Theoretical Linear Algebra

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Selected theorems from Sheldon Axler's Linear Algebra Done Right[1]

Contents

1 Vector Spaces

Definition 1 (vector space)

A **vector space** is a set V where the following properties hold

1. Commutativity

$$u + v = v + u$$

2. Associativity

$$(u + v) + w = u + (v + w) \quad \text{and} (ab)v = a(bv)$$

3. Additive Identity

$$\text{there exists } \mathbf{0} \in V \text{ such that } v + \mathbf{0} = v \text{ for all } v \in V$$

4. Additive Inverse

$$\text{for every } v \in V \text{ there exists } w \in V \text{ such that } v + w = \mathbf{0}$$

5. Identity

$$1v = v \text{ for all } v \in V$$

6. Distributive

$$a(u + v) = au + av \text{ and } (a + b)u = au + bu \text{ for all } a, b \in \mathbb{K} \text{ and all } u, v \in V$$

Where $\mathbb{K}$ refers to field which is equal $\mathbb{R}$ or $\mathbb{C}$ and that we make a distinction between the scalar 0 and the $\mathbf{0}$ additive identity(zero vector)

Fact 2

When we say two subspaces/vector spaces are disjoint we can only say

$$V_1 \cap V_2 = \{0\}$$

which in proper terms the **trivial intersection** and not

$$V_1 \cap V_2 = \emptyset$$

because every vector space must contain a zero vector(identity inverse) by definition. There is a clear difference between the empty set and the zero vector set

Proposition 3

Every element in a vector space has a unique **additive indentity** and **additive inverse**

Proof. Suppose 0 an $0'$ are additive identities. Then

$$0' = 0' + 0 = 0$$

Suppose w and w' are additive inverses of v. Then

$$w = w + 0 = w + (v + w') = (v + w) + w' = 0 + w' = w'$$

Proposition 4

$0v = \mathbf{0}$ for all $v \in V$

Proof.

$$0v = (0 + 0)v = 0v + 0v$$

adding the additive inverse of $0v$ to both sides gets the desired conclusion

Proposition 5

$a\mathbf{0} = \mathbf{0}$ for every $a \in \mathbb{K}$

Proof.

$$a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) = a\mathbf{0} + a\mathbf{0}$$

adding the additive inverse of $a\mathbf{0}$ to both sides gets the desired conclusion

Proposition 6

$(-1)v$ is the additive inverse of v

Proof.

$$(-1)v + v = 0v = \mathbf{0}$$

Definition 7 (Subspace)

Let V be a vector space. A subset $W \subseteq V$ is a **subspace** of V if for all $w_1, w_2 \in W$ and $\lambda_1, \lambda_2 \in \mathbb{K}$ we have

$$\lambda_1 w_1 + \lambda_2 w_2 \in W$$

(that is closed under linear combinations) and has an **additive identity**

Definition 8 (Sum of Subspaces)

The **sum of subspaces** as denoted by $W_1 + \ldots + W_k$ is the smallest subspace that contains all subspaces of all W_i

$$W_1 + \ldots + W_k = \{v \in V | v = w_1 + \ldots + w_k, w_i \in W_i\}$$

Definition 9 (Direct Sum)

If $W_1 + \ldots + W_k$ is the sum of subspaces W_i where all W_i are *independent* meaning no sum satisfies

$$w_1 + \ldots + w_k = 0$$

where $w_i \in W_i$ except when $w_i = \mathbf{0}$ for all i. Then such a sum is called a **direct sum** which we denote as

$$V = W_1 \oplus \ldots \oplus W_k$$

Proposition 10

Every element of $V = W_1 \oplus \ldots \oplus W_k$ can be written uniquely as a sum of $w_1 + \ldots + w_k = 0$ where $w_i \in W_i$

Proof. Suppose

$$v = u_1 + \ldots + u_k = v_1 + \ldots + v_k$$

Then

$$0 = (u_1 - v_1) + \ldots + (u_k - v_k)$$

By definition of independence the only solution is where $u_k - v_k = 0$ for all k and hence $u_k = v_k$

Lemma 11

If $v = (-1)v$ if and only if $v = \mathbf{0}$

For the forward direction, adding v on both sides yields the conclusion. For the backward direction adding the additive inverse to both sides yields the conlusion.

