Introduction to Partial Differential Equations

Ian Poon

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Selected theorems from `MIT 18.152 Introduction to PDEs` course notes.

Contents

1 Introduction

Fact 1 (PDE Notation)
Note the following notation

$$\frac{\partial}{\partial x^i} = u_{x^i} = \partial_i u$$

and

$$\frac{\partial^2 u}{\partial x^i x^j} = u_{x^i x^j} = \partial_i \partial_j u$$

or

$$\partial_i \partial_i u = \partial_i^2 u$$

Definition 2
Let $u = u(x^1, x^2, \ldots, x^n)$. A PDE in a single unknown u is an equation involving u and its partial derivatives in the form

$$F(u, u_{x^1}, \ldots, u_{x^n}, u_{x^1 x^1} \ldots, u_{x^{i_1} \ldots x^{i_N}}, x^1, x^2, \ldots, x^n)$$

where $i_1, \ldots, i_N \in \{1, 2, \ldots, n\}$ here N is the **order** of the PDE is the degree of the highest partial derivative

Example 3
Let $u = u(t, x)$

$$-\partial_t^2 + (1 + \cos u)\partial_x^3 u = 0$$

is a **non-linear** 3rd order PDE. Because the dependent variable u occurs in a product with a partial derivative.

Example 4
Let $u = u(t, x)$

$$-\partial_t^2 u + 2\partial_x^2 u + u = t$$

is a **linear** 2nd order PDE. It is a *constant coefficient linear* PDE

Example 5
Let $u = u(t, x)$

$$\partial_t u + 2(1 + x^2)\partial_x^3 u + u = t$$

is a **linear** 3rd order PDE. It is a *variable coefficient linear* PDE

Definition 6 (Linear PDE)
A PDE is **linear** if it can be written as

$$\mathcal{L}u = f(x^1, \ldots, x^n)$$

if $f = 0$ we say that the PDE is **homogenous**. Otherwise it is **inhomogenous**

Definition 7 (C^k norms)

Let f be defined on domian $\Omega \subset \mathbb{R}$

$$||f||_{C^k(\Omega)} = \sum_{a=0}^{k} \sup_{x\in\Omega} \left| f^{(a)}(x) \right|$$

where $f^{(a)}$ is the ath order derivative of $f(x)$

Example 8

When f is a function of multiple variables, take for example $f = (t, x)$

$$||f||_{C^{1,2}} = \sum_{a=0}^{1} \sup_{(t,x)\in\mathbb{R}^2} |\partial_t^a f(t,x)| + \sum_{a=0}^{2} \sup_{(t,x)\in\mathbb{R}^2} |\partial_x^a f(t,x)|$$

where 1 is for t and 2 is for x, essentially follows the order given in the definition

Definition 9 (Vectorfield)

A vectorfield $\mathbf{F}$ on $\Omega \subset \mathbb{R}^n$ is an $\mathbb{R}^n$ given by

$$\mathbf{F} : \Omega \to \mathbb{R}^n$$

$$\mathbf{F}(x^1, \ldots, x^n) = \left(F^1(x^1, \ldots, x^n), \ldots, F^n(x^1, \ldots, x^n) \right)$$

where each F^i are scalar valued functions on $\mathbb{R}^n$

Definition 10 (Divergence)

$$\nabla \cdot \mathbf{F} = \sum_{i=1}^{n} \partial_i F^i$$

like the laplacian(see below) the symbol ∂_i by definition implies the divergence operator only applies to spatial variables/cartesian coordinates x_i. We will assume this for the rest of the notes too

Theorem 11 (Divergence Theorem)

Let $\Omega \subset \mathbb{R}^3$ be a domain with a boundary that we denote by $\partial\Omega$

$$\int_\Omega \nabla \cdot \mathbf{F}(x,y,z) dxdydz = \int_{\partial\Omega} \mathbf{F}(\sigma) \cdot \hat{\mathbf{N}}(\sigma) d\sigma$$

where $\hat{\mathbf{N}}(\sigma)$ is the **unit outward** normal bector to $\partial\Omega$ where $\partial\Omega = \{(x,y,z) | z = \phi(x,y)\}$ then

$$d\sigma = \sqrt{1 + |\nabla\phi(x,y)|^2} dxdy$$

where $\nabla\phi = (\partial_x\phi, \partial_y\phi)$ is the gradient of ϕ and $|\nabla\phi| = \sqrt{(\partial_x\phi)^2 + (\partial_y\phi)^2}$ is the *euclidean length* of $\nabla\phi$(obviously also only applies to spatial variables)

Definition 12 (Laplacian)
The **laplacian** operator is defined in terms of *cartesian coordinates* by

$$\Delta = \sum_{i=1}^{n} \partial_i^2$$

Note that means suppose we a function defined in terms of space time coordinates $u(t,x) = u(t, x_1, \ldots, x_n)$ then the laplacian is one that applies only to spatial variables that is

$$\Delta u = \sum \partial_i^2 u = \sum \frac{\partial^2 u}{\partial x_i x_i}$$

We will assume this for the rest of the notes

Proposition 13

$$\nabla \cdot \nabla u = \Delta u$$

Proof. See that

$$\nabla \cdot \nabla u = \nabla \cdot [\partial_x u, \partial_y u] = \partial_{xx} u + \partial_{yy} u = \Delta u$$

1.1 well-posedness

Fact 14
We say a problem is **well-posed** if that our solution for a PDE is able to depend "continuously" on data conditions

Definition 15 (Dirichet Boundary Conditions)

$$\begin{cases} \partial_t u - D\partial_x^2 u = 0 & (t,x) \in (0,T) \times (0,L) \\ u(0,x) = g(x) & x \in [0,L] \\ u(t,0) = h_0(t) & u(t,L) = h_L(t), t > 0 \end{cases}$$

Definition 16 (Neumann Boundary Conditions)

$$\begin{cases} \partial_t u - D\partial_x^2 u = 0 & (t,x) \in (0,T) \times (0,L) \\ u(0,x) = g(x) & x \in [0,L] \\ -\partial_x u(t,0) = h_0(t) & \partial_x u(t,L) = h_L(t) \end{cases}$$

Definition 17 (Robin Boundary Conditions)

$$\begin{cases} \partial_t u - D\partial_x^2 u = 0 & (t,x) \in (0,T)\times(0,L) \\ u(0,x) = g(x) & x \in [0,L] \\ -\partial_x u(t,0) + \alpha u(t,0) = h_0(t) & \partial_x u(t,L) + \alpha h_L(t) = h_L(t) \end{cases}$$

Which is a linear combination of both **dirichet** and **neumann** boundary conditions(see the last line). As you can see the last 2 lines of each boundary condition describe the effect of a change in x when $t = 0$ and change in t when $x = 0$ respectively.

1.2 useful multivariable results

Definition 18

A **radial function** is a function whose value at each point depends only on the distance between that point and the origin. For example a radial function in $\mathbb{R}^2$ has the form

$$\Phi(x,y) = \phi(r), \quad r = \sqrt{x^2+y^2}$$

The euclidean norm itself is another example

Proposition 19

Along $\partial B_r(0)$ we have that

$$\partial_r u = \nabla u \cdot \hat{N} = \nabla_{\hat{N}} u$$

where $\hat{N}(\sigma)$ is the unit normal ot $\partial B_r(0)$

Proof. Consider the case for $\mathbb{R}^2$, then in polar coordinates (r,θ) we have

$$\frac{\partial u}{\partial r}(r\cos\theta, r\sin\theta) = \nabla u \cdot \frac{\partial}{\partial r}\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \nabla u \cdot \hat{N}$$

As that matrix simply represented the position vector relative to the origin at the centre of the sphere. Clearly the position vector is normal to the surface of the sphere and is pointing outwards so it positively oriented.

Proposition 20

We know that for spherically symmetric functions we have $\Phi(x) = \Phi(r)$ in $\mathbb{R}^n$ where $r = |x|$. Show that $\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r$ when $r > 0$ for such functions

Proof. The Laplacian in Cartesian coordinates is given by:

$$\Delta\Phi = \sum_{i=1}^{n} \frac{\partial^2\Phi}{\partial(x^i)^2}.$$

Since $\Phi(x) = \Phi(r)$, where $r = \|x\| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 \ldots}$, the function Φ depends only on r, and not directly on the individual coordinates $x^1, x^2, x^3 \dots$. Therefore, we can express the derivatives of Φ with respect to x^i using the chain rule.

By the chain rule, the first derivative of Φ with respect to x^i is:

$$\frac{\partial \Phi}{\partial x^i} = \frac{d\Phi}{dr} \cdot \frac{\partial r}{\partial x^i}.$$

Now, calculate $\frac{\partial r}{\partial x^i}$:

$$\frac{\partial r}{\partial x^i} = \frac{\partial}{\partial x^i} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 \dots} = \frac{x^i}{r}.$$

Thus,

$$\frac{\partial \Phi}{\partial x^i} = \frac{d\Phi}{dr} \cdot \frac{x^i}{r} = \Phi'(r) \cdot \frac{x^i}{r}.$$

Next, compute the second derivatives of Φ:

$$\frac{\partial^2 \Phi}{\partial (x^i)^2} = \frac{\partial}{\partial x^i} \left(\Phi'(r) \cdot \frac{x^i}{r} \right).$$

Apply the product rule:

$$\frac{\partial^2 \Phi}{\partial (x^i)^2} = \Phi''(r) \cdot \left(\frac{x^i}{r} \right)^2 + \Phi'(r) \cdot \frac{\partial}{\partial x^i} \left(\frac{x^i}{r} \right).$$

The first term is:

$$\Phi''(r) \cdot \frac{(x^i)^2}{r^2}.$$

For the second term, compute $\frac{\partial}{\partial x^i} \left(\frac{x^i}{r} \right)$:

$$\frac{\partial}{\partial x^i} \left(\frac{x^i}{r} \right) = \frac{r^2 - (x^i)^2}{r^3} = \frac{1}{r} \left(1 - \frac{(x^i)^2}{r^2} \right).$$

Thus, the second derivative is:

$$\frac{\partial^2 \Phi}{\partial (x^i)^2} = \frac{\Phi''(r)(x^i)^2}{r^2} + \frac{\Phi'(r)}{r} \left(1 - \frac{(x^i)^2}{r^2} \right).$$

Sum over all $i = 1, 2, 3, \dots$ to obtain the full Laplacian:

$$\Delta \Phi = \sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial (x^i)^2} = \Phi''(r) \cdot \sum_{i=1}^{n} \frac{(x^i)^2}{r^2} + \frac{\Phi'(r)}{r} \cdot \sum_{i=1}^{n} \left(1 - \frac{(x^i)^2}{r^2} \right).$$

Note that $\sum_{i=1}^{n} (x^i)^2 = r^2$, so:

$$\Delta \Phi = \Phi''(r) \cdot \frac{r^2}{r^2} + \frac{\Phi'(r)}{r} \cdot \sum_{i=1}^{n} \left(1 - \frac{(x^i)^2}{r^2} \right).$$

Simplify the second term:

$$\sum_{i=1}^{n} \left(1 - \frac{(x^i)^2}{r^2} \right) = n - 1$$

Thus, the Laplacian becomes:

$$\Delta \Phi = \Phi''(r) + \frac{n-1}{r} \Phi'(r).$$

Theorem 21 (Green's First identity)
Essentially multivariable integration by parts

$$\int_D \nabla v \cdot \nabla u + \int_D v\Delta u = \int_{\partial D} v\frac{\partial u}{\partial \mathbf{n}}$$

where $\mathbf{n}$ is the unit normal

Proof. Recall from above that we denote $u_i = \partial_{x_i} u$ and $u_{ii} = \partial_{x_i}\partial_{x_i} u$. Now consider the 3 dimensional case

$$\begin{aligned}\nabla \cdot \underbrace{(v\nabla u)}_{\text{scalar product}} &= \nabla \cdot (v\frac{\partial u}{\partial x_1}, v\frac{\partial u}{\partial x_2}, v\frac{\partial u}{\partial x_3}) \\ &= \frac{\partial}{\partial x_1}(v\frac{\partial u}{\partial x_1}) + \frac{\partial}{\partial x_2}(v\frac{\partial u}{\partial x_2}) + \frac{\partial}{\partial x_3}(v\frac{\partial u}{\partial x_3}) \\ &= v_1u_1 + vu_{11} + v_2u_2 + vu_{22} + v_3u_3 + vu_{33}\end{aligned}$$

where the last line follows by chain rule on each $\frac{\partial}{\partial x_i}(v\frac{\partial u}{\partial x_i})$ therefore since

$$\nabla v \cdot \nabla u = (v_1, v_2, v_3)\cdot(u_1, u_2, u_3) = v_1u_1 + v_2u_2 + v_3u_3$$

and

$$v\nabla u = vu_{11} + vu_{22} + vu_{33}$$

hence we have

$$\nabla \cdot (v\nabla u) = \nabla v \cdot \nabla u + v\nabla u$$

By the divergence theorem we have

$$\int_D \nabla \cdot (v\nabla u)dx = \int_{\partial D} (v\nabla u)\cdot \mathbf{n} dS$$

recalling from `analysis and manifolds` that

$$(v\nabla u)\cdot \mathbf{n} = (vD_1u, vD_2u, vD_3u)\cdot \mathbf{n} = vDu\cdot \mathbf{n} = v\frac{\partial u}{\partial \mathbf{n}}$$

and the theorem clearly follows

Theorem 22 (Green's Second identity)

$$\int_{\partial\Omega} (-v\frac{\partial u}{\partial \mathbf{n}} + u\frac{\partial v}{\partial \mathbf{n}})dS = \int_\Omega (u\Delta v - v\Delta u)dx$$

where $\mathbf{n}$ is the unit normal

Proof. Consider from **green's 1st identity** we have

$$\int_D \nabla v \cdot \nabla u + \int_D v\Delta u = \int_{\partial D} v\frac{\partial u}{\partial \mathbf{n}}$$

and

$$\int_D \nabla u \cdot \nabla v + \int_D u\Delta v = \int_{\partial D} u\frac{\partial v}{\partial \mathbf{n}}$$

then combining we have and noticing that dot products in our euclidean case are order invariant we have

$$\int_D \nabla u\cdot\nabla v = -\int_D u\Delta v + \int_{\partial D} u\frac{\partial v}{\partial \mathbf{n}} = -\int_D v\Delta u + \int_{\partial D} v\frac{\partial u}{\partial \mathbf{n}}$$

and the theorem clearly follows

Theorem 23 (n-sphere polar coordinates)
The polar coordinates in n-dimensions can be written as

$$\begin{cases} x_1 = r\prod_{k=1}^{n-1}\sin\theta_k \\ x_m = r\cos\theta_{m-1}\prod_{k=m}^{n-1}\sin\theta_k \quad 2 \le m \le n-1 \\ x_n = r\cos\theta_{n-1} \end{cases}$$

Proof. See appendix in good notes

Theorem 24 (Change of coordinates formula for n-sphere)
Jacobian determinant for n-dimensional polar coordates

$$J_n = \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(r, \theta_1, \theta_2, \dots, \theta_{n-1})}$$

and for $n \ge 3$

$$\det J_n = (-1)^{n-1} r^{n-1} \prod_{k=2}^{n-1} \sin^{k-1}\theta_k$$

Remark 25. *You should already from basic multi-variable calculus that for $n = 2$ we have*

$$J_2 = \frac{\partial(x_1, x_2)}{\partial(r, \theta_1)} = -r$$

Also notice that $n = 3$ is essentially polar coordinates. Subbing in $n = 3$ into the equation gets you your familiar $dA = R^2\sin\theta d\theta d\phi$

Proof. See appendix in good notes

Definition 26
Consider

$$B_r(0) = \{w \in \mathbb{R}^n : |w - 0| < r\}$$

$$\partial B_r(0) = \{w \in \mathbb{R}^n : |w - 0| = r\}$$

Note that for the n-sphere $B_r(0)$ with radius r

$$dw = (-1)^{n-1} r^{n-1} \prod_{k=2}^{n-1} \sin^{k-1}\theta_k d^{n-1}\theta$$

Example 27

Consider

$$B_1(0) = \{w \in \mathbb{R}^n : |w - 0| < 1\}$$

and

$$B_R(x) = \{w \in \mathbb{R}^n : |w - x| < R\} = \{\sigma \in \mathbb{R}^n : |\sigma| < R\}$$

then on the boundary we have

$$\partial B_1(0) = \{\sigma \in \mathbb{R}^n : |w| = 1\}$$

$$\partial B_R(x) = \{\sigma \in \mathbb{R}^n : |\sigma| = R\}$$

therefore on the boundary by the previou defintion

$$d\sigma = R^{n-1} dw$$

for the same angular coordinates

Theorem 28

Prove that for surface/volume integration over spheres we have

$$\partial_r \int_{B_r(x)} h(y) d^3y = \partial_r \int_0^r \int_{w \in \partial B_1(0)} p^2 h(p, x + pw) dw dp = \int_{w \in \partial B_1(0)} r^2 h(r, x + rw) dw = \int_{\partial B_r(x)} h(\sigma) d\sigma$$

Proof. The first equality follows by making the change of coordinates where $w \in \partial B_1(0)$ so $y = x + pw$ where p is the radius in which we take the integral over. So the 2nd term here is really just another way of writing

$$\int_0^r \int_{v \in \partial B_p(x)} p^2(p, v) dv dp$$

which is just the volume as desired. The 3rd term follows by **leibniz integral rule**. Notice

$$\partial_r \int_0^r \int_{w \in \partial B_1(0)} p^2 h(p, x + pw) dw dp = \partial_r r \int_{w \in \partial B_1(0)} r^2 h(r, x + rw) dw - \underbrace{\partial_r 0 \int_{w \in \partial B_1(0)} 0^2 h(0, x + 0w) dw}_{=0}$$

$$+ \underbrace{\int_0^r \partial_r \int_{w \in \partial B_1(0)} p^2 h(p, x + pw) dw dp}_{=0}$$

The final term follows the reverse reasoning as for term 2.

