Principle of Analytical Mechanics

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Abstract

I. CENTRAL FORCE PROBLEM

A. Central-Force Problem

Establish Lagrangian:

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \\
V &= U(r) \\
L &= T - V \\
&= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)
\end{aligned} \tag{1}
$$

Solve Lagrange equations by means of r, θ:

$$
m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial U}{\partial r} = 0 \tag{2}
$$

$$
\frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{3}
$$

End with:

$$
mr^2\dot{\theta} = l \tag{4}
$$

$$
m\ddot{r} - \frac{l^2}{mr^3} + \frac{\partial U}{\partial r} = 0 \tag{5}
$$

l is a constant in Eq. 4, called angular momentum. Eq. 5 represents the correct motion equation.

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B. A wrong method

Note that Eq. 4 can eliminate one degree of freedom. A very straightforward idea is to plug Eq. 4 into Lagrangian before taking the variation:

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} - U(r) = L \tag{6}$$

$$m\ddot{r} + \frac{l^2}{mr^3} + \frac{\partial U}{\partial r} = 0 \tag{7}$$

Eq. 5 and Eq. 7 are different.

C. Effective potential

There is a way that can eliminate the coordinate θ called effective potential[1], say:

$$V_{eff} = U(r) + \frac{1}{2}\frac{l^2}{mr^2} \tag{8}$$

and now the Lagrangian is written as:

$$L = T - V = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}\frac{l^2}{mr^2} - U(r) \tag{9}$$

Taking variation, the motion equation is:

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial U}{\partial r} = 0 \tag{10}$$

We can see under the effective potential Eq. 10 is right. But such method is not general enough. Generally, the reason why Eq. 5 is wrong is that we put into a nonholonomic constraint Eq. 4 before taking variations. Such operation is forbidden in Analytical mechanics.

II. PRINCIPLE OF CONSTRAINTS

A. holonomic constraints

Now we will verify that holonomic constraints can be plugged into Lagrangian before taking variations. Assume there's a Lagrangian:

$$L(q_1, ..., q_n; \dot{q_1}, ..., \dot{q_n}; t)$$

Where $q_1, ..., q_n$ are genearal coordinates. A holonomic constraint in terms of q_i:

$$q_1 = q_1(q_2, ..., q_n; t)$$

$$f(q_1, q_2, ..., q_n; t) = q_1 - q_1(q_2, ..., q_n; t) = 0$$

On the one hand, known from Variations(using Lagrange multiplier) that the Lagrange equation is [2]

$$-\frac{d}{dt}(\frac{\partial L}{\partial \dot{q_j}}) + (\frac{\partial L}{\partial q_j}) + \lambda_i(t)\frac{\partial f_i}{\partial q_j} = 0$$

Where i represents constraints from 1 to m, j represents coordinates from 1 to n. And λ_i can be solved from equations whose $j = 1, ..., m$ when $\frac{D(f_1, f_2, ..., f_m)}{D(y_1, y_2, ..., y_m)} \neq 0$. There is a summation over i(Einstein peace agreement, we would use this for i and j Throughout the article)

Consider these constraints one by one. If there's one holonomic constraint, solve λ from $j = 1$:

$$0 = -\frac{d}{dt}(\frac{\partial L}{\partial \dot{q_1}}) + (\frac{\partial L}{\partial q_1}) + \lambda(t)\frac{\partial f}{\partial q_1}$$
$$\lambda = [\frac{d}{dt}(\frac{\partial L}{\partial \dot{q_1}}) - (\frac{\partial L}{\partial q_1})]$$

While the last equations:

$$-\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j})+(\frac{\partial L}{\partial q_j})+\lambda(t)\frac{\partial f}{\partial q_j}=0 \tag{11}$$

$$-\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j})+(\frac{\partial L}{\partial q_j})+[\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_1})-(\frac{\partial L}{\partial q_1})](-\frac{\partial q_1}{\partial q_j})=0 \tag{12}$$

$$-\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j})-\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_1})\frac{\partial q_1}{\partial q_j}+\frac{\partial L}{\partial q_j}+\frac{\partial L}{\partial q_1}\frac{\partial q_1}{\partial q_j}=0 \tag{13}$$

Eq. 13 is the right equation gained by Lagrange multipliers.