Proposition 12

Suppose U, W are subspaces of V. Then $V = U \oplus W$ if and only if $U \cap V = \{\mathbf{0}\}$

From the forward direction, suppose $V = U \oplus W$ and that $v + w = \mathbf{0}$ for some $v \in V$ and $w \in W$. Since $v + (-v) = 0$ and that representation of any element in a direct sum is unique, we must have $w = -v \in W$.Since every element in a vector space has an additive identity, v must also exist in W. Therefore $U \cap V = \{\mathbf{0}\}$ or there will exist $v, (-1)v \in U \cap W$ that are non zero vectors that satisfy the above relation which is a contradiction since the only time where $v = (-1)v = \mathbf{0}$ is when $v = \mathbf{0}$.

For the backward direction suppose $U \cap V = \{\mathbf{0}\}$. From the above knowing that $v, w \in V \cap W$, then the only candidates we can choose for v, w is $\mathbf{0}$ which implies $V = U \oplus W$.

2 Finite Dimensional Vector Spaces

Lemma 13 (Linear Dependence Lemma)

If $(v_1, \ldots, v_m)$ is linearly dependent in V and $v_1 \neq 0$ then there exists $j \in \{2, \ldots, m\}$ such that

(a) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$

(b) If the jth term is removed from $(v_1, \ldots, v_m)$ then the span of the remaining list is still the same

Proof. Suppose there exist

$$a_1 v_1 + \ldots + a_m v_m = 0$$

where all not $a_i \in \mathbb{K}$ equal zero. Then there exist $a_j \neq 0$ $v_j = \frac{a_1}{a_j} v_1 - \ldots - \frac{a_{j-1}}{a_j}$ proving (a).

For (b) Consider any u spanned by $(v_1, \ldots, v_{j-1})$

$$u = c_1 v_1 + \ldots c_m v_m$$

Then we may substitute $v_j = \frac{a_1}{a_j} v_1 - \ldots - \frac{a_{j-1}}{a_j}$ into the above such that u still remains in the span of the list. Thus the span of this list without v_j must be the same as that of the original.

Definition 14 (Basis)

A list of **linearly independent** vectors in V that spans V

Theorem 15

Every spanning list can be reduced to the basis of a finite-dimensional vector space

Proof. Simply invoke the *linear independence lemma* and enumerate through elements of the list. If the span is unaffected, delete the element if not remove it.

Theorem 16

Any basis of a subspace can be extended to that of the finite dimensional vector space

Proof. First consider any list that spans the vector space. Then invoke *linear independence lemma* again, enumerate through this list and add every vector to the basis of the subspace unless the addition does not change the span of subspace.

Theorem 17

If V is finite dimensional then every spanning basis of V has same length $\dim V$

Proof. Let B_1 and B_2 be any two bases of V. Since vectors in B_1 are linearly independent by 15 we have that $\dim B_1 \leq \dim B_2$. But by the same argument we have $\dim B_1 \leq \dim B_2$ so equality follows

Theorem 18

If $U \subset V$ then $\dim U \leq \dim V$

Proof. Any basis of U can be extended to the basis of V hence $\dim U \leq \dim V$. However the basis of V can span V and thus U so no such extension is required.

