1) M PSD

$$x^T M x = 0$$

Solution 1:

$$x^T M x = \frac{1}{2} x^T (M+M^T) x = 0$$

$(M+M^T)$ symmetric real-valued $\Rightarrow$ Diagonalizable

$$(M+M^T) = P D P^T$$

P unitary $(P^T P = P P^T = I)$

D diagonal

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

$$x^T P D P^T x = 0$$

Let $y = P^T x$, $x = Py$

$$y^T D y = \sum_{i=1}^{n} y_i^2 d_i$$

for each $i \in \{1, \dots, n\}$ either

$y_i = 0$ or $d_i = 0$ $\Rightarrow$ $Dy = 0$

$\Rightarrow PDP^T y = 0 \Rightarrow (M+M^T) x = 0$

2.) M PSD

z is Solution to $LCP(M,q)$

If $(M+M^T)d = 0$

$z+d \geq 0$

$M(z+d)+q \geq 0$

$d^Tq = 0$

Then $z+d$ Solution.

Proof!

Need to show!

$$(z+d)^T\left(M(z+d)+q\right) = 0$$

$$z^T(Mz+q) + d^T(Mz+q) + z^TMd + d^TMd$$

$$= 0 + d^TMz + d^Tq + z^TMd + \tfrac{1}{2}d^T(M+M^T)d$$

$$= 0 + z^TM^Td + 0 + z^TMd + 0$$

$$= z^T(M+M^T)d = 0 \quad \checkmark$$