MIT 8.012(Classical Mechanics I)

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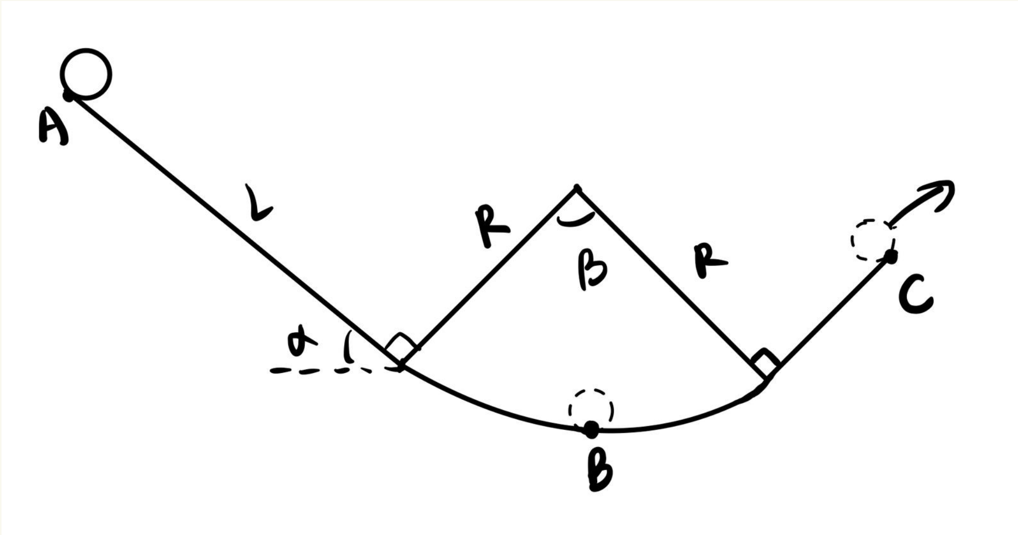
The content can be encapsulated by my university mechanics coursework here. I also added extra derivations for tensor of inertia and parallel axes theorem from the OCW readings

Contents

1 Kinematics

Problem 1 (**1:** *Apply.* Category A.)

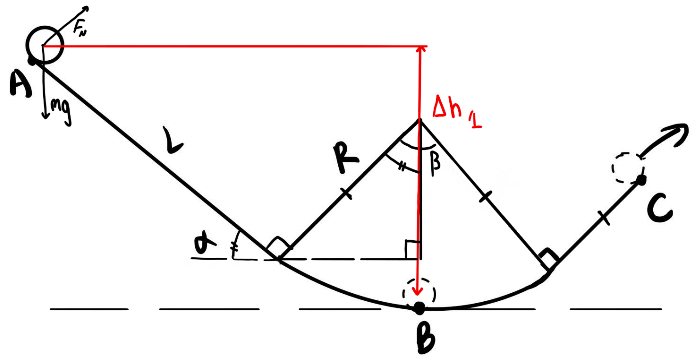
An athlete of mass m skies down a slope of length L and an α with respect to the horizon, which is tangent to a circular section at the bottom of the track. The circular arc has a radius, R, and angle, β. Furthermore it is tangent to the ground. Tangent to the other end of the ark is a ramp of length R. Assume no friction or air resistance while the skier is on the ramp.



1. Using $L = 100m$, $R = 50m$, $\alpha_1 = \frac{1}{6}\pi$, and $\beta = \frac{1}{3}\pi$, calculate the velocity at V_b.
2. Find an angle, α, to make the the new value of V_c equal to the previously calculated value of V_b.
3. After leaving the track the skier experiences an retarding force $\vec{F}_r = -kv$. Find the expression for horizontal and vertical displacement.

Solution.

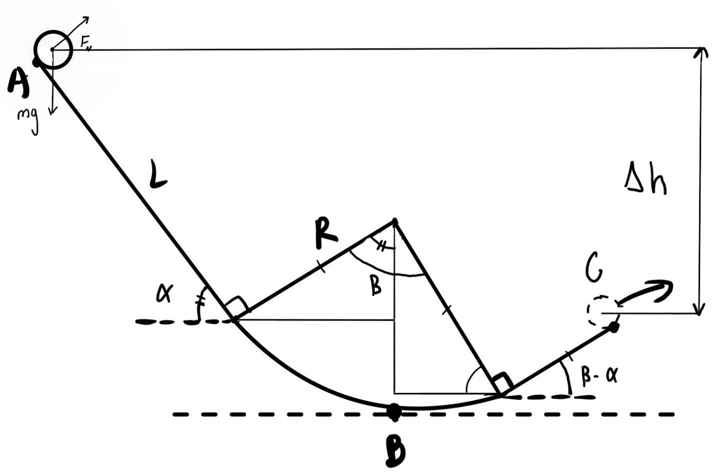
1. The first step is drawing the free body diagram:



Solving via conservation of energy we can set the potential energy of the skier from the height of the slope equal to his kinetic energy at the bottom of the slope:

$$
\begin{aligned}
U_i + E_i + \int_{r_i}^{r_f} (F_{app} + F_{nc}) \cdot dr &= U_f + E_f \\
mgh_1 &= \frac{1}{2}mv^2 \\
h_1 &= L\sin(\alpha_1) + R - R\cos(\alpha_1) \\
v_{b1} &= \sqrt{2g(L\sin(\alpha_1) + R - R\cos(\alpha_1))} \\
v_{b1} &= 33.35\frac{m}{s}
\end{aligned}
\tag{1}
$$

2. For problem B we have to calculate the result using the previously derived equation (1). Next draw the diagram of the new ramp:



Then we calculate Δh:

$$\Delta h = L\sin(\alpha) + R - R\cos(\alpha) - (R - R\cos(\beta - \alpha) + R\sin(\beta - \alpha))$$
$$\Delta h = L\sin(\alpha) - R\cos(\alpha) + R\cos(\beta - \alpha) - R\sin(\beta - \alpha)$$

Applying the work-energy theorem we can once again find the velocity and equate it to the value found in part

A.

$$
\begin{aligned}
U_i + E_i + \int_{r_i}^{r_f} (F_{app} + F_{nc}) \cdot dr &= U_f + E_f \\
mg\Delta h &= \frac{1}{2}mv^2 \\
v_{b1} = v_c &= \sqrt{2g(L\sin(\alpha) - R\cos(\alpha) + R\cos(\beta-\alpha) - R\sin(\beta-\alpha))} \\
\frac{v_{b1}^2}{2g} &= (L\sin(\alpha) - R\cos(\alpha) \\
&\quad + R\cos(\beta-\alpha) - R\sin(\beta-\alpha)) \\
Apply \quad & \sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b) \\
and \quad & cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b) \\
\frac{v_{b1}^2}{2g} &= L\sin(\alpha) - R\cos(\alpha) + R\cos(\beta)\cos(\alpha) + R\sin(\beta)\sin(\alpha) \\
&\quad - R\sin(\beta)\cos(\alpha) + R\cos(\beta)\sin(\alpha) \\
&= \sin(\alpha)(L + R\sin(\beta) + R\cos(\beta)) \\
&\quad + \cos(\alpha)(R\cos(\beta) - R\sin(\beta) - R) \\
\frac{v_{b1}^2}{2g} - \cos(\alpha)(R\cos(\beta) - R\sin(\beta) - R) &= \\
&\quad \sqrt{1-\cos^2(\alpha)}(L + R\sin(\beta) + R\cos(\beta))
\end{aligned}
$$

Simplify by substituting for constants $A = R\cos(\beta) - R\sin(\beta) - R$ and $B = L + R\sin(\beta) + R\cos(\beta)$

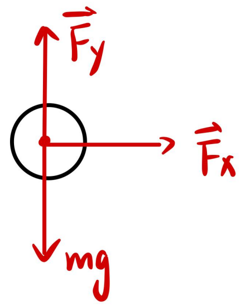
$$
\begin{aligned}
(\frac{v_{b1}^2}{2g} - A\cos\alpha)^2 &= (B\sqrt{1-\cos^2\alpha})^2 \\
A^2\cos^2(\alpha) - \frac{v_{b1}^2}{g}A\cos(\alpha) + (\frac{v_{b1}^2}{2g})^2 &= (1-\cos^2\alpha)(B)^2 \\
0 &= (A^2+B^2)\cos^2(\alpha) - \frac{v_{b1}^2}{g}A\cos(\alpha) + (\frac{v_{b1}^2}{2g})^2 - B^2
\end{aligned}
$$

Finally apply the quadratic formula to solve for the new value of α.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{(b)^2 - 4ac}}{2a} \\
\cos(\alpha) &= \frac{(\frac{Av_{b1}^2}{g}) \pm \sqrt{(\frac{Av_{b1}^2}{g})^2 - 4(A^2+B^2)((\frac{v_{b1}^2}{2g})^2 - B^2)}}{2(A^2+B^2)} \\
\alpha &= \arccos(\frac{(\frac{Av_{b1}^2}{g}) \pm \sqrt{(\frac{Av_{b1}^2}{g})^2 - 4(A^2+B^2)((\frac{v_{b1}^2}{2g})^2 - B^2)}}{2(A^2+B^2)}) \\
\therefore \alpha &= 40.3^\circ, \quad 183.9^\circ
\end{aligned}
$$

Since 183.9° is out of the range we ignore this value, leaving us with 40.3°.

3. Start with the free body diagram:



Next set up the equation for the net force:

$$\vec{F}_{net} = mg\hat{e}_y - kv_y\hat{e}_y - kv_x\hat{e}_x \tag{2}$$

We can solve for the x and y components independently:
For $\hat{e}_x$:

$$\begin{aligned}
\vec{F} &= -m\vec{g} - k\vec{v}_y \\
\vec{F} &= m\ddot{y} \\
\frac{dv}{dt} &= -g - \frac{k}{m}v_y \\
-\frac{dv}{g + \frac{k}{m}v} &= dt \\
&\text{Integrate both sides} \\
-\int \frac{dv}{g + \frac{k}{m}v} &= \int dt \\
-\frac{m}{k}\ln\left(\left|\frac{k}{m}v + g\right|\right) &= t + C \\
\ln\left(\left|\frac{k}{m}v + g\right|\right) &= -\frac{m}{k}t + C \\
\frac{k}{m}v + g &= Ae^{-\frac{k}{m}t} \\
\frac{k}{m}v &= Ae^{-\frac{k}{m}t} - g \\
v_y &= Ae^{-\frac{k}{m}t} - \frac{mg}{k}
\end{aligned}$$

$$\text{Apply initial conditions}$$
$$\text{when } t = 0, \quad v_y = v_c \sin(\beta - \alpha)$$
$$\therefore A = V_c \sin(\beta - \alpha) + \frac{mg}{k}$$
$$v_y = (v_c \sin(\beta - \alpha) + \frac{mg}{k})e^{-\frac{k}{m}t} - \frac{mg}{k}$$
$$\text{Next solve for y:}$$
$$\frac{dy}{dt} = (v_c \sin(\beta - \alpha) + \frac{mg}{k})e^{-\frac{k}{m}t} - \frac{mg}{k}$$
$$\int dy = \int ((v_c \sin(\beta - \alpha) + \frac{mg}{k})e^{-\frac{k}{m}t} - \frac{mg}{k})dt$$
$$y = -\frac{m}{k}(v_c \sin(\beta - \alpha) + \frac{mg}{k})e^{-\frac{k}{m}t} - \frac{mg}{k}t + C_2$$
$$\text{Apply initial conditions}$$
$$\text{when } t = 0, \quad y = R\sin(\beta - \alpha) + R - R\cos(\beta - \alpha)$$
$$\therefore C_2 = R\sin(\beta - \alpha) + R - R\cos(\beta - \alpha) + \frac{m}{k}(v_c \sin(\beta - \alpha) + \frac{mg}{k})$$
$$y = \frac{m}{k}(v_c \sin(\beta - \alpha) - \frac{mg}{k})(1 - e^{-\frac{k}{m}t}) - \frac{mg}{k}t + R\sin(\beta - \alpha) + R - R\cos(\beta - \alpha)$$

