PROPER PUSHFORWARD OF INTEGRAL CYCLES ON ALGEBRAIC STACKS

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Abstract. Artin stacks of finite type over a field which can be stratified by global quotients admit an intersection theory with integral coefficients, including a pushforward operation for projective morphisms. We show that the pushforward operation for integral coefficients may be extended to proper representable morphisms. Additionally, rational cycles may be pushed forward across morphisms of relative Deligne-Mumford type.

1. Introduction

It is known that Artin stacks of finite type over a field which can be stratified by locally closed substacks which are global quotients admit an intersection theory with integral coefficients. The theory includes a pushforward operation for projective morphisms [Kre99]. Here we show that the pushforward operation for integral coefficients may be extended to proper representable morphisms. If the coefficients are rational, there is also a pushforward for morphisms of relative Deligne-Mumford type. All stacks are Artin stacks of finite type over a fixed base field.

These results simplify definitions which previously relied on projective pushforwards, e.g., the definition of Mumford classes [Ful10].

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2. Notation and terminology

Let a base field k be fixed. All stacks are Artin stacks of finite type over k. All morphisms are morphisms over k.

A morphism of stacks $X \to Y$ is projective if it can be factored up to 2-isomorphism through a closed immersion into the relative Proj $\boldsymbol{P}(\mathcal{E}) \to Y$ for a coherent sheaf $\mathcal{E}$ of $\mathcal{O}_Y$-modules.

3. Chow groups

We briefly recall the construction of the Chow groups of an algebraic stack X of finite type over k. A representative of a d-cycle is a pair (f, α) for a projective morphism $f : Y \to X$ from a stack Y and a naive cycle $\alpha \in A^\circ_{d+\mathrm{rk}\ E}(E)$ on a vector bundle $E \to Y$ of constant rank. The Chow groups of X are

$$A_d(X) = \varinjlim_{Y \in \mathfrak{A}_X} \left[\widehat{A}_d Y / \widehat{B}_d Y \right]$$

where

$$\begin{aligned}
\widehat{A}_d Y &= \varinjlim_{E \in \mathfrak{B}_Y} A^\circ_{d+\mathrm{rk}\ E} E \\
A^\circ_d E &= Z_d(E)/\delta Rat_{d+1}(E) \\
Z_d(E) &= d\text{-dimensional cycles} \\
Rat_{d+1}(E) &= \oplus_Z k(Z)^*, Z \subset E \text{ integral, } \dim Z = (d+1) \\
Ob(\mathfrak{A}_X) &= \{X' \to X \mid \text{ projective}\} \\
Mor(X', Y') &= \{X' \to Y' \mid X\text{-isom. onto conn. components of } Y'\} \\
Ob(\mathfrak{B}_X) &= \{E \to X \mid \text{ vector bundle of constant rank}\} \\
Mor(E, F) &= \{E \to F \mid \text{ surjective vector bundle morphism }\}
\end{aligned}$$

The subgroup $\widehat{B}_d X'$ is defined to be

$$\widehat{B}_d X' = \coprod_{\substack{p_1, p_2 : W \to X' \\ g \circ p_1 \cong g \circ p_2}} \ \coprod_{E, F \in \mathfrak{B}_{X'}} Z_{E,F}$$

$$Z_{E,F} = \left\{ p_{2*}\beta_2 - p_{1*}\beta_1 \ \middle| \ \begin{array}{c} \beta_1 \in A^\circ_{d+\mathrm{rk}\ E}(p_1^* E), \quad \beta_2 \in A^\circ_{d+\mathrm{rk}\ F}(p_2^* F) \\ \beta_1 \sim \beta_2 \text{ in } \widehat{A}_d(W) \end{array} \right\}$$

where the first union is over *projective* morphisms and introduces a dependence on the morphism $g : X' \to X$ through the constraint $g \circ p_1 \cong g \circ p_2$. This accomplishes in a single step what the original definition [Kre99, Definition 2.1.4(iii)] does in two by assuming the vector bundles to be of constant rank.

