PROPER PUSHFORWARD OF INTEGRAL CYCLES ON ALGEBRAIC STACKS

JONATHAN SKOWERA

Abstract. Artin stacks of finite type over a field which can be stratified by global quotients admit an intersection theory with integral coefficients, including a pushforward operation for projective morphisms. We show that the pushforward operation for integral coefficients may be extended to proper representable morphisms. Additionally, rational cycles may be pushed forward across morphisms of relative Deligne-Mumford type.

1. Introduction

It is known that Artin stacks of finite type over a field which can be stratified by locally closed substacks which are global quotients admit an intersection theory with integral coefficients. The theory includes a pushforward operation for projective morphisms [Kre99]. Here we show that the pushforward operation for integral coefficients may be extended to proper representable morphisms. If the coefficients are rational, there is also a pushforward for morphisms of relative Deligne-Mumford type. All stacks are Artin stacks of finite type over a fixed base field.

These results simplify definitions which previously relied on projective pushforwards, e.g., the definition of Mumford classes [Ful10].

Acknowledgments. We record here an argument communicated by A. Kresch, discussed with D. Rydh and extended by the author.

2. Notation and terminology

Let a base field k be fixed. All stacks are Artin stacks of finite type over k. All morphisms are morphisms over k.

Date: April 27, 2019.

2010 *Mathematics Subject Classification.* Primary 14A20; Secondary 14C15, 14C25.

We acknowledge financial support by the German Science Foundation (grant CH 843/2-1), the Swiss National Science Foundation (incl. grants PP00P2_128455, 20CH21_138799 (CHIST-ERA project CQC)), the Swiss National Center of Competence in Research 'Quantum Science and Technology (QSIT)', the Swiss State Secretariat for Education and Research supporting COST action MP1006 and the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement no. 337603.

A morphism of stacks $X \to Y$ is projective if it can be factored up to 2-isomorphism through a closed immersion into the projective bundle $\boldsymbol{P}(\mathcal{E}) \to Y$ for a coherent sheaf $\mathcal{E}$ of $\mathcal{O}_Y$-modules.

3. Chow groups

We briefly recall the construction of the Chow groups of an algebraic stack Y of finite type over k. A representative of a d-cycle is a pair (f, α) for a projective morphism $f : X \to Y$ from a stack X and a naive cycle $\alpha \in A^\circ_{d+\operatorname{rk} E}(E)$ on a vector bundle $E \to X$ of constant rank. The Chow groups of Y are

$$A_d(Y) = \varinjlim_{X \to Y \in \mathfrak{A}_Y} \left[\widehat{A}_d X / \widehat{B}_d X\right]$$

where

$$\begin{aligned}
\widehat{A}_d X &= \varinjlim_{E \in \mathfrak{B}_X} A^\circ_{d+\operatorname{rk} E} E \\
A^\circ_d E &= Z_d(E)/\delta Rat_{d+1}(E) \\
Z_d(E) &= d\text{-dimensional cycles} \\
Rat_{d+1}(E) &= \oplus_Z k(Z)^*, Z \subset E \text{ integral}, \dim Z = (d+1) \\
Ob(\mathfrak{A}_Y) &= \{X \to Y \mid \text{ projective}\} \\
Mor(X, X') &= \{X \to X' \mid Y\text{-isom. onto conn. components of } X'\} \\
Ob(\mathfrak{B}_X) &= \{E \to X \mid \text{ vector bundle of constant rank}\} \\
Mor(E, F) &= \{E \to F \mid \text{ surjective vector bundle morphism }\}
\end{aligned}$$