For $\hat{e}_y$

$$\vec{F} = -k\vec{v}_x$$
$$\vec{F} = m\ddot{x}$$
$$\frac{dv}{dt} = -\frac{k}{m}v_x$$
$$-\frac{dv}{\frac{k}{m}v} = dt$$
$$\text{Integrate both sides}$$
$$-\int \frac{dv}{\frac{k}{m}v} = \int dt$$
$$-\frac{m}{k}\ln\left|\frac{k}{m}v\right| = t + C$$
$$\ln\left|\frac{k}{m}v\right| = -\frac{k}{m}t + C$$
$$\frac{k}{m}v = Ae^{-\frac{k}{m}t}$$
$$v_x = Ae^{-\frac{k}{m}t}$$

Apply initial conditions

when $t = 0$,

$$v_x = v_c \cos(\beta - \alpha)$$

$$\therefore A = v_c \cos(\beta - \alpha)$$

$$v_x = v_c \cos(\beta - \alpha) e^{-\frac{k}{m} t}$$

Next solve for x :

$$\frac{dx}{dt} = v_c \cos(\beta - \alpha) e^{-\frac{k}{m} t}$$

$$\int dx = \int v_c \cos(\beta - \alpha) e^{-\frac{k}{m} t} dt$$

$$x = -\frac{m v_c}{k} \cos(\beta - \alpha) e^{-\frac{k}{m} t} + C_2$$

Apply initial conditions

when $t = 0, \quad x = 0$

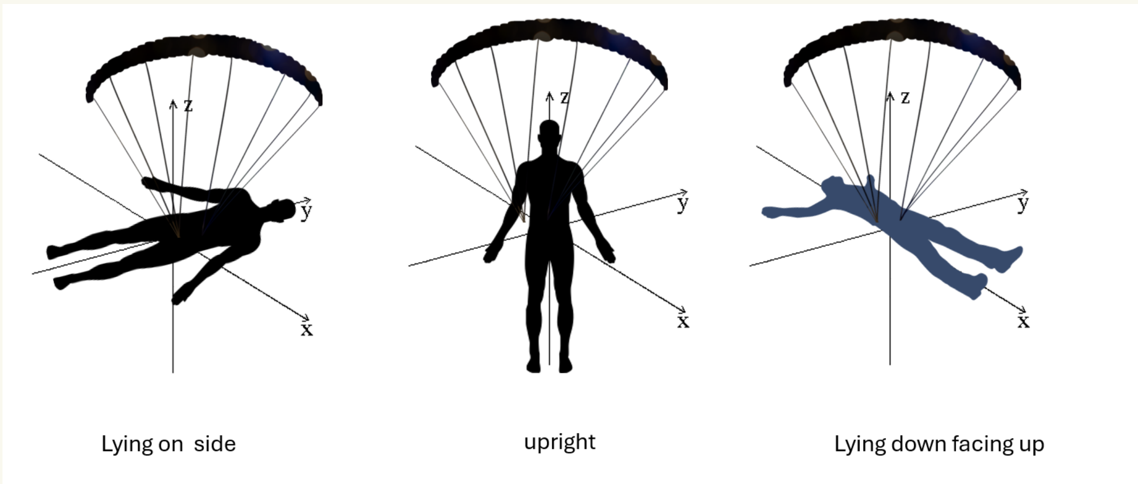
$$\therefore C_2 = \frac{m v_c}{k} \cos(\beta - \alpha)$$

$$x = \frac{m v_c}{k} \cos(\beta - \alpha)(1 - e^{-\frac{k}{m} t})$$

2 Rigid Body Motion

Problem 2 (**2:** *Transfer.* Category C.)

The 2024 Olympics opening ceremony is set to kick off with a marvelous skydiving show. Use knowledge of rigid bodies to derive the intermediate axis theorem (see appendix). Initially the descending skydivers are oriented in different positions along the principle axes and are spinning with a relatively large angular velocity (w_s) about the z axis.

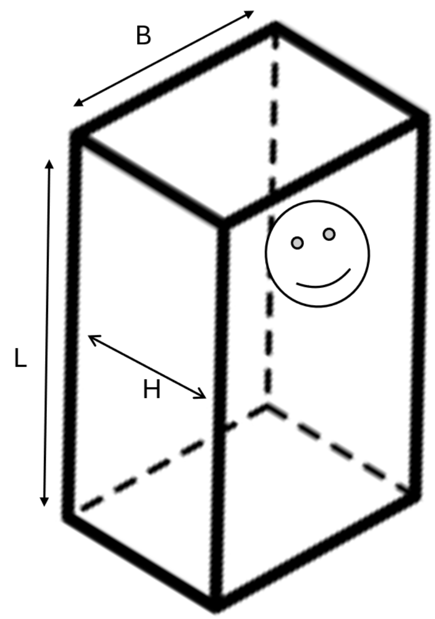


During descent the skydiver's descent might be perturbed by random gusts of wind such that they are no longer aligned with principle axes. Which position should a skydiver not choose to ensure stable descent? Assume torque free conditions after the perturbations, small angles, and no air resistance. The performer can be modeled by a uniform density rectangular plank with thickness (H), breadth (B), length (L), where $L > B > H$.

