Summer School Alpbach: Multiple Zeta-Values[1]

September 26, 2012

[1]The workshop was organized by J. Ayoub, S. O. Gorchinskiy and G. Wüstholz (Chair) within the ETH Zurich ProDoc module in Arithmetic and Geometry. The speakers were Mario Huicochea, Roland Paulin, Fritz Hörmann, Alberto Vezzani, Thomas Weissschuh, Sergey Rybakov, Javier Fresán, Rafael von Känel, Konrad Völkel, Martin Gallauer, Simon Pepin Lehalleur, Lars Kühne, Joseph Ayoub and Sergey Gorchinskiy. The notes were recorded by Jonathan Skowera.

Contents

About these notes

These are notes from the Workshop on Multiple Zeta-Values in Alpbach, Austria organized by the ProDoc Arithmetic and Geometry module of the ETH and the Universität Zürich and meeting from September 2nd – 7th, 2012. The workshop aims to study the motivic approach to multiple zeta values (MZV's) included recent advances due to Francis Brown. We especially rely on the presentation of P. Deligne in and *Multizetas, d'apres Francis Brown* and refer to the paper *Groupes fondamentaux motiviques de Tate mixte* by P. Deligne and A. Goncharov for many basic facts.

Please attribute all errors first to the scribe.

Preliminary Talk: Unipotence, periods and motivations

Brent Doran on August 27th, 2012.

Our introductory talks will concern themselves with three main questions.

1. What are periods?
2. How do they classically arise in Hodge theory?
3. How do they arise in our motivic context?

Toplogical and geometric motivations

We start at the beginning, in order not to lose sight of the roots. Fix a field k, of characteristic 0 if necessary.

Definition 1 (Algebraic group)**.** An algebraic group over k is a k-variety with morphisms of varieties, $m : G \times G \to G$ and $i : G \to G$, satisfying the usual group axioms for multiplication, inversion and identity.

Example 2. Examples of algebraic groups are finite groups, the multiplicative group of k ($\mathbb{G}_m$), the additive group of k ($\mathbb{G}_a$), elliptic curves, etc.

Definition 3 (Group extension)**.** A group G is an *extension* of a group Q by N if there is a short exact sequence

$$1 \to N \to G \to Q \to 1$$

It is called a *central extension* if N lies in the center of G.

Proposition 4 (Key Fact)**.** *Every unipotent algebraic group over a field k of characteristic 0 is an iterated central extension of $\mathbb{G}_a$.*

How general are these among algebraic groups?

Classifying algebraic groups

The classification proceeds according to the following plan.

$$\begin{tikzcd}
 & & \text{algebraic groups} \arrow[d] & \\
 & & \text{connected} \arrow[dl] \arrow[dr] & \\
 & \text{affine} \arrow[d, equal] & & \text{abelian variety} \\
 & \text{linear} \arrow[dl] \arrow[dr] & & \\
\text{unipotent} \arrow[d, equal] & & \text{reductive} \arrow[d] & \\
\text{strictly upper triang.} & & \text{semisimple} &
\end{tikzcd}$$

G/G^0 finite group

Chevalley

Linearize

Levi decomp.

torus

Classification of representations of reductive groups proceeds by the theory of weights in the co-character lattice of maximal torus. Classification of representations of unipotent groups is hard.

Connected algebraic groups

Remark 5. Let G^0 be the connected component of the identity. Then $G^0 \triangleleft G$, and G^0 is finite index. The quotient G/G^0 can be chosen to be an arbitrary finite group, so when we classify, we can simply consider connected G. The rest of the groups will be extensions of finite groups by these connected groups.

Reducing to affine and projective groups

Definition 6 (Affine algebraic group)**.** An algebraic group is *affine* if its underlying k-variety is affine.

Definition 7 (Projective algebraic group)**.** Similarly, an algebraic group is *projective* if its underlying k-variety is projective.

For example, abelian varieties are projective algebraic groups.

Remark 8. Non-affine, non-projective algebraic groups arise naturally as Jacobians of singular curves and universal additive extensions of abelian varieties.

Remark 9. But all algebraic groups are quasi-projective.

Theorem 10 (Chevalley)**.** *An algebraic group G admits a unique normal affine subgroup H such that*

$$1 \to H \to G \to A \to 1$$

for an abelian variety A.

The above theorem justifies studying affine and projective groups separately. We focus here on affine groups.

Proposition 11 (Linearize a G-variety)**.** *Let G be an affine algebraic group acting on an affine variety X. Then there exists a unique linearization, i.e., a G-equivariant closed immersion $X \hookrightarrow V$ into a finite dimensional G-representation V.*

Corollary 12. *Affine algebraic groups are in fact linear algebraic groups, i.e., admit inclusions $G \hookrightarrow GL(V)$.*

Unipotent groups

The definition of a unipotent group follows rests on the definition of unipotence for linear transformations.

Definition 13 (Jordan decomposition)**.** Given $g \in GL_N$, there exists a unique decomposition $g = g_s \cdot g_u$ such that

- g_s is semisimple, i.e., diagonalizable over $\overline{k}$.
- g_u is unipotent, i.e., all eigenvalues are 1.
- $g_s g_u = g_u g_s$.

Proposition 14. *If $f : G \to H$ is a homomorphism of linear algebraic groups, then it preserves the Jordan decomposition*

$$f(g)_s = f(g_s) \qquad f(g)_u = f(g_u)$$

By the Corollary 12, the notion of semisimplicity and unipotence is defined for elements of affine algebraic groups.

Definition 15 (Unipotent group)**.** An affine algebraic group G is *unipotent* if all its elements are unipotent.

Remark 16 (Warning!). The analogous is NOT true for the definition of semisimple groups.

Proposition 17. *Any connected unipotent group is isomorphic to a subgroup of strictly upper triangular matrices.*

Definition 18 (Derived group)**.** Let G be an algebraic group. Then its *derived groups* are given by $G' := \overline{[G, G]}$, $G'' := \overline{[G', G']}$, etc.

Definition 19 (Solvable group)**.** If this sequence terminates, then G is said to be *solvable.*

Theorem 20 (Lie-Kolchin)**.** *Every solvable algebraic group G over an algebraically closed dield can be embedded into some GL_N as upper triangular matrices.*

Example 21. The group $\mathbb{G}_m \ltimes U$ is pro-unipotent.

Definition 22 (Unipotent radical)**.** Let G be affine and connected. The *unipotent radical* $r_u(G)$ of G is the maximal connected unipotent normal subgroup of G.

Definition 23 (Reductive algebraic group)**.** An affine, connected algebraic group G is *reductive* if $r_u(G) = 0$. It is called redutive because its representations always decompose into a direct sum of irreducible representations. Hence to understand $Rep(G)$ for a reductive group G, one need only understand the irreducible representation, a feat accomplished by the theory of highest weights.

Definition 24 (Radical)**.** let G be affine and connected. Then the *radical* $r(G)$ of G is the maximal connected solvable normal subgroup of G.

Then $r_u(G) \subset r(G)$.

Definition 25 (Semisimple algebraic group)**.** An affine, connected algebraic group G is *semisimple* if $r(G) = 0$.

Definition 26 (Torus)**.** An affine, connected algebraic group G is a *torus*, if there is a linearization $G \hookrightarrow \boldsymbol{GL}_N$ such that all of its elements map to diagonal matrices.

Proposition 27. *Every reductive group is an extension of a semisimple group by a torus, and torii are rigid (their morphisms cannot be deformed) and classified by a discrete invariant.*

Theorem 28 (Levi decomposition)**.** *Let G be an affine algebraic group. Then there exists a reductive subgroup H such that*

$$G = r_u(G) \rtimes H$$

The subgroup H is unique up to conjugation and called the Levi factor*.*

Example 29. If G is commutative, the Levi decomposition has the form $G = U \times T = (\mathbb{G}_a)^k \times T$.

So what can we say about representation theory? We study reductive and unipotent groups separately. There are only discrete choices with arbitrary dimensional moduli which are arbitrary badly behaved, a situation known as Murphy's Law.

The classification of unipotent groups themselves is very hard. What can be said:

- As a variety, every unipotent group is isomorphic to $\mathbb{A}^n$.
- The orbits of a unipotent group are all closed subvarieties.
- Abelian unipotent groups in characteristic zero are of the form $\mathbb{G}_a^k$.

- Unipotent groups in characteristic zero are iterated extensions by $\mathbb{G}_a$'s.

Definition 30 (Lower central series)**.** Given a group G (not necessarily algebraic), its *lower central series* is

$$G_1 = G, \qquad G_{i+1} = [G_i, G]$$

For example, consider the following unipotent group and its lower central series.

$$G = G_1 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in k \right\} \qquad G_2 = \left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} | t \in k \right\}$$

and $G_3 = 1$. Because $G_3 = 1$, we say that G has nilpotency class 2, or that G is a step 2 nilpotent group. More generally, the $n \times n$ strictly upper triangular matrices have nilpotency class $n - 1$.

Remark 31*.* Affine algebraic groups G are determined by $\mathrm{Rep}(G)$ in a Tannakian manner.

Discrete groups

There exists a useful formalism for affine algebraic groups, but not for general discrete groups. Given a finitely generated discrete group Γ, e.g., $\boldsymbol{GL}(n, \mathbb{Z})$, what can we say? If Γ were an algebraic group, we might study it by looking for a variety X and an embedding of $\Gamma \subset \mathrm{Aut}(X)$. In the discrete case, we find a manifold M with $\Gamma = \pi_1(M)$. This leads to covering spaces! But a direct algebro-geometric interpretation of such Γ is difficult, e.g., not every Γ is realized as a profinite completion.

Definition 32. A $\mathbb{Q}$-*local system* on a manifold M is a locally constant sheaf[1] of finite dimensional $\mathbb{Q}$-vector spaces.

Local systems "linearize" the theory of covering spaces:

$$\begin{aligned} \mathrm{Rep}_{\mathbb{Q}}(\Gamma) \quad &\longleftrightarrow \quad \mathbb{Q}\text{-local system on } M \\ \text{monodromy group is } \rho(\Gamma) \quad &\longleftrightarrow \quad \text{monodromy group actions of } \pi_1(M) \end{aligned}$$

There are a number of issues with the scheme:

1. There is no reason these local systems underlie a variation of Hodge structure. Most do not. (See the next lecture for Hodge structures, Definition 64.)

2. Representations of discrete Γ can be arbitrarily bad. But they relate to representations of algebraic groups for which we have the Tannakian formalism for concrete understanding.

[1] A locally constant sheaf $\mathcal{F}$ is a sheaf such that $\mathcal{F}(U) \cong V$ for all connected U in the same connected component. For example $\underline{\mathbb{Z}}$ or $\underline{\mathbb{Q}}$.

We could try to overcome these issues by taking the closure of the discrete group Γ in $\boldsymbol{GL}(V)$ for various Γ-representations V. That is sort of what we'll do. From a topologist's viewpoint, the homotopy theory of an arbitrary M is hard. So we would like to "linearize" from homotopy theory to cohomology, and add additional structure to it, giving it a more geometric nature. We try "rational homotopy theory" (Sullivan, Deligne, Griffiths, Morgan, et al.) which is more manageable and is in a sense the "rational cohomology" of Γ with homotopy tools (e.g., Postaikov towers). Then everything reduces to the study of a de Rham complex. Here arises a dg-algebra structure (Cf. Definition 81).

There is no obvious way to encode all of $Rep_{\mathbb{Q}}(\Gamma)$. However, Sullivan, et al. realized quite early that it is quite powerful to encode all unipotent representations of Γ via Malcev pro-unipotent completion of Γ.

Remark 33. For a compactifiable Kähler manifold, this linearization of homotopy admits a mixed Hodge structure. Ultimately, Deligne-Gonehovov lift this structure to show it is a "motive", and hence the period formalism, etc. can be used.

Proposition 34 (Morgan)**.** *If M is compact Kähler, then the Malcev algebra, i.e.,* $\mathrm{Lie}\pi_1(M)^{un}$ *is generated by quadratic terms; Kähler groups.*

Many groups do not arise in this way. Even the free group on two generators does not!

By design, $\pi_1(M)^{un}$ encodes unipotent $\mathbb{Q}$-local systems on M. In other words, it encodes unipotent monodromy representations. Hence it encodes iterated extensions of trvial local systems.

Why should unipotence be relevant to Hodge theory and periods?

Maybe we never get unipotent $\mathbb{Q}$-local systems under variation of Hodge structure. Actually, it's almost the opposite: locally we always get, in effect, unipotent $\mathbb{Q}$-local systems.

The basic picture of periods is found in the Lefschetz degeneration. For example, a torus degenerates to a pinched torus and a cylinder to a cone which produces nodes in both cases.

Let us first associate a local system $\mathcal{L}$. Then $\pi_1(\Delta^*)$, where Δ^* is the punctured disk, is generated by the obvious loop. Then,

$$\mathcal{L}_+ = H^1(E_t), t \neq 0 \qquad H^1(E_t, \mathbb{Q}) = Q^2, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

There is a Picard-Lefschetz formula for the general case of local monodromy.

Period picture

E has a unique 1-form ω and period vector $w(t) = S_\gamma$. The cocyles ω_i and γ_i are basis vectors for $H^1(E, \mathbb{Q})$. Then $\Gamma = \rho(\pi_1(\Delta^*))$ acts on $H^1(E_t, \mathbb{Q})$. There is a map

$$\Delta^* \to \Gamma\text{-orbits of periods}, \qquad t \mapsto \Gamma w(t)$$

Then

$$\mathbb{P}(H^1(E_t, \mathbb{Q})) \cong \mathbb{P}^1 = \{[\omega_1 : \omega_2]\} = \{[1_\epsilon : \tau]\}$$

The period domain is $\mathfrak{h} = H^1$ which is isomorphic to the Poincaré unit disk.

Theorem 35 (Monodromy theorem). *Let a local system $\mathcal{L}$ underlie a variation in Hodge structure. Then the monodromy operator T is quasi-unipotent, i.e., its eigenvalues are roots of unity.*

Theorem 36 (Deligne's finiteness theorem). *Fix a compactifiable B and an integer N. There exist at most finitely many conjugacy classes of rational maps of $\pi_1(B)$ of dimension N giving local systems that occur as direct factor of a variation in mixed Hodge structure.*

Remark 37. There are far too many unipotent representations, and very few variations of Hodge structure have unipotent monodromy.

So there exists two examples of "competing factors" with unipotent local systems. On the one hand there are $\mathbb{Q}$-local systems with Deligne finiteness; on the other hand, there is global monodromy. Global monodromy is "typically" not unique, while local monodromy "is" unipotent.

Hence there is an incomplete "motivic π" theory. It was proved 40 years ago that π_1^{un} admits a mixed Hodge structure. The proof that π_1^{un} is a $\mathbb{Q}$-motive brought this into a modern period formalism.

Nilpotent groups

What is a pro-nilpotent completion of π_i?

It exists for any discrete group. It is convenient to reduce to a finitely generated group in characteristic 0. We'll be more concrete and conceptual than the formal way.

Definition 38 (Nilpotent group). A group Γ (not necessarily algebraic) is *nilpotent* if is lower central series, $F_{i+1} = [\Gamma_i, \Gamma]$, stabilizes in finitely many steps at the trivial group.

Remark 39. Over a field of characteristic 0, unipotent groups are nilpotent.

Definition 40 (Pro-nilpotent completion). The *pro-nilpotent completion* of a group Γ is a group Γ^{nil} satisfying the universal property:

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[dr, "\forall"'] & \Gamma^{nil} \arrow[d, "\exists!"] \\ & N \end{tikzcd} \qquad N \text{ nilpotent}$$

It can be constructed as the limit of the projective system formed by morphisms $\Gamma \to N$ for nilpotent groups N. The construction can be simplified by restricting to the system of lower central series quotients, $\Gamma \to \Gamma/\Gamma_{n+1}$, which satisfy universal propreties for morphisms into step n nilpotent groups. Then Γ^{nil} can also be calculated as the natural projection of Γ to the limit of

$$\cdots \to \Gamma/\Gamma_3 \to \Gamma/\Gamma_2$$

Lemma 41. *Let N be a nilpotent group. The set of torsion elements* $\operatorname{Tor} N$ *is a normal subgroup. If N is finitely generated, then* $\operatorname{Tor} N$ *is finite.*

Torsion is no big deal, so we define torsion-free pro-nilpotent completion in a similar way.

Definition 42 (Torsion-free pro-nilpotent completion)**.**

$$\begin{tikzcd} \Gamma \arrow[r, "j_0"] \arrow[dr] & \Gamma_0^{nil} \arrow[d, "\exists!"] \\ & N_0 = N/\operatorname{Tor} N \end{tikzcd}$$

It can similarly be calculated as a limit of $\cdots \to (\Gamma/\Gamma_3)_0 \to (\Gamma/\Gamma_2)_0$.

Example 43. A crucial example is a when G is a finitely generated abelian group. Then the pro-nilpotent completion is $id : G \to G$. (This is obvious: G is step 1 nilpotent and the identity satisfies the universal property.) The torsion-free pro-nilpotent completion is the quotient $G \to G/\operatorname{Tor} G$.

Definition 44 (Pro-unipotent group)**.** A pro-unipotent group is an algebraic group over a field such that every representation has an increasing unipotent filtration.

Definition 45 (Pro-unipotent completion)**.** Let Γ be a finitely generated group. Then its pro-unipotent completion Γ^{un} (also known as the Malcev completion) is the pro-unipotent algebraic group Γ^{un} over $\mathbb{Q}$ satisfying the universal property:

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[dr, "\forall"'] & \Gamma^{un}(\mathbb{Q}) \arrow[d, "\exists!"] \\ & U(\mathbb{Q}) \end{tikzcd} \qquad U \text{ unipotent alg. group over } \mathbb{Q}$$

Alternatively, the pro-unipotent completion can be calculated as

$$\Gamma^{un} = \operatorname{Spec} \varinjlim_{n} (\mathbb{Q}[\Gamma]/I^n)^\vee.$$

where $I \subset \mathbb{Q}[\Gamma]$ is the augmentation ideal which will be defined later. This will be explained in the lecture on pro-unipotent completion **??**.

Preliminary Talk: Cohomology, Hodge structures and Hopf algebras

Sergey Gorchinskiy on September 30th, 2012.

Fundamental groups and groupoids

Definition 46 (Fundamental groups and sets)**.** Let M be a path-connected topological space and $a, b \in M$. The *fundamental group* $\pi_1(M; a)$ of M is the group of homotopy classes of loops based at a where multiplication denotes concatenation of loops. We use the convention that $\gamma_1 \circ \gamma_2$ denotes going along γ_1 and then γ_2.

Let $\pi_1(M; a, b)$ denote the set of homotopy classes of paths from a to b. It has no group structure, but is a left torsor under $\pi_1(M; a)$ and a right torsor under $\pi_1(M; b)$.

These torsors combine to form the *fundamental groupoid* with a multiplication of the form $\pi_1(M; a, b) \times \pi_1(M; b, c) \to \pi_1(M; a, c)$:

$$\begin{tikzcd} & \coprod_{a,b} \pi_1(M; a, b) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ M & & M \end{tikzcd}$$

Filtered vector space

Let V be a k-vector space. F^iV will denote a decreasing filtration of sub-k-vector spaces of V, e.g.,

$$V \supset \cdots \supset F^{-1}V \supset F^0V \supset F^1V \supset \cdots,$$

and F_iV, an increasing one. For example, a graded vector space $V = \bigoplus_{i \in I} V^i$ induces filtrations $F^pV = \bigoplus_{i \geq p} V^i$ and $F_pV = \bigoplus_{i \leq p} V^i$.

Definition 47 (Associated Graded Vector Space)**.** Given a decreasing filtration F^iV on a vector space V, there is an associated graded vector space with a decreasing filtration,

$$\mathrm{gr}_F V = \bigoplus_{i \in \mathbb{Z}} F^iV/F^{i+1}V, \qquad \mathrm{gr}_F^n = \bigoplus_{i \geq n} F^iV/F^{i+1}V.$$

Given an increasing filtration $F_i V$, there is an analogous associated graded vector space $\mathrm{gr}^F V$ with graded parts $\mathrm{gr}_n^F V = \bigoplus_{i \leq n} F^{i+1}V/F^iV$.

Definition 48 (Exhaustive filtration)**.** A filtration F on a vector space V is *exhaustive* if $\cup_{i\in\mathbb{Z}} F^i V = V$.

Definition 49 (Separated filtration)**.** A filtration F on a vector space V is *separated* if $\cap_{i\in\mathbb{Z}} F^i V = \{0\}$.

Definition 50 (Completion of a filtered vector space)**.** The *completion* of a filtered vector space is

$$\widehat{V} = \varprojlim_i V/F^iV$$

Definition 51 (Morphism of filtered vector spaces)**.** A *morphism* $f : V \to U$ of filtered vector spaces satisfies $f(F^iV) = F^iU$. Its kernel is a filtered vector space with filtration,

$$F^i \ker f := \ker f \cap F^i V.$$

The filtration on $\mathrm{coker} f$ is similarly defined.

Let f be an endomorphism of the filtered vector space V. If the filtration F^iV is exhaustive and separated, then f is an isomorphism of filtered vector spaces (use $\mathrm{gr}^i f : \mathrm{gr}^i V \to \mathrm{gr}^i V$). Note that

$$\ker(\mathrm{coker}) \neq \mathrm{coker}(\ker)$$

Proposition 52. *Filtered vector spaces with strict morphisms form an abelian category.*

Betti and de Rham cohomology

Betti cohomology

Definition 53 (Singular complex)**.** Let M be a topological space. The *singular complex* of M is the complex of freely generated abelian groups,

$$\cdots \to S_1(M) \to S_0(M) \to 0, \qquad S_i(M) := \langle f : \Delta^i \to M \rangle_{\mathbb{Z}},$$

with alternating sums of faces for boundary maps. The singular cocomplex is the dual complex $S^i(M) := \mathrm{Hom}(S_i(M), \mathbb{Z})$.

Definition 54 (Betti cohomology)**.** Given a topological space M, its Betti homology and cohomology is the

$$H_i^B(M) := H_i(S_\bullet(M)), \qquad H_B^i(M) := H_i(S^\bullet(M))$$

The Betii homology with coefficients in a $\mathbb{Z}$-module F is

$$H_i^B(M, F) = H_i(S_\bullet(M)) \otimes_{\mathbb{Z}} F$$

Proposition 55. *Let M be a (nice enough?) topological space and $a \in M$ a point. Then*

$$\pi_1(M;a)^{ab} := \pi_1(M;a)/[\pi_1(M;a), \pi_1(M;a)] \cong H_1(M, \mathbb{Z})$$

Definition 56 (Relative cohomology)**.** Let M be a topological space, and let $N \hookrightarrow M$ be a closed subspace. Let F be a $\mathbb{Z}$-module. Form the relative complex and cocomplex

$$S_i(M,N) := S_i(M)/S_i(N), \qquad S^i(M,N) = \ker(S^i(M) \to S^i(N))$$

Then the cohomology of M relative to N is

$$H^i_B(M,N;F) := H^i(S,(M,N;F))$$

There is an alternative formulation of relative cohomology using sheaves:

Definition 57 (Relative cohomology by sheaves)**.** Let M be a topological space, and let $N \hookrightarrow M$ be a closed subspace. Let F be a $\mathbb{Z}$-module. Form the relative cohomology

$$H^i_B(M,N;F) := H^i(M, j_! F|_{M \setminus N})$$

where F is the constant sheaf of abelian groups on M and $j_!$, where $j : N \hookrightarrow M$, extends the sheaf by 0 over the closed subspace N.

de Rham Cohomology

Definition 58 (de Rham complex)**.** Let M be a smooth manifold. Define the de Rham complex to be the complex $\mathbb{C}$-vector spaces,

$$A^i_M := \{\text{smooth } i\text{-forms on } M\},$$

with the usual differential d.

Definition 59 (de Rham cohomology)**.** Let M be a smooth manifold. Its de Rham cohomology is the complex of $\mathbb{C}$-vector spaces:

$$H^i_{dR}(M, \mathbb{C}) := H^i(A^\bullet_M)$$

Theorem 60 (De Rham's theorem)**.** *Let M be a smooth manifold. Then there is an isomorphism,*

$$H^i_B(M, \mathbb{C}) \cong H^i_{dR}(M, \mathbb{C}),$$

arising from the pairing

$$\begin{array}{rcl} S_i(M, \mathbb{C}) \otimes_{\mathbb{C}} A_M & \to & \mathbb{C} \\ (\sigma, f) \otimes \omega & \mapsto & \int_\sigma f^* \omega \end{array}$$

Mixed Hodge structures

Definition 61. Let H be a finite dimensional $\mathbb{Q}$-vector space. A $\mathbb{Q}$-pure Hodge structure of weight n on H is a decreasing filtration, $F^\bullet H_\mathbb{C}$, of $H_\mathbb{C} = H \otimes_\mathbb{Q} \mathbb{C}$ such that

$$\bigoplus_{p\in\mathbb{Z}} H_\mathbb{C}^{p,n-p} \xrightarrow{\sim} H_\mathbb{C},$$

where

$$H_\mathbb{C}^{p,n-p} := F^p H_\mathbb{C} \cap \overline{F^{n-p} H_\mathbb{C}}.$$

Theorem 62 (Hodge Theorem)**.** *Let X be a smooth complex variety and $X(\mathbb{C})$ its corresponding topological space. Then $H_B(X(\mathbb{C}),\mathbb{Q})$ admits a $\mathbb{Q}$-pure Hodge structure.*

Proof. Using the isomorphisms $H_B^n(X(\mathbb{C}),\mathbb{Q})_\mathbb{C} = H_B^n(X(\mathbb{C}),\mathbb{C}) = H_{dR}^n(X(\mathbb{C}),\mathbb{C}) = H^n(A_{X(\mathbb{C})})$ ($A^n_{X(\mathbb{C})}$ is the $\mathbb{C}$-vector space of rank n smooth, complex differential forms on $X(\mathbb{C})$), it suffices to find a filtration on $H^n(A_{X(\mathbb{C})})$. Define

$$A^{p,q}_{X(\mathbb{C})} = f(z) dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$$

These form a double complex

$$\begin{array}{ccccc}
A_M^{1,0} & \longrightarrow & A_M^{1,1} & \longrightarrow & A_M^{1,2} \\
\partial\Big\uparrow & & \partial\Big\uparrow & & \Big\uparrow \\
A_M^{0,0} & \xrightarrow{\overline{\partial}} & A_M^{0,1} & \xrightarrow{\overline{\partial}} & A_M^{0,2}
\end{array}$$

where the differentials satsify $\partial\overline{\partial} = \overline{\partial}\partial$, $\partial^2 = \overline{\partial}^2 = 0$, and $d = \partial + \overline{\partial}$. Then

$$A^n_{X(\mathbb{C})} = \bigoplus_{p+q=n} A^{p,n-q}_{X(\mathbb{C})}.$$

Hence $A^{p,q}_{X(\mathbb{C})}$ defines a Hodge structure. ☐

Compare this to the Hodge structure $H^{p,q} = H^q(X,\Omega^p)$, where Ω^p is the sheaf of holomorphic p-forms on X. There is an inclusion

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \to \mathcal{A}^{p,1} \to \cdots \to \mathcal{A}^{p,d} \to 0$$

where $\mathcal{A}^{p,q}$ is the sheaf of smooth (p,q)-forms on the underlying manifold.

Mixed Hodge Structure

Let $X \subset \overline{X}$ be a subvariety of a smooth projective curve over $\mathbb{C}$, where $X = \overline{X} \setminus D$ for a non-empty divisor D.

Notation 63. We let $\mathbb{Q}(-1) = H^2(\mathbb{P}^1)$.

There is an exact sequence:

$$0 \to H^1(\overline{X}, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \to \bigoplus_{p \in D} \mathbb{Q}(-1) \to \mathbb{Q}(-1) \to 0$$

Define a filtration on $H(\overline{X}, \mathbb{Q})_{\mathbb{C}}$ by

$$F^n H^1(\overline{X}, \mathbb{Q})_{\mathbb{C}} := \begin{cases} H^2(\overline{X}, \mathbb{Q})_{\mathbb{C}} & i < 1 \\ H^0(\overline{X}, \Omega^1_{\overline{X}}\langle D\rangle) & i = 1 \\ 0 & i > 1 \end{cases}$$

Then

$$\text{im } (F^1 H^1(X) \xrightarrow{r} \bigoplus \mathbb{Q}_{\mathbb{C}} = H^0(D, \mathbb{C}) = \left\{ (a_x)_{x \in D} \middle| \sum_{x \in D} a_x = 0 \right\}$$

Hence the Tate twist.

Definition 64 (Mixed Hodge Structure)**.** Given a $\mathbb{Q}$-vector space H, a mixed Hodge structure on H consists of two filtrations:

$W_\bullet H$: The weight filtration, an increasing filtration on H

$F^\bullet H_{\mathbb{C}}$: The Hodge filtration, a decreasing filtration on $H_{\mathbb{C}}$

These must satisfy the condition that the vector space associated to the weight filtration, $\mathrm{gr}^W H$ (Cf. Definition 47), is a pure $\mathbb{Q}$-Hodge structure of weight n with respect to the strict subquotient filtration induced by $F^\bullet H$. In other words,

$$\mathrm{gr}^W_n H \cap F^n H_{\mathbb{C}} = \bigoplus_{r \geq n} W_r H \cap F^n H_{\mathbb{C}}$$

is a pure $\mathbb{Q}$-Hodge structure of weight n. A morphism of mixed Hodge structures H and H' is a $\mathbb{Q}$-linear map $f : H \to H'$ which respects the filtrations, i.e., such that $f(W_n H) \subset W_n H'$ and $f_{\mathbb{C}}(F^n H_{\mathbb{C}}) \subset F^n H'_{\mathbb{C}}$.

Corollary 65. *The category* MHS *of Q-vectors spaces with mixed Hodge structures is a tensor abelian category.*

Theorem 66 (Deligne)**.** *Let X be an arbitrary complex algebraic variety. Then $H^n(X(\mathbb{C}), \mathbb{Q})$ has a canonical mixed Hodge structure, and given a morphism $\phi : X \to Y$ of complex varieties, the induces morphism on cohomology $\phi : H^n(Y(\mathbb{C})) \to H^(X(\mathbb{C}))$ is a morphism of mixed Hodge structures.*

Example 67. How many mixed Hodge structures are there on $H \cong \mathbb{Q}^2$ fitting into the exact sequence in MHS:

$$0 \to \mathbb{Q}(1) \xrightarrow{\iota} H \xrightarrow{p} \mathbb{Q} \to 0?$$

The extension are classified by

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q},\mathbb{Q}(1)) = \mathbb{C}^*/\mathrm{Tor}\ \mathbb{C}^* \xrightarrow[\sim]{\exp(2\pi i t)} \mathbb{C}/\mathbb{Q}$$

The weight filtrations can be easily calculated,

$$\begin{array}{llllllllll}
-3 & 0 & \to & 0 & \to & W_{-3}H & \to & 0 & \to 0 & W_{-3}H = 0 \\
-2 & 0 & \to & \mathbb{Q}(1) & \to & W_{-2}H & \to & 0 & \to 0 & W_{-2}H = \mathbb{Q}(1) \\
-1 & 0 & \to & \mathbb{Q}(1) & \to & W_{-1}H & \to & 0 & \to 0 & W_{-1}H = \mathbb{Q}(1) \\
0 & 0 & \to & \mathbb{Q}(1) & \to & W_0H = H & \to & \mathbb{Q} & \to & W_0H = H \\
1 & 0 & \to & \mathbb{Q}(1) & \to & W_1H & \to & \mathbb{Q} & \to 0 & W_1H = H
\end{array}$$

as well as the Hodge filtration (note the reversed direction!),

$$\begin{array}{llllllllll}
1 & 0 & \to & 0 & \to & F^1H_{\mathbb{C}} & \to & 0 & \to 0 & F^1H_{\mathbb{C}} = 0 \\
0 & 0 & \to & \mathbb{C} & \to & F^0H_{\mathbb{C}} & \to & 0 & \to 0 & F^0H_{\mathbb{C}} = \mathbb{C} \\
-1 & 0 & \to & \mathbb{C} & \to & F^{-1}H_{\mathbb{C}} & \to & 0 & \to 0 & F^{-1}H_{\mathbb{C}} = \mathbb{C} \\
-2 & 0 & \to & \mathbb{C} & \to & F^{-2}H_{\mathbb{C}} = H_{\mathbb{C}} & \to & \mathbb{C} & \to 0 & F^{-2}H_{\mathbb{C}} = H_{\mathbb{C}}
\end{array}$$

This shows that the mixed Hodge structure on H is determined by a 1-dimensional subspace $F^H_{\mathbb{C}} \subset H_{\mathbb{C}}$ such that

$$\iota(\mathbb{Q}(1)_{\mathbb{C}}) \cap F = 0.$$

Let $e \in \mathbb{Q}(1)$ and $f \in H$ be defined by $\langle e\rangle_{\mathbb{Q}} = \mathbb{Q}(1)$ and $\langle p(f)\rangle_{\mathbb{Q}} = \mathbb{Q}$. Then $H = \langle e, f\rangle_{\mathbb{Q}}$ and $F = \langle ae + bf\rangle_{\mathbb{Q}} = \langle ae + f\rangle_{\mathbb{C}}$ for $b \neq 0$ and $a \in \mathbb{C}/\mathbb{Q}$.

Example 68. Let $X = \mathbb{G}_m \setminus \{a'\}$ for some $a' \in \mathbb{C}^*$. Then $H^1(X) \xrightarrow{\sim} \mathbb{Q}(-1) \oplus \mathbb{Q}(-1)$, $a' = \exp(a)$.

Example 69. $a' \neq 1$ $(a + a' = 1) = \mathbb{G}_m/\{a' = 1\} = X$. Hence

$$0 \to \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q} \to H^1(\mathbb{G}_m, \{1, a'\}) \to H^1(\mathbb{G}_m) \cong \mathbb{Q}(-1) \to 0$$

The extension corresponding to $a \in \mathbb{C}/\mathbb{Q} \cong \mathbb{C}^*/\mathrm{Tor}\ \mathbb{C}^*$.

$$0 \to \mathbb{Q}(1) \to H^1(\mathbb{G}_m, \{1, a'\})(1) \to \mathbb{Q} \to 0$$

$H(i) := H \otimes \mathbb{Q}(i)$.

Algebraic de Rham cohomology

Let X be a smooth projective algebraic variety over C. Let Ω_X be the sheaf of algebraic forms on X

Proposition 70 (GAGA).

$$H^p(X, \Omega^q_X) \cong H^p(X(\mathbb{C}), \Omega^q_{X,an})$$

Let

$$\Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d_X$$

be a complex of Zariski sheaves on X.

$$H^n_{dR}(X) := H^n(X, \Omega^*_X) \xrightarrow[\sim]{\alpha} H^n_{dR}(X(\mathbb{C}), \mathbb{C})$$

There is a filtration on $H^n_{dR}(X(\mathbb{C}), \mathbb{C})$ induced by the filtration $F^p\Omega^*_X = \bigoplus_{r\geq p} \Omega^r_X$.

Let k be a subfield of $\mathbb{C}$ and X a k-variety. Then

$$H^n_{dR}(X) \otimes_k \mathbb{C} \cong H^n_{dR}(X_{\mathbb{C}}) \cong H^n_{dR}(X(\mathbb{C}), \mathbb{C})$$

Remark 71. The same is true for any smooth algebraic variety X over k, i.e., there exist filtrations $W_\bullet H^\bullet_{dR}(X)$ and $F^\bullet H^n_{dR}(X)_{\mathbb{C}}$.

Definition 72 (Period isomorphism)**.** Given a field $k \hookrightarrow \mathbb{C}$ and a smooth k-variety X, the period isomorphism is

$$H^n_B(X, \mathbb{Q})_{\mathbb{C}} \xrightarrow{\sim} H^n_{dR}(X) \otimes_k \mathbb{C}$$

After fixing $\mathbb{Q}$-basis $\delta_1, \ldots, \delta_m$ of $H_n(X, \mathbb{Q})$ and k-basis $\omega_1, \ldots, \omega_m$ of $H^n(X)_{\mathbb{C}}$, the isomorphism is specified by a choice of matrix $[M] \in GL_m(k)$
$GL_m(\mathbb{C})/GL_m(\mathbb{Q})$.

Example 73.

$$\mathbb{Q}(-1) = H^(\mathbb{P}^1) \in \mathrm{Q}, \qquad \mathbb{P}^1/\mathbb{Q} \subset \mathbb{C}, \qquad \mathbb{C}^*/\mathbb{Q}^* \ni [2\pi i]$$

Example 74. Let $X = \{y^2 = x^3 + ax + b\}$ be a (chart on a) torus. Then $H^1_{dR}(X) \supset F^1 H^1_{dR}(X) = H^0(X, \Omega^1_X) = \langle \frac{dx}{y} \rangle_k$. Let $H^B_1(X, \mathbb{Q}) = \langle \sigma_1, \sigma_2 \rangle$ be a basis. Then the period matrix is

$$\begin{pmatrix} \int_{\sigma_1} \frac{dx}{y} & \int_{\sigma_1} \frac{xdx}{y} \\ \int_{\sigma_2} \frac{dx}{y} & \int_{\sigma_2} \frac{xdx}{y} \end{pmatrix}$$

Hopf algebras

In the section, let G be a linear algebraic group and $A = \mathcal{O}(G)$ its ring of regular functions. The multiplication $G \times G \to G$ induces a comultiplication $\Delta : A \to A \otimes_k A$. Similarly, $e \in G$ induces $\epsilon : A \to k$ and g^{-1} induces $c : A \to A$.

Definition 75 (Category of representations)**.** Given a group G, its category of *finite dimensional* representations with G-equivariant linear maps between them will be denoted $\mathrm{Rep}(G)$.

Definition 76 (Category of comodules)**.** Given an commutative, associative, unital k-algebra A, its category comodules $\mathrm{Comod}(A)$ is the category of diagrams

$$\begin{aligned} \mathrm{Comod}(A) &= \{\, V \xrightarrow{\Delta} V \otimes_k A \underset{1\otimes\Delta}{\overset{\Delta\otimes 1}{\rightrightarrows}} V \otimes_k A \otimes_k A \,\} \\ &= \{\text{matrices with coefficients in } A\} \end{aligned}$$

There is an equivalence of categories

$$\mathrm{Rep}(G) \cong \mathrm{Comod}(A)$$

Example 77. If $G = \mathbb{G}_m$, then G maps under the above equivalence to $A = k[t, t^{-1}]$ with $\Delta(t) = t \otimes t$.

Example 78. If $G = \mathbb{G}_a$, then G maps under the above equivalence to $A = k[t]$ with $\Delta(t) = 1 \otimes t + t \otimes 1$.

Definition 79 (Lie algebra)**.** Given an algebraic group G, its Lie algebra is

$$\mathrm{Lie}(G) = \mathfrak{m}/\mathfrak{m}^2, \quad \mathfrak{m} := \ker(\epsilon)$$

where $\epsilon : \mathcal{O}(G) \to k$, as above.

Remark 80. There are such things as filtered and graded Hopf algebras. They are as you might guess.

dg-algebras

Definition 81 (dg-algebra)**.** Given a field k, a dg-algebra over k is a complex A of k-vector spaces with an associative product $\cdot : A \otimes_k \otimes \to A$ compatible with the differential of the complex by the Leibniz rule,

$$d(a \cdot b) = da \cdot b + (-1)^{\deg(a)} a \cdot db.$$

Definition 82 (Commutative dg-algebra)**.** A dg-algebra (A, d) is commutative if

$$a \cdot b = (-1)^{\deg a \deg b} b \cdot a$$

Remark 83. Given a dg-algebra A, its cohomology $H(A)$ is a graded $H^0(A)$-algebra.