On the other hand, plugging holonomic constraints into Lagrangian before, the Lagrangian becomes:

$$L=L(q_1,...,q_n;\dot{q}_1,...,\dot{q}_n;t)$$

$$L'=L'(q_2,...,q_n;\dot{q}_2,...,\dot{q}_n;t)$$

Before taking variation, using the property of holonomic constraints, write down the form of q_1, $\dot{q}_1$:

$$q_1=q_1(q_2,...,q_n;t) \tag{14}$$

$$\dot{q}_1=\frac{\partial q_1}{\partial q_j}\dot{q}_j+\frac{\partial q_1}{\partial t} \tag{15}$$

$$(\frac{\partial q_1}{\partial q_j})=\frac{\partial q_1}{\partial q_j}(q_2,...,q_n;t) \tag{16}$$

$$\dot{q}_1=\dot{q}_1(q_2,...,q_n;\dot{q}_2,...,\dot{q}_n;t) \tag{17}$$

Eq. 15 imples:

$$\frac{\partial \dot{q}_1}{\partial \dot{q}_j}=\frac{\partial q_1}{\partial q_j} \tag{18}$$

And beginning with Eq. 17,differentiate the formula:

$$\frac{\partial \dot{q}_1}{\partial q_j}=\dot{q}_s\frac{\partial \frac{\partial q_1}{\partial q_s}}{\partial q_j}+\frac{\partial q_1}{\partial q_s}\frac{\partial \dot{q}_j}{\partial q_j}+\frac{\partial \frac{\partial q_1}{\partial t}}{\partial q_j} \qquad (s=2,...,n) \tag{19}$$

Here q_j and $\dot{q}_j$ are independent, so:

$$\frac{\partial \dot{q}_1}{\partial q_j} = \dot{q}_s \frac{\partial \frac{\partial q_1}{\partial q_s}}{\partial q_j} + \frac{\partial \frac{\partial q_1}{\partial t}}{\partial q_j} \tag{20}$$

In the real world, such holonomic constraint is second-order derivable, so exchange the subscript s, j:

$$\frac{\partial \dot{q}_1}{\partial q_j} = \dot{q}_s \frac{\partial \frac{\partial q_1}{\partial q_j}}{\partial q_s} + \frac{\partial \frac{\partial q_1}{\partial q_j}}{\partial t} \qquad (s = 2, ..., n) \tag{21}$$

Differentiating t on $\frac{\partial q_1}{\partial q_j}$:

$$\frac{d}{dt}(\frac{\partial q_1}{\partial q_j}) = \frac{\partial \frac{\partial q_1}{\partial q_j}}{\partial q_s} \dot{q}_s + \frac{\partial \frac{\partial q_1}{\partial q_j}}{\partial t} \tag{22}$$

Obviously Eq. 21 and Eq. 22 are the same. So:

$$\frac{d}{dt}(\frac{\partial q_1}{\partial q_j}) = \frac{\partial \dot{q}_1}{\partial q_j} \tag{23}$$

With the help of Eq. 18,Eq. 23, now take variations on L':

$$-\frac{d}{dt}(\frac{\partial L'}{\partial \dot{q}_j}) + \frac{\partial L'}{\partial q_j} = 0 \tag{24}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial \dot{q}_j} + \frac{\partial L}{\partial q_1}\frac{\partial q_1}{\partial \dot{q}_j}) - (\frac{\partial L}{\partial q_j} + \frac{\partial L}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial q_j} + \frac{\partial L}{\partial q_1}\frac{\partial q_1}{\partial q_j}) = 0 \tag{25}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial \dot{q}_1}\frac{\partial q_1}{\partial q_j}) - \frac{\partial L}{\partial q_j} - \frac{\partial L}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial q_j} - \frac{\partial L}{\partial q_1}\frac{\partial q_1}{\partial q_j} = 0 \tag{26}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) + \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_1})\frac{\partial q_1}{\partial q_j} + \frac{\partial L}{\partial \dot{q}_1}\frac{d}{dt}(\frac{\partial q_1}{\partial q_j}) - \frac{\partial L}{\partial q_j} - \frac{\partial L}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial q_j} - \frac{\partial L}{\partial q_1}\frac{\partial q_1}{\partial q_j} = 0 \tag{27}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) + \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_1})\frac{\partial \dot{q}_1}{\partial \dot{q}_j} + \frac{\partial L}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial q_j} - \frac{\partial L}{\partial q_j} - \frac{\partial L}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial q_j} - \frac{\partial L}{\partial q_1}\frac{\partial q_1}{\partial q_j} = 0 \tag{28}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) + \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_1})\frac{\partial q_1}{\partial q_j} - \frac{\partial L}{\partial q_j} - \frac{\partial L}{\partial q_1}\frac{\partial q_1}{\partial q_j} = 0 \tag{29}$$