Hint:

1. Model deviations from the original position as independent rotations about each axes

Solution. Begin by drawing the model of the skydiver



L: Height from head to shoulders
B: Shoulder to shoulder distance
H: Front torso to back torso distance

We can state that infinitesimal rotations are commutative because small angles are used. For example consider x and y rotation matrices(recall your derivation in contiuum mechanics):

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$
$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Using small angle, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, one can verify that $R_x R_y = R_y R_x$.
therefore we can split angular velocity into independent components So using these relations that we have now

$$w = w_x\hat{i} + w_y\hat{j} + w_z\hat{k}$$
$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

we can do a frustratingly humongous amount of cross multipication:

Lemma 3 (Tensor of Inertia)
We have

$$\mathbf{I} = \sum m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j)\mathbf{I} - \mathbf{r}_j\mathbf{r}_j^T\right]$$

Proof. The angular momentum of a system of point masses is given by:

$$\mathbf{L} = \sum m_j \mathbf{r}_j \times \mathbf{v}_j.$$

Since the velocity is given by $\mathbf{v}_j = \mathbf{w} \times \mathbf{r}_j$, we substitute:

$$\mathbf{L} = \sum m_j \mathbf{r}_j \times (\mathbf{w} \times \mathbf{r}_j).$$

Using the vector triple product identity(recall derivation in your tensor calculus notes),

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C},$$

with $\mathbf{A} = \mathbf{r}_j$, $\mathbf{B} = \mathbf{w}$, and $\mathbf{C} = \mathbf{r}_j$, we get:

$$\mathbf{r}_j \times (\mathbf{w} \times \mathbf{r}_j) = (\mathbf{r}_j \cdot \mathbf{r}_j)\mathbf{w} - (\mathbf{r}_j \cdot \mathbf{w})\mathbf{r}_j.$$

Substituting this into $\mathbf{L}$:

$$\mathbf{L} = \sum m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j)\mathbf{w} - (\mathbf{r}_j \cdot \mathbf{w})\mathbf{r}_j\right].$$

Rewriting the second term using the dyadic product:

$$\sum m_j (\mathbf{r}_j \cdot \mathbf{w})\mathbf{r}_j = \sum m_j \left(\mathbf{r}_j \mathbf{r}_j^T\right) \mathbf{w},$$

This is because expanding the dot product:

$$\mathbf{r}_j \cdot \mathbf{w} = \sum_k r_{jk} w_k.$$

Substituting this into the left-hand side:

$$\sum m_j (\mathbf{r}_j \cdot \mathbf{w})\mathbf{r}_j = \sum m_j \sum_k r_{jk} w_k \mathbf{r}_j.$$

Expressing $\mathbf{r}_j$ in component form:

$$\mathbf{r}_j = \sum_l r_{jl} \hat{e}_l,$$

we obtain:

$$\sum m_j \sum_k r_{jk} w_k \sum_l r_{jl} \hat{e}_l.$$

Rearranging the summations:

$$\sum_l \sum_k \left(\sum m_j r_{jk} r_{jl}\right) w_k \hat{e}_l.$$

Recognizing the dyadic product:

$$\sum m_j r_{jk} r_{jl} = \sum m_j \left(\mathbf{r}_j \mathbf{r}_j^T\right)_{lk},$$

we rewrite:

$$\sum_l \sum_k \left(\sum m_j \left(\mathbf{r}_j \mathbf{r}_j^T\right)_{lk}\right) w_k \hat{e}_l.$$

Reinterpreting as a matrix-vector multiplication:

$$\sum m_j \left(\mathbf{r}_j \mathbf{r}_j^T\right) \mathbf{w}.$$

we obtain:

$$\mathbf{L} = \sum m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j)\mathbf{I} - \mathbf{r}_j \mathbf{r}_j^T\right] \mathbf{w}.$$

Thus, the inertia tensor is:

$$\boxed{\mathbf{I} = \sum m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j)\mathbf{I} - \mathbf{r}_j \mathbf{r}_j^T \right]}$$

□

to get the tensor of inertia($\mathbf{I}$) from the form $\mathbf{L} = \mathbf{I}w$ we evaluate

$$\mathbf{I} = \sum m_j \left[(x_j^2 + y_j^2 + z_j^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} x_j^2 & x_j y_j & x_j z_j \\ y_j x_j & y_j^2 & y_j z_j \\ z_j x_j & z_j y_j & z_j^2 \end{pmatrix} \right].$$

$$L = \sum r_j \times m_j v_j$$
$$L = \sum r_j \times m_j (w_j \times r_j)$$
$$L = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

where the products of inertia (the non diagonal terms) and moment of inertia about respective principle axes x,y,z (diagonal terms) are calculated like so in this example (the 1st row of $\mathbf{I}$). m_j is just a point mass with coordinates (x_j, y_j, z_j). If you like you could use integration to replace the sum function too.

$$I_{xx} = \sum m_j (y_j^2 + z_j^2)$$
$$I_{xy} = -\sum m_j x_j y_j$$
$$I_{xz} = -\sum m_j x_j z_j$$

Consider I_{xy}. Because the human which is modelled as a symmetrical plank is aligned with the principle axes, for every point mass with (x_j, y_j) there exists another with $(x_j, -y_j)$ likewise for I_{xz}. So without the square power terms like in the diagonal terms, the non diagonal terms will cancel and become zero. You can derive similarly for other rows too. Now we write out our simplified tensor of inertia $\mathbf{I}$

$$\mathbf{I} = \begin{pmatrix} I_3 \\ I_2 \\ I_1 \end{pmatrix} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$
$$\begin{pmatrix} L_3 \\ L_2 \\ L_1 \end{pmatrix} = \begin{pmatrix} I_3 \\ I_2 \\ I_1 \end{pmatrix} \begin{pmatrix} w_3 \\ w_2 \\ w_1 \end{pmatrix}$$