Definition 84 (dg-Hopf algebra)**.** A dg-Hopf algebra is a commutative dg-algebra equipped with a coproduct $\Delta : A \to A \otimes_k A$, a counit $\epsilon : A \to k$ and coinverse $s : A \to A$ satisfying the usual commutative diagrams.

Remark 85. Given a dg-Hopf algebra, its zeroth cohomology $H^0(A)$ is a Hopf algebra.

Introduction to Multiple Zeta Values

Roland Paulin and Mario Huicochea on the September 2nd, 2012.

In this talk we define and give some basic properties of the multiple zeta values. In the first part we define give some motivation to study the multiple zeta values; In the second part we talk about the $\mathbb{Q}$-vector space generated by the multiple zeta values; In the third part, we define the stuffle product; In the fourth part we define the shuffle product and in the last part we give some applications of the stuffle product, shuffle product and the regularized relations.

Multiple Zeta Values

Euler studied the Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and he obtained a series of results:

- $\zeta(2) = \frac{\pi^2}{6}$
- $\zeta(2n) = \frac{-(2\pi i)^{2n}}{4n} B_{2n}$ for all $n \in \mathbf{N}$ where the $B_k \in \mathbb{Q}$ are Bernoulli numbers defined by the equation $\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{n=2}^{\infty} B_n \frac{t^n}{n!}$
- $\mathbf{Q}[\zeta(2), \zeta(4), \ldots] = \mathbf{Q}[\pi^2]$.

However, much less is known about $\zeta(2n+1)$ for $n \in \mathbf{N}$. It is known, for example that:

- $\zeta(3)$ is irrational. (Apéry 1978)
- $\{\zeta(2n+1)\}_{n \in \mathbf{N}}$ has infinitely many irrational values. (Rivoal 2000)

A famous conjecture in this direction is the following.

Conjecture 86. $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ *are algebraically independent over* $\mathbf{Q}$.

Definition 87 (Zagier, Hoffman 1992)**.** Let $n_1, \ldots, n_k \in \mathbf{N}$ with $n_1 \geq 2$ and $\overline{n} = (n_1, \ldots, n_k)$. The real numbers

$$\zeta(\overline{n}) = \sum_{i_1 > \cdots > i_k > 0} \frac{1}{i_1^{n_1} \cdots i_k^{n_k}}$$

are known as the *multiple zeta values.* The *weight of* $\zeta(\overline{n})$ is $|\overline{n}| = n_1 + \cdots + n_k$ and k is the *length.*

Definition 88. A weight $\overline{n} = (n_1, \ldots, n_k)$ for $n_1, \ldots, n_k \in \mathbf{N}$ is *admissible* if $n_1 \geq 2$. Let $\mathcal{W} \subset \coprod_{r=1}^{\infty} \mathbb{N}^r$ denote the set of admissible weights.

The multiple zeta values were discovered and studied by Euler for $k \leq 2$. Hoffman and Zagier defined the multiple zeta values for arbitrary $k \in \mathbf{N}$ independently in 1992.

Definition 89. Let

- $\mathcal{Z}$ be the $\mathbf{Q}$-vector space generated by the MZV's.
- $\mathcal{Z}_n$ be the $\mathbf{Q}$-vector space generated by the MZV's of weight $n \geq 2$.
- $\mathcal{F}^n Z$ be the $\mathbb{Q}$-vector space generated by the MZV's of length $\leq k$.
- $\mathcal{F}^k \mathcal{Z}_n$ be the $\mathbb{Q}$-vector space generated by the MZV's of weight n and length $\leq k$ for $2 \leq k + 1 \leq n$.

Remark 90. If $2 \leq k + 1 \leq n$ then $\mathcal{F}^k \mathcal{Z}_n \subset \mathcal{F}^k \mathcal{Z} \cap \mathcal{Z}_n$.

Conjecture 91. $\mathcal{F}^k \mathcal{Z}_n = \mathcal{F}^k \mathcal{Z} \cap \mathcal{Z}_n$.

Conjecture 92. *The weight defines a gradation* $\mathcal{Z} = \oplus_{n \geq 2} \mathcal{Z}_n$. *In particular* $\mathcal{Z}_2 \cap \mathcal{Z}_3 = \{0\}$ *so* $\zeta(3)/\pi^2 \notin \mathbb{Q}$.

Let $d_0 = 1, d_1 = 0$ and $d_n = \dim_{\mathbb{Q}} \mathbb{Z}_n$ be the dimension of the nth graded part of $\mathcal{Z}$ for all $n \in \mathbf{N}_{>2}$.

- $d_2 = 1$ since $\mathcal{Z}_2 = \mathbb{Q}\zeta(2)$.
- $d_3 = 1$ since $\mathcal{Z}_3 = \mathbb{Q}\zeta(2, 1) + \mathbb{Q}\zeta(3)$ and $\zeta(2, 1) = \zeta(3)$.
- $d_4 = 1$ since $\mathcal{Z}_4 = \mathbb{Q}\zeta(2, 1, 1) + \mathbb{Q}\zeta(2, 2) + \mathbb{Q}\zeta(3, 1) + \mathbb{Q}\zeta(4)$ and $\zeta(2, 1, 1) = \zeta(4), \zeta(2, 2) = \frac{3}{4}\zeta(4), \zeta(3, 1) = \zeta(4)/4$.

These are the unique known d_n. There are some upper bounds known for the rest. For example, d_5 is the dimension of a space generated by $\{\zeta(3, 2), \zeta(2, 3)\}$ and d_6 by $\{\zeta(2, 2, 2), \zeta(3, 3)\}$, so they are both at most 2.

A main result about the linear relations among multiple zeta values was proved by Goncharov and Terasoma.

Theorem 93 (Goncharov-Terasoma). *If $D_n \in \mathbf{N} \cup \{0\}$ for all $n \in \mathbf{N}$ such that $D_0 = D_2 = 1$, $D_1 = 0$ and $D_n = D_{n-2} + D_{n-3}$ for all $n \geq 3$ then $d_n \leq D_n$.*

The equality is still a conjecture.

Conjecture 94 (Zagier 1992). *For all $n \geq 3, n \in \mathbf{N}$*

$$d_n = d_{n-2} + d_{n-3}$$

The conjecture is proved for $n = 3, 4$ and can be stated as

$$\sum_{n=0}^{\infty} d_n x^n = \frac{1}{1 - x^2 - x^3}$$

Conjecture 95 (Hoffman 1997). *Let $n \in \mathbf{N}$ and $S_n = \{(s_1, \ldots, s_n) \mid s_i \in \{2, 3\}, \sum_{j=1}^{k} s_j = n\}$. The set $\{\zeta(\overline{s})\}_{\overline{s} \in S_n}$ is a basis of Z_n.*

A main result about multiple zeta values is Brown's theorem:

Theorem 96 (Brown). *All multiple zeta values are linear combinations of $\zeta(\overline{n})$ with $n_i \in \{2, 3\}$ for all n_i in $\overline{n}$.*

Words and Products of Words

Definition 97 (Words). Let X be the set of words formed by the letters $\{X_0, X_1\}$. Let 1 be the empty word $\varnothing \in X$, and let $Y_n := X_0^{n-1} X_1 \in X$ for all $n \in \mathbb{N}$.

Definition 98. Let $\mathfrak{H} = \mathbb{Q}\langle x_0, x_1 \rangle$ be the $\mathbb{Q}$-algebra of polynomials in the two non-commutative variables $\{x_0, x_1\}$ which is graded by the degree ($\deg x_0 = \deg x_1 = 1$). Let $\mathfrak{H}_1$ be the $\mathbb{Q}$-vector space of polynomials generated by $\{1\} \cup \{Y_n\}_{n \in \mathbb{N}}$ and $\mathfrak{H}_0$ be the $\mathbb{Q}$-vector space of polynomials generated by $\{1\} \cup \{Y_n\}_{n \geq 2}$.

Definition 99 (Word associated to a vector). We associate to an admissible vector a word

$$w : \begin{array}{rcl} \mathcal{W} & \to & X^k \\ (n_1, \ldots, n_k) = \overline{n} & \mapsto & Y_{\overline{n}} = (Y_{n_1}, \ldots, Y_{n_k}) \end{array}$$

Definition 100. We define a function from words associated to admissible vectors, $w(\mathcal{W})$, to multiple zeta values by

$$\widehat{\zeta} : \begin{array}{rcl} w(\mathcal{W}) & \to & \mathcal{Z} \\ Y_{\overline{n}} & \mapsto & \zeta(\overline{n}) \end{array}$$

and let $|Y_{\overline{n}}| = |\overline{n}|$. The function $\widehat{\zeta}$ can be restricted to $\overline{n}$ with $n_i > 2$ for all i and then extended by $\mathbb{Q}$-linearity to define a function

$$\widehat{\zeta} : \mathfrak{H}_0 \to \mathcal{Z}.$$

Remark 101. $\mathfrak{H}_0 \subset \mathfrak{H}_1 \subset \mathfrak{H}$

The stuffle product

Definition 102 (Stuffle product)**.** The stuffle product (also known as the harmonic product) $* : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$ on non-commutative polynomials is defined by extending by linearity from three rules:

1. $1 * w = w * 1 = w$ for all $w \in X$.
2. $Y_n w_1 * Y_m w_2 = Y_n(w_1 + Y_m w_2) + Y_m(Y_n w_1 + Y_2) + Y_{n+m}(w_1 * w_2)$ for all $w_1, w_2 \in X$ and all $m, n \in \mathbb{N}$.
3. $X_0^n + W = W + x_0^n = W x_0^n$ for all $n \in \mathbb{N}, w \in X$.

Remark 103. The stuffle product is commutative and associative. Together with addition it forms a graded $\mathbb{Q}$-algebra $(\mathfrak{H}, +, *)$ with grading $\mathfrak{H}^n = \mathbb{Q}\langle\text{words of length } n\rangle$. $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are subalgebras.

Example 104.

$$\begin{aligned} Y_2 * Y_2 = Y_2 1 * Y_2 1 &= Y_2(1 * Y_2 1) + Y_2(Y_2 1 * 1) + Y_4(1 * 1) \\ &= Y_2 Y_2 + Y_2 Y_2 + Y_4 \\ &= 2 Y_2 Y_2 + Y_4 \end{aligned}$$

Example 105.

$$\begin{aligned} Y_2 * Y_3 Y_2 = Y_2 1 * Y_3 Y_2 &= Y_2(1 * Y_3 Y_2) + Y_3(Y_2 * Y_2) + Y_5(1 * Y_2) \\ &= Y_2 Y_3 Y_2 + Y_3(2 Y_2 Y_2 + Y_4) + Y_5 Y_2 \\ &= Y_2 Y_3 Y_2 + 2 Y_3 Y_2 Y_2 + Y_3 Y_4 + Y_5 Y_2. \end{aligned}$$

Proposition 106 (Nielsen reflexion Formula)**.** $Y_n + Y_m = Y_n Y_m + Y_m Y_n + Y_{n+m}$ *for all* $m, n \geq 2$.

Theorem 107. *The function* $\widehat{\zeta} : (\mathfrak{H}_0, *) \to (\mathbb{R}, \cdot)$ *is a homomorphism for the stuffle product* $*$*, i.e., for all* $Z_1, Z_2 \in \mathfrak{H}_0$*,*

$$\widehat{\zeta}(Z_1 * Z_2) = \widehat{\zeta}(Z_1)\widehat{\zeta}(Z_2).$$

So, for example, Example 104 implies

$$2\zeta(2,2) + \zeta(4) = \widehat{\zeta}(Y_2 * Y_2) = \zeta(2)^2$$

and Example 105 implies

$$\zeta(2,3,2) + 2\zeta(3,2,2) + \zeta(3,4) + \zeta(5,2) = \widehat{\zeta}(Y_2 * Y_3 Y_2) = \zeta(2)\zeta(3,2).$$

The shuffle product

Definition 108 (Shuffle product)**.** Let S_n denote the symmetric group of permutations on n letters. Let subsets of S_n be defined by

$$S_{p,q} = \{\sigma \in S_{p+q} \mid \sigma(1) < \cdots < \sigma(p) \text{ and } \sigma(p+1) < \cdots < \sigma(p+q)\} \subset S_{p+q}.$$

These are permutations which preserve the ordering of the first p and the last q letters while allowing shuffling between them. The shuffle product of $p+q$ words $u_1, \ldots, u_{p+q}$ is defined by

$$\text{Ш} : \begin{array}{ccc} \mathfrak{H} \times \mathfrak{H} & \to & \mathfrak{H} \\ (u_1 \cdots u_p, u_{p+1} \cdots u_{p+q}) & \mapsto & \sum_{\sigma \in S_{p+q}} u_{\sigma^{-1}(1)} \cdots u_{\sigma^{-1}(p+q)} \end{array}$$

Example 109. For example, consider the shuffle product of two monomials,

$$\frac{3}{2}Y_2 \text{ Ш } \frac{-1}{6}Y_2 = \frac{3}{2}X_0X_1 \text{ Ш } \frac{-1}{6}X_0X_1 = \frac{-1}{4}\left(4X_0^2X_1^2 + 2X_0X_1X_0X_1\right) = -Y_3Y_1 - \frac{1}{2}Y_2Y_2.$$

Theorem 110. *The function* $\widehat{\zeta} : (\mathfrak{H}_0, \text{Ш}) \to (\mathbb{R}, \cdot)$ *is also a homomorphism for Ш, i.e.,*

$$\widehat{\zeta}(Z_1 \text{ Ш } Z_2) = \widehat{\zeta}(Z_1)\widehat{\zeta}(Z_2)$$

The regularized relation

Corollary 111 (Regularized relation)**.** *For all* $w_1, w_2 \in \mathfrak{H}_0$*,*

$$\widehat{\zeta}(w_1 \text{ Ш } w_2 - w_1 * w_2) = 0.$$

Let the shuffle algebra, $\mathfrak{H}_{\text{Ш}}$, be the commutative $\mathbb{Q}$-algebra $(\mathfrak{H}, +, \text{Ш})$; then $\mathfrak{H}^0_{\text{Ш}} \subset \mathfrak{H}^1_{\text{Ш}} \subseteq \mathfrak{H}_{\text{Ш}}$ are subalgebras. Similarly, let the stuffle algebra, $\mathfrak{H}_*$, be the commutative $\mathbb{Q}$-algebra $(\mathfrak{H}, +, *)$; then $\mathfrak{H}^0_* \subset \mathfrak{H}^1_* \subseteq \mathfrak{H}_*$ are subalgebras. The subalgebras are related by

$$\mathfrak{H}^1_{\text{Ш}} = \mathfrak{H}^0_{\text{Ш}}[x_1] \qquad \mathfrak{H}_{\text{Ш}} = \mathfrak{H}^1_{\text{Ш}}[x_0],$$

and therefore $\mathfrak{H}_{\text{Ш}} = \mathfrak{H}^0_{\text{Ш}}[x_0, x_1]$. Similar relations hold for the stuffle algebras.

$$\mathfrak{H}^1_* = \mathfrak{H}^0_*[x_1] \qquad \mathfrak{H}_* = \mathfrak{H}^1_*[x_0] = \mathfrak{H}^0_*[x_0, x_1].$$

Example 112. The regularized relation implies that $\widehat{\zeta}(Y_1 \text{ Ш } Y_2 - Y_1 * Y_2) = 0$, and calculation shows that

$$Y_1 \text{ Ш } Y_2 = Y_1Y_2 + 2Y_2Y_1 \qquad Y_1 * Y_2 = Y_1Y_2 + Y_2Y_1 + Y_3.$$

Since the difference $Y_2Y_1 - Y_3 \in \mathfrak{H}_0$, this implies a relation for multiple zeta values, namely that $\zeta(2,1) = \zeta(3)$.

Conjecture 113. *The stuffle, shuffle and regularized relations generate all relations among the multiple zeta values. (A strong version restricts the regularized relations to those with* $w_1 = X_1$*.)*

Theorem 114 (Sum Theorem)**.** *For all* $p \geq 2$ *and* $1 \leq \ell \leq p-1$,

$$\zeta(p) = \sum_{(n_1,\ldots,n_\ell)\in\mathcal{W}} \zeta(n_1,\ldots,n_\ell)$$

Exercise 115. Derive from the regularized relation the following formula:

$$\zeta(s) = \sum_{\substack{i+j=s\\ i\geq 2}} \zeta(i,j)$$

Chen's theorem: Iterated integrals

Fritz Hörmann on September 3rd, 2012.

Differential forms on the path-space

This talk will explain that multiple zeta values are (limits of) iterated integrals. This is the starting point for establishing that they are, in fact, periods of mixed motives, and thus for using motivic methods to investigate them.

Iterated integrals, however, have been originally invented by Chen [**?**] to establish a de Rham theory for the path-space PM of a f.d. real manifold M.

In these notes we let $\sigma_n = \{0 \leq t_1 \leq \cdots \leq t_n \leq 1 \mid t_i \in \mathbb{R}\}$ be the standard n-simplex.

Recall:

Definition 116. Let M be a f.d. real manifold. Then the **path-space** of M is defined as

$$PM = \{f : \sigma_1 \to M \mid f \text{ is smooth}\}.$$

It comes equipped with two maps

$$\begin{tikzcd} & PM \arrow[dl, "s"'] \arrow[dr, "d"] & \\ M & & M \end{tikzcd}$$

$s =$ starting point and $d =$ endpoint of path. We denote $PM_{a,b}$ to be the fiber over (a, b), that is, the set of paths from a to b.

The first problem is to specify what a differential form on PM should be. Since any reasonable structure on PM would yield an infinite dimensional object, one cannot use charts in the usual sense. A nice way to circumvent this problem is to define instead a notion of smooth map from (finite dimensional) manifolds to PM, so called **plots**, and to consider the set of all such maps as the additional structure. This is basically a functorial description of PM in the style of Grothendieck, cf. [**?**, Chapter 8].

Definition 117. An (infinite dimensional) real manifold is a set P together with a collection of **plots** $\phi : U \to P$ from f.d. real manifolds to P, such that

1. if $\phi : U \to P$ is a plot and $\psi : U' \to U$ is smooth, $\phi \circ \psi$ is a plot,
2. constant maps are plots,
3. if for a map $\phi : U \to P$, and an open cover $U = \bigcup_i U_i$, each restriction $\phi_i : U_i \to P$ is a plot, ϕ itself is a plot.

Remark 118. By condition 1. and 3., one is even restricted to consider open subsets $U \subseteq \mathbb{R}^n$.

Remark 119. This defines in the obvious way a category which contains the category of (finite dimensional) real manifolds as a full subcategory.

The structure of infinite dimensional manifold on PM is then easy to define:

Definition 120. A function $\phi : U \to PM$ is called a plot if

$$\begin{array}{rcl} \widetilde{\phi} : U \times \sigma_1 & \to & M \\ (u,t) & \mapsto & \phi(u)(t) \end{array}$$

is smooth.

Differential forms can now be defined in literally the same way as for manifolds:

Definition 121. Let P be an (infinite dimensional) real manifold. A differential form γ on P is a collection

$$\{\gamma_\phi \in A^*(U)\}_\phi$$

indexed by all plots $\phi : U \to P$, satisfying the following compatibility condition: For a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d, "\alpha"'] & PM \\ U' \arrow[ur, "\phi'"'] & \end{tikzcd}$$

where α is smooth, we must have $\alpha^*\gamma_{\phi'} = \gamma_\phi$.

This defines a complex

$$A^\bullet(P)$$

of differential forms on P. Observe that if P was itself a manifold, with the obvious notion of plot, we get the same complex as in the classical definition.

In the case of PM, $A^\bullet(PM)$ is too big, in the sense that it does not compute the right cohomology of PM. Chen considered therefore a subcomplex of $A^*(PM)$, which is more closely related to differential forms on M, the subcomplex of **iterated integrals**.

Let $\omega_1, \dots, \omega_n \in A^\bullet(M)$ be differential forms of degree $k_1, \dots, k_n$. We define their iterated integral to be the following differential form $\int \omega_1 \otimes \cdots \otimes \omega_n \in A^k(PM)$, where $k = \sum_{i=1}^n (k_i - 1)$.

For a plot $\phi : U \to PM$, write

$$\widetilde{\phi}^*\omega_i = \beta_i + \mathrm{d}t \wedge \gamma_i$$

for the canonical decomposition, where β_i and γ_i do not contain $\mathrm{d}t$. We define

$$\int_\phi \omega_1 \otimes \cdots \otimes \omega_n := \int_{\sigma_n} \gamma_1(t_1, u) \wedge \cdots \wedge \gamma_n(t_n, u)\mathrm{d}t_1 \cdots \mathrm{d}t_n \quad \in A^k(U)$$

where the integral is an elementary integral over the t_i, not an integral of the forms γ_i.

We extend this linearly to sums of tensors, i.e. to $\omega \in \bigoplus_{k=0}^{\infty} A^\bullet(M)^{\otimes k}$. For the special case that in the summands $\omega_1 \otimes \cdots \otimes \omega_k$ of ω all $\deg(\omega_i) = 1$, we get an actual function

$$\gamma \mapsto \int_\gamma \omega$$

on the path-space. This function is invariant under deformations of the path if and only if

$$\mathrm{d} \int \omega = 0.$$

The derivative of iterated integrals in general is easy to compute: We have

Proposition 122.

$$\mathrm{d} \int \omega_1 \otimes \cdots \otimes \omega_n = \mathrm{d}' \int \omega_1 \otimes \cdots \otimes \omega_n + \mathrm{d}'' \int \omega_1 \otimes \cdots \otimes \omega_n$$

where

$$\begin{aligned} \mathrm{d}' \textstyle\int \omega_1 \otimes \cdots \otimes \omega_n &= \textstyle\sum_{i=1}^{n} (-1)^i \int \omega_1' \otimes \cdots \omega_{i-1}' \otimes \mathrm{d}\omega_i \otimes \omega_{i+1} \otimes \cdots \otimes \omega_n \\ \mathrm{d}'' \textstyle\int \omega_1 \otimes \cdots \otimes \omega_n &= \textstyle\sum_{i=1}^{n-1} (-1)^{i-1} \int \omega_1' \otimes \cdots \otimes \omega_{i-1}' \otimes (\omega_i' \wedge \omega_{i+1}) \otimes \omega_{i+2} \otimes \cdots \otimes \omega_n \\ & \quad -s^*\omega_1 \wedge (\textstyle\int \omega_2 \otimes \cdots \otimes \omega_n) + (-1)^n (\int \omega_1' \otimes \cdots \otimes \omega_{n-1}') \wedge d^*\omega_n, \end{aligned}$$

where $\omega' = (-1)^{\deg \omega}\omega$. Recall that the maps s and d are given by begin and endpoint of paths.

Remark 123. The formula is also true if any of the k_i is 0, that is $\gamma_i = 0$, and hence, if the left hand side is 0. It still gives non-trivial information.

Remark 124. If we restrict the iterated integral to a differential form on $PM_{a,b}$, $s^*\omega$, resp. $d^*\omega$, will be zero, unless ω is of degree 0, i.e. a function.

Before proving the proposition we will analyze the structure of the formula. It is convenient to make the following

Definition 125.

$$A_{iter}^{-j,i}(PM_{a,b}) = \{\text{degree } i \text{ elements in } (A^\bullet(M))^{\otimes j}\}$$

The formulas appearing in the above proposition turn these groups into a double complex:

$$
\begin{tikzcd}[column sep=small]
\ddots & & A_{iter}^{-2,0} \arrow[dl, "\mathrm{d}'"'] \arrow[dr, "\mathrm{d}''"] & & \iddots & \\
& A_{iter}^{-1,0} \arrow[dl, "\mathrm{d}'"'] \arrow[dr, "\mathrm{d}''"] & & A_{iter}^{-2,1} \arrow[dl, "\mathrm{d}'"'] \arrow[dr, "\mathrm{d}''"] & & \\
A_{iter}^{0,0} \arrow[dr, "0"] & & A_{iter}^{-1,1} \arrow[dl] \arrow[dr, "\mathrm{d}''"] & & A_{iter}^{-2,2} \arrow[dl, "\mathrm{d}'"'] \arrow[dr, "\mathrm{d}''"] & \\
& 0 \arrow[dr] & & A_{iter}^{-1,2} \arrow[dl] \arrow[dr, "\mathrm{d}''"] & & A_{iter}^{-2,3} \arrow[dl, "\mathrm{d}'"'] \\
& & 0 \arrow[dr] & & A_{iter}^{-1,3} \arrow[dl] & \\
& & & 0 & & \ddots
\end{tikzcd}
$$

and the proposition may be restated as:

Corollary 126.

$$\mathrm{Tot}^{\oplus}(A_{iter}^{\bullet,\bullet}(PM_{a,b})) \to A^{\bullet}(PM_{a,b})$$

is a homomorphism of complexes.

We will denote $H^i(\mathrm{Tot}^{\oplus}(\cdots))$ also by $\mathbb{H}^i(\cdots)$ in the sequel.

By the very definition of iterated integral, this map factorizes via the quotient by tensors $\omega_1 \otimes \cdots \otimes \omega_n$, where one of the ω_i has degree zero, and their differentials. We will recognize $A_{iter}^{\bullet,\bullet}(PM_{a,b})$, resp. its quotient, as a **bar complex**, resp. a **reduced bar complex** in talk 4. The point of introducing this special kind of differential forms on PM is the following

Theorem 127 (Chen)**.** $\mathrm{Tot}^{\oplus}(A_{iter}^{\bullet,\bullet}(PM_{a,b}))$ *(and also its image under the above map) computes the cohomology of* $PM_{a,b}$*.*

We will see a proof of the special case for $H^0(PM_{a,b},\mathbb{C}) = \mathbb{C}[\pi_1(M,a,b)]^{\vee}$ of this theorem in the course of this seminar.

Example 128. Assume that ω is a sum of tensors of forms of degree 1. We have seen, that its iterated integral is a *function* on $PM_{a,b}$, which is a homotopy invariant, i.e. a function on $\pi_1(M,a,b)$, if and only if $\mathrm{d}\omega = 0$. In low degrees, this formula explicitly means the following:
If $\deg\omega \leq 1$, then $\omega = \omega_1 + c$. Hence $\mathrm{d}\omega = 0$ if and only if $\mathrm{d}\omega_1 = 0$.
If $\deg\omega \leq 2$, then $\omega = \omega_1 \otimes \omega_2 + \omega_{12} + c$ and hence $\mathrm{d}\omega = 0$ if and only if

$$\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0 \quad \text{and} \quad \omega_1 \wedge \omega_2 = \mathrm{d}\omega_{12}.$$

Example 129. Let ω_i be forms of degree 1. If $\mathrm{d}\omega_i = 0$ and $\omega_i \wedge \omega_{i+1} = 0$, then $\mathrm{d}(\omega_1 \otimes \cdots \otimes \omega_n) = 0$.

Proof of the proposition. Recall $\widetilde{\phi}^*\omega_i = \beta_i + \mathrm{d}t \wedge \gamma_i$. We will need this decomposition also for the forms $\omega_i \wedge \omega_{i+1}$ and $\mathrm{d}\omega_i$ respectively occurring in the claimed formula. We have

$$\begin{aligned} \widetilde{\phi}^*\mathrm{d}\omega_i &= \mathrm{d}\widetilde{\phi}^*\omega_i = \mathrm{d}_u\beta_i - \mathrm{d}t \wedge \frac{\partial}{\partial t}\beta_i - \mathrm{d}t \wedge \mathrm{d}\gamma_i && (0.0.1)\\ \widetilde{\phi}^*(\omega_i \wedge \omega_{i+1}) &= \beta_i \wedge \beta_{i+1} + \mathrm{d}t \wedge (\gamma_i \wedge \beta_{i+1} + \beta_i' \wedge \gamma_{i+1}) && (0.0.2) \end{aligned}$$

Now we start calculating the derivative. Unless otherwise specified, we substitute $t \to t_i$ in γ_i and β_i, respectively.

$$\begin{aligned} & \mathrm{d}\int_{\sigma_n} \gamma_1 \wedge \cdots \wedge \gamma_n \mathrm{d}t_1 \cdots \mathrm{d}t_n \\ = & \sum_{i=1}^n \int_{\sigma_n} \gamma_1' \wedge \cdots \gamma_{i-1}' \wedge \mathrm{d}\gamma_i \wedge \gamma_{i+1} \wedge \cdots \wedge \gamma_n \mathrm{d}t_1 \cdots \mathrm{d}t_n \\ = & \sum_{i=1}^n (-1)^i \int_\phi \omega_1' \otimes \cdots \otimes \omega_{i-1}' \otimes \mathrm{d}\omega_i \otimes \omega_{i+1} \otimes \cdots \otimes \omega_n \\ & + \int_{\sigma_n} \gamma_1' \wedge \cdots \wedge \gamma_{i-1}' \wedge \frac{\partial}{\partial t_i}\beta_i \wedge \gamma_{i+1} \wedge \cdots \wedge \gamma_n \mathrm{d}t_1 \cdots \mathrm{d}t_n \\ = & \sum_{i=1}^n (-1)^i \int_\phi \omega_1' \otimes \cdots \otimes \omega_{i-1}' \otimes \mathrm{d}\omega_i \otimes \omega_{i+1} \otimes \cdots \otimes \omega_n \\ & + \sum_{i=1}^n \int_{\sigma_{n-1}} \gamma_1' \wedge \cdots \wedge \gamma_{i-1}' \wedge (\beta_i(t_{i+1}, u) - \beta_i(t_{i-1}, u)) \wedge \gamma_{i+1} \wedge \cdots \wedge \gamma_n \mathrm{d}t_1 \cdots \mathrm{d}t_{i-1}\mathrm{d}t_{i+1} \cdots \mathrm{d}t_n \end{aligned}$$

where we set $t_0 = 0$ and $t_{n+1} = 1$. We get:

$$\begin{aligned} = & \sum_{i=1}^n (-1)^i \int \omega_n' \otimes \cdots \omega_{i-1}' \otimes \mathrm{d}\omega_i \otimes \omega_{i+1} \otimes \cdots \otimes \omega_n \\ & + \sum_{i=1}^{n-1} (-1)^{i-1} \int \omega_1' \otimes \cdots \otimes \omega_{i-1}' \otimes (\omega_i' \wedge \omega_{i+1}) \otimes \omega_{i+2} \otimes \cdots \otimes \omega_n \\ & - s^*\omega_1 \wedge (\int \omega_2 \otimes \cdots \otimes \omega_n) + (-1)^n (\int \omega_1' \otimes \cdots \otimes \omega_{n-1}') \wedge d^*\omega_n, \end{aligned}$$

$\square$

Proposition 130. *The iterated integrals have the following properties:*

1. $(\int \omega_1 \otimes \cdots \otimes \omega_n) \wedge (\int \omega_{n+1} \otimes \cdots \otimes \omega_{n+m}) = \sum_{\sigma \in S_{n,m}} \mathrm{sgn}(\sigma) \int \omega_{\sigma^{-1}(1)} \otimes \cdots \otimes \omega_{\sigma^{-1}(n+m)}$, *where $S_{n,m}$ is the set of shuffles introduced in talk 1, and $\mathrm{sgn}(\sigma)$ is a sign depending on σ and the degrees of the ω_i. It is 1, if all of them have degree 1.*

2. If $d\phi_1 = s\phi_2$, we have

$$\int_{\phi_1\circ\phi_2} \omega_1 \otimes \cdots \otimes \omega_n = \sum_{i=0}^{n} \left(\int_{\phi_1} \omega_1 \otimes \cdots \otimes \omega_i \right) \left(\int_{\phi_2} \omega_{i+1} \otimes \cdots \otimes \omega_n \right).$$

Recall that $\phi_1 \circ \phi_2$ means (according to our convention) that the path $\phi_1(u)$ is taken first.

3. $\int_{\phi^{-1}} \omega_1 \otimes \cdots \otimes \omega_n = \pm \int_\phi \omega_n \otimes \cdots \otimes \omega_1$.

Remark 131. 1. will later yield the shuffle formula for the multiple zeta values (cf. 138).

Corollary 132. *Under the pairing:*

$$\int : \mathbb{C}[\pi_1(M,a,b)] \times \mathbb{H}^0(A^{\bullet,\bullet}_{iter}(PM_{a,b})) \to \mathbb{C}$$

the product $\mathbb{C}[\pi_1(M,a,b)] \times \mathbb{C}[\pi_1(M,b,c)] \to \mathbb{C}[\pi_1(M,a,c)]$ is dual to the following (deconcatenation) coproduct:

$$\begin{array}{rcl} \mathbb{H}^0(A^{\bullet,\bullet}_{iter}(PM_{a,c})) & \to & \mathbb{H}^0(A^{\bullet,\bullet}_{iter}(PM_{a,b})) \otimes \mathbb{H}^0(A^{\bullet,\bullet}_{iter}(PM_{b,c})) \\ \omega_1 \otimes \cdots \otimes \omega_n & \mapsto & \sum_{i=0}^{n} (\omega_1 \otimes \cdots \otimes \omega_i) \otimes (\omega_{i+1} \otimes \cdots \otimes \omega_n). \end{array} \tag{0.0.3}$$

Proof of proposition 130. 1. and 2. of the proposition itself follow immediately from the following combinatorial Lemma about decompositions of (products) of simplexes. For 1., note that reordering the γ_i in the integral gives a sign which is the one denoted $\mathrm{sgn}(\sigma)$ in the formula. However, if all k_i are equal to one, the γ_i are functions. 3. is an easy calculation which we leave as an exercise. □

Lemma 133. *1. There is a bijection (on an open dense subset):*

$$\begin{array}{rcl} \sigma_n(0,1) \times \sigma_m(0,1) & \overset{\sim}{\dashrightarrow} & \bigcup_{\sigma \in S_{m,n}} \sigma_{n+m}(0,1) \\ \{0 \le t_1 \le \cdots t_n \le 1\} \times \{0 \le t_{n+1} \le \cdots \le t_{n+m} \le 1\} & \mapsto & \{0 \le t_{\sigma^{-1}(1)} \le \cdots \le t_{\sigma^{-1}(n+m)} \le 1\} \end{array}$$

2. There is a bijection (on an open dense subset):

$$\begin{array}{rcl} \sigma_n(0,1) & \overset{\sim}{\dashrightarrow} & \bigcup_{i=0}^{n} \sigma_i(0,\frac{1}{2}) \times \sigma_{n-i}(\frac{1}{2},1) \\ \{0 \le t_1 \le \cdots \le t_n \le 1\} & \mapsto & \{0 \le t_1 \le \cdots \le t_i \le \frac{1}{2}\} \times \{\frac{1}{2} \le t_{i+1} \le \cdots \le t_n \le 1\} \end{array}$$

Proof. 1. The shuffle σ in the first line is chosen such that

$$0 \le t_{\sigma^{-1}(1)} \le \cdots \le t_{\sigma^{-1}(n+m)} \le 1$$

holds true. It is uniquely determined whenever all t_i are different from each other. There is an obvious inverse to this map which just forgets the σ. 2. The map is given by choosing an i such that $t_i \le \frac{1}{2} \le t_{i+1}$. It is uniquely determined whenever all t_i are different from each other and from $\frac{1}{2}$. There is an obvious inverse to this map which just forgets the i. □

Multiple zeta values as iterated integrals

Let $M = \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$. The differential forms $\omega_0 = \frac{dz}{z}$ and $\omega_1 = \frac{dz}{1-z}$ generate $H^1_{dR}(M)$.

Recall from the first talk, that there is a bijection

$$\begin{array}{rcl} \{(s_1, \dots, s_n) \in \mathbb{N}^n \mid s_n \geq 2\} & \xrightarrow{\sim} & \{\text{admissible words in } 0, 1\} \\ (s_1, \cdots, s_n) & \mapsto & (1\underbrace{0\cdots0}_{s_1-1}1\underbrace{0\cdots0}_{s_2-1}\cdots1\underbrace{0\cdots0}_{s_n-1}). \end{array}$$

Recall that admissible means: starting by 1 and ending by 0. This bijection mimics the way multiple zeta values are expressed as iterated integrals:

Proposition 134.

$$\begin{aligned} \zeta(s_1, \dots, s_n) &\overset{Def.}{=} \zeta((1\underbrace{0\cdots0}_{s_1-1}1\underbrace{0\cdots0}_{s_2-1}\cdots1\underbrace{0\cdots0}_{s_n-1})) \\ &= \int_0^1 \omega_1 \otimes \underbrace{\omega_0 \otimes \cdots \otimes \omega_0}_{s_1-1} \otimes \omega_1 \otimes \underbrace{\omega_0 \otimes \cdots \otimes \omega_0}_{s_2-1} \otimes \cdots \otimes \omega_1 \otimes \underbrace{\omega_0 \otimes \cdots \otimes \omega_0}_{s_n-1} \end{aligned}$$

This is an improper integral, which is, however, convergent as soon as $s_n \geq 2$.

Proof. This follows by induction from Lemma 137 because $\zeta(s_1, \dots, s_n) = \mathrm{Li}(s_1, \dots, s_n; 1)$. $\square$

Putting $\mathrm{S}\omega := \int_0^{z'} \omega|_{z'=z}$, we have obviously:

Lemma 135. *For $\omega_1, \dots, \omega_n$ 1-forms*

$$\int_0^z \omega_1 \otimes \cdots \otimes \omega_n = \mathrm{S}(\cdots \mathrm{S}(\mathrm{S}\omega_1 \cdot \omega_2) \cdot \omega_3) \cdots \omega_n)$$

(which explains, by the way, the name 'iterated integral').

Definition 136. The multiple polylogarithm function is defined as

$$\mathrm{Li}(s_1, \dots, s_n; z) = \sum_{0 < i_1 < \cdots < i_n} \frac{z^{i_n}}{i_1^{s_1} \cdots i_n^{s_n}}$$

Lemma 137. *1.* $\mathrm{S}(\mathrm{Li}(s_1, \dots, s_n; z) \cdot \frac{dz}{z}) = \mathrm{Li}(s_1, \dots, s_n + 1; z)$.

2. $\mathrm{S}(\mathrm{Li}(s_1, \dots, s_n; z)\frac{dz}{1-z}) = \mathrm{Li}(s_1, \dots, s_n, 1; z)$.

3. $\mathrm{S}(\frac{dz}{1-z}) = \mathrm{Li}(1; z) = -\log(1 - z)$.

Proof. The proof of 1. is straightforward from the definitions.

The proof of 2. follows by the calculation:

$$\int_0^{z'} \left(\sum_{0<i_1<\cdots<i_n} \frac{z^{i_n}}{i_n^{s_1}\cdots i_n^{s_n}} \right) \left(\sum_{k=0}^{\infty} z^k \right) \mathrm{d}z = \int_0^{z'} \sum_{0<i_1<\cdots<i_{n+1}} \frac{z^{i_{n+1}-1}}{i_1^{s_1}\cdots i_n^{s_n}} \mathrm{d}z = \sum_{0<i_1<\cdots<i_{n+1}} \frac{(z')^{i_{n+1}}}{i_1^{s_1}\cdots i_n^{s_n} i_{n+1}}$$

where $i_{n+1} := i_n + k + 1$. □

Corollary 138 (Shuffle formula for multiple zeta values)**.** *For* $(x_1 \cdots x_n)$ *and* $(x_{n+1} \cdots x_{n+m})$ *admissible words, we have*

$$\zeta((x_1 \cdots x_n)) \cdot \zeta((x_{n+1} \cdots x_{n+m})) = \sum_{\sigma \in S_{n,m}} \zeta((x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n+m)})).$$

Corollary 139. *For* $(x_1 \cdots x_n)$ *an admissible word, we have*

$$\zeta((x_1 \cdots x_n)) = \zeta((1 - x_n \cdots 1 - x_1)).$$

Example 140.

$$\zeta(3) = \zeta((100)) = \zeta((110)) = \zeta(1,2).$$

Monodromy interpretation

Iterated integrals are useful to express the monodromy of vector bundles with connection, and can even be so characterized. To that end, let M be again a real manifold and let $V = \mathbb{C}^k \times M \to M$ be a trivial vector bundle. Let a strictly upper triangular matrix of 1-forms be given:

$$N = \begin{pmatrix} 0 & \omega_{12} & \cdots & \omega_{1k} \\ & & \vdots & \\ 0 & 0 & \cdots & \omega_{(k-1)k} \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{End}(\mathbb{C}^k) \otimes A^1(M).$$

It defines a connection

$$\begin{aligned} \nabla : V &\to V \otimes A^1(M) \\ v &\mapsto \mathrm{d}v + Nv \end{aligned}$$

on V.