Example 29

28 can be seen by letting $h(y) = 1$(the 1 function)

$$\partial_r (\frac{4}{3} \pi r^3) = 4\pi r^2$$

so differential with respect to r of a volume of a sphere is just the surface area

1.3 separation of variables

Example 30
Consider the heat equation with vanishing dischlet conditions

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & (t,x) \in (0,T) \times (0,1) \\ u(0,x) = 0 & x \in [0,1] \\ u(t,0) = 0 & u(t,1) = 0, t \in (0,T) \end{cases}$$

Show our solution is the form

$$u(t,x) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x)$$

Solution. To standard method to this is by **separation of variables**. But please use the generalized separation of variables method using **spectral theory** instead.Don't waste all that hard work learning it... refer to `functional analysis`.

2 Diffusion: The heat equation

Definition 31
Let $\Omega \subset \mathbb{R}^n$ be a bounded *spatial* domain.(open connected subset of $\mathbb{R}^n$) and let $T > 0$ be time.
We define the corresponding **spacetime cylinder** $Q_T \subset \mathbb{R}^{1+n}$ by

$$Q_T \equiv (0,T) \times \Omega$$

We denote of the **closure** of Ω by

$$\overline{\Omega} = \Omega \cup \partial\Omega$$

where $\partial\Omega$ is the boundary of the spatial domain.The union of the lower extrema in both time and space $\{t = 0\} \times \overline{\Omega}$ and the boundary points of the space-time cylinder $S_T = (0,T] \times \partial\Omega$ gives the **parabolic boundary** which we denote as

$$\partial_p Q_T = (\overline{\Omega} \times \{0\}) \cup S_T$$

2.1 uniqueness

Fact 32 (Lebniz Integral Rule)
Let f be continuously differentiable real functions on some region R of the (x,t) plane. Then for all $(x,y) \in \mathbb{R}$

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t)dx = f(b(t),t)\frac{db}{dt} - f(a(t),t)\frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dx$$

If a and b are constants then the first 2 terms are zero. We also call this **differentiation under the integral**. See the proof in your `Introduction to Manifolds & Analysis` workbook.

Theorem 33
Solutions $u \in C^{1,2}(\overline{Q}_T)$ to the inhomogenous heat equation

$$\partial_t u - D\partial_x^2 u = f(t,x)$$

are *unqiue* under Dirichlet,Nuemann,Robin or mixed conditions

Remark 34. *$u \in C^{1,2}(\overline{Q_T})$ means the time derivatives of $u(t,x)$ up to order 1(the first index) are continuously differentiable on Q_T and extend continuously to the closure of Q_T. Analagously for the 2nd index.*

Proof. We do the proof for Dirichlet and Cauchy conditions in the case when $D = 1$. So if we take any 2 arbituary solutions and take their difference w, it will clearly satisy the below conditions since

$$\partial_t u_1 - D\partial_x^2 u_1 - (\partial_t u_2 - D\partial_x^2 u_2) = f(t,x) - f(t,x) = 0$$

then by linearity of homoenous linear PDE we have.

$$\partial_t(u_1 - u_2) - D\partial_x^2(u_1 - u_2) = 0$$

The conditions in question are:

$$\begin{cases} \partial_t w - \partial_x^2 w = 0 & (t,x) \in (0,T) \times (0,L) \\ w(0,x) = 0 & x \in [0,L] \\ w(t,0) = 0 & w(t,L) = 0, t \in (0,T) \end{cases}$$

And to prove uniqueness we need to show $w = 0$. A common strategy is to multiply both sides by w then take their integral(reinmann since closed and continuous)

$$\int_{[0,L]} w\partial_t w dx = \int_{[0,L]} w\partial_x^2 w dx$$

Then do **differentiation under the integral** noticing that we can rexpress the LHS by

$$\frac{d}{dt}\frac{1}{2}\int_{a(t)=0}^{b(t)=L} w^2(t,x) = \underbrace{\frac{db}{dt}w^2(L,t)\frac{1}{2} - \frac{da}{dt}w^2(0,t)}_{\text{=0 since L and 0 are constants}} + \int_0^L \frac{d}{dt}w^2(x,t)\frac{1}{2} = \int_{[0,L]} w\partial_t w dx$$
$$= \int w\partial_x^2 w dx$$

And then complete the integration on the RHS using integration by parts.

$$\int_{[0,L]} w\partial_x^2 w dx = \underbrace{-\int_{[0,L]} (\partial_x w(t,x))^2 dx}_{\leq 0} + \underbrace{w(t,x)\partial_x w(t,x)|_{x=0}^{x=L}}_{\text{=0 by boundary conditions}}$$

You will find that letting the "energy" be $E(t) = \int w^2(t,x)dx$ we will get

$$\frac{d}{dt}E(t) \leq 0$$

meaning that it non-strictly decreasing over the interval $(0,T)$ but at boundary conditions $E(0) = 0$ so it must be that $E(t) = 0$ so $w = 0$ as desired.

2.2 Maximum and Comparison Principles

Theorem 35 (Weak Maximum Principle)
Recall that $\Delta = \sum_i^n \partial_i^2$ from 12
Let $w \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ be a solution to

$$w_t - D\Delta w \leq 0$$

where $f \geq 0$. Then $w(t,x)$ obtains its max in the region $\overline{Q}_T$ on $\partial_p Q_T$(parabolic boundary). Thus if w is strictly negative on $\partial_P Q_T$ then it also is on $\overline{Q}_T$

Corollary 36
Suppose that v, w are solution to the heat equation

$$v_t - Dv_{xx} = f$$
$$w_t - Dw_{xx} = g$$

Then

1. (Comparison) If $v \geq w$ on $\partial_q Q_T$ and $f \geq g$ then $v \geq w$ on all of Q_T
2. (Stability) $\max_{\overline{Q}_T} \leq \max_{\partial_p Q_T} |v - w| + T \max_{\overline{Q}_T} |f - g|$

2.3 The fundamental Solution

Definition 37
Recall that $\Delta = \sum_i^n \partial_i^2$ from 12
The **fundamental solution** $\Gamma_D(t,x)$ to the inhomogenous heat equation

$$u_t - D\Delta u = f$$

where as usual $\Delta u = u_{xx}$ and D is a constant is defined by

$$\Gamma_D(t,x) = \frac{1}{(4\pi Dt)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4Dt}}$$

where $x = (x_1, \ldots, x_n)$ and $|x|^2 = \sum_{i=1}^n (x_i)^2$ which is just the euclidean norm

Lemma 38
The fundamental solution satisfies

1. If $x \neq 0$ then $\lim_{t \to 0} \Gamma_D(t,x) = 0$
2. $\lim_{t \to 0^+} \Gamma_D(t,0) = \infty$
3. $\int_{\mathbb{R}^n} \Gamma_D(t,x) d^n x = 1$ for all $t \geq 0$

Proof. (1) and (2) is obvious by inspection. Just consider

$$\frac{1}{0}e^{-\infty} = (\infty)(0) = 0$$

for (1) and

$$\frac{1}{0}e^{0} = (\infty)(1) = \infty$$

for (2). For (3) recall `complex analysis`

Lemma 39

$\Gamma_D(t,x)$ is a solution to the homogenous heat equation, that is when $f = 0$ in the above equation, for $x \in \mathbb{R}^n$, $t > 0$

Proof. Consider

$$u_t - D\Delta u = \partial_t u - D(\partial_{tt}u + \partial_{xx}u)$$

Then computing using elementary calculus we have

$$\partial_t \Gamma_D(t,x) = \partial_t \frac{1}{(4\pi Dt)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4Dt}} = \left(\frac{1}{(4\pi Dt)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4Dt}}\right)\left(\frac{|x|^2}{4Dt^2})\right) - \frac{2\pi Dn}{(4\pi Dt)^{\frac{n}{2}+1}} e^{-\frac{|x|^2}{4Dt}}$$

Do similarly for the rest..

Definition 40

The **delta distribution** δ acts on suitable functions $\phi(x)$ as follows

$$\langle \delta, \phi \rangle = \phi(0)$$

where we $\langle .,. \rangle$ refers to the L^2 inner product, which is if you recall

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)g(x)d^n x$$

Fact 41

$$\int_{\mathbb{R}^n} d^n x = \int_{\mathbb{R}^n} r^{n-1} dr d^{n-1}\theta = C_n \int_{\mathbb{R}^n} r^{n-1} dr$$

which is basically repeated application of $dxdy = rdrd\theta$ and that we can factor out $d^{n-1}\theta$ if we integrate over the entire space which in this case is $\mathbb{R}^n$. For example in 2D $C_n = 2\pi$ which is the term associated circumference of the circle and $C_n = 4\pi$ in 3D which is the term associated with the surface area of a circle. Note that $d^n x = dx_1 \ldots dx_n$ while $d^{n-1}\theta = d\theta_1 \ldots d\theta_{n-1}$

Theorem 42 (AM GM Inequality)

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdots a_n},$$

with equality if and only if

$$a_1 = a_2 = \cdots = a_n$$

Proof. Just do induction...

Lemma 43

Suppose that $\phi(x)$ is a continuous function on $\mathbb{R}^n$ and that there exists constants $a, b > 0$ such that

$$|\phi(x)| \leq ae^{b|x|^2}$$

Then

$$\lim_{t\to 0^+} \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)d^n x = \phi(0)$$

or equivalently

$$\lim_{t\to 0^+} \langle \Gamma_D(t,.), \phi(.)\rangle = \langle \delta(.), \phi(.)\rangle = \phi(0)$$

which is say Γ_D acts like the delta function as $t \to 0^+$ recall 40

$$\lim_{t\to 0^+} \Gamma_D(t,x) = \delta(x)$$

Proof. First we begin with the simple relation

$$\phi(0) = \int_{\mathbb{R}^n} \Gamma(t,x)_D \phi(0) d^n x = \int_{\mathbb{R}^n} \Gamma(t,x)_D \phi(x) d^n x + \int_{\mathbb{R}^n} \Gamma(t,x)_D (\phi(0) - \phi(x)) d^n x$$

then

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \Gamma_D(t,x)(\phi(0) - \phi(x)) d^n x \right| &\leq \int_B \Gamma(t,x)_D |\phi(0) - \phi(x)| d^n x + \int_{B^c} \Gamma_D(t,x)|\phi(0) - \phi(x)| d^n x && (1)\\
&\leq \int_B \Gamma_D(t,x)\varepsilon d^n x + \int_{B^c} \Gamma_D(t,x)|\phi(0)| d^n x + + \int_{B^c} \Gamma_D(t,x)|\phi(x)| d^n x && (2)\\
& && (3)
\end{aligned}$$

(1) follows if you recall that $\left|\int f\right| \leq \int |f|$ for reinmann/lebesgue integrals.(2) follows because ϕ is continuous so for on ball B centred at 0 with radius δ can surely find $(\phi(0) - \phi(x)) \leq \varepsilon$. The rest of the proof follows when we make we use 41 and do change of variables $r = p(\frac{1}{4DT} - b)^{-\frac{1}{2}}$ such that the RHS becomes

$$C_n' \int_{p=R\sqrt{\frac{1}{4Dt} - b}}^{\infty} p^{n-1} e^{-p^2} dp$$

where $C_n' > 0$ is yet another constant. Note that the lower limit of the integral, $p \to \infty$ as $t \to 0^+$ clearly. Therefore this term goes to zero under such conditions as desired.

Definition 44

If f, g are two functions on $\mathbb{R}^n$ then we define their convulation $f * g$ to be

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)(x - y) d^n y$$

recall from `fourier analysis` that they are commutative and associative.

Earlier we have proven that Γ_D solves the homogenous heat equation. However we know that if $x \neq 0$ then $\lim_{t\to 0} \Gamma_D(t,x) = 0$ and $\lim_{t\to 0^+} \Gamma_D(t,0) = \infty$. But what if we require our solution to the homogenous solution to satisfy other well posed condition like cauchy conditions? We will now show that there exists a solution to the homogenous heat equation other than $u = \Gamma_D$ where unlike the fundamental solution we can achieve these well-posed conditions. All we have to do is to take the convolution of the fundamental solution with our desired bounded function defined in our initial conditions.

Theorem 45 (Solving the global Cauchy problem via the fundamental solution)
Assume that $g(x)$ is a continuous function on $\mathbb{R}^n$ that verifies the bounds $|g(x)| \leq ae^{b|x|^2}$ where $a, b > 0$ are constants. Then there exists a solution $u(t, x)$ to the homogenous heat equation

$$\begin{aligned} u_t - D\Delta u &= 0, \quad (t > 0, x \in \mathbb{R}^n) \\ u(0, x) &= g(x), \quad x \in \mathbb{R}^n \end{aligned}$$

existing for $(t, x) \in [0, T) \times \mathbb{R}^n$ where

$$T \equiv \frac{1}{4Db}$$

Furthermore $u(t, x)$ can be represented as

$$\begin{aligned} u(t, x) = [g(.) * \Gamma_D(t, .)](x) &= \int_{\mathbb{R}^n} g(y)\Gamma_D(t, x - y)d^n y \\ &= \frac{1}{(4\pi D)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(y)e^{-\frac{|x-y|^2}{4Dt}} d^n y \end{aligned}$$

The solution $u(t, x)$ is of regularity $C^\infty((0, \frac{1}{4Db}) \times \mathbb{R}^n)$. Finally for each compact subinterval $[0, T'] \subset [0, T)$ there exists constants $A, B > 0$ depending on the subinterval such that

$$|u(t, x)| \leq Ae^{B|x|^2}$$

for all $(t, x) \in [0, T'] \times \mathbb{R}^n$.

