Functional Analysis Workbook

Ian Poon

July 2024

Selected theorems from `Lin Andrew's introduction to functional analysis` notes at `MIT`

Contents

1 Banach spaces

Definition 1 (Norm)

A **norm** on a vector space V is function $||.|| : V \to [0, \infty)$ satisfying

1. (Definiteness) $||v|| = 0$ if and only if $v = 0$
2. (Homogenity) $||\lambda v|| = |\lambda| \, ||v||$
3. (Triangle Inequality) $||v_1 + v_2|| \leq ||v_1|| + ||v_2||$

A **seminorm** is a function $||.|| : V \to [0, \infty)$ which satisfies (2) and (3) but not necessarily (1). A vector space equipped with a norm is called a **normed space**

Example 2

Let the norm that assigns the real number $\sup |f'|$ to a function f. This satisfies homegenity and triangle inequality but not definiteness because the derivative of a constant function is zero and clearly not all constant functions must be zero functions.

recall

Fact 3 (full form of triangle inequality)

By the triangle inequality for any $a, b \in \mathbb{R}$ the following is always true

$$|a \pm b| \leq |a| + |b|$$

Proof. For the negative case

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$$

Definition 4 (Metric)

Recall the definition of a **metric** $d : X \times X \to [0, \infty)$

1. (Identification) $d(x, y) = 0$ if and only if $x = y$
2. (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
3. (Triangle Inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Proposition 5

All norms and metrics are continuous. That is to say

$$\lim_{n \to \infty} ||u_n|| = \left|\left| \lim_{n \to \infty} u_n \right|\right|$$

$$\lim_{n \to \infty} d(u_n, y) = d(\lim_{n \to \infty} u_n, y)$$

Proof. Consider by triangle inequality

$$||x|| = ||(x - y) + y|| \leq ||x - y|| + ||y||$$

so rearranging we have

$$||x|| - ||y|| \leq ||x - y|| < \varepsilon$$

so for every ε we can just let $\delta = \varepsilon$. The same applies for metric

Proposition 6

The norm on vector space V is defines a metric on V

$$d(x, y) = ||v - w||$$

We call this possible metric defined by the norm "the metric induced by the norm"

Proof. One can easily show that properties of a norm imply properties of a metric

Definition 7

The **euclidean norm** on $\mathbb{R}^n$ or $\mathbb{C}^n$ is given by

$$||x||_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$$

But it is just one of the norms in $\mathbb{R}^n$ or $\mathbb{C}^n$ which in general is defined by

$$||x||_p = \begin{cases} \max_{1 \leq i \leq n} |x| & p = \infty \\ \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} & 1 \leq p < \infty \end{cases}$$

Definition 8

Let X be a metric space. Then the vector space C_∞ is defined as

$$C_\infty = \{f : X \to \mathbb{C} : f \text{ continuous and bounded}\}$$

Proposition 9

for any metric space X we can define the norm on C_∞ via

$$||u||_\infty = \sup_{x \in X} |u(x)|$$

Proof. (Definiteness). If u is the zero function, then value of all mapped elements is simply zero. so the sup over them is zero too. (Homogenity).$||\lambda U||_\infty = \lambda \sup |u(x)| = \lambda \, ||U||_\infty$. (Triangle inquality) Recall from previous basic `real analysis` courses that $\sup(x + y) \leq \sup x + \sup y$

Definition 10

The **ℓ^p space** is the space of infinite sequences where

$$\ell^p = \{\{a_j\}_{j=1}^{\infty} : ||a||_p < \infty\}$$

and we define the **ℓ^p norm** to be

$$||a||_p = \begin{cases} \sup_{1 \le i \le \infty} |a_j| & p = \infty \\ \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}} & 1 \le p < \infty \end{cases}$$

Definition 11

A normed space is a **Banach space** if it is complete with respect to the metric induced by the norm

Fact 12

The absolute value is a norm on the $\mathbb{R}$ and $\mathbb{C}$ space. It is clear that $|.|$ satisfies all properties of a norm. When it is used to define the metric we are familiar with $d(x, y) = |x, y|$, we can prove that $\mathbb{R}$ and $\mathbb{C}$ are complete recall `rudin`. In fact the same can be said for **euclidean norm** which when used to define distance in R^k, we can likewise prove that it is complete. Hence $\mathbb{R}, \mathbb{C}, R^k$ are all banach spaces!!

Theorem 13

For any metric space X, the space of bounded, continuous functions on X is complete and thus C_∞ is a Banach space.

First by assumption, our metric(distance function "$d(x, y)$") used will be the norm. We now have to prove that C_∞ is complete and hence a Banach space too. Therefore the approach is consider any cauchy sequence $\{u_n\}$ in C_∞ and show that is converges in C_∞. Specifically $u = \lim_{n\to\infty} u_n$ must (1)*exist*, be (2)*bounded* and (3)*continuous*.