Define the following function on the path-space PM:

$$\begin{aligned} \int N^{\otimes} : PM &\to \mathrm{End}(\mathbb{C}^k), \\ \gamma &\mapsto \int_\gamma \sum_{n=0}^{\infty} N^{\otimes n}. \end{aligned}$$

Here $N^{\otimes n}$ involves the usual matrix product using the tensor product of forms, for example:

$$N = \begin{pmatrix} 0 & \omega_1 & 0 \\ 0 & 0 & \omega_2 \\ 0 & 0 & 0 \end{pmatrix} \qquad N^{\otimes 2} = \begin{pmatrix} 0 & 0 & \omega_1 \otimes \omega_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad N^{\otimes 3} = 0.$$

Proposition 141. *The following are equivalent*

1. *N gives a flat connection, i.e., $\nabla^2 = 0$.*
2. *$dN + N \wedge N = 0$.*
3. *The function $\int N^{\otimes}$ is homotopy invariant.*

Recall that, given a vector bundle (E, ∇) with flat connection, any vector $v \in E_a$ in the fibre above a point a can be extended uniquely to a flat section $\overline{v} : U \to E|_U$ (i.e. $v_a = v$) in a small neighborhood $U \ni a$. Performing the unique extension along a path yields a map (**parallel transport**):

$$\pi_1(M, a, b) \to \mathrm{Hom}(E_a, E_b),$$

which, for $a = b$, yields a representation of $\pi_1(M, a)$, the **monodromy representation**.

The main result of this section is

Proposition 142. *Assume any one of the three conditions of 141 (and hence all) holds. Then the parallel transport of (V, ∇) is given by the map*

$$\begin{aligned} \pi_1(M, a, b) &\to \mathrm{Hom}(V_a, V_b) = \mathrm{End}(\mathbb{C}^k), \\ \gamma &\mapsto \int_\gamma N^{\otimes}. \end{aligned}$$

Example 143. The above holds true even if N is not nilpotent: Let $M = \mathbb{R}$ and $N = \mathrm{d}x$. Then

$$\int_\gamma \mathrm{d}x^{\otimes n} = \frac{x^n}{n!} \qquad \int_\gamma \mathrm{d}x^{\otimes} = \exp(x),$$

where γ is any path from 0 to x. $\int N^{\otimes}$, therefore, may be seen as a generalization of the exponential series.

Partial proof of proposition 141. It is well-known that 1. and 2. are equivalent. We will show that 2. implies 3. Consider a plot $\phi : U \to PM$ with fixed endpoints. Using Proposition 122, we get

$$\begin{aligned} \mathrm{d} \sum_{n=0}^{\infty} \int_\phi N^{\otimes n} &= -\sum_{n=0}^{\infty} \sum_{i=1}^{n} \int_\phi N^{\otimes i-1} \otimes \mathrm{d}N \otimes N^{\otimes n-i} \\ &\quad - \sum_{n=0}^{\infty} \sum_{i=1}^{n-1} \int_\phi N^{\otimes i-1} \otimes (N \wedge N) \otimes N^{\otimes n-i-1} \\ &= 0 \end{aligned}$$

by property 2. $\square$

Proof of proposition 142. Consider two points $a, b \in M$. Let $U \subset M$ be a small open subset around a point b. Choose a smooth morphism $\widetilde{\phi} : U \times \sigma_1 \to M$ such that $\widetilde{\phi}(u, 0) = a$ and $\widetilde{\phi}(u, 1) = u$. By definition, it determines a plot $\phi : U \to PM$. The statement follows, if we can show that

$$\nabla(\int_\phi N^\otimes)v = 0$$

for all $v \in \mathbb{C}^k$. Note that $\int_\phi N^\otimes$ is a function on U with values in $\mathrm{End}(\mathbb{C}^k)$. Using Proposition 122 and the previous calculation, we get

$$\begin{aligned}\nabla \int_\phi N^\otimes &= \sum_{n=1}^{\infty}(-1)(\int_\phi N^{\otimes n-1}) \cdot (d \circ \phi)^* N + \sum_{n=0}^{\infty} N \cdot (\int_\phi N^{\otimes n}) \\ &= 0.\end{aligned}$$

Observe, that $d \circ \phi = \mathrm{id}$ by construction, and that N commutes with $\int_\phi N^{\otimes n}$. □

Remark 144. Even if condition 2. on N is not true, i.e. if ∇ is not flat, there is still a notion of parallel transport, which depends on the particular element in the homotopy class of the path, though. The formula above remains true in this case. We will not need this.

Cohomological interpretation

Let X be a smooth variety and $a, b \in X$.

Definition 145. Define the union of hyperplanes

$$Z^n_{a,b} = \{x_1 = a\} \cup \{x_1 = x_2\} \cup \cdots \cup \{x_n = b\} \subset X^n = \underbrace{X \times \cdots \times X}_{n \text{ times}}.$$

Consider the following cosimplicial scheme

$$\{\cdot\} \rightrightarrows X \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; X^2 \cdots$$

where $\delta_{1,0} = a$, $\delta_{1,1} = b$, $\delta_{2,0} = a \times \mathrm{id}$, $\delta_{2,1} = \Delta$, $\delta_{2,2} = \mathrm{id} \times b$, etc. and the degeneracies are just the projections forgetting one of the factors. Note that the joint image of the δ_n-maps is just $Z^n_{a,b}$.

Taking **de Rham complexes** of the entries of this cosimplicial scheme induces a simplicial object in the category of complexes of Abelian groups

$$\mathbb{C} \leftleftarrows A^{-1,\bullet}_{iter}(PX_{a,b}) \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; A^{-2,\bullet}_{iter}(PX_{a,b}) \cdots$$

whose associated "alternating face map"-complex is just the double complex $A^{\bullet;\bullet}_{iter}(PX_{a,b})$ (bar complex) introduced before, such that the alternating face map becomes d$''$. Its normalized complex $\widetilde{A}^{\bullet;\bullet}_{iter}$ (dividing out images of the degeneracies) has the same homology and also the map 'iterated integration' factors through it. This is not the reduced bar complex, however. It lies in between the bar complex and the reduced one.

Dually, taking **chain complexes** (with rational coefficients) induces a cosimplicial object in the category of complexes of Abelian groups

$$\mathbb{C} \rightrightarrows C^{iter}_{-1,\bullet}(PX_{a,b}) \rightrightarrows C^{iter}_{-2,\bullet}(PX_{a,b}) \cdots ,$$

i.e.

$$C^{iter}_{-j,i}(PX_{a,b}) := C_i(X^j, \mathbb{Q}).$$

In the 4th talk, it will be shown that the total complex of *the truncation* of the normalized complex

$$\mathbb{C} \xleftarrow{\mathrm{d}''} \widetilde{A}^{-1,\bullet}_{iter}(PX_{a,b}) \xleftarrow{\mathrm{d}''} \widetilde{A}^{-2,\bullet}_{iter}(PX_{a,b}) \xleftarrow{\mathrm{d}''} \cdots \xleftarrow{\mathrm{d}''} \widetilde{A}^{-n,\bullet}_{iter}(PX_{a,b})$$

denoted $\widetilde{A}^{\bullet \geq -n,\bullet}_{iter}$ actually computes the relative cohomology group $\mathbb{C}_{a,b} \oplus H^n(X^n, Z^n_{a,b})$[2].

Dually, the total complex of the truncated normalized complex $\widetilde{C}^{iter}_{\bullet \geq -n,\bullet}(PX_{a,b})$ computes the homology $\mathbb{Q}_{a,b} \oplus H_n(X^n, Z^n_{a,b})$. We may and will normalize this complex by dividing out the images of the $\delta_{n,i}$ for $i < n$.

The first step towards Chen's theorem is to construct a map

$$\widetilde{c}_n : \mathbb{Q}[\pi_1(X,a,b)] \to \mathbb{H}_0(\widetilde{C}^{iter}_{\bullet \geq -n,\bullet}(PX_{a,b})) \cong \mathbb{Q}_{a,b} \oplus H_n(X^n, Z^n_{a,b}).$$

To this end, let γ be a path from a to b. It induces a chain $\gamma_n \in C_n(X^n)$ by restricting the map

$$\gamma^n : \sigma^n_1 \to X^n$$

to the simplex σ_n. The sequence

$$(\gamma_0, \ldots, \gamma_n)$$

is a *cycle* in $\mathbb{H}_0(\widetilde{C}^{iter}_{\bullet \geq -n,\bullet})(PX_{a,b})$ because the boundary $\mathrm{d}'\gamma_n$ is minus the image $\mathrm{d}''\gamma_{n-1}$ of γ_{n-1} under the alternating face map. Here $\gamma_0 = 1 = \mathrm{aug}(\gamma)$.

Let $\delta : \sigma^2_1 \to X$ be a homotopy from γ_1 to γ_2. We have then

$$\mathrm{d}(\delta_0, \ldots, \delta_n) = (\gamma_{2,0}, \ldots, \gamma_{2,n}) - (\gamma_{1,0}, \ldots, \gamma_{1,n})$$

where δ_n is the chain[3]:

$$\begin{gathered} \delta_n : \sigma_1 \times \sigma_n \to X^n \\ t, (t_1, \ldots, t_n) \mapsto (\delta(t,t_1), \ldots, \delta(t,t_n)). \end{gathered}$$

In other words, we get a well-defined map

$$\begin{array}{rcl} \widetilde{c}_n : \mathbb{Q}[\pi_0(X,a,b)] & \to & \mathbb{H}_0(\widetilde{C}^{iter}_{\bullet \geq -n,\bullet}(PX_{a,b})) \quad (\cong \mathbb{Q}_{a,b} \oplus H_n(X^n, Z^n_{a,b})) \\ \gamma & \mapsto & (\gamma_0, \ldots, \gamma_n). \end{array}$$

[2] $\mathbb{C}_{a,b}$ is $\mathbb{C}$, if $a = b$ and 0 otherwise.

[3] decomposed into a sum of simplexes in any way you like

By definition of iterated integral, we have for a form $\omega \in \mathbb{H}^0(\widetilde{A}^{\bullet;\bullet}_{iter}(PX_{a,b}))$

$$\int_\gamma \omega = \langle \widetilde{c}_n(\gamma), \omega \rangle.$$

In particular the product $\mathbb{Q}[\pi_0(X,a,b)] \times \mathbb{Q}[\pi_0(X,b,c)] \to \mathbb{Q}[\pi_0(X,a,c)]$ is compatible with the dual of the coproduct (0.0.3), i.e. with the product

$$(\alpha_0, \dots, \alpha_n) \cdot (\beta_0, \dots, \beta_n) = (\alpha_0 \cdot \beta_0, \alpha_0 \cdot \beta_1 + \alpha_1 \cdot \beta_0, \alpha_0 \cdot \beta_2 + \alpha_1 \cdot \beta_1 + \alpha_2 \cdot \beta_0, \dots). \quad (0.0.4)$$

Its k-th entry is $\sum_{i=0}^k \alpha_i \cdot \beta_{k-i}$. The product '$\cdot$' in this formula is the usual Cartesian product of chains $C^i(X^n) \times C^j(X^m) \to C^{i+j}(X^{n+m})$ (formed, say, using the canonical decomposition of Cartesian products 133, 1. into subsimplices).

Recall the following

Definition 146. Let M be a path-connected topological space and $a \in M$. Let

$$\begin{array}{rcl} \mathrm{aug}: \mathbb{Q}[\pi_1(M;a)] & \to & \mathbb{Q} \\ \sum_i a_i \gamma_i & \mapsto & \sum_i a_i \end{array}$$

be the augmentation. Its kernel in $\mathbb{Q}[\pi_1(M;a)]$

$$I = \ker \mathrm{aug}.$$

is called the augmentation ideal.

The augmentation ideal is generated by elements of the form $1 - \gamma$ for $\gamma \in \pi_1(M;a)$.

Corollary 147. *The map $\widetilde{c}_n$ factors via*

$$c_n : \mathbb{Q}[\pi_0(X,a,b)]/I^{n+1}\mathbb{Q}[\pi_0(X,a,b)] \to \mathbb{H}_0(\widetilde{C}^{iter}_{\bullet \geq -n, \bullet}(PX_{a,b})).$$

Proof. The image of the augmentation ideal has the property, that the first entry α_0 is 0. If we define a filtration $F^i\mathbb{H}_0(\widetilde{C}_{\bullet \geq -n, \bullet}(PX_{a,b}))$ by the property that the first i-entries be zero, we have by formula (0.0.4) that

$$F^i\mathbb{H}_0(\widetilde{C}_{\bullet \geq -n, \bullet}(PX_{a,b})) \cdot F^j\mathbb{H}_0(\widetilde{C}_{\bullet \geq -n, \bullet}(PX_{b,c})) \subseteq F^{i+j}\mathbb{H}_0(\widetilde{C}_{\bullet \geq -n, \bullet}(PX_{a,c})).$$

In paricular, if $\gamma \in I^m$, $\widetilde{c}_n(\gamma) = (\gamma_0, \dots, \gamma_n)$ lies in $F^m\mathbb{H}_0(\widetilde{C}_{\bullet \geq -n, \bullet}(PX_{a,a}))$. Since trivially

$$F^{n+1}\mathbb{H}_0(\widetilde{C}_{\bullet \geq -n, \bullet}(PX_{a,b}) = 0,$$

the statement follows. $\square$

Theorem 148 (Chen)**.** *c_n is an isomorphism.*

Proof. This will be proven later in Theorem 198, relating the two sides to unipotent representations of π_1 and bundles with nilpotent connection, respectively. We prove here that c_n is surjective by induction on n: It suffices to consider the case $a = b$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[d] & \\
I^{n+1}/I^n \arrow[d] & H_1(X,\mathbb{Q})^{\otimes n} \arrow[l, dotted, "\rho"'] \arrow[d, "\epsilon"] \\
\mathbb{Q}[\pi_0(M,a)]/I^{n+1} \arrow[r, "c_n"'] \arrow[d] & \mathbb{H}_0(\widetilde{C}^{iter}_{\bullet\geq -n,\bullet}(PX_{a,b})) \arrow[d] \\
\mathbb{Q}[\pi_0(M,a)]/I^n \arrow[r, two heads, "c_{n-1}"'] \arrow[d] & \mathbb{H}_0(\widetilde{C}^{iter}_{\bullet\geq -n+1,\bullet}(PX_{a,b})) \arrow[d] \\
0 & 0
\end{tikzcd}$$

The map ϵ is defined as follows. It maps $\gamma_1 \otimes \cdots \otimes \gamma_n$ to $(0, \ldots, 0, \gamma_1 \cdot \cdots \cdot \gamma_n)$. The map ρ sends $\gamma_1 \otimes \cdots \otimes \gamma_n$ to $(1 - \gamma_1) \cdot \cdots \cdot (1 - \gamma_n)$. (Note that $H_1(X, \mathbb{Q}) \cong I/I^2$ via $\gamma \mapsto 1 - \gamma$.) The lower horizontal map is surjective by the induction hypothesis.

We claim that the right column is an exact sequence. The lower vertical map is surjective because c_{n-1} is an surjective. For the exactness in the middle, observe that an element in the kernel of the truncation is represented (modulo boundaries) by an element of the form $(0, \ldots, 0, \alpha_n)$ with $\mathrm{d}'\alpha_n = 0$ (usual d of chains), i.e. where α_n is a cycle. By Künneth, modulo boundaries again, which do not affect the zeros, we may write $\alpha_n = \sum_i \alpha_{i,1} \cdot \cdots \cdot \alpha_{i,n}$ where each summand has total degree n. An element $\alpha_{i,1} \cdot \cdots \cdot \alpha_{i,n}$ with $\deg \alpha_{i,j} = 0$ for some j, however, lies in the sum of the images of the $\delta_{n,i}$ with $i < n$. The claim follows.

Formula (0.0.4) shows that the top square of the above diagram is commutative (up to sign). A small diagram chase shows that c_n is surjective, too. □

Addendum: Multizetas are (mixed) periods!

We have seen that iterated integrals are periods of relative motives $h^n(X^n, Z^n_{a,b})$, and that multiple zeta values can be expressed as iterated integrals on $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The problem is, of course, that we integrate over a simplex which is not entirely contained in $X_{\mathbb{C}}$. This will be resolved later by introducing tangential base points and quite indirectly. It can also be resolved "manually" by blowing up X^n. We will illustrate this in the example $n = 2$.

Recall the differential forms

$$\omega_0 = \frac{\mathrm{d}z}{z} \qquad \omega_1 = \frac{\mathrm{d}z}{1 - z}$$

which generate $H^1_{dR}(X)$.

We consider the iterated integral

$$\zeta(2) = \int_\gamma \omega_1 \otimes \omega_0$$

where γ is the straight path from 0 to 1. Recall that this iterated integral is, by definition, the integral of

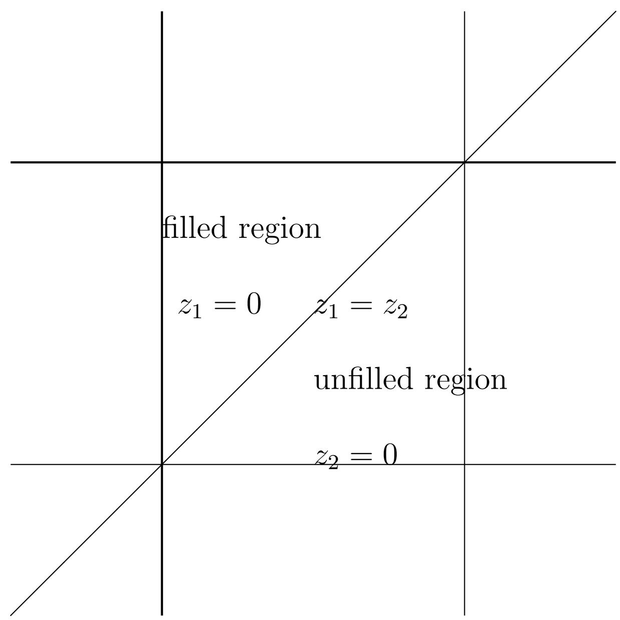
$$\omega = \omega_1 \boxtimes \omega_0 = \frac{dz_1}{1-z_1} \wedge \frac{dz_2}{z_2}$$

over the simplex

$$\sigma_2 = \{0 \leq t_1 \leq t_2 \leq 1 \mid t_i \in \mathbb{R}\},$$

which we consider as subset of $(\mathbb{P}^1_{\mathbb{C}})^2$.

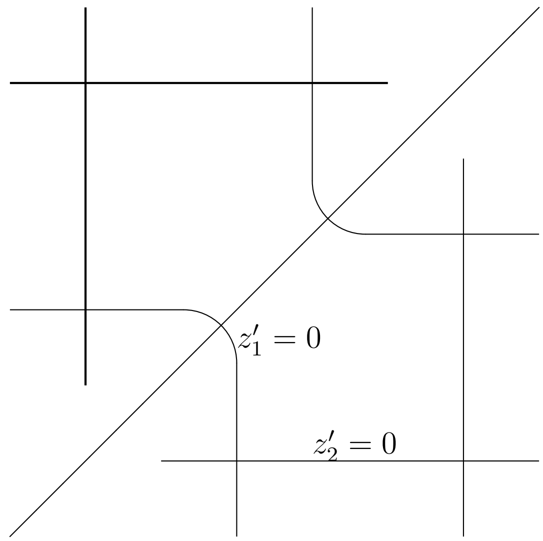
The following picture visualizes the simplex (filled region) and the singularities of ω.



Observe that ω is actually a differential form not only on M^2 but on

$$(\mathbb{P}^1)^2 \setminus (\{z_1 = 1\} \cup \{z_2 = 0\} \cup \{z_1 = \infty\} \cup \{z_2 = \infty\}).$$

Still, σ_2 is not contained in this larger space. However, if we consider the blow-up $\widetilde{M}$ of $(\mathbb{P}^1)^2$ at $(0,0)$, $(1,1)$ and (∞,∞), we get the following picture:



In the blown-up coordinates z_1', z_2' transformed by

$$z_1 = z_1' z_2' \qquad z_2 = z_2'$$

the differential form becomes

$$\omega = \frac{\mathrm{d}z_1' z_2' + \mathrm{d}z_2' z_1'}{1 - z_2' z_1'} \wedge \frac{\mathrm{d}z_2'}{z_2'} = \frac{\mathrm{d}z_1' \mathrm{d}z_2'}{1 - z_2' z_1'}$$

This means that the singularities of ω occur only at the strict transforms of $\{z_1 = 1\}$ and $\{z_2 = 0\}$ and the preimage of $\{z_1 = \infty\}$ and $\{z_2 = \infty\}$. A slightly more refined argument shows that actually

$$\begin{aligned} \omega \in H^n(\widetilde{M} \setminus (\{z_1 = 1\} \cup \{z_2 = 0\} \cup \{z_1 = \infty\} \cup \{z_2 = \infty\} \cup E_{\infty,\infty}), \\ (\{z_1 = 0\} \cup \{z_2 = 1\} \cup \{z_1 = z_2\} \cup E_{0,0} \cup E_{1,1}) \cap \cdots). \end{aligned}$$

where, by abuse of notation, we denoted the strict transform of e.g. $\{z_1 = 1\}$ again by the same term. Alike, σ_2, appropriately reparametrized, is now an element of

$$\begin{aligned} H_n(\widetilde{M} \setminus (\{z_1 = 1\} \cup \{z_2 = 0\} \cup \{z_1 = \infty\} \cup \{z_2 = \infty\} \cup E_{\infty,\infty}), \\ (\{z_1 = 0\} \cup \{z_2 = 1\} \cup \{z_1 = z_2\} \cup E_{0,0} \cup E_{1,1}) \cap \cdots). \end{aligned}$$

and the period pairing $\langle \sigma_2, \omega \rangle$ is still $\zeta(2)$.

Actually, we have the following connection to the theory of moduli of curves:

- $X^2 \setminus \Delta$ is the moduli space of **smooth** genus 0 curves with 5 distinct marked points.
- $\widetilde{M}$ is the moduli space of **stable** genus 0 curves with 5 distinct marked points.

A point $(z_1, z_2) \in X^2 \setminus \Delta$ classifies the curve $C = \mathbb{P}^1$ with $(0, 1, \infty, z_1, z_2)$ marked. Every smooth genus 0 curve with 5 marked points is isomorphic to one of those.

The boundary divisor of $\widetilde{M}$ has the following interpretation, where we write, for instance, $(0, z_1, \infty | 1, z_2)$ for a chain of 2 $\mathbb{P}^1$'s with 3 points $0, z_1, \infty$ marked in the first and 2 points $1, z_2$ marked in the second. The symbols $0, 1, \infty$ are considered as *abstract* symbols here.

boundary component	moduli of singular marked curves of type
$\{z_1 = 0\}$	$(1, z_2, \infty \| 0, z_1)$
$\{z_2 = 0\}$	$(1, z_1, \infty \| 0, z_2)$
$\{z_1 = 1\}$	$(0, z_2, \infty \| 1, z_1)$
$\{z_2 = 1\}$	$(0, z_1, \infty \| 1, z_2)$
$\{z_1 = \infty\}$	$(0, 1, z_2 \| \infty, z_1)$
$\{z_2 = \infty\}$	$(0, 1, z_1 \| \infty, z_2)$
$\{z_1 = z_2\}$	$(0, 1, \infty \| z_1, z_2)$
$E_{0,0}$	$(0, z_1, z_2 \| 1, \infty)$
$E_{1,1}$	$(1, z_1, z_2 \| 0, \infty)$
$E_{\infty,\infty}$	$(\infty, z_1, z_2 \| 0, 1)$

These correspond precisely to the $\binom{5}{2}$ possible ways of putting $\{0, 1, \infty, z_1, z_2\}$ into 2 boxes such that none of those contains less than 2 elements. The intersections correspond to maximally singular stable curves, for example:

boundary component	*the* singular marked curve of type
$\{z_1 = 0\} \cap E_{0,0}$	$(1, \infty \| z_2 \| 0, z_1)$
$\{z_1 = z_2\} \cap E_{\infty,\infty}$	$(z_1, z_2 \| \infty \| 0, 1)$
$\vdots$	$\vdots$

The ultimate generalization of the above is the

Theorem 149 (Goncharov, Manin)**.** *$\zeta(n_1, \ldots, n_k)$ is a period of $h^n(\overline{\mathcal{M}_{0,n+3}} \setminus E_1, E_2 \cap (\cdots))$, where n is the total weight of $(n_1, \ldots, n_k)$.*

Here $\overline{\mathcal{M}_{0,n+3}}$ is the moduli space of stable genus 0 curves with $n + 3$ marked points and E_1 and E_2 are each the union over one of two disjoint sets of boundary divisors, whose combinatorial description as above can be given explicitly in terms of $(n_1, \ldots, n_k)$.

The pro-unipotent completion

Alberto Vezzani on September 3rd, 2012.

The aim of these notes is to give an overview of Quillen's construction of the pro-unipotent completion of an abstract group (or a Lie algebra). Some consequences of the formulas and special cases are explained in more detail.

Introduction

We start by presenting the work of Quillen [5, Appendix A], and translating it into the setting of algebraic groups, following the approach of [2]. We also follow Cartier [1] for specific facts on Hopf algebras. From now on, we work over the base field $\mathbb{Q}$.

Definition 150. Given an abstract group Γ [resp. a Lie algebra $\mathfrak{g}$], the pro-unipotent completion Γ^{un} [resp. $\mathfrak{g}^{un}$] is the universal pro-unipotent algebraic group G endowed with a map $\Gamma \to G(\mathbb{Q})$ [resp. $\mathfrak{g} \to \operatorname{Lie} G$].

Let us focus on the case of groups. The meaning of the definition is that there is a map $u : \Gamma \to \Gamma^{un}(\mathbb{Q})$ such that for any map $f : \Gamma \to G(\mathbb{Q})$ to the $\mathbb{Q}$-points of a pro-unipotent algebraic group G, there exists a unique map $\phi : \Gamma^{un} \to G$ such that $f = \phi(\mathbb{Q}) \circ u$. In other words, we are looking for a left adjoint to the functor $G \mapsto G(\mathbb{Q})$ defined from pro-unipotent algebraic groups to abstract groups. Sadly enough, we anticipate that we will need to restrict to a subcategory of abstract groups in order to find such a functor.

The category of pro-unipotent algebraic groups is a full subcategory of the category of pro-affine algebraic algebraic groups (the category of formal filtered limits of affine algebraic groups over quotients). This category is clearly equivalent to the opposite category of Hopf algebras (not necessarily finitely presented). What we need to do is therefore to associate to an abstract group a particular commutative Hopf algebra over $\mathbb{Q}$. There are some well-known examples of adjoint pairs which are close to reaching this aim.

Proposition 151. *There is an adjoint pair of functors*

$$\mathbb{Q}[\cdot]\colon \mathbf{Gps} \rightleftarrows \mathbb{Q}\text{-}\mathbf{Alg} :(\cdot)^{\times}$$

between the category of abstract groups and (not necessarily commutative) $\mathbb{Q}$-algebras.

Any group algebra $\mathbb{Q}[\Gamma]$ can be endowed with the structure of a Hopf algebra with respect to the maps

$$\Delta : g \mapsto g \otimes g \qquad S : g \mapsto g^{-1} \qquad \epsilon : g \mapsto 1$$

for all $g \in \Gamma$. Therefore the functor $\mathbb{Q}[\cdot]$ factors over the category of Hopf algebras. Also this new functor has an adjoint:

Proposition 152. *There is an adjoint pair of functors*

$$\mathbb{Q}[\cdot]\colon \mathbf{Gps} \rightleftarrows \mathbf{HA} :\mathcal{G}$$

between the category of abstract groups and (not necessarily commutative) $\mathbb{Q}$-Hopf algebras where $\mathcal{G}$ associates to a Hopf algebra R the set of group-like elements:

$$\mathcal{G}R := \{x \in R^\times : \Delta x = x \otimes x\}$$

endowed with the product inherited from R.

Proof. This comes from the previous proposition. Indeed, given a map $\mathbb{Q}[\Gamma] \to R$ induced by $f : \Gamma \to R^\times$, the diagram

$$\begin{tikzcd}
\mathbb{Q}[\Gamma] \arrow[r] \arrow[d] & \mathbb{Q}[\Gamma] \otimes \mathbb{Q}[\Gamma] \arrow[d] \\
R \arrow[r] & R \otimes R
\end{tikzcd}$$

commutes if and only if all images of the elements of G are group-like. Moreover, any group-like element x satisfies $\epsilon(x) = 1$, hence also the augmentation is preserved. □

Similarly for Lie algebras:

Proposition 153. *(i) There is an adjoint pair of functors*

$$\mathcal{U}\colon \mathbf{LA} \rightleftarrows \mathbb{Q}\text{-}\mathbf{Alg} : for$$

between the category of Lie algebras and (not necessarily commutative) $\mathbb{Q}$-algebras. The functor for sends a $\mathbb{Q}$-algebra R to the Lie algebra structure over R induced by commutators.

(ii) The left adjoint factors over the category of Hopf algebras by endowing the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ with the structure of a Hopf algebra with respect to the maps

$$\Delta : x \mapsto x \otimes 1 + 1 \otimes x \qquad S : x \mapsto -x \qquad \epsilon : x \mapsto 0$$

for all $x \in \mathfrak{g}$.

(iii) *There is an adjoint pair of functors*

$$\mathcal{U}\colon \mathbf{LA} \rightleftarrows \mathbf{HA} : \mathcal{P}$$

between the category of Lie algebras and (not necessarily commutative) $\mathbb{Q}$-Hopf algebras where $\mathcal{P}$ associates to a Hopf algebra R the set of primitive elements:

$$\mathcal{P}R := \{x \in R : \Delta x = x \otimes 1 + 1 \otimes x\}$$

endowed with the Lie bracket induced by commutators.

Note that $\mathbb{Q}[\Gamma]$ and $\mathcal{U}\mathfrak{g}$ are cocommutative, but not necessarily commutative (this happens iff Γ or $\mathfrak{g}$ is abelian). Since our initial aim was to associate to a group (or to a Lie algebra) a commutative, but not necessarily cocommutative Hopf algebra, the natural idea is now to "take duals". Taking duals of vector spaces is a delicate operation whenever the dimension is not finite. Hence, we will need to restrict to a particular case where the situation is self-reflexive as in the finite-dimensional case.

Definition 154. A topological vector space V is linearly compact if it is homeomorphic to $\varprojlim V/V_i$, where V/V_i are discrete and finite dimensional, and the maps in the diagram are quotients.

We will denote by $(\cdot)^\vee$ the dual space and by $(\cdot)^*$ the topological dual.

Example 155. If V is a discrete vector space, we will always enodow its dual $V^\vee$ with the linearly compact topology $\varprojlim W_i^\vee$ by letting W_i vary among the subvector spaces of V which are finite dimensional.

Proposition 156.

1. *If V is discrete [resp. linearly compact], then $(V^\vee)^* \cong V$ [resp. $(V^*)^\vee \cong V$].*
2. *If V is discrete [resp. linearly compact], then $(V \otimes V)^\vee \cong V^\vee \hat{\otimes} V^\vee$ [resp. $(V \hat{\otimes} V)^* \cong V^* \otimes V^*$].*

In particular, duality defines an equivalence of categories between commutative Hopf algebra and the category of linearly compact Hopf algebras.

Our attempt is now to use these dualities in order to obtain a commutative and cocommutative Hopf algebra out of $\mathbb{Q}[\Gamma]$ or $\mathcal{U}\mathfrak{g}$. By what just stated, we need to get a complete topological Hopf algebra. Any Hopf algebra R is augmented by the counit ϵ. Let I denote the augmentation ideal. We can endow R with the I-adic topology, and complete it with respect to it.

Definition 157. A complete Hopf algebra is a complete topological augmented algebra $\epsilon : R \to R/I \cong \mathbb{Q}$, homeomorphic to $\varprojlim R/I^k$ and endowed with a map $\Delta : R \to R \hat{\otimes} R$ that fit in the usual diagrams of Hopf algebras. We denote the category of complete Hopf algebras by **CHA**.

We remark that our definition differs slightly from the one of [5] since Quillen introduces also the choice of a filtration.

Example 158. If R is a Hopf algebra, then its I-adic completion $\hat{R}$ is a complete Hopf algebra. In particular, $\Gamma \mapsto \widehat{\mathbb{Q}[\Gamma]}$ and $\mathfrak{g} \mapsto \widehat{\mathcal{U}\mathfrak{g}}$ define functors to the category **CHA**.

The following proposition is a formal consequence of the previous ones.

Proposition 159. *There are adjoint pairs of functors*

$$\hat{\mathbb{Q}}[\cdot] \colon \mathbf{Gps} \rightleftarrows \mathbf{CHA} : \mathcal{G}$$

$$\hat{\mathcal{U}} \colon \mathbf{LA} \rightleftarrows \mathbf{CHA} : \mathcal{P}$$

where $\mathcal{G}$ and $\mathcal{P}$ are defined like before.

We can now isolate in **CHA** the full subcategory **C** of those algebras R which are also linearly compact. Since $R \cong \varprojlim R/I^k$, this condition is equivalent to asking that R/I^k is finite dimensional for all k. Since this is obviously true for $k = 1$, by induction we conclude that this is equivalent to the finite dimensionality of all I^k/I^{k+1}. Multiplication defines a surjection $(I/I^2)^{\otimes k} \to I^k/I^{k+1}$, and therefore this is equivalent to imposing I/I^2 finite dimensional.

Example 160.

1. Let Γ be an abstract group. Then

$$I_{\hat{\mathbb{Q}}[\Gamma]}/I^2_{\hat{\mathbb{Q}}[\Gamma]} \cong I_{\mathbb{Q}[\Gamma]}/I^2_{\mathbb{Q}[\Gamma]} \cong \Gamma^{ab} \otimes_Z \mathbb{Q}$$

   where Γ^{ab} is the abelianization of Γ, and where the last isomorphism is induced by $(g - e) \mapsto g$. In particular, if Γ is such that $\Gamma^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$ has finite rank, then $\hat{\mathbb{Q}}[\Gamma]$ is linearly compact. We denote by $\widetilde{\mathbf{Gps}}$ the full subcategory of **Gps** of objects satisfying this property.

2. Let $\mathfrak{g}$ be a Lie algebra. Then

$$I_{\hat{\mathcal{U}}\mathfrak{g}}/I^2_{\hat{\mathcal{U}}\mathfrak{g}} \cong I_{\mathcal{U}\mathfrak{g}}/I^2_{\mathcal{U}\mathfrak{g}} \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$$

   where the last isomorphism is induced by $x \mapsto x$. In particular, if $\mathfrak{g}$ is such that $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ has finite rank, then $\hat{\mathcal{U}}\mathfrak{g}$ is linearly compact. We denote by $\widetilde{\mathbf{LA}}$ the full subcategory of **LA** of objects satisfying this property.

By duality, the category **C** is equivalent to a full subcategory of $\mathbf{HA}^{op}$, and hence to a subcategory of pro-affine algebraic groups.

Proposition 161. *Let $G = \operatorname{Spec} R$ be a pro-affine algebraic group. Then $R^\vee \in \mathbf{C}$ if and only if $\mathcal{P}R$ is finite dimensional and the "conilpotency filtration"*

$$0 \subset C_0 := \operatorname{Ann}_R I \subset \ldots \subset C_k := \operatorname{Ann}_R I^{k+1} \subset \ldots \tag{0.0.5}$$

is exhaustive, i.e. if $R = \bigcup C_i$, where I is the augmentation ideal of $R^\vee$.

Proof. We know that $R^\vee$ lies in $\mathbf{C}$ if I/I^2 is finite dimensional, and if $R^\vee \cong \varprojlim R^\vee/I^k$. The dual space $(R^\vee/I^k)^*$ coincides with C_k. In particular, $C_0 = \mathbb{Q}$ and $C_1 = \mathbb{Q}^{op} lus \mathcal{P}R$. Indeed, an element x of R lies in $\mathrm{Ann} I^2$ if and only if $\Delta x = y \otimes 1 + 1 \otimes z$, and by using the axioms of Hopf algebra, this turns out to be equivalent to $\Delta x = x \otimes 1 + 1 \otimes x$ if $\epsilon(x) = 0$.

We then conclude that I/I^2 is finite dimensional if and only if $(I/I^2)^* = (\ker(R/I^2 \to R/I))^* = C_1/C_0 = \mathcal{P}R$ is finite dimensional, and that $R^\vee \cong \varprojlim R^\vee/I^k$ if and only if $R = \varinjlim (R^\vee/I^k)^* = \varinjlim C_k$. □

We remark that the conilpotency filtration $\{C_i\}$ just defined coincides with the one of Cartier [1, 3.8 (A)]. This is part of the following proposition, whose proof comes by induction from the previous one.

Proposition 162. *Let $G = \mathrm{Spec}\ R$ be a pro-affine algebraic group and let $\bar{R}$ be its augmentation ideal. The elements of the filtration* (0.0.5) *can be defined equivalently in the following ways:*

1. $C_i = \mathbb{Q}^{op} lus \ker \bar{\Delta}_n$, *where* $\bar{\Delta} : \bar{R} \to \bar{R} \otimes \bar{R}$ *maps* x *to* $\Delta x - x \otimes 1 - 1 \otimes x$ *and* $\bar{\Delta}_n : \bar{R} \to \bar{R}^{\otimes n}$ *maps* x *to* $(\bar{\Delta} \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id})(\bar{\Delta}_n x)$.
2. C_{i+1}/C_i *is the trivial subrepresentation of* G *inside* R/C_i.

Definition 163. A pro-unipotent algebraic group is a pro-affine algebraic group Spec R such that the conilpotency filtration (0.0.5) is exhaustive. A unipotent algebraic group is a pro-unipotent algebraic group Spec R such that R is finitely presented. The category defined by [pro-]unipotent algebraic groups will be denoted with **UAG** [resp. p**UAG**].

In particular, a unipotent algebraic group G such that the Lie algebra $\mathcal{PO}(G)$ is finite dimensional defines an object $\mathcal{O}(G)^\vee$ of $\mathbf{C}$.

Our definition is different from the "standard" one. We now prove the equivalence of the two notions. Recall that $\mathbf{UT}_n$ is the subgroup of $\boldsymbol{GL}_n$ defined by upper-triangular matrices, which have 1's on the main diagonal.

Proposition 164. *Let G be a pro-affine algebraic group. The following are equivalent:*

(i) The group G is pro-unipotent.

(ii) For every non-zero representation V of G, there exists a non-zero vector $v \in V$ such that $G \cdot v = v$.

In case G is an algebraic group, the previous conditions are equivalent to:

(iii) G is isomorphic to a subgroup of $\mathbf{UT}_n$ for some n.

Proof. Let $G = \mathrm{Spec}\ R$. Suppose (i) is satisfied. Then any representation $\rho : V \to V \otimes R$ admits an exhaustive filtration $\{V_k\}$ where $V_k := \{v \in V \otimes C_k\}$. In particular, V_0 is a trivial subrepresentation since if $v \in V_0$, then $\rho v = v \otimes 1$. We now prove (ii) by showing that $V_k = 0$ implies $V_{k+1} = 0$.