Applying Euler equations we now solve for the equations of motion in the form $\tau = \frac{dL}{dt} = \mathbf{I}\dot{w}$

$$\tau_1 = \frac{dL_1}{dt} = \frac{I_1 dw_1}{dt} + (I_3 - I_2) w_3 w_2 \tag{3}$$
$$\tau_2 = \frac{dL_2}{dt} = \frac{I_2 dw_2}{dt} + (I_1 - I_3) w_1 w_3 \tag{4}$$
$$\tau_3 = \frac{dL_3}{dt} = \frac{I_3 dw_3}{dt} + (I_2 - I_1) w_2 w_1 \tag{5}$$

Initially $w_1 = w_s$ and $w_3 = w_2 = 0$, and, due to the use of small angles, after perturbation $w_1 \approx w_s$ and w_2 and w_3 are negligible (≈ 0)

Now under torque free conditions (equilibrium conditions, $\tau = 0$) and the above constraints applied, we reduce Euler's equations to:
From equation 3

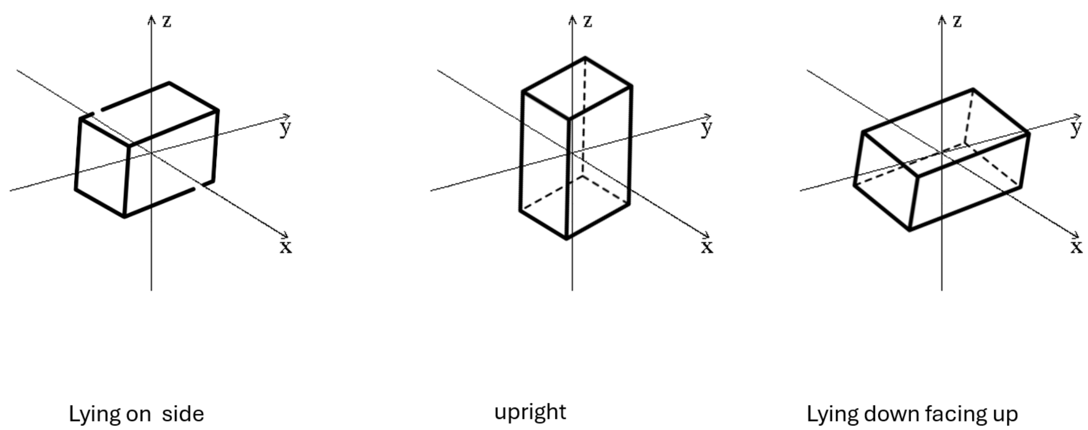
$$\frac{I_1 dw_1}{dt} = 0$$

From equation 4, keeping in mind w_1 is constant differentiate the below:

$$\frac{I_2 dw_2}{dt} + (I_1 - I_3) w_1 w_3 = 0$$

then substitute the value of $\frac{dw_3}{dt}$ into equation 5 obtaining an equation that resembles simple harmonic motion (SHM):

$$\begin{aligned} I_2 \frac{d^2 w_2}{dt^2} - \frac{(I_1 - I_3)(I_2 - I_1)}{I_3} w_1^2 w_2 = 0 \\ \frac{d^2 w_2}{dt^2} = -\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3} w_1^2 w_2 \\ \frac{d^2 w_2}{dt^2} = -A w_2 \end{aligned}$$



for facing up

$$\begin{aligned} I_1 &= \frac{1}{12}(L^2 + B^2) \\ I_2 &= \frac{1}{12}(H^2 + B^2) \\ I_3 &= \frac{1}{12}(H^2 + L^2) \\ A &> 0 \end{aligned}$$

for upright

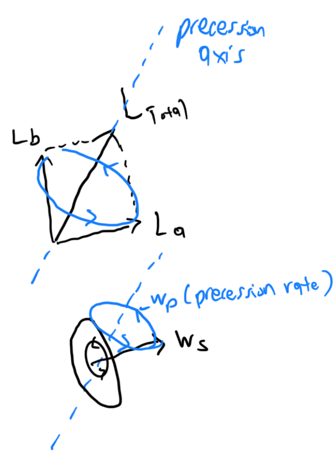
$$I_1 = \frac{1}{12}(H^2 + B^2)$$
$$I_2 = \frac{1}{12}(H^2 + L^2)$$
$$I_3 = \frac{1}{12}(L^2 + B^2)$$
$$A > 0$$

for lying on side

$$I_1 = \frac{1}{12}(H^2 + L^2)$$
$$I_2 = \frac{1}{12}(H^2 + B^2)$$
$$I_3 = \frac{1}{12}(B^2 + L^2)$$
$$A < 0$$

Similar equations and results can be derived in a similar manner for w_3. Clearly, if I_1 is not the largest or smallest the SHM equation will not be fulfilled. Thus when laying on side the human will flip up and down the y or z axis instead of precessing about a fixed axis which is a relatively more "stable" movement. (Precession = rotation of rotational axis)

So how does precession work and why does it occur here? When w_2 and w_3 are bound by equations of SHM, their corresponding angular momentum vector(vector sum $I_3 w_3 + I_2 w_2$) sweeps around the TOTAL angular momentum vector in a circular motion. Recall that if you plot the x and y displacements of a swinging pendulum in SHM you will get a circular plot.Then the radial position vector which is similarly a vector sum of x and y is said to have sweeped out a circle. The same reasoning applies here.
Now, consider this illustration which should help explain why the axis of circular motion must be the total angular momentum



As you can see the 2 orthogonal components of total angular momentum in this illustration(L_a and L_b) sweep around the total angular momentum(the vector sum of L_a and L_b). Let $L_a = I_3 w_3 + I_2 w_2$ and $L_b = I_1 w_s$. (1.) Because L_b is the other orthogonal component of L_{total} it must also demonstrate the same circular motion as discussed earlier like L_a. (2.) Because the total angular momentum vector remains constant in magnitude and in fixed direction under the assumption of torque free conditions it must be the axis of precession like so. It cannot move in a circular motion like its components.