Theorem 19

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$$

Theorem 20

If

$$V = U_1 + \ldots + U_m$$

and

$$\dim V = \dim U_1 + \ldots + \dim U_m$$

then

$$V = U_1 \oplus \ldots + U_n$$

3 Linear Operators

Definition 21 (Linear Operator)
Let the space of linear maps $T : V \to W$ be defined by $\mathcal{L}(V, W)$. Like a vector space it satisfies both **additivity** and **homogeneity**

$$T(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 T w_1 + \lambda_2 T w_2$$

And has a **additive identity**

$$0v = \mathbf{0}$$

and has a **identity**

$$Iv = v$$

where $I, 0 \in \mathcal{L}(V, W)$ and $\mathbf{0} \in W$

Definition 22 (Nullspace)

$$\ker T = \{v \in V : Tv = 0\}$$

Proposition 23
Nullspace is a subspace

Proof. Let $T(w_1) = T(w_2) = \mathbf{0}$

$$T(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 T w_1 + \lambda_2 T w_2 = \mathbf{0}$$

Now consider

$$T(0) = T(0 + 0) = T(0) + T(0)$$

Now subtract $T(0)$ on both sides(similar to how we did previously), showing that it has an additive identity.

Remark 24. *An important point to note is that $T(0) = 0$ for any $\mathcal{L}(V, W)$, this is analagous to vector spaces where $v0 = \mathbf{0}$*

Definition 25 (Range)

$$\operatorname{Im} T = \{Tv : v \in V\}$$

Definition 26 (Injective)
For any $u, v \in V$

$$Tu = Tv \Rightarrow u = v$$

Proposition 27 (injectivity vs nullspace)
Let $T \in \mathcal{V}, \mathcal{W}$ then T is injective if and only if $\ker T = \emptyset$

Proof.

$$T(v) = 0 = T(0)$$

hence v can only be 0

Definition 28 (Surjective)
Linear operator T **Surjective** if $\operatorname{Im} T = W$ for

$$\mathcal{L}(V, W)$$

Theorem 29 (Rank Nullity Theorem)
If V is finite dimensional and $T \in \mathcal{L}(V, W)$ then

$$\dim V = \dim \ker T + \dim \operatorname{Im} T$$

Proof. Let $(u_1, \ldots, u_m)$ be a basis of $\ker T$ and because it is a subset of V it can be extended to the basis of V like so

$$v = a_1 u_1 + \ldots + a_m u_m + b_1 w_1 + \ldots + b_n w_n$$

Then taking T on all sides we have

$$Tv = 0 + b_1 T w_1 + \ldots + b_n T w_n$$

This implies that Tv is spanned by $(Tw_1, \ldots, Tw_n)$. It now remains to prove that this is linearly independent and we are done. Consider

$$0 = c_1 T w_1 + \ldots + c_n T w_n = T(c_1 w_1 + \ldots + c_n w_n)$$

Hence $(c_1 w_1 + \ldots + c_n w_n) \in \ker T$ so

$$c_1 w_1 + \ldots + c_n w_n = d_1 u_1 + \ldots + d_m u_m$$
$$c_1 w_1 + \ldots + c_n w_n - d_1 u_1 - \ldots - d_m u_m = 0$$

So all c_i and c_j must be equal zero.

Remark 30. *Clearly this theore is also valid for linear operators $\mathcal{L}(V, V)$ there was no restrictions on what vector space W should be . Moreover you could see this as an alternative proof to 41. We have a 1 to 1 correspondence with the basis vectors of the range and the basis vectors in V that map to them. But if nullspace contains a non-zero vector, then there exists a subset in W, that is the empty set that is mapped by non-zero vectors in V.*

Fact 31 (Relation of dimensional formula to ODE)
Consider **rank nullity theorem** for $T \in \mathcal{L}(V, W)$.Suppose V has basis $(v_1 \ldots, v_n, u_1, \ldots, u_m)$

$$v = a_1 v_1 + \ldots a_n v_n + b_1 u_1 + \ldots + b_m u_m \in V$$

and that $(Tv_1, \ldots, Tv_n)$ is a basis for the range and $(u_1, \ldots, u_m)$ is the basis for the nullspaces.Hence the basis vectors $(v_1 \ldots, v_n)$ which is an extension from the basis of the nullspace to V spans a the subspace disjoint from the nullspace where only the zero vector is in their intersection recall 2. This means it spans the disjoint union $\{V/\ker T\} \cup \{0\}$ because $\ker T$ contains the 0 vector by definition of vector space so the set difference does not have the zero vector and thus is not a vector space! Hence we have to add back the zero vector to get a valid vector space that can be spanned.