It can be explicitly seen that $\Delta C_i \subset \sum_{a+b=i} C_a \otimes C_b$. Therefore if $x \in V_{k+1}$, then $(1\otimes\Delta)(\rho x)$ lies in $\sum_{a+b=k+1} V \otimes C_a \otimes C_b$. Since a and b can't be both bigger than k, if follows that V_{k+1} is mapped to 0 via the composite map

$$V \to V \otimes R \overset{1\otimes\Delta}{\to} V \otimes R \otimes R \overset{\pi}{\to} V \otimes R/C_k \otimes R/C_k$$

On the other hand, the previous map coincides (by the axioms of comodules) with

$$V \to V \otimes R \overset{\rho\otimes 1}{\to} V \otimes R \otimes R \overset{\pi}{\to} V \otimes R/C_k \otimes R/C_k$$

which is an injection since $V_k = 0$. Viceversa, if any representation V has a non-zero trivial subrepresentation, by induction one can define an ascending filtration $\{V_i\}$ such that V_{i+1}/V_i is the trivial subrepresentation of V/V_i. Since any element of V generates a finite dimensional subrepresentation, it follows that this filtration is exhaustive. The conilpotency filtration corresponds to the filtration associated to the representation defined on R itself. This proves $(i) \Leftrightarrow (ii)$.

If V is finite dimensional, then (by induction on its dimension, since $V_0 \neq 0$) it is an extension of trivial representations. It follows in particular that, with respect to a suitable basis, $\rho : G \to \boldsymbol{GL}_V$ factors over $\mathbf{UT}_n$. If G is an algebraic group, one can apply this fact to a faithful finite dimensional representation to prove (iii). □

Being pro-unipotent is closed under quotients (using the condition (ii) for example). Hence, if Spec R is a pro-unipotent algebraic group, then any sub-Hopf algebra R' of R defines a pro-unipotent algebraic group. It follows that the category of p**UAG** coincides with the pro-objects of **UAG** and **C** is a subcategory of it.

We remark that our definition is slightly different from the one of Cartier [1, end of p. 53], since we do not impose that R has countable dimension.

Example 165.

1. Let Γ be an abstract group such that $\Gamma^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$ has finite rank. Then $\hat{\mathbb{Q}}[\Gamma]$ endowed with the I-adic topology is linearly compact, since it is homeomorphic to $\varprojlim \hat{\mathbb{Q}}[\Gamma]/I^k$ and all I_k have finite codimension. In particular, Spec $(\hat{\mathbb{Q}}[\Gamma]^*)$ is pro-unipotent.

2. Similarly, if $\mathfrak{g}$ is a Lie algebra such that $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ has finite rank, then Spec $(\hat{\mathcal{U}}\mathfrak{g}^*)$ is pro-unipotent.

3. Consider $G = \mathbf{G}_a$. The Hopf algebra is $R = \mathbb{Q}[t]$, its dual vector space is $\mathbf{pro}d\mathbb{Q}\epsilon_k$ where $\epsilon_k(t^i) = \delta_{k,i}$. By duality, the augmentation ideal is $I = \ker((\mathbb{Q} \to R)^\vee) = \{\phi : R \to \mathbb{Q} : \phi(1) = 0\} = \langle \epsilon_k \rangle_{k>0}$. The product is defined via duality from the coproduct of R which sends t to $t \otimes 1 + 1 \otimes t$, so that

$$(\epsilon_h \cdot \epsilon_k)(t^i) = (\epsilon_h \otimes \epsilon_k)(\Delta t^i) = (\epsilon_h \otimes \epsilon_k)(\sum_{\alpha+\beta=i} t^\alpha \otimes t^\beta) = \delta_{h+k,i}$$

and hence $\epsilon_h \cdot \epsilon_k = \epsilon_{h+k}$.

Therefore, I is generated as an ideal by $\epsilon := \epsilon_1$. In particular, $R^\vee \cong \mathbb{Q}[[\epsilon]]$, which is I-adically complete. We conclude that $\mathbf{G}_a$ is unipotent.

4. Consider $G = \mathbf{G}_m$. In this case, the coproduct on $R = \mathbb{Q}[t, t^{-1}]$ sends t to $t \otimes t$. Therefore

$$(\epsilon_h \cdot \epsilon_k)(t^i) = (\epsilon_h \otimes \epsilon_k)(\Delta t^i) = (\epsilon_h \otimes \epsilon_k)(t^i \otimes t^i) = \delta_{h,k,i}$$

We conclude in particular that $I^2 = I$, and hence $\mathbf{G}_m$ is not unipotent.

Let's now consider the functors we have obtained from p**UAG** to **Gps** and to **LA**.

Proposition 166. *Let $G = \operatorname{Spec} R$ be a pro-affine algebraic group. Then $\mathcal{G}(R^\vee) \cong G(\mathbb{Q})$ and $\mathcal{P}(R^\vee) \cong \operatorname{Lie} G$.*

Proof. The unit of $R^\vee$ is the counit ϵ. Also, for any $\phi \in R^\vee$ and any $x, y \in R$, $(\Delta\phi)(x \otimes y) = \phi(xy)$. Therefore

$$\Delta\phi = \phi \otimes \phi \Leftrightarrow \phi(xy) = \phi(x)\phi(y)$$

and

$$\Delta\phi = \phi \otimes 1 + 1 \otimes \phi \Leftrightarrow \phi(xy) = \phi(x)\epsilon(y) + \epsilon(x)\phi(y) \Leftrightarrow \phi(I^2) = \phi(R/I) = 0$$

so that $\mathcal{G}R^\vee \cong G(\mathbb{Q})$ and $\mathcal{P}R^\vee \cong (I/I^2)^\vee \cong \operatorname{Lie} G$. □

Proposition 167. *If G is a unipotent algebraic group, then $G(\mathbb{Q})^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\operatorname{Lie} G$ have finite rank.*

Proof. This is true for $\mathbf{UT}_n$, hence for any unipotent algebraic group. □

Recall that we have denoted by $\widetilde{\mathbf{Gps}}$ [resp. by $\widetilde{\mathbf{LA}}$] the subcategory of **Gps** [resp. of **LA**] constituted by groups Γ such that $\Gamma^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$ has finite rank [resp. by Lie algebras $\mathfrak{g}$ such that $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ has finite rank]. Let $\mathbf{pro}\widetilde{\mathbf{Gps}}$ [resp. $\mathbf{pro}\widetilde{\mathbf{LA}}$] denote the associated category of pro-objects. It is equivalent to the category of topological groups Γ which are homeomorphic to $\varprojlim \Gamma/\Gamma_i$, with Γ/Γ_i discrete and lying in $\widetilde{\mathbf{Gps}}$ [resp. topological Lie algebras $\mathfrak{g}$ which are homeomorphic to $\varprojlim \mathfrak{g}/\mathfrak{g}_i$, with $\mathfrak{g}/\mathfrak{g}_i$ discrete and lying in $\widetilde{\mathbf{LA}}$].

By our construction, we have therefore obtained adjunction pairs

$$\begin{aligned} \operatorname{Spec}((\hat{\mathbb{Q}}[\cdot])^*) \colon \mathbf{pro}\widetilde{\mathbf{Gps}} &\rightleftarrows \mathrm{p}\mathbf{UAG} : (\mathbb{Q}) \\ \operatorname{Spec}(\hat{\mathcal{U}}(\cdot)^*) \colon \mathbf{pro}\widetilde{\mathbf{LA}} &\rightleftarrows \mathrm{p}\mathbf{UAG} : \operatorname{Lie} \end{aligned} \tag{0.0.6}$$

which are actually what we were looking for from the very beginning!

Corollary 168 (Quillen's formula)**.**

1. *Let Γ be an object of $\widetilde{\mathbf{Gps}}$ (e.g. if Γ is finitely generated). Then $\operatorname{Spec}((\hat{\mathbb{Q}}[\Gamma])^*) \cong \Gamma^{un}$.*

2. *Let $\mathfrak{g}$ be an object of $\widetilde{\mathbf{LA}}$ (e.g. if $\mathfrak{g}$ is finite dimensional). Then $\operatorname{Spec}((\hat{\mathcal{U}}\mathfrak{g})^*) \cong \mathfrak{g}^{un}$.*

Proof. This follows formally from the previous adjunctions. We focus on the case of groups. Suppose that $G = \text{Spec }(\mathcal{O}(G))$ is in p**UAG**. Then G is a filtered limit of unipotent algebraic groups G_i with $\mathcal{PO}(G_i)$ finite-dimensional. In particular, $\mathcal{O}(G_i)^\vee$ as well as $\hat{\mathbb{Q}}[\Gamma]$ lie in $\mathbf{C}$ and therefore:

$$\begin{aligned}\mathrm{Hom}(\Gamma, G(\mathbb{Q})) = \varprojlim_i \mathrm{Hom}(\Gamma, G_i(\mathbb{Q})) = \varprojlim_i \mathrm{Hom}(\Gamma, \mathcal{GO}(G_i)^\vee) = \varprojlim_i \mathrm{Hom}(\hat{\mathbb{Q}}[\Gamma], \mathcal{O}(G_i)^\vee) = \\ = \varprojlim_i \mathrm{Hom}(\text{Spec }((\hat{\mathbb{Q}}[\Gamma])^*), G_i) = \mathrm{Hom}(\text{Spec }((\hat{\mathbb{Q}}[\Gamma])^*), G).\end{aligned}$$

□

We conclude our panorama on adjunctions by the following remark. There are well known adjunctions from **Set** to **Gps** and from **Set** to **LA**. We wonder whether they are compatible with the rest of the diagram. In what follows we are crucially using the fact that we are working in characteristic 0.

Let R be in **CHA**. Suppose that x lies in the augmentation ideal. Then the series $\sum \frac{x^k}{k!}$ has a limit which we denote by $\exp x$.

Proposition 169. *The adjunction diagram*

$$\begin{tikzcd} & \mathbf{Set} \arrow[dl, shift left] \arrow[dr, shift left] & \\ \mathbf{Gps} \arrow[ur, shift left] \arrow[r, shift left] & \mathbf{CHA} \arrow[l, shift left] \arrow[r, shift left] & \mathbf{LA} \arrow[l, shift left] \arrow[ul, shift left] \end{tikzcd}$$

commutes up to an equivalence of functors induced by the exponential map.

Proof. It suffices to prove that the two right adjoints are equivalent. The claim then follows from the following lemma. □

Lemma 170. *Let R be an object of* **CHA**. *Then $x \in \mathcal{P}R \Leftrightarrow \exp x \in \mathcal{G}R$.*

Proof. This follows from the equalities

$$\Delta x = x \otimes 1 + 1 \otimes x \Leftrightarrow \Delta \exp x = \exp(\Delta x) = \exp(x \otimes 1 + 1 \otimes x) = \exp(x) \otimes \exp(x)$$

which come from the definition of the exponential. □

Quillen's theorem and corollaries

Theorem 171 (Quillen)**.** *Let* **MGps** *[resp.* **MLA***] be the subcategory of* $\widetilde{\mathbf{Gps}}$ *[resp.* $\widetilde{\mathbf{LA}}$*] constituted by nilpotent, uniquely divisible groups [resp. nilpotent algebras]. Then the adjunctions* (0.0.6) *induce equivalence of categories:*

$$\mathbf{pro\,MGps} \underset{\sim}{\rightleftarrows} \mathrm{p}\mathbf{UAG} \underset{\sim}{\rightleftarrows} \mathbf{pro\,MLA}$$

We devote the rest of the section to sktching the proof of this theorem.

We begin with a useful fact from category theory. It is a generalization of well-known cases (Galois correspondences, closures of subsets, algebraic sets etc.) which usually involve ordered sets rather than general categories.

Proposition 172. *Let $F\colon \mathbf{C} \rightleftarrows \mathbf{D} : U$ be an adjunction. The following are equivalent*

1. *$FUF \to F$ is an isomorphism of functors.*

2. *$U \to UFU$ is an isomorphism of functors.*

Moreover, if the previous conditions are satisfied, then the adjoint pair decomposes into three adjoint pairs

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "UF"] & \mathbf{C}^{UF} \arrow[l, shift left] \arrow[r, shift left, "F"] & \mathbf{D}^{FU} \arrow[l, shift left, "U", "\simeq"'] \arrow[r, shift left] & \mathbf{D} \arrow[l, shift left, "FU"] \end{tikzcd}$$

where $\mathbf{C}^{UF}$ [resp. $\mathbf{D}^{FU}$] is the full subcategory of $\mathbf{C}$ [resp. $\mathbf{D}$] constituted by the objects X such that $X \to UFX$ is an isomorphism [resp. $FUX \to X$ is an isomorphism], and where the pair in the center is an equivalence of categories.

Proof. The first part is standard category theory (e.g. [3, Lemma 4.3]), the second is a straightforward exercise. $\square$

In particular, in order to prove the theorem we are left to prove the following facts:

(i) $UF \cong \mathrm{id}$.

(ii) $\Gamma \in \mathbf{pro\,MGps} \Leftrightarrow \Gamma \in \widetilde{\mathbf{proGps}}^{UF}$.

(iii) $\mathfrak{g} \in \mathbf{pro\,MLA} \Leftrightarrow \mathfrak{g} \in \widetilde{\mathbf{proLA}}^{UF}$.

where U is either $(\mathbb{Q})$ or Lie and F is its respective left adjoint.

Corollary 173. *In order to prove the theorem, it suffices to prove*

(I) If $G \in \mathrm{p}\mathbf{UAG}$, then $G(\mathbb{Q}) \in \mathbf{pro\,MGps}$ and $\mathrm{Lie}\, G \in \mathbf{pro\,MLA}$.

(II) $\Gamma \in \mathbf{pro\,MGps} \Rightarrow \Gamma \in \widetilde{\mathbf{proGps}}^{UF}$.

(III) $\mathfrak{g} \in \mathbf{pro\,MLA} \Rightarrow \mathfrak{g} \in \widetilde{\mathbf{proLA}}^{UF}$.

(IV) $(\mathbb{Q})$ and Lie reflect isomorphisms.

Proof. The only non-trivial fact is the proof of condition (i). Let X be in $\mathrm{p}\mathbf{UAG}$. Then by (I) and (II), $UX \to UFUX$ is an isomorphism. Because the compostion $UX \to UFUX \to UX$ is the identity, we also conclude that $UFUX \to UX$ is an isomorphism. By (IV), we conclude $FUX \cong X$, as wanted. $\square$

Sketch of the proof of Quillen's theorem. Conditions (II), (III), (IV) are proved by Quillen at the level of **Gps** $\leftrightarrows$ **CHA** $\leftrightarrows$ **LA** (see [5, Theorem A.3.3]). Condition (I) comes from the fact that if G is unipotent then Lie G and $G(\mathbb{Q})$ are nilpotent (it suffices to check this for $\mathbf{UT}_n$), Lie G is finite dimensional and $G(\mathbb{Q})^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$ has finite rank (already remarked), and $G(\mathbb{Q})$ is uniquely divisible (it is isomorphic to $\exp \mathrm{Lie}\, G$ by Proposition 169). □

Corollary 174. *1. Let Γ be in $\widetilde{\mathbf{Gps}}$. Then Γ^{un} is characterized by the fact that it is pro-unipotent and $\Gamma^{un}(\mathbb{Q})$ is the universal pro-nilpotent uniquely divisible group associated to Γ.*

2. Let $\mathfrak{g}$ be in $\widetilde{\mathbf{LA}}$. Then $\mathfrak{g}^{un}$ is characterized by the fact that it is pro-unipotent and Lie $\mathfrak{g}^{un}$ *is the universal pro-nilpotent Lie algebra associated to Γ.*

Proof. This comes from Quillen's theorem and the lateral adjunctions of Proposition 172. □

Corollary 175. *The category of unipotent algebraic group is equivalent to the category of finite dimensional nilpotent Lie algebras.*

Proof. The functor from **UAG** to nilpotent Lie algebras is fully faithful by Quillen's theorem. It is essentially surjective by [4, Theorem 3.27]. □

From the previous corollary and Proposition 169, we can deduce a similar equivalence between unipotent algebraic groups and abstract groups which are exponentials of nilpotent, finite dimensional Lie algebras.

The free case

We now consider the free pro-unipotent group G_S associated to a finite set $S = \{e_0, \ldots, e_n\}$. By the commutativity of the adjunction of Proposition 169, it is isomorphic to the pro-unipotent completion of the free group over S, and of the free Lie algebra over S.

By what we already proved, $G_S \cong \mathrm{Spec}\,(((\mathcal{U}LS)^\wedge)^*)$, where L is the free Lie algebra functor. The functor $\mathcal{U}L$ from **Set** to $\mathbb{Q}$-**Alg** is left adjoint to the forgetful functor, and therefore $\mathcal{U}LS \cong \mathbb{Q}\langle e_1, \ldots, e_n\rangle$, the algebra of non-commutative polynomials in n variables. It is straightforward to check that $(\mathcal{U}LS)^\wedge \cong \mathbb{Q}\langle\langle e_1, \ldots, e_n\rangle\rangle$, the formal non-commutative power series in n variables. Its coproduct is defined via the relations $e_i \mapsto e_i \otimes 1 + 1 \otimes e_i$. If $I = (i_1, \ldots, i_k)$ is a multi-index, we indicate with e_I the product $e_1 \cdot \ldots \cdot e_k$. By induction, it follows $\Delta e_I = \sum_\sigma e_J \otimes e_K$, where I, J vary among multi-indices and σ varies among the permutations $\mathrm{Sym}(|J|, |K|)$ such that $\sigma(J, K) = I$. We clarify this with an example.

Example 176.

$$\Delta(e_1^2 e_2) = e_1^2 e_2 \otimes 1 + e_1^2 \otimes e_2 + 2e_1 e_2 \otimes e_1 + 2e_1 \otimes e_1 e_2 + e_2 \otimes e_1^2 + 1 \otimes e_1^2 e_2.$$

This algebra is graded with respect to the degree, isomorphic to $\mathbf{prod} V^{\otimes k}$, where V is the free vector space generated by S. It follows that

$$((\mathcal{U}LS)^\wedge)^* \cong \bigoplus (V^{\otimes k})^\vee \cong \bigoplus (V^\vee)^{\otimes k} \cong T(V^\vee). \tag{0.0.7}$$

We now investigate its Hopf operations. We denote by ϵ_I the dual of e_I.

From the formulas

$$(\epsilon_I \cdot \epsilon_J)(e_K) = (\epsilon_I \otimes \epsilon_J)(\Delta e_K) = (\epsilon_I \otimes \epsilon_J)\left(\sum_\sigma e_M \otimes e_N\right)$$

we deduce that $(\epsilon_I \cdot \epsilon_J)(e_K) = 1$ if there is a way to shuffle I and J to form K and is 0 otherwise. Hence, the product of $T(V^\vee)$ is the shuffle product ш.

From the formula

$$(\Delta \epsilon_I)(e_J \otimes e_K) = \epsilon_I(e_{JK})$$

we deduce that $(\Delta\epsilon_I)(e_J \otimes e_K) = 1$ if $JK = I$ and is 0 otherwise. Therefore, $\Delta\epsilon_I = \sum_{JK=I} \epsilon_J \otimes \epsilon_K$, the so-called deconcatenation coproduct.

By Corollary 174, we get that Lie G_S is the universal pro-nilpotent algebra associated to LS, i.e. its completion by the lower central series. Also, by what we proved in the first part, $G_S(\mathbb{Q}) = \mathcal{G}\mathbb{Q}\langle\langle e_1, \ldots, e_n\rangle\rangle$.

Malcev original construction

We now give an explicit description of another special case, originally studied by Malcev. We refer to [6] for the group theory facts we need here. Suppose Γ is nilpotent and finitely generated. In this case, the torsion elements constitute a subgroup H. Since any uniquely divisible group has no torsion, by Corollary 169 we conclude that $\Gamma^{un} \cong (\Gamma/H)^{un}$. We can therefore suppose that Γ has no torsion.

Let

$$\Gamma = \Gamma_1 \geq \Gamma_2 \geq \ldots \geq \Gamma_k = 1$$

be the lower central series ($\Gamma_i = [\Gamma_{i-1}, \Gamma]$). Each factor Γ_i/Γ_{i+1} is abelian and finitely generated since $[\Gamma_i, \Gamma_i] \leq [\Gamma_i, \Gamma] = \Gamma_{i+1}$. We can then refine the lower central series to obtain another central series with cyclic quotients. A group with such a central series is called polycyclic. Quotients and subgroup of polycyclic ones are again polycyclic (by studying the induced filtrations). In particular, the quotients of the upper central series ($Z_i/Z_{i+1} = Z(G/Z_{i+1})$)

$$\Gamma = Z_1 \geq Z_2 \geq \ldots \geq Z_k = 1$$

are polycyclic. They are also without torsion by the next lemma.

Lemma 177. *If Γ is nilpotent and without torsion, then all quotients Γ/Z_i are without torsion.*

Proof. Since $\Gamma/Z_{i+1} \cong (G/Z_i)/(Z_{i+1}/Z_i) \cong (G/Z_i)/(Z(\Gamma/Z_i))$, it suffices to prove that if Γ is nilpotent and without torsion, then $\Gamma/Z(\Gamma)$ is without torsion.

Suppose x^m is central. We need to prove that also x is. If x^m is central, then for any y we have $(y^{-1}xy)^m = x^m$. It suffices to prove uniqueness of roots in a torsion-free nilpotent group. We make induction on the nilpotency class, being the case of an abelian group trivial.

Let $a^m = b^m$ in a torsion-free nilpotent group Γ. In order to prove $a = b$, it suffices to prove $[a, b] = 1$ since Γ is torsion-free. Since $b^{-1}ab = a[a, b]$, both $b^{-1}ab$ and a lie in the subgroup $H = \langle [\Gamma, \Gamma], a\rangle$, which has a stricly lower nilpotency class (see [6, Proposition 2.5.5]). By induction, from the equality $(b^{-1}ab)^m = b^{-1}a^m b = b^m = a^m$, we conclude $[a, b] = 1$ as wanted. □

In conclusion, the upper central series can be enriched into a filtration

$$\Gamma = \Gamma^1 \geq \Gamma^2 \geq \dots \Gamma^{s+1} = 1$$

such that $\Gamma^i/\Gamma^{i+1} \cong \langle e_i\rangle \cong \mathbb{Z}$. The set $\{e_1, \dots, e_s\}$ is called a Malcev basis for Γ. We can associate to any element $g \in \Gamma$ a unique set of s coordinates $t_i(g) \in \mathbb{Z}$ such that $g = \mathbf{pro}de_i^{t_i(g)}$, and Γ^i coincides with the subset $\{g \in \Gamma : t_j(g) = 0, j < i\}$. This defines a bijection from Γ to $\mathbb{Z}^s$. We now recover also the product in terms of the coordinates.

Proposition 178. *Let Γ be finitely generated, nilpotent and without torsion. Let $\{e_1, \dots, e_s\}$ be a Malcev basis for Γ and $t_i(g)$ the Malcev coordinates of an element g.*

1. *The product is polynomial in the coordinates, i.e. there exist polynomials P_i with rational coefficients such that*
$$t_i(gh) = t_i(g) + t_i(h) + P_i(t_j(g), t_j(h)).$$
Moreover, the polynomial P_i depends only on t_j's with $j < i$.

2. *The inverse is polynomial in the coordinates, i.e. there exist polynomials $Q_{i,k}$ with rational coefficients such that*
$$t_i(g^k) = kt_i(g) + Q_{i,k}(t_j(g)).$$
Moreover, the polynomial $Q_{i,k}$ depends only on t_j's with $j < i$.

Proof. Make induction on the cardianlity of the Malcev basis. Details in [6, Propriété 3.1.5]. □

Since the polynomials P_i and $Q_{i,k}$ have rational coefficients, they define an algebraic group $G : R \mapsto (R^s, \cdot)$ where $\cdot$ is the product defined using the above formulas.

Proposition 179. *The group G is unipotent.*

Proof. Since $\mathcal{O}(G)$ is a polynomial ring, $G(\mathbb{Q})$ is dense in G. Therefore, if we prove that there is a faithful finite dimensional representation of G such that $G(\mathbb{Q}) \to \mathbf{GL}_V(\mathbb{Q})$ is made of unipotent morphisms (i.e. , for all $g \in G(\mathbb{Q})$, $(g - \mathrm{id})^n = 0$ for $n \gg 0$), we conclude that, with respect to a suitable basis, $G(\mathbb{Q})$ factors through $\mathbf{UT}_n(\mathbb{Q})$. By density, we can isomorphically embed G in $\mathbf{UT}_n$, as wanted. Because the regular representation contains all representation, we can equivalently prove that all endomorphisms of $G(\mathbb{Q})$ are unipotent with respect to it (i.e. $(g-\mathrm{id})^n = 0$ for $n \gg 0$ when restricted to any subrepresentation of finite dimension).

This representation sends g to the endomorphism $T_i \mapsto t_i(g)+T_i+P_i(t_j,T_j)$. By the formulas above, it follows that $(g-\mathrm{id})$ it sends a monomial $T^I = T_1^{i_1}\cdot\ldots T_s^{i_s}$ to a linear combination of monomials which are stricly smaller with respect to the lexicographic order. In particular, for any multi-index I, $(g-\mathrm{id})^n(T^I) = 0$ for $n \gg 0$. $\square$

We remark that the map $\Gamma \to G(\mathbb{Q})$ is induced by the inclusion $\mathbb{Z}^s \to \mathbb{Q}^s$. It satisfies a universal property:

Proposition 180. *The abstract group $G(\mathbb{Q})$ is the nilpotent, uniquely divisible closure of Γ.*

Proof. Using the formulas and induction on i, it is straightforward to see that $G(\mathbb{Q})$ is uniquely divisible and if $x \in G(\mathbb{Q})$, then $x^n \in \Gamma$ for $n \gg 1$. $\square$

Corollary 181. $\Gamma^{un} \cong G$.

Proof. This comes from the previous propositions and Corollary 169. $\square$

Torsors

Let Γ be an abstract group. We remark that the functors we used to define Γ^{un} make sense more generally for Γ-sets:

$$\Gamma\text{-}\mathbf{Set} \overset{\mathbb{Q}[\cdot]}{\to} \mathbb{Q}[\Gamma]\text{-}\mathbf{Mod} \overset{\wedge}{\to} \hat{\mathbb{Q}}[\Gamma]\text{-}\mathbf{Mod} \overset{\mathrm{Spec}\ ((\cdot)^*)}{\to} \Gamma^{un}\text{-}\mathbf{Var}$$

and we denote again their composition with $S \mapsto S^{un}$. Moreover, all these functors are tensorial with respect to the tensors defined in each category. The first and the last one are tensorial with respect to the cartesian product. It follows that if $S \in \Gamma\text{-}\mathbf{Set}$ is a torsor, i.e. if

$$\Gamma \times S \to S \times S \qquad (g,s) \mapsto (g\cdot s, s)$$

is an isomorphism, then also

$$\Gamma^{un} \times S^{un} \cong (\Gamma \times S)^{un} \to (S \times S)^{un} \cong S^{un} \times S^{un}$$

is an isomorphism. Therefore, $S \mapsto S^{un}$ maps torsors to torsors.

The Tannakian approach

Proposition 182. *Let Γ be an abstract group such that Γ^{un} il well defined. Then Γ^{un} is the pro-affine algebraic group associated to the Tannakian category of unipotent representations of Γ.*

Proof. The functor $\Gamma^{un}\mathrm{Rep} \to \Gamma\mathrm{Rep}$ sending $\Gamma^{un} \to \mathbf{GL}_V$ to $\Gamma \to \Gamma^{un}(\mathbb{Q}) \to \mathbf{GL}_V(\mathbb{Q})$ factors over unipotent representations of Γ since Γ^{un} is pro-unipotent. Viceversa, if $\Gamma \to \mathbf{GL}_V(\mathbb{Q})$ is unipotent then, with respect to a suitable basis, it factors over $\mathbf{UT}_n$. It follows that the subgroup H of $\mathbf{GL}_V$ generated by Γ is isomorphic to a subgroup of $\mathbf{UT}_n$, hence unipotent. By the universal property, the map $\Gamma \to H(\mathbb{Q})$ then induces a map $\Gamma^{un} \to H \to \mathbf{GL}_V$, as wanted. $\square$

Since we have proved the existence (and Quillen's construction) of Γ^{un} only for groups Γ with nice properties (i.e. $\Gamma^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$ has finite dimension), we wonder if this proposition gives a more general construction of Γ^{un}, i.e. if the Tannaka dual of unipotent representations of Γ satisfies the universal property of the pro-unipotent completion.

More on the conilpotency filtration

We now present a last corollary of the first section and Quillen's paper. Suppose G is unipotent, and let $\mathfrak{g}$ be its Lie algebra. We have $\mathfrak{g} \cong \mathcal{P}\mathcal{O}(G)^\vee$, and it inherits the I-adic filtration from $\mathcal{O}(G)^\vee$:

$$\mathcal{P}\mathcal{O}(G)^\vee = \mathcal{P}\mathcal{O}(G)^\vee \cap I \supset \mathcal{P}\mathcal{O}(G)^\vee \cap I^2 \supset \dots$$

On the other hand, we can consider the graded Hopf algebra $\mathrm{gr}^\bullet\mathcal{O}(G)^\vee$, obtained via the I-adic filtration. Its primitive elements constitute a graded subset $\mathcal{P}\mathrm{gr}^\bullet\mathcal{O}(G)^\vee$.

Proposition 183. *[5, Proposition A.2.14] The natural map* $\mathrm{gr}^\bullet\mathcal{P}\mathcal{O}(G)^\vee \to \mathcal{P}\mathrm{gr}^\bullet\mathcal{O}(G)^\vee$ *is an isomorphism.*

In particular, we deduce $\mathrm{gr}^1\mathfrak{g} \cong I/I^2 \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. This abelian Lie algebra has a natural map to $\mathrm{gr}^\bullet\mathcal{P}\mathcal{O}(G)^\vee \cong \mathcal{P}\mathrm{gr}^\bullet\mathcal{O}(G)^\vee$. It is a quotient of the free Lie algebra LS generated by a chosen basis S of $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. By adjunction, we obtain a map $\hat{\mathcal{U}}LS \to \mathrm{gr}^\bullet\mathcal{O}(G)^\vee$, and by duality a map $(\mathrm{gr}^\bullet\mathcal{O}(G)^\vee)^* \to (\hat{\mathcal{U}}LS)^*$, and a map $\mathrm{Spec}\ ((\hat{\mathcal{U}}LS)^*) \to \mathrm{Spec}\ ((\mathrm{gr}^\bullet\mathcal{O}(G)^\vee)^*)$.

We recall that the conilpotency filtration on $\mathcal{O}(G)$

$$0 \subset C_1 \subset C_2 \subset \dots$$

is dual to the I-adic filtration on $\mathcal{O}(G)^\vee$

$$\mathcal{O}(G)^\vee \supset I \supset I^2 \supset \dots$$

in the sense that $(\mathrm{gr}^i\mathcal{O}(G)^\vee)^* \cong \mathrm{gr}_i\mathcal{O}(G)$. Hence in particular, $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]^* \cong (I/I^2)^* \cong (\mathrm{gr}_1\mathcal{O}(G))$ and $(\mathrm{gr}^\bullet\mathcal{O}(G)^\vee)^* \cong \mathrm{gr}_\bullet\mathcal{O}(G)$.

We remark that $\mathrm{Spec}\ ((\hat{\mathcal{U}}LS)^*)$ is a free pro-unipotent group generated by S. By Equation 0.0.7, $(\hat{\mathcal{U}}LS)^* \cong T(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]^\vee) \cong T(\mathrm{gr}_1\mathcal{O}(G))$.

Proposition 184. *The map* $\mathrm{gr}_\bullet\mathcal{O}(G) \to T(\mathrm{gr}_1\mathcal{O}(G))$ *is injective.*

Proof. We prove that the dual is surjective. It is the map $\hat{\mathcal{U}}LS \to \mathrm{gr}^{\bullet}\mathcal{O}(G)^{\vee} \cong \mathrm{gr}^{\bullet}\hat{\mathcal{U}}\mathfrak{g}$, where the last equality follows from the unipotency of G.

The graded algebra $\mathrm{gr}^{\bullet}\hat{\mathcal{U}}\mathfrak{g}$ is generated by its first graded piece $\mathrm{gr}^1\hat{\mathcal{U}}\mathfrak{g} = \mathcal{P}\mathrm{gr}^1\hat{\mathcal{U}} \cong \mathrm{gr}^1\mathcal{P}\hat{\mathcal{U}} \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, hence $\hat{\mathcal{U}}LS \to \mathrm{gr}^{\bullet}\hat{\mathcal{U}}\mathfrak{g}$ is surjective, as claimed. $\square$

Chen's Theorem: The Bar Complex

Thomas Weißschuh on September 3rd, 2012.

The (unreduced) bar complex

Let $A^\bullet$ be a differential graded algebra, $\mathbb{Z}$-graded with a differential $\partial : A^k \to A^{k+1}$. The degree of an element in A is for short denoted by $| - |$. Let also $a, b : A^\bullet \to k$ be augmentations. Consider the infinite direct sum

$$\bigoplus_{r \geq 0} A^{\otimes r} = k \oplus A \oplus A^{\otimes 2} \oplus \dots$$

generated by elements written as

$$[a_1 \mid \dots \mid a_r] \in A^{\otimes r}.$$

On this set, one has two graduations: First the usual degree on the tensor product of graded complexes, given by

$$\deg_{tensor} : [a_1 \dots a_r] \mapsto |a_1| + \dots + |a_r|$$

and the second, so-called simplicial graduation, given by negative length of the tensor,

$$\deg_{simpl} : [a_1 \dots a_r] \mapsto -r.$$

Splitting the above sum up with respect to these to degrees leads to the following array.

$$\begin{tikzcd}
 & -2 & -1 & 0 & \deg_{simpl} / \deg_{tensor} \\
{} \arrow[r] & (A^{\otimes 2})^2 \arrow[r, "\delta"] & A^2 \arrow[r, "\delta"] & 0 & 2 \\
{} \arrow[r] & (A^{\otimes 2})^1 \arrow[r, "\delta"] \arrow[u, "\partial"] & A^1 \arrow[r, "\delta"] \arrow[u, "\partial"] & 0 & 1 \\
{} \arrow[r] & (A^{\otimes 2})^0 \arrow[r, "\delta"] \arrow[u, "\partial"] & A^0 \arrow[r, "0"] \arrow[u, "\partial"] & k & 0 \\
{} \arrow[r] & (A^{\otimes 2})^{-1} \arrow[r, "\delta"] \arrow[u, "\partial"] & A^{-1} \arrow[r, "\delta"] \arrow[u, "\partial"] & 0 & -1
\end{tikzcd}$$

Here the differentials are given by

$$
\begin{aligned}
\partial : [a_1 \mid \ldots \mid a_r] &\mapsto \sum_{i=1}^{r} (-1)^{|a_1|+\ldots+|a_{i-1}|+i} [\ldots \mid \partial a_i \mid \ldots] \\
\delta : [a_1 \mid \ldots \mid a_r] &\mapsto \sum_{i=0}^{r} (-1)^{|a_1|+\ldots+|a_i|+i+1} [\ldots \mid a_i a_{i+1} \mid \ldots]
\end{aligned}
$$

where in the definition of δ the terms for $i = 0, r$ are defined as follows

$$
\begin{aligned}
i = 0 :&\ \ -a(a_1)[a_2 \mid \ldots \mid a_r] \\
i = r :&\ \ (-1)^{|a_1|+\ldots+|a_r|+r+1} [a_1 \mid \ldots \mid a_{r-1}] b(a_r)
\end{aligned}
$$

Especially for $r = 1, 2$, one gets

$$
\delta : \begin{array}{c} [a_1] \mapsto -a(a_1) + (-1)^{|a_1|} b(a_1) \\ [a_1|a_2] \mapsto -a(a_1)[a_2] + (-1)^{|a_1|}[a_1 \mid a_2] - (-1)^{|a_1|+|a_2|} b(a_2)[a_1] \end{array}
$$

With these definitions, $\partial^2 = 0$, $\delta^2 = 0$ and $\partial\delta + \delta\partial = 0$, so this defines a double complex.

Let $B(A, a, b)$ be the associated $\oplus$-Total complex formed by summing diagonals of slope -1 in the above diagram and indexing by the x-intercept.

Hopf algebra structure on the bar complex

In what follows the bar complex, up to now defined as a vector space, is equipped with an hopf algebra structure.

Definition 185. The bar complex admits a product

$$
\nabla : [a_1 \mid \ldots \mid a_r] \otimes [a_{r+1} \mid \ldots \mid a_{r+s}] \mapsto \sum_{\mu \in S_{r,s}} (-1)^{\sigma(\mu,a)} [a_{\mu^{-1}(1)} \mid \ldots \mid a_{\mu^{-1}(r+s)}]
$$

where

$$
\sigma(\mu, a) = \sum_{\substack{i<j \\ \mu(i)>\mu(j)}} (|a_i| - 1)(|a_j| - 1) \tag{0.0.8}
$$

ist the graded signum of the tupel $a = [a_1 \ldots a_r]$ with respect to the total degree.

Remark that the definition of the product doesn't make use of the augmentations. To get compatibility with the simplicial differential one assumes that the product is defined on $B(A, a, c) \otimes B(A, c, b)$ and takes values in $B(A, a, b)$. Here a, b, c are arbitrary augmentations of A.

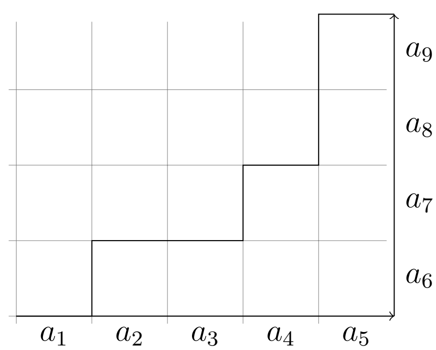
This product is

- Associative
- Graded-commtutative
- Unital with unit given by inclusion $k \to B(A, a, b)$
- Compatible with the total differential, if A is graded-commutative.

Example 186. For elements a, b in A one has

- $\nabla([a] \otimes 1) = [a] = \nabla(1 \otimes [a])$
- $\nabla([a] \otimes [b]) = [a \mid b] + (-1)^{(|a|-1)(|b|-1)}[b \mid a]$
- If a is of degree 1, $[a]^n = n!\underbrace{[a \mid \ldots \mid a]}_{n}$

Remark 187. A (p, q) shuffle μ can be thought of as a way in a rectangle. If one places the $a_1 \ldots a_p$ horizontally and the $a_{p+1} \ldots a_{p+q}$ vertically, then μ is given by shuffling the two paths together.



The signum of the shuffle can be found as the area below this graph. To be more precise, it is -1 power the area, which is calculated by giving the segment of a_i the length $|a_i| - 1$.

Definition 188. There is also a coproduct on $B(A, a, b)$:

$$\Delta : B(A, a, b) \to B(A, a, c) \otimes B(A, c, b)$$
$$[a_1 \mid \ldots \mid a_r] \mapsto \sum_{i=0}^{r} [a_1 \mid \ldots \mid a_i] \otimes [a_{i+1} \mid \ldots \mid a_r].$$

In the case $i = 0$, this should be read as $1 \otimes [a_1 \mid \ldots \mid a_r]$ and similar for $i = r$.

Example 189.

