Zippers and Derivatives

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Outline

Overview

- Zipper = context type, which helps moving through and "modifying" a functional data structure [1]

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- Deriving context types vs. differentiating real-valued functions [2]
- Relation with combinatorial species [3]

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  - `p = begin(l)`
  - `prev(p)`
  - `next(p)`
  - `*p := a`

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- `begin (Cons x xs) = (Nil, x, xs)`

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- ```
    left (a, (llc, lrc, IsRight p (Node b rlc rrc) anc))
       = (b, (rlc, rrc, IsLeft  p (Node a llc lrc) anc))
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- Simpler definition:

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'a btree_pointer = 'a * 'a bree_context
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'a tree_context = 'a tree list
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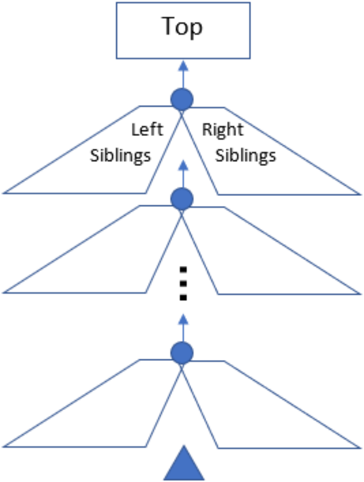
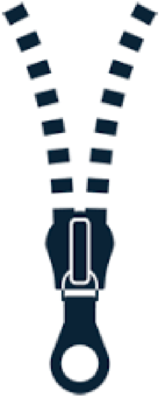
Zipper![1]





[1] Image source: www.pacifictrimming.com

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  ```
- Math-ly notation, e.g. $\mathbf{list}(a) = 1 + a \times \mathbf{list}(a)$
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- Math-ly notation, e.g. $\textbf{list}(a) = 1 + a \times \textbf{list}(a)$
- Context for an arbitrary algebraic data type?
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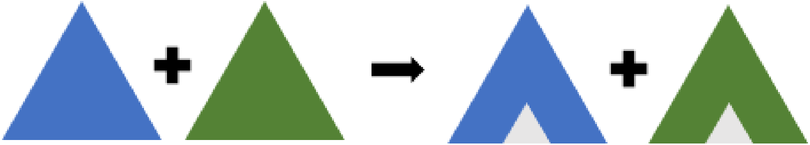
  - e.g. $C[a](\textbf{list}(a)) = \textbf{list\_context}(a) = \textbf{list}(a) \times \textbf{list}(a)$
- Context of type a inside type 1 (i.e. `unit`): impossible!
  - $C[a](1) = 0$
- Context of type a inside type a: dummy unit
  - $C[a](a) = 1$

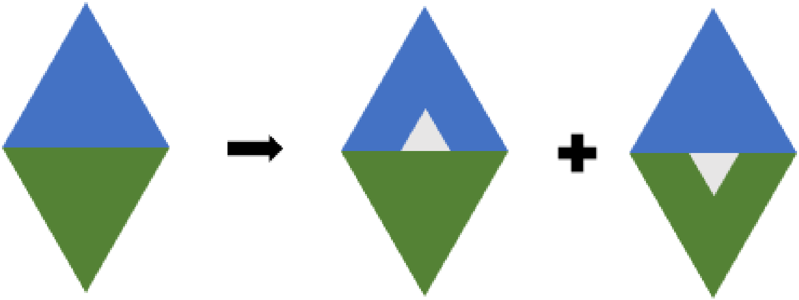
Context of Sum Type

- Inside $T_1 + T_2$, type a occurs in either of them
  - $C[a](T_1 + T_2) = C[a](T_1) + C[a](T_2)$



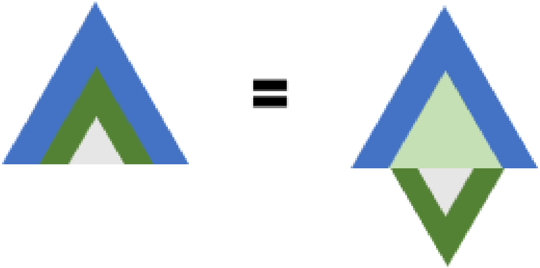
Context of Product Type

- Inside $T_1 \times T_2$, type a occurs in one of them, while the other must be carried in the context
  - $C[a](T_1 \times T_2) = C[a](T_1) \times T_2 + T_1 \times C[a](T_2)$ content