Hence the descent will be far less smooth for the lying on side position because it cannot precess.

Theorem 4 (Parallel axis thereom)
About the new shifted axes we for moment of inertia

$$I'_{xx} = I_{xx}^{\text{CM}} + M(d_y^2 + d_z^2).$$

Proof. The parallel axis theorem states that the moment of inertia about an axis displaced by a vector $\mathbf{d} = (d_x, d_y, d_z)$ from the center of mass is given by:

$$I' = I_{\text{CM}} + Md^2,$$

where I_{CM} is the moment of inertia about the center of mass, M is the total mass, and d^2 is the square of the perpendicular distance from the new axis. The moment of inertia about the x-axis is:

$$I_{xx} = \sum m_j(y_j^2 + z_j^2).$$

Let the new coordinate system be shifted by $\mathbf{d} = (d_x, d_y, d_z)$, so that the new coordinates are:

$$y'_j = y_j - d_y, \quad z'_j = z_j - d_z.$$

Substituting into the inertia formula:

$$I'_{xx} = \sum m_j \left[(y_j - d_y)^2 + (z_j - d_z)^2\right].$$

Expanding:

$$I'_{xx} = \sum m_j \left[y_j^2 - 2d_y y_j + d_y^2 + z_j^2 - 2d_z z_j + d_z^2\right].$$

Rearranging:

$$I'_{xx} = \sum m_j(y_j^2 + z_j^2) - 2d_y \sum m_j y_j - 2d_z \sum m_j z_j + \sum m_j(d_y^2 + d_z^2).$$

Since the center of mass satisfies:

$$\sum m_j y_j = M y_{\text{CM}}, \quad \sum m_j z_j = M z_{\text{CM}},$$

we obtain:

$$I'_{xx} = I_{xx}^{\text{CM}} + M(d_y^2 + d_z^2) - 2Md_y y_{\text{CM}} - 2Md_z z_{\text{CM}}.$$

If the new origin is at the center of mass ($y_{\text{CM}} = 0, z_{\text{CM}} = 0$), we get the final parallel axis theorem for I_{xx}:

$$I'_{xx} = I_{xx}^{\text{CM}} + M(d_y^2 + d_z^2).$$

□

Corollary 5
About the new shifted axes we for moment of inertia

$$I'_{xy} = I_{xy}^{\text{CM}} - Md_x d_y.$$

Proof. Parallel Axis Theorem for I_{xy} The product of inertia is defined as:

$$I_{xy} = -\sum m_j x_j y_j.$$

Applying the coordinate shift:

$$x_j' = x_j - d_x, \quad y_j' = y_j - d_y,$$

we substitute into the product of inertia:

$$I_{xy}' = -\sum m_j (x_j - d_x)(y_j - d_y).$$

Expanding:

$$I_{xy}' = -\sum m_j (x_j y_j - d_x y_j - d_y x_j + d_x d_y).$$

Splitting the sum:

$$I_{xy}' = -\sum m_j x_j y_j + d_x \sum m_j y_j + d_y \sum m_j x_j - d_x d_y \sum m_j.$$

Using the center of mass properties:

$$\sum m_j x_j = M x_{\text{CM}}, \quad \sum m_j y_j = M y_{\text{CM}},$$

we get:

$$I_{xy}' = I_{xy}^{\text{CM}} + M d_x y_{\text{CM}} + M d_y x_{\text{CM}} - M d_x d_y.$$

If the new origin is at the center of mass ($x_{\text{CM}} = 0, y_{\text{CM}} = 0$), we obtain:

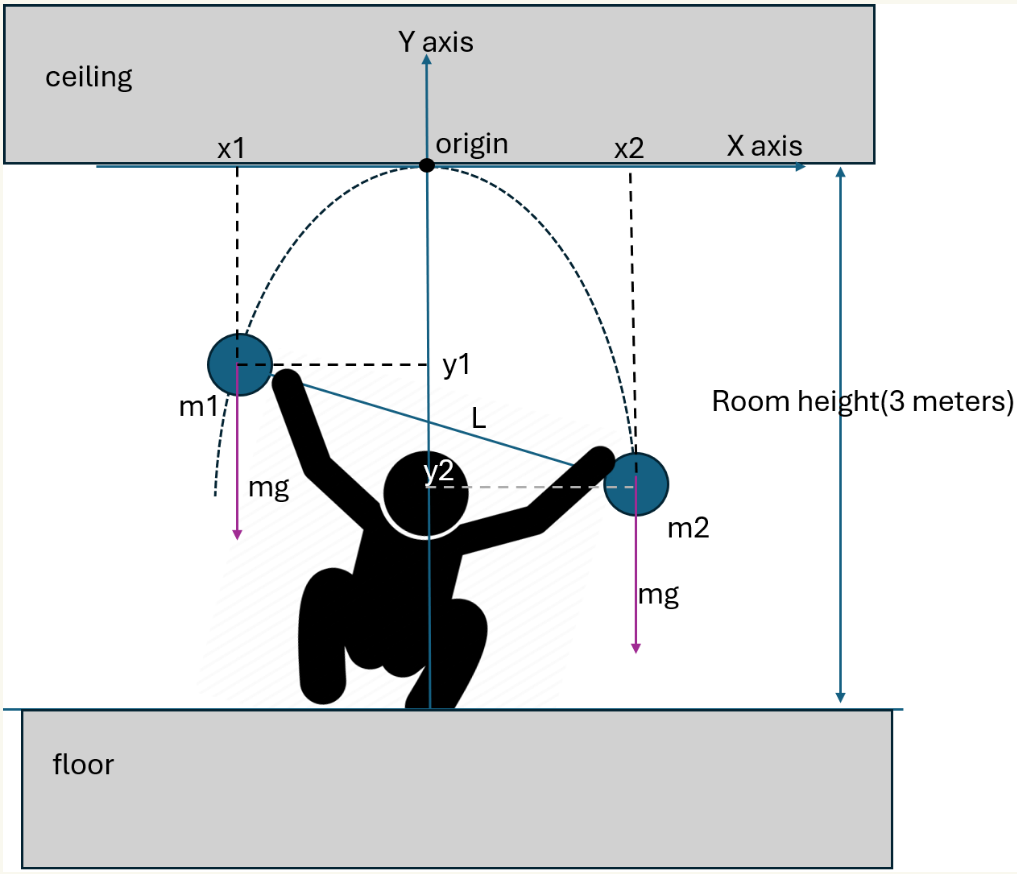
$$I_{xy}' = I_{xy}^{\text{CM}} - M d_x d_y.$$