The subgroup $\widehat{B}_d X$ is defined to be

$$\widehat{B}_d X = \coprod_{\substack{p_1, p_2 : W \to X \\ g \circ p_1 \cong g \circ p_2}} \coprod_{E, F \in \mathfrak{B}_X} Z_{E,F}$$

$$Z_{E,F} = \left\{ p_{2*}\beta_2 - p_{1*}\beta_1 \;\middle|\; \begin{array}{c} \beta_1 \in A^\circ_{d+\operatorname{rk} E}(p_1^* E), \quad \beta_2 \in A^\circ_{d+\operatorname{rk} F}(p_2^* F) \\ \beta_1 \sim \beta_2 \text{ in } \widehat{A}_d(W) \end{array} \right\}$$

where the first union is over *projective* morphisms and introduces a dependence on the morphism $g : X \to Y$ through the constraint $g \circ p_1 \cong g \circ p_2$. This accomplishes in a single step what the original definition [Kre99, Definition 2.1.4(iii)] does in two by assuming the vector bundles to be of constant rank.

The above definition facilitates projective pushforwards: the cycle (g, α) pushes forward under f to $(g \circ f, \alpha)$. A projective morphism $S \to U$ to an open substack $U \subset Y$ can be realized as the pullback of a projective morphism to Y [Kre99, Cor 2.3.2]. It is unknown whether this property can be extended to proper morphisms.

Definition 3.1. Let $f : X \to Y$ be a morphism of algebraic stacks. We further define restricted Chow groups formed from the pullbacks of vector bundles,

$$A_d^f X = \varinjlim_{Y' \to Y \in \mathfrak{A}_Y} \left[\widehat{A}_d^{f'}(X') / \widehat{B}_d^{f'}(X') \right],$$

where $f' : X' \to Y'$ is the pullback of f by the projective morphism $Y' \to Y$. Recall that the restricted Edidin-Graham-Totaro Chow groups [Kre99, Definition 2.1.4(iv)] are the groups

$$\widehat{A}_d^{f'} X' = \varinjlim_{E \in \mathfrak{B}_{Y'}} A^{\circ}_{d+\operatorname{rk} E} f'^{*} E. \tag{3.1}$$

The quotienting group,

$$\widehat{B}_d^{f'} X' = \coprod_{\substack{p_1, p_2 : W \to Y' \\ g \circ p_1 \cong g \circ p_2}} \coprod_{E, F \in \mathfrak{B}_{Y'}} Z^f_{E,F}$$

$$Z^f_{E,F} = \left\{ p'_{2*}\beta_2 - p'_{1*}\beta_1 \;\middle|\; \begin{array}{c} \beta_1 \in A^{\circ}_{d+\operatorname{rk} E}(p'^{*}_1 f'^{*} E), \beta_2 \in A^{\circ}_{d+\operatorname{rk} F}(p'^{*}_2 f'^{*} F) \\ \beta_1 \sim \beta_2 \text{ in } \widehat{A}_d^{f''}(W') \end{array} \right\}$$

depends on the notation,

$$\begin{tikzcd} W' \arrow[r, shift left, "p'_1"] \arrow[r, shift right, "p'_2"'] \arrow[d, "f''"'] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ W \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Recall that we only consider vector bundles of constant rank. So a cycle in $A_d^f X$ is represented by (g, α'), $\alpha' \in A^{\circ}_{d+\operatorname{rk} E'}(E')$ as in the pullback diagram:

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "f''"] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f\ \text{rep}"] \\ E \arrow[r, "\pi"'] & Y' \arrow[r, "g\ \text{proj}"'] & Y \end{tikzcd} \tag{3.2}$$

There is a natural morphism from the restricted Chow groups to the usual Chow groups, $\iota_f : A_d^f X \to A_d X$.

When f is proper and representable, the cycles in $A_d^f X$ can be pushed forward between naive Chow groups by direct generalization of the definition of pushforward for Deligne-Mumford stacks. [Vis89, Def 3.6] If E is a vector bundle on Y', then f_* pushes the class represented by a cycle $\alpha \in A^{\circ}E'$ forward to $f''_*(\alpha) \in A^{\circ}E$ in the notation of diagram (3.2).