Context of Composed Type

- Inside composed type $T(U(a))$, type a occurs in one of U, which resides somewhere in T
  - $C[a](T(U(a))) = C[b](T(b))|_{b=U(a)} \times C[a](U(a))$



Context as Derivative

- Rules for context:
  - $C[a](1) = 0$
  - $C[a](a) = 1$
  - $C[a](T_1 + T_2) = C[a](T_1) + C[a](T_2)$
  - $C[a](T_1 \times T_2) = C[a](T_1) \times T_2 + T_1 \times C[a](T_2)$
  - $C[a](T(U(a))) = C[b](T(b))|_{b=U(a)} \times C[a](U(a))$
- Rules for derivative:
  - $\frac{\partial}{\partial x} c = 0$
  - $\frac{\partial}{\partial x} x = 1$
  - $\frac{\partial}{\partial x}(f + g) = \frac{\partial}{\partial x} f + \frac{\partial}{\partial x} g$
  - $\frac{\partial}{\partial x}(f \cdot g) = \frac{\partial}{\partial x} f \cdot g + f \cdot \frac{\partial}{\partial x} g$
  - $\frac{\partial}{\partial x}(f \circ g)|_{x_0} = \frac{\partial}{\partial u} f|_{u=g(x_0)} \cdot \frac{\partial}{\partial x} g|_{x=x_0}$
- $\frac{\partial}{\partial \mathrm{a}}(T) \triangleq C[a](T)$

Context of List, Revisit

$$\frac{\partial}{\partial \mathsf{a}}\,\textbf{list}(a) = \frac{\partial}{\partial \mathsf{a}}(1 + a \times \textbf{list}(a))$$

Context of List, Revisit

$$
\begin{aligned}
\frac{\partial}{\partial \mathsf{a}} \textbf{list}(a) &= \frac{\partial}{\partial \mathsf{a}}(1 + a \times \textbf{list}(a)) \\
&= 0 + \frac{\partial}{\partial \mathsf{a}}(a \times \textbf{list}(a))
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Context of List, Revisit

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&= \textbf{list}(a) + a \times \frac{\partial}{\partial \mathsf{a}} \,\textbf{list}(a)
\end{aligned}$$

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&= \mathbf{list}(a) + a \times \frac{\partial}{\partial \mathsf{a}} \mathbf{list}(a) \\
\frac{\partial}{\partial \mathsf{a}} \mathbf{list}(a) &= \mathbf{list}(a) \times \mathbf{list}(a)
\end{aligned}$$

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&= \textbf{list}(a) + a \times \frac{\partial}{\partial \mathsf{a}} \textbf{list}(a) \\
\frac{\partial}{\partial \mathsf{a}} \textbf{list}(a) &= \textbf{list}(a) \times \textbf{list}(a) \\
&= \textbf{list\_context}(a)
\end{aligned}
$$

Binary Tree Context Revisit

$$\frac{\partial}{\partial \mathsf{a}} \textbf{btree}(a) = \frac{\partial}{\partial \mathsf{a}}(1 + a \times \textbf{btree}^2(a))$$

Binary Tree Context Revisit

$$\begin{aligned}\frac{\partial}{\partial \mathsf{a}} \mathbf{btree}(a) &= \frac{\partial}{\partial \mathsf{a}}(1 + a \times \mathbf{btree}^2(a)) \\ &= 0 + \frac{\partial}{\partial \mathsf{a}}(a \times \mathbf{btree}^2(a))\end{aligned}$$

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&= \mathbf{btree}^2(a) + a \times 2 \times \mathbf{btree}(a) \times \frac{\partial}{\partial \mathsf{a}} \mathbf{btree}(a)
\end{aligned}
$$

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\frac{\partial}{\partial \mathsf{a}} \mathbf{btree}(a) &= \mathbf{btree}^2(a) \times \mathbf{list}(2 \times a \times \mathbf{btree}(a))
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\end{aligned}$$

Context of Tree, Revisit

$$\frac{\partial}{\partial \mathsf{a}} \textbf{tree}(a) = \frac{\partial}{\partial \mathsf{a}}(1 + a \times \textbf{list}(\textbf{tree}(a)))$$