□

3 Principle of virtual work

Problem 6 (**3:** *Transfer.* Category B)

Consider a weightlifter trying to balance a barbell that consists mass m_1 and m_2 connected by a massless rod of length L which slides on a frictionless inverted oval hoop securly fastened to the ceiling during a training session for the Olympics. This is an exercise to train the hip flexion ability of the weighlifter who might have to rebalance the barbell in awkard positions where the barbell is inclined rather than horizontal. The inverted oval hoop therefore serves as a guide to practice rebalancing given a specific range of allowed motion.



As shown, y is the displacement of the weightlifter from the origin. (x_1, y_1) and (x_2, y_2) are the positions of the centre of the 2 uniform spherical masses m_1 and m_2 respectively. The parabolic plot represents the inverted oval hoop these masses slide on while the x axis is the ceiling.

In typical free weight lifting exercises, there is typically a spotter(see appendix) to "catch" the barbell and prevent it from injuring the weightlifter should the weightlifter get exhausted and lets go of the bar. Provide convincing evidence that such a spotter may be unnecessary.

You may assume the following

1. masses m_1 and m_2 are equal
2. the parabola $y = -\frac{1}{2}x^2$ corresponds to the equation of the hoop
3. room height is 3 meters and its infinitely wide
4. the weightlifter is tall enough to touch the ceiling
5. the hoop stretches all the way to the ground

Hints:

1. What specific relations between x_1 and x_2 allow the freely sliding dumbell to be in static equilibrium on the hoop should the weightlifter let go?

2. Use Lambert's principle of virtual work rather than trying to find to equation for COM

Solution. The position vectors of mass 1 and 2 are

$$r_1 = x_1\hat{i} - \frac{1}{2}x_1^2\hat{j}$$
$$r_2 = x_2\hat{i} - \frac{1}{2}x_2^2\hat{j}$$

At equilibrium by D'Alambert Principle

$$\delta W = \vec{F} \cdot \delta\vec{r} = (F^a + F^c) \cdot \delta r = 0$$

where F^a are the applied forces and F^c are the forces of constraint(the force that confines the motion of the barbell to this parabola).

Since the forces of constraint in this case is the normal contact force between the ball and hoop), it is normal to the virtual displacements along the parabola hence virtual work done by forces of constraint is zero. Thus

$$\begin{aligned}
\delta W &= \vec{F^a} \cdot \delta\vec{r} = 0 \\
\delta W &= -mg\hat{j} \cdot (\delta r_1 + \delta r_2) = 0 \\
\delta r_1 &= (\hat{i} - x_1\hat{j})\delta x_1 \\
\delta r_2 &= (\hat{i} - x_2\hat{j})\delta x_2 \\
\delta W &= x_1\delta x_1 + x_2\delta x_2 = 0 \\
\frac{x_1}{x_2} &= \frac{\delta x_2}{\delta x_1}
\end{aligned}$$

Clearly we still need to get rid of δx. We use the other constraint as follows.

Initial:

$$|r_1 - r_2| = L$$

After virtual displacements:

$$|(r_1 + \delta r_1) - (r_2 + \delta r_2)| = L$$

Since after virtual displacements in fixed time, the length of rod which is fixed should not change

$$L^2 = |r_1 - r_2|^2 + 2(r_1 - r_2) \cdot (\delta r_1 - \delta r_2) + |\delta r_1 - \delta r_2|^2$$

Ignoring the last term as they are products of infinitesimally small displacements

$$\begin{aligned}
L^2 + 2(r_1 - r_2) \cdot (\delta r_1 - \delta r_2) &= L^2 \\
(r_1 - r_2) \cdot (\delta r_1 - \delta r_2) &= 0 \\
(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(-x_1\delta x_1 + x_2\delta x_2) &= 0
\end{aligned}$$

substituting in $y_1 = -\frac{1}{2}x_1^2$ and $y_2 = -\frac{1}{2}x_2^2$ we obtain

$$\delta x_1 \left(1 + \frac{1}{2}(x_1 + x_2)x_1\right) = \delta x_2 \left(1 + \frac{1}{2}(x_1 + x_2)x_2\right)$$

now with 2 equations and 4 variables $\delta x_1, x_2$ and x_1, x_2 we can reduce to only x_1, x_2 only