Eq. 13 and Eq. 29 are the same. So plugging the holonomic constraint is correct.

B. nonholonomic constraints

III. LAGRANGE MULTIPLIER

When there's constraint, Lagrange multipliers can sustain the original differential relations:

$$
\begin{aligned}
L' &= L + \lambda_\alpha f_\alpha \\
0 &= \delta \int_{t_1}^{t_2} L' dt \\
&= \int \{-[\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - (\frac{\partial L}{\partial q_i})] + \{\lambda_\alpha[\frac{\partial f_\alpha}{\partial q_i} - \frac{d}{dt}(\frac{\partial f_\alpha}{\partial \dot{q}_i})] - \frac{d\lambda_\alpha}{dt}\frac{\partial f_\alpha}{\partial \dot{q}_i}\}\}\delta q_i dt + (\frac{\partial L'}{\partial \dot{q}_i}\delta q_i)\Big|_{initial}^{last}
\end{aligned}
\tag{30}
$$

And the Variation to λ apply the constraint f. All degrees of freedom has been preserved, so the differential relation basically hasn't changed(for the first term). So the meaning about Lagrange multiplier in variations is the same as other math fields —- sustain the original relations, add the restrictions as a result of solving equations. Here is a picture explaining it visually[3]:

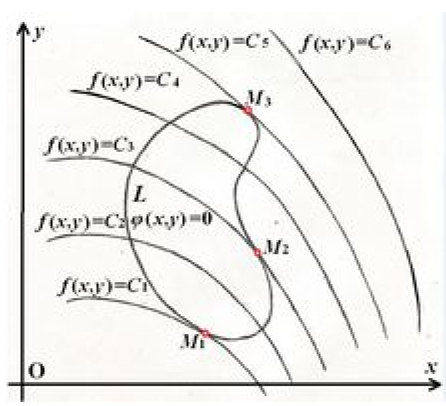

FIG. 1. consider Lagrange multiplier visually

IV. CONDITIONS FOR THE ESTABLISHMENT OF ANALYTICAL MECHANICS

For Analytical Mechanics, it establish on the basis of Hamilton's Principle:

$$\delta \int L dt = 0 \tag{31}$$

$$= \int [-\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) + \frac{\partial L}{\partial q_i}]\delta q_i dt + (\frac{\partial L}{\partial \dot{q}_i}\delta q_i)\Big|_{initial}^{last} \tag{32}$$

$$\delta \int (p_i \dot{q}_i - H) dt = 0 \tag{33}$$

$$= \int [(\dot{q}_i - \frac{\partial H}{\partial p})\delta p - (\dot{p}_i + \frac{\partial H}{\partial q})\delta q] dt + p_i \delta q_i\Big|_{initial}^{last} \tag{34}$$

And there's restrictions say:

$$\begin{cases} \text{all of } \delta q_i \text{(in Hamilton Mechanics, together with } \delta p_i\text{) should be independent} \\ \text{all paths have fixed start and end points} \end{cases} \tag{35}$$

Only over the restrictions Con. 35 can we derive the correct differential equations.