In the context of ODEs, the differential is basically a linear transformation . By definition of solve, we must find the corresponding complete solution in V for a particular result in W. That is we have to express our solution as a span of all possible basis vectors of V including those in nullspace because any result $w \in W$ even if non-zero it can be expressed as $w + 0 = w$ which exists by existence of additive inverse in vector spaces.The standard way of solving ODE is by trial and error some linear combination of basis vectors which are typically of the form e^{kt} that could give our result in W. However instead of solving

$$w = T(a_1 v_1 + \ldots a_n v_n, b_1 u_1 + \ldots + b_m u_m) \tag{1}$$

for arbituary $w \in W$, zero or not directly, we can actually separately solve for **particular solution**(equation 2) and the **general solution**(equation 3) to make life easier. First consider the particular solution

$$w = T(a_1 v_1 + \ldots a_n v_n) = T(V/\ker T), w \neq 0, \text{at least 1 of the } a_i \neq 0 \tag{2}$$

To restrict span $(v_1 \ldots, v_n)$ to span only $\{V/\ker T\}$ and not the original $\{V/\ker T\} \cup \{0\}$. We need to exclude the linear combination which gives the zero vector. By linear independence that will only be when all $a_i = 0$, the trivial linear combination.Therefore, we conclude that our trial and error linear combination basis vector cannot be equal to the zero vector for solving this case. That is to say a non-zero result must have a non-zero solution provided that the solution exists. Next, the general solution corresponds to

$$0 = T(b_1 u_1 + \ldots + b_m u_m) = T(\ker T) \tag{3}$$

We now prove that we can indeed directly add the solutions together because by linearity of ODE differentials we have:

$$w = w + 0 = T(a_1 v_1 + \ldots a_n v_n) + T(b_1 u_1 + \ldots + b_m u_m) = T(a_1 v_1 + \ldots a_n v_n, b_1 u_1 + \ldots + b_m u_m) \tag{4}$$

Therefore our solution for $w \neq 0$ includes all eigenvectors as desired. If $w = 0$ we only have to solve equation (3) since it already includes all possible basis vectors for the nullspace as desired and that we don't have to try any basis vectors $(v_1 \ldots, v_n)$ since all a_i is gauranteed to be zero in our linear combination. So with this we don't have to try so many eigenvectors at once and worry less about missing solutions because we cases to account for.

Remark 32. *for particular solution, the coefficients scalars for the basis vectors are known because as the name implies they correspond to one non-zero particular solution in the range. For the general solution, our solution must account/span the whole nullspace. So obviously we have general arbitruary coeffients in front of the basis vectors. Thus the name "general solution"*

Corollary 33

If $\dim V > \dim W$ then no linear map from V to W is injective

Proof.

$$\begin{aligned} \dim \ker T &= \dim V - \dim \operatorname{Im} T \\ &\geq \dim V - \dim W \\ &> 0 \end{aligned}$$

Corollary 34

If $\dim V < \dim W$ then no linear map from V to W is surjective

Proof. use 29 again.

Definition 35 (Matrix of Linear Map)

We define linear map T from V to K using a transformation matrix by

$$T_{v_k} = a_{1,k} w_1 + \ldots + a_{n,k} w_m$$

Example 36

The map of T_{v_1} is defined by

$$\begin{array}{c|c} T & \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} \\ \hline \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} & \begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \ldots & a_{m,n} \end{bmatrix} \end{array}$$

$$T_{v_1} = a_{1,1} w_1 + \ldots + a_{n,1} w_m$$

Definition 37 (Invertible)

A linear map $T \in \mathcal{L}(V, W)$ is called **invertible** if there exists

$$ST = I \text{ and } TS = I$$

where I is the identity matrix

Proposition 38

If A is a square matrix that has a right inverse R where $AR = I$ and a left inverse where $LA = I$ then $R = L$

Proof. $R = IR = (LA)R = L(AR) = LI = L$

Proposition 39

A linear map is invertible if and only if it is injective and surjective. In other words, bijective

Proof.