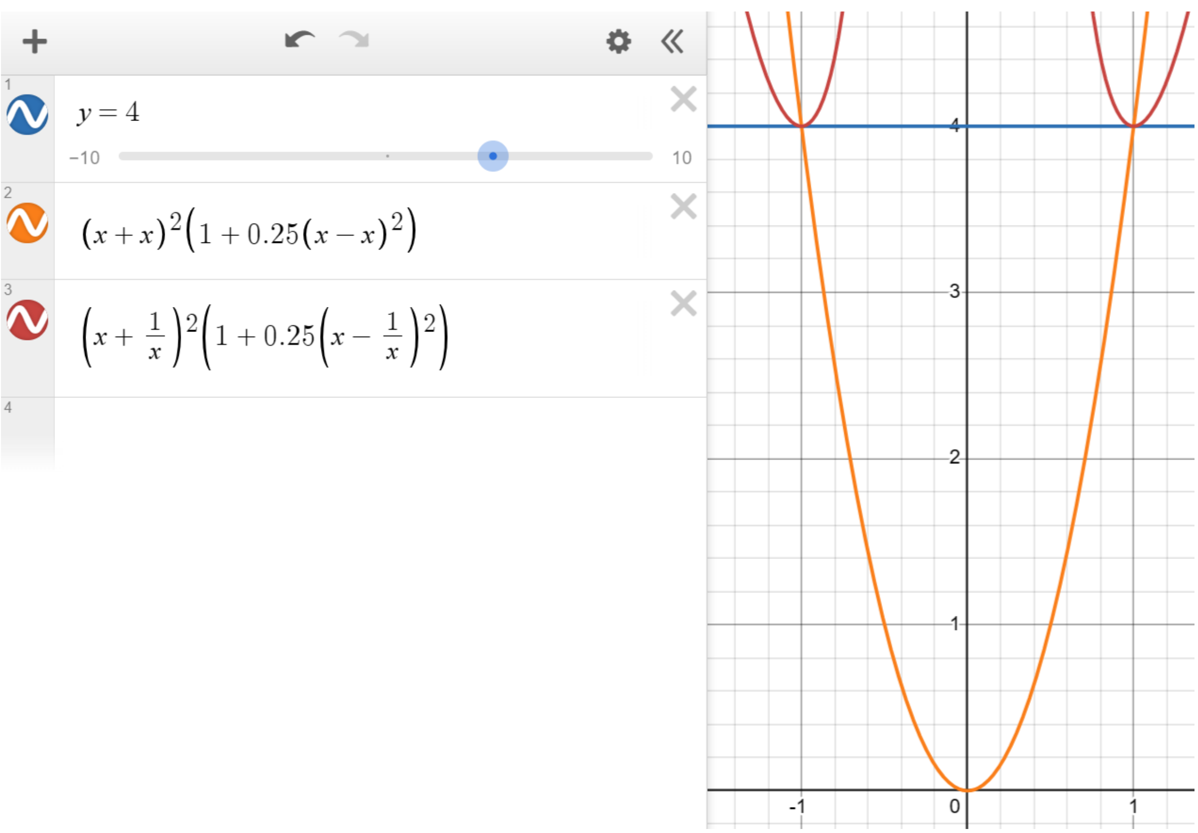
$$\delta x_1 \left(1 + \frac{1}{2}(x_1 + x_2)x_1\right) = \delta x_2 \left(1 + \frac{1}{2}(x_1 + x_2)x_2\right)$$
$$\delta x_1 \left(1 + \frac{1}{2}(x_1 + x_2)x_1\right) = \delta x_1 \left(\frac{x_1}{x_2}\right) \left(1 + \frac{1}{2}(x_1 + x_2)x_2\right)$$
$$x_2^2(x_1) + x_2(1 + x_1^2) + x_1 = 0$$

Apply the quadratic formula

$$x_2 = -\frac{-x_1^2 - 1 \pm \sqrt{1 - 2x_1^2 + x_1^4}}{2x_1}$$
$$x_2 = -x_1 \text{ or } x_2 = -\frac{1}{x_1}$$

Now given a choice of 2 solutions, it is only fair to check against the initial conditions to see when they are valid or should be rejected. Substituting our solutions into equation for L and plotting them graphically we have

$$L^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$L^2 = (x_1 - x_2)^2 + \frac{1}{4}(-x_1^2 + x_2^2)$$



Where the orange plot corresponds to $x_2 = -x_1$ and the red plot corresponds to $x_2 = -\frac{1}{x_1}$. We see that the second solution is only valid for $L \geq 2$. As for the 1st solution, it means the rod's COM is aligned with the y axis and perfectly horizontal. As seen, it is valid for all non-zero values of L(length cant be zero). Analytically we could verify these solutions by finding the minimum points via differentiation.

Finally we need to account for room height. Consider the maximum value x of a barbell mass for which the room height is reached.

$$\frac{1}{2}x^2 = 3$$
$$x = \sqrt{6}$$

Reject the negative square root because height is positive. Thus the solution for $x_2 = -x_1$ is

$$\{x_2 = -x_1; |x_1| \leq \sqrt{6}; L > 0\}$$

Note when $x_1 = \sqrt{6}$ in this case, both masses on the dumbbell are basically resting on the floor so the barbell is definitely in equilibrium. Now for the case of $x_2 = -\frac{1}{x_1}$ it is

$$\{x_2 = -\frac{1}{x_1}; |x_1| < \sqrt{6}; L \geq 2\}$$

It does not include $\sqrt{6}$ as only 1 mass on the barbell is subject to the normal contact forces between the floor and mass.This will disrupt the equilibrium of the system which we calculated equilibrium conditions for as we did not account for contact of the mass with the floor.

Thus we now have obtained the relations between x_1 and x_2 for which the barbell is in static equilibrium on the hoop where it will not descend to a lower position and possibly hit the weightlifter.