Context of Tree, Revisit

$$\begin{aligned}\frac{\partial}{\partial \mathsf{a}} \mathbf{tree}(a) &= \frac{\partial}{\partial \mathsf{a}}(1 + a \times \mathbf{list}(\mathbf{tree}(a))) \\ &= 0 + \frac{\partial}{\partial \mathsf{a}}(a \times \mathbf{list}(\mathbf{tree}(a)))\end{aligned}$$

Context of Tree, Revisit

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\frac{\partial}{\partial \mathsf{a}} \textbf{tree}(a) &= \frac{\partial}{\partial \mathsf{a}}(1 + a \times \textbf{list}(\textbf{tree}(a))) \\
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&= \textbf{list}(\textbf{tree}(a)) + a \times \frac{\partial}{\partial \mathsf{a}}(\textbf{list}(\textbf{tree}(a)))
\end{aligned}
$$

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&= \mathbf{list}(\mathbf{tree}(a)) + a \times \mathbf{list}^2(\mathbf{tree}(a)) \times \frac{\partial}{\partial \mathsf{a}}(\mathbf{tree}(a))
\end{aligned}
$$

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\end{aligned}
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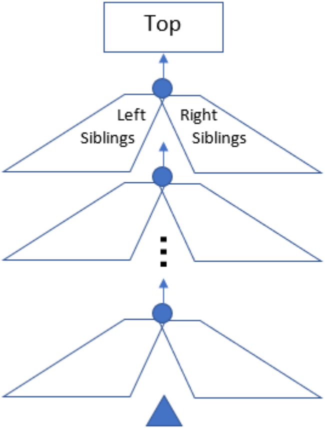
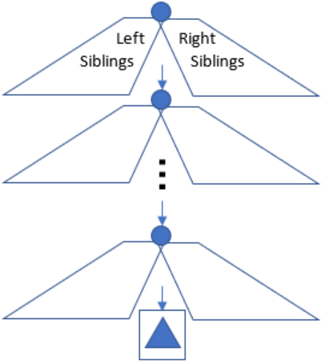
\frac{\partial}{\partial \mathsf{a}} \textbf{tree}(a) &= \textbf{list}(\textbf{tree}(a)) \times \textbf{list}(a \times \textbf{list}^2(\textbf{tree}(a))) \\
&= \textbf{tree\_context}(a)
\end{aligned}
$$

Huet's Zipper, Revisit

- $\mathbf{ltree}(a) = a + \mathbf{list}(\mathbf{ltree}(a))$
- Differentiating against non-basic type is a bit tricky
- $\frac{\partial}{\partial\,\mathbf{ltree}}(\mathbf{ltree}) = 1$?
- $\frac{\partial}{\partial\,\mathbf{ltree}}(\mathbf{ltree}) = \frac{\partial}{\partial\,\mathbf{ltree}}(a + \mathbf{list}(\mathbf{ltree}))$?
- A hack: $$\begin{aligned}\frac{\partial}{\partial\,\mathbf{ltree}} &= ((\mathbf{1})) + \frac{\partial}{\partial\,\mathbf{ltree}}(a + \mathbf{list}(\mathbf{ltree})) \\ &= \mathbf{1} + \frac{\partial}{\partial\,\mathbf{ltree}}(\mathbf{list}(\mathbf{ltree})) \\ &= \mathbf{1} + \mathbf{list}^2(\mathbf{ltree}) \times \frac{\partial}{\partial\,\mathbf{ltree}}(\mathbf{ltree}) \\ &= \mathbf{ltree\_context}\end{aligned}$$

- $\frac{\partial}{\partial a}\,\mathbf{tree}(a) = \mathbf{list}(\mathbf{tree}(a)) \times \mathbf{list}(a \times \mathbf{list}^2(\mathbf{tree}(a)))$
- Reversed interpretation of the recursion path:





Subtraction and Division?

- $\textbf{list}(a) = 1 + a \times \textbf{list}(a)$

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Subtraction and Division?