- $\Delta(1) = 1 \otimes 1$
- $\Delta([a_1]) = 1 \otimes [a_1] + [a_1] \otimes 1$

It is

- Co-associative $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.
- Compatible with $\epsilon =$ Projection to k. ϵ is the co-unit of the coalgebra.
- Δ is a morphism of complexes ie. it satisfies the co-Leibniz rule.

Remark 190. The last statement is the reason that the two inner augmentations in the definition of the coproduct have to be equal.

The algebra and coalgebra structure are compatible with each other, which can be expressed by saying that the coproduct and counit are homomorphisms of unitary algebras. In other words, one has a bialgebra-structe on the bar complex.

Definition 191. Define the antipode of the bar complex by

$$\begin{aligned} S : B(A, a, b) &\to B(A, a, b) \\ [a_1|\dots|a_r] &\mapsto (-1)^{r+\sigma(s,a)}[a_r|\dots a_1] \end{aligned}$$

where $\sigma(s, a)$ is the graded sign of the permutation $[a_1 \dots a_r] \mapsto [a_r \dots a_1]$, see 0.0.8.

This antipode gives the bar complex the structure of a hopf algebra. This means that the following diagram (with $B = B(A, a, b)$ for short) commutes.

$$\begin{tikzcd} & B \otimes B \arrow[rr, "S\otimes id"] & & B \otimes B \arrow[dr, "\nabla"] & \\ B \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "u"] & & B \\ & B \otimes B \arrow[rr, "id\otimes S"] & & B \otimes B \arrow[ur, "\nabla"'] & \end{tikzcd}$$

The reduced bar complex

For this section, we assume that A vanishes in negative degree.

Let $D(A, a, b)$ be the subcomplex generated by $[a_1|\dots|a_r]$ with at least one $a_i \in A^0$ of degree 0. As a vectorspace, it is generated by elements $[a_1|\dots|a_r]$, some $a_i \in A^0$, and their total differentials.

Definition 192 (Reduced bar complex)**.**

$$\overline{B}(A, a, b) = \frac{B(A, a, b)}{D(A, a, b)}$$

The subcomplex $D(A, a, b)$ forms an ideal (resp. coideal) for ∇ (resp. for Δ) in $B(A, a, b)$, and consequently gives rise to well defined morphisms of the reduced bar complex.

To see this, one first observe that the respective property hold for homogenous elements $x \in D(A, a, b)$ with at least one component in degree 0. Then use the compatibility of ∇ resp. Δ with the total differential to see the properties for $\nabla(dx \otimes y)$ and $\Delta(dx)$ respectively (use the first case).

An explicit definition of the reduced bar complex.

On the reduced bar complex, we have two relations. The first relation is

$$[\dots a_{i-1}|h|a_{i+1}\dots] = 0, \quad h \in A^0$$

Letting $A^+ = A/A^0$ be the positive-degree-part of A, this enables one to write

$$\overline{B}(A, a, b) = B(A^+, a, b) \mod \text{ additional relations}$$

with relations arising as differentials of the above elements. To be precise, one has the relations

$$0 = d[\dots|h|\dots]$$

for each $h \in A^0$ (and every possible substitute for $\dots$). Explicit description of the total differential gives - modulo the first relation(s) - the additional relations

$$[\dots a_{i-1}|\partial h|a_{i+1}\dots] = [\dots|a_{i-1}|ha_{i+1}|\dots] - [\dots|a_{i-1}h|a_{i+1}|\dots]$$

These are all relations that occur.

The schematic picture of the reduced bar complex one should have in mind is

$$\begin{tikzcd}
-4 & -3 & -2 & -1 & 0 & \deg_{simpl} / \deg_{tensor} \\
(A^1)^{\otimes 4} \arrow[r, "\delta"] & ((A^+)^{\otimes 3})^4 \arrow[r, "\delta"] & ((A^+)^{\otimes 2})^4 \arrow[r, "\delta"] & A^4 \arrow[r] & 0 & 4 \\
 & (A^1)^{\otimes 3} \arrow[u, "\partial"] \arrow[r, "\delta"] & ((A^+)^{\otimes 2})^3 \arrow[u, "\partial"] \arrow[r, "\delta"] & A^3 \arrow[u, "\partial"] \arrow[r] & 0 & 3 \\
 & & (A^1)^{\otimes 2} \arrow[u, "\partial"] \arrow[r, "\delta"] & A^2 \arrow[u, "\partial"] \arrow[r] & 0 & 2 \\
 & & & A^1 \arrow[u, "\partial"] \arrow[r] & 0 & 1 \\
 & & & & k \arrow[u, "0"] & 0
\end{tikzcd}$$

From this, the 0th cohomology of the reduced bar complex is easy to describe. Consider a 0-cocycle x with x_i it's part in simplicial degree i. Then one has $\delta x_i = -\partial x_{i+1}$. Now every summand in δx_i and ∂x_{i+1} is an tensor with exactly 1 degree-2 component and degree-1-components otherwise. Comparing the components shows that the tensor-differential decomposes, i.e. one has $\partial a = a'a''$ for each component a which shows up as a component in a summand of x. Hence every class in $H^0\overline{B}(A, a, b)$ is represented by a sum of $[a_1 \mid \dots \mid a_r]$ with all $|a_i| = 1$ and

$$\partial a_i \in A^1 \cdot A^1 \quad \text{or} \quad \partial a_i = 0.$$

For an element $x = [a_1 \dots a_r]$ concentrated in a single simplicial degree, one has explicit necessary and sufficient conditions to represent a cohomology class in $H^0\overline{B}(A, a, b)$, namely

$$x \text{ is cocycle} \iff \text{all } a_i \text{ closed, and } a_i a_{i+1} = 0 \quad \forall i$$

Example 193. Let V be a vector space and $A^\bullet = k \oplus V[-1]$ be the complex $(k \xrightarrow{0} V)$, with k set in degree 0. The trivial product gives $A^\bullet$ an DGA structure with augmentation the projection onto k. Then one has $A^+ = V[-1]$ and the reduced Barkomplex by definition the associated total complex of

$$\dots V[-1]^{\otimes 3} \to V[-1]^{\otimes 2} \to V[-1] \to k \to 0$$

This as a set equals the set of tensors $T(V)$. The total degree of every element $[v_1| \dots |v_r]$ equals 0, so the Barcomplex and $T(V)$ are the same as graded vectorspaces. The (total) differential vanishes and one has the hopf algebra structure given by product

$$\nabla([v_1 \dots v_r] \otimes [v_{r+1} \dots v_{r+1}]) = \sum_{\mu \in S_{r,s}} [v_{\mu^{-1}} \dots v_{\mu^{-1}(r+s)}]$$

since $\sigma(\mu, a) = \sum (1-1)(1-1) = 0$. The antipode is

$$S[v_1 \dots v_r] = (-1)^r [v_r \dots v_1]$$

and the coproduct formula doesn't specialize.

Comparison reduced/unreduced bar complex

We now compare the two definitions of the bar complex. For this we assume $A^\bullet$ to be cohomologically connected i.e. graded in non-negative degree's and $H^0(A) = k$. The plan is to split the quotient map from the bar-complex to the reduced bar-complex as a composition of two quasi-isomorphism

$$B(A, a, b) \to \widetilde{B}(A, a, b) \to \overline{B}(A, a, b).$$

Step 1. The maps $s_i : [a_1 \mid \dots \mid a_r] \mapsto [a_1 \dots a_{i-1} \mid 1 \mid a_i \dots a_r]$ which insert a 1 in position i, together with the simplicial differential, endow the bar-doublecomplex with the structure of a simplicial abelian group. Using the sign-trick and modifying the tensor differential, one even gets the structure of a simplicial complex, i.e. the structure maps are compatible with the tensor differential.

$$\begin{array}{c} B(A,a,b)^2 \\ \downarrow \; \uparrow \; \downarrow \; \uparrow \; \downarrow \\ B(A,a,b)^1 \\ \downarrow \; \uparrow \; \downarrow \\ B(A,a,b)^0 \end{array}$$

By theory of normalization, the image $\widetilde{D}(A,a,b)$ of all degeneration maps s_i (with varying r) is acyclic. Now define $\widetilde{B}(A,a,b)$ as the total complex of the normalization with respect to this simplicial structure, i.e. the total complex of the quotient of the bar-doublecomplex by $\widetilde{D}(A,a,b)$. Sind $\widetilde{D}$ is acyclic, $\widetilde{B}(A,a,b)$ is quasi-isomorphic to the unreduced bar complex.

Step 2. Now consider the projection $\widetilde{B} \to \overline{B}$. Filter both complexes by columns. The associated spectral sequence converges to the cohomology of the respective complexes (since they are defined as $\oplus$-totalcomplexes). So it suffices to show, that the above projection is a quasi-isomorphism columnwise. By kunneth, it is enought to look at the tensor-degree-1 case (kunneth is compatible with relations). So one needs $A/k \to A/(A^0 \to dA^0)$ to be an quasi-isomorphism. But its kernel is the complex $(A^0/k \to dA^0)$, which is acyclic since A is asumed to be cohomologically connected.

The bar complex and iterated integrals

Let M be a smooth connected manifold and $A^\bullet_M$ the differential graded algebra of complex valued differential forms on M. Assume that two points $a, b \in M$ are given. They induce augmentations $a, b : A^\bullet_M \to k$ by setting

$$a : \begin{array}{ccc} \omega & \mapsto & 0 \\ f & \mapsto & f(a) \end{array}$$

Definition 194 (Iter)**.** The properties of iterated integrals discussed in talk 2 show that the map

$$\text{iter} : \begin{array}{ccc} \mathbb{Q}[\pi_1(M,a,b)] & \to & H^0\overline{B}(A,a,b)^\vee = H^0B(A,a,b)^\vee \\ \gamma & \mapsto & \left([a_1 \mid \ldots \mid a_r] \mapsto \int_\gamma a_1 \ldots a_r\right) \end{array}$$

is well defined.

It is not just a map of vector spaces but a map of Hopf algebras, i.e., it respects the product, co-product and antipode. Its dual is given by

$$\text{iter}^\vee : \begin{array}{ccc} H^0\overline{B}(A,a,b) & \to & \mathbb{C} \otimes \mathcal{O}(\pi_1(M,a,b)^{un}) \\ [a_1 \mid \ldots \mid a_r] & \mapsto & \left(\gamma \mapsto \int_\gamma a_1 \ldots a_r\right) \end{array}$$

An equivalent version of Chen's theorem is

Theorem 195 (Chen)**.** *The morphism* iter$^\vee$ *is an isomorphism.*

The bar complex and relative cohomology

Let n be a natural number, and let

$$Y_i = \begin{cases} \{(x_1, \ldots, x_n) \in M^n \mid x_i = x_{i+1}\}, & i = 1, \ldots, n-1 \\ \{(a, x_2, \ldots, x_n) \in M^n\}, & i = 0 \\ \{(x_1, \ldots, x_{n-1}, b) \in M^n\}, & i = n \end{cases}$$

Then

$$H^0 B(A,a,b) \to \mathbb{C} \otimes_{a,b} \lim_n H^n(M^n, Z_{ab})$$

where $Z_{ab} = \cup_{i=0}^n Y_i$. We calculate the singular cohomology of M^n relative to Z_{ab}, which is by definition the cohomology of the kernel of the restriction of singular cochains to Z_{ab},

$$\begin{aligned} H^*(M^n, Z_{ab}) &= H^* \ker(S^* M^n \to S^* Z_{ab}) \\ &= H^* \mathrm{Tot}(S^* M^n \to \oplus_i S^*(Y_i) \to \oplus_{i<j} S^*(Y_i \cap Y_j) \to \cdots S^*(Y_0 \cap \cdots \cap Y_n)) \end{aligned}$$

The second equality can be seen as follows. Restriction gives a quasi-isomorphism $S^\bullet(Z_{ab}) \to S^\bullet(Y_0 + \ldots + Y_n)$ which goes into the group of singular cochains defined over singular chain whose image lies completely in one of the Y_i. This essentially follows from the barycentric subdivision of chains. Now the Mayer-Vietoris exact sequence allows to replace $S^\bullet(Y_0 + \ldots + Y_n)$ by the above complex, with differentials given by alternating restrictions.

By integrating one gets a quasi-Isomorphism between complexes

$$\begin{tikzcd} S^* M^n \arrow[r] & \oplus_i S^*(Y_i) \arrow[r] & \oplus_{i<j} S^*(Y_i \cap Y_j) & \cdots & S^*(Y_0 \cap \cdots \cap Y_n) \\ A_{M^n} \arrow[r] \arrow[u] & \oplus A_{Y_i} \arrow[r] \arrow[u] & \oplus_{i<j} A_{Y_i \cap Y_j} \arrow[u] & \cdots & A_{Y_0 \cap \ldots \cap Y_n} \arrow[u] \end{tikzcd}$$

inducing an quasi-isomorphism on total complexes. Observe that $Y_{i_1} \cap \ldots \cap Y_{i_r} \cong M^{n-r}$. So by the Künneth formula, one finally has an isomorphism

$$H^* \mathrm{Tot}(A_M^{\otimes n} \to \oplus A_M^{\otimes n-1} \to \oplus_{i<j} A_M^{\otimes n-2} \to \cdots) \xrightarrow{\cong} H^*(M^n, Z_{ab})$$

which is essentially just the integration map.

There is a short exact sequence of complexes,

$$\begin{tikzcd} 0 \arrow[d] & & & \\ \widetilde{K}_n^- \arrow[d] & A_M^{\otimes n} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \text{kernel} \arrow[d] \\ \widetilde{K}_n \arrow[d] & A_M^{\otimes n} \arrow[r] & \cdots \arrow[r] & \oplus_{n+1} \mathbb{C} \arrow[d] \\ \mathbb{C}_{a,b}[-n] \arrow[d] & 0 \arrow[r] & \cdots \arrow[r] & \mathbb{C}_{a,b} \\ 0 & & & \end{tikzcd}$$

where *kernel* denotes the kernel of the rightmost map in [—], which in the above diagram is the map to $\mathbb{C}_{a,b}$. Note that *kernel* sits in degree n and so by definition $\widetilde{K}_n^-$ calculates $H^n(M^n, Z_{ab})$.

Furthermore the cohomology of $\widetilde{K}_n$ and $\widetilde{K}_n^-$ agree in degree $> n$. Taking the long exact sequence associated to the short exact sequence gives,

$$H^{n-1}(\mathbb{C}[-n]) \to H^n(\widetilde{K}_n^-) \to H^n(\widetilde{K}_n) \to H^n(\mathbb{C}_{a,b}[-n]) \xrightarrow{0} \cdots$$

The rightmost map is zero, because the following morphism is an identity. Hence

$$\mathbb{C}_{a,b} \oplus H^n(M^n, Z_{ab}) = H^n(\widetilde{K}_n)$$

Fact: "Doing the summation" gives an quasi-isomorphism of complexes $\widetilde{K}_n \to F^{-n}\widetilde{B}(A, a, b)$. Here $F^{-n}\widetilde{B}(A, a, b)$ denotes the truncated subcomplex of elements simplicial degree $\geq -n$. Taking inductive limit (which is compatible with taking cohomology) gives

$$H^0 B(A_M, a, b) = \varinjlim_n F^{-n}\widetilde{B}(A_M, a, b) = \varinjlim_n \mathbb{C}_{a,b} \oplus H^n(M^n, Z_{ab})$$

Remark 196. It seems that the isomorphism constructed above makes the following diagram commute. So Chen's theorem is equivalent for $\mathrm{iter}^\vee$ to be an isomorphism.

$$\begin{tikzcd}
H^0\overline{B}(A_M, a, b) \arrow[rr] \arrow[dr, "\mathrm{iter}^\vee"'] & & \mathbb{C}_{a,b} \oplus H^n(M^n, Z_{ab}) \arrow[dl, "c^\vee"] \\
& \mathbb{C} \otimes \mathcal{O}[\pi_1(M, a, b)]^\vee &
\end{tikzcd}$$

Hodge structure: Proof of Chen's theorem

Sergey Rybakov on September 4th, 2012.

Recall that M is a smooth connected manifold, A_M is the de Rham complex of rings of complex differential forms, and $B(A_M; a, b)$ is the bar complex.

The theorem

Recall the definition of iter (Definition 194) and its dual. As explained in the previous lecture, the dual morphism,

$$\mathrm{iter}^\vee : H^0(B(A_M; a, b)) \to \mathbb{C} \otimes_{\mathbb{Q}} \mathcal{O}(\pi_1(M; a, b)^{un}),$$

is an isomorphism of algebras and respects coproducts. Recall additionally that for an algebraic group G, G^{un} denotes its pro-unipotent completion (Cf. Definition 45).

Remark 197. Both algebras are torsors under their corresponding Hopf algebras for $a = b$.

Theorem 198 (Chen's theorem)**.** *Let $H(M, a) = H^0(B(A^1_M; a, a))$. Then*

$$\mathrm{iter}^\vee : H(M, a) \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Q}} \mathcal{O}(\pi_1(M, a)^{un})$$

is an isomorphism of Hopf algebras.

Theorem 199. *The morphism*

$$\Phi : \begin{array}{rcl} \mathrm{Comod}\ H(M, a) & \longrightarrow & \mathrm{Rep}^{un}(\pi_1(M, a)) \\ V & \mapsto & \left(\gamma \mapsto \left(V \to V \otimes H(M, a) \xrightarrow{1 \otimes \int_\gamma} V\right)\right) \end{array}$$

is an equivalence of categories.

Lemma 200 (Key fact)**.** *The theory of Tannakian categories shows that ϕ is an equivalence of categories if and only if* iter$^\vee$ *is an isomorphism.*

To prove Theorem 199, we construct an inverse functor:

$$\begin{tikzcd} \mathrm{Rep}^{un}(\pi_1(M,a)) \arrow[r, "\Phi"] \arrow[dr, "RH"'] & \mathrm{Comod}(H(M,a)) \\ & \mathrm{Conn}^{un}(M) \end{tikzcd}$$

Riemann-Hilbert correspondence

Notation 201. We denote by Conn M the category of complex vector bundles with flat connection. There are adjoint functors

$$\mathrm{Conn}\ M \rightleftarrows \mathrm{Rep}\ \pi_1(M,a)\ .$$

Given a representation V of $\pi_1(M,a)$, form $L = V \times \widetilde{M}/\pi_1(M,a)$, where $\widetilde{M}$ denotes the universal cover of M. Then

$$V \mapsto L \otimes_{\mathbb{C}} \mathcal{A}^0_M = \mathcal{E}$$

where $\mathcal{E}$ is the sheaf of sections of E.

$$L \otimes_{\mathbb{C}} \mathcal{A}^0_M \overset{\nabla - \bar{d} \otimes_{\mathbb{C}} d}{\longrightarrow} L \otimes_{\mathbb{C}} \mathcal{A}^1_M \cong L \otimes_{\mathbb{C}} \mathcal{A}^0_M \otimes_{\mathcal{A}^0_M} \mathcal{A}^1_M \cong \mathcal{E} \otimes_{\mathcal{A}^0_M} \mathcal{A}^1_M$$

Proposition 202. *The Riemann-Hilbert correspondence takes uniipotent representations to unipotent connections. In particular, it is an equivalence,*

$$RH : \mathrm{Rep}^{un}(\pi_1(M,a)) \overset{\sim}{\longrightarrow} \mathrm{Conn}^{un}(M)$$

Construction of the functor G from unipotent connections to comodules

Let $(E,\nabla) \in \mathrm{Conn}^{un}(M)$. The sheaf of sections of E with respect to a trivialization is

$$\phi : \mathcal{E} \overset{\sim}{\longrightarrow} E_a \otimes_{\mathbb{C}} \mathcal{A}^0_M$$

Choose ϕ such that ϕ induces id_{E_a}.

$\nabla = d - N$ where $N \in \mathrm{End}_{\mathbb{C}} E \otimes A'_M$. N is nilpotent.

$$N^{\mathrm{rk}\ E} = 0 \in \mathrm{End}_{\mathbb{C}} E \otimes (A'_M)^{\otimes n}$$

Definition 203. Let

$$P_E = 1 + N + N^{\otimes 2} + \cdots \in (\mathrm{End}_{\mathbb{C}} E_a) \otimes_{\mathbb{C}} B(A_M, a)^0$$

This is a finite sum.

Proposition 204. *Given a vector bundle E on M, P_E is a cocycle.*

Proof. [Room for improvement] □

Corollary 205. *[Room for improvement]*

Recall that

$$\begin{aligned} d'(\omega_1 \otimes \cdots \otimes \omega_n) &= -\sum_{i=1}^{n} \omega_1 \otimes \cdots \otimes d\omega_i \otimes \cdots \otimes \omega_n \\ d''(\omega_1 \otimes \cdots \otimes \omega_n) &= -\sum_{i=1}^{n-1} \omega_1 \otimes \cdots \otimes \omega_i \wedge \omega_{i+1} \otimes \cdots \otimes \omega_n \end{aligned}$$

Proposition 206 (Flatness criteria)**.** *Given a vector bundle with connection (E, ∇), ∇ is flat if and only if $dN = N \wedge N$.*

Remark 207. Note that $d'N^{\otimes n} = d''N^{\otimes (n+1)}$.

Definition 208. Given a vector space V, we define an associated bialgebra $T(V) := \bigoplus_{n\geq 0} V^{\otimes n}$ with product given by shuffle and coproduct given by deconcatenation.

Let $f : (E, \nabla_E) \to (F, \nabla_F)$ be a morphism in $\mathrm{Conn}^{un}(M)$.

Proposition 209. *The morphism*

$$P_F \cdot f_a \to f_a \cdot P_E$$

is a coboundary in $\mathrm{Hom}_C(E_a, F_a) \otimes_{\mathbf{C}} B(A_M, a)^0$.

For an application, consider the case when $E = F$ and $\phi_1, \phi_2 : \mathcal{E} \to E_a \otimes \mathcal{A}_M^0$. Denote by P_i the sum which come from ϕ_i for $i = 1, 2$. Since the ϕ_i are 1_{E_a}, $P_1 - P_2$ is a coboundary in $\mathrm{End}(E_a) \otimes_{\mathbb{C}} B(A_m^1, a)^0$. Hence $[P_1] = [P_2]$.

Proof. Consider the following isomorphism of complexes, independent of ∇:

$$\mathrm{Hom}_{\mathbb{C}}(E_a, F_a) \otimes_{\mathbb{C}} B(A_M, a) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}_M^0}(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{A}_M^i} B(A_M, a)$$

The differential of the right hand side is

$$\delta : M \otimes \omega \mapsto dM \otimes_{cA_M^0} \omega + M \otimes d\omega$$

where $d\omega$ is a differential in $B(A_M, a)$ and $dM \in \mathrm{Hom}_{\mathcal{A}_M^0}(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{A}_M^0} \mathcal{A}_M^1$. The key formula

$$P_E f_a - f_a P_F = \delta \left(\sum_{n\geq 0} \sum_{i=0}^{n} N_F^{\otimes i} \otimes f \otimes N_E^{\otimes (n-i)} \right)$$

where the terms of the right hand sum are elements of $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{A}_M^0} B(A_M, a)$.

[Room for improvement]

□

Definition 210. Concatenation of P_E is the tensor square of P_E. Hence define

$$G : \mathrm{Conn}^{un}(M) \to H(M,a)\text{-Comod}$$

to be the functor induced by $[P_E] \in \mathrm{End}(E_a) \otimes \mathrm{Hom}(H,a)$.

Proposition 211. *The composition,*

$$\Phi \circ \Psi = 1$$

is the identity on $\mathrm{Rep}^{un}(\pi_1(M;a))$.

Proof. Let V be a representation of $\pi_1(M;a)$. Let $\Psi(V) = G(RH(V))$ where $RH(V)$ is given by some nilpotent N.

$$P = \sum_{n=0}^{\infty} N^{\otimes n}$$

By Proposition 142 the monodromy of V is given by $\gamma \mapsto \int_\gamma P$. □

Proposition 212. *The functor Ψ is essentially surjective.*

Corollary 213. *The composition,*

$$\Psi \circ \Psi = 1,$$

is the identity on $H(M,a)$-Comod.

Proof. The functor Ψ send a unipotent connection described by a unipotent matrix to a comodule whose coproduct is described by a matrix. To show essential surjectivity, we begin with a matrix describing a coproduct and try to produce a connection that maps to the coproduct.

Decompose $A^0_M = V \oplus dA^0_M$. Then A_M is quasi-isomorphic to the dg-algebra

$$\widetilde{A}_M = (0 \to \mathbb{C} \xrightarrow{\sigma} V \xrightarrow{d} A^2_M \xrightarrow{d} \cdots)$$

There is a subcoalgebra

$$H^0B(A^0_M,a)) \xrightarrow{\sim} H^0(B(\widetilde{A}_M,a)) \xrightarrow{\sim} H^0(\overline{B}(\widetilde{A}_M,a)) \subset T(V)$$

where $T(V)$ is the bialgebra described above in Definition 208. Any comodule C over $H(M,a)$ is also a comodule over $T(V)$. Any comodule over $T(V)$ comes from a comodule over $T(U)$ for $U \subset V$ with $\dim U < \infty$ because $\dim C < \infty$. So $U \subset A^1_M$ and C is a comodule over $T(U)$.

Let $e_1, \ldots, e_n$ be a basis of $U^\vee$ and $f_1, \ldots, f_n$ a basis of U.Then $T(U)^\vee = \mathbb{C}\langle\langle \phi_1, \ldots, \phi_n \rangle\rangle$. $\Gamma = \langle \gamma = \exp e_i \rangle$ is a free group.

Then γ_i gives a unipotent matrix $M_i \in \mathrm{End}_{\mathbb{C}} C$.

$$N_i = \log M_i$$

$$N = \sum_i N_i \otimes f_i \in \mathrm{End} C \otimes A'_M$$

$$\exp e_j \cdot \left(\sum_{i=1}^n N_i \otimes f_i \right)^{\otimes k} = \frac{e_i^{\otimes k}}{k!} N_i^k \times f_j^{\otimes k} = \frac{N_i^k}{k!}$$

$$\gamma_i(P) = \exp(e_i)(P) = \exp N_i = M_i$$

□

Hodge structure: Mixed Hodge structure of the fundamental group

Javier Fresán on September 4th, 2012.

Our goal will be to understand all the relations among multiple zeta values $S(n_1, \dots, n_k) \in \mathbb{R}$. For example,

- Stuffle: $S(2)^2 = 2S(2,2) + S(4)$
- Shuffle: $S(2)^2 = 2S(2,2) + 4S(1,3)$
- Unknown: $28S(3,9) + 150S(5,7) + 168S(7,5) = \frac{5197}{691}S(12)$

Formal multiple zeta values

We must replace multiple zeta values by formal multiple zeta values (Cf. Definition 89:

$$\begin{array}{rcl} Z & \longleftrightarrow & \mathfrak{H}_0 \subset \mathbb{Q}\langle x_0, x_1\rangle \supset \mathfrak{H}_0 \\ S(n_1, \dots, n_k) & \longleftrightarrow & x_0^{n_1-1}x_1 \dots x_0^{n_k-1}x_1 \end{array}$$

Some of the relations between MZV's are explained by the shuffle and stuffle relations between formal MZV's.

The next step will be a motivic interpretation of MZV's as periods of mixed Tate motives. Much of the theory is based on mixed Hodge theory, so we will first see how to interpret a MZV as a period of a mixed Hodge structure on π_1^{un}.

Recall from Definition 46 that, for a smooth, connected complex manifold M and $a, b \in M$, $\mathbb{Q}[\pi_1(M; a, b)]$ is the $\mathbb{Q}$-vector space generated by homotopy classes of paths from a to b and a free $\mathbb{Q}[\pi_1(M; a)]$-module of rank one.

Dualizing the Chen isomorphism from Proposition **??** gives

$$c_n^\vee : \mathbb{Q}_{a,b} \oplus H^n(M^n, Z^n_{a,b}) \xrightarrow{\sim} (\mathbb{Q}[\pi_1(X; a, b)]/I^{n+1})^\vee$$

These form a directed system.

Let $\mathbf{H}$ denote the left-hand side. Then $\mathbf{H}$ has an algebra structure ($\mathbf{H} \otimes \mathbf{H} \to \mathbf{H}$) and a mixed Hodge structure. We aim to show that these two structures are compatible.

[Room for improvement]

Mixed Hodge structure

Definition 214. A $\mathbb{Q}$-*mixed Hodge structure* (Cf. Definition 61) is a triple $H = (H, W_\bullet, F^\bullet)$ such that

1. H is a $\mathbb{Q}$-vector space.
2. $W_\bullet$ is an ascending filtration on H called the weight filtration.
3. $F^\bullet$ is a descending filtration on $H \otimes \mathbb{C}$ and called the Hodge filtration.

$$\mathrm{gr}_\ell^W H = W_\ell H / W_{\ell-1} H$$

And $F^\bullet_{ud}$ is a pure Hodge structure of weight ℓ.

Then H is defined over $k \subset \mathbb{C}$. If there exists a k-vector space H_{dR} and a filtration $T^\bullet$ on H_{dR} and an isomorphism

$$\alpha : H_{dR} \otimes_k \mathbb{C} \xrightarrow{\sim} H \otimes_{\mathbb{Q}} \mathbb{C}$$

compatible with the filtration. This forms $(H, H_{dR}, W_\bullet, F^\bullet, \alpha)$

$$\begin{tikzcd} MH(k) \arrow[r, "W_{dR}"] \arrow[dr, "w_B"'] & H_{dR} \in \mathrm{Vect}(k) \\ & H \in Vect(\mathbb{Q}) \end{tikzcd}$$

Definition 215 (Comparison isomorphism)**.** The comparison morphism is

$$c_n : \mathbb{C} \otimes w_{dR} \xrightarrow{\sim} \mathbb{C} \otimes w_B$$

The comparison morphism encodes the periods.

Definition 216 (Period matrix)**.** Fix bases for the comparison isomorphism. Then the period matrix is the matrix of the comparison isomorphism in those bases, and it is generally chosen from a coset in $GL_n(k)GL_n(\mathbb{C})/GL_n(\mathbb{Q})$.

Proposition 217. MH(k) *is a ???, symmetric tensor category with tensor product*

$$W_p(H \otimes H) = \sum W_u H \otimes W_{p-u} H'$$

The unit is $H^0(\mathrm{Spec}\ k) = \mathbb{Q}(0)_H = \mathbb{Q}^{0,0}$ *pure of weight* 0.

Examples

1. $H^2(\mathbb{P}^1_k) = \mathbb{Q}(-1)_H = (2\pi i)\mathbb{Q} = \mathbb{Q}(-1)_H$ pure of weight -2, $H_{dR} = k$. Period is $\frac{1}{2\pi i}$. $\mathbb{Q}(n)_H$ is pure of weight $-2n$. Recall that $\mathbb{Q}(-1) = H^2(\mathbb{P}^1)$ (Cf. 63).

2. (Deligne) Any algebraic variety over k has a k-Mixed Hodge structure. Assume the variety X is smooth and includes into a proper variety $j : X \hookrightarrow \overline{X}$ such that $D = \overline{X} \setminus X$ is a simple normal crossing divisor. Then $\Omega^\bullet_{\overline{X}}(\log D) \to Rj_*\Omega_X$.

   We can compute the cohomology of X by computing the hypercohomology:

   $$\mathbf{H}^k(\overline{X}, \Omega^\bullet_{\overline{X}}(\log D))^{\otimes\mathbb{C}} \to H^k(X, \mathbb{C})$$

   The period is $2\pi i$.

   Definition 218 (Weight filtration and Hodge filtration)**.** Given a smooth variety X with (possibly singular) closure $\overline{X}$, its *weight filtration* is defined over $\mathbb{Q}$ and given by

   $$W_m\Omega^p_{\overline{X}}(\log D) = \begin{cases} 0 & m \leq 0 \\ \Omega^m_{\overline{X}}(\log D) \wedge \Omega^{p-m}_{\overline{X}} & 0 \leq m \leq p \\ \Omega^p_{\overline{X}}(\log D) & m \geq p \end{cases}$$

   Its Hodge filtration is

   $$F^p\Omega^\bullet_{\overline{X}}(\log D) = \Big(0 \to \cdots \to 0 \to \Omega^p_{\overline{X}}(\log D) \to \cdots\Big)$$

   $$W_kH^m(X, \mathbb{C}) = \mathrm{Im}H^m(X, W_{k-m}\Omega^\bullet(\log D))$$

   To show: defined over $\mathbb{Q}$.

3. Singular cohomology and relative cohomology. $Z^n_{a,b}$ and $H^n(X, Y)$ and mixed Hodge structure. $\mathbb{Q}_{a,b} \oplus \varinjlim H^n(M^n, Z^n_{a,b}) \in \text{Ind MH}(k)$. (Recall that $\mathbb{Q}_{a,b}$ is defined to be $\mathbb{Q}$ if $a \neq b$ and 0 otherwise.)

   Definition 219 (Ind-category)**.** Fix a field k. Given a tensor category $(\mathcal{C}, \otimes)$ with a tensor functor $w : \mathcal{C} \to \text{Vect}$, it has an Ind-category Ind $\mathcal{C}$.

objects : Directed systems of k-varieties $(X_\alpha)_{\alpha\in A}$.

morphisms : $\mathrm{Hom}((X_\alpha), (Y_\beta)) = \varprojlim_\alpha \varinjlim_\beta \mathrm{Hom}(X_\alpha, Y_\beta)$

   It is a tensor category.

   Definition 220. A commutative algebra is ??? $M \in \text{Ind}\,\mathcal{C}$ and $m : M \otimes M \to M$, $u : \mathbf{1} \to M$ and satisfies compatibility conditions. This can be defined analogously for commutative Hopf algebras.

Definition 221. A $\mathcal{C}$-affine scheme Spec $(M) = \mathcal{O}(X)$ is an object in the opposite category $(\text{Ind } \mathcal{C})^{op}$.

$$\begin{tikzcd} MH(k) \arrow[r] \arrow[dr] & \mathrm{Vect}(\mathbb{Q}) \\ & \mathrm{Vect}(k) \end{tikzcd}$$

[Room for improvement]

Proposition 222. *If $A \to B$ is a quasi-isomorphism of dg-algebras, and a, b abused notation for compatible augmentations*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d] & k \\ B \arrow[ur, "a"] & \end{tikzcd}$$

then $B(A; a, b)$ and $B(B; a, b)$ are also quasi-isomorphic.

1. $H^0(B(A_M; a, b))$ is a $\mathbb{Q}$-vector space.
2. Singular cochains $C^0_{X(\mathbb{C})}$ with augmentations $\mathbb{C}^0_{X(\mathbb{C})} \to \mathbb{Q}$. $H^0(B(C^0_{X(\mathbb{C})}; a, b))$ is a $\mathbb{Q}$-vector space.
3. Let $\Omega^\bullet_{\overline{X}}(\log D)$ be the Godement resolution and $A^\bullet_X = R\Gamma(\overline{X}, \Omega^\bullet_{\overline{X}}(\log D))$ be the derived complex. $H^0(B(A_X; a, b))$ is a k-vector space.

Definition 223 (Mixed Hodge complex)**.** $(H^\bullet_{\mathbb{C}}, W_\bullet, F^\bullet)$.

$$\begin{array}{ll} (H^\bullet, W_\bullet) & \text{complex of } \mathbb{Q}\text{-vector spaces} \\ (H^\bullet_{dR}, W_\bullet, F^\bullet) & \text{complex of } k\text{-vector spaces} \\ \alpha^\bullet : H^\bullet \otimes \mathbb{C} \to H^\bullet_{\mathbb{C}} & \text{quasi-isomorphisms} \\ \beta^\bullet : H^\bullet_{dR} \otimes \mathbb{C} \to H^\bullet_{\mathbb{C}}) & \text{quasi-isomorphisms} \end{array}$$

Proposition 224 (Hain, Journal of K-theory 1987)**.** *The bar construction applied to a mixed Hodge complex gives a mixed Hodge complex.*

Geometric origin:

$$\begin{array}{ccc} \mathbf{H} & \to & \mathbf{H} \otimes \mathbf{H} \\ [\omega_1 \mid \cdots \mid \omega_n] & \mapsto & \sum [\omega_1 \mid \cdots \mid \omega_i] \otimes [\omega_{i+1} \mid \cdots \mid \omega_k] \end{array}$$

To perform the bar construction, we take tensor powers of $A^\bullet_M$:

$$(A^*_M)^{\otimes n} \to (A^*_M)^{\otimes (n-1)} \to \cdots$$

$$A^*_{M^n} \to A^*_{M^{n-1}} \to \cdots$$

$$M^n \overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}} M^{n-1} \longleftarrow$$

$\mathcal{O}(\pi_1(X;a,b)_H)$ is an object in Ind MH(k).

$$\pi_1(X;a,b)_B = \text{Spec } H^0(B(C^*_{X(\mathbb{C})};a,b)) \qquad \text{(Betti)}$$
$$\pi_1(X;a,b)_{dR} = \text{Spec } H^0(B(A^*_X;a,b)) \qquad \text{(de Rham)}$$

The comparison between realizations gives a mapping

$$\mathcal{O}(\pi_1(M;a,b)_{dR}) \otimes_k \mathbb{C} \xrightarrow{\sim} \mathcal{O}(\pi_1(M;a,b)_p) \otimes_{\mathbb{Q}} \mathbb{C} \underset{\text{iter}^\vee}{\longleftarrow} H^0(\overline{B}(A_{X(\mathbb{C})};a,b)) \cong H^0(B(C_{X(\mathbb{C})};a,b))$$

$w_1,\ldots,w_n$ closed differential 1-forms such that $\omega_i \wedge \omega_{i+1} = 0$. $\omega_1 \otimes \cdots \otimes \omega_n \in \mathcal{O}(\pi_1(X;a,b)_{dR}) \otimes \mathbb{C}$.

$$\text{comp}(\omega_1 \otimes \cdots \otimes \omega_n)(\gamma) = \int_\gamma \omega_1 \cdots \omega_n, \qquad \gamma \in \pi_1(X;a,b)(\mathbb{Q})$$

Hence we see that iterated integrals are periods of mixed Hodge structures.

Later we would like to apply to this to punctured $\mathbb{P}^1$'s, but the punctures cause problems with the paths.

Unipotent filtration

Definition 225 (Unipotent filtration)**.** Let G be an algebraic group. The unipotent filtration is an ascending filtration on $\mathcal{O}(G)$ given by

$$\begin{aligned} N_0\mathcal{O}(G) &= \mathbb{Q} \\ \subset N_1\mathcal{O}(G) &= \{f \in \mathcal{O}(G) \mid \Delta(f) = f \otimes 1 + 1 \otimes f + (\text{const})1 \otimes 1\} \\ \subset N_2\mathcal{O}(G) &= \{f \in \mathcal{O}(G) \mid \Delta^2(f) \in \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes \mathcal{O}(G)\} \end{aligned}$$

Definition 226. A group G is unipotent if and only if $\mathcal{O}(G) = \cup N_\ell \mathcal{O}(G)$

We are concerned with the particular case $G = \mathcal{O}(\pi_1(X;a)^{un})$.

There is a sub-mixed Hodge structure:

$$\text{gr}^N_m \mathcal{O}(\pi_1(X;a)^{un}) \subset (\text{gr}^N_1 \mathcal{O}(\pi_1(X;a)^{un})^{\otimes m} = \text{Hom}(\pi_1(X;a)^{ab}, \mathbb{Q}) = H^(X)$$

For example, $H^1(X)$ is pure of weight 2 if and only if $H^1(X)$ is pure of type $(1,1)$ if and only if $H^1(\overline{X}, \mathcal{O}) = 0$ if and only if $H^1_{dR}(\overline{X}) = 0$.

$$0 \to H^1_{dR}(\overline{X}) \to H^1(X) \to \mathbb{Q}(-1)^{\oplus 2}$$

Remark 227. In this case, the unipotent filtration is the weight filtration, i.e., $N_k = W_{2k}$.

