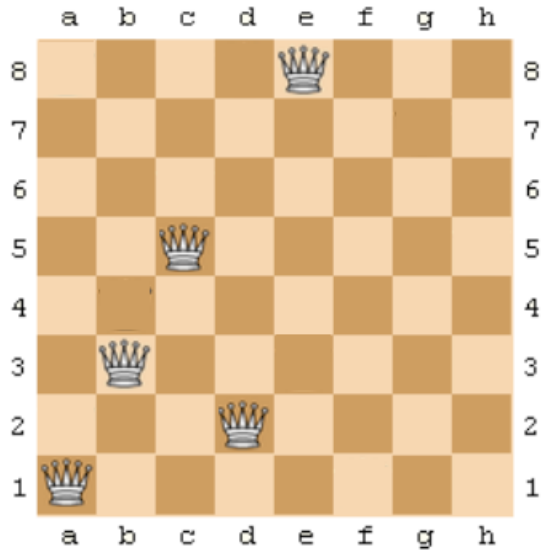
Problem: 8 Queens Problem



$Q[e] = 4$, but feasibility fails in column f.

Examples

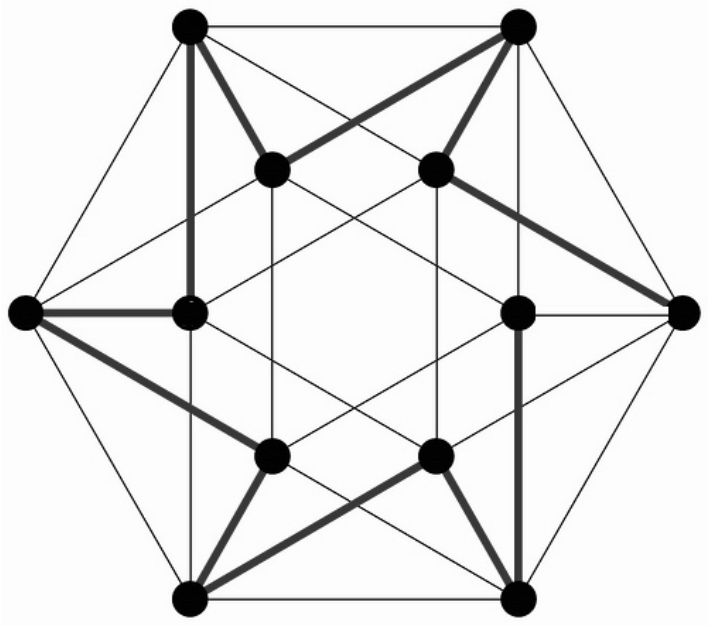
Problem: 8 Queens Problem



$Q[e] = 8$, but feasibility fails in column f. Backtrack!

Examples

Problem: Hamiltonian path with starting and ending node



Given a graph $G = (V, E)$ and two nodes s and t, determine whether a Hamiltonian path exists between node s and t.

Examples

```
Function HamPath(v)
  if v = t then                                          {base case}
    if sum(visit) = N then
      return true
    else                                                 {rejection test}
      return false
  for each {v, i} ∈ E do                                 {(implicit) rejection test}
    if visit[i] = 0 then
      visit[i] = 1                                       {mark as visited}
      if HamPath(i) = true then                          {recursive step}
        return true
      visit[i] = 0                                       {mark as unvisited}
  return false
```

- At each step, the insertion of edge $\{v, i\}$ is tested.
- Array `visit` marks the nodes that are visited.
- In the first call, mark node s and call $HamPath(s)$.

Examples

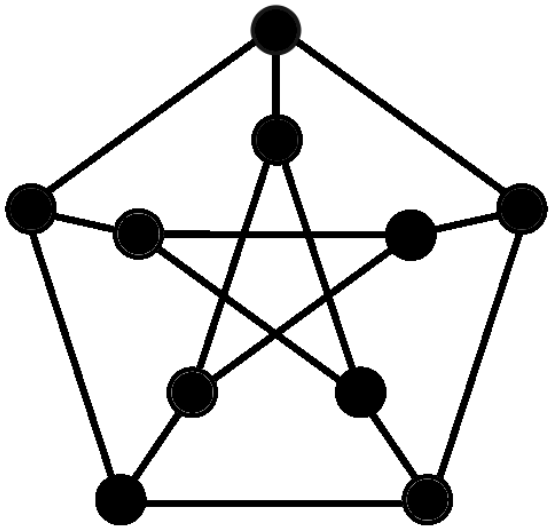
Problem: Hamiltonian cycle in graph $G = (V, E)$