Proof. We prove the case when $n = 1$. We first aim to prove the bound property of u. Noticing that u is expressed as a convolution of x, y, we then try to express the express the bound of g in relation to this.That is we have $|g(x - y)| \leq ae^{b|x-y|^2}$. However we notice that our desired bound $Ae^{B|x|^2}$ is independent of x so we should attempt to separate e^x out of the integral. Moreover $\int e^{-y^2} dy < \infty$(recall Gaussian integral from `complex analysis`) First Applying **AM-GM ineqality** we have

$$\frac{\epsilon^{-1}x^2 + \epsilon y^2}{2} \geq \sqrt{\epsilon^{-1}x^2 \cdot \epsilon y^2}$$

which upon simplifying we get

$$|2xy| \leq \varepsilon^{-1}x^2 + \varepsilon y^2$$

and hence

$$|x - y|^2 = x^2 - 2xy + y^2 \leq (1 + \varepsilon^{-1})x^2 + (1 + \varepsilon)y^2$$

Now we can apply it like so

$$|g(x - y)| \leq ae^{b|x-y|^2} \leq ae^{(1+\varepsilon^{-1})b|x|^2} e^{(1+\varepsilon)b|y|^2}$$

Now applying this to our equation for $u(t, x)$ we have

$$\begin{aligned} |u(t, x)| &\leq \int |g(x - y)| \Gamma_D(t, y)dy \\ &\leq ae^{1+\varepsilon^{-1})b|x|^2} \int e^{(1+\varepsilon)b|y|^2} \Gamma_D(t, y)dy \\ &\leq ae^{(1+\varepsilon^{-1})b|x|^2} \int \frac{1}{\sqrt{4\pi D}} t^{-\frac{1}{2}} e^{-[\frac{1}{4\pi Dt} - (1+\varepsilon)b]y^2} dy \\ &= Ae^{(1+e^{-1})b|x|^2} \end{aligned}$$

Secondly we aim to prove it satisfies the intial conditions given. Let

$$\mathcal{L} = \partial_t - D\partial_x^2$$

and

$$u(t,x) = [g(.) * \Gamma_D(t,.)](x) = \frac{1}{(4\pi D)^{\frac{n}{2}}} \int g(y) e^{-\frac{|x-y|^2}{4Dt}} dy$$

Now **differenting under the integral** we have

$$\mathcal{L}u(t,x) = \int_{\mathbb{R}} g(y) \underbrace{\mathcal{L}\Gamma_D(t, x-y)}_{=0 \text{ by } 39} dy = 0$$

and the first 2 terms on the RHS follows as they are constants with respect to y moroever

$$\lim_{y\to\infty} g(y) e^{-\frac{|x-y|^2}{4Dt}} = 0$$

Finally by 43 we have that

$$\lim_{t\to 0^+} u(t,x) = g(x)$$

Naturally the next task is to find out how to solve the inhomogenous heat equation given well-posed conditions. Now consider

Theorem 46 (Duhamel's principle)

Let $g(x)$ and $T \equiv \frac{1}{4Db}$ like in 45. Aloso assume that $f(t,x), \partial_i \partial_j f(t,x)$ are continuous and bounded functions on $[0,T) \times \mathbb{R}^n$ for $1 \le i,j \le n$. Then there exists a unique solutions $u(t,x)$ to the *inhomogenous heat equation*

$$u_t - D\Delta u = f(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}$$

$$u(0,x) = g(x), \quad x \in \mathbb{R}$$

exisiting for $(t,x) \in [0,T) \times \mathbb{R}$. Furthermore $u(t,x)$ can be represented as

$$u(t,x) = (\Gamma_D(t,.) * g)(x) + \int_0^t (\Gamma_D(t-s,.) * f(s,.))(x) ds$$

The solutions has the following regularity properties: $u \in C^0([0,T) \times \mathbb{R}]) \cap C^{1,2}((0,T) \times \mathbb{R})$

The idea is that $u(x)$ is clearly the sum of the general solution and particular solution(which is 0 when $t = 0$). Additionally because Δ is a linear operator on spatial variables only there exists solution in this form as we will prove in the problems below

Problem 47

Let $\mathcal{L} = \partial_t + \tilde{\mathcal{L}}$ where $\tilde{\mathcal{L}}$ is a linear differential operator acting on spatial variables alone(for example $\tilde{L} = -\Delta = \partial_x \partial_x$ as in the heat equation). Suppose we want to solve the inhomogenous problem

$$\mathcal{L}v(t,x) = f(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^n$$

$$v(0,x) = 0, \quad x \in \mathbb{R}^n$$

Show that the solution to the above is

$$v(t,x) = \int_{s=0}^{t} v_{(s)}(t-s,x)ds$$

where each $v_{(s)}$ is the solution the following homogenous intial value problem

$$\mathcal{L}v_{(s)}(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n$$

$$v_{(s)}(0,x) = f(t,x), \quad x \in \mathbb{R}^n$$

Remark 48. *The last 2 lines allows us to make use 45 to rexpress the solution $v(t,x)$ in terms of convolution with f using*

$$v_{(s)}(t-s,x) = (\Gamma_D(t-s,\cdot) * f(s,\cdot))(x)$$

in the case of the heat solution

Solution. Consider the operator $\mathcal{L} = \partial_t + \tilde{\mathcal{L}}$, where $\tilde{\mathcal{L}}$ is a linear differential operator acting on the spatial variables. The goal is to solve the inhomogeneous problem:

$$\mathcal{L}v(t,x) = f(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^n,$$

with the initial condition $v(0,x) = 0$.

We express $v(t,x)$ as an integral of solutions to homogeneous problems with varying initial conditions:

$$v(t,x) = \int_0^t v_{(s)}(t-s,x)\,ds,$$

where $v_{(s)}$ is the solution to the homogeneous problem:

$$\mathcal{L}v_{(s)}(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n,$$

with the initial condition $v_{(s)}(0,x) = f(s,x)$.

Differentiating $v(t,x)$ with respect to t, we obtain:

$$\partial_t v(t,x) = \partial_t \left(\int_0^t v_{(s)}(t-s,x)\,ds \right).$$

Using the Leibniz rule, this becomes:

$$\partial_t v(t,x) = \int_0^t \partial_t v_{(s)}(t-s,x)\,ds + v_{(s)}(0,x).$$

Since $v_{(s)}(t-s,x)$ satisfies $\mathcal{L}v_{(s)}(t-s,x) = 0$, we have:

$$\partial_t v_{(s)}(t-s,x) = -\tilde{\mathcal{L}}v_{(s)}(t-s,x),$$

so

$$\partial_t v(t,x) = -\int_0^t \tilde{\mathcal{L}} v_{(s)}(t-s,x)\,ds + f(t,x).$$

Now, applying the operator $\mathcal{L}$ to $v(t,x)$, we get:

$$\mathcal{L}v(t,x) = \partial_t v(t,x) + \tilde{\mathcal{L}} v(t,x).$$

Substituting the expressions for $\partial_t v(t,x)$ and $\tilde{\mathcal{L}} v(t,x)$:

$$\mathcal{L}v(t,x) = -\int_0^t \tilde{\mathcal{L}} v_{(s)}(t-s,x)\,ds + f(t,x) + \tilde{\mathcal{L}} \int_0^t v_{(s)}(t-s,x)\,ds.$$

Since $\tilde{\mathcal{L}}$ acts only on the spatial variables, it can be brought inside the integral:

$$\mathcal{L}v(t,x) = \int_0^t \left[-\tilde{\mathcal{L}} v_{(s)}(t-s,x) + \tilde{\mathcal{L}} v_{(s)}(t-s,x)\right] ds + f(t,x).$$

The terms inside the integral cancel out, leaving:

$$\mathcal{L}v(t,x) = f(t,x).$$

Thus, the solution $v(t,x)$ to the inhomogeneous problem is given by:

$$v(t,x) = \int_0^t v_{(s)}(t-s,x)\,ds$$

where $v_{(s)}(t,x)$ is the solution to the homogeneous initial value problem with initial condition $v_{(s)}(0,x) = f(s,x)$. □

In the case of where $\tilde{\mathcal{L}} = -\Delta$ as in the heat equation as mentioned in the earlier remark that we have

$$v(t,s) = \int_0^t (\Gamma_D(t-s,\cdot) * f(s,\cdot))(x)\,ds$$

We will now use this to finally prove Durhamel Principle for the inhomogenous heat equation

Problem 49

Finally, use the previous problem to show that a solution to the inhomogenous heat equation

$$\partial_t u(t,x) - D\Delta u = f(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^n$$

$$u(0,x) = g(x), \quad x \in \mathbb{R}^n$$

is

$$u(t,x) = (\Gamma_D(t,.) * g)(x) + \int_0^t (\Gamma_D(t-s,.) * f(s,.))(x) ds$$

where $\Gamma_D(t,x)$ is the fundamental solution.

Hint: Show that the solution to the above equation can be split into two pieces: $u = u_{hom} + u_{inhom}$ where u_{hom} solves the homogenous heat equaith with corrrect data $u_{hom}(0,x) = g(x)$ and u_{inhom} solves the correction equation $\partial_t u_{inhom} = f(t,x)$ with data $u_{inhom}(0,x) = 0$. Then handle each piece separately.

Solution. To find the solution, we split $u(t,x)$ into two parts:

$$u(t,x) = u_{hom}(t,x) + u_{inhom}(t,x),$$

So that we have

$$(\partial_t u_{inhom}(t,x) - D\Delta u_{inhom}(t,x)) + (\partial_t u_{hom}(t,x) - D\Delta u_{hom}(t,x)) = f(t,x) + 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n,$$

with the initial condition:

$$u_{inhom}(0,x) + u_{hom}(0,x) = 0 + g(x), \quad x \in \mathbb{R}^n.$$

where $u_{hom}(t,x)$ solves the homogeneous heat equation(RHS 0) with the initial condition $u_{hom}(0,x) = g(x)$, and $u_{inhom}(t,x)$ solves the inhomogeneous equation(RHS $f(t,x)$) with initial condition $u_{inhom}(0,x) = 0$.

From our previous problem and theorem 45,substituting the expressions for $u_{hom}(t,x)$ and $u_{inhom}(t,x)$ into u, we get:

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma_D(t, x-y) g(y)\, d^n y + \int_0^t \int_{\mathbb{R}^n} \Gamma_D(t-s, x-y) f(s,y)\, d^n y\, ds.$$

This can be written as:

$$u(t,x) = (\Gamma_D(t,\cdot) * g)(x) + \int_0^t (\Gamma_D(t-s,\cdot) * f(s,\cdot))(x)\, ds.$$

Lemma 50 (Invariance of solution to heat equation under translations and parabolic dilations)
Suppose that $u(t,x)$ is a solution to the heat equation

$$\partial_t u(t,x) - D\Delta_x u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n$$

Then the *amplified and translated* function

$$u^*(t,x) = Au(t - t_0, x - x_0)$$

where $A, t_0, x_0 \in \mathbb{R}$ be constants is also a solution. Similarly if $\lambda > 0$ is a constant then the *amplified and parabolically scaled* function

$$u^*(t,x) = Au(\lambda^2 t, \lambda x)$$

is also a solution

Proof. Consider the latter case and recalling that $\Delta_x = \partial_x \partial_x$ we have

$$\partial_t u^*(t,x) - \Delta u^*(t,x) = \lambda^2 A \left((\partial_t u)(\lambda^2 t, \lambda, x) - (D\Delta u)(\lambda^2 t, \lambda x) \right) = 0$$

where we notice the terms in the brackets is basically the our original solution. The proof for the other case is similar.

Definition 51
We define the **total thermal energy** $\mathcal{T}(t)$ at time t associated to $u(t,x)$ by

$$\mathcal{T}(t) = \int_{\mathbb{R}^n} u(t,x) d^n x$$

It is important to note that for rapidly-spatially decaying solutions to the heat equation, $\mathcal{T}(t)$ is constant as required by our conditions. We will show that our definition using u satisfies this.

Lemma 52

Let $u(t,x) \in C^{1,2}([0,\infty)\times\mathbb{R}^n)$ be a solution to the heat equation $-\partial_t u(t,x) + \Delta u(t,x) = 0$. Assume that at each fixed t, $\lim_{|x|\to\infty} |x|^{n-1} |\nabla_x u(t,x)| = 0$ uniformly in x. Furthermore assume that there exists a function $f(x) \geq 0$ not depending on t such that $|\partial_t u| \leq f(x)$ and $\int_{\mathbb{R}^n} f(x) d^n x < \infty$. Then the total thermal energy of $u(t,x)$ is costant in time

$$\mathcal{T}(t) = \mathcal{T}(0)$$

Proof. First differentiate under the integral

$$\frac{d}{dt}\mathcal{T}(t) = \int_{\mathbb{R}^n} \partial_t u(t,x) d^n x = \int_{\mathbb{R}^n} \Delta u(t,x) d^n x = \lim_{R\to\infty} \int_{B_R(0)} \Delta u(t,x) d^n x \tag{1}$$

since $-\partial_t u(t,x) + \Delta u(t,x) = 0$.First $d\sigma$ and dw is the surface measure on $\partial B_r(0)$ and $\partial B_1(0)$ respectively.Meaning we have

- $\partial B_R(0) = \{\sigma \in \mathbb{R}^n : |\sigma| = R\}$
- $\partial B_1(0) = \{w \in \mathbb{R}^n : |w| = 1\}$

since

$$|\sigma| = R\,|w|$$

so by 26 we have

$$d\sigma = R^{n-1} dw$$

and

$$\sigma = \underbrace{Rw}_{\text{scalar product}}$$

So we conclude that via the divergence thoerem

$$\lim_{R\to\infty} \int_{B_R(0)} \Delta u(t,x) d^n x = \lim_{R\to\infty} \int_{B_R(0)} \nabla\cdot\nabla u(t,x) d^n x \tag{2}$$

$$= \lim_{R\to\infty} \int_{\partial B_R(0)} \nabla u(t,\sigma)\cdot \hat{N} d\sigma \tag{3}$$

$$= \lim_{R\to\infty} \int_{\partial B_R(0)} \nabla_{\hat{N}} u(t,\sigma) d\sigma \tag{4}$$

$$= \lim_{R\to\infty} \int_{\partial B_1(0)} R^{n-1} \nabla_{\hat{N}} u(t,Rw) dw \tag{5}$$

$$= \int_{\partial B_1(0)} R^{n-1} \lim_{R\to\infty} \nabla_{\hat{N}} u(t,Rw) dw = 0 \tag{6}$$

where In (6) we also managed to bring to the limit in as we have $\lim_{|x|\to\infty} |x|^{n-1} |\nabla_x u(t,x)| = 0$ uniformly in x. Recall `rudin`, that we can do so.

Finally having proven the properties of the heat equation above, we now look into how they were derived in the first place to achieve such properties.

Fact 53

We first begin with the goal of finding fundamental solutions of the form

$$\Gamma_D(t,x) = \frac{1}{\sqrt{Dt}} V(\zeta)$$

where

$$\zeta \equiv \frac{x}{\sqrt{Dt}}$$

See that the choice of ζ is motivated by 50, which shows that our solutions should be invariant under parabolic dilations since

$$\frac{x}{\sqrt{Dt}} = \frac{\lambda x}{\sqrt{D\lambda^2 t}}$$

3 Laplace and Poisson Equations

Definition 54

The **laplace equation** is

$$\Delta u(x) = 0$$

whlie the **poisson equation** is the inhomogenous equation

$$\Delta u(x) = f(x)$$

Functions $u \in C^2(\Omega)$ verifying the Laplace equation is said to be harmonic.

3.1 uniqueness

Theorem 55

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Then under Dirichet, Robin or mixed boundary conditions theres is at most one solution of regularity $u \in C^2(\Omega) \cap C^2(\overline{\Omega})$ to the **poisson equation**

Proof. Use the same energy method as for the heat equation. Namely taking the difference of solutions $w = u_1 - u_2$ plugging it into the Poisson equation $\Delta w = 0$ then multiplying this by w and integrating it over Ω. That is

$$\begin{aligned} 0 = \int_\Omega w \nabla w d^n x &= -\int_\Omega \nabla w \cdot \nabla w d^n x + \int_{\partial\Omega} w \nabla_{\hat{N}} w d\sigma \\ &= -\int_\Omega |\nabla w|^2 d^n x + \int_{\partial\Omega} w \nabla_{\hat{N}} w d\sigma \end{aligned}$$

which follows by **Green's First identity** from above if you recall. In the case of Dirchihelt data $w|\partial\Omega = 0$ so

$$\int_\Omega |\nabla w|^2 = 0$$

so therefore w is constant on $\overline{\Omega}$. However $\overline{\Omega}$ contains $\partial\Omega$ where we know $w = 0$. Because it is a constant w is constant then it can only be 0 as desired.

Fact 56

The symbol $B_r(x)$ denotes the open ball in $\mathbb{R}^n$ with radius r and center at x that is

$$B_r(x) = \{y \in \mathbb{R}^n; |x - y| < r\}$$

The volume of $B_r(x)$ and the area of its boundary $\partial B_r(x)$ are given by

$$|B_r| = \frac{w_n}{n} r^n$$

and

$$|\partial B_r| = w_n r^{n-1}$$

respectively. w_n is the surface area of the unit sphere ∂B_1 in $\mathbb{R}^n$. In particular

$$w_2 = 2\pi, \quad w_3 = 4\pi$$

See the proof in appendix of good notes too

A special property about harmonic functions is that they satisfy mean value properties.