Definition 228 (Mixed Tate type)**.** A mixed Hodge structure H is of *mixed Tate type* if all $\text{gr}^W_\ell H$ are direct sums of $\mathbb{Q}(n)_H$.

Proposition 229. *Under these conditions, $\mathcal{O}(\pi_1(X;a,b)_H)$ is of mixed Tate type.*

Hodge structure: The case of the projective line with punctures

Rafael von Känel on September 4th, 2012.

We will use the following notation.

Notation 230. $\overline{X} := \mathbb{P}^1_{\mathbb{C}}$

D is a divisor on $\overline{X}$ with $D(\mathbb{C}) = \{\infty, a_1, \ldots, a_r\}$ such that $r \geq 0$ and $a_1, \ldots, a_r \in \overline{X}(k)$.

$X := \overline{X} \setminus D$.

k is a field of characteristic 0 with an inclusion $k \hookrightarrow \mathbb{C}$.

$a, b \in X(k)$.

$M = X(\mathbb{C})$ is a manifold.

Explicit Definition of $\mathcal{O}(\pi_n(X; a, b)_H)$

Fix an integer $n > 0$.

The first goal will be to define $\mathcal{O}(\pi_1(X; a, b)_H) = \varinjlim_n V_n$.

1. **Define the weight filtration on V_n.** Let V_n be the $\mathbb{Q}$-vector space $\mathbb{Q}_{a,b} \oplus H^n(X^n, Z^n_{a,b})$, (Cf. Definition 145) and let $V^M_{n+1} = \mathbb{Q}[\pi_1(M; a, b)]/I^{n+1}$.

   Dualizing Chen's isomorphism c_n of Theorem 148 gives an isomorphism of $\mathbb{Q}$-vector spaces,

   $$c_n^\vee : V_n \to V^M_{n+1}.$$

   This has weight filtration

   $$W_{2m}V_n = \begin{cases} 0 & m \leq -1 \\ (c_n^v)^{-1} V'_m & 0 \leq m \leq n-1 \\ V_n & n \leq m \end{cases}$$

2. **Define Hodge filtration on $V_n \otimes_{\mathbb{Q}} \mathbb{C}$.**

   Let Ω be the k-vector space of algebraic differential 1-forms on $\overline{X}$ with at most first order poles along D, i.e.,

   $$\Omega = H^0(\overline{X}, \Omega^1_{\overline{X}}(\log D)) \cong H^1_{dR}(X)$$

   and

   $$\Omega = \mathrm{span}_K \left\{ \frac{dz}{z-a_1}, \dots, \frac{dz}{z-a_r} \right\}$$

   Recall that the path space of M is

   $$PM = \{\text{piecewise smooth functions } [0,1] \to M\}.$$

   Definition 231 (Chains)**.** Let the chains $Ch(\Omega)$ of Ω be the k-vector space generated by the constant functions and the iterated integrals

   $$\int \omega_1, \dots, \omega_\ell : PM \to \mathbb{C}, \qquad \ell > 0, \quad \omega_1, \dots, \omega_\ell \in \Omega$$

   It is a k-algebra with multiplication given by the shuffle product of iterated integrals.

   Definition 232 (Length filtration on Chains)**.** The *length* of the integral (which depends on how we've written it) is $\mathrm{length}(\int \omega_1 \cdots \omega_\ell) := \ell$. The length function defines a filtration on $Ch(\Omega)$ (and $CH(\Omega)_{\mathbb{C}}$).

   $$L_n Ch(\Omega) = \left\langle c, \int \omega_1 \cdots \omega_\ell \mid 1 \le \ell \le n, c \in k \right\rangle_k$$

   Definition 233 (Weight filtration on Chains)**.** The weight filtration on the chains is

   $$W_{2m} = \begin{cases} 0 & m \le -1 \\ (Ch)^{-1} V_m^M & 0 \le m \le n-1 \\ V_n & n \le m \end{cases}$$

   Lemma 234. *There is an isomorphism $\varphi : Ch(\Omega)_{\mathbb{C}} \to \mathcal{O}(\pi_1(C; a, b)_B)$ such that $\varphi(L_n Ch(\Omega)) = V_n \otimes_{\mathbb{Q}} \mathbb{C}$.*

   Proof. (Idea)

   $$T(\Omega) = \mathcal{O}(\pi_1(X; a, b)_{dR}$$

   $$\begin{array}{rcll} \mathcal{O}(\pi_1(X;a,b)_H) \otimes_{\mathbb{Q}} \mathbb{C} & = & \mathcal{O}(\pi_1(X;a,b)_B) \otimes_{\mathbb{Q}} \mathbb{C} & c^\vee \\ & \cong & \mathcal{O}(\pi_1(X;a,b)_{dR}) \otimes_{\mathbb{Q}} \mathbb{C} & \text{comp} \\ & \cong & H^0(B(A_X); a, b) \otimes_{\mathbb{Q}} \mathbb{C} & \\ & \cong & T(\Omega) \otimes_k \mathbb{C} & \text{Equation 2} \\ & \cong & Ch(\Omega)_{\mathbb{C}} & \end{array}$$

We calculate the bar complex associated this dg-algebra arising from the affine variety X. There is a quasi-isomorphism, (see Thomas' talk [Room for improvement])

$$A_X \cong k \oplus \Omega[-1].$$

where A_X is the algebraic de Rham complex. $0 \to k \to \Omega \to 0$. Let $\varphi : Ch(\Omega)_{\mathbb{C}} \to \mathcal{O}(\pi_1(X;a,b)_H) \otimes_{\mathbb{Q}} \mathbb{C}$ be the resulting isomorphism. One can check that $\varphi(L_n Ch(\Omega)) = V_n \otimes_{\mathbb{Q}} \mathbb{C}$, hence the statement. $\square$

Remark 235. The map length defines a grading on $Ch(\Omega)$ (and $Ch(\Omega)_{\mathbb{C}}$), and φ is a morphism of (weight) graded $\mathbb{C}$-algebras. Then φ induces an integral,

$$\mathrm{int} : \begin{array}{ccc} Ch(\Omega)_{\mathbb{C}} & \to & \mathcal{O}(\pi_1(X;a,b,)^{un}) \\ \int \omega_1 \cdots \omega_n & \mapsto & \{[\gamma] \mapsto \int_\gamma \omega_1 \cdots \omega_l\} \end{array}$$

and

$$\varphi = (c^v)^{-1} \circ \mathrm{int}$$

Definition 236 (Hodge filtration on Chains)**.** The Hodge filtration is

$$F^p(V_n \otimes_{\mathbb{Q}} \mathbb{C}) = \bigoplus_{i=p}^{n} \varphi(L_i Ch(\Omega)_{\mathbb{C}})$$

The morphisms $V_n \to V_{n+1}, n \in \mathbb{Z}_{\geq 1}$ form a direct system in MH and we denote by $\pi_1(X;a,b)_H$ the resulting affine MH-scheme.

3. **Take the inductive limit**

Lemma 237. *The above filtrations make $Ch(\Omega)$ into a mixed Tate-Hodge structure over k.*

Proof. **Betti realization**

The Betti realization is $\pi_1(X;a,b)^{un} \cong \pi_1(X;a,b)_B$ (Cf. previous talk [Room for improvement]). Since X is a punctured sphere, $\Gamma = \pi_1(X;a,b)$ is a finitely generated free group of rank r. As seen in Alberto's talk [Room for improvement], this gives a $\mathbb{Q}$-vector space V of dimension r such that,

$$\mathcal{O}(\pi_1(X;a)) \cong T(V).$$

Combining gives $\pi_1(X;a,b)_B \cong \operatorname{Spec} T(V)$. $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ implies that $\mathcal{O}(\pi_1(X;a,b)_B) = \mathcal{O}(\pi_1(X;a,b)^{un}) \cong T(V)$. choose generators.

$$V^\vee = \langle e_1, \ldots, e_r \rangle_{\mathbb{Q}}$$

De Rham Realization

The de Rham realization is $\mathcal{O}(\pi_1(X;a,b))_{dR} = T(\Omega)$.

$$F^pT(\Omega) = \oplus_{k\geq p}\Omega^{\otimes k}$$

$$W_{2n}T(\Omega) = \oplus_{k\leq n}\Omega^{\otimes k}$$

It remains to show that $\mathcal{O}(\pi_1(X;a,b)_H)$ is of mixed Tate type. $\overline{X}$ is a rational variety, hence $H^1(\overline{X},\mathcal{O}_{\overline{X}}) = 0$. Thus the example of the last talk implies $\mathcal{O}(\pi_1(X;a,b)_H)$ is of mixed Hodge type. (use 2 times unipotent filtr. = weight filtration on $\mathcal{O}(\pi_1(X;a,b)_H)$ where unipotent is induced by the action of $\pi_1(X;a)_H$). [Room for improvement]

Periods

The comparison isomorphism is

$$\mathbb{C}\otimes_k T(\Omega) \xrightarrow{\sim} \mathbb{C}\otimes_Q T(V) = \left(\mathbb{C}\otimes\left(\prod_{i\geq 0}(V^{vee})^{\otimes i}\right)\right)^{\vee} \cong \mathbb{C}\otimes\mathbb{Q}\langle\langle e_1,\ldots,e_n\rangle\rangle$$

To specify an element of $\mathbb{C}\otimes_{\mathbb{Q}} T(V)$ it suffices to specify a function on $\pi_1(X;a)$, becuase of the morphism $\pi_1(X;a)\to\mathbb{C}\otimes_{\mathbb{Q}} T(V)$. In turn, to specify such a function, it suffices to give its value on generators, namely, on simple loops γ_i around a_i and based at a.

By the universal property of the pro-nilpotent completion, there is a map

$$\begin{array}{rcl} \pi_1(X;a) & \to & (\text{Spec }\mathcal{O}(V))(\mathbb{Q}) \\ \gamma_i & \mapsto & \exp(e_i)\in\prod_{i\geq 0}(V^{vee})^{\otimes i} \end{array}$$

A point on this Spec is a special function on functions.

$$(1-\gamma_{i_1})\cdots(1-\gamma_{i_n})\mapsto(1-\exp(e_{i_1}))\cdots(1-\exp(e_{1_n}))\in\mathbb{Q}\langle\langle e_1,\ldots,e_r\rangle\rangle$$

The period of $\mathrm{gr}^W_{2n}\pi_1(X;a)_H$ are given by

$$\int_{(1-\gamma_{i_1})\cdots(1-\gamma_{i_k})}\omega_1\cdots\omega_n \overset{\text{calc}}{=} \prod_{j=1}^{n}\int_{\gamma_{i_j}}\omega_j$$

Hence non-zero periods are $(2\pi i)^n$. □

Another example (interlude by Sergey)

Let $D = \{0, \infty\}$ so $X = \mathbb{G}_m$.

Proposition 238. *There is an isomoprhism in* Ind MTH(k).

$$\mathcal{O}(\pi_1(X; a, b)_H) \cong \bigoplus_{n \geq 0} \mathbb{Q}(-n) = T(\mathbb{Q}(-1))$$

Let $V = \mathbb{Q} \cdot e$ and $a = b$. Let $\gamma \in \pi_1(X; a)$ be (the class of) a simple loop around 0.

$$\begin{array}{rcl} ccc\text{de Rham} & & \text{Betti} \\ \mathbb{C} \otimes_k (T(\Omega) = k[\omega]) & \xrightarrow{\sim} & \mathbb{C} \otimes_{\mathbb{Q}} (T(V) = \mathbb{Q}\langle\omega\rangle^{un}) \\ \Omega = \langle \omega = \dfrac{dz}{z} \rangle_k & & \pi_1(X; a) = \langle \gamma \rangle \end{array}$$

Define

$$\text{comp}(\omega^n) := (2\pi i)^n e^n.$$

To check:

$$\text{comp}(\omega^n)(\gamma) = \int_\gamma \omega^n$$

The check proceeds as:

$$\frac{(2\pi i)^n}{n!} = \langle (2\pi i)^n e^n, \exp(\sigma) \rangle \stackrel{?}{=} \int_\gamma \omega^n = \frac{1}{k!}(\int_\gamma \omega)^n = \frac{1}{n!}(2\pi i)^n$$

where the right side follows from the shuffle product formula.

$$\int_\gamma \omega \cdot \int_\gamma \omega = 2 \int_\gamma \omega^2$$

We use the notation

$$\mathbb{Q}(1)_H := \text{Spec } (T(\mathbb{Q}(-1))_H)$$

Definition of affine MTH(k)-schemes

We will be considering paths which leave and arrive along fixed tangent vectors in order to [Room for improvement]. We will define $\pi_1(X; u, v)$ with u, v tangents as those paths which leave along u and arrive along v. Fixing these tangent vectors results in no loss of generality, because changing the tangent vector results in a canonically isomorphic construction. The construction was originally defined by Deligne for curves, by Hain for X and by Deligne-Goncharov for unirational varieties.

Let

$$\begin{array}{rl} x, y \in D & \\ u \in T_x\overline{X} & \text{non-zero tangent vector} \\ v \in T_x\overline{(}Y) & \text{non-zero tangent vector} \end{array}$$

Definition 239 (Paths tangent to given vectors)**.** Given tangents on $\overline{X}$ as above,

$P_{u,v} = \{\gamma : [0,1] \to \overline{X}(\mathbb{C}) \mid \gamma \text{ piecewise smooth paths}, \gamma(]0,1[) \subset M, \gamma(0) = x, \gamma(1) = y, \gamma'(x) = u, \gamma'(y) = -v\}$

Define $\pi_1(M; u, v)$ as the homotopy classes of $P_{u,v}$.

Compose two path $\gamma_1 \circ \gamma_2$ by following γ_1 until ϵ-close to its endpoint, and then circle around an ϵ-ball to arrive at the point where γ_2 first exits it? [Room for improvement]

Let E be a vector bundle on $\overline{X}$ with a unipotent connection ∇ with at most first order poles along D. As in Sergey's talk [Room for improvement], we get $\nabla = d - N$ for a nilpotent matrix N with values in Ω.

E is an extension of a trivial vector bundle.

For $\epsilon > 0$ and $[\gamma] \in \pi_1(M; u, v)$, let γ_ϵ start in ϵ and end in $1 - \epsilon$ as in Fritz' talk [Room for improvement].

Definition 240.

$$\operatorname{res}(\nabla) := \operatorname{res}(-N) \text{ and } \epsilon^N = \exp(\log(\epsilon) N)$$

Define monodromy along γ by the regularization:

$$\int_\gamma \nabla := \lim_{\epsilon \to 0} \epsilon^{\operatorname{res}_y(\nabla)} \circ \int_{\gamma_\epsilon} \circ \epsilon^{-\operatorname{res}_x(\nabla)}$$

It is indeed well-defined. For $\omega_1, \ldots, \omega_n \in \Omega$, define $\int_\gamma \omega_1 \cdots \omega_n$ by the monodromy interpretation

$$N = \begin{pmatrix} 0 & \omega_1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & \omega_n \\ & & & 0 \end{pmatrix}$$

Explicitly,

$$\int_\gamma \omega_1 \cdots \omega_n = \lim_{\epsilon \to 0} \sum_{0 \le i \le j \le n} \frac{(-1)^i}{i!(n-j)!} \prod_{l=1}^{i} \operatorname{res}_y(\omega_l) \cdot \int_{\gamma_\epsilon} \omega_{i+1} \cdots \omega_{i+j} \cdot \prod_{l=j+1}^{n} \operatorname{res}_x(\omega_l) \log(\epsilon)^{i+n-j}$$

The iter morphism (Cf. Definition 194) is well defined for paths from $\pi_1(M; u, v)$. Two steps will define $\pi_1(X; u, v)_H$.

1. In the first case, suppose that $u = a \in X(k)$.

   Note that $\pi_1(M; a, v)$ is a $\pi_1(M; a)$-torsor. By Alberto's talk [Room for improvement], this implies $\pi_1(M; a, v)^{un}$ is a $\pi_1(M; a)^{un}$-torsor. Recall that

   $$\mathcal{O}(\pi_1(M; a, v)^{un}) = \lim_{n \to \infty} \mathbb{Q}[\pi_1(M; a, v)/I^{n+1}\mathbb{Q}[\pi_1(M; a, v)]^v.$$

Then iter factors through quotients by powers of the augmentation ideal I^{n+1}. Dualizing gives an isomorphism $\mathrm{iter}^\vee$. Taking the limit as n approaches infinity and tensoring with $\mathbb{C}$ gives a morphism of $\mathbb{C}$-vector spaces $\mathrm{iter}^\vee : \mathcal{O}(\pi_1(M;a,v)^{un}) \otimes_{\mathbb{Q}} \mathbb{C} \to T(\Omega) \otimes_k \mathbb{C}$.

One can check that $\mathrm{iter}^\vee$ is a $\mathbb{C}$-algebra morphism using the composition of path formula by the monodromy interpretation, and that it respects the actions of $\mathcal{O}(\pi_1(X;a)^{un})$ on $\mathcal{O}(\pi_1(X;a,v)^{un})$ and $\mathcal{O}(\pi_1(X;u,a))$ by the following diagram from torsor theory:

$$\begin{tikzcd}
\pi_1(X;a)^{un} \arrow[r, "\mathrm{comp}"] \arrow[d] & T(U) \otimes_k \mathbb{C} \arrow[d] \\
\pi_1(X;a,v)^{un} & T(U) \otimes_k \mathbb{C} \arrow[l, "\mathrm{iter}^\vee"]
\end{tikzcd}$$

Hence there is a morphism of torsors $\mathrm{Spec}\ \mathrm{iter}^\vee : \pi_1(M;a,v)^{un} \times_{\mathbb{Q}} \mathbb{C} \to \mathrm{Spec}\ T(\Omega) \times_k \mathbb{C}$ which respects isomorphisms of pro-unipotent groups $un \to B \overset{comp}{\to} dR \to T(\Omega)$. $\pi_1(M;a)^{un} \times_{\mathbb{Q}} \mathbb{C} \cong \mathrm{Spec}\ T(\Omega) \times_k \mathbb{C}$. Therefore $\mathrm{Spec}\ \mathrm{iter}^\vee$ is an isomorphism. The weight filtration on $\mathcal{O}(\pi_1(M;a,v)^{un})$ is two times the unipotent filtration [Room for improvement]induced by action of $\pi_1(M;a)^{un}$. There is a Hodge filtration by the isomorphism $\mathrm{Spec}\ \mathrm{iter}^\vee$. This mixed Hodge structure is in fact a mixed Tate-Hodge structure using the unipotent filtration imitating the previous example, hence $\mathcal{O}(\pi_1(M;a,v)^{un}) \in ind(\mathrm{MTH}(k))$.

2. In the second case, suppose that both $u, v \in D(\mathbb{C})$. iter and $\mathrm{iter}^\vee$ give morphisms as before. Let

$$\pi_1(X;u,v) := \pi_1(X;u,v)^{un} \qquad \pi_1(X;u,v)_{dR} = T(\Omega)$$

To show that $\mathrm{iter}^\vee$ is an isomorphism, form the product of right and left $\pi_1(X;a)_B$-torsors quotiented by the relation $(g_1 g, g_2) = (g, g_1 g_2)$ gives

$$\pi_1(M;u,v)_B \cong \pi_1(X;u,a)_B \times \pi_1(X;a,v)_B / \sim$$

Now use the comparison isomorphism between the Betti and de Rham realizations of $\pi_1(M;a,v)$. This leads to an affine $\mathrm{MTH}(k)$-scheme $\pi_1(X;u,v)_H$ with a groupoid structure and with non-negative weights.

Lemma 241. $\mathbb{Q}(1)_H \to \pi_1(X;u)_H$ *of affine* $\mathrm{MTH}(k)$*-schemes.*

Multiple zeta values are periods

The multiple zeta values are periods of a mixed Tate-Hodge structure over $\mathbb{Q}$ with non-negative weights.

Let $[\gamma_x] \in \pi_1(X;u)$ be a simple loop around x. Write $\mathbb{Q}(1)_H$ also for $\pi_1(\mathbb{P}^1 \setminus \{0,\infty\}, a)_H$ where $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0,\infty\}$.

$$\int_{\gamma_x} \nabla = \exp(2\pi i \mathrm{res}_x(\nabla))$$

$$\mathbb{Q}(1)_H = \pi_1(X)$$

Lemma 242. *The loop γ_x induces a morphism,*

$$\mathbb{Q}(1)_H \to \pi_1(X;u)_H,$$

in $\mathrm{MHT}(k)$.

Idea. The Betti realization $n \mapsto \gamma_x^n \in \pi_1(M;u)$ induces

$$\mathcal{O}(\mathbb{Q}(1)_B) = \mathcal{O}(\mathbb{Z}^{un}) \to \mathcal{O}(\pi_1(M;u)^{un}) = \mathcal{O}(\pi_1(X;u)_B)$$

where $\mathbb{Z}^{un} = \mathbb{G}_a$. The de Rham realization $\mathrm{res}_x : \Omega \to k$ induces

$$\mathcal{O}(\pi(X;u)_{dR}) = T(\Omega) \to T(k) = \mathcal{O}(\mathbb{Q}(1)_{dR}).$$

One can check that $\mathcal{O}(\mathbb{Q}(1)_B) \to \mathcal{O}(\pi_1(X;u)_B)$ and $\mathcal{O}(\pi_1(X;u)_{dR}) \to \mathcal{O}(\mathbb{Q}(1)_{dR})$ commute vertically. Using the definition of iterated integrals via the monodromy interpretation and the fact that the morphism comp is induced by $\mathrm{iter}^\vee$ gives

$$\mathcal{O}(\mathbb{Q}(1)_H) = T(H_2(\mathbb{P}^1)) = T(\mathbb{Q}(-1)) \qquad \pi_1(\mathbb{P} \setminus \{0,\infty\}, a)_{dR} = \mathbb{G}_a$$

$\square$

Now let $D = \{0, 1, \infty\}$, $k = \mathbb{Q}$ and $\Omega = \frac{dz}{z}\mathbb{Q} \otimes \frac{dz}{z-1}\mathbb{Q}$. Let $\overline{n} = (n_1, \ldots, n_k), n_k \geq 2$.

As as in talk 1 (Cf. Definition 97), this corresponds to a word $w(\overline{n})$ in 0 and 1 starting in 0 and ending in 1. As in Definition **??**, let $\overline{w} = w(\overline{n})$ sequence in $w_0 = dzz$ and $w_1 = \frac{dz}{1-z}$.

Definition 243 (The path dch)**.** Let dch denote the real interval [0,1] viewed as a path from 0 to 1. Hence $[(dch)] \in \pi_1(X; 0, 1) := \pi_1(X; \overrightarrow{01}, \overrightarrow{10})$.

By Fritz' talk [Room for improvement],

$$\begin{array}{rcll} \zeta(\overline{n}) & = & \lim_{\epsilon \to 0} S_{dch_\epsilon} \overline{w} & \qquad \text{Proposition 134} \\ & = & S_{dch}^{-1} \overline{w} & \\ & = & \mathrm{comp}(\overline{w})(\mathrm{dch}) & \end{array}$$

Motivic structure: Tannakian categories

Konrad Völker on September 6th, 2012.

After reminding you of the basic properties of Tannakian categories, along some examples, we will work out a strategy to prove upper bounds on periods of mixed Tate motives, which will be used by the following talk, and which will finally lead to the theorem of Goncharov-Terasoma. The strategy consists of defining a weigth filtration on the real periods, obtained as fixed points of all periods under complex conjugation, and then to show that it's a quotient of a certain explicit graded $\mathbb{Q}$-subalgebra $\mathcal{O}(I(\omega,\eta))_+^\epsilon \subseteq \mathcal{O}(I(\omega,\eta))$, whose Poincaré series can be computed. In the proof, we exhibit the structure of graded k-algebra

$$\mathcal{O}(I(\omega,\eta))_+^\epsilon \simeq \mathcal{O}(\mathbb{G}_m \rtimes U) = k[t,t^{-1}] \otimes_k T\left(\bigoplus_{n>0} Ext^1_{MTM(\mathbb{Z})}\left(\mathbb{Q}(0),\mathbb{Q}(n)\right)\right).$$

Since we don't have mixed Tate motives over $\mathbb{Z}$ at hand at this time, we will do this in a very abstract setting, using mixed Tate Hodge structures as main example. We will have to make several assumptions for the proofs, all of which will be proven in the case of mixed Tate motives over $\mathbb{Z}$ in the next talk. Mixed Tate Hodge structures don't suffice for the proof of Goncharov-Terasoma, since the Ext-groups are too large.

Basic notions

Definition 244. A *tensor category* over k is an abelian category $\mathcal{C}$ enriched over k-vector spaces, equiped with a "tensor" product k-bifunctor $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and an "identity" object $1\in\mathcal{C}$ together with natural isomorphisms $(\cdot)\otimes 1\cong \mathrm{id}$ and $1\otimes(\cdot)\cong\mathrm{id}$ that make the identity object worth its name, and natural isomorphisms $\alpha_{A,B,C} : (A\otimes B)\otimes C \cong A\otimes(B\otimes C)$, that are called *associators*, which have to obey the pentagon and the triangle identites. From MacLane's coherence theorem, this implies all further combinations of associators and identities that can be isomorphic, are isomorphic. It is called *symmetric* if there are symmetry isomorphisms $A\otimes B\cong B\otimes A$. These conditions are often abbreviated to ACU (for associativity, commutativity, unit). We require furthermore that a tensor category is *rigid*, i.e. internal Homs exist, and $End(1)\simeq k$.

A neutral Tannakian category over k is a rigid abelian tensor category over k, together with a k-tensor functor to k-vector spaces, that is exact and faithful. A Tannakian category is a category which admits a functor to k-vector spaces that makes it neutral Tannakian (in other words: we don't fix the fiber functor).

One can do the same not only for k-vector spaces but for quasicoherent sheaves over a k-scheme S. Deligne discusses this in detail, but we won't need the more general theory right now.

Example 245. Let G be a linear pro-algebraic group over k, e.g. $\boldsymbol{SL}_\infty = \lim \boldsymbol{SL}_n$ over $\mathbb{Q}$, then the category of all finite-dimensional k-rational representations is a rigid abelian tensor category over k, and the forgetful functor to k-vector spaces is exact and faithful, making $\operatorname{Rep}(G)$ a neutral Tannakian category.

Proposition 246. *Let $\omega : \operatorname{Rep}(G) \to \operatorname{Vect}(k)$ be the forgetful functor. Then the tensor-automorphisms $G_\omega := \mathcal{A}ut_k^\otimes(\omega)$ form a linear pro-algebraic group, called* fundamental group*. There is a canonical isomorphism $G \to G_\omega$.*

Proof. Let R be a k-algebra. The R-points of G_ω are the automorphisms $\mathcal{A}ut_k^\otimes(\omega)(R)$, i.e.,

$$G_\omega(R) = \left\{ (\lambda_X)_{X \in \operatorname{Rep}(G)} \;\middle|\; \begin{array}{c} \lambda_X \in \operatorname{Aut}(X \otimes R) \qquad \lambda_X \ R\text{-linear} \qquad \lambda_{X \otimes Y} = \lambda_X \otimes \lambda_Y \\ \lambda_1 = \mathrm{id}_R \text{ and for all } G\text{-equivariant } \alpha : X \to Y \\ \lambda_Y \circ (\alpha \otimes 1) = (\alpha \otimes 1) \circ \lambda_X : X \otimes R \to Y \otimes R \end{array} \right\}$$

so we can map $G(R) \to G_\omega(R)$, since every $g \in G(R)$ acts on every G-representation X tensored with R. This gives $G \to G_\omega$ and is in fact an isomorphism of functors of k-algebras:

We restrict to the full subcategory $\mathcal{C}_X$ of subquotients of any sum of tensor powers of X and $X^\vee$ for a fixed object $X \in \mathcal{C}$. Then $\operatorname{Aut}^\otimes(\omega|_{\mathcal{C}_X})(R)$ can be considered as a subgroup of $\boldsymbol{GL}(X \otimes R)$ by $\lambda \mapsto \lambda_X$. Let G_X be the image of G in $\boldsymbol{GL}_X$, which is a closed algebraic subgroup, then we have

$$G_X(R) \subseteq \operatorname{Aut}^\otimes(\omega|_{\mathcal{C}_X})(R) \subseteq \boldsymbol{GL}_X(R) = \boldsymbol{GL}(X \otimes R).$$

If $V \in \mathcal{C}_X$ and $t \in V^G$, then the 1-parameter group $\alpha : k \to V,\ a \mapsto at$ is G-equivariant, and so $\lambda_V(t \otimes 1) = t \otimes 1$. Thus $\operatorname{Aut}^\otimes(\omega|_{\mathcal{C}_X})$ is the subgroup of $\boldsymbol{GL}_X$ fixing all tensors in representations of G_X fixed by G_X, which implies that $G_X = \operatorname{Aut}^\otimes(\omega|_{\mathcal{C}_X})$.

Now this works for all X, and we can take a limit construction to get the general result. $\square$

Theorem 247 (Tannakian Reconstruction)**.** *Let $\mathcal{C}$ be a neutral Tannakian category with fiber functor ω. Then the representation category of the fundamental group $G_\omega := \mathcal{A}ut_k^\otimes(\omega)$, as a neutral Tannakian category* $(\operatorname{Rep}(G_\omega), \omega_{forget})$*, is canonically equivalent to* $(\mathcal{C}, \omega)$*.*

Vague proof idea: We can write down a functor $(\mathcal{C}, \omega) \to (\operatorname{Rep}(G_\omega), \omega_{\text{forget}})$ by mapping any $S \in \mathcal{C}$ to $\omega(S)$ with the G_ω-action given by

$$\operatorname{Aut}_k^\otimes(\omega) \times \omega(S) \to \omega(S), \qquad (\alpha, v) \mapsto \alpha_S(v).$$

Definition 248. Let $I(\omega, \eta) := \mathrm{Iso}_k^{\otimes}(\omega, \eta)$ be the isomorphism scheme from one fiber functor $\omega : \mathcal{C} \to \mathrm{Vect}(k)$ to another η. The S-points of this scheme (for $u : S \to k$ a k-scheme) consists of the set of isomorphisms of the fiber functor $u^*\omega$ with $u^*\eta$.

The scheme $I(\omega, \eta)$ is a right torsor under G_ω and a left torsor under G_η. If $\mathcal{C}$ is tensor generated by a single object S, then $I(\omega, \eta)$ is a closed subscheme of $\mathrm{Iso}_k(\omega(S), \eta(S))$, the relations corresponding to the coherence constraints on the tensor product.

Examples

Example 249. The category of $\mathbb{Z}$-graded vector spaces over a field k is neutral Tannakian with fiber functor the forgetful functor to ungraded k-vector spaces. A $\mathbb{Z}$-grading can be thought of as the weight grading of a $\mathbb{G}_m$-representation, where $\lambda \in \mathbb{G}_m(k)$ acts as multiplication with λ^{-n} on the nth graded part. This shows that the category of $\mathbb{Z}$-graded vector spaces over a field k is equivalent to the category of k-rational $\mathbb{G}_m$-representations, with forgetful functors on both sides corresponding to each other.

Example 250. Take an abstract group G and the subcategory of all finite-dimensional representations in k-vector spaces where G acts by unipotent matrices. This is a Tannakian subcategory (sums and tensor products of unipotent representations are still unipotent). Its fundamental group wrt. the forgetful functor is then the solution to the universal problem of a group over which unipotent representations factor, and it is called the unipotent completion of G. It coincides with the Malcev completion because it satisfies the same universal property. In short:

$$\mathrm{Rep}^{un}(G) \simeq \mathrm{Rep}(G^{un}).$$

Example 251. The category $MTH(\mathbb{Q})$ carries at least two interesting fiber functors: deRham realization

$$\omega_{dR} : MTH(\mathbb{Q}) \to \mathrm{Vect}(\mathbb{Q}), \qquad (H, H_{dR}, W_\bullet, F^\bullet, \alpha) \mapsto H_{dR})$$

and Betti realization

$$\omega_B : MTH(\mathbb{Q}) \to \mathrm{Vect}(\mathbb{Q}), \qquad (H, H_{dR}, W_\bullet, F^\bullet, \alpha) \mapsto H.$$

There is a comparison isomorphism over $\mathbb{C}$, i.e. a $\mathbb{C}$-point of $I(\omega_{dR}, \omega_B)$, given by $\alpha : H_{dR} \otimes \mathbb{C} \cong H \otimes \mathbb{C}$.

Upper bounds on periods

Definition 252. Let K/k be a field extension, $S \in \mathcal{C}$ an ind-object in a Tannakian category $\mathcal{C}$ with two fiber functors $\omega, \eta : \mathcal{C} \to \mathrm{Vect}(k)$ and a point $p \in I(\omega, \eta)(K)$. Then a *period* of this data is an element of the k-vector space $P \subset K$ of *periods* generated by the numbers $\langle \alpha, p^\vee \beta \rangle$ for all $\alpha \in \omega(S)$ and all $\beta \in \eta(S)^\vee$.

We can use this to define various maps, in particular the main actor of this talk:

Definition 253. Let S be an object in $\mathcal{C}$. We define a k-linear map

$$\psi : \omega(S) \otimes_k \eta(S)^\vee \to \mathcal{O}(I(\omega, \eta))$$

which assigns to every $\gamma \otimes \sigma \in \omega(S) \otimes_k \eta(S)^\vee$ the function on $I(\omega, \eta)$ that sends a point $p \in I(\omega, \eta)(K)$ to the value of $\gamma \otimes p^\vee(\sigma) \in \omega(S)_K \otimes_K \omega(S)_K^\vee$ under the canonical pairing (i.e. evaluation map).

Proposition 254. *Periods $P \subset K$ for fixed $(k, K, \mathcal{C}, S, \omega, \eta, p)$ are a quotient of the subset of the Hopf algebra $\mathcal{O}(I(\omega, \eta))$ which is generated by the image of $\omega(S) \otimes_k \eta(S)^\vee$ under ψ.*

Proof. With the point $p \in I(\omega, \eta)(K)$ we can define an evaluation map $p^* : \mathcal{O}(I(\omega, \eta)) \to K$, whose concatenation with ψ gives a generating set for the period k-vector space $P \subset K$ of $(k, K, \mathcal{C}, S, \omega, \eta, p)$. The evaluation map is k-linear and the map $\mathrm{Span}_k \psi\left(\omega(S) \otimes_k \eta(S)^\vee\right) \to P$ is surjective by definition of P. $\square$

Lemma 255. *Let $c : K \to K$ be a field involution over k, assume* char $k \neq 2$. *Suppose c extends (not necessarily uniquely) to an involution $\tilde{c}$ of $I(\omega, \eta)$ over k that commutes with $p : \mathrm{Spec}\ (K) \to I(\omega, \eta)$, i.e. $c \circ p^* = p^* \circ \tilde{c}$. Then we have not only the c-fixed periods P^c, but also the $\tilde{c}$-fixed space $\mathcal{O}(I(\omega, \eta))^{\tilde{c}}$, and $\mathcal{O}(I(\omega, \eta))^{\tilde{c}} \to P^c$ is still surjective.*

In particular, The c-fixed periods P^c are a subquotient of $\mathcal{O}(I(\omega, \eta))^{\tilde{c}}$.

Proof. Let $x \in P^c$, then there is a preimage $y \in \mathcal{O}(I(\omega, \eta))$, and $(y + \tilde{c}(y))/2 \in \mathcal{O}(I(\omega, \eta))^{\tilde{c}}$ is a preimage of x (which fails for characteristics 2). $\square$

In the application, the involution c of $I(\omega, \eta)$ will be complex conjugation on the $\mathbb{C}$-points of varieties, which is a natural choice.

Pro-unipotent groups

Now we work only in characteristics 0 (since unipotent groups in positive characteristics behave worse than in characteristics 0). Furthermore, we assume in this section the fundamental group G_ω of our Tannakian category $(\mathcal{C}, \omega)$ to be pro-unipotent.

Lemma 256. *The fundamental group G_ω of a neutral Tannakian category $(\mathcal{C}, \omega)$ is a pro-unipotent algebraic group if and only if every object $S \in \mathcal{C}$ has a filtration such that the subquotients are isomorphic to the unit object $1 \in \mathcal{C}$.*

Proof. If every object has such a filtration, then in particular $\mathcal{O}(G_\omega)$ has one, and therefore is pro-unipotent.

If, on the other hand, G_ω is pro-unipotent, then every S has a unipotent filtration, and this can be refined to one with the properties of the lemma. $\square$

Lemma 257. *Whenever a morphism of pro-unipotent groups $f : G' \to G$ is surjective on $H_1(-, k)$, it is already surjective.*

Proof. Since $H_1(G,k)^\vee \simeq H^1(G,k)$, the fact that $H_1 f$ is surjective implies that $H^1 f$ is injective. Also, surjectivity of f is equivalent to injectivity of $f^\natural : \mathcal{O}(G) \to \mathcal{O}(G')$.

We know from a previous talk that $V := H^1(G,k) \simeq \mathrm{gr}_1^N \mathcal{O}(G)$, since G is pro-unipotent. There is a canonical morphism $\mathrm{gr}_\bullet^N \mathcal{O}(G) \hookrightarrow T(\mathrm{gr}_\bullet^1 \mathcal{O}(G)) = T(V)$, which commutes with the morphisms $\mathrm{gr}_\bullet f$ and $T(\mathrm{gr}_\bullet^1 f^\natural)$, so the former is injective:

$$\begin{tikzcd}
\mathrm{gr}_\bullet^N \mathcal{O}(G) \arrow[r, "\mathrm{gr}_\bullet f^\natural"] \arrow[d, "\mathrm{can}"] & \mathrm{gr}_\bullet^N \mathcal{O}(G') \arrow[d, "\mathrm{can}"] \\
T(V) \arrow[r, "T(\mathrm{gr}_1 f^\natural)"'] & T(V')
\end{tikzcd}$$

The result follows from the fact that a morphism which is injective on $\mathrm{gr}_\bullet$ is already injective, since the kernel has $\mathrm{gr}_\bullet = 0$, so it must vanish as well. $\square$

Proposition 258. *Suppose $V := \mathrm{Ext}^1_{\mathcal{C}}(1,1)$ has k-dimension $r < \infty$ and $\mathrm{Ext}^2_{\mathcal{C}}(1,1) = 0$. Then there is an isomorphism of Hopf algebras $\mathcal{O}(G_\omega) \cong T(V)$.*

Proof. The proof proceeds in two steps: First we construct a surjective morphism $\alpha : \mathrm{Spec}\, T(V) \to G_\omega$, then we show it splits and must be an isomorphism.

Step 1: Construction of α

We remember that $T(V)$ is the Hopf algebra of the pro-unipotent completion of a free group in r generators.

Choose $\gamma_1, \dots, \gamma_r \in G_\omega(\mathbb{Q})$ minimal such that their image is a basis for

$$G_\omega(\mathbb{Q})^{ab} = H_1(G_\omega) = H^1(G_\omega; k)^\vee = V^\vee.$$

Then we have a map from the free group generated by the γ_i to $G_\omega(\mathbb{Q})$. From the universal property of the pro-unipotent completion of this free group we get a morphism $\alpha : \langle \gamma_1, \dots, \gamma_r \rangle^{un} \to G_\omega$ (which depends on the choice of the γ_i). It is also surjective on H_1 by construction, so from the previous lemma, it is surjective.

Step 2: α splits and is an isomorphism

From $0 = \mathrm{Ext}^2_{\mathcal{C}}(1,1) = H^2(G_\omega; k)$ classifying the extensions of G_ω by $\mathbb{G}_a$, we see that all these must be split extensions.