Function *HamCycle*(v)
  if $sum(visit) = N$ **then** {base case}
    if $(v, 1) \in G$ **then**
      return *true*
  for each $\{v, i\} \in E$ **do** {(implicit) rejection test}
    if $visit[i] = 0$ **then**
      $visit[i] = 1$ {mark as visited}
      if *HamCycle*$(i) =$ `true` **then** {recursive step}
        return `true`
      $visit[i] = 0$ {mark as unvisited}
  return `false`

- In the first call, mark node 1 and call *HamCycle*(1).
- Check if it is a cycle at the base case.

Examples

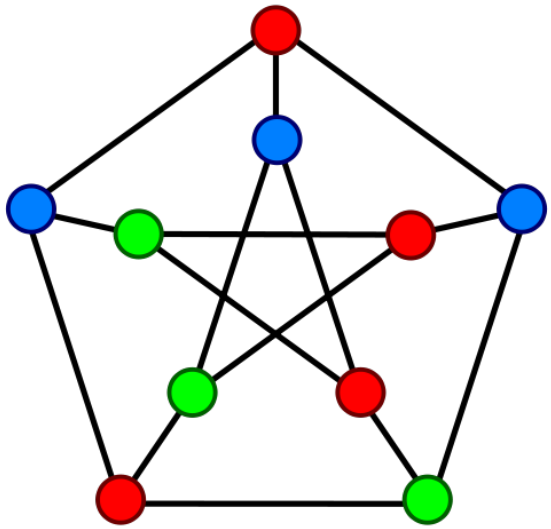
Problem: Graph coloring problem



Given a graph $G = (V, E)$ and K colors, find whether you can color the nodes such that no two adjacent nodes have the same color.

Examples

Problem: Graph coloring problem



Given a graph $G = (V, E)$ and K colors, find whether you can color the nodes such that no two adjacent nodes have the same color.

Examples

```
Function gcp(v)
  if v = N + 1 then                                   {base case}
    return true
  for i = 1 to K do                                   {for all colors}
    feasible = true
    for each {v, j} ∈ E do                            {rejection test}
      if color[j] = i then
        feasible = false
        break
    if feasible = true then
      color[v] = i                                    {assignment}
      if gcp(v + 1) = true then                       {recursive step}
        return true
      color[v] = 0                                    {remove assignment}
  return false
```

Examples

- At each step, a feasible color is assigned to a node.
- $color[v] = 0$ means that node v has no color yet.
- In the first call, color the first node and call $gcp(2)$.
- And if the goal is to minimize the number of colors?

Examples

Problem: The trip of Mr. Rowan

Mr. Rowan plans to make a walking tour of Paris. However, since he is a little lazy, he wants to take the shortest path that goes through all the places he wants to visit. He plans to take a bus to the first place and another one back from the last place, so he is free to choose the starting and ending places. Can you help him?

Examples

Problem: The trip of Mr. Rowan

Mr. Rowan plans to make a walking tour of Paris. However, since he is a little lazy, he wants to take the shortest path that goes through all the places he wants to visit. He plans to take a bus to the first place and another one back from the last place, so he is free to choose the starting and ending places. Can you help him?

It is the shortest Hamiltonian path problem (in a complete graph)

Examples

```
Function ShortPath(v, len)
  if len ≥ best then                                  {1st rejection test}
    return
  if sum(visit) = N and len < best  then              {base case}
    best = len
    return
  for each {v, i} ∈ E do                              {2nd rejection test}
    if visit[i] = 0 then
      visit[i] = 1                                    {mark as visited}
      ShortPath(i, len + M[v][i])                     {recursive step}
      visit[i] = 0                                    {mark as unvisited}
```

- Array M has the distance between every pair of locations.
- Variable *best* has the value of the shortest path found so far. It should be initialized with a large value.