Theorem 57 (Mean Value Properties)

Let $u(x)$ be harmonic in the domain $\Omega \subset \mathbb{R}^n$ and let $B_r(x) \subset \Omega$ be a ball of radius R centered at point x. Then the following mean value formulas hold

$$u(x) = \frac{n}{w_n R^n} \int_{B_R(x)} u(y) d^n y \tag{1}$$

$$u(x) = \frac{1}{w_n R^{n-1}} \int_{\partial B_R(x)} u(\sigma) d\sigma \tag{2}$$

3.2 Maximum and Comparison Principles

3.3 The fundamental solution

Definition 58

The fundamental solution Φ corresponding to the operator Δ is

$$\Phi(x) \equiv \begin{cases} \frac{1}{2\pi} \ln |x| & n = 2 \\ -\frac{1}{w_n |x|^{n-2}} & n \geq 3 \end{cases}$$

where as usual $|x| = \sqrt{\sum_{n=1}^{\infty} (x^i)^2}$ and w_n is the surface area of the unit ball in $\mathbb{R}^n$. It is clearly to see that the fundmanetal solution is spherically symmetric recall 18 since $\phi(x) = \phi(r)$

Lemma 59

If $x \neq 0$ then $\Delta\Phi(x) = 0$

Proof. In the case when $n = 3$. We have $\Delta = \partial_r^2 + \frac{2}{r}\partial_r$ when $r > 0$ for spherically symmetric functions as proven in the above lemma. Therefore we have from 58 and 20

$$\Delta\Phi = \partial_r^2\Phi + \frac{2}{r}\partial_r\Phi = \frac{-2}{w_3 r^3} + \frac{2}{w_3 r^4} = 0$$

It is easy to show for the other cases as well.

Theorem 60 (Solution to Poisson Equation in $\mathbb{R}^n$)
Let $f(x) \in C_0^\infty(\mathbb{R}^n)$(recall this means $f(x)$ is a smooth compactly supported function on $\mathbb{R}^n$). Then for $n \geq 3$ the laplace equation $\Delta u(x) = f(x)$ has a unique smooth solution $u(x)$ that tends to 0 as $|x| \to \infty$. For $n = 2$ the solution is unique under the assumptions $\frac{u(x)}{|x|} \to 0$ as $|x| \to \infty$ and $|\nabla u(x)| \to 0$ as $|x| \to \infty$. Furthermore these unique solutions can be represented as

$$u(x) = (\Phi * f)(x) = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|y|\, f(x-y)d^2y & n = 1 \\ -\frac{1}{w_n}\int_{\mathbb{R}^n} \frac{1}{|y|^{n-2}} f(x-y)d^n y & n \geq 3 \end{cases}$$

Furthermore there exists constants $C_n \geq 0$ such that the following decay estimate holds for the solution as $|x| \to \infty$

$$|u(x)| \leq \begin{cases} C_2 \ln|x| & n = 2 \\ \frac{C_n}{|x|^{n-2}} & n \geq 3 \end{cases}$$

Remark 61. *If we can show there indeed exists solution u to the poisson equation $\Delta u = f$ that can be represeted as $\Phi * f$ as given in the proposition here we will have*

$$f = \Delta u = \Delta(\phi * f) = (\Delta\phi) * f$$

which implies $\Delta\phi = \delta$ recall 40

Proof. Consider the case for $n = 3$. Using the fact that $\Delta_x f(x-y) = \Delta_y f(x-y)$ we derive

$$\Delta_x u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|y|}\Delta_x f(x-y)d^3y = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|y|}\Delta_y f(x-y)d^3y$$

To show that the RHS of the above equal to $f(x)$ we will split the integral into

$$\Delta_x u(x) = \underbrace{-\frac{1}{4\pi}\int_{B_\varepsilon(0)} \frac{1}{|y|}\Delta_x f(x-y)d^3y}_{I} \underbrace{-\frac{1}{4\pi}\int_{B_\varepsilon^c(0)} \frac{1}{|y|}\Delta_y f(x-y)d^3y}_{II}$$

We first show that I goes to 0 as $\varepsilon \to 0^+$. First let

$$M = \sup_{y\in\mathbb{R}^3} |f(y)| + |\nabla f(y)| + |\Delta_y f(y)|$$

which exists since f is smooth and $B_\varepsilon(0)$ is clearly bounded. Then we have

$$|I| \leq \int_{B_\varepsilon(0)} \left|\frac{1}{|y|}\Delta_y f(x-y)\right| d^3y \leq M\int_{r=0}^{\varepsilon}\int_{\partial B_1(0)} r\,dw\,dr = 2\varepsilon^2\pi M$$

To see why first note that since we have $d^3y = r^2 dw dr$ in which case

$$\int_{r=0}^{\varepsilon}\int_{\partial B_r(0)} r^2 dw dr = \frac{4\pi}{3}\varepsilon^3$$

which is the volume. However we used the fact that

$$\left|\frac{1}{|y|}\Delta_y f(x-y)\right| \leq \left|\frac{1}{|y|}\right| |\Delta_y f(x-y)| \leq rM$$

so we replaced r^2 with r in the integral. Clearly this result goes to zero when $\varepsilon \to 0^+$. So now our only hope is to prove that

$$|f(x) - II| \to 0$$

as $\epsilon \to 0^+$ so we can get our desired conclusion. Using **green's identity** and that $d^3y = r^2 dw$ where $w \in \partial B_1(0) \subset \mathbb{R}^3$ and so $dw = \sin\theta d\theta d\phi$ recall 26. So we have

$$\frac{1}{4\pi}\int_{B_\varepsilon^c(0)} -\frac{1}{|y|}\Delta_y f(x-y) + f(x-y)\underbrace{\Delta_y \frac{1}{|y|}}_{=0} d^3y = \frac{1}{4\pi}\int_{\partial B_\varepsilon^c(0)} \frac{1}{|\sigma|}\nabla_{\hat{N}(\sigma)} f(x-\sigma) - f(x-\sigma)\nabla_{\hat{N}(\sigma)}\frac{1}{|\sigma|} d\sigma$$

where $\Delta_y \frac{1}{|y|} = 0$ because function $\frac{1}{|y|}$ is **radially symmetric** recall 18 and so by 59 its laplacian is zero. Also we have swapped the signs on the LHS because $\nabla_{\hat{N}(\sigma)}$ is the unit outward radial derivative onn the sphere $\partial B_\varepsilon(0)$ which is exactly the opposite of that of ∂B_ε^c. However they both separated by the same boundary

$$\partial B_\varepsilon(0) = \partial B_\varepsilon^c(0)$$

.First define

- $B_r(0) = \{\sigma \in \mathbb{R}^n : |\sigma| = 1\}$ and $B_1(0) = \{w \in \mathbb{R}^n : |w| = 1\}$
- $\hat{N}(\sigma) = \frac{\sigma - 0}{r}$
- $\nabla_{\hat{N}(\sigma)}\frac{1}{|\sigma|} = -\frac{1}{|\sigma|^2}$ since recall $\partial_r u = \nabla_{\hat{N}} u$ for radially symmetric functions
- $|\sigma| = \varepsilon$ on the boundary
- $d\sigma = \varepsilon^2 dw$ and $\sigma = \varepsilon w$ since recall 52

so we have

$$\frac{1}{4\pi}\int_{B_\varepsilon^c(0)} -\frac{1}{|y|}\Delta_y f(x-y) d^3y = \frac{1}{4\pi}\int_{\partial B_1(0)} \varepsilon w \cdot (\nabla f)(x - \varepsilon w) dw + \frac{1}{4\pi}\int_{\partial B_1(0)} f(x - \varepsilon x) dw$$

the first integral on the RHS can be bound by $4\pi M\varepsilon$ in which case when $\varepsilon \to 0^+$ it goes to zero too.But for the 2nd integral when $\varepsilon \to 0^+$,

$$\frac{1}{4\pi} f(x) \int_{\partial B_1(0)} 1 dw = f(x)$$

in which case our conclusion follows

Definition 62

Let $\Omega \subset \mathbb{R}^n$ be a domain. A **Green function** in Ω is defined to be a function of $(x, y) \in \Omega \times \Omega$ verifying the following considitions for each $x \in \Omega$

$$\Delta_y G(x, y) = \delta(x), \quad y \in \Omega$$
$$G(x, \sigma) = 0, \quad \sigma \in \partial\Omega$$

Proposition 63

Let Φ be the fundamental solution 58 for $\Delta \in \mathbb{R}^n$ and let $\Omega \in \mathbb{R}^n$ be a domain. Then the Green function can be decomposed as

$$G(x,y) = \Phi(x-y) - \phi(x,y)$$

where for each $x \in \Omega, \phi(x,y)$ solves the *Dirichelet Problem*

$$\begin{aligned} \Delta_y \phi(x,y) &= 0, \quad y \in \Omega \\ \phi(x,\sigma) &= \Phi(x-\sigma), \quad \sigma \in \partial\Omega \end{aligned}$$

Proof. recall that we have proven via 60 that $\Delta\Phi = \delta$. Moreover by the dischelet conditions here we have

$$\Delta_y(\Phi(x-y) - \phi(x,y)) = \Delta_y \Phi(x-y) - \underbrace{\Delta_y \phi(x,y)}_{=0, y\in\Omega} = \delta(x-y)$$

On the other hand when $\sigma \in \partial\Omega$ we have

$$\Phi(x-\sigma) - \phi(x,\sigma) = 0$$

which precisely matches the properties defined for the green function as desired

Proposition 64 (Representation formula for u)

Let Φ be the fundamental solution in 58 for $\Delta \in \mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u \in C^2(\overline{\Omega})$. Then for every $x \in \Omega$ we have the following representation formula for $u(x)$:

$$u(x) = \int_\Omega \Phi(x-y)\Delta_y u(y) d^n y - \underbrace{\int_{\partial\Omega} \Phi(x-\sigma)\nabla_{\hat{N}(\sigma)} d\sigma}_{\text{single layer potential}} + \underbrace{\int_{\partial\Omega} u(\sigma)\nabla_{\hat{N}(\sigma)}\Phi(x-\sigma)d\sigma}_{\text{double layer potential}}$$

Proof. We will do the proof for $n=3$ in which case $\Phi(x) = -\frac{1}{4\pi|x|}$. Let $B_\varepsilon(x)$ be a ball of radius ε centred at x and let $\Omega_\varepsilon = \Omega / B_\varepsilon(x)$. Note that $\partial\Omega_\varepsilon = \partial\Omega - \partial B_\varepsilon(x)$. Then by **green identity** we have

$$\int_{\Omega_\varepsilon} \frac{1}{|x-y|}\Delta u(y) d^3 y = \int_{\partial\Omega_\varepsilon} \frac{1}{|x-\sigma|}\nabla_{\hat{N}}(\sigma) - u(\sigma)\nabla_{\hat{N}}\left(\frac{1}{|x-\sigma|}\right) d\sigma$$

which can then by split into those on $\partial\Omega_\varepsilon$ and $\partial\Omega$ like so

$$\begin{aligned} \int_{\Omega_\varepsilon} \frac{1}{|x-y|}\Delta u(y) d^3 y = &\int_{\partial\Omega} \frac{1}{|x-\sigma|}\nabla_{\hat{N}}(\sigma) - \int_{\partial\Omega} u(\sigma)\nabla_{\hat{N}}\left(\frac{1}{|x-\sigma|}\right) d\sigma \\ &- \int_{\partial B_\varepsilon} \frac{1}{|x-\sigma|}\nabla_{\hat{N}}(\sigma) + \int_{\partial B_\varepsilon} u(\sigma)\nabla_{\hat{N}}\left(\frac{1}{|x-\sigma|}\right) d\sigma \end{aligned}$$

We label the above terms in corresponding order by

$$L = R1 + R2 + R3 + R4$$

So in order to attain the representation formula we need to show as $\varepsilon \downarrow 0$(which is possible since ε is abritruary) we have

- $L \to -4\pi \int_\Omega \Phi(x-y)\Delta_y u(y) d^3 y$
- $R1 \to 4\pi\times$ single layer potential

- $R2 \to -4\pi\times$ double layer potential
- $R3 \to 0$
- $R4 \to -4\pi u(x)$

and then rearranging gets form the proposition as desired.For L consider by triangle inequality and breaking up rienmann integrals on disjoint sets we have

$$\begin{aligned}\left|\int_\Omega \frac{1}{|x-y|}\Delta u(y)d^3y - \int_{\Omega_\varepsilon}\frac{1}{|x-y|}\Delta u(y)d^3y\right| &\le \int_{B_\varepsilon}\frac{1}{|x-y|}|\Delta u(y)|d^3y \\ &\le M\int_{B_\varepsilon}\frac{1}{|x-y|}d^3y\end{aligned}$$

we can clearly bound $\sup|\Delta u| = M$ since $u \in C^2(\overline{\Omega})$ and then the integral on the RHS goes to zero as $\varepsilon \to 0$ like in the last step above. The proofs for the rest are similar... $\square$

Definition 65

The boundary value Poisson problem is described by

$$\begin{aligned}\Delta u(x) &= f(x), \quad \Omega \subset \mathbb{R}^n \\ u(x) &= g(x), \quad x \in \partial\Omega\end{aligned}$$

Theorem 66 (Representation formula for solutions to the boundary value Poisson equation)

The solution to the *boundary value poisson problem* can be represented as

$$u(x) = -\int_\Omega f(y)G(xx,y)d^n - \underbrace{\int_{\partial\Omega} g(\sigma)\nabla_{\hat{N}}G(x,\sigma)d\sigma}_{\text{Poisson Kernel}}$$

Proof. apply 64 $\square$

We now use a technque known as **method of images** that works for special domains. One such application is to derive a representation formula for solutions to the Laplace equation on a ball

Theorem 67 (Poisson formula)

Let $B_r(p) \subset \mathbb{R}^3$ be a ball of radius R centered at $p = (p^1,p^2,p^3)$ and let $x = (x^1,x^2,x^3)$ denote a point in $\mathbb{R}^3$. Let $g \in C(\partial B_R(p))$ then the unique soution $u \in C^2(B_r(p)) \cup C(\overline{B}_R(p))$ of the PDE

$$\begin{cases}\Delta u(x) = 0 & x \in B_r(p) \\ u(x) = g(x) & x \in \partial B_r(p)\end{cases}$$

can be represented by the possion formula

$$u(x) = \frac{R^2 - |x-p|^2}{4\pi R}\int_{\partial B_r(p)}\frac{g(\sigma)}{|x-\sigma|^3}$$

4 Wave Equation

4.1 D'Alembert formula

Definition 68 (Wave equation)
The standard wave equation for a function $u(t,x)$ where $t \in \mathbb{R}, x \in \mathbb{R}^n$ is

$$-c^{-2}\partial_t^2 u(t,x) + \Delta u(t,x) = 0$$

in other words u is defind in $1 \times n$ dimensions. Is second order and linear. THe constant $c > 0$ is called **speed**. We note that if f and g are differentiable functions then $f(x - c\tau)$ and $g(x + c\tau)$ are solutions to the above i.e they are equal to $u(x,\tau)$

Definition 69
First from the **wave equation** make a change of variables $t = c\tau$. That we are assuming the speed $c = 1$. So now have one of the most often studied well-posed problem for the wave equation is the **Global Cauchy Problem** in $1 + n$ spacetime dimensions defined by

$$\begin{aligned} -\partial_t^2 u(t,x) + \Delta_x u(t,x) &= 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0,x) &= f(x), \quad x \in \mathbb{R}^n \\ \partial_t u(0,x) &= g(x), \quad x \in \mathbb{R}^n \end{aligned}$$

If we further simplify and assume we are working in 1×1 dimensions and within a certain range of (t,x). That is $t \in \mathbb{R}$ and $x \in \mathbb{R}$. Then we can just rewrite Δ_x as ∂_x^2 and replace all $\mathbb{R}^n$ with $\mathbb{R}$ above to get the global cauchy problem in $1 + 1$ dimensions like so

$$\begin{aligned} -\partial_t^2 u(t,x) + \partial_x^2 u(t,x) &= 0, \quad (t,x) \in \mathbb{R} \times [0,L] \\ u(0,x) &= f(x), \quad x \in [0,L] \\ \partial_t u(0,x) &= g(x), \quad x \in [0,L] \end{aligned}$$

Theorem 70 (d'Alembert's Formula)
Assume that $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$. Then the unique solution $u(t,x)$ to the **global cauchy problem** in $1+1$ dimensions satisfies $u \in C^2([0,\infty) \times \mathbb{R}])$ and can be represented by **d'Alembert's formula**

$$u(t,x) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int_{z=x-t}^{z=x+t} g(z)dz$$