Since $\mathrm{Spec}\, T(V)$ is pro-unipotent, we have an extension

$$U \hookrightarrow \mathrm{Spec}\, T(V) \twoheadrightarrow G_\omega$$

with U pro-unipotent as well. We look at the abelianization of U

$$U/[U,U] = \mathbb{G}_a^m \hookrightarrow \mathrm{Spec}\, T(V)/[U,U] \twoheadrightarrow G_\omega$$

and this extension splits, giving a section $s : G_\omega \to \mathrm{Spec}\, T(V)/[U,U]$. Since $s \circ \alpha$ on $\{\gamma_1, \dots, \gamma_r\} \subset \mathrm{Spec}\, T(V)$ is the identity, s is an isomorphism, and in particular $U/[U,U] = 0$, so $U = 0$ and we're finished. $\square$

Semi-direct product of a pro-unipotent group with Gm

From now on, we assume that $\mathcal{C}$ is generated by extensions of tensor powers of a fixed rank one object L (think: line bundle) and its dual $L^{-1} := L^\vee$. Note that G_ω is no longer pro-unipotent, but we will show that it is still almost pro-unipotent.

In other words: every object $S \in \mathcal{C}$ carries an increasing filtration $w_n S$, $n \in \mathbb{Z}$, whose nth adjoint quotient $gr_n^w = w_n S / w_{n-1} S$ is a direct sum of several copies of $L^{\otimes(-n)}$. We assume this filtration to be exact (in particular, gr_n^w is exact), respecting morphism in $\mathcal{C}$ and the tensor structure of $\mathcal{C}$, in particular $\mathrm{Hom}_{\mathcal{C}}(1, L^{\otimes n}) = 0$ for all $n \neq 0$ and

Proposition 259. $\mathrm{Ext}^1_{\mathcal{C}}(1, L^{\otimes(-n)}) = 0$ *for* $n \geq 0$

Proof. Let $L^{\otimes(-n)} \hookrightarrow S \twoheadrightarrow 1$ be an extension, then apply gr_0^w to get $\mathrm{gr}_0^w S \simeq 1$, apply gr_n^w to get $\mathrm{gr}_n^w S \simeq L^{\otimes(-n)}$. We have $1 = w_0 S \hookrightarrow S$ and $L^{\otimes(-n)} \hookrightarrow S$, so we can form the direct sum and get an exact sequence

$$\ker \hookrightarrow w_0 S \oplus L^{\otimes(-n)} \to S \twoheadrightarrow \mathrm{coker}$$

where ker is pure of weight n and coker is pure of weight 0. Applying gr_n and gr_0 to the sequence show then that $\ker = 0$ and $\mathrm{coker} = 0$, so the extension splits. □

Definition 260. In this setting, one has a *canonical fiber functor*

$$\omega : S \mapsto \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}\left(L^{\otimes(-n)}, \mathrm{gr}_n^w S\right)$$

from $\mathcal{C}$ into $\mathbb{Z}$-graded k-vector spaces. This defines a dual morphism of Tannakian fundamental groups $\mathbb{G}_m \to G_\omega$.

Proof. (that it is indeed a fiber functor) The functor ω is k-linear, since gr_n^w, covariant Hom and $\bigoplus$ are k-linear. It is a $\otimes$-functor, since the weight filtration respects the $\otimes$-structure. It is also exact, since for $S' \hookrightarrow S \twoheadrightarrow S''$ we have $\mathrm{Ext}^1_{\mathcal{C}}(L^{\otimes(-n)}, \mathrm{gr}_n^w S') = 0$ □

Lemma 261. *The corresponding fundamental group has the form $G_\omega \simeq \mathbb{G}_m \rtimes U$ with U a pro-unipotent group. The category $\mathcal{C}$ is equivalent to the category of graded comodules over $\mathcal{O}(U)$.*

Proof. Look at $\mathcal{C}' \subset \mathcal{C}$, the subcategory $\otimes$-generated by L. It has no non-trivial extensions, and a natural grading, given by the tensor powers of L that appear. This makes it a category of $\mathbb{G}_m$-representations, which gives us a dual morphism of Tannakian fundamental groups $G_\omega \twoheadrightarrow \mathbb{G}_m$, with kernel U a pro-unipotent group, since L is the trivial $\mathbb{G}_m$-module, and $\mathcal{C}$ contains all iterated extensions of this trivial $\mathbb{G}_m$-module. The morphism induced by ω splits $G_\omega \twoheadrightarrow \mathbb{G}_m$, since ω is a retraction of the full inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$. This shows $G_\omega \simeq \mathbb{G}_m \rtimes U$. □

Example 262. Let $\mathcal{C} = MTH(k)$ and $L = \mathbb{Q}(1)_H$, with corresponding weight filtration $w_n = W_{2n}$. Then the canonical fiber functor coincides with the deRham realization functor $\omega = \omega_{dR}$, which amounts to

$$H^n_{dR} = \mathrm{Hom}_{MTH(k)}(\mathbb{Q}(-n), \mathrm{gr}^W_{2n} H),$$

which is a reformulation of the definition, that the n-th weight-graded part of H is a pure Hodge structure H^n_{dR} over k.

We have $G_H = \mathbb{G}_m \rtimes U_H$ and $MTH(k)$ is equivalent to the category of graded $\mathcal{O}(U_H)$-comodules.

Definition 263. We consider the grading on the tensor algebra $T(V)$ of the graded vector space $V = \bigoplus_{n>0} V_n$ with $V_n = \mathrm{Ext}^1_{\mathcal{C}}(1, L^{\otimes n})$ to be such that $v \otimes w$ has degree $|v| + |w|$ (rather than $|v| + |w| + 2$, which would also give a graded algebra).

We would love to have something like this (which, by the way, doesn't make any sense):

Lemma 264. *The graded Hopf algebra $T(V)$ is the universal graded pro-unipotent group such that every morphism of graded groups from a free group in r_n generators of degree n into the $\mathbb{Q}$-points of a graded pro-unipotent group G induces a morphism of graded pro-unipotent groups $T(V) \to G$.*

Instead, we have to express what we need in terms of Lie algebras, to get a graded version:

Lemma 265. *Let U be a graded pro-unipotent group with finitely many nonzero graded components. Then there is a surjective graded morphism* $\mathrm{Spec}\, T(V) \twoheadrightarrow U$.

Let U be a graded pro-unipotent group, then there is a surjective graded morphism $\mathrm{Spec}\, T(V) \twoheadrightarrow U$.

Proof. The second statement follows from the first by a limit process, where we use $\mathrm{Spec}\, T(V) = \lim \mathrm{Spec}\, T(\bigoplus_{i=0}^n V_i)$ and $U = \lim U_n$ is the standard limit description.

The first statement comes from a graded re-statement of the last result of talk 3:

Let $F(V^\vee)$ be the free Lie algebra on $V^\vee$, where $V = H^1(U) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. We take a lift of $V^\vee$ to $\mathfrak{g}$, called $\tilde{V^\vee}$ (that is a choice, the same choice that the $\gamma_1, \ldots, \gamma_r$ were before). So we get a morphism of graded Lie algebras $F(\tilde{V^\vee}) \to \mathfrak{g}$, which induces a morphism of enveloping algebras $\mathcal{U}(F(\tilde{V^\vee})) \to \mathcal{U}(\mathfrak{g})$. The latter one is a pro-unipotent Lie algebra, so the map factors through the pro-unipotent completion to a map $\mathcal{U}(F(\tilde{V^\vee}))^\wedge \to \mathcal{U}(\mathfrak{g})$. Dualizing gives us $T(V) \leftarrow \mathcal{O}(G)$ - as graded Hopf algebras, since everything respected the grading. $\square$

Proposition 266. *Assume for any n, the k-vector space V_n has finite dimension r_n and all $\mathrm{Ext}^2_{\mathcal{C}}(1, L^{\otimes n})$ vanish. Then $r_n = 0$ for $n \leq 0$ and $V := \bigoplus_{n>0} V_n$ is a graded vector space whose graded tensor algebra $T(V)$, with V_n put in degree n, such that we have an isomorphism of graded Hopf algebras*

$$\mathcal{O}(U) \cong T(V) = T\left(\bigoplus_{n>0} \mathrm{Ext}^1_{\mathcal{C}}(1, L^{\otimes n})\right).$$

Proof. Same argument as before, now with grading:

We get a morphism of graded pro-unipotent groups $T(\bigoplus_{n>0} V_n) \twoheadrightarrow U$. It splits, as before, and we have an isomorphism of graded Hopf algebras. $\square$

Periods in the graded pro-unipotent case

Now we assume, in addition to the canonical fiber functor ω, to have a fiber functor η.

Suppose the involution $\tilde{c} : I(\omega, \eta) \to I(\omega, \eta)$ is given by the action of an order 2 element $\epsilon \in G_\eta$ (with respect to the G_η-torsor structure on $I(\omega, \eta)$).

Lemma 267. *The element $\epsilon \in G_\eta$ is conjugate to $-1 \in \mathbb{G}_m \subset G_\eta$.*

Proof. Look at the commutative diagram

$$\begin{tikzcd}
G_\omega \times I(\omega,\eta) \arrow[r, "\text{action}"] \arrow[d, "ad(\xi)\times\xi\cdot"'] & I(\omega,\eta) \arrow[d, "\xi\cdot"] \\
G_\omega \times I(\omega,\eta) \arrow[r, "\text{action}"'] & I(\omega,\eta)
\end{tikzcd}$$

for $\xi \in G_\omega$ any element. It commutes, since $\xi x \xi^{-1} \xi = \xi x$. This shows that conjugation in G_η corresponds to multiplication in $I(\omega, \eta)$.

In $G_\eta = \mathbb{G}_m \rtimes U$, the multiplication is

$$(\lambda, u) \cdot (\lambda', u') = (\lambda\lambda', u\lambda u' \lambda^{-1})$$

so if $\epsilon = (\lambda, u)$, we can multiply with $(-\lambda^{-1}, \lambda^{-1}u^{-1}\lambda)$ to get

$$\epsilon \cdot (-\lambda^{-1}, \lambda^{-1}u^{-1}\lambda) = (-1, 1).$$

□

Definition 268. The filtration on S defines a filtration on $\omega(S)$. Putting a trivial filtration on $\eta(S)^\vee$, this gives a filtration on $\omega(S) \otimes_k \eta(S)^\vee$ and thus a filtration on periods P. This filtration induces a filtration on the c-fixed points P^c.

Definition 269. Let $\mathcal{O}(I(\omega, \eta))_+ \subset \mathcal{O}(I(\omega, \eta))$ be the subspace generated (as k-algebra) by the image of all $\omega(S) \otimes_k \eta(S)^\vee$ under ψ, for S of positive weight, i.e. $\mathrm{gr}^w_n S = 0$ for all $n < 0$.

Theorem 270.

- *The involution $\tilde{c}$, resp. $\epsilon \in G_\eta$, respects the subspace $\mathcal{O}(I(\omega, \eta))_+$.*
- *The real periods P^c, as filtered k-vector space, are the image of $\mathcal{O}(I(\omega, \eta))^\epsilon_+$ under $p^* : \mathcal{O}(I(\omega, \eta)) \to K$.*
- *There is a graded Hopf algebra isomorphism $\mathcal{O}(I(\omega, \eta))^\epsilon_+ \simeq k[t^2] \otimes_k \mathcal{O}(U)$.*
- *In particular, P^c is a quotient of $k[t^2] \otimes_k T\left(\bigoplus_{n>0} \mathrm{Ext}^1_{\mathcal{C}}(1, L^{\otimes n})\right)$.*

Proof. The first is almost by definition, since any $\xi \in G_\eta$ respects the image of $\omega(S) \otimes \eta(S)^\vee$ in $\mathcal{O}(I(\omega, \eta))$, in particular those with only positive weights. In other words: translations respect the subalgebra of positive weights. The second statement also follows from this, by the definition of P. From our knowledge about $\mathcal{O}(U)$, we only need to prove the third statement, to get the fourth one. We will do this in four steps.

Step 1: $\omega = \eta$

First, we know that all G_ω-torsors are trivial (since $H^1(k, \mathbb{G}_a) = 0 = H^2(k, \mathbb{G}_m)$ and G_ω is an iterated extension of $\mathbb{G}_a$ and $\mathbb{G}_m$) so we know $\omega = \eta$.

Step 2: the positive part

Now we have the morphism

$$\mathbb{G}_m \times U \to G_\omega,\ (a, u) \mapsto a \cdot u$$

which defines an isomorphism of the corresponding graded k-algebras

$$\mathcal{O}(G_\omega) \cong k[t, t^{-1}] \otimes_k \mathcal{O}(U),$$

the grading on $\mathcal{O}(G_\omega)$ being induced by right translations of $\mathbb{G}_m \subset G_\omega$.

Since we consider only S with $gr^n_W(S) = 0$ for $n < 0$, the action of G_ω on the image of $\omega(S) \otimes \eta(S)^\vee$ factors through the monoid Spec $(k[t] \otimes_k \mathcal{O}(U)) = \mathrm{Ao}_k \times_k U$. Put differently, $\mathcal{O}(I(\omega, \eta))_+ \simeq k[t] \otimes_k \mathcal{O}(U)$, as k-algebra.

Step 3: invariants under conjugation

Since $\mathcal{O}(\mathbb{G}_m)^{-1}$, the invariants of $\mathcal{O}(\mathbb{G}_m)$ under -1, is isomorphic to $k[t^2, t^{-2}]$, we have $\mathcal{O}({}_{-1}\backslash G_\omega) = k[t^2, t^{-2}] \otimes_k \mathcal{O}(U)$.

Step 4: putting stepts 2 and 3 together

We have $\mathcal{O}(I(\omega, \eta))_+ \simeq k[t] \otimes_k \mathcal{O}(U)$ and $\mathcal{O}(I(\omega, \eta))^\epsilon \simeq k[t^2, t^{-2}] \otimes_k \mathcal{O}(U)$ and altogether we get

$$\mathcal{O}(I(\omega, \eta))^\epsilon_+ \simeq k[t^2] \otimes_k \mathcal{O}(U). \qquad \square$$

Motivic structure: Mixed Tate motives over the integers

Martin Gallauer on September 6th, 2012.

Contruction of the category of rational motives

Let k be a field, and let Sm/k be the category of smooth schemes over k.

Rough idea behind the construction

We would like to produce categories and functors to produce the diagram.

$$\begin{tikzcd}
\mathrm{Sm}/k \arrow[r, dotted] \arrow[d, "T"] & \mathrm{MM}(k) \arrow[dl, dotted, "\text{realization}"] \arrow[d, dotted, "\text{realization}"] \\
A \arrow[d] & \mathrm{D}^b(\mathrm{MM}(k)) \\
\mathrm{D}^b(A) &
\end{tikzcd}$$

Step 1: Correspondences

Definition 271 (Elementary correspondence)**.** Let $X, Y \in \mathrm{Sm}/k$ and let $\alpha \subset X \times Y$ be a closed integral variety in the product. We call α a *elementary correspondence* if it the projections onto the connected components of X are finite and surjective.

Definition 272 (Finite correspondence)**.** An element of the $\mathbb{Q}$-vector space

$$c(X,Y) = \mathbb{Q}[\{\alpha \subset X \times Y \mid \alpha \text{ is an elementary correspondence}\}$$

i.e., a $\mathbb{Q}$-linear combination of elementary correspondences.

Example 273. The graph Γ_f of a morphism of schemes, $f : X \to Y$, is a finite correspondence.

Remark 274. When Grothendieck was originally considering motives, he defined correspondences without the above restrictions. Composition could be motivated as "composition of multi-valued functions", and relies on the ability to integrate the values associated to a particular $x \in X$. But if these values do not form a finite set (or at least a compact one), there is no clear way to compose them. Here we restrict to finite correspondences as Voevodsky did.

Definition 275 (Composition of correspondences)**.** Let $\alpha \subset X \times Y$ and $\beta \subset Y \times Z$ be elementary correspondences. Then their composition is

$$\beta \circ \alpha := (\alpha \times Z) \cdot (X \times \beta)$$

Definition 276 (Category of smooth correspondences)**.** We denote by $\mathrm{SmCorr}(k)$ the category of smooth correspondences. Its objects are exactly the objects of Sm/k and for $X, Y \in \mathrm{Sm}/k$, them morphisms from X to Y are the finite correspondences

$$Mor(X, Y) := c(X, Y).$$

Remark 277.

- The category of smooth correspondences $\mathrm{SmCorr}(k)$ is $\mathbb{Q}$-linear tensor category. Its sum is the coproduct, $\coprod$, and its zero object is the empty set, $\emptyset$. The tensor product is the direct product $\times$.
- The functor $\mathrm{Sm}/k \to \mathrm{SmCorr}(k)$ is a tensor functor.

Step 2: Triangulation

Definition 278. Let $K^b(k)$ be the category of bounded complexes in $\mathrm{SmCorr}(k)$ up to chain homotopy. In other words, complexes

$$\cdots \to 0 \to X_i \to X_{i+1} \cdots \to X_j \to 0 \to \cdots$$

of smooth varieties over k with finite correspondences for morphisms which eventually are eventually 0 to both the left and the right. Two complexes, X and Y, define the same object in $K^b(k)$ if there are chain morphisms $\alpha : X \to Y$ and $\beta : Y \to X$ such that $\alpha \circ \beta \sim 1_Y$ and $\beta \circ \alpha \sim 1_X$, where the homotopy equivalence is given by chain homotopy, e.g., a morphism $\partial : X \to X[-1]$ such that

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & X_{-1} \arrow[r, "d"] \arrow[d, shift right, "\beta\circ\alpha"'] \arrow[d, shift left, "1_X"] & X_0 \arrow[r, "d"] \arrow[d, shift right, "\beta\circ\alpha"'] \arrow[d, shift left, "1_X"] \arrow[dl, "\partial"'] & X_1 \arrow[r, "d"] \arrow[d, shift right, "\beta\circ\alpha"'] \arrow[d, shift left, "1_X"] \arrow[dl, "\partial"'] & \cdots \arrow[dl, "\partial"'] \\
\cdots \arrow[r, "d"'] & X_{-1} \arrow[r, "d"'] & X_0 \arrow[r, "d"'] & X_1 \arrow[r, "d"'] & \cdots
\end{tikzcd}$$

such that

$$d\partial + \partial d = \beta \circ \alpha - 1_X$$

Remark 279. The category $K^b(k)$ is $\mathbb{Q}$-linear, tensor and triangulated. The distinguished triangles are of the form $A \to B \to \mathrm{cone}(f)$.

Step 3: Impose relations

Definition 280. Let $K^b(k)_{\mathbb{A}^1,MV}$ be the category $K^b(k)$ with the following relations imposed:

- The structure morphism
$$[A^1_X] \xrightarrow{\sim} [X]$$
is an isomorphism.
- If $X = U \cup V$, then

$$\begin{tikzcd}
{[U \cap V]} \arrow[r, "\alpha"] & {[U] \oplus [V]} \arrow[r, "{(+,-)}"] \arrow[dr] & {[X]} \arrow[r] & \cdots \\
& & \mathrm{Cone}(\alpha) \arrow[u, "\cong"'] &
\end{tikzcd}$$

Remark 281. $K^b(k)_{\mathbb{A}^1,MV}$ is a $\mathbb{Q}$-linear triangulated tensor category.

Step 4: Pseudo-abelian Hull

Definition 282 (Pseudo-abelian category)**.** An additive category $(\mathcal{C}, \otimes)$ is called *pseudo-abelian* if for all objects $X \in \mathcal{C}$ and for all idempotents $p : X \to X \in \mathcal{C}$,

$$X = \ker(p) \oplus \ker(id - p)$$

Definition 283. Let
$$\mathrm{DM}^{\mathrm{eff}}(k) := \mathrm{Psa}(\mathrm{K^b(k)_{\mathbb{A}^1,MV}})$$
the pseudo-abelian hull of $K^b(k)_{\mathbb{A}^1,MV}$, which is the triangulated category of effective mixed motives.

Remark 284. (Balmer / Schlichtling) The category of effective mixed motives, $\mathrm{DM}^{\mathrm{eff}}(k)$, is a $\mathbb{Q}$-linear triangulated tensor category.

Let
$$M : \begin{array}{rcl} \mathrm{Sm}/k & \to & \mathrm{DM}^{\mathrm{eff}}(k) \\ X & \mapsto & (\cdots \to 0 \to [X] \to 0 \to \cdots) \end{array}$$
where $[X]$ lies in degree 0 of the complex. This is a tensor functor. We occasionally write $[X]$ for $M(X)$.

Example 285. The motive of the tensor identity is,

$$\mathbf{1}_{\otimes} = M(\mathrm{Spec}\ k) = \mathbb{Q}(0) =: \mathbb{Q}.$$

Example 286. To compute the motive of P^1, one examines its cohomology. We would like the notation to work out so that $\mathbb{Q}(1) = h_2\mathbb{P}^1$. Hence define,

$$\mathbb{Q}(1) := \widehat{M}(\mathbb{P}^1)[-2].$$

Then,

$$M(\mathbb{P}^1) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2].$$

Definition 287 (Reduced motive)**.**

$$\widehat{M}(X) := [X_0 \to \text{Spec } k]$$

$$\widehat{M}(X) \to M(X) \to \mathbb{Q}(0) \to +d.t.$$

We call $\widehat{M}(X)$ the *reduced motive* of X.

Let $x : \text{Spec } k \to \mathbb{P}^1$ be a rational point. Then the composition

$$p : \mathbb{P}^1 \to \text{Spec } k \xrightarrow{x} \mathbb{P}^1$$

is idempotent, i.e., $p \circ p = p$. Then

$$M(\mathbb{P}^1) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2]$$

$$\begin{tikzcd}
\widehat{\mathbb{P}^1} \arrow[r] & M(\mathbb{P}^1) \arrow[r] \arrow[d, equal] & \mathbb{Q}(0) \arrow[r] \arrow[d, equal] & + \\
{} \arrow[r] & M(\mathbb{P}^1) \arrow[r, "q"] & \mathbb{Q}(0) \arrow[r] & +
\end{tikzcd}$$

[Room for improvement]

Step 5: Towards rigidity

Definition 288. Let

$$\text{DM}(k) := \text{DM}^{\text{eff}}(k)[\mathbb{Q}(1)^{-1}]$$

Remark 289. The category $\text{DM}(k)$ is a $\mathbb{Q}$-linear triangulated tensor category. It remains pseudo-abelian.

Notation 290. Now M will denote the functor $(\text{Sm}/k) \to \text{DM}(k)$.

Basic properties

For the remainder, we assume char $k = 0$.

Functorial properties of the category of mixed motives

There exists an extension of M to $Vect/k \to \mathrm{DM}(k)$. MV d.h., lily invovea, Künneth formula, blow-up formula, projective akjskd.

Example 291. Consider $\mathbb{G}_m^n$. If $n = 1$, then there is a useful exact sequence

$$\widehat{M}(\mathbb{G}_m) \to \underbrace{\widehat{M}(\mathbb{A}^1) \oplus \widehat{M}(\mathbb{A}^1)}_{0} \to \widehat{M}(\mathbb{P}^1) \to 1 \qquad d.t.$$

Hence

$$\widehat{M}(\mathbb{G}_m) = \widehat{M}(\mathbb{P})[-1] = \mathbb{Q}(1)[1]$$

If $n \geq 1$.

$$M(\mathbb{G}_m^n) = \otimes_n M(\mathbb{G}_m) = \otimes_n(\mathbb{Q}(0) \oplus \mathbb{Q}(1)[1]) = \oplus_{i=0}^n \binom{n}{2} \mathbb{Q}(i)[i]$$

A nontrivial theorem of Voevodsky says the following.

Theorem 292. *If the endofunctor*

$$- \otimes \mathbb{Q}(1) : \mathrm{DM}(k) \to \mathrm{DM}(k)$$

is fullly faithful, then the inclusion

$$\mathrm{DM}^{\mathrm{eff}}(k) \hookrightarrow \mathrm{DM}(k)$$

is a full embedding.

$$v : \mathrm{DM}(k)^p \to \mathrm{DM}(k)$$

autobidioyl and motived $\mathrm{DM}(k)$ is rigid.

Notation 293. Let

$$\mathbb{Q}(-1) := \mathbb{Q}(1)^{-1} = \mathbb{Q}(1)^\vee$$

and given a mixed motive $M \in \mathrm{DM}(k)$, let

$$M(n) := M \otimes \mathbb{Q}(n) = M \otimes Q(\pm 1)^{\otimes n}$$

If X is a smooth projective variety over k, then the dual of its motive has the simple form

$$M(X)^\vee = M(X)(-d)[-2d].$$

In this case, $M(X)^\vee$ is a chydra object in $\mathrm{DM}(k)$.

Lemma 294.

$$M(X)^\vee \otimes M(X)^\vee \cong (M(X) \otimes M(X))^\vee \cong (M(X \times X)^{vee} \to M(X)^\vee)$$

[Room for improvement]

Realization functors

Example 295. There is a weight filtration and a Hodge filtration (tensor)

$$\omega_{dR} : \mathrm{DM}(k) \to \mathrm{D}^b(k - n + 1)$$

$$\begin{tikzcd}
\mathrm{Sm}/k \arrow[r] \arrow[d, "\text{dR-Hodge}"'] & \mathrm{DM}(k) \arrow[dl, "\omega_{dR}"] \\
\mathrm{D}^b(\mathrm{Var}/k) &
\end{tikzcd}$$

Let k be a subfield of $\mathbb{C}$ with embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then there is an associated functors

$$\omega_{B,\sigma} : \mathrm{DM}(k) \to \mathrm{D}^b(\mathbb{Q} - \mathrm{Mod})$$

Let ℓ be a prime. $\omega_\ell : \mathrm{DM}(k) \to \mathrm{D}^b(\mathbb{Q}_\ell - \mathrm{Mod})$ is a geometric realization. Action of Galois group $\omega_{dR} \otimes \mathbb{C} \cong \omega_{B,\sigma} \otimes \mathbb{C}$. Hence the Hodge realization

$$\omega_H : \mathrm{DM}(k) \to \mathrm{D}^b(\mathrm{MHC}(k))$$

Example 296. $\omega_H(\mathbb{Q}(1)) = \mathbb{Q}(1)_H$

Motivic cohomology

Definition 297 (Motivic cohomology)**.** Let X be a smooth projective variety over k. Then the motivic cohomology of X with $\mathbb{Q}$-coefficients is

$$H^{i,r}(X, \mathbb{Q}) := \mathrm{DM}(k)(M(X), \mathbb{Q}(r)[i])$$

A nontrivial theorem is

Proposition 298. *Let X be a smooth projective variety over k. Then*

$$H^{i,r}(X, \mathbb{Q}) = K_{2r-i}(X)^{(r)}_{\mathbb{Q}}$$

Example 299. Let $X = \mathrm{Spec}\ k$. Then

$$H^{i,r}(X, \mathbb{Q}) \overset{!}{=} 0 \quad \forall r \geq 0 \forall i < 0$$

Theorem 300 (Borel)**.** *Let $k = \mathbb{Q}$. Then*

$$K_{2r-1}(\mathbb{Q})^{(}_{\mathbb{Q}}r) = K_{2r-1}(k)_{\mathbb{Q}} = \begin{cases} \mathbb{Q}^\times \otimes \mathbb{Q} & r = 1 \\ \mathbb{Q} & r > 1 \ \textit{odd} \\ 0 & \textit{otherwise} \end{cases}$$

$$K_{2r}(\mathbb{Q})_{\mathbb{Q}} = 0$$

Hence $(B - S)_{\mathbb{Q}}$ is true.

Construction of the category of mixed Tate motives over the integers

Definition 301 (Mixed Tate motive)**.** A mixed Tate motive is an iterated extension[4] in $\mathrm{DM}(k)$ of motives of the form $\mathbb{Q}(n), n \in \mathbb{Z}$.

Definition 302. The category of mixed Tate motives, $\mathrm{MTM}(k)$, is the full subcategory of $\mathrm{DM}(k)$ spanned by mixed Tate motive.

Definition 303. Let $\mathrm{DTM}(k)$ be the full triangulated subcategory of $\mathrm{DM}(k)$ generated by mixed Tate motives.

Remark 304. Hence $\mathrm{MTM}(k) \subset \mathrm{DTM}(k) \subset \mathrm{DM}(k)$. While forming $\mathrm{MTM}(k)$, we only admit as objects the iterated extensions. For $\mathrm{DTM}(k)$, we require the objects to be closed under shifts and triangles. $\mathbb{G}_m$ is a distinguishing example:

$$M(\mathbb{G}_m) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[1] \notin \mathrm{MTM}(k) \quad (\text{but} \in \mathrm{DTM}(k))$$

Theorem 305. *If $(B-S)_k$, then*

- *There exists a t-structure on* $\mathrm{DTM}(k)$ *whose heart is* $\mathrm{MTM}(k)$*.*
- $\mathrm{MTM}(k)$ *is abelian and closed under extensions.*
- $\mathrm{D}^b(\mathrm{MTM}(k)) \to \mathrm{DMT}(k)$*.*
- *The $\mathbb{Q}$-structure on* $\mathrm{DM}(k)$ *restricts to* $\mathrm{MTM}(k)$ *and makes it rigid. More generally, the t-structure is compatible with the tensor structure.*
- *There exists a finite, functorial, increasing $\mathbb{Z}$-filtration, W, on* $\mathrm{DMT}(k)$ *which induces an exact filtration on $MTM(k)$.*

$$\mathrm{gr}^W_{2n} M = \oplus_{i=1}^N \mathbb{Q}(-n_i), \qquad \mathrm{gr}^W_n \otimes \mathrm{gr}^W_m = \mathrm{gr}^W_{n+m}$$

- *Let*
- *The functor*

$$\omega_W : \begin{array}{rcl} \mathrm{DMT}(k) & \to & k\text{-}Mod \\ \hline M & \mapsto & \oplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{MTM}(k)}(\mathbb{Q}(-n), \mathrm{gr}^W_{2n} M) \end{array}$$

is an exact, faithful, tensor functor.

[4]An iterated extension of $\mathbb{Q}(n)$ is a motive M which fits into an exact sequence $0 \to I \to M \to \mathbb{Q}(n) \to 0$ where I is an iterated extension of $\mathbb{Q}(n)$.

- *The* Ext*-groups satisfy*

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{MTM}(k)}(M,N) &= \mathrm{Hom}^1_{\underline{\mathrm{DMT}(k)}}(M,N)\\ \mathrm{Ext}^2_{\mathrm{MTM}(k)}(M,N) &\hookrightarrow \mathrm{Hom}^2_{\underline{\mathrm{DMT}(k)}}(M,N)\end{aligned}$$

where, for objects A and B in a triangulated category C, we define

$$\mathrm{Hom}^n_C(A,B) := \mathrm{Hom}_C(A,B[n]).$$

Corollary 306. *If $k=\mathbb{Q}$, then we can compute the* Ext*-groups as*

$$\mathrm{Ext}^1_{\mathrm{MTM}(k)}(\mathbb{Q}(0),\mathbb{Q}(n)) = \begin{cases}\mathbb{Q}^\times\otimes\mathbb{Q} & n=1\\ \mathbb{Q} & n\,odd\\ 0 & otherwise\end{cases}$$

$$\mathrm{Ext}^2_{\mathrm{MTM}(k)}(\mathbb{Q}(0),\mathbb{Q}(n)) = 0$$

Remark 307. The n-homology of a motive $M\in\mathrm{DTM}(k)$ is defined by the functor

$$H_n : \mathrm{DTM}(k)\to\mathrm{MTM}(k)$$

Definition 308. Let $f: Y\to X$. Then

$$M(X,Y) := M([Y_1\to X_0]\in\mathrm{DTM}(k)$$

Example 309 (Kummer motives)**.** Let $t\in k^*/\{1\}$.

$$\{1,t\}\hookrightarrow\mathbb{P}^1\setminus\{0,\infty\}$$

There is a short exact sequence in $\mathrm{MTM}(k)$,

$$0\to\mathbb{Q}(1)\to H_1(\mathbb{G}_m,\{1,t\})\to\mathbb{Q}(0)\to 0,$$

which gives a class in $\mathrm{Ext}^1_{\mathrm{MTM}(k)}(\mathbb{Q}(0),\mathbb{Q}(1))\in\mathrm{MTM}(k)$.

Realization

Let ℓ be a prime.

$$0\to\mathbb{Z}_\ell(1)\to E\to\mathbb{Z}/\ell^n\mathbb{Z}\to 0$$

$$0\to M_{\ell^n}(\overline{k})\to E_n\xrightarrow{f} Z/\ell^n\mathbb{Z}\to 0$$

The ℓ^n-th roots of t in $\overline{k}$.

Proposition 310. *The ℓ-adic realization is unramified at v if and only if $t\in k_v^*$.*

Mixed Tate Motives over $\mathbb{Z}$

Corollary 311. *The category* $(\mathrm{MTM}(\mathbb{Q}), \omega_1, \otimes ..)$ *is a neutral Tannakian category.*

Definition 312 (Mixed Tate Motives over $\mathbb{Z}$)**.** The category of mixed Tate motives over $\mathbb{Z}$, $\mathrm{MTM}(\mathbb{Z})$ is the full subcategory of $\mathrm{MTM}(\mathbb{Q})$ spanned by motive $\underline{M}$ such that for some subquotient N of M, there exists a short exact sequence,

$$0 \to \mathbb{Q}(n+1) \to N \to \mathbb{Q}(n) \to 0$$

Hence the sequence splits.

Why are these called mixed Tate motives over $\mathbb{Z}$? The problem was large Ext^1-groups, so here we take motives with trivial Ext^1-groups.

Remark 313.

- The category $\mathrm{MTM}(\mathbb{Z})$ is a Tannakian subcategory of $\mathrm{MTM}(\mathbb{Q})$.
- $$\mathrm{Ext}^1_{\mathrm{MTM}(\mathbf{Z})}(\mathbb{Q}, \mathbb{Q}(n)) = \begin{cases} \mathbb{Q}^* \otimes \mathbb{Q} & \text{n=1} \\ \mathbb{Q} & n > 1 \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$
- $\mathrm{Ext}^2_{\mathrm{MTM}(\mathbf{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0.$
- Let M be a motive in $\mathrm{MTM}(\mathbb{Q})$. Then M is also in the subcategory $\mathrm{MTM}(\mathbf{Z})$ if and only if for all primes p, there exists a prime $\ell \neq p$, such that $\omega_\ell(M)$ is unramified at p.

Example 314. Let X be a smooth scheme over k such that $M(X) \in \mathrm{MTM}(\mathbb{Q})$, where X has good reduction at all primes[5]. Then $M(X) \in \mathrm{MTM}(\mathbb{Z})$.

Applications

The fundamental group and periods

Let G_M be and algebraic group over Q such that (Cf. Konrad's talk)

$$\mathrm{Rep}(G_M) = (\mathrm{MTM}(\mathbb{Z}), \omega_{con}, \omega_{dR})$$

$$\omega_{con} \cong \omega_{dR}$$
$$\mathrm{gr}^W_{-2n} M = \bigoplus \mathbb{Q}(n)$$

ω_{2n} is exact, functorial and compatible with the tensor product. Hence $G_M = \mathbb{G}_m \ltimes U_M$ for a pro-unipotent group U_M and $\mathcal{O}(G_M) = \mathbb{Q}[t, t^{-1}] \otimes T(\bigoplus_{n>0} \mathbb{Q} e_{2n+1})$

$$\begin{tikzcd} \mathrm{MTM}(\mathbb{Q}) \arrow[rr, "\omega_H"] \arrow[dr, "\omega_{dR}"'] & & \mathrm{MTH}(\mathbb{Q}) \arrow[dl, "\omega_{dR}"] \\ & \mathbb{Q}\text{-Mod} & \end{tikzcd}$$

[5]A scheme X over $\mathrm{Spec}\ \mathbb{Z}$ has good reduction at all primes if...

We get a morphism of fundamental groups, and hence $G_H \to G_M$ (or equivalently, $\mathbb{G}_m \ltimes U_H \to \mathbb{G}_m \ltimes U_M$ is a Tannakian dual.

Degree-wise the associated morphism of Lie algebras $\mathrm{Lie} U_H \to \mathrm{Lie} U_M$ is dual to 1.

The morphism $H^1(U_M) \to H^1(U_H)$ is isomorphic to the Borel-Beilinson regulator. 257

Definition 315 (Borel-Beilinson regulator)**.**

$$\bigoplus_{n\in\mathbb{Z}} \left(\mathrm{Ext}^1_{\mathrm{MTM}(\mathbb{Z})}(\mathbb{Q}, \mathbb{Q}(n)) \to \mathrm{Ext}^1_{\mathrm{MTM}(\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}(n)_H)\right)$$

Since the regulator is injective, the morphism $G_H \to G_M$ is surjective.

$$\mathrm{Rep(G_M)} = \{\phi \in \mathrm{Rep}(G_H) \mid \phi \text{ factors through } G_M\}$$

So $\mathrm{Rep}(G_M)$ is a full subcategory of $\mathrm{Rep}(G_H)$ and closed under subquotients.

Upperbounds on periods

Let X be a variety over $\mathbb{Q}$. Complex conjugation defines an involutive automorphism of ω_B as a tensor functor

$$\mathrm{MTM}(\mathbf{Z}) \to \mathrm{D}^b(\mathbb{Q}\text{-Mod})$$

Hence there is an involution ϵ on $\underline{\mathrm{Isom}}^{\otimes}_{\mathbb{Q}}(\omega_{dR}, \omega_B)$

$$\begin{tikzcd}
\underline{\mathrm{Isom}}^{\otimes}_{\mathbb{Q}}(\omega_{dR}, \omega_B) \arrow[r, "\epsilon"] & \underline{\mathrm{Isom}}^{\otimes}_{\mathbb{Q}}(\omega_{dR}, \omega_B) \\
\mathrm{Spec}\ \mathbb{C} \arrow[u, "\mathrm{comp}"] \arrow[r, "c"] & \mathrm{Spec}\ \mathbb{C} \arrow[u, "\mathrm{comp}"']
\end{tikzcd}$$

From the previous talk, this is compatible with having the involution be...

By Theorem 270, the real periods, $P^c_{\mathrm{comp},t}$, filter a quotient of $\mathbb{Q}[t^2] \otimes T(\bigoplus_{n>0} \mathbb{Q}e_{2n+1})$.

$$\begin{aligned}
\phi &= \frac{1}{1-x^2} \sum_{i\geq 0} \frac{x^3}{1-x^3} \\
&= \frac{1}{1-x^2} \cdot \frac{1}{1-\frac{x^3}{1-x^2}} \\
&= \frac{1}{1-x^2} \cdot \frac{1-x^2}{1-x^2-x^3} \\
&= \frac{1}{1-x^2-x^3}
\end{aligned}$$

Hence

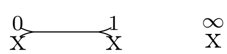
$$D_n = D_{n-2} + D_{n-3}, \qquad D_0 = D_2 = 1, D_1 = 0$$

Let $d_n := \dim_{\mathbb{Q}} \mathrm{gr}_n P^c_{\mathrm{comp},t}$. Hence $d_n \leq D_n$ for all n.

Motivic structure: Motivic fundamental group

Simon Pepin Lehalleur on September 6th, 2012.

- Let $X = \mathbb{P}^1_k \setminus D$ for $k \hookrightarrow \mathbb{C}$.
- Let $a, b \in X(k)$.
- $(\pi_1(X; a, b)_H)_{a,b}$ MH(k)-affine schemes and groupoid structure
- $x, y \in D(k), u \in T_x\mathbb{P}^1 \setminus \{0\}, v \in T_y\mathbb{P}^1 \setminus \{0\}$
- $\pi_1(X; a, u)_H$ and $\pi_1(X; u, v)_h$ MH($\mathbb{Q}$)-schemes with composition of paths.
- Multiple zeta values are real periods of



- $\pi_1(\mathbb{P}^1 \setminus \{0, 1\infty\}, u, v)$

Section 1

First, under hypothesis $k = \mathbb{Q}$, we lift all of these to MTM($\mathbb{Q}$).