A. Constraints and form of Lagrangian

The exsitence of restrictions Con. 35 can be explained. The independence of q_i comes from freedom. Holonomic constraints reduce the degrees of freedom immediately. Nonholonomic constraint breaks the differential relations between q_i. The other restriction, the fixed beginning and ending, comes from the original $q_{t=0}, \dot{q}_{t=0}$. Right differential equations not only needs restrictions Con. 35, but also needs the standard form of Lagrangian, which symbolizes a right functional in variations. But can we gain right motion equations without sustain the form of Lagrangian?
Inspired by Examples, there's a way to get motion equation when there're conservations by changing the form of Lagrangian.

V. SOLUTIONS FOR CONSERVATIONS

A. Sustain differential relations from conservations

Hamilton's Principle has determined the correct form of Lagrangian whose variation corresponds Newton's law, the correct differential relations. Constraints and Lagrange multiplier absorb restrictions of the space. That's the frame of Analytical Mechanics.

But for some conditions like section (1.2)(1.5), something like momentum conservation can be regarded as an obvious restriction before it is gained from Lagrange equations, which can really simplify the solving process.

The reason why section (1.2) lead out wrong variational relation can be performed by comparing Lagrange equations:

For section(1.1), correct variational relation is:

$$\begin{aligned} \delta \int L dt = 0 \\ = \int [-\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) + \frac{\partial L}{\partial q_i}] \delta q_i dt + (\frac{\partial L}{\partial \dot{q}_i}\delta q_i)\Big|_{initial}^{last} \end{aligned} \tag{36}$$

By Hamilton's Principle, the correct equation is:

$$-\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) + \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, 2..., n)$$

Performed as:

$$-\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) + \frac{\partial L}{\partial r} = 0 \qquad (i = 1, 2..., n) \tag{37}$$

For section (1.2), after plugging into an nonholonomic constraint, the variational relation perform as:

$$
\begin{aligned}
L' &= L'(r, \dot{r}, \dot{\theta}(r, \dot{r})) \\
\delta \int L' dt &= \int [-\frac{d}{dt}(\frac{\partial L'}{\partial \dot{r}}) + \frac{\partial L'}{\partial r}]\delta r dt + (\frac{\partial L'}{\partial \dot{r}}\delta r)\Big|_{initial}^{last} \\
&= \int [-\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}} + \frac{\partial L}{\partial \dot{\theta}}\frac{\partial \dot{\theta}}{\partial \dot{r}}) + (\frac{\partial L}{\partial r} + \frac{\partial L}{\partial \dot{\theta}}\frac{\partial \dot{\theta}}{\partial r})]\delta r dt + (\frac{\partial L'}{\partial \dot{r}}\delta r)\Big|_{initial}^{last}
\end{aligned}
$$

If we extract Lagrange equation from it:

$$
-\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}} + \frac{\partial L}{\partial \dot{\theta}}\frac{\partial \dot{\theta}}{\partial \dot{r}}) + (\frac{\partial L}{\partial r} + \frac{\partial L}{\partial \dot{\theta}}\frac{\partial \dot{\theta}}{\partial r}) = 0 \tag{38}
$$

(11) differs from (10). In order to make use of the conservation and sustain the differential relations, there're other ways to deal with it coming from Routh procedure.[1]

B. Change Lagrangian to sustain differential relations

For the wrong method in section (1.2), we have:

$$
\begin{aligned}
p_\theta &= constant = l \\
&= mr^2\dot{\theta} \\
L' &= L'(r, \dot{r}, \dot{\theta}(r, \dot{r}))
\end{aligned} \tag{39}
$$

Apply such transformation on the Lagrangian:

$$
\begin{aligned}
L''(r, \dot{r}) &= L'(r, \dot{r}, \dot{\theta}(r, \dot{r})) - p_\theta\dot{\theta} \\
&= \frac{1}{2}m(\dot{r}^2 + \frac{l^2}{m^2r^2}) - U(r) - l(\frac{l}{mr^2}) \\
&= \frac{1}{2}m\dot{r}^2 - \frac{l^2}{2mr^2} - U(r)
\end{aligned}
$$

And from this Lagrangian, there's only one degree of freedom in r, who is the only variable in Varitional Method:

$$\delta \int_{t_1}^{t_2} L'' dt = 0 \tag{40}$$

$$\begin{aligned} 0 &= -\frac{\partial L}{\partial r} + \frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) \\ 0 &= m\ddot{r} - \frac{l^2}{mr^3} + \frac{\partial U}{\partial r} \end{aligned} \tag{41}$$

And Eq. 40 is correct compared with Eq. 5. From such an example we can detect that, we can derive the correct equation without using the formal form of Lagrangian.