Remark 71. *Making of change of variables back to the wave equation(in 1×1 dimensions)*

$$-c^{-2}\partial_t^2 u(t,x) + \partial_x^2 u(t,x) = 0$$

then u is replaced by

$$u(t,x) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2}\int_{z=x-ct}^{z=x+ct} g(z)dz$$

Proof. We first consider

Lemma 72

Suppose we introduct a change of variables $(t, x) \to (s, q)$ known as **null coordinates**

$$q = t - x$$
$$s = t + x$$

The chain rule implies the following relationships between partial derivatives

$$\partial_q = \frac{1}{2}(\partial_t - \partial_x) \qquad \partial_x = \frac{1}{2}(\partial_t + \partial_x)$$
$$\partial_t = \partial_q + \partial_s \qquad \partial_x = \partial_s - \partial_q$$

Proof. Just consider

$$\frac{\partial q}{\partial t} = 1, \quad \frac{\partial q}{\partial x} = -1$$
$$\frac{\partial s}{\partial t} = 1, \quad \frac{\partial s}{\partial x} = 1$$

And we know that for any $f(s, q)$ we can get

$$\frac{\partial}{\partial t} = \frac{\partial q}{\partial t}\frac{\partial}{\partial q} + \frac{\partial s}{\partial t}\frac{\partial}{\partial s} = \partial_q + \partial_s$$

$$\frac{\partial}{\partial x} = \frac{\partial q}{\partial x}\frac{\partial}{\partial q} + \frac{\partial s}{\partial x}\frac{\partial}{\partial s} = -\partial_q + \partial_s$$

Thus, we have:

$$\partial_t = \partial_q + \partial_s$$
$$\partial_x = \partial_s - \partial_q$$

Then solving for ∂_q and ∂_s Adding and subtracting ∂_t and ∂_x, we solve for ∂_q and ∂_s:

$$\partial_t + \partial_x = 2\partial_s \quad \Rightarrow \quad \partial_s = \frac{1}{2}(\partial_t + \partial_x)$$

$$\partial_t - \partial_x = 2\partial_q \quad \Rightarrow \quad \partial_q = \frac{1}{2}(\partial_t - \partial_x)$$

□

Back to our proof of d'Alembert formula we see note that the significance of null coordinates is that we are able to get either the t or x slot to be equal zero so that we can make use of results in boundary conditions.For some reason it works when it is able to turn equations like those in the form of the wave equation in $1+1$ dimensions into some $\partial_s\partial_q u = 0$. Specifically consider that we can rewrite our global cauchy problem like so

$$0 = -\partial_t^2 u(t, x) + \partial_x^2 u(t, x) = (\partial_x + \partial_t)(\partial_x - \partial_t)u(t, x) = -\frac{1}{4}(\partial_q\partial_s)$$

so we have

$$\partial_s\partial_q = 0$$

so we can define

$$\partial_q u = H(q)$$

Now consider a pair of cartesian spacetime pints (τ, y) recall from the previous lemma how the variables are related,we know that we

$$(\tau, y) \to p = \tau - y$$

$$(0, y - \tau) \to p = 0 - \tau + y$$

so in fact (τ, y) is equivalent to $(0, \tau - y)$ to our function H which is only in terms of p. Moreover by assumption in the conditions in 69

$$\partial_t u(0, y - \tau) = g(y - \tau)$$

and

$$\partial_x u(0, y - \tau) = f'(g - \tau)$$

so putting this althogehter we have

$$H = \partial_q u(\tau, t) = \partial u(0, y - \tau) = \left(\frac{1}{2}(\partial_t - \partial_x)u(0, y - \tau)\right) = \frac{1}{2}\left(g(y - \tau) - f'(y - \tau)\right)$$

we can do the same by now letting $\partial_s u = H(s)$ after swapping $\partial_q \partial_s = 0$(recall from `Munkres Manifolds Analysis` the order of partial derivatives doesn't matter). In which we will get

$$H = \partial_s u(\tau, t) = \frac{1}{2}\left(g(y + \tau) - f'(y + \tau)\right)$$

because for $u(t, x) \to u(s, q)$ we have

$$(\tau, y) \to s = \tau + y$$

$$(0, y - \tau) \to s = 0 + \tau + y,$$

so in fact (τ, y) is equivalent to $(0, \tau + y)$ to our function H which is only in terms of s now. Combining our results we have

$$\partial_t u(t, x) = \frac{1}{2}\left(f'(x + t) - f'(x - t) + g(x + t) + g(x - t)\right)$$

Now integrating with respect to t from 0 to t our conclusion follows $\square$

Now consider the following corllary which will prove to be useful in future when we extend this to $1+3$ dimensions. The big idea is given a solution u to the $1+1$ dimensional problem where $x \in \mathbb{R}^+$ now instead, considering $\tilde{u}$, the odd extension of u we may use d'Alembert's formula to solve for the odd extension and hence recover the expression of u afterwards as desired since $\tilde{u}|\mathbb{R}^+ = u$

Corollary 73
Let $f \in C^2([0,\infty))$, $g \in C^1([0,\infty))$ and assume that $f(0) = g(0) = 0$. The unique solution to the following $1+1$ dimensional initial + oundary problem

$$\begin{aligned} -\partial_t^2 u(t,x) + \partial_x^2 u(t,x) &= 0 \\ u(t,0) &= 0 \\ u(0,x) &= f(x) \\ \partial_t u(0,x) &= g(x) \end{aligned}$$

then satisfies $u \in C^2([0,\infty) \times [0,\infty))$ and can be represented as

$$u(t,x) = \begin{cases} \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{z=|x-t|}^{z=x+t} g(z)dz & 0 \le t \le x \\ \frac{1}{2}\left(f(x+t) - f(t-x)\right) + \frac{1}{2}\int_{z=|x-t|}^{z=x+t} g(z)dz & 0 \le x \le t \end{cases}$$

Remark 74. *note the additional boundary value problem $u(t,0) = 0$ compared to the previous theorem and that x is restricted to $\mathbb{R}^+$ here as compared to $\mathbb{R}$ from above.*

Proof. Given u that solves the above initial + boundary problem defined, let us consider the odd extension in x.

$$\tilde{w}(t,x) = \begin{cases} u(t,x) & t \ge 0, x \ge 0 \\ -u(t,-x) & t \ge 0, x \le 0 \end{cases}$$

$$\tilde{f}(t,x) = \begin{cases} f(x) & t \ge 0, x \ge 0 \\ -f(-x) & t \ge 0, x \le 0 \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x) & t \ge 0, x \ge 0 \\ -g(-x) & t \ge 0, x \le 0 \end{cases}$$

We say this is an extension because for example $\tilde{u}|\mathbb{R}^+ = u|\mathbb{R}^+$. But $\tilde{u}$ is defined to work not just in $\mathbb{R}^+$ but in $\mathbb{R}$ as well. Likewise for $\tilde{f}$ and $\tilde{g}$. So in fact is our solution solves

$$\begin{aligned} -\partial_t^2 \tilde{u}(t,x) + \partial_x^2 \tilde{u}(t,x) &= 0 \\ \tilde{u}(t,0) &= 0 \\ \tilde{u}(0,x) &= \tilde{f}(x) \\ \partial_t \tilde{u}(0,x) &= \tilde{g}(x) \end{aligned}$$

where $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$ and assume that $\tilde{f}(0) = g\tilde{(0)} = 0$. Notice that this are exactly the conditions in our previou theorem 70 and so we can use let is solution to be our $\tilde{u}$ in which case we can express it as

$$\tilde{u}(t,x) = \frac{1}{2}\left(\tilde{f}(x+t) + \tilde{f}(x-t)\right) + \frac{1}{2}\int_{z=x-t}^{z=x+t} \tilde{g}(z)dz$$

now we convert the odd extension functions back to the original f, g, u in which case we obtain

$$u(t,x) = \begin{cases} \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{z=|x-t|}^{z=x+t} g(z)dz & 0 \le t \le x \\ \frac{1}{2}\left(f(x+t) - f(t-x)\right) + \frac{1}{2}\int_{z=|x-t|}^{z=x+t} g(z)dz & 0 \le x \le t \end{cases}$$

as desired. We see that for the case where $0 \le x \le t$ the absolute value applies because

$$\int_{x-t}^{x+t} \tilde{g} = \int_{x-t}^{0} \tilde{g} + \int_{0}^{t-x} \tilde{g} + \int_{t-x}^{x+t} \tilde{g} = -\int_{0}^{t-x} g + \int_{0}^{t-x} g + \int_{t-x}^{x+t} g$$

so it is evident the first two terms are zero and the theorem follows knowing $|x-t| = t-x$

4.2 Kirchoff formula

We would now like to derive an analog of **d'Alembert's formula** for the $1+3$ dimensional case which is known as **Kirchoff's formula**. We will do so by the **method of spherical means**.

The big idea is given a solutionn $u(t,x)$ to the $1+3$ dimensional wave equation we may define a spherical average of u centered at x. For a fixed x we will show that a slight modification of this spherical average we can use d'Almbert formula to solve the $1+1$ dimensional wave equation in the unknowns (t,r). Then taking the limit of $r \to 0$ of this spherical average recovers the value of $u(t,x)$ as desired. In summary taking spherical averages reduces our $1+3$ dimensional problem to that of a $1+1$ dimensional problem.

Proposition 75 (Spherical Averages)
Let $u(t,x) \in C^2([0,\infty) \times \mathbb{R}^3)$ be a solution to the $1+3$ dimensional global cauchy problem

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3 \tag{1}$$
$$u(0,x) = f(x), \quad x \in \mathbb{R}^3 \tag{2}$$
$$\partial_t u(0,x) = g(x), \quad x \in \mathbb{R}^3 \tag{3}$$

For each $r > 0$ define the spherically averaged quantities. That is for every x we are now integrating every $u(t,\sigma)$ over a sphere centered around x. So clearly $r \to 0$ the sphere contracts back to $u(t,x)$ where we recover our solution.

$$U(t,r,x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} u(t,\sigma)d\sigma = \frac{1}{4\pi}\int_{w \in \partial B_1(0)} u(t,x+rw)dw \tag{4}$$
$$F(r,x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} f(\sigma)d\sigma \tag{5}$$
$$G(r,x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} g(\sigma)d\sigma \tag{6}$$

and their related modifications

$$\tilde{U}(t,r,x) = rU(t,r,x) \tag{7}$$
$$\tilde{F}(r,x) = rF(r,x) \tag{8}$$
$$\tilde{G}(r,x) = rG(r,x) \tag{9}$$

Ten $\tilde{U}(t,r,x) \in C^2([0,\infty) \times [0,\infty))$ is a solution to the following initial + boundary value problem for the *one-dimensional* wave equation

$$-\partial_t^2 \tilde{U}(t,r,x) + \partial_r^2 \tilde{U}(t,r,x) = 0 \tag{10}$$
$$\tilde{U}(t,0,x) = 0 \tag{11}$$
$$\tilde{U}(0,r,x) = \tilde{F}(r,x) \tag{12}$$
$$\partial_t \tilde{U}(0,r,x) = \tilde{G}(r,x) \tag{13}$$

Furthermore

$$\lim_{r \to 0} U(t,r,x) = u(t,x)$$

Proof. The natural approach is to use $\tilde{U} = rU$ then differentiate under the integral to show that $\partial_t^2 \tilde{U} = \partial_t^2 \tilde{U}$. We see that $\partial_r^2(rU) = \partial_r(U + r\partial_r U) = 2\partial_r U + r\partial_r^2 U$ by chain rule. Hence it makes sense to first study what happens when we take $\partial_r U$. Observe that $\sigma \in \partial_r B(x)$ and $w \in \partial_1 B(0)$. Now differentiating (4) under the integral we have

$$\partial_r[u(t,x+rw)]dw = (\nabla u)(t,x+rw) \cdot w dw = \frac{1}{r^2}\nabla_{\hat{N}(\sigma)} u(t,\sigma)d\sigma$$

where $\hat{N}(\sigma)$ is the outward unit normal to $\partial B_r(x)$. We know the 1st equality follows by chain rule since

$$\partial_r[u(t,x+rw)] = \partial_{x_1}\partial_r(t) + \partial_{x_2}\partial_r(x+rw) = 0 + \partial_{x_2} w$$

But we also know

$$\hat{N}(\sigma) = \frac{\sigma - x}{r}$$

and

$$\sigma = x + rw$$

therefore $w = \hat{N}(\sigma)$. We also know that

$$d\sigma = r^2 dw$$

since on the boundary $|\sigma| = r\,|w|$ since w is a unit vector and that even though the spheres are centered at different points, the corresponding angular coordinates are still the same(invariant under translation clearly as simply integrating in a circle with respect to the new centre point). so this follows by 26. Therefore applying divergence theorem we have

$$\partial_r U = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} \nabla_{\hat{N}(\sigma)} u(t,\sigma) = \frac{1}{4\pi r^2}\int_{B_r(x)} \Delta_y u(t,x) d^3 y$$

Then making use of this result and 28 we have

$$\partial_r(r^2\partial_r U) = \frac{1}{4\pi}\partial_r \int_{B_r(x)} \Delta_y u(t,y) d^3 y = \frac{1}{4\pi}\int_{\partial B_r(x)} \Delta u(t,\sigma) d\sigma$$

We will see why we took this last step(specifically why we want to express the integral in terms of the boundary surface measure) when we find out how $\partial_t^2(rU)$ looks like in the next step

It sufficies to just study $\partial_r^2 U$ afterall r is independent to t and be just shifted out.Without further ado, let us again differentiate (4) under the integral this time with respect t. But notice this time we could make use of boundary conditions as listen in (1) which says that $-\partial_t^2 u = \Delta u$. So we do

$$\partial_t^2 U(t,r,x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} \partial_t^2(t,\sigma) d\sigma = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} \Delta u(t,\sigma) d\sigma$$

Notice now that we can relate $\partial_r U$ with $\partial_r(r^2\partial_t^2 U)$! Thats why did that last step previously mentioned. Moreover we know by chain rule is just $\partial_r^2 U + \frac{2}{r}\partial_r U$. By comparison we hence have

$$\partial_t^2 U = \frac{1}{r^2}\partial_r(r^2\partial_r U) = \partial_r^2 U + \frac{2}{r} U_r$$

but recall that from the start we had the result $\partial_r^2(rU) = \partial_r(U + r\partial_r U) = 2\partial_r U + r\partial_r^2 U$. In other words when multiply our latest result on all sides by r notice that we have proven

$$\partial_t^2(rU) = \partial_r^2(rU)$$

as desired! Now using the corollary to d'Alembert formula we have

$$\tilde{U} = rU = \frac{1}{2}(\tilde{F}(r+t,x) - \tilde{F}(r-t,x)) + \frac{1}{2}\int_{p=-r+t}^{p=r+t} \tilde{G}(p,x) dp$$

for $0 \le r \le t$. Then taking the expression of U and taking limits $r \to 0$ recovers u as desired.

Theorem 76 (Kirchoff Formula)
Let $u(t,x) \in C^2([0,\infty) \times \mathbb{R}^3)$ be a solution to the $1+3$ dimensional global cauchy problem

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3 \tag{14}$$

$$u(0,x) = f(x), \quad x \in \mathbb{R}^3 \tag{15}$$

$$\partial_t u(0,x) = g(x), \quad x \in \mathbb{R}^3 \tag{16}$$

Then it can be represented as

$$u(t,x) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} f(\sigma) d\sigma + \frac{1}{4\pi t} \int_{\partial B_t(x)} \nabla_{\hat{N}(\sigma)} f(\sigma) d\sigma + \frac{1}{4\pi t} \int_{\partial B_t(x)} g(\sigma) d\sigma$$

Proof. We know from previoulsy that

$$\begin{aligned} u(t,x) &= \lim_{r\to 0^+} U(t,r,x) = \lim_{r\to 0^+} \frac{\tilde{U}(t,r,x)}{r} \\ &= \lim_{r\to 0^+} \frac{\tilde{F}(r+t,x) - \tilde{F}(r-t,x)}{2r} + \frac{1}{2r} \int_{p=-r+t}^{p=r+t} \tilde{G}(p,x) dp \\ &= \partial_t \tilde{F}(t,x) + \tilde{G}(t,x) \end{aligned}$$

We see that by chain rule

$$\partial_t(\tilde{F}(t,x)) = \partial_t(tF(t,x)) = F + t\partial_t(F(t,x))$$

So we just need to find out how $\partial_t(F(t,x))$ looks like and we will have our representation formula. Noticing that we done something similar in the previous theorem(refer if necessary),specifically we can differentiate under the integral and apply chain rule like so

$$\begin{aligned} \partial_t\left(\frac{1}{4\pi t^2} \int_{\partial B_t(x)} f(\sigma) d\sigma\right) &= \partial_t\left(\frac{1}{4\pi t^2} \int_{\partial B_1(0)} f(x+tw) dw\right) \\ &= \frac{1}{4\pi t^2} \int_{\partial B_1(0)} \partial_t[f(x+tw)] dw \\ &= \frac{1}{4\pi t^2} \int_{\partial B_1(0)} \nabla_{\hat{N}(x+tw)} f(x+tw) dw \\ &= \frac{1}{4\pi t^2} \int_{\partial B_t(x)} \nabla_{\hat{N}(\sigma)} f(\sigma) d\sigma \end{aligned}$$

Now putting everything together we can get the desired expression.