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v$$

shows invertibility while

$$w = T(T^{-1}(Tw))$$

shows surjectivitity. To prove the other direction suppose $T(Sw) = w$ which exists due to surjectivity which implies every w came from a map and due to injectivity that the map is unique we cant have $T(Sw) \neq w$. Then we can show that S is the inverse by

$$T(STv) = (TS)(Tv) = I(Tv) = Tv$$

hence $TS = I$ and $ST = I$ as desired. Finally we show that it is indeed a linear operator by

$$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2) = w_1 + w_2 \Rightarrow S(w_1 + w_2) = Sw_1 + Sw_2$$
$$T(aSw) = aT(Sw) = aw \Rightarrow S(aw) = aSw$$

which basically follows from the linearity of T

Proposition 40

If V, W are finite dimensional then $\mathcal{L}(V, W)$ is finite dimensional and $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$

Proof. Consider that $\mathcal{L}$ can be seen as $V \times W$, 2 element tuples consisting of a value and its corresponding map.

Proposition 41

If V is finite dimensional and $T \in \mathcal{L}(V)$ then the following are equivalent

(a) T is invertiable

(b) T is injective

(c) T is surjective

Proof. To break down what this means, we first recall have already proven the case for $\mathcal{L}(U, W)$ in 39, now we are just proving for the $W = V$. The difference here is we don't need to *both* surjective and injective to conclude invertible. Just either surjective or injective and the rest follows.

4 Polynomials

Definition 42 (Zero of a polynomial)

A number $\lambda \in \mathbb{K}$ is called a **zero/root** of a polynomial if $p \in \mathcal{P}(\mathbb{K})$ if

$$p(\lambda) = 0$$

Proposition 43

Each zero of a polynomial corresponds to a degree one factor where

$$p(z) = (z - \lambda)q(z)$$

$p(z)$ has degree m while $q(z)$ has degree $m - 1$

Proof. Suppose $p(\lambda) = 0$. Let $p \in \mathcal{F}(\mathbb{K})$ be defined by

$$p(z) = a_0 + a_1 z + \ldots + a_m z^m$$

for all $z \in \mathbb{K}$ Then

$$p(z) - p(\lambda) = a_1(z - \lambda) + \ldots + a_m(z^m - \lambda^m)$$

However each term in the brackets can be factorized by **binomial theorem** like so. To prove from the other direction it is clear

$$p(\lambda) = (\lambda - \lambda)q(\lambda) = 0$$

$$z^k - \lambda^k = (z - \lambda)\sum_{j=1}^{k} \lambda^{j-1} z^{k-j}$$

Proposition 44

degree m implies at most m zeroes

Proof. by induction...

5 Eigenvalues and Eigenvectors

Definition 45 (Invariant)

U is **invariant** under T if $u \in U$ implies $Tu \in U$. We denote the space of such operators by $\mathcal{L}(V)$ which means $\mathcal{L}(V, V)$

Example 46

If $T \in \mathcal{L}(V)$ then $\ker T$ is an invariant subspace. Consider that if $u \in \ker T$ then $Tu = 0 \in \ker T$

Example 47

If $T \in \mathcal{L}(V)$ then then $\operatorname{Im} T$ is an invariant subspace. Consider that if $u \in \operatorname{Im} T$ then $u \in V$ since $T : V \to V$. Then by definition of Range $Tu \in \operatorname{Im} T$.

Fact 48

Consider a 1 dimensional subspace U of V (spanned by single vector):

$$U = \{au : a \in \mathbb{K}\}$$

where u is some non-zero vector in V. Then if U is invariant under $T \in \mathcal{L}(V)$ then it must have the form since $T(au) = \lambda(au) \to Tu = \lambda u$. Thus u is the basis vector for some 1 dimensionsal invariant subspace under T and there is a $\lambda \in \mathbb{K}$ associated with it. This has important applications as we will see so we give these some special names below.