Lemma 3.2. *If $f : X \to Y$ is a proper, representable morphism, then there is a proper pushforward $f_* : A_d^f X \to A_d Y$ for all d.*

A morphism $f : X \to Y$ of stacks is *of relative Deligne-Mumford type* if it pulls back Deligne-Mumford stacks over Y to Deligne-Mumford stacks over X. Note that representable morphisms are of relative Deligne-Mumford type. *Rational* cycles on Deligne-Mumford stacks can be pushed forward [Vis89, Prop 3.7], leading to the following.

Lemma 3.3. *If $f : X \to Y$ is a proper morphism of relative Deligne-Mumford type, then there is a proper pushforward $f_* : A^f_d(X, \boldsymbol{Q}) \to A_d(Y, \boldsymbol{Q})$ for all d.*

4. Properties of restricted Chow groups

Definition 4.1 ([Tot99])**.** An approximating vector bundle of X in codimension d is a vector bundle $E \to X$ such that $E \setminus S$ is an algebraic space for a closed substack S with $\operatorname{codim}_E S > d$. It is called approximating, because $A_j(X) \xrightarrow{\sim} A^\circ_{j+\operatorname{rk} E}(E)$ [Kre99, Cor 2.4.9] for large enough j.

Let $X = [\widehat{X}/G]$ be a global quotient stack. Then for any d, X has an approximating vector bundle in codimension d which is the pullback of $\operatorname{Hom}_k(k^{N+n}, W)$ from BG for a faithful representation W of G of dimension n and N large. [Tot99, Remark 1.4]

Lemma 4.2. *Let $f : X \to Y$ be representable. Let (g, α) be a cycle in $A_d X$ or $A^f_d X$ and F an approximating vector bundle on X, respectively the pullback by f of an approximating vector bundle on Y. Then $(g, \alpha) \sim (1_X, \delta)$ for some naïve cycle $\delta \in A^\circ_{d+\operatorname{rk} F} F$.*

Proof. We consider the case of $A_d X$. The case of $A^f_d X$ is analogous. Let $\alpha \in A^\circ_{d+\operatorname{rk} E} E$ be a representative of a cycle in $A_d X$, and let pullbacks be defined.

$$\begin{tikzcd}
E \oplus H \arrow[r, "q"] \arrow[d, "p"'] & H \arrow[r, "s"] \arrow[d] & F \arrow[d] \\
E \arrow[r, "\text{v.b.}"] & X' \arrow[r, "g\ \text{proj}"] & X
\end{tikzcd}$$

Then $\beta = p^*\alpha$ is a cycle on $E \oplus H$. Let $r = \operatorname{rk} F$. Since H is an algebraic space in dimension $d + r$, the pullback q^* is surjective. Let $\beta = q^*\gamma$ and $\delta = s_*\gamma$ so that $(g, \alpha) \sim (g, \gamma)$. But $(g, \gamma) \sim (1_X, \delta)$ as in [Kre99, Rem 2.1.16]. $\square$

Proposition 4.3. *If Y is a global quotient stack and $f : X \to Y$ is representable, then for all d the natural morphism $\iota_f : A^f_d X \to A_d X$ is an isomorphism.*

Proof. In the notation of (3.2), let F be an approximating vector bundle pulled back from Y. Then ι_f factors as $A^f_d X \xrightarrow{\sim} A^\circ_{d+\operatorname{rk} F} F \xrightarrow{\sim} A_{d+\operatorname{rk} F} F \xleftarrow{\sim} A_d X$. The association $(g, \alpha) \mapsto (1_X, \delta)$ of Lemma 4.2 defines the first isomorphism. Its inverse is the inclusion $A^\circ_{d+\operatorname{rk} F} F \to A^f_d X$. The second map is an isomorphism because F is an algebraic space in dimension $d + \operatorname{rk} F$. [Kre99, Thm 2.1.12(i)] The third is an isomorphism by homotopy invariance. [Kre99, Cor 2.4.9].

