Functional Programming

Functors, Applicatives, and Parsers

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WS 2022/23

Introduction

- Functors and applicatives are concepts from category theory
- A very general and abstract theory about structures and maps between them
- So general that mathematicians call it "general abstract nonsense"
- Yields very useful abstractions for functional programming
- After a brief review we specialize for Haskell

Plan

1. Categories
2. Functors
3. Applicatives
4. Parsers

(Small) Categories

Definition (part 1)

A small category is given by

- a set of objects,
- for each pair of objects A, B, a set $Hom(A, B)$ of arrows (morphisms) between A and B,
- for each pair of arrows $f \in Hom(A, B)$ and $g \in Hom(B, C)$ (for objects A, B, C), there is an arrow $(f ; g) \in Hom(A, C)$, the composition of f and g (alternativel, write $g \circ f$).

Moreover, the following laws are expected to hold

(Small) Categories

Definition (part 2: laws)

- For each object A there is a designated identity arrow $i_A \in Hom(A, A)$ which behaves as an identity with respect to composition:
  - for each $f \in Hom(A, B)$, $i_A; f = f$,
  - for each $g \in Hom(B, A)$, $g; i_B = g$.
- Composition of arrows is associative, that is:

$$f; (g; h) = (f; g); h$$

for all $f \in Hom(A, B)$, $g \in Hom(B, C)$, and $h \in Hom(C, D)$ and objects A, B, C, D.

Examples of categories (not small)

Set

Objects are sets and morphisms are total functions.

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Group, Ring, Vect

Objects are groups (rings, vector spaces), morphisms are group (ring, vector space) homomorphisms

Smaller categories

FinSet (only essentially small)

Objects are finite sets and morphisms are total functions.

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Partially ordered sets

Every poset $(A, \leq)$ gives rise to a category with objects $a \in A$ and a single morphism m_{ab} for each $a, b \in A$ such that $a \leq b$.

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Graphs

Every graph (N, E) gives rise to category with objects $n \in N$ and morphisms paths in N.

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Hask (small?)

Objects are Haskell types, morphisms are Haskell functions.

Plan

Functors (on Hask)

Definition

A Functor is a mapping `f` between types such that for every pair of type `a` and `b` there is a function `fmap :: (a -> b) -> (f a -> f b)` such that the functorial laws hold:

1. the identity function on `a` is mapped to the identity function on `f a`:
   `fmap id fx == id fx`, for all `fx` in `f a`
2. `fmap` is compatible with function composition
   `fmap (f . g) == fmap f . fmap g`, for all `f :: b -> c` and `g :: a -> b`

Functors (on Hask)

Definition

A Functor is a mapping f between types such that for every pair of type a and b there is a function fmap :: (a -> b) -> (f a -> f b) such that the functorial laws hold:

1. the identity function on a is mapped to the identity function on f a:
   fmap **id** fx == **id** fx, for all fx in f a
2. fmap is compatible with function composition
   fmap (f . g) == fmap f . fmap g, for all f :: b -> c and g :: a -> b

Functions on types

- **Int**, **Bool**, **Double** etc are types.
- parameterized types like [a], BTree a, **IO** a can be considered as a type constructor (i.e., [], BTree, **IO**) applied to a type
- We can express that formally by writing kindings: **Int** :: $*$, **Bool** :: $*$, **Double** :: $*$, but [] :: $* \rightarrow *$, BTree :: $* \rightarrow *$, **IO** :: $* \rightarrow *$

Functors in Haskell

The functor class

```haskell
class Functor f where
  fmap :: (a -> b) -> (f a -> f b)
```

- Recall `f` is a type variable that can stand for **type constructors** (ie, functions on types) like `IO`, `[]`, and others. So `f :: * -> *`!

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Good news

We already know a couple of functors!

List is a functor

- To make list an instance of functor, we need to instantiate the type `f` in the type of `fmap` by `[]`, the list type constructor

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- It remains to check the functorial laws on **map**

Functorial laws for list

fmap **id** fx == **id** fx

fx is a list, so we must proceed by induction

- **map id** [] == [] == **id** []
- **map id** (x:xs) == **id** x : **map id** xs == x : xs == **id** (x : xs)

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fmap (f . g) == fmap f . fmap g

Must hold when applied to any list fx

- **map** (f . g) [] == [] == **map** f (**map** g [])
- **map** (f . g) (x : xs) == (f . g) x : **map** (f . g) xs
  == f (g x) : (**map** f . **map** g) xs by function composition and induction
  == f (g x) : **map** f (**map** g xs) by function composition
  == **map** f (g x : **map** g xs) by **map** f
  == **map** f (**map** g (x : xs)) by **map** g
  == (**map** f . **map** g) (x : xs)

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- Second equation could return **Nothing**, but that would violate the functorial laws
- It remains to check the functorial laws on **mapMaybe**

Functorial laws for Maybe

fmap id fx == id fx

fx is a Maybe, so we must proceed by induction (cases)

- **mapMaybe id Nothing** == **Nothing** == **id Nothing**
- **mapMaybe id** (**Just** x) == **Just** x == **id** (**Just** x)

Functorial laws for Maybe

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- **mapMaybe id Nothing** == **Nothing** == **id Nothing**
- **mapMaybe id** (**Just** x) == **Just** x == **id** (**Just** x)

fmap (f . g) == fmap f . fmap g

Must hold when applied to any Maybe fx

- **mapMaybe** (f . g) **Nothing** == **Nothing** == **map** f (**map** g **Nothing**)
- **mapMaybe** (f . g) (**Just** x)
  == **Just** ((f . g) x)
  == **Just** (f (g x)) by function composition
  == **mapMaybe** f (**Just** (g x)) by **map** f
  == **mapMaybe** f (**mapMaybe** g (**Just** x)) by **map** g
  == (**mapMaybe** f . **mapMaybe** g) (**Just** x)