Definition 49

An **eigenvector** v of a linear operator T is a *non-zero* vector such that

$$Tv = \lambda v$$

for some $\lambda \in \mathbb{K}$ which is known as an **eigenvalue**

We require that the eigenvector v be non-zero or λ will have infinite solution since $\lambda 0 = 0, \forall \lambda \in \mathbb{K}$

Proposition 50

λ is an eigenvalue of T if and only if T is not injective

Proof. Consider

$$Tu = \lambda u \Rightarrow (T - \lambda I)v = 0$$

. If v were to be an eigenvalue, then the nullspace of $(T - \lambda I)$ must contains a non-zero vector(the eigenvector), which therefore implies non-injectivivity by 41.

Theorem 51

Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding non-zero eigenvectors. Then $(v_1, \ldots, v_m)$ are linearly independent

Proof. By contradiction, suppose that $(v_1, \ldots, v_m)$ are linearly dependent. Then by the *linear dependence lemma* there exists $v_k \in \text{span}(v_1, \ldots, v_{k-1})$

$$v_k = a_1 v_1 + \ldots + a_m v_{k-1} \tag{1}$$

Taking T on both sides we have

$$\lambda_k v_k = \lambda_1 a_1 v_1 + \ldots + \lambda_{K-1} a_{K-1} v_{k-1} \tag{2}$$

Then we have from (1) and (2) that

$$0 = a_1(\lambda_k - \lambda_1)v_1 + \ldots + a_{K-1}(\lambda_k - \lambda_{K-1})v_{k-1}$$

But all $\lambda_k \neq \lambda_1, \ldots, \lambda_{k-1}$ because we assumed eigenvalues are distinct. Also we assumed non-zero eigenvectors too. Hence this implies $a_1, \ldots, a_{k-1}$ are all zero but this contradicts linear dependence. $\square$

Corollary 52
Each operator on V has at most $\dim V$ distinct eigenvalues

Proof. Suppose with have m distinct eigenvalues corresponding to m linear independent eigenvectors. Then by 16 we can always extend the number of linearly independent vectors to the basis of the vector space so we have $m \leq \dim V$ as desired.

Fact 53
Polynomials applied to operators

1. $T^n v = T \circ T \ldots \text{n times} \ldots \circ T v$
2. $(aT + bT^2)v = aTv + bT^2 v$

Theorem 54
Every operator $T \in \mathcal{L}(V)$ on a finite-dimensional, non-zero complex vector space V has an eigenvalue

Proof. Suppose V is a complex vector space with dimension n and $v \neq \mathbf{0}$, lest $T^k \mathbf{0} = \mathbf{0}$ for any k which makes our following relations meaningless. Recall this from 23. Now consider the $n+1$ vectors as shown here. Then there exists complex numbers $a_1, \ldots, a_n$ not all 0 such that

$$0 = a_0 v + a_1 T v + \ldots + a_n T^n v$$

because each $a_i T^i v$ above is a vector in V given $T \in \mathcal{L}(v)$ and there can only exist a maximum of $\dim V = n$ linearly independent vectors in V by *linear dependence lemma* but we have $n+1$. Therefore using the result of *fundamental theorem of algebra* applied to the space of operators

$$\begin{aligned} 0 &= a_0 v + a_1 T v + \ldots a_n T^n v && (1)\\ &= (a_0 I + a_1 T + \ldots + a_n T^n) v && (2)\\ &= c(T - \lambda_1 I) \ldots (T - \lambda_m I) v && (3) \end{aligned}$$

where c is a non-zero complex number.We know that at least one of the $(T - \lambda_k I)$ is not injective(has a non-zero vector in the nullspace). To show this, suppose that $(T - \lambda_k I)$ is injective for all $k = 1, \ldots, m$. That is for every $v \neq \mathbf{0}$, $(T - \lambda_k I)v \neq \mathbf{0}$ for every k. However considering the composition of $(T - \lambda_1 I) \circ \ldots \circ (T - \lambda_m I)v$ if that is true then it would contradict (1).