$\square$

Proposition 4.4. *If $f : X \to Y$ is of relative Deligne-Mumford type and Y is a global quotient stack, then the natural morphism $\iota_f : A_d^f(X, \boldsymbol{Q}) \to A_d(X, \boldsymbol{Q})$ is an isomorphism for all d.*

Proof. By the fact that $A_d^\circ(X, \boldsymbol{Q}) \simeq A_d(X, \boldsymbol{Q})$ for any Deligne-Mumford stack X [Kre99, Thm 2.1.12(ii)], the morphism ι_f factors in the same way as ι_f in Prop 4.3 over $\boldsymbol{Q}$. $\square$

Consider the fiber diagram.

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ W \arrow[r, "h"'] & Y \end{tikzcd}$$

For each d, the groups $A_d^f(X)$ satisfy the following properties:

(i) Let the above morphism $h : W \to Y$ be a flat morphism of relative dimension r. Then there is a functorial pullback homomorphism $A_d^f X \to A_{d+r}^{\tilde{f}} V$.

(ii) Let the above morphism $h : W \to Y$ be a projective morphism, then there is a functorial pushforward homomorphism $A_d^{\tilde{f}} V \to A_d^f X$.

(iii) The homomorphisms (i) and (ii) are compatible with the natural morphism $A_d^f X \to A_d X$.

Proposition 4.5 (Excision)**.** *Let $i : Z \to Y$ be a closed substack, and $j : U \to Y$ its complement. Let $Z' \to X$ be the pre-image of Z under f, and let U' be its complement. Let $f' = f|_{Z'}$ and $\widetilde{f} = f|_{U'}$. Then for each d the flat pullback and projective pushforward fit together into an exact sequence*

$$A_d^{f'} Z' \xrightarrow{i_*} A_d^f X \xrightarrow{j^*} A_d^{\widetilde{f}} U' \longrightarrow 0.$$

Proof. This is [Kre99, Prop. 2.3.6] where constructions are performed in the lower level (Y, Z, U) and pulled back by f. $\square$

Definition 4.6. Recall that the excision sequence for Chow groups may be extended using underlined Chow groups defined in [Kre99, Corollary 4.1.10]. Here we define their restricted analogues. Let

$$\underline{A}_d^f X = \varinjlim_{Y' \in \mathfrak{A}_Y} \left[\widehat{\underline{A}}_d^{f'} (X') / \widehat{\underline{B}}_d^{f'} (X') \right]$$

where $\widehat{\underline{A}}_d^{f'}$ is defined in analogy with equation (3.1). There is a natural homomorphism $\iota_f : \underline{A}_d^f X \to \underline{A}_d X$.

Proposition 4.7. *Let $p : E \to Y$ be a vector bundle on a Deligne-Mumford stack. Then there is a surjection $p^* : \underline{A}^\circ_d(Y, \boldsymbol{Q}) \to \underline{A}^\circ_{d+\operatorname{rk} E}(E, \boldsymbol{Q})$ for all d.*

Proof. We first reduce to the case of quotient stacks of the form $[W/G]$ for an algebraic space W and a finite group G. Assuming the proposition holds for such quotient stacks, let $U \subset Y$ be such a stack, non-empty and open in Y [LMB00, Corollaire 6.1.1]. By naturality of the long exact sequence [Kre99, (4.2.1)], there are morphisms,

$$\begin{tikzcd}
\underline{A}^\circ_*(Z, \boldsymbol{Q}) \arrow[r] \arrow[d] & \underline{A}^\circ_*(Y, \boldsymbol{Q}) \arrow[r] \arrow[d] & \underline{A}^\circ_*(U, \boldsymbol{Q}) \arrow[r, "\delta"] \arrow[d, "\sim"] & A^\circ_*(Z, \boldsymbol{Q}) \arrow[d, "\sim"] \\
\underline{A}^\circ_*(E|_Z, \boldsymbol{Q}) \arrow[r] & \underline{A}^\circ_*(E, \boldsymbol{Q}) \arrow[r] & \underline{A}^\circ_*(E|_U, \boldsymbol{Q}) \arrow[r] & A^\circ_*(E|_Z, \boldsymbol{Q})
\end{tikzcd}$$

where $Z = Y \setminus U$. The rightmost morphism is an isomorphism by homotopy invariance of Deligne-Mumford stacks with rational coefficients. By noetherian induction on Y, the leftmost vertical morphism is surjective. Hence $\underline{A}^\circ_*(Y, \boldsymbol{Q}) \to \underline{A}^\circ_*(E, \boldsymbol{Q})$ is a surjection by the four lemma.