The above definition makes projective pushforwards tautological: the cycle (g, α) pushes forward under f to $(g \circ f, \alpha)$. Projectivity cannot be replaced by properness in the definition without losing excision. A projective morphism $S \to U$ to an open substack $U \subset X$ can be realized as the pullback of a projective morphism to X. It is unknown whether this property can be extended to proper morphisms.

Definition 3.1. Let $f : Y \to X$ be a morphism of algebraic stacks. We further define restricted Chow groups formed from the pullbacks of vector bundles,

$$A_d^f Y = \varinjlim_{X' \in \mathfrak{A}_X} \left[\widehat{A}_d^{f'}(Y') / \widehat{B}_d^{f'}(Y') \right],$$

where $f' : Y' \to X'$ is the pullback of f by the projective morphism $X' \to X$. Recall that the restricted Edidin-Graham-Totaro Chow groups [Kre99, Definition 2.1.4(iv)] are the groups

$$\widehat{A}_d^{f'} Y' = \varinjlim_{E \in \mathfrak{B}_{X'}} A^\circ_{d+\mathrm{rk}\ E} f'^* E. \tag{3.1}$$

The quotienting group,

$$\widehat{B}_d^{f'} Y' = \coprod_{\substack{p_1, p_2 : W \to X' \\ g \circ p_1 \cong g \circ p_2}} \coprod_{E, F \in \mathfrak{B}_{X'}} Z^f_{E,F}$$

$$Z^f_{E,F} = \left\{ p'_{2*}\beta_2 - p'_{1*}\beta_1 \;\middle|\; \begin{array}{c} \beta_1 \in A^\circ_{d+\mathrm{rk}E}(p'^*_1 f'^* E), \beta_2 \in A^\circ_{d+\mathrm{rk}F}(p'^*_2 f'^* F) \\ \beta_1 \sim \beta_2 \text{ in } \widehat{A}_d^{f''}(W') \end{array} \right\}$$

depends on the notation,

$$\begin{tikzcd} W' \arrow[r, shift left, "p'_1"] \arrow[r, shift right, "p'_2"'] \arrow[d, "f''"'] & Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ W \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X' \arrow[r, "g"'] & X \end{tikzcd}$$

Recall that we only consider vector bundles of constant rank. So a cycle in $A_d^f Y$ is represented by (g, α'), $\alpha' \in A^\circ_{d+\mathrm{rk}\ E'}(E')$ as in the pullback diagram:

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "f''"] & Y' \arrow[r] \arrow[d, "f'"] & Y \arrow[d, "f \text{ rep}"] \\ E \arrow[r] & X' \arrow[r, "g \text{ proj}"'] & X \end{tikzcd} \tag{3.2}$$

There is a natural morphism from the restricted Chow groups to the usual Chow groups, $\iota : A_d^f Y \to A_d Y$.

The cycles in $A_d^f Y$ can be pushed forward by direct generalization of the definition of pushforward for varieties. If E is a vector bundle on X', then f_* pushes the class represented by a cycle $\alpha \in A^\circ E'$ forward to $f''_*(\alpha) \in A^\circ E$ in the notation of diagram (3.2).

Lemma 3.2. *If $f : Y \to X$ is a proper representable morphism, then there is a proper pushforward $f_* : A_d^f Y \to A_d X$ for all d.*

A morphism $f : Y \to X$ of stacks is *of relative Deligne-Mumford type* if it pulls back Deligne-Mumford stacks over X to Deligne-Mumford stacks over Y. *Rational* cycles on Deligne-Mumford stacks can be pushed forward [Vis89], leading to the following.