5 geometric estimates

Definition 77

In the following discussions we denote the standard rectangular coordinates $\mathbb{R}^{1+n}$ by $(x^0, x^1, \ldots, x^n)$ where $x^0 = t$. The **Minkowski metric** on $\mathbb{R}^{1+n}$ which we denote as m emobodies **lorentzian geometry** at the heart of *Einstein theory of special relativity*. The components of m take on the form realtive to a standard coordinate system like so

$$m_{\mu\nu} = (m^{-1})^{\mu\nu} = \text{diag}(-1, \underbrace{1, 1, \ldots, 1}_{\text{n copies}})$$

where diag refers to the diagonal matrix. That is we can see $m_{\mu\nu}$ as a $(1+n) \times (1+n)$ matrix of real numbers whose elements are all zero except the diagonal. In this case $m_{00} = -1, m_{11} = 1, m_{22} = 1, m_{33} = 1 \ldots$ etc. We can also tell m is symmetric that is $m_{\mu\nu} = m_{\nu\mu}$.

Definition 78

A covector is essentially a $1-$form or the **metric dual**. Recall for example the map $a : V \to K$ then $a \in V^*$(dual space). If X is a vector in $\mathbb{R}^{1+n}$ with components $X^\mu (0 \leq \mu \leq n)$ we contract a vector with a metric to obtain a covector

$$X_\mu = \sum_{a=0}^{3} m_{\mu a} X^a$$

This is what we call **lowering the index** where you go from subscript to superscript

Similary we contract a vector with the inverse metric to obtain a vector.

$$Y^\mu = \sum_{a=0}^{3} (m^{-1})^{\mu a} Y_a$$

This is what we call **raising the index** where you go from superscript to subscript

Definition 79

From now on we use the **Einstein Summation convention** that is defined by

$$\underbrace{X_a Y^a}_{\text{with convention}} = \sum_{a=0}^{3} X_a Y^a = \sum_{a=0}^{3} \sum_{b=0}^{3} (m_{ab} X^b) Y^a = \underbrace{m_{ab} X^b Y^a}_{\text{with convention}}$$

where the RHS according to the convention represents the repeated inner summation over b then the outer repeated summation over a But due to symmetry $m_{ab} = m_{ba}$

$$m_{ba} X^a Y^b = m_{ab} X^b Y^a = m_{ba} X^b Y^a$$

where here the order of summation has been swapped. So putting all these together we have

$$X_a Y^a = m_{ab} X^a Y^b$$

What is with the subscripts superscripts? How is say e_i a dual/covector? Consider the following examples to see why

Example 80 (Inner product)

$$v = v^i e_i = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}$$

See the covector e clearly is the matrix map on the column vector v here. The e_i subscript denotes elements of a row vector while v_i denotes elements of a column vector.

Example 81 (Matrix Vector Product)

$$u_i = (Av)_i = \sum_{j-1}^{N} A_{ij} v_j$$

in einstein summation is

$$u^i = A^i_j v^j$$

Example 82 (Matrix Product)

$$C_{ik} = (AB)_{ik} = \sum_{j-1}^{N} A_{ij} B_{jk}$$

in einstein summation is

$$C^i_k = A^i_j B^j_k$$

Example 83 (Raising and lowering indices)

$$g_{\mu\sigma} T^{\sigma}_{\beta} = T_{\mu\beta}$$

and

$$g^{\mu\sigma} T^{\alpha}_{\sigma} = T^{\mu\alpha}$$

In summary subscript to superscript represents left multiplication

$$\mathbf{A} X = Y$$

contract while superscript to subscript represents right multiplication

$$B\mathbf{P} = B'$$

Definition 84

Let

1. **Timelike** vectors: $m(X,X) = m_{ab}X^aX^b < 0$
2. **Spacelike** vectors: $m(X,X) > 0$
3. **Null** vectors: $m(X,X) = 0$
4. **Casual** vectors: Timelike vectors $\cup$ Null vectors

A vector $X \in \mathbb{R}^n$ is **future-directed** if $X^0 > 0$

Definition 85 (Lorentz Transformation)

A **lorentz transformation** is a linear transformation Λ^μ_ν(a matrix) that preserves the form of the *Minkowski metric* $m_{\mu\nu} = (m^{-1})^{\mu\nu} = \text{diag}(-1, \underbrace{1, 1, \ldots, 1}_{\text{n copies}})$. That is

$$\Lambda^a_\mu \Lambda^b_\nu m_{ab} = m_{\mu\nu}$$

and

$$\Lambda^T m \Lambda = m$$

Definition 86

The **Minkowskian inner product** is defined by

$$m(X,Y) = m_{ab}X^aY^b$$

and ΛX is the vector whose components are $(\Lambda X)^\mu = \Lambda^\mu_a X^a$ so we see that

$$m(\Lambda X, \Lambda Y) = m(X,Y)$$

since

$$m(\Lambda X, \Lambda Y) = m_{ab}(\Lambda X)^a(\Lambda Y)^b = m_{ab}(\Lambda^a_i X^i)(\Lambda^b_j X^j) = m_{ij}X^iY^j = m(X,Y)$$

Corollary 87

If X is timelike and Λ is a lorentz transformation then ΛX is also timelike. Same for spacelike or null.

Proof. For timelike see that $m(\Lambda X, \Lambda) = m(X,X) < 0$. Do similary for the rest.

Fact 88

It can be checked that the lorentz transformations form a group. In particular

- if Λ is a lorentz transformation then so is Λ^{-1}
- If Λ and Υ are lorentz transformations then so is their matrix product $\Lambda\Upsilon$ which has components $(\Lambda\Upsilon)^\mu_\nu = \Lambda^\mu_a \Upsilon^a_\nu$

See problems below

Definition 89

The following shorthand notation if osften usef for the "linear wave operator associated to m"

$$\Box_m = (m^{-1})^{ab}\partial_a\partial_b$$

Under this notation the wave equation $-\partial_t^2 + \Delta\phi = 0$ can be expressed as

$$\Box_m\phi = 0$$

This is easy to see by they definitions above. Because $(m^{-1})^{00} = -1$ and $X^0 = t$ thats why we have $-\partial_t^2$

Definition 90

The **energy momentum tensor** associated to equation is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_{\mu\nu}(m^{-1})^{ab}\partial_a\phi\partial_b\phi$$

Note that $T_{\mu\nu}$ is symmetric that is

$$T_{\mu\nu} = T_{\nu\mu}$$

this is obvious by observation. See that mixed partial derivatives are just scalar fields while $m_{\mu\nu}$ and its inverse are symmetric.

Proposition 91 (Null Frame Decomposition)

If $\{L, \underline{L}, e_{(1)}, \ldots, e_{(n-1)}\}$ is a null frame then we can decompose

$$m_{\mu\nu} = -\frac{1}{2}L_\mu\underline{L}_\nu - \frac{1}{2}\underline{L}_\mu L_\nu + \not{m}_{\mu\nu}$$

where $\not{m}_{\mu\nu}$ is positive definite on the m-orthgonal complexment of span$(L, \underline{L})$ and $\not{m}_{\mu\nu}$ vanishes on span $(L, \underline{L})$. Similary by raising each index on both sides of the above with m^{-1} we have that

$$(m^{-1})^{\mu\nu} = -\frac{1}{2}L^\mu\underline{L}^\nu - \frac{1}{2}\underline{L}^\mu L_\nu + \not{m}^{\mu\nu}$$

Lemma 92 (Dominant Energy Condition for $T_{\mu\nu}$)

$T(X, Y) = T_{ab}X^aY^b \geq 0$ if X, Y are both timemlike and future-directed or timelike and past directed

Proof. See the problems below

Problem 93

Assume tha the matrices Λ and Ω are lorentz tranformations. Recall that this means thaat

$$\Lambda^T m \Lambda = m$$
$$\Omega^T m \Omega = m$$

where $m_{\mu\nu} = \text{diag}(-1, 1, 1 \ldots, 1)$ denotes the Minkowsi metric on $\mathbb{R}^{1+n}$ and T denotes the matrix transpose.

(a) Using elementary properties of the determinant show that $|\det \Lambda| = 1$

(b) Show that the matrix product $\Lambda\Omega$ is also a lorentz transformation

(c) Show that the matri inverse Λ^{-1} is also a lorentz transformation

Solution. Let Λ and Ω be Lorentz transformations. This means they satisfy:

$$\Lambda^T m \Lambda = m$$

$$\Omega^T m \Omega = m$$

where $m_{\mu\nu} = \text{diag}(-1, 1, 1, \ldots, 1)$ is the Minkowski metric in $\mathbb{R}^{1+n}$.

(a) Show that $|\det \Lambda| = 1$

To show $|\det \Lambda| = 1$, we use the fact that Λ is a Lorentz transformation, which preserves the Minkowski metric. Consider the determinant of both sides of the Lorentz transformation condition:

$$\Lambda^T m \Lambda = m.$$

Taking the determinant on both sides, we get:

$$\det(\Lambda^T m \Lambda) = \det(m).$$

Using the property of determinants that $\det(AB) = \det(A)\det(B)$ and $\det(A^T) = \det(A)$, we have:

$$\det(\Lambda^T)\det(m)\det(\Lambda) = \det(m).$$

Since $\det(\Lambda^T) = \det(\Lambda)$ and $\det(m) = -1$ (for the Minkowski metric in $\mathbb{R}^{1+n}$), we get:

$$\det(\Lambda)^2 \det(m) = \det(m).$$

Thus:

$$\det(\Lambda)^2(-1) = -1.$$

So:

$$\det(\Lambda)^2 = 1,$$

which implies:

$$|\det \Lambda| = 1.$$

(b) Show that the matrix product $\Lambda\Omega$ is also a Lorentz transformation

To show that $\Lambda\Omega$ is a Lorentz transformation, we need to verify that:

$$(\Lambda\Omega)^T m (\Lambda\Omega) = m.$$

Compute the transpose and use the fact that both Λ and Ω are Lorentz transformations:

$$(\Lambda\Omega)^T m(\Lambda\Omega) = \Omega^T(\Lambda^T m\Lambda)\Omega.$$

Since Λ and Ω are Lorentz transformations, we know:

$$\Lambda^T m\Lambda = m \quad \text{and} \quad \Omega^T m\Omega = m.$$

Thus:

$$\Omega^T(\Lambda^T m\Lambda)\Omega = \Omega^T m\Omega = m.$$

So:

$$(\Lambda\Omega)^T m(\Lambda\Omega) = m,$$

which confirms that $\Lambda\Omega$ is indeed a Lorentz transformation.

(c) Show that the matrix inverse Λ^{-1} is also a Lorentz transformation

To show that Λ^{-1} is a Lorentz transformation, we need to check:

$$(\Lambda^{-1})^T m\Lambda^{-1} = m.$$

Starting from the condition for Λ:

$$\Lambda^T m\Lambda = m,$$

take the inverse on both sides:

$$(\Lambda^T m\Lambda)^{-1} = m^{-1}.$$

Using the property that $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^T)^{-1} = (A^{-1})^T$, we get:

$$(\Lambda^{-1})^T m^{-1}\Lambda^{-1} = m^{-1}.$$

Since $m^{-1} = m$ (because m is its own inverse in the metric tensor sense), we have:

$$(\Lambda^{-1})^T m\Lambda^{-1} = m.$$

Thus, Λ^{-1} satisfies the condition for a Lorentz transformation, confirming that Λ^{-1} is indeed a Lorentz transformation.

Problem 94

Consider the following "boost linear transformation in the x^1 direction" on $\mathbb{R}^{1+3}$

$$\begin{bmatrix} -\gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $v \in (-1, 1)$ is a "velocity" and $\gamma = \sqrt{\frac{1}{1-v^2}}$. Show that Λ is a lorentz transformation

Solution. To show that the given matrix

$$\Lambda = \begin{bmatrix} -\gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a Lorentz transformation, we need to verify that it satisfies the Lorentz condition:

$$\Lambda^T \Lambda = m,$$

where m is the Minkowski metric, which in $\mathbb{R}^{1+3}$ is given by

$$m = \text{diag}(-1, 1, 1, 1).$$

Let's compute $\Lambda^T \Lambda$.

1. Calculate Λ^T:

The transpose of Λ is:

$$\Lambda^T = \begin{bmatrix} -\gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that in this case, Λ is symmetric, so $\Lambda^T = \Lambda$.

2.Calculate $\Lambda^T \Lambda$:

Now, we compute $\Lambda\Lambda$:

$$\Lambda^T \Lambda = \begin{bmatrix} -\gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Computing the elements, we get:

$$\Lambda^T \Lambda = \begin{bmatrix} \gamma^2(1 - v^2) & 0 & 0 & 0 \\ 0 & \gamma^2(1 - v^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Recall that $\gamma = \frac{1}{\sqrt{1-v^2}}$, so:

$$\gamma^2 = \frac{1}{1 - v^2}.$$

Therefore:

$$\gamma^2(1 - v^2) = 1.$$

Hence:

$$\Lambda^T \Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is exactly the Minkowski metric m in $\mathbb{R}^{1+3}$.

Since $\Lambda^T \Lambda = m$, the matrix Λ satisfies the Lorentz condition and hence is a Lorentz transformation.

Example 95 (Example of a Lorentz Boost)

Consider two observers: one in a stationary frame S, and another in a frame S' moving at velocity v along the x-axis relative to S. Suppose an event happens at coordinates (t, x^1, x^2, x^3) in the stationary frame S. The coordinates of this event in the moving frame S', denoted as (t', x'^1, x'^2, x'^3), are related by the Lorentz boost as follows:

$$\begin{bmatrix} t' \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

Multiplying the matrix with the vector, we get the transformed coordinates:

$$t' = \gamma(t - vx^1)$$

$$x'^1 = \gamma(x^1 - vt)$$

$$x'^2 = x^2$$

$$x'^3 = x^3$$

This shows how both time and space coordinates are mixed for an observer in motion relative to the original frame.

Definition 96

Lorentz Factor γ The **Lorentz factor** γ is defined as:

$$\gamma = \frac{1}{\sqrt{1 - v^2}}$$

where v is the velocity of the moving frame relative to the speed of light (assuming units where $c = 1$).

For small velocities ($v \ll 1$), γ is approximately 1, meaning that time and space are not significantly altered. This is consistent with our everyday experience of slow-moving objects.

As v approaches the speed of light ($v \to 1$), γ grows very large. This results in significant time dilation and length contraction, which are key predictions of special relativity.

Problem 97

Let X be a future directed time-like vector in $1 + n$ dimensional minkowski spacetime where $m_{\mu\nu} = (m^{-1})^{\mu\nu} = \text{diag}(-1, 1, 1, \dots, 1)$. Recall that this means $m(X, X) < 0$ and $X^0 > 0$. Show that there exists a proper lorentz transofmration("proper" means that its determinant is equal to 1 rather than -1) Λ such that $(\Lambda X)^\mu = (c_1, 0, 0 \dots, 0)$ for some number $c_1 > 0$. This show that if X is future directed and timemlike, you can always pick a standard rectangular Minkowskian coordinate system in which $m_{\mu\nu} = \text{diag}(-1, 1, 1, \dots, 1)$ such that X is parallel to the t axis. Hint: First briefly aruge that there exists a product R of spatial rotations (which are Lorentz tranformations) such that $(RX)^\mu = (c_1, c_2, 0, 0, \dots, 0)$ then construct a lorentz boost B such that $(BRX)^\mu = (d, 0, 0, 0, \dots, 0)$. The transformation of interest is therefore $\Lambda = BR$

Solution. Let X be a future-directed time-like vector in $(1+n)$-dimensional Minkowski spacetime, where the Minkowski metric $m_{\mu\nu}$ is given by:

$$m_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \dots, 1).$$

We know that $m(X, X) < 0$ and $X^0 > 0$, which means X is a future-directed time-like vector.