For the base case, consider a quotient $[W/G]$ with vector bundle E. Then E has the form $[V/G]$ for a vector bundle $V \to W$. There is a diagram,

$$\begin{tikzcd}
A_*(W;1)_{\boldsymbol{Q}} \arrow[r, "\sim"] & A_*(V;1)_{\boldsymbol{Q}} \\
\underline{A}^\circ_*(W, \boldsymbol{Q}) \arrow[r, "\sim", "s^*"'] \arrow[u, "\sim"] \arrow[d, shift left, "q_*"] & \underline{A}^\circ_*(V, \boldsymbol{Q}) \arrow[u, "\sim"'] \arrow[d, shift left, "r_*"] \\
\underline{A}^\circ_*([W/G], \boldsymbol{Q}) \arrow[r, "p^*"'] \arrow[u, shift left, "q^*"] & \underline{A}^\circ_*(E, \boldsymbol{Q}) \arrow[u, shift left, "r^*"]
\end{tikzcd}$$

The upper square vertical isomorphisms follow from the natural isomorphism for algebraic spaces $\underline{A}^\circ_*(-) \xrightarrow{\sim} A_*(-;1)$. [Kre99, Prop 4.1.7] The lower squares exist, because the quotient morphisms are flat and proper, and they commute by compatibility of pullback and pushforward. The morphism r_* is surjective, because $r_* \circ r^*$ is multiplication by $|G|$. But $r_* = p^* \circ q_* \circ s^{*-1}$, so p^* must also be surjective. By similar reasoning p^* is injective, hence an isomorphism. $\square$

Using Proposition 4.7, we see that the proof of Lemma 4.2 works for (g, α) in $\underline{A}_d X$ or $\underline{A}_d(X, \boldsymbol{Q})$ with F' an approximating bundle or, respectively, a Deligne-Mumford stack away from a closed substack of high codimension.

Proposition 4.8. *If $f : X \to Y$ is representable and Y is a global quotient stack, then the natural morphism $\iota_f : \underline{A}^f_d X \to \underline{A}_d X$ is an isomorphism for all d.*

Proof. Let F be an approximating vector bundle on X pulled back from Y. In this case, the association $(g, \alpha) \mapsto (1_X, \delta)$ from Lemma 4.2 defines an isomorphism from $\underline{A}^f_d X$ whose inverse is the inclusion $j_f : \underline{A}^\circ_{d+\operatorname{rk} F'} F' \to$

$\underline{A}_d^f X$. Likewise, it defines an isomorphism from $\underline{A}_d X$ whose inverse is the inclusion $j : \underline{A}^{\circ}_{d+\mathrm{rk}\, F'} F' \to \underline{A}_d X$. Then $j = \iota_f \circ j_f$, so ι_f is an isomorphism. □

Proposition 4.9. *If $f : X \to Y$ is of relative Deligne-Mumford type and Y is a global quotient stack, the natural morphism $\iota_f : \underline{A}_d^f(X, \boldsymbol{Q}) \to \underline{A}_d(X, \boldsymbol{Q})$ is a surjection for all d.*

Proof. Let $(g, \alpha) \in \underline{A}_d(X, \boldsymbol{Q})$. By Lemma 4.2, $(g, \alpha) \sim (1_X, \delta)$ for some naïve cycle δ on the pullback of a vector bundle on Y by f. Then $(1_X, \delta)$ maps to (g, α) under ι_f. □

Proposition 4.10. *With the notation of Proposition 4.5, there is a connecting homomorphism δ_f fitting into a commutative diagram with the connecting homomorphism of* [Kre99, (4.2.2)],

$$\begin{array}{ccc} \underline{A}_d^{\tilde{f}} U' & \xrightarrow{\delta_f} & A_d^{f'} Z' \\ \downarrow & & \downarrow \\ \underline{A}_d U' & \xrightarrow{\delta} & A_d Z' \end{array}$$

Proof. The construction of δ in [Kre99, Equation 4.2.2] goes through for restricted Chow groups. □

Propositions 4.5 and 4.10 fit together in an exact sequence.