Lemma 3.3. *If $f : Y \to X$ is a proper morphism of relative Deligne-Mumford type, then there is a proper pushforward $f_* : A_d^f(Y, \boldsymbol{Q}) \to A_d(X, \boldsymbol{Q})$ for all d.*

4. Properties of restricted Chow groups

Proposition 4.1. *If $f : Y \to X$ is representable, then for all d the natural morphism $A_d^f Y \to A_d Y$ is an isomorphism when X is a global quotient stack.*

Proof. Since $X = [\widehat{X}/G]$ is a global quotient stack, it can be approximated by a vector bundle $F \to X$ [Tot99, Remark 1.4] which is pulled back from BG. This vector bundle leads to a factorization $A_d^f Y \xrightarrow{\sim} \widehat{A}_d Y \xrightarrow{\sim} A_d Y$. The second isomorphism is defined as in [Kre99, Remark 2.1.17]. To see that the first isomorphism can be defined in the same way, we must note that the approximating vector bundle on Y is chosen to be the pullback of an approximating vector bundle on X, and that the equivalence $(f, \alpha) = (1_Y, f'_*\alpha')$ (in Kresch's notation) is witnessed by objects pulled back from X. □

Proposition 4.2. *If $f : Y \to X$ is of relative Deligne-Mumford type and X is a global quotient stack, then the natural morphism $A_d^f(Y, \boldsymbol{Q}) \to A_d(Y, \boldsymbol{Q})$ is an isomorphism for all d.*

Proof. The result follows directly from the existence of approximating vector bundles, and the fact that the pullback to a vector bundle on a Deligne-Mumford stack induces an isomorphism of naive Chow groups with $\boldsymbol{Q}$-coefficients. □

Consider the fiber diagram.

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\tilde{f}"'] & Y \arrow[d, "f"] \\ V \arrow[r, "h"'] & X \end{tikzcd}$$

For each d, the groups $A_d^f(Y)$ satisfy the following properties:

(i) Let the above morphism $h : V \to X$ be a flat morphism of relative dimension r. Then there is a functorial pullback homomorphism $A_d^f Y \to A_{d+r}^{\tilde{f}} W$.

(ii) Let the above morphism $h : V \to X$ be a projective morphism, then there is a functorial pushforward homomorphism $A_d^{\tilde{f}} W \to A_d^f Y$.

(iii) The homomorphisms (i) and (ii) are compatible with the natural morphism $A_d^f Y \to A_d Y$.

Proposition 4.3 (Excision)**.** *Let $i : Z \to X$ be a closed substack, and $j : U \to X$ its complement. Let $Z' \to Y$ be the pre-image of Z under f, and let U' be its complement. Let $f' = f|_{Z'}$ and $\widetilde{f} = f|_{U'}$. Then for each d the flat pullback and projective pushforward fit together into an exact sequence*

$$A_d^{f'} Z' \xrightarrow{i_*} A_d^f Y \xrightarrow{j^*} A_d^{\widetilde{f}} U' \longrightarrow 0.$$

Proof. The proof is completely analogous to the proof of [Kre99, Prop. 2.3.6]. □

Definition 4.4. Recall that the excision sequence for Chow groups may be extended using underlined Chow groups defined in [Kre99, Corollary 4.1.10]. Here we defined their restricted analogues. Let

$$\underline{A}_d^f Y = \varinjlim_{X' \in \mathfrak{A}_X} \left[\widehat{\underline{A}}_d^{f'}(Y') / \widehat{\underline{B}}_d^{f'}(Y') \right]$$

where $\widehat{\underline{A}}_d^{f'}$ is defined in analogy with equation (3.1). There is a natural homomorphism $\underline{A}_d^f Y \to \underline{A}_d Y$.

Proposition 4.5. *Let $p : E \to X$ be a vector bundle on a Deligne-Mumford stack. Then there is a surjection $p^* : \underline{A}_d^\circ(X, \boldsymbol{Q}) \to \underline{A}_{d+\mathrm{rk}\ E}^\circ(E, \boldsymbol{Q})$ for all d.*