Our goal is to show that there exists a proper Lorentz transformation Λ such that $(\Lambda X)^\mu = (c_1, 0, 0, \dots, 0)$ for some $c_1 > 0$. In other words, we want to find a Lorentz transformation that makes X align with the time axis in a standard Minkowski coordinate system.

1. Find a spatial rotation R to simplify X:

We start by constructing a product of spatial rotations R such that RX has the form $(c_1, c_2, 0, 0, \dots, 0)$.

Since X is a time-like vector, its spatial components X^i (for $i = 1, 2, \dots, n$) are not all zero, but we can use spatial rotations to align X in such a way that only the first two components are non-zero.

Specifically, a rotation in the spatial x^1-x^2 plane can be used to eliminate all but two non-zero components of X. Let R be such a rotation that gives:

$$RX = (c_1, c_2, 0, 0, \dots, 0).$$

2. Apply a Lorentz boost B to align RX:

Next, we apply a Lorentz boost in the x^1 direction to further simplify RX. The boost matrix B in the x^1 direction is given by:

$$B = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 & \cdots & 0 \\ -\gamma\beta & \gamma & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where β and γ are given by:

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

We want to choose v such that $B(RX) = (d, 0, 0, \dots, 0)$ where d is a positive constant. We apply the boost to the vector $RX = (c_1, c_2, 0, 0, \dots, 0)$. After the boost, we want the resulting vector BRX to have the form:

$$BRX = (d, 0, 0, \dots, 0),$$

where d is a positive constant. The Lorentz boost mixes the time and x^1 components as follows:

$$t' = \gamma(c_1 - \beta c_2),$$

$$x'^1 = \gamma(c_2 - \beta c_1).$$

We want $x'^1 = 0$, which gives the condition:

$$\gamma(c_2 - \beta c_1) = 0.$$

This implies:

$$c_2 = \beta c_1 \quad \Rightarrow \quad \beta = \frac{c_2}{c_1}.$$

Now that we have $\beta = \frac{c_2}{c_1}$, we can calculate the Lorentz factor γ as:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\left(\frac{c_2}{c_1}\right)^2}} = \frac{c_1}{\sqrt{c_1^2 - c_2^2}}.$$

The boost parameter β can be determined by:

$$\gamma = \frac{c_1}{\sqrt{c_1^2 - c_2^2}}.$$

By applying this boost, the new vector BRX will have the form:

$$BRX = (d, 0, 0, \dots, 0).$$

where $d = \sqrt{c_1^2 - c_2^2}$.

3. Construct the overall Lorentz transformation Λ:

The overall Lorentz transformation Λ is then given by:

$$\Lambda = BR.$$

This matrix Λ is a proper Lorentz transformation (its determinant is $+1$) that aligns X with the time axis, as desired.

Thus, we have shown that if X is a future-directed time-like vector, there exists a proper Lorentz transformation Λ such that:

$$(\Lambda X)^\mu = (c_1, 0, 0, \dots, 0)$$

for some $c_1 > 0$. This confirms that we can always choose a standard Minkowski coordinate system where X is aligned with the time axis.

Problem 98

Let X, Y be future directed(i.e. $X^0 > 0, Y^0$) timelike vectors in $1+n$ dimensional minkowski spacetime. Recall that by defintion null vectors are vectors L such that $m(L, L) = 0$ show that there exists a pair of future directed null vectors $L, \underline{L}$ normalized by $m(L, \underline{L}) = -2$ and positive constants a, b, c, d such that $X = aL + b\underline{L}, Y = cL + d\underline{L}$. Hint: By the previous problem you can assume that $X^\mu(X^0, 0, \dots, 0)$. By performing a further sequence of spatial rotations R argue that you can also assume that $Y^\mu = (Y^0, Y^1, 0, 0, \dots, 0)$. You can therefore reduce the problem to case of $1+1$ spaceimte dimensions where you can explicitly find an L and an $\underline{L}$

Solution. Let X and Y be future-directed time-like vectors in $(1+n)$-dimensional Minkowski spacetime. We want to show that there exist future-directed null vectors L and $\underline{L}$, normalized such that $m(L, \underline{L}) = -2$, and positive constants a, b, c, d such that:

$$X = aL + b\underline{L}, \quad Y = cL + d\underline{L}.$$

1. Align X with the time axis:

From the previous problem, we know that we can always find a proper Lorentz transformation Λ such that X aligns with the time axis. That is, there exists a Lorentz transformation Λ such that:

$$(\Lambda X)^\mu = (X^0, 0, 0, \dots, 0).$$

For simplicity, let us assume this transformation has already been applied so that:

$$X^\mu = (X^0, 0, 0, \dots, 0).$$

2. Use spatial rotations to simplify Y:

Next, we apply a sequence of spatial rotations R in the x^1-x^2-plane to align Y such that it has the form $(Y^0, Y^1, 0, \dots, 0)$. The spatial rotations will only affect the spatial components and not the time component. Thus, after applying these rotations, we assume:

$$Y^\mu = (Y^0, Y^1, 0, \dots, 0).$$

3. Reduce the problem to $1+1$ dimensions:
With X and Y now in the form:

$$X^\mu = (X^0, 0), \quad Y^\mu = (Y^0, Y^1),$$

we can effectively reduce the problem to $1+1$ dimensional Minkowski spacetime with coordinates (x^0, x^1). Here, the metric $m_{\mu\nu}$ simplifies to:

$$m = \text{diag}(-1, 1).$$

4. Find null vectors L and $\underline{L}$ in $1+1$ dimensions:
In $1+1$ dimensions, a null vector L satisfies:

$$m(L, L) = -L^0L^0 + L^1L^1 = 0.$$

For simplicity, choose:

$$L^\mu = (1, 1), \quad \underline{L}^\mu = (1, -1).$$

We verify:

$$m(L, L) = -(1)^2 + (1)^2 = 0,$$

$$m(\underline{L}, \underline{L}) = -(1)^2 + (-1)^2 = 0,$$

$$m(L, \underline{L}) = -(1 \cdot 1) + (1 \cdot -1) = -2.$$

5. Express X and Y in terms of L and $\underline{L}$:
Since:

$$X^\mu = (X^0, 0),$$

we can write:

$$X^\mu = \frac{X^0}{2}(L^\mu + \underline{L}^\mu).$$

Here, $a = \frac{X^0}{2}$ and $b = \frac{X^0}{2}$.

Similarly, for Y:

$$Y^\mu = (Y^0, Y^1),$$

we can decompose:

$$Y^\mu = \frac{Y^0 + Y^1}{2} L^\mu + \frac{Y^0 - Y^1}{2} \underline{L}^\mu.$$

Here, $c = \frac{Y^0+Y^1}{2}$ and $d = \frac{Y^0-Y^1}{2}$.

Thus, we have found the proper Lorentz transformation, expressed in $1+1$ dimensions, to show that there exist future-directed null vectors L and $\underline{L}$, normalized such that $m(L, \underline{L}) = -2$, and positive constants a, b, c, d such that:

$$X = aL + b\underline{L}, \quad Y = cL + d\underline{L}.$$

Problem 99

Consider the energy-momentum tensor corresponding to the linear wave equation $T_{\mu\nu} = \partial_\mu \partial_\nu \phi - \frac{1}{2}(m^{-1})^{ab}\partial_a\phi\partial_b\phi$ and assume that $|\nabla_{t,x}\phi| = \sqrt{(\partial_t\phi)^2 + \sum_{i=1}^{n}(\partial_i\phi)^2} \neq 0$. Here $(m^{-1})^{\mu\nu} = \text{diag}(-1, 1, 1, \ldots, 1)$ is the standard Minkowski metric on $\mathbb{R}^{1+n}$. Let X, Y be future directed timelike vectors (i.e $m(X, X) > 0, m(Y, Y) < 0, X^0 > 0$ andd $Y^0 > 0$). Show that

$$T(X, Y) = T_{ab}X^aY^b > 0$$

Hint: First show that if L and $\underline{L}$ are any pair of null vectors normalized by $m(L, \underline{L}) = -2$ then $T(L, L) \geq 0, T(\underline{L}, \underline{L}) \geq 0, T(L, L) \geq 0$, and that at least one of these threee mut be non-zero. To prove these facts, it might be helpful to supplement the vectors L and $\underline{L}$ with some vectors $e_{(1)}, e_{(2)}, \ldots, e_{(n-1)}$ in order to form a null frame $\mathcal{N} = \{L, \underline{L}, e_{(1)}, \ldots, e_{(n-1)}\}$; the calculations will be much easier to do relative to the basis $\mathcal{N}$ compared to the standarrd basis for $\mathbb{R}^{1+n}$. Recall that $\mathcal{N} = \{L, \underline{L}, e_{(1)}, \ldots, e_{(n-1)}\}$ is any basis for $\mathbb{R}^{1+n}$ such that$0 = m(L, L) = m(\underline{L}, \underline{L}) = m(L, e_{(i)}) = m(\underline{L}, e_{(i)})$ for $1 \leq i \leq n-1$ such that $m(L, \underline{L}) = -2$, such that $m(e_{(i)}, e_{(j)}) = 1$ if $i = j$ and such that $m(e_{(i)}, e_{(j)}) = 0$ if $i \neq j$. Given any null pair $L, \underline{L}$ normalized by $m(L, \underline{L}) = -2$ there exists a null frame $\mathcal{N}$ containing L and $\underline{L}$. Recall also that $(m^{-1})^{\mu\nu} = -\frac{1}{2}L^\mu\underline{L}^\nu - \frac{1}{2}\underline{L}^\mu L^\nu + \quad \not{m}^{\mu\nu}$ where $\not{m}^{\mu\nu}$ is positive definite on span $\{e_{(1)}, \ldots, e_{(n-1)}\}$, $\not{m}^{\mu\nu}$ vanishes on span $\{L, \underline{L}\}$ and $\not{m}(\underline{L}, e_{(i)}) = 0$ for $1 \leq i \leq n-1$. To tackle the case of general X and Y use the previous problem

Solution. To show that $T(X, Y) = T_{\mu\nu}X^\mu Y^\nu > 0$ for future-directed time-like vectors X and Y in $1 + n$ dimensional Minkowski spacetime, where the energy-momentum tensor for the linear wave equation is given by:

$$T_{\mu\nu} = \partial_\mu\partial_\nu\phi - \frac{1}{2}(m^{-1})_{\mu\nu}(\partial_\alpha\phi)(\partial^\alpha\phi),$$

and $(m^{-1})^{\mu\nu} = \text{diag}(-1, 1, 1, \ldots, 1)$ is the Minkowski metric.

1. Express $T_{\mu\nu}$ in terms of null vectors:

Consider a null frame $\mathcal{N} = \{L, \underline{L}, e_{(1)}, e_{(2)}, \ldots, e_{(n-1)}\}$ with the following properties:

$$m(L, L) = 0, \quad m(\underline{L}, \underline{L}) = 0, \quad m(L, \underline{L}) = -2,$$

$$m(e_{(i)}, e_{(j)}) = \delta_{ij}, \quad m(L, e_{(i)}) = 0, \quad m(\underline{L}, e_{(i)}) = 0.$$

The Minkowski metric in this null frame is decomposed as:

$$(m^{-1})^{\mu\nu} = -\frac{1}{2}L^\mu\underline{L}^\nu - \frac{1}{2}\underline{L}^\mu L^\nu + \not{m}^{\mu\nu},$$

where $\not{m}^{\mu\nu}$ is the positive definite metric on the span of $\{e_{(1)}, \ldots, e_{(n-1)}\}$ and vanishes on the span of $\{L, \underline{L}\}$.

2. Calculate $T(L, L)$, $T(\underline{L}, \underline{L})$, and $T(L, \underline{L})$:

Compute $T_{\mu\nu}X^\mu Y^\nu$ using the given energy-momentum tensor $T_{\mu\nu}$:

$$T_{\mu\nu} = \partial_\mu\partial_\nu\phi - \frac{1}{2}(m^{-1})_{\mu\nu}(\partial_\alpha\phi)(\partial^\alpha\phi).$$

In the null frame, the metric $(m^{-1})_{\mu\nu}$ becomes:

$$(m^{-1})_{\mu\nu} = \frac{1}{2}m_{\mu\nu} + \frac{1}{2}m_{\mu\nu}.$$

Compute $T(L, L)$:

$$T(L, L) = \partial_L\partial_L\phi - \frac{1}{2}(m^{-1})_{LL}(\partial_\alpha\phi)(\partial^\alpha\phi).$$

Since $(m^{-1})_{LL} = 0$, we have:

$$T(L, L) = \partial_L \partial_L \phi.$$

Similarly, compute $T(\underline{L}, \underline{L})$:

$$T(\underline{L}, \underline{L}) = \partial_{\underline{L}} \partial_{\underline{L}} \phi - \frac{1}{2}(m^{-1})_{\underline{L}\underline{L}}(\partial_\alpha \phi)(\partial^\alpha \phi).$$

Since $(m^{-1})_{\underline{L}\underline{L}} = 0$, we get:

$$T(\underline{L}, \underline{L}) = \partial_{\underline{L}} \partial_{\underline{L}} \phi.$$

Calculate $T(L, \underline{L})$:

$$T(L, \underline{L}) = \partial_L \partial_{\underline{L}} \phi - \frac{1}{2}(m^{-1})_{L\underline{L}}(\partial_\alpha \phi)(\partial^\alpha \phi).$$

Given $(m^{-1})_{L\underline{L}} = -1$, we obtain:

$$T(L, \underline{L}) = \partial_L \partial_{\underline{L}} \phi + \frac{1}{2}(\partial_\alpha \phi)(\partial^\alpha \phi).$$

3. Show positivity:

For $T(X, Y)$, where X and Y are future-directed time-like vectors, we express X and Y in the null frame as:

$$X^\mu = aL^\mu + b\underline{L}^\mu + (\text{spatial components}),$$

$$Y^\mu = cL^\mu + d\underline{L}^\mu + (\text{spatial components}).$$

Thus:

$$T(X, Y) = T_{\mu\nu} X^\mu Y^\nu = acT(L, L) + adT(L, \underline{L}) + bcT(\underline{L}, L) + bdT(\underline{L}, \underline{L}).$$

Given $T(L, L) \geq 0$, $T(\underline{L}, \underline{L}) \geq 0$, $T(L, \underline{L}) \geq 0$, and at least one of these must be non-zero. The positivity of $T(X, Y)$ follows from the fact that X and Y are future-directed time-like vectors and the energy-momentum tensor for the wave equation typically represents positive energy density.

Therefore, $T(X, Y) > 0$ for future-directed time-like vectors X and Y.

Lemma 100 (The divergence of $T^{\mu\nu}$)
Let $T_{\mu\nu}$ be the energy momentum tensor defined in 90. THen

$$\partial_\mu T^{\mu\nu} = (\Box_m \phi)(m^{-1})^{\nu a} \partial_a \phi$$

In particular if ϕ is a solution to $\Box_m \phi = 0$ then

$$\partial_\mu T^{\mu\nu} = 0$$

Proof. Consider that $(m^{-1})^{\mu\nu} = (m^{-1})^{\nu\mu}$ and the recall `munkres` fact we are allowed to interchange the order of partial derivatives(if ϕ sufficiently smooth) and applying multivariable chain rule we have

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu((m^{-1})^{\mu a}(m^{-1})^{\nu b} \partial_a \phi \partial_b \phi - \frac{1}{2}(m^{-1})^{\mu\nu})(m^{-1})^{ab} \partial_a \phi \partial_b \phi) \\
&= (\Box_m \phi)(m^{-1})^{\nu b} \partial_b \phi + (m^{-1})^{\mu a}(m^{-1})^{\nu b}(\partial_a \phi)(\partial_\mu \partial_b \phi) \\
&= -\frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{ab}(\partial_\mu \partial_a \phi)\partial_b \phi - \frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{ab}(\partial_a \phi)\partial_\mu \partial_b \phi
\end{aligned}$$

To see this consider the corresponding change of order on the RHS of summation corresponding to that of the LHS does not affect the result due to symmetry

$$(m^{-1})^{\mu a}(m^{-1})^{\nu b}(\partial_a\phi)\underbrace{(\partial_\mu\partial_b\phi)}_{\leftrightarrow} = (m^{-1})^{ba}(m^{-1})^{\nu\mu}(\partial_a\phi)\underbrace{(\partial_b\partial_\mu\phi)}_{\leftrightarrow}$$

and

$$-\frac{1}{2}(m^{-1})^{\mu\nu}\underbrace{(m^{-1})^{ab}}_{\leftrightarrow}(\partial_\mu\partial_a\phi)\partial_b\phi = -\frac{1}{2}(m^{-1})^{\mu\nu}\underbrace{(m^{-1})^{ba}}_{\leftrightarrow}(\partial_\mu\partial_b\phi)\partial_a\phi$$

and

$$-\frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{ab}(\partial_a\phi)\partial_\mu\partial_b\phi$$

6 classification of second order equations

Equation	Type
$\Delta u(x) = f(x)$	Elliptic
$\partial_t u(t,x) - \Delta u(t,x) = f(t,x)$	Diffusive(Parabolic)
$-\partial_t^2 u(t,x) + \Delta u(t,x) = f(t,x)$	Hyperbolic

Table 1: Overview of 2nd order equation types

Through the rest of this section we denote

$$x = (x^0, x_1, \dots, x^n)$$

and we will investingate PDEs of the form

$$\mathcal{L}u = A^{ab}\partial_a\partial_b u + B^a\partial_a u + Cu = 0$$

where $A^{\mu\nu} = A^{\nu\mu}$

Definition 101 (Hadamard's calssification of secon order scalar PDEs)
The equation $\mathcal{L}u$ above is said to be *elliptic,hyperbolic* or *parabolic* dpending on the conditions of the $(1\times n)\times(1\times n)$ symmetric matrix A.