Definition 55 (Upper Triangular)
A matrix is **upper triangular** if it is in the form of

$$\begin{bmatrix} * & \ldots & * \\ & \ddots & \vdots \\ & & * \end{bmatrix}$$

Where anything below the diagonal is equal zero while $*$ there is an element at the position in the matrix. Its value is immaterial it could be zero or non-zero. We simply treat it as some $a_{i,j} \in \mathbb{K}$ at row i and column j

Proposition 56

Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V. Then the following are equivalent

(a) the matrix of T with respect to $(v_1, \ldots, v_n)$ is upper triangular

(b) $T_{v_k} \in \operatorname{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$

(c) $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$

Proof. The (a) and (b) imply each other given that for any v_k, there exists elements $a_{i,k}$ where $i = 1, \ldots, k$ in column k.In other words there are elements the 1st k rows in the kth column while the rest are just 0. Hence we can express

$$T_{v_k} = a_{1,k}v_1 + \ldots + a_{n,k}v_k$$

as given by the defintion of the matrix of a linear map. Likewise (c) implies (b) as it directly means

$$T(\operatorname{span}(v_1, \ldots, v_k)) \in \operatorname{span}(v_1, \ldots, v_k)$$

for $k = 1, \ldots, n$ which certainly means for every k (b) is true. We now need to show that (b) implies (c). Given that

$$\begin{aligned} &Tv_1 \in \operatorname{span}(v_1) \\ &Tv_2 \in \operatorname{span}(v_1, v_2) \\ &\qquad \vdots \\ &Tv_k \in \operatorname{span}(v_1, \ldots, v_k) \end{aligned}$$

By taking linear combinations of the above for up to $\operatorname{span}(v_1, \ldots, v_k)$. We see that,

$$T(\operatorname{span}(v_1, \ldots, v_k)) \in \operatorname{span}(v_1, \ldots, v_k)$$

6 Operators on Complex spaces

Proposition 57

Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T. Then the set of generalized eigenvectors of T corresponding to λ equals $\ker(T - \lambda I)^{\dim V}$

Theorem 58

Let $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then fr every basis of V with respect to which T has an upper triangular matrix, λ appears on the diagonal of the matrix T precisely $\dim\ker(T - \lambda I)^{\dim V}$ times

Proposition 59

The multiplicity of an eigenvalue λ of T is equal to the dimension of the subspace of generalized eigenvectors corresponding to λ

Proposition 60

If V is a complex vector space and $T \in \mathcal{L}(V)$ then the sum of all multiplicities of all the eigenvalues of T equals $\dim V$

Lemma 61

If $N \in \mathcal{L}(V)$ is nilpotent then there exists vectors $v_1, \ldots, v_k \in V$ such that

(a) $\big((v_1, Nv_1 \ldots, N^{m(v_1)}v_1), \ldots, (v_k, Nv_k \ldots, N^{m(v_k)}v_k)\big)$ is a basis of V

(b) $(N^{m(v_1)}v_1 \ldots, N^{m(v_k)}v_k)$ is a basis of $\ker N$

Proof. Suppose N is nilpotent. Then N is not injective and thus $\dim \operatorname{Im} N < \dim V$

Theorem 62

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T and let $U_1, \ldots, U_m$ be the corressponding subspaces of generalized eigenvectors. Then

(a) $V = U_1 \oplus \ldots \oplus U_m$

(b) each U_j is invariant under T

(c) each $(T - \lambda_j I)_{U_j}$ is nilpotent

Proof. See your notes in `artin algebra(pg 124)` on Jordan from □

References

[1] Sheldon Jay Axler. *Linear algebra done right*. 2nd ed. Undergraduate texts in mathematics. New York: Springer, 1997. ISBN: 978-0-387-98259-5 978-0-387-98258-8.