Proof. We first reduce to the case of quotient stacks of the form $[W/G]$ for an algebraic space W and a finite group G. Assuming the proposition holds for such stacks, let $U \subset X$ be such a stack, non-empty and open in X [LMB00, Corollaire 6.1.1]. By naturality of the long exact sequence [Kre99, (4.2.1)], there are morphisms,

$$\begin{tikzcd}
\underline{A}^\circ_*(Z,\boldsymbol{Q}) \arrow[r] \arrow[d] & \underline{A}^\circ_*(X,\boldsymbol{Q}) \arrow[r] \arrow[d] & \underline{A}^\circ_*(U,\boldsymbol{Q}) \arrow[r, "\delta"] \arrow[d, "\sim"] & A^\circ_*(Z,\boldsymbol{Q}) \arrow[d, "\sim"] \\
\underline{A}^\circ_*(E|_Y,\boldsymbol{Q}) \arrow[r] & \underline{A}^\circ_*(E,\boldsymbol{Q}) \arrow[r] & \underline{A}^\circ_*(E|_U,\boldsymbol{Q}) \arrow[r] & A^\circ_*(E|_Z,\boldsymbol{Q})
\end{tikzcd}$$

where $Z = X \setminus U$. By noetherian induction on X, the leftmost vertical morphism is surjective. Hence $\underline{A}^\circ_*(X,\boldsymbol{Q}) \to \underline{A}^\circ_*(E,\boldsymbol{Q})$ is a surjection by the four lemma.

In the case of a vector bundle E on a quotient $[W/G]$, E has the form $[V/G]$ for a vector bundle $V \to W$. There is a diagram,

$$\begin{tikzcd}
A_*(W;1)_{\boldsymbol{Q}} \arrow[r, "\sim"] & A_*(V;1)_{\boldsymbol{Q}} \\
\underline{A}^\circ_*(W,\boldsymbol{Q}) \arrow[r, "\sim", "s^*"'] \arrow[u, "\sim"] \arrow[d, shift left, "q_*"] & \underline{A}^\circ_*(V,\boldsymbol{Q}) \arrow[u, "\sim"] \arrow[d, shift left, "r_*"] \\
\underline{A}^\circ_*([W/G],\boldsymbol{Q}) \arrow[r, "p^*"'] \arrow[u, shift left, "q^*"] & \underline{A}^\circ_*(E,\boldsymbol{Q}) \arrow[u, shift left, "r^*"]
\end{tikzcd}$$

The upper square isomorphisms follow from the natural isomorphism for algebraic spaces $\underline{A}^\circ_*(W) \xrightarrow{\sim} A_*(W;1)$. The lower squares exist since the quotient morphisms are flat and proper, and they commute by compatibility of pullback and pushforward. The morphism r_* is surjective, because $r_* \circ r^*$ is multiplication by $|G|$. But $r_* = p^* \circ q_* \circ s^{*-1}$, so p^* must also be surjective. By similar reasoning p^* is injective, hence an isomorphism. $\square$

Proposition 4.6. *If $f : Y \to X$ is representable and X is a global quotient stack, then the natural morphism $\underline{A}^f_d Y \to \underline{A}_d Y$ is an isomorphism for all d.*

Proof. This follows immediately from the existence of approximating vector bundles for global quotient stacks and the fact that a pullback to a vector bundle on an algebraic space induces isomorphism of restricted Chow groups. $\square$

Proposition 4.7. *If $f : Y \to X$ is of relative Deligne-Mumford type and X is a global quotient stack, the natural morphism $\underline{A}^f_d(Y,\boldsymbol{Q}) \to \underline{A}_d(Y,\boldsymbol{Q})$ is a surjection for all d.*

Proof. Let $\alpha \in \underline{A}^{\circ}_{d+\mathrm{rk}\ E'}(E', \boldsymbol{Q})$ be a representative of a cycle in $\underline{A}^{f}_{d}(Y, \boldsymbol{Q})$.