Proposition 4.11. *With the notation of Proposition 4.5, let U be a global quotient stack. Then there is an exact sequence,*

$$\underline{A}_d^{\tilde{f}} U' \xrightarrow{\delta_f} A_d^{f'} Z' \to A_d^f X \to A_d^{\tilde{f}} U' \to 0.$$

Proof. The proof is completely analogous to the proof of [Kre99, Prop. 4.2.1]. □

5. Main theorems

Recall that a stack admits a stratification by global quotient stacks if and only if every geometric stabilizer is affine [Kre99, Prop 3.5.9].

Proposition 5.1. *Let Y be a stack stratified by global quotient stacks and let $f : X \to Y$ be representable. Then $A_d^f X \to A_d X$ is an isomorphism for all d. If, alternatively, f is of relative Deligne-Mumford type, then $A_d^f(X, \boldsymbol{Q}) \to A_d(X, \boldsymbol{Q})$ is an isomorphism for all d.*

Proof. The proof proceeds by Noetherian induction. Using the same notation as Proposition 4.5, let U be a global quotient. There is a morphism from the exact sequence of Proposition 4.11 to the exact sequence of [Kre99, Prop 4.2.1],

$$\begin{tikzcd}
\underline{A}_d^{\widetilde{f}}U' \arrow[r, "\delta_f"] \arrow[d, "c'"] & A_d^{f'}Z' \arrow[r] \arrow[d, "a"] & A_d^f X \arrow[r] \arrow[d, "b"] & A_d^{\widetilde{f}}U' \arrow[r] \arrow[d, "c"] & 0 \\
\underline{A}_d U' \arrow[r, "\delta"] & A_d Z' \arrow[r] & A_d X \arrow[r] & A_d U' \arrow[r] & 0
\end{tikzcd}$$

Morphism a is an isomorphism by the induction hypothesis, and morphism c by Proposition 4.3. Then c' is an isomorphism by Proposition 4.8 if f is representable, and is a surjection by Proposition 4.9 if f is of relative Deligne-Mumford type where all Chow groups are taken with $\boldsymbol{Q}$-coefficients. By the five lemma, b is also an isomorphism. □

Theorem 5.2. *Let Y be a stack stratified by global quotients, and let $f : X \to Y$ be a proper, representable morphism. Then there is a proper pushforward $f_* : A_d X \to A_d Y$ for all d. If, instead, f is proper and of relative Deligne-Mumford type, then there is a proper pushforward $f_* : A_d(X, \boldsymbol{Q}) \to A_d(Y, \boldsymbol{Q})$ for all d.*

Proof. The pushforward arises from the factorization

$$\begin{tikzcd}
A_d^f X \arrow[rr] \arrow[dr, "\sim"'] & & A_d Y \\
& A_d X \arrow[ur] &
\end{tikzcd}$$

In case f is of relative Deligne-Mumford type, Chow groups should be taken with $\boldsymbol{Q}$-coefficients. □

References

[Ful10] D. Fulghesu, *Tautological classes of the stack of rational nodal curves*, Comm. Alg. **38** (2010), 2677–2700.

[Kre99] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. **138** (1999), no. 3, 495–536.

[LMB00] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 39, Springer-Verlag, Berlin (2000).

[Tot99] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory: AMS-IMS-SIAM Joint Summer Research Conference on Algebraic K-Theory, July 13-24, 1997, University of Washington, Seattle, Proceedings of Symposia in Pure Mathematics, **67** (1999) 249–282.

[Vis89] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces.* Invent. Math. 97 (1989), no. 3, 613-670.