C. Explaination about the changing in Lagrangian

Under the discussion in section(2.1), it's easy to find:

$$\begin{aligned} \delta \int_{t_1}^{t_2} L'' dt &= \delta \int_{t_1}^{t_2} (L'(r, \dot{r}, \dot{\theta}(r, \dot{r})) - p_\theta \dot{\theta}) dt \\ &= \int [-\frac{d}{dt}(\frac{\partial (L'(r, \dot{r}, \dot{\theta}(r, \dot{r})) - p_\theta \dot{\theta})}{\partial \dot{r}}) + \frac{\partial (L'(r, \dot{r}, \dot{\theta}(r, \dot{r})) - p_\theta \dot{\theta})}{\partial r}] \delta r dt + (\frac{\partial L''}{\partial \dot{r}} \delta r)\Big|_{initial}^{last} \\ &= \int [-\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}} + \frac{\partial L}{\partial \dot{\theta}} \frac{\partial \dot{\theta}}{\partial \dot{r}} - p_\theta \frac{\partial \dot{\theta}}{\partial \dot{r}}) + (\frac{\partial L}{\partial r} + \frac{\partial L}{\partial \dot{\theta}} \frac{\partial \dot{\theta}}{\partial r} - p_\theta \frac{\partial \dot{\theta}}{\partial r})] \delta r dt + (\frac{\partial L''}{\partial \dot{r}} \delta r)\Big|_{initial}^{last} \end{aligned} \tag{42}$$

Because:

$$\frac{\partial L}{\partial \dot{\theta}} = p_\theta \tag{43}$$

So:

$$\delta \int_{t_1}^{t_2} L'' dt = \int [-\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) + \frac{\partial L}{\partial r}] \delta r dt + (\frac{\partial L''}{\partial \dot{r}} \delta r)\Big|_{initial}^{last}$$

With the restriction Eq. 35, we can get:

$$-\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) + \frac{\partial L}{\partial r} = 0 \tag{44}$$

Which is correct compared to Eq. 5.

Because we need to use Eq. 43 to eliminate the items, such progress can only appear at the relation Eq. 4, but other nonholonomic constraints.

VI. PRINCIPLE ABOUT ROUTH'S PROCEDURE

A. Out of the Classical Hamilton Principle

Consider about a system having n degrees of freedom and s degrees of noncyclic coordinites:

$$L = L(q_1, q_2..., q_s; \dot{q}_1, \dot{q}_2..., \dot{q}_n; t)$$
$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) = 0 \qquad (i = s+1, ..., n) \tag{45}$$

Rewrite Eq. 45 into nonholonomic constraints:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$
$$= C_i \qquad (i = s+1, ..., n) \tag{46}$$

Pulg Eq. 46 into Lagrangian:

$$L' = L'(q_1, ..., q_s; \dot{q}_1, ..., \dot{q}_s; t) \tag{47}$$

Using Eq. 46, apply Routh's Procedure:

$$R(q_1, ..., q_n; \dot{q}_1, ..., \dot{q}_s; p_{s+1}, ..., p_n; t) = p_j \dot{q}_j - L'$$

Where j represents the summation over cyclic coordinates.

$$R(q_1, ..., q_n; \dot{q}_1, ..., \dot{q}_s; p_{s+1}, ..., p_n; t) =$$
$$H_{cycl}(p_{s+1}, ..., p_n) - L'_{noncycl}(q_1, ..., q_s; \dot{q}_1, ..., \dot{q}_s)$$

And for the s nonignorable coordinates, the Lagrange equations:

$$-\frac{d}{dt}(\frac{\partial R}{\partial \dot{q}_i}) + \frac{\partial R}{\partial q_i} = 0 \qquad i = 1, ..., s \tag{48}$$

Notice that R contains the Lagrangian Eq. 47 which has q_i only. But Eq. 47 is the Lagrangian who has been plugged by nonholonomic constraints Eq. 46. So L' isn't L at all. Can we use L' to derive Lagrange equations in form of the Lagrange equations derived from L? It does, same principle as section(4.2).