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- To make BTree an instance of functor, we need to instantiate the type f in the type of fmap by the type constructor BTree
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```
mapBTree g Leaf = Leaf
mapBTree g (Node l a r) = Node (mapBTree g l) (g a) (mapBTree g r)
```

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- It remains to check the functorial laws on mapBTree, but we'll leave this inductive proof to you.

Remark

- Many of the predefined type constructors have `Functor` instances
- Some of them may be unexpected
- For instance **instance Functor** ((,) a) makes the pair type into a functor by defining `fmap` on the second component
- Mapping on the first component would also define a (different) functor!

Plan

Applicatives

- An applicative (functor) is a special kind of functor
- It has further operations and laws
- We motivate it with a couple of examples

Applicative

Example 1: sequencing IO commands

```
sequence :: [IO a] -> IO [a]
sequence [] = return []
sequence (io:ios) = do x <- io
                       xs <- sequence ios
                       return (x:xs)
```

Applicative

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sequence :: [IO a] -> IO [a]
sequence [] = return []
sequence (io:ios) = do x <- io
                       xs <- sequence ios
                       return (x:xs)
```

Alternative way

```
sequence [] = return []
sequence (io:ios) = return (:) `ap` io `ap` sequence ios

return :: Monad m => a -> m a
ap :: Monad m => m (a -> b) -> m a -> m b
```

Applicative

Example 2: transposition

```
transpose :: [[a]] -> [[a]]
transpose [] = repeat []
transpose (xs:xss) = zipWith (:) xs (transpose xss)
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Rewrite

```
transpose [] = repeat []
transpose (xs:xss) = repeat (:) `zapp` xs `zapp` transpose xss

zapp :: [a -> b] -> [a] -> [b]
zapp fs xs = zipWith ($) fs xs
```

Applicative Interpreter

A datatype for expressions

```
data Exp v
  = Var v -- variables
  | Val Int -- constants
  | Add (Exp v) (Exp v) -- addition
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Standard interpretation

```
eval :: Exp v -> Env v -> Int
eval (Var v) env = fetch v env
eval (Val i) env = i
eval (Add e1 e2) env = eval e1 env + eval e2 env

type Env v = v -> Int
fetch :: v -> Env v -> Int
fetch v env = env v
```

Applicative Interpreter

Alternative implementation

```
eval' :: Exp v -> Env v -> Int
eval' (Var v) = fetch v
eval' (Val i) = const i
eval' (Add e1 e2) = const (+) `ess` (eval' e1) `ess` (eval' e2)

ess a b c = (a c) (b c)
```

Applicative

Extract the common structure

```
class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

Applicative

Laws

- Identity

```
pure id <*> v == v
```

- Composition

```
pure (.) <*> u <*> v <*> w = u <*> (v <*> w)
```

- Homomorphism

```
pure f <*> pure x = pure (f x)
```

- Interchange

```
u <*> pure y = pure ($ y) <*> u
```

Instances of Applicative

- List, Maybe, and IO are also applicatives

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Lists

```haskell
instance Applicative [] where
  -- pure :: a -> [a]
  pure a = [a]
  -- (<*>) :: [a -> b] -> [a] -> [b]
  fs <*> xs = concatMap (\f -> map f xs) fs
```

Instances of Applicative

- List, Maybe, and IO are also applicatives

Lists

```haskell
instance Applicative [] where
  -- pure :: a -> [a]
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```

Maybe

```haskell
instance Applicative Maybe where
  -- pure :: a -> Maybe a
  pure a = Just a
  -- (<*>) :: Maybe (a -> b) -> Maybe a -> Maybe b
  Just f <*> Just a = Just (f a)
  _ <*> _ = Nothing
```

Plan

An interesting example for Applicatives

Simple arithmetic expressions

```haskell
data Term = Con Integer
          | Bin Term Op Term
            deriving (Eq, Show)

data Op = Add | Sub | Mul | Div
            deriving (Eq, Show)
```

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Simple arithmetic expressions

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data Term = Con Integer
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```

Parsing expressions

- Read a string like `"3+42/6"`
- Recognize it as a valid term
- Return `Bin (Con 3) Add (Bin (Con 42) Div (Con 6))`

Parsing

The type of a simple parser

```
type Parser token result = [token] -> [(result, [token])]
```

Combinator parsing

Primitive parsers

```
pempty :: Parser t r
succeed :: r -> Parser t r
satisfy :: (t -> Bool) -> Parser t t
msatisfy :: (t -> Maybe a) -> Parser t a
lit :: Eq t => t -> Parser t t
```

Combinator parsing II

Combination of parsers

```
palt :: Parser t r -> Parser t r -> Parser t r
pseq :: Parser t (s -> r) -> Parser t s -> Parser t r
pmap :: (s -> r) -> Parser t s -> Parser t r
```

A taste of compiler construction

A lexer

A lexer partitions the incoming list of characters into a list of tokens. A token is either a single symbol, an identifier, or a number. Whitespace characters are removed.

Underlying concepts

Parsers have a rich structure

- parsing illustrates functors, applicatives, as well as monads that we already saw in the guise of IO instructions

Parsing is . . .

A functor

Check the functorial laws!

An applicative

Check applicative laws!

A monad

Check the monad laws (upcoming)!

Consequence

Can use `do` notation for parsing!

Parsers are Applicative!

```
instance Applicative (Parser' token) where
  pure = return
  (<*>) = ap

instance Alternative (Parser' token) where
  empty = mzero
  (<|>) = mplus
```

Wrapup

- what if there are multiple applicatives?

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- applicatives cannot express dependency
- enable more clever parsers