Case of interior base points

The idea is to perform the bar construction on $M(X)^\vee$ to produce a motivic bar complex $B^*_M(X; a, b) \in \mathrm{Ind}(\mathrm{DM}(k))$. The idea meets the following obstructions.

1. DM(k) is too small. Therefore we truncate.
2. There is no total functor $Tot : K^b(\mathrm{DM}(k)) \to \mathrm{DM}(k)$.
3. Duality is only defined on DM(k). So we take duals last of all.

$$\mathrm{CPX}_*^{a,b} = \left[\begin{array}{ccccccc} \mathrm{Spec}\ k & \xrightarrow{d_0} & X & \xrightarrow{d_1} & X^2 & \xrightarrow{d_2} & \cdots \\ 0 & & -1 & & -2 & & \end{array}\right] \in K(\mathrm{SmCorr}(k))$$

where

$$d_0 = a - b$$

$$d_i = \sum_{j=0}^{i} (-1)^j d_i^j, \qquad d_i^j(x_1, \ldots, x_i) = \begin{cases} (a, x_1, \ldots, x_i), & j = 0 \\ (x_1, \ldots, x_j, x_j, \ldots, x_i), & 0 < j < i \\ (x_1, \ldots, x_i, b), & j = i \end{cases}$$

$$\sigma_{\geq -n}\mathrm{CPX}_*^{a,b} = [\mathrm{Spec}\ k \to X \to \cdots \to X^n] \in K^b(\mathrm{SmCorr}(k))$$
$$B_M^{\geq -n}(X; a, b) = M([\sigma_{1 \geq -n}\mathrm{CPX}_*^{a,b}])^\vee \in \mathrm{DM}(k)$$

Then

$$B_M(X; a, b) = \varinjlim_n B_M^{\geq -n}(X; a, b) \text{``=''} \cdots \to \overset{-2}{M(X^2)^\vee} \to \overset{-1}{M(X)^\vee} \to \overset{0}{\mathrm{Spec}\ k}$$

Operations

We need a coproduct which encodes the shuffle and a product which encodes the composition of paths.

Shuffle product

Because of v, we need various shuffle coproducts:

$$\sigma_{\geq -n}\mathrm{CPX}_*^{a,b} \to \sigma_{\geq -n}\mathrm{CPX}_*^{a,b} \otimes \sigma_{-n}\mathrm{CPX}_*^{a,b}$$

The shuffle coproduct before truncations:

$$\begin{array}{rcl} \mathrm{CPX}_*^{a,b} & \to & \mathrm{CPX}_*^{a,b} \otimes \mathrm{CPX}_*^{a,b} \\ (x_1, \ldots, x_n) & \mapsto & \sum_{\sigma \in \sigma_{p,q}} \mathrm{sgn}(\sigma)(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) \otimes (x_{\sigma(p,n)}, \ldots, x_{\sigma(n)})_{p+q=n} \end{array}$$

This is the same as Thomas's talk (Ref missing). Everywhere there was multiplication we apply diagonal map. Every place there was a coproduct, we apply the embedding of points into a variety.

Composition of paths

$$\begin{array}{rcl} \mathrm{CPX}_*^{a,b} \otimes \mathrm{CPX}_*^{a,b} & \to & \mathrm{CPX}_*^{a,b} \\ (x_1, \ldots, x_k) \otimes (g_1, \ldots, g_l) & \mapsto & \sum_{i=1} \begin{cases} (x_1, x_2, y_1, y_2) \\ (x_1, b, y_1, y_2) \\ (x_1, b, b, y_2) \\ (x_1, x_2, b, y_2) \end{cases} \end{array}$$

where the sum has to do with

$$\begin{array}{cccccccc} x & x & x & x & x & x & x & c \\ a & x_1 & x_2 & b & y_1 & y_2 & & \end{array}$$

Proposition 316. *Let $M(X) \in \mathrm{DTM}(k)$. Then $B_M^*(X;a,b) \in \mathrm{Ind}(\mathrm{DTM}(k))$.*

Proof. By induction on truncations. □

Proposition 317. *Assume $(B-S)_k$. Then there is a t-structure on* $\mathrm{DTM}(k)$

$$H_0 = H^0 : \mathrm{DTM}(k) \to \mathrm{MTM}(k)$$

(Here $H^n = H_{-n}$.)

Since we know the tensor product is exact on $\mathrm{DTM}(k)$...

$$H^0(B_M(X;a,b)) \in \mathrm{Ind}(\mathrm{MTM}(k))$$

Because of $\otimes$ t-exact and some arguments? Operations on $H^0(B_M^*(X;a,b))$.

Definition 318 (Motivic fundamental group)**.** Let X be a smooth variety over k. Then its motivic fundamental group is

$$\mathrm{Spec}\ (H^0 B_M^*(X;a,b))$$

The $M(X)^\vee$ has positive weights. Hence $\pi_1(X;a,b)$ has positive weights.

Realizations

We prove that $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, u, v)$ is an $\mathrm{MTM}(\mathbf{Z})$-scheme.

$$\omega_H : \mathrm{DTM}(k) \to \mathrm{D}^b(\mathrm{MTH}(k)) \qquad \omega_H \circ H^0 : \mathrm{MTM}(k) \to \mathrm{MTH}(k)$$

$$\begin{aligned} \omega_{H*}\pi_1(X;a,b) &= \mathrm{Spec}\ \left(H^0 \omega_H B_M^*(X;a,b)\right) \\ &= \varprojlim_n \mathrm{Spec}\ \left(H^0 \omega_H B_M^{\geq -n}(X;a,b)\right) \\ &= \varprojlim_n \mathrm{Spec}\ \left(H^0 \omega_M \left[M(\sigma_{\leq n}\mathrm{CPX}_*^{a,b})\right]^\vee\right) \end{aligned}$$

Then

$$X_0 \to X_{-1} \to \cdots \to X_{-n} \in K^b(\mathrm{SmCorr}(k))$$

Replace each X_i by $R\Gamma(X_i)$ and take the total complex.

$$\omega_H(\pi_1(X;a,b)_M) = \mathrm{Spec}\ H^0(B_H^*(X;a,b)) = \pi_1(X;a,b)_H$$

Brown's proof: Strategy of the proof

Sergey Gorchinskiy on September 7th, 2012.

Statements

Write $\zeta(2//3)$ to mean $\{\zeta(a_1, \ldots, a_n) \mid n \in \mathbb{N}, a_i = 2 \text{ or } 3\}$.

Conjecture 319 (Hoffman)**.** *The set $\zeta(2//3)$ forms a $\mathbb{Q}$-basis for $\mathcal{Z} = \langle \zeta(\overline{n}) \rangle_{\mathbb{Q}} \subset \mathbb{R}$. where $\overline{n} = (n_1, \ldots, n_r), n_r \geq 2$.*

Conjecture 320 (Brown's theorem)**.** *Let $\zeta(2//3)$ $\mathbb{Q}$-linearly generate $\mathcal{Z}$.*

Theorem 321. *The mixed Tate motive $cO(\pi_1(X; 0, 1)_M)$ generates* MTM(**Z**) *under the tensor product.*

Theorem 322. *The morphism $\mathcal{O}(I(dR, B))_+ \twoheadrightarrow \mathcal{Z}$ is a strict quotient of a filtered algebra (not just a subquotient).*

Question 323. Does Theorem 322 imply stronger upper bounds on d_n? For example

$$d_n \leq d_{n-2} + d_{n-3}$$

Motivic lift of Hodge class

Set-up

- $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$
- $\pi_1(X; 0, 1)$-motivic scheme of paths
- $\pi_1(X; 0, 1)_{dR} = T(\Omega)_1$.
- $\Omega = \langle \frac{dz}{z} = \omega^0, \frac{dz}{1-z} = \omega^1 \rangle_{\mathbb{Q}}$
- dch $\in \pi_1(X; 0, 1)_B(\mathbb{Q})$[6]

[6] Recall from Definition 243 that dch denotes the unit path $[0, 1]$.

- $$1 \longrightarrow U_M \longrightarrow G_M \rightleftarrows \mathbb{G}_m \longrightarrow 1$$

- $I := I(\omega_{dR}, \omega_B)$ and $\mathcal{O}(01)_{dR} := \mathcal{O}(\pi_1(X;0,1)_{dR}) \cong T(\Omega)$ are in $\mathrm{Ind}(\mathrm{Rep}(G_M))$

- $\mathrm{comp} \in I(\mathbb{C})$.

Definition 324. The path dch leads to the composition:

$$\begin{tikzcd} \mathcal{O}(01)_{dR} \arrow[r] \arrow[rr, bend right, "g"'] & \mathcal{O}(01)_{dR} \otimes \mathcal{O}(01)_B^\vee \arrow[r] & \mathcal{O}(I) \arrow[d, "\mathrm{comp}^*"] \\ & & \mathbb{C} \end{tikzcd}$$

Definition 325. Let

$$\mathcal{Z}_M := \mathrm{im}\ g$$

$\zeta_M(\overline{n}) \to \zeta(\overline{n}) := \mathrm{im}\ (\tau_0(\tau_1))$

Remark 326. All maps above are mixed motives of G_M representations. In particular, $\mathcal{Z}_M$ is a G_M-representation. Furthermore, all of the above vector spaces are canonically graded with respect to $\mathbb{G}_m \hookrightarrow G_M$.

- $\mathrm{comp}^*(\mathcal{Z}_M) = \mathcal{Z}$

- $\mathcal{Z}_M \subset \mathcal{O}(I)_+^\epsilon$ [7]

- $\epsilon \in G_{\omega_B}(\mathbb{Q})$ is given by complex conjugation and correspondence to non-negative weights using the fact that $\mathrm{dch} \subset X(\mathbb{R})$.

Remark 327. Let LI denote the proposition that $\zeta_M(2//3)$ is linearly independent in $\mathcal{O}(I)$. Then HC implies LI. Via upper bounds on $\mathcal{O}(I)_+$, this implies in turn Brown's theorem, as well as Theorems 321 and 322. Use that $\mathbb{Q}(-1)_M$ is a subquotient of $\mathcal{O}(01)$ by weight applied to $\mathcal{O}(0,1)_{dR}$.

Since $\mathcal{Z}_M$ is graded, it is enough to prove LI_n for each weight $n \geq 0$.

The guiding principle

$$\mathcal{Z}_M \xrightarrow{\mathrm{comp}^*} \mathcal{Z}$$

$$\mathcal{O}(I) \xrightarrow{\mathrm{comp}^*} \mathbb{C}$$

The left side is algebro-geometric and has functions. The right side is analytic and has numbers.

The Kontsevich-Zagier conjecture implies that comp^* is injective.

Examples of relations of multiple zeta values.

[7] The + denotes taking the positive degree functions.

Weight	Relations
0	1
1	0
2	$\zeta_M(2)$
3	$\zeta_M(3)$
4	$\zeta_M(2,2)$
5	$\zeta_M(3,2), \zeta_M(2,3)$

Kontsevich-Zagier implies that

$$\begin{pmatrix} \zeta_M(2,3) \\ \zeta_M(3,2) \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -2 \\ -\frac{11}{2} & 3 \end{pmatrix} \begin{pmatrix} \zeta_M(5) \\ \zeta_M(2,3,2) \end{pmatrix}$$

The Lie algebra $\mathfrak{u}_M := \text{Lie } U_M$ acts by derivations on regular functions in order to reduce to $\zeta_M(2), 1$. It is pro-nilpotent.

$$\begin{array}{rcl} \mathfrak{u}_M \to \mathfrak{u}_M^{ab} = \mathfrak{u}_M|_{[\mathfrak{u}_M,\mathfrak{u}_M]} & \cong & \coprod_{r\geq 1} \text{Ext}_{\text{MTM}(\mathbf{Z})}(\mathbb{Q}(0), \mathbb{Q}(2r+1))^\vee \\ \partial_{2r+1} & \mapsto & \mathbb{Q}[\partial_{2r+1}] \end{array}$$

$$\text{comp}^*(\zeta_M(2,\ldots,2)) = \zeta(\underbrace{2,\ldots,2}_{m}) \sim \pi^{2m}$$

We need to derviate $\zeta_M(2,\ldots,2)$. Apply Kontsevich-Zagier formula again to infer that $\zeta_M(\underbrace{2,\ldots,2}_{n})$ is a function of the period of $\mathbb{Q}(2n)$.

As $\mathbb{Q}(1)_{dR}$ is a G_M representation factors through $G_M \to \mathbb{G}_m$. Hence $\partial_{2r+1}(\zeta(2,\ldots,2)) = 0$.

Aside on algebraic groups

Definition 328 (Linear function)**.** Let G be an algebraic group acting algebraically on a scheme X over k. Then a function $f \in \mathcal{O}(X)$ is *linear* with respect to G if for all $\partial \in \text{Lie }(G)$, $\partial(f)$ is constant on X.

For example, if $G = \mathbb{G}_a$ acts on $X = \mathbb{G}_a$, then linear functions g are those such that $dg \leq 1$.

Remark 329. Any linear $f \in \mathcal{O}(X)$ defines a morphism $X_f : g \to k$ out of χ_f. We get

$$0 \to 1 \to V_f \to 1 \to 0 \qquad V_f = \begin{pmatrix} 0 & \chi_f \\ 0 & 0 \end{pmatrix}$$

where V_f are the representations of g.

Lemma 330. *Let* $\mathcal{C} \overset{\omega}{\underset{\eta}{\rightrightarrows}} \text{Vect}(k)$.

$$0 \to L_1 \to S \to L_2 \to 0$$

rk $L_i = 1$*,* $S \in \mathcal{C}$*,* $f \in \mathcal{O}(I(\omega,\eta))$ *any period of* S *of kind* $\begin{pmatrix} * & f \\ 0 & * \end{pmatrix}$*. Then* f *is linear with respect to* $U = r_U(G_M)$*, where* r_U *denotes the unipotent radical (Cf. Definition 22). In addition,* $V_f \cong \omega(S)$ *as* U*-representations.*

Borel's result on ζ-values as regulators and the Borel-Beilinson comparison theorem together imply that the periods of $M_{2r+1} = \begin{pmatrix} (2\pi i)^{2r+1} & \zeta(2r+1) \\ 0 & 1 \end{pmatrix}$

Kontsevich-Zagier and the lemma together imply that $\zeta_M(2r+1)$ is linear with respect to U_M and

$$\partial_{2r+1}(\zeta_M(2s+1)) \sim \delta_{r,s}$$

Why is this proportional? Take the linear function ζ it gives you an extension which gives you a character of the Lie algebra, and mapping over to the Ext group, we see that it vanishes.

So we've deduced three formulas from Kontsevich-Zagier: Equations , and .

Application

$$(\partial_\epsilon + \partial_s) \begin{pmatrix} \zeta_M(2,3) \\ \zeta_M(3,2) \end{pmatrix} = A \begin{pmatrix} 1 \\ \zeta_M(2) \end{pmatrix}$$

where $A := \begin{pmatrix} 3/2 & -2 \\ -11/2 & 3 \end{pmatrix}$ is the matrix of . The matrix A is invertible. Indeed, we may multiple the first column by 2. Then it is an integer matrix. Examine it modulo 2 and it becomes lower triangle with ones on the diagonal, and hence it is invertible. This will be a general tactic for see that a matrix is invertible.

"So", $\zeta_M(2,3)$ and $\zeta_M(3,2)$ are linearly independent.

After applying these derivations we are still in the subspace generated by $\zeta_M(3)$. By the way if you apply the derivations to $(1\zeta_M(2))^T$, you get zero.

Strategy of the proof

Definition 331. We define

$$3_\ell \mathcal{Z}_M := \langle \zeta_M(2//3) \mid \text{at most } \ell \text{ threes appear} \rangle$$

Remark 332. The filtration 3_ℓ commutes with W_n.

- $3_\ell \mathcal{Z}_M$ are stable under $\mathfrak{u}_M \ni \partial = \sum_{k=1}^{\infty} \partial_{2k+1}$.
- $\partial : \mathrm{gr}_e^{3_\ell} \mathcal{Z}_M \to \mathrm{gr}_{e-1}^{3_\ell} \mathcal{Z}_M$. But we need to prove the dependency in each (weight) degree. So consider the n-part. They all have negative weights, so in degree n we have

$$(\mathrm{gr}_e^{3_\ell} \mathcal{Z}_M)_n \xrightarrow{\partial} \bigoplus_{r \geq 1} \left(\mathrm{gr}_{e-1}^{3_\ell} \mathcal{Z}_M\right)_{n-(2r+1)}$$

Explicit bases:

$$\begin{aligned} \{\zeta_M(2//3) \mid \text{with } m \text{ 2's and } \ell \text{ 3's where } 2m+3\ell = n\} &\to \{\zeta_M(2//3) \mid \text{the same for various weights}\} \\ \zeta_M(2//3, \ldots, 2//3)(3,2,\ldots,2)) &\mapsto \zeta_M((2//3, \ldots, 2//3)) \end{aligned}$$

By the main theorem, there exists an *invertible* matrix A such that

$$\partial(\text{LHS basis}) = A \cdot \partial(\text{RHS basis})$$

Proposition 333. *M.th. implies LI.*

Proof. Induction on ℓ. The induction step is M.th. The base step is $\ell =$. In the base case, $\zeta_M(2, \ldots, 2) \neq 0$. $\mathbb{R} \ni \zeta(2, \ldots, 2) \geq 0$. $\square$

Brown's proof: Part 2

Sergey Gorchinskiy on September 7th, 2012.

Plan of the proof

1. Prove the property of ∂ and $3_\ell \mathcal{Z}_M$. "Then" $\partial_{2r+1}(\zeta_M(2,\ldots,2)) = 0$.
2. $\partial_{2r+1}(\zeta_M(2s+1)) \sim \delta_{rs}$.
3. Case $\ell = 1$: Motivic lift of Zeta function.
4. Case $\ell \geq 2$: Recurrency process.

Step 1 is algebro-geometric. Steps 2, 3 and 4 require step 1.5, the use of Goncharov's formula which gives restrictions on these derivations $\partial_{2r+1}(\zeta_M(2s+1))$. This combinatorial results will be applied to show Steps 3 and 4.

Step 1

Motivic origin of $3_\ell \mathcal{Z}_M$

We would like to show that $3_\ell \mathcal{Z}_M \subset \mathcal{Z}_M$ is a G_M-subrepresentation and moreover, that

$$3_\ell \mathcal{O}(0,1)_{dR} = 3_\ell T(\Omega) := \langle\ \overline{\omega} \text{ that contain only } \omega^1\omega^0\omega^1\cdots\omega^1_{\leq \ell}\omega^0\cdots\omega^0\rangle_{\mathbb{Q}}$$

Proposition 334.

$$3_{ell}\mathcal{O}(0,1)_{dR} \subset \mathcal{O}(0,1)_{dR}$$

The action of G_M on that path space is a difficult object. But we have a geometric interpretation of all this. By the Tannakian theory, we see that subrepresentations of G_M correspond to mixed motives.

$$\mathrm{MTM}(\mathbf{Z}) \xrightarrow{\sim} \mathrm{Rep}(G_M)$$

Equivalently, we want that $3_\ell \mathcal{O}(0,1)_{dR}$ is motivic, i.e., that there exists a subobject in $\mathcal{O}(0,1)_{dR}$ whose $\omega_{dR}(-)$ is $3_{ell}\mathcal{O}(0,1)_{dR}$. To do this, we prove two lemmas.

We will first show that the subspace of admissible $\overline{\omega}$ are admissible, and then we will show a lemma to extend the result to all $\overline{\omega}$ in $3_\ell \mathcal{O}(0,1)_{dR}$.

Lemma 335. *The span*

$$\langle \overline{omega} \mid \overline{\omega} = \omega \otimes \cdots \omega^0 \textit{ is admissible}\rangle_{\mathbb{Q}} \subset \mathcal{O}(0,1)_{dR}$$

is motivic.

Proof. Consider

$$\mathcal{O}(\mathbb{Q}(1) \setminus \pi_1(X;0,1)_M/\mathbb{Q}(1)) \subset \mathcal{O}(\pi_1(X;0,1)_M)$$

The left $\mathbb{Q}(1)$ corresponds to a loop around 0 exiting and reentering the point along the tangent vector pointing toward 1. The right $\mathbb{Q}(1)$ similarly corresponds to a loop around 1 exiting and reentering the point along the tangent vector pointing toward 0. The paths of the π_1 exit the point 0 along the vector pointing toward 1 and enter the point 1 along the vector pointing from 0. This geometric picture shows that

$$\omega_{dR}(\mathcal{O}(\mathbb{Q}(1) \setminus \pi_1(X;0,1)_M/\mathbb{Q}(1))) = aconcavediamond$$

$$\begin{aligned} \overline{\omega} \quad &\mapsto \quad \sum \overline{\omega}_{i_1} \otimes \overline{\omega}_{i_2} \otimes \overline{\omega}_{i_3} \\ &\mapsto \quad \sum \mathrm{res}_0(\overline{\omega}_{i_1}) \otimes \overline{\omega}_{i_2} \otimes \mathrm{res}_1(\overline{\omega}_{i_3}) \\ &\stackrel{?}{=} \overline{\omega} \end{aligned}$$

∎

The difference is $\sum \mathrm{res}_0(\overline{\omega} \otimes \overline{\omega}_{i_2} \otimes \mathrm{res}_1(\overline{\omega}_{i_3})$. The first factor of the tensor product has positive coefficients. The third factor does as well. Hence $\omega_1 = \omega^1$ and $\omega_n = \omega^0$.

Lemma 336. *Fix p.m. Then*

$$\heartsuit = \left\langle \overline{\omega} \mid \overline{\omega} \textit{ does not contain } p \textit{ blocks of } \omega^0\textit{'s of total length } m \right\rangle_{\mathbb{Q}} \subset \mathcal{O}(0,1)_{dR} = T(\Omega)$$

is motivic.

Corollary 337. $3_\ell\mathcal{O}(0,1)$ *is motivic.*

The proof of the corollary is left as an exercise, but we consider the example of canceling those series of 0's of length greater than 4. Lemma 335 implies

$$\underbrace{10\cdots0}_{n_1}\underbrace{10\cdots0}_{n_2}10\cdots01\cdots\underbrace{10\cdots0}\,n_n$$

Apply Lemma 336 with $p = 1$ and $m = 3$. For a general 3_ℓ, $p = \ell$ and $m = 2\ell$ apply Lemma 336 with reg'd by 1 $p = 1$, $m = 2$.

Proof of Lemma 336. Define $S := \pi_1(X;0)_M^{\times p} \times \pi_1(X;0,1)_M$. Consider

$$f : \mathbb{Q}(1) \times S \to \pi_1(X;0,1)_M$$

The Betti realization is $(n, \gamma_1, \dots, \gamma_0, \gamma) \mapsto (\gamma_1\gamma_2 \dots \gamma_p \gamma)$.

Then $\heartsuit$ is ω_{dR}.

$$(f^*)^{-1}((\mathbb{Q}(0) \oplus \mathbb{Q}(-1) \oplus \cdots \oplus \mathcal{O}(-n)) \otimes \mathcal{O}(s)) \subset \mathcal{O}(0,1)$$

Blocks correspond to ∂^n's. The degree of polynomials is exactly the length. $\square$

Proposition 338 (Step 1)**.** *$\mathfrak{u}_M$ acts trivially on the graded pieces $\mathrm{gr}^3_\ell \mathcal{Z}_M$.*

Proof.

$$\partial_{2r+1}(\zeta_M(\dots,3,2,\dots,2,3,\dots) \mapsto \sum \zeta_M(2,\dots,2,3,\dots,3,2,\dots,2)$$

...missing... $\square$

This immediately implies Step 1.

Step 2

Consider the action of $\partial_{\geq r+1}$ on $\mathcal{O}(0,1)_{dR} = T(\Omega)$. We identify tensors $(\omega^1\omega^0 \cdots \omega^0)$ with words $(10 \cdots 0)$.

Theorem 339 (Goncharov's Formula)**.** *Let $\partial \in \mathfrak{u}_M$ be a derivation. Then the following relation between words holds in $\mathcal{O}(01)_{dR}$.*

$$\partial(w) = \sum_{\substack{v \neq \varnothing \\ (0v1)\subset(0w1)}} ct(\partial(v)) \cdot (w \setminus v) + \sum_{\substack{v \neq \varnothing \\ (1v0)\subset(0w1)}} (-1)^{|v|} ct(\partial(v^*)) \cdot (w \setminus w)$$

where v^* denotes the inverse word to v.

There is nothing motivic behind this formula. There is some Lie algebra canonically acting on $T(\Omega)$, and a chain of reasoning proves it, but we omit it.

$\partial_{2r+1}(\zeta_M(2s+1))$

Lemma 340. *The images of $\underbrace{0\dots0}_{m} = (\omega^0)^{\otimes m}$ under $\mathcal{O}(0,1)_{dR} \twoheadrightarrow \mathcal{Z}_M$ vanish. The same is true when 0 is replaced by 1, i.e., $(\omega^1)^{\otimes m} \mapsto 0$.*

Proof. Consider $\mathrm{gr}^w_{0,1}\mathcal{O}(0,1)_M$. Looking at the de Rham realization, we see that

$$\begin{aligned} \mathrm{gr}^w_0\mathcal{O}(0,1)_M &= \mathbb{Q}(0) \\ \mathrm{gr}^w_1\mathcal{O}(0,1)_M &= \mathbb{Q}(-1)\oplus\mathbb{Q}(-1) \end{aligned}$$

$\mathrm{Ext}^{1,0}_{\mathrm{MTM}(\mathbf{Z})}(\mathbb{Q}(0),\mathbb{Q}(1)) = 0$. Hence there is a canonical splitting, so that $\omega_1\mathcal{O}(0,1) \cong \mathbb{Q}(0)\otimes(\mathbb{Q}(1)\otimes\mathbb{Q}(-1))$ by a canonical isomorphism. Also, $\omega_{dR} = \mathbb{Q}\otimes\Omega$.

So the images of $\omega^{0/1}$ in $\mathcal{Z}_M \subset \mathcal{O}(I)$ are G_M-eigenfunctions of I. Evaluating at comp $\in I(\mathbb{C})$ gives

$$\langle \mathrm{comp}(\overline{\omega})(\gamma) = \int_\gamma \overline{\omega}\rangle$$

$$\int_{\mathrm{dch}} \omega^{0/1} = 0$$

Since G_M acts transitively on I, we see that

$$\begin{aligned} \mathcal{O}(0,1)_{dR} &\to \mathcal{O}(I) \\ \omega^{0/1} &\mapsto 0\in\mathcal{O}(I) \end{aligned}$$

Since $\int_\gamma(\omega^0)^{\otimes n} = \frac{1}{m!}\int_\gamma \omega^0$, $(\omega^0)^{\otimes m}\mapsto 0^m = 0$. □

Proposition 341.

$$\partial_{2r+1}(\zeta_M(2s+1)) \sim \delta_{r,s}$$

Proof. **Proof Part 1**

We show that $\zeta_M(2s+1)\in\mathcal{O}(I)$ is a linear function with respect to U_M. The proof proceeds by cases.

In the case $r>s$, $\partial_{2r+1}(\zeta_M(2s+1)) = 0$.

In the case $r=s$, $\partial_{2r+1}(1\underbrace{0\ldots0}_{2r}) = ct(\partial_{2r+1}(10\cdots0)$. $(0v1)\subset(0|1\underbrace{0\ldots0}_{2r}|1)$. $|r| = 2r+1$. $(1v0)\subset(0|10\ldots0)|1)\to 0$.

In the case $r<s$, $\partial_{2r+1}(1\underbrace{0\ldots0}_{2s} = ct(\partial_{2r+1}(\underbrace{0\ldots0}_{2r+1}))(1\,\underbrace{0\ldots0}_{2s-2r-1}) - ct()$. Because the map $\mathcal{O}(0,1)_{dR}\to\mathcal{O}(I)$ is graded, the term $\partial_{2r+1}(0\cdots0)$ vanishes.

$$\begin{aligned} &(0v1)\subset(0|1\underbrace{0\ldots0}_{2s}|1) \quad |v| = 2r+1 \\ &(1v0)\subset(0|1\underbrace{0\ldots0}_{2s}|1) \quad |v| = 2r+1 \end{aligned}$$

Proof Part 2

To show: $\partial_{2r+1}(\zeta_M(2r+1)) \neq 0$. Suppose the converse. Then $\zeta_M(2r+1) \in \mathcal{O}(I)^{U_M}$ as computed by Konrad (Cf. missing).

$$\mathcal{O}(I)^\epsilon_+ \cong \mathbb{Q}[t^2] \otimes_\mathbb{Q} T(\bigoplus_{r\geq 1} \mathbb{Q}e_{2r+1})$$

is graded and $\deg t = 1$. So $\mathcal{O}(I)^\epsilon_+)^{U_M} \cong \mathbb{Q}[t^2]$ has only even components. Hence $\zeta_M(2r+1) = 0$. Contradiction! We already know that $\zeta_M(2r+1) > 0$. □

This completes Step 2.

Step 3

Remark 342. Now we fix a normalization ∂_{2r+1} so that $\partial_{2r+1}(\zeta_M(2r+1)) = 1$.

Proposition 343. *The embedding $\mathcal{Z} \hookrightarrow \mathcal{O}(I)^\epsilon_+$ of U_M-representations induces equalities*

$$\begin{array}{rcl} N_0\mathcal{Z}_M & = & N_0\mathcal{O}(I)^\epsilon_+ \\ \{V_M\text{-}linear\}\mathcal{Z}_M & = & \{U_M\text{-}linear\}\mathcal{O}(I)^\epsilon_+ \\ N_1\mathcal{Z}_M & = & N_1\mathcal{O}(I)^\epsilon_+ \end{array}$$

Proof. The proof is by calculation using the explicit description of $\mathcal{O}(I)^\epsilon_+$ given in Equation . Then

$$N_0\mathcal{Z}_M \subset N_0\mathcal{O}(I)^\epsilon_+.$$

The left hand side equals $\langle 0 \neq \zeta_M(\underbrace{2,\ldots,2}_{m})\rangle_\mathbb{Q}$ and the right hand side, $\mathbb{Q}[t^2]$. But these are equal, hence $N_0\mathcal{Z}_M$ is the improper subset.

Remark 344. It follows that $\zeta_M(\underbrace{2,\ldots,2}_{m}) \sim \zeta_M(2)^m$. Hence $\zeta(\underbrace{2,\ldots,2}_{m}) \sim \zeta(2)^m \sim \mathrm{comp}^*(t)^{2m}$. Then t is the period of $\mathbb{Q}(-1)$, so $\mathrm{comp}^*(t) \sim 2\pi i$. Hence $\zeta(\underbrace{2,\ldots,2}_{m}) \sim \pi^{2m}$.

Note that

$$\begin{array}{ccc} \{\text{linear in } \mathcal{Z}_M\} & \subset & \{\text{linear in } \mathcal{O}(I)^\epsilon_+\} \\ \| & & \| \\ \langle \zeta_M(2r+1) \neq 0, \zeta_M(\underbrace{2,\ldots,2})\rangle_\mathbb{Q} & \subset & \langle t^{2m}, e_{2r+1}\rangle_\mathbb{Q} \end{array}$$

Since $\zeta_M(2r+1) \sim e_{2r+1}$, the bottom subset relation is actually an equality, and hence the top one is too.

$$\begin{array}{ccc} N_1\mathcal{Z}_M & \subset & N_1\mathcal{O}(I) \\ \cup & & \| \\ \langle \zeta_M(\underbrace{2,\ldots,2}_{m}), \zeta_M(2r+1)\rangle & \subset & \langle t^{2m} \cdot e_{2r+1}\rangle_\mathbb{Q} \end{array}$$

□

Corollary 345. *There exists a Zagier type formula:*

$$\zeta_M(2,\ldots,2,3,2,\ldots,2) = \sum_{r\geq 1} c\zeta_M(2,\ldots,2)\cdot\zeta_M(2r+1)$$

where $c \in \mathbb{Q}$ is a constant.

Proof. The Lie algebra $\mathfrak{u}_M$ acts trivially on $\mathrm{gr}^3_\ell \mathcal{Z}_M$. Hence

$$3_1\mathcal{Z}_M \subset N_1\mathcal{Z}_M.$$

□

The Goncharov formula and combinatorial part

The Ihara group: An explanation of the Gondcharov formula

Remark 346. The action of G_M preserves the following structure.

- The groupoid structure of $\pi_1(X;0)_{dR}$, $\pi_1(X;0,1)_{dR}$, and $\pi_1(X;1)_{dR}$.
- The morphisms $\mathbb{Q}(1)_{dR} \to \pi_1(X;0)_{dR}$ and $\mathbb{Q}(1)_{dR} \to \pi_1(X;1)_{dR}$.

The action of U_M leaves the images of

$$\mathbb{Q}(1)_{dR} \to \pi_1(X;0)_{dR} \qquad \mathbb{Q}(1)_{dR} \to \pi_1(X;1)_{dR}$$

Definition 347 (Ihara group)**.** Given a smooth scheme X over k, its Ihara group is

$IH := \{(\phi_1,\phi_2,\phi_3) \in \mathrm{Aut}(\pi_1(X;0)_{dR}, \pi_1(X;0,1)_{dR}, \pi_1(X;1)_{dR}) \mid \phi_1$, ϕ_2 and ϕ_3 preserve the groupoid structu

The action of U_M on $\pi(X;0,1)_{dR}$ factors through $U_M \to IH$. The Goncharov formula holds for $\partial \in$ Lie (IH).

Proposition 348.

$$\begin{array}{ccc} IH & \xrightarrow{\sim} & \pi_1(X;0,1)_{dR} \\ g & \mapsto & g(dc_{dR}) \end{array}, \qquad dc_{dR} \in \pi_1(X;0,1)_{dR}(\mathbb{Q}) \textit{ corresponding to ct}$$

The explicit description of IH begins with the free group on two generators, $\Gamma := \langle\gamma_1,\gamma_2\rangle$. We associate to Γ three its pro-unipotent completion Γ^{un} and two sets, Γ^{un}_l and Γ^{un}_r which are left and right torsors under Γ^{un}.

$IH \cong \{\phi \in \mathrm{Aut}(\Gamma^{un}_l, \Gamma^{un}, \Gamma^{un}_r) \mid \phi$ preserves $\gamma_1 \in \Gamma^{un}_l$, $\gamma_2 \in \Gamma^{un}_r$ and the product groupoid structure$\}$

Proposition 349.

$$\begin{array}{ccc} IH & \xrightarrow{\sim} & \Gamma^{un} \\ g & \mapsto & g(1) \end{array}$$

To prove the proposition, one does this for $(\Gamma_l, \Gamma, \Gamma_r)$.

Moreover, one obtains a new group structure $*$ on Γ and Γ^{un} defined as

$$\gamma * \gamma' = \gamma(\gamma') \cdot \gamma$$

where $\gamma' \mapsto \gamma(\gamma')$ is a group automorphism of Γ such that

$$\gamma_1 \mapsto \gamma_1, \quad \gamma_2 \mapsto \gamma\gamma_2\gamma^{-1}$$

From this one gets a new

$$\Delta^* : \mathcal{O}(\Gamma^{un}) \to \mathcal{O}(\Gamma^{un}) \otimes_{\mathbb{Q}} \mathcal{O}(\Gamma^{un})$$

But $\mathcal{O}(\Gamma^{un}) \cong T(\Omega)$. In "our Goncharov formula"

Example 350. We would like to find the coefficients

$$\zeta_M(2,3) = c_1\zeta_M(5) + c_2\zeta_M(2) \cdots \zeta_M(3)$$

Then $\partial_3\zeta_M(2,3) = \underbrace{ct(\partial_3(100)) \cdot (10)}_{\zeta_M(2)} - \underline{ct(\partial_3(010)) \cdot (10)}_{-2\zeta_M(2)} = 3\zeta_M(2)$. $(0|10100|)$, $|v| = 3$.

The trick is

$$\begin{array}{rcl} \mathcal{O}(01) & \to & \mathcal{Z}_M \\ (0) = \omega^0 & \mapsto & 0 \end{array} \qquad \begin{array}{ccccc} (0) \cdot (10) & = & (010) & + & 2(100) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & = & (010) & + & 2\zeta_M(3) \end{array}$$

$$\partial_5\zeta_M(2,3) = ct(\partial_5(10100)) = ct(\partial_5\zeta_M(2,3))$$

$(0|10100|1) \supset (0v1)$ for $|v| = 5$ implies $v = w$. Hence

$$\zeta_M(2,3) - 3\zeta_M(2) \cdot \zeta_M(3) = c \cdot \zeta_M(5)$$

for some $c \in \mathbb{Q}$. Apply the morphism comp* and Zagier's formula to see that $c = -11/2$.

Proposition 351. *Zagier's formula holds for $\zeta_M(2//3)$.*

Proof. Apply ∂_{2r+1} to $\zeta_M(2\ldots232\ldots2)$ such that $2r+1 < n$, where n is the weight of $\zeta_M(2\ldots232\ldots2)$. One obtains words v that are either $(10)^?(100)(10)^?$, in which case $n = 2r+1$ or $0(10)^?$, in which case the weight $n > 2r+1$. Examining the weights gives $ct(\partial_{2r+1}(v))$. Then

$$\zeta_M(2\ldots232\ldots2) = c \cdot \zeta(n) + \epsilon$$

Apply comp to get $c = c_{\mathcal{Z}}$. ☐

This completes Step 3.

Step 4

Consider the case $\ell \geq 2$. We are interested in describing $\partial = \sum_{r\geq 1} \partial_{2r+1}$ from $\mathrm{gr}^3_\ell \mathcal{Z}_\ell \to \mathrm{gr}^3_{\ell-1}\mathcal{Z}_M$. That is, we want to describe $\partial_{2r+1}\zeta_M(\underbrace{2\ldots 2332\ldots 3\ldots 3}_{\ell 3's})$ modulo $\zeta_M(2//3)$ with at most $\ell - 2$ 3's. Apply Goncharov's formula. We get many v's and those which matter are in $3_1\mathcal{Z}_M$. Apply motivic Zagier and do an explicit calculation. By the explicit Zagier formula, one knows v_2.

If you look more precisely at the Zagier formula. We need to show that the matrix is invertible. The point is that the coefficients in Goncharov's formula (339), the only non-integrality comes out of $ct(\partial(v))$. But if you look in the explicit Zagier formula, there are only powers of two in the denominators.

Proposition 352. *For all $\ell \geq 2$. Let A be the matrix for ∂. There exists a way to multiply columns by a power of 2 such that $A(mod 2)$ is lower triangular with ones along the diagonal. Hence it is invertible.*

We use induction on ℓ where the base case, $\ell = 1$ follows from Zagier's formula.

This completes Step 4 and the proof.

Bibliography

[1] P. Cartier. A primer of hopf algebras. *Frontiers in Number Theory, Physics, and Geometry II*, pages 537–615, 2007.

[2] T. Ekedahl and S. Merkulov. Lecture notes on the Grothendieck-Teichmüller group. `http://www2.math.su.se/~sm/Papers/GT.pdf`.

[3] J. Lambek and P.J. Scott. *Introduction to higher order categorical logic*, volume 7. Cambridge Univ Pr, 1988.

[4] J.S. Milne et al. Algebraic groups, lie groups, and their arithmetic subgroups, 2010.

[5] D. Quillen. Rational homotopy theory. *The Annals of Mathematics*, 90(2):205–295, 1969.

[6] E. Suisse. Groupes nilpotents sans torsion et complétés de Mal'cev. Mémoire. `http://www-magistere.u-strasbg.fr/IMG/pdf/memoire_esther_suisse.pdf`, 2007.