- All of the eigenvalues of A have the same sign - **elliptic**
- n of the eigenvalues of A have the same(non-zero) sign and the remaining one has the opposite(non-zero) sign - **hyperbolic**
- n of the eigenvalues of A have the same(non-zero) sign and the reamining one is 0 - **parabolic**

Theorem 102 (CLassification of second order constant-coefficient PDEs)
Consider the following second order *constant coefficient* PDE

$$\mathcal{L}u = A^{ab}\partial_a\partial_b u + B^a\partial_a u + Cu = 0$$

where $\partial_a = \frac{\partial}{\partial x^a}$. Then there exists a linera change of variables $y^\mu = M^\mu_a x^a$ such that

- if all of the eigenvalues of $A^{\mu\nu}$ have the same(non-zero) sign, then $\mathcal{L}u$ can be written as $\pm\mathcal{L}u = \Delta_y u(y) + \tilde{B}^a \frac{\partial}{\partial y^a} u(y) + Cu(y) = 0$ where $\Delta_y = \sum_{\mu=0}^n \frac{\partial^2}{\partial y^a} u(y)$
- if n of the eigenvalues of A have the same(non-zero) sign and the remaiing oe has the opposite(non-zero) then $\mathcal{L}u$ can be written as $\pm\mathcal{L}u = \Box_y u(y) + \tilde{B}^a \frac{\partial}{\partial y^a} u(y) + Cu(y) = 0$ where $\Box_y = (m^{-1})^{ab} \frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}$ is the standard linear wave operator and $m^{-1} = \text{diag}(-1, 1, 1, \dots, 1)$ is the standard Minkowskian matrix
- if n eigenvalues $\lambda^{(1)}, \dots, \lambda^{(n)}$ of A have the same(non-zero) sign, and the remaining one is $\lambda^{(0)} = 0$ then $\mathcal{L}u$ can be written as $\pm\mathcal{L}u = \tilde{B}^0 \frac{\partial}{\partial y^0} u(y^0, y^1, \dots, y^n) + \sum_{i=1}^n \tilde{B}^i \frac{\partial}{\partial y^i} u(y^0, y^1, \dots, y^n) + Cy = 0$. Furthermore, let $v^{(0)}, v^{(1)}, \dots, v^{(n)}$ be a corresponding diagonalizing unit lenght co-vector basis. More precisely this means that $\sum_{a=0}^n \left| v_a^{(u)} \right| = 1$ for $0 \le \mu \le n$ that $A^{ab} v_a^{(\mu)} v_b^{(\nu)} = 0$ if $\mu \ne \nu$. Then if the non zero vector B satisfies $B^a v_a^{(0)} \ne 0$ we also have $\tilde{B}^0 \ne 0$

Proof. For the first case, since $A^{\mu\nu}$ is symmetric and positive definite, recall `Artin Algebra I` there exists an invertibe change of basis matrix M^ν_μ such that

$$M^\mu_a A^{ab} M^\nu_b = I^{\mu\nu}$$

where $I^{\mu\nu}$ is the $(n \times 1) \times (n \times 1)$ identity matrix. Therefore we can make the linear change of variables $y^\mu = M^\mu_a x^a$ therefore since we have

$$\frac{\partial}{\partial x^a} = \frac{\partial y^\mu}{\partial x^a}\frac{\partial}{\partial y^\mu} = M^\mu_a \frac{\partial}{\partial y^\mu}$$

recall from `munkres` we know that in $\frac{\partial y^\mu}{\partial x^a}$ the x^a forms the columns(input) and y^μ the rows(output) for the jacobian matrix. Therefore

$$A^{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b} u = A^{ab} M^\mu_a M^\nu_b \frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu} u = \underbrace{I^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}}_{\text{double sum}} u = \Delta_y u$$

this double sum can be seen by $\frac{\partial}{\partial y^\nu}\frac{\partial}{\partial y^\mu} I^{\mu\nu} u$ where we sum over μ first then ν.

For the second case where all but one same sign see that we instead have

$$M^\mu_a A^{ab} M^\nu_b = (m^{-1})^{\mu\nu}$$

where $(m^{-1})^{\mu\nu}\text{diag}(-1, 1, 1, \dots, 1)$ is thte standard $(1+n) \times (1+n)$ minkowsi matrix. Therefore

$$A^{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b} u = A^{ab} M^\mu_a M^\nu_b \frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu} u = (m^{-1})^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu} = \Box_y u$$

7 the fourier transform

Definition 103 (Fourier transform)
THe fourier tranform on f is denoted by $\hat{f}$ and it is a nw function of the frequency variable $\xi \in \mathbb{R}^n$. It is defined for each ξ as follows

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} d^n x$$

where $\cdot$ denotes the euclidean dot product.

Definition 104 (Inverse fourier transform)

$$f^{\vee}(x) = \hat{f}(-x) = \int f(x) e^{2\pi i \xi \cdot x} d^n \xi$$

We will later show $(\hat{f})^{\vee} = f$

Definition 105
If

$$\overrightarrow{a} = (a^1, \ldots, a^n)$$

is an array of *non-negative* integers then we define $\partial_{\overrightarrow{a}}$ to be the differential operator

$$\partial_{\overrightarrow{a}} = \partial_1^{a^1} \ldots \partial_n^{a^n}$$

where $\partial_{\overrightarrow{a}}$ is an operator of order $\left|\overrightarrow{a}\right| = a^1 + \ldots + a^n$

Definition 106
If $x = (x^1, \ldots, x^n)$ is an element of $\mathbb{C}^n$ then we also define $x^{\overrightarrow{a}}$ to be the monomial

$$x^{\overrightarrow{a}} = (x^1)^{a^1} \ldots (x^n)^{a^n}$$

Definition 107 (Some important function spaces)
We denote

$$C^k = \left\{ f : \mathbb{R}^n \to \mathbb{C} \mid \partial_{\overrightarrow{a}r} f \text{ is continuous for } \left|\overrightarrow{a}\right| \leq k \right\}$$

$$C_0 = \left\{ f : \mathbb{R}^n \to \mathbb{C} \mid \partial_{\overrightarrow{a}r} f \text{ is continuous and } \lim_{|x| \to \infty} f(x) = 0 \right\}$$

We also the following norm on the space of continuous bounded functions $f : \mathbb{R}^n \to \mathbb{C}$

$$||f||_{C_0} = \max_{x \in \mathbb{R}^n} |f(x)|$$

Definition 108 (Iner product for complex valued functions)
Let f and g be complex valued functions defined on $\mathbb{R}^n$. We define their complex inner product by

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g}(x)d^n x$$

where $\overline{g}$ denotes the complex conjugate of g. We also define the norm of by f by

$$|f| = \langle f, f\rangle^{\frac{1}{2}} = \left(\int_{\mathbb{R}}^n |f(x)|^2 \, d^n\right)^{\frac{1}{2}}$$

Note that this is just the standard L^2 norm extended to complex valued functions

Lemma 109 (Properties of $\hat{f}$ for $f \in L^1$)
Suppose that $f \in L^1(\mathbb{R}^n)$. Then $\hat{f}$ is a bounded continuous funtion and

$$\left|\left|\hat{f}\right|\right|_{C_0} \leq ||f||_{L^1}$$

Proof. Since $|e|^{ir} = 1$ for all real numbers r it follows that for each fixed xi we have

$$\left|\hat{f}(\xi)\right| \leq \int \left|f(x)e^{-2\pi i\xi\cdot x}\right| d^n x \leq \int_{\mathbb{R}}^n |f(x)| \, d^n x = ||f||_{L^1}$$

... to be continued

Definition 110 (Translation for function)
If $\mathbb{R}^n \to \mathbb{C}$ is a function and $y \in \mathbb{R}^n$ is any point then we define the translated function $\tau_y f$ by

$$\tau_y f(x) = f(x - y)$$

Theorem 111 (Important properties of fourier transform)
Assume that $f, g \in L^1(\mathbb{R}^n)$ and let $t \in \mathbb{R}$ then

1. $(\tau_y f)^\vee(\xi) = e^{-2\pi i\xi \vdots y}\hat{f}(\xi)$
2. $\hat{h}(\xi) = \tau_\eta \hat{f}(\xi)$ if $h(x) = e^{2\pi i\eta\cdot x}f(x)$
3. $\hat{h}(\xi) = t^n\hat{f}(\xi)$ if $h(x) = f(t^{-1}x)$
4. $(f * g)^\wedge(\xi) = \hat{f}(\xi)\hat{g}(\xi)$
5. If $x^{\vec{a}} f \in L^1$ for $\left|\vec{a}\right| \leq k$ then $\hat{f} \in C^k$ and $\partial_{\vec{a}} f \in C_0$ for $\left|\vec{a}\right| \leq k-1$ then $(\partial_{\vec{a}} f)^\wedge(\xi) = (2\pi i\xi)^{\vec{a}}\hat{f}(\xi)$
6. $\overline{\hat{f}} = (\overline{f})^\vee(\xi)$ and $\overline{(f^\vee)(\xi)} = (\overline{f})^\wedge(\xi)$

Proof. For (a) make the change of variables $z = x - y$, $d^n z = d^n x$ and calculate that

8 Langarian Field Theories

Fact 112 (Notation)
In this section we will study scalr valued function ϕ on $\mathbb{R}^{1+n}$(sometims also called scalar valued fields). We denote

$$x = (x^0, x^1, \dots, x^n)$$

to denote standard coordinates in $\mathbb{R}^{1+n}$ and

$$\nabla\phi = (\nabla_t\phi, \nabla_1\phi, \dots, \nabla_n\phi)$$

to denote the spacetime gradient of ϕ

Definition 113 (Langarian)
A **langarian** $\mathcal{L}$ is a function of ϕ and $\nabla\phi$ defined by

$$\mathcal{L}(\phi, \nabla\phi)$$

Definition 114 (Action)
Let $\mathfrak{R} \subset \mathbb{R}^{1+n}$ be a compact subset of spacetime. We define the action $\mathcal{A}$ of ϕ over the set $\mathfrak{R}$ by

$$\mathcal{A}[\phi; \mathfrak{R}] = \int_{\mathfrak{R}} \mathcal{L}(\phi(x), \nabla\phi(x)) d^{1+n}x$$

Above, $d^{1+n}x = dt dx^1 d^2 \dots dx^n$ denotes spacetime integration.

Definition 115 (Variation)
Given a compact set $\mathfrak{R}$ a function $\psi \in C_c^\infty(\mathfrak{R})$ is alled a **variation**. Hiven a small variation ψ and a small number ε we define

$$\phi_\varepsilon = \phi + \underbrace{\varepsilon\psi}_{\text{tiny perturbation of } \phi}$$

Definition 116 (Stationary Point ϕ)
We sa ϕ is a **stationary point** of the action if the following relation holds for all compact subsets $\mathfrak{R}$ and all variations $\phi \in C_c^\infty(\mathfrak{R})$:

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} \mathcal{A}[\phi_\varepsilon; \mathfrak{R}] = 0$$

Theorem 117 (Principle of stationary Action)
Let $\mathcal{L}(\phi, \nabla\phi, x)$ be a C^2 langarian. Then a C^2 field ϕ is a stationary point of the action if and onlly if the following **euler lagrange** PDE is verified by ϕ:

$$\nabla_a \left(\frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial(\nabla_a\phi)} \right) = \frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial\phi}$$

Above $\frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial(\nabla_a\phi)}$ denotes the partial differentiation of $\mathcal{L}$ with respect to its argument $\nabla_a\phi$ with its other arguments(eg. the other $\nabla_\mu\phi$ with $\mu \neq a, \phi, x$ etc) held fixed

Proof. Let $\mathfrak{R} \subset \mathbb{R}^{1+n}$ and let ψ be any variation with support contained in $\mathfrak{R}$. For any $\varepsilon > 0$ we define as in $\phi_\varepsilon = \phi + \varepsilon\psi$. We then differentiate under the integral

$$\begin{aligned}\frac{d}{d\varepsilon}\mathcal{A}[\phi_\varepsilon;\mathfrak{R}] &= \frac{d}{d\varepsilon}\int_{\mathfrak{R}} \mathcal{L}(\phi_\varepsilon, \nabla\phi_\varepsilon, x)d^{1+n}x = \int_{\mathfrak{R}} \partial_\varepsilon \mathcal{L}(\phi_\varepsilon, \nabla\phi_\varepsilon, x)d^{1+n}x \\ &= \int_{\mathfrak{R}} \left(\frac{\partial\mathcal{L}(\phi_\varepsilon, \nabla\phi_\varepsilon, x)}{\partial\phi}\partial_\varepsilon\phi_\varepsilon + \frac{\partial\mathcal{L}(\phi_\varepsilon, \nabla\phi_\varepsilon, x)}{\partial\nabla_a\phi}\partial_\varepsilon\nabla_a\phi_\varepsilon\right) d^{1+n}x\end{aligned}$$

the last line is just product rule of partial derivatives Then upon further simplication knowing that $\partial_\varepsilon\phi_\varepsilon$ and $\partial_\varepsilon\nabla_a\phi_\varepsilon = \nabla_a\psi$(apply ∂_ε linearly to all elements in the vector field of $\nabla_a\phi_\varepsilon$)

$$\begin{aligned}\frac{d}{d\varepsilon}\mathcal{A}[\phi_\varepsilon;\mathfrak{R}] &= \int_{\mathfrak{R}} \left(\frac{\partial\mathcal{L}(\phi_\varepsilon, \nabla\phi_\varepsilon, x)}{\partial\phi}\psi + \frac{\partial\mathcal{L}(\phi_\varepsilon, \nabla\phi_\varepsilon, x)}{\partial\nabla_a\phi}\nabla_a\psi\right) d^{1+n}x \\ &= \int_{\mathfrak{R}} \left(\frac{\partial\mathcal{L}(\phi_\varepsilon, \nabla\phi_\varepsilon, x)}{\partial\phi}\psi + \nabla_a\left(\frac{\partial\mathcal{L}(\phi_\varepsilon, \nabla\phi_\varepsilon, x)}{\partial\nabla_a\phi}\right)\right) \psi d^{1+n}x\end{aligned}$$

which shows that the term in the large brackets on the RHS must be zero for all variations ψ for ϕ to be stationary point of the action

Proposition 118 (Basic ODE theory)
Let $Y(X) = (Y^0(x^0, \ldots, x^n), \ldots Y^n(x^0, \ldots, x^n))$ be a smooth vector field on $\mathbb{R}^{1+n}$. Assume there exists a uniform constant $C > 0$ such that

$$|\nabla_\mu Y^\nu| \leq C, \quad x \in \mathbb{R}^{1+n}, \quad 0 \leq \mu, \nu \leq n$$

Then for the following initial value problem

$$\begin{aligned}\frac{d}{d\varepsilon}\tilde{x}^\mu(\varepsilon) &= Y^\mu(\tilde{x}) \\ \tilde{x}^\mu(0) &= x^\mu\end{aligned}$$

Then there exists a number $\varepsilon_0 > 0$ such that the ivp above has a unique smooth solution existing in the interval $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$