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- $\textbf{list}(a) \leftrightarrow \frac{1}{1-a}$?
- $\frac{\partial}{\partial \mathrm{a}} \textbf{list}(a) \leftrightarrow \frac{\partial}{\partial \mathrm{a}}(\frac{1}{1-a}) = \frac{1}{(1-a)^2} \leftrightarrow \textbf{list}^2(a)$?!

Outline

Species

- General definition: endofunctor on the category of finite sets

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  - ...
- Species, as functors, preserve identity arrow and composition of arrows
  - should be obvious for *regular* species

Regular Species

- Composition of 0, 1, X, $+$, $\cdot$ and least fix-point

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- $n \triangleq \underbrace{1 + (1 + (\dots + 1))}_{n}$: $\{\} \to \{\{\}_1, \{\}_2, \dots, \{\}_n\}$

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- $X^n \triangleq \underbrace{X \cdot (X \cdot (\dots \cdot X))}_{n}$: $\{1, 2, \dots, n\} \to \{\text{permutations}\}$
- $L \triangleq 1 + X \cdot L = 1 + X \cdot (1 + X \cdot (\dots)) \simeq 1 + X + X^2 + \dots$

Exponential Generating Function

- Counting function: $C_F : |L| \to |F(L)|$ for species F
- Exponetial Generating Function:
  $E_F(x) = C_F(0) + C_F(1) * \frac{x}{1!} + C_F(2) * \frac{x^2}{2!} + \dots$
- $E_0(x) = 0$
- $E_1(x) = 1$
- $$\begin{aligned} E_{F+G}(x) &= (C_F(0) + C_G(0)) + (C_f(1) + C_G(1)) * \frac{x}{1!} + \dots \\ &= E_F(x) + E_G(x) \end{aligned}$$
- $$\begin{aligned} C_{F \cdot G}(i) * \frac{x^i}{i!} &= \sum_{j=0}^{i} \frac{x^i}{i!} * \binom{i}{j} * C_F(j) * C_G(i-j) \\ &= \sum_{j=0}^{i} \frac{x^j}{j!} * C_F(j) * \frac{x^{i-j}}{(i-j)!} * C_G(i-j) \end{aligned}$$
- $E_{F \cdot G}(x) = E_F(x) * E_G(x)$
- E_F and F share the same expression!

Derivative of Regular Species

- $F' : S \to F(S \cup \{\Box\})$
  - e.g. $(X^2)' : \{1\} \to X^2(\{1, \Box\}) = \{(1, \Box), (\Box, 1)\}$
- Derivative rules apply:
  - $(F + G)' = F' + G'$
  - $(F \cdot G)' = F' \cdot G + F \cdot G'$
  - $\ldots$
- $(X^n)' = n * X^{n-1}$
- $E_{(X^n)'}(x) = E_{n \cdot X^{n-1}}(x) = n * x^{n-1} = (x^n)' = (E_{X^n}(x))'$
- $E_{F'} = (E_F)'$
- Derivative preserves the consistency between regular species and its counting function!

List, Revisit

- $L = 1 + X \cdot L$

List, Revisit

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- $E_L(x) = 1 + x \cdot E_L(x)$

List, Revisit

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List, Revisit

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- $E_L(x) = 1 + x \cdot E_L(x)$
- $E_L(x) = \frac{1}{1-x}$
- $L = \frac{1}{1-X} =?$

List, Revisit

- $L = 1 + X \cdot L$
- $E_L(x) = 1 + x \cdot E_L(x)$
- $E_L(x) = \frac{1}{1-x}$
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List, Revisit

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- $E_L(x) = \frac{1}{1-x}$
- $L = \frac{1}{1-X} = (\textbf{THE } F.\ E_F(x) = \frac{1}{1-x})$
  - $E_F = E_G \Longrightarrow F \simeq G$

Outline

Overview

Zipper Examples

Context as a Derivative

Combinatorial Species

Conclusion

Conclusion

- "Functional pointer": context type
- The structure of context of ordered tree resembles a zipper
- Differentiating an algebraic datatype
- Combinatorial species and its EGF
- Regular species vs. algebraic datatype?

Thank you for listening!

Reference

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2. McBride, Conor. "The derivative of a regular type is its type of one-hole contexts." Unpublished manuscript (2001): 74-88.
3. Yorgey, Brent. "Functional Pearl: Species and Functors and Types, Oh My!."