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ E \arrow[r, "\text{v.b.}"] & W \arrow[r, "\text{proj}"] & X \end{tikzcd}$$

Let F approximate X away from high codimension, i.e., be an algebraic space away from high codimension. Let pullbacks be formed,

$$\begin{tikzcd} E' \oplus F'' \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d] & F' \arrow[r] \arrow[d] & F \arrow[d] \\ E' \arrow[r] & W' \arrow[r] & Y \arrow[r, "f"] & X \end{tikzcd}$$

Since f is of Deligne-Mumford type, F' is Deligne-Mumford away from high codimension. The cycle α can be pulled back to $E' \oplus F''$ and by the surjectivity of Proposition 4.5, it comes from a cycle on F'' and finally pushes forward to F'. $\square$

Proposition 4.8. *With the notation of Proposition 4.3, there is a connecting homomorphism δ_f fitting into a commutative diagram with the connecting homomorphism of* [Kre99, (4.2.2)],

$$\begin{tikzcd} \underline{A}^{\tilde{f}}_{d}U' \arrow[r, "\delta_f"] \arrow[d] & A^{f'}_{d}Z' \arrow[d] \\ \underline{A}_{d}U' \arrow[r, "\delta"] & A_{d}Z' \end{tikzcd}$$

Proof. The morphism of δ_f arises analogously and is well-defined under equivalence of cycles for which the main technical ingredient is [Kre99, Cor. 2.3.5]. $\square$

Propositions 4.3 and 4.8 fit together in an exact sequence.

Proposition 4.9. *With the notation of Proposition 4.3, let U be a global quotient stack. Then there is an exact sequence,*

$$\underline{A}^{\tilde{f}}_{d}U' \xrightarrow{\delta_f} A^{f'}_{d}Z' \to A^{f}_{d}Y \to A^{\tilde{f}}_{d}U' \to 0.$$

Proof. The proof follows analogously from the proof of [Kre99, Prop. 4.2.1]. $\square$

5. Main theorems

Recall that a stack admits a stratification by global quotient stacks if and only if every geometric stabilizer is affine [Kre99, Propk 3.5.9].

Proposition 5.1. *Let X be a stack stratified by global quotient stacks and let $f : Y \to X$ be representable. Then $A^f_d Y \to A_d Y$ is an isomorphism for all d. If, alternatively, f is of relative Deligne-Mumford type, then $A^f_d(Y, \boldsymbol{Q}) \to A_d(Y, \boldsymbol{Q})$ is an isomorphism for all d.*

Proof. The proof proceeds by Noetherian induction. Using the same notation as Proposition 4.3, let U be a global quotient. There is a morphism from the exact sequence of Proposition 4.9 to the exact sequence of [Kre99, Prop 4.2.1],

$$\begin{tikzcd}
\underline{A}^{\widetilde{f}}_d U' \arrow[r, "\delta_f"] \arrow[d, "c'"] & A^{f'}_d Z' \arrow[r] \arrow[d, "a"] & A^f_d Y \arrow[r] \arrow[d, "b"] & A^{\widetilde{f}}_d U' \arrow[r] \arrow[d, "c"] & 0 \\
\underline{A}_d U' \arrow[r, "\delta"] & A_d Z' \arrow[r] & A_d Y \arrow[r] & A_d U' \arrow[r] & 0
\end{tikzcd}$$

Morphism a is an isomorphism by the induction hypothesis, and morphism c by Proposition 4.1. Then c' is an isomorphism by Proposition 4.6 if f is representable, and is a surjection by Proposition 4.7 if f is of relative Deligne-Mumford type where all Chow groups are taken with $\boldsymbol{Q}$-coefficients. By the five lemma, b is also an isomorphism. □

Theorem 5.2. *Let X be a stack stratified by global quotients, and let $f : Y \to X$ be a proper, representable morphism. Then there is a proper pushforward $f_* : A_d Y \to A_d X$ for all d. If, instead, f is proper and of relative Deligne-Mumford type, then there is a proper pushforward $f_* : A_d(Y, \boldsymbol{Q}) \to A_d(X, \boldsymbol{Q})$ for all d.*

Proof. The pushforward arises from the factorization

$$\begin{tikzcd}
A^f_d Y \arrow[rr] \arrow[dr, "\sim"'] & & A_d X \\
& A_d Y \arrow[ur] &
\end{tikzcd}$$

In case f is of relative Deligne-Mumford type, Chow groups should be taken with $\boldsymbol{Q}$-coefficients. □

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