B. Explaination about Routh procedure

Same progress in section (4.1) can be generalized. For Eq. 48, write down the Hamilton Principle:

$$\delta\int_{t_1}^{t_2} Rdt = \delta\int_{t_1}^{t_2}(p_j\dot{q}_j - L'(q_1,...,q_s;\dot{q}_1,...,\dot{q}_s;t))dt \tag{49}$$

$$\begin{aligned}
&= \int[-\frac{d}{dt}(\frac{\partial R}{\partial\dot{q}_i}) + \frac{\partial R}{\partial q_i}]\delta q_i dt + (\frac{\partial R}{\partial\dot{q}_i}\delta q_i)\Big|_{initial}^{last} \\
&= \int[\frac{d}{dt}(\frac{\partial L}{\partial\dot{q}_i} + \frac{\partial L}{\partial\dot{q}_j}\frac{\partial\dot{q}_j}{\partial\dot{q}_i} - p_j\frac{\partial\dot{q}_j}{\partial\dot{q}_i}) - (\frac{\partial L}{\partial q_i} + \frac{\partial L}{\partial\dot{q}_j}\frac{\partial\dot{q}_j}{\partial q_i} - p_j\frac{\partial\dot{q}_j}{\partial q_i})]\delta q_i dt + (\frac{\partial R}{\partial\dot{q}_i}\delta q_i)\Big|_{initial}^{last} \\
&= \int[\frac{d}{dt}(\frac{\partial L}{\partial\dot{q}_i}) - (\frac{\partial L}{\partial q_i})]\delta q_i dt + (\frac{\partial R}{\partial\dot{q}_i}\delta q_i)\Big|_{initial}^{last} \qquad (i=1,...,s;j=s+1,...,n)
\end{aligned} \tag{50}$$

q_i remains independent. So the "Hamilton Principle" which has replaced L by R in form:

$$\delta\int_{t_1}^{t_2} Rdt = 0 \tag{51}$$

Can leads to the correct equation:

$$\frac{d}{dt}(\frac{\partial L}{\partial\dot{q}_i}) - (\frac{\partial L}{\partial q_i}) = 0 \tag{52}$$

Which is the true principle about Routh's Procedure. The meaning of such operating is that, "Hamilton's principle" is the variation about "formal Lagrangian". Such Lagrangian can lead to right differential equations correspond to Newton's law.

Now the Routh procedure R, less rigorously speaking, "replaced the Lagrangian of cyclic coordinates into H_{cycle}". So Eq. 51 is not a real "Hamilton Principle" since for q_i, R contains nonholonomic constraints and H_{cycle}. Routh's Procedure is just a special method for Analytical Mechanics, which is specifically aim to cyclic coordinates.

VII. DISCUSSION ABOUT INDEPENDENCE

One of the two restrictions is the independence of δq_i. Further, q_i and p_i are independent with wach other, especially in Hamilton mechanics.

A. Meaning about independence

Functional means a function of function. The variables are function forms. In Analytical Mechanics, the basic element is t. The variables are q_i, the functions of t. As long as the form of q_i has been determined, the form of $\dot{q}_i$ has, too. So they are not truly independent. The true meaning about "independence" is the conditions of the establishment of D'Alembert principle and Lagrange equations. Genearally:

$$\bar{r} = \bar{r}(q_1, ..., q_n, t) \tag{53}$$

$$\dot{\bar{r}} = \frac{\partial \bar{r}}{\partial q_i}\dot{q}_i + \frac{\partial \bar{r}}{\partial t} \tag{54}$$

$$\frac{\partial \dot{\bar{r}}}{\partial \dot{r}} = \frac{\partial \bar{r}}{\partial r} \tag{55}$$

$$\frac{d}{dt}\left(\frac{\partial \bar{r}}{\partial r}\right) = \frac{\partial \dot{\bar{r}}}{\partial r} \tag{56}$$

Eq. 54 implies:

$$\dot{\bar{r}} = \dot{\bar{r}}(\dot{q}_1, ..., \dot{q}_n, t) \tag{57}$$

Eq. **??** implies:

$$\dot{\bar{r}} = \dot{\bar{r}}(\dot{q}_1, ..., \dot{q}_n) \tag{58}$$

Eq. 53 Eq. 57,Eq. **??** Eq. 58 showed the "independence" between q_i and $\dot{q}_i$. Which means, the function form of $\bar{r}$ is just determined by q_i, while the function form of $\dot{\bar{r}}$ is just determined by $\dot{q}_i$.

B. constraints and independence

And from section(3.1), substitution about holonomic constraints doesn't effect these independence:

$$f(q_1, q_2, ..., q_n, t) = 0 \tag{59}$$

$$\begin{aligned}\bar{r} &= \bar{r}(q_1, ..., q_n, t) \\ &= \bar{r}(q_1(q_2, ..., q_n, t), ..., q_n, t) \\ &= \bar{r}'(q_2, ..., q_n, t) \end{aligned} \tag{60}$$

$$\begin{aligned}\dot{\bar{r}} &= \frac{\partial \bar{r}}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}}{\partial q_1}\frac{\partial q_1}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}}{\partial t} \\ \dot{\bar{r}} &= \dot{\bar{r}}'(\dot{q}_2, ..., \dot{q}_n, t)\end{aligned} \tag{61}$$

Eq. 60 Eq. 61 verified the independence under holonomic constraints, which is the reason why the form of Lagrangian remains the same.
For nonholonomic constraints:

$$\begin{aligned}\bar{r} &= \bar{r}(q_1, ..., q_n, t) \\ &= \bar{r}(q_1(q_2, ..., q_n, \dot{q}_p, t), ..., q_n, t) \\ \dot{\bar{r}} &= \frac{\partial \bar{r}}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}}{\partial q_1}\frac{\partial q_1}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}}{\partial q_1}\frac{\partial q_1}{\partial \dot{q}_p}\ddot{q}_p + \frac{\partial \bar{r}}{\partial t}\end{aligned} \tag{62}$$

Eq. 62 implies that it can't sustain the independence.

C. Consider Lagrange Mechanics in the view of constraints

Solving the form of q_i, which is the aim of Mechanics, is determined by variations and the form of Lagrangian. But the Lagrangian is not only constructed by q_i, but also $\dot{q}_i$. q_i and $\dot{q}_i$ are not truly independent mathematically but just fit some relations Eq. 57 Eq. 58.

In the view of constraints, assume there're 2n coordinates independent with each other:

$$q_{11}, q_{12}, ..., q_{1n}; q_{21}, q_{22}, ..., q_{2n}$$

And there're n nonholonomic constraints:

$$q_{2i} = \dot{q}_{1i} \qquad f = q_{2i} - \dot{q}_{1i}$$

There's a function:

$$L = L(q_{11}, q_{12}, ..., q_{1n}; q_{21}, q_{22}, ..., q_{2n}) \tag{63}$$

With the help of Lagrange multiplier, the Hamilton's principle becomes:

$$\begin{aligned}
0 &= \delta \int_{t_1}^{t_2} (L + \lambda_\alpha f_\alpha) dt && (64)\\
&= \int_{t_1}^{t_2} \Sigma[\frac{\partial L}{\partial q_{1i}} \delta q_{1i} + \frac{\partial L}{\partial q_{2i}} \delta q_{2i} + \Sigma\delta(\lambda_j(q_{2j} - \dot{q_1}_j))]dt && (65)\\
&= \int_{t_1}^{t_2} \Sigma[\frac{\partial L}{\partial q_{1i}} \delta q_{1i} + \frac{\partial L}{\partial q_{2i}} \delta q_{2i} + \Sigma(\lambda_j q_{2j} - \frac{d}{dt}(\lambda_j q_{1j}) + \dot{\lambda}_j q_j)]dt && (66)\\
&= \int_{t_1}^{t_2} \Sigma[(\frac{\partial L}{\partial q_{1i}} + \dot{\lambda}_i)\delta q_{1i} + (\frac{\partial L}{\partial q_{2i}} + \lambda_i)\delta q_{2i}]dt - \Sigma\lambda_i \delta q_{1i}\Big|_{t_1}^{t_2}
\end{aligned}$$

$i = 1, 2..., n$. These n functions determined n λ. Because of the independence in $q_\alpha i$:

$$\dot{\lambda}_i = -\frac{\partial L}{\partial q_{1i}}$$

$$\lambda_i = -\frac{\partial L}{\partial q_{2i}}$$

Restore out the nonholonomic constraints and it leads to:

$$q_{2i} = \dot{q}_{1i}$$

$$\frac{d}{dt}\frac{\partial L}{\partial q_{2i}} = \frac{\partial L}{\partial q_{1i}}$$

Which determined the last Lagrange equations:

$$\begin{aligned}
0 &= \delta \int L dt && (67)\\
&= \int [-\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) + \frac{\partial L}{\partial q_i}]\delta q_i dt + (\frac{\partial L}{\partial \dot{q}_i}\delta q_i)\Big|_{initial}^{last}
\end{aligned}$$

Which is the Hamilton's Principle.

This method highlights the importance of nonholonomic constraints(differential constraints) by the way of the eatablishment of Lagrange Mechanics. You can say that, in Analytical mechanics, the differential relation determines the motion equation. Right differential relation is that q and $\dot{q}$ is One-to-one correspondence:

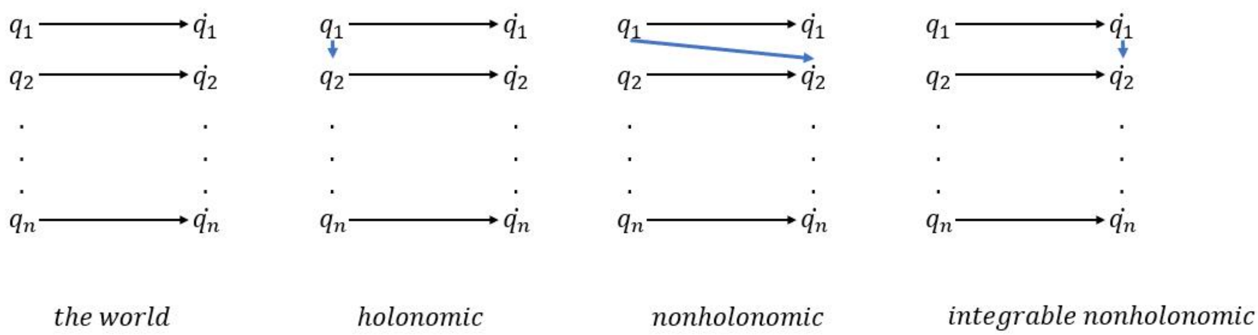

FIG. 2. Comparison between different situations

D. Hamilton Mechanics

In Hamilton's Mechanics, things got concise:

$$\delta \int (p_i \dot{q}_i - H) dt = 0 \tag{68}$$

$$= \int [(\dot{q}_i - \frac{\partial H}{\partial p}) \delta p - (\dot{p}_i + \frac{\partial H}{\partial q}) \delta q] dt + p_i \delta q_i \Big|_{initial}^{last} \tag{69}$$

There's no priorirelations in the definition of q_i, p_i. They are totally independent and perform the same.

E. freedom

By the way of section(7.3), in Lagrange Mechanics, there are (2n) variables originally, and (n) nonholonomic constraints. So there are 2n-n=n freedom in this system and 2n "independent variables". So n Lagrange equations coming from variations can solve out n functionals.

While in Hamilton mechanics there are (2n) coordinates originally, no more. So there are (2n) degrees of freedom, (2n) independent variables. So 2n Hamilton Canonical equations coming form variations can solve out 2n functionals.

In conclusion, the number of freedom is the number of independent variables. Lagrange Mechanics has n independent variables say $q_1, ..., q_n$; Hamilton Mechanics has 2n independent variables say $q_1, ..., q_n; p_1, ..., p_n$.

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[1] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics*, 3rd ed. (Higher Education Press).

[2] D. Lao, *Fundamentals of the Calculus of Variations*, 3rd ed. (National Defense Industry Press).

[3] C. P. Science, *Lagrange multiplier*, 1st ed. (Baidu Baike).