By definition of cauchy sequence, there exists some N where

$$|u_n(x) - u_m(x)| \le \sup_x |u_n(x) - u_m(x)| = ||u_n - u_m|| = d(u_n, u_m) < \varepsilon$$

is valid as long as $n, m \ge N$. Firstly we can infer that this holds true for *all* $x \in X$ too as implied by sup. To put it explicitly, for every ε there exists a pair of $n, m > N$ such that

$$|u_n(x) - u_m(x)| < \varepsilon$$

is true for *all* $x \in X$. Secondly for each x for the same fixed ε, n, m we can see that $u_n(x), u_m(x)$ are values on $\mathbb{C}$ and that $u_n(x)$ is a cauchy sequence on $\mathbb{C}$. Therefore since $\mathbb{C}$ is complete, u_n is pointwise convergent on X. This means

$$u(x) = \lim_{n\to\infty} u_n(x)$$

exists for every $x \in X$. Thus Objective (1) complete.

$$||u_n - 0||_\infty \le ||u_n - u_N||_\infty + ||u_N - 0||_\infty \le \varepsilon + ||u_N||_\infty$$

Hence for any n which could be either $n \geq N_0$ or $n < N_0$ we have

$$||u_n||_\infty \leq \max(||u_1||_\infty, ||u_2||_\infty, \ldots, ||u_N - 1||_\infty, \varepsilon + ||u_N||_\infty)$$

which exists as there are finite terms and each $C_\infty : ||.|| \in \mathbb{R}$ Thus $||u_n||_\infty$ is bounded so for some B we have

$$|u_n(x)| \leq \sup_x |u_n| = ||u_n||_\infty \leq B$$

which implies $u_n(x)$ is bounded as well. Therefore taking

$$|u(x)| = \lim_{n\to\infty} |u_n(x)| = \limsup |u_n(x)| \leq B$$

shows that $u(x)$ is bounded as well. Objective (2) complete.

Finally returning to our cauchy sequence at the start we can have,

$$|u_n(x) - u_m(x)| \leq \sup_x |u_n(x) - u_m(x)| = ||u_n - u_m|| = d(u_n, u_m) < \varepsilon$$

notice the sup implies for any $x \in X$

$$|u_n(x) - u_m(x)| < \varepsilon$$

as long as $n, m \geq N$. Taking the limit for m alone(a.k.a treating u_n like a constant) which exists because u_m converges pointwise on X as proven earlier

$$\lim_{m\to\infty} |u_n(x) - u_m(x)| = \limsup |u_n(x) - u_m(x)| = |u_n(x) - u(x)| < \varepsilon$$

again this applies to any $x \in X$ as long as $n > N$,

$$\sup |u_n(x) - u(x)| < \varepsilon$$

. Thus for each ε we have an N that works for all $x \in X$ thus proving *uniform convergence*. Recall that the uniform limit of continuous functions is continuous. Thus Objective (3) is complete

Corollary 14 (Cauchy Criterion for Uniform Convergence)
Let $\{f_n\}$ be a sequence of functions where each $f_n : E \to R^k$. Then it converges uniformly on E if and only if there exists N for every $\varepsilon > 0$ such that

$$|f_n(x) - f_m(x)| \leq \varepsilon$$

for $n, m \geq N$

Proof. As seen above, every Cauchy sequence in a metric space is bounded and from the above definition clearly f_n is pointwise convergent. Thus using the same method as above(taking the limits of 1 variable), we have uniform convergence. From the other direction it is easy to see that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ and $|f_m(x) - f(x)| < \frac{\varepsilon}{2}$ for $n, m \geq N$. Now apply triangle inequality.

Definition 15
Let $\{v_n\}_n$ be a sequence of points in V. Then the series $\sum_{n=1}^{\infty} V_n$ is **summable** if $\{\sum_{n=1}^{m} V_n\}_m$ converges and $\{v_n\}_n$ is **absolutely summable** if $\{\sum_{n=1}^{m} ||V_n||\}_m$ converges

Lemma 16

If a subsequence of a cauchy sequence converges then the whole cauchy sequence converges

Proof. Let $\varepsilon > 0$ Take $N > 0$ such that for all $n, m > N$ we have

$$d(x_n, x_m) < \varepsilon/2$$

By hypothesis, we can take also $K > 0$ such that for all $n_k > K$ we have

$$d(x_{n_k}, x) < \varepsilon/2$$

Put $M = \max(N, K)$ Therefore, for all $n, m, n_k > M$ we have

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Theorem 17

A normed vector space V is a Banach space if and only if every absolutely summable series is summable

Proof. For the forward direction, suppose that V is Banach, then V is complete under the metric induced by the norm. Since every absolutely summable series is convergent, it is cauchy. In a complete space, this implies its series converges, hence it is summable.

For the other direction suppose that every absolutely summable series is summable. First we construct a cauchy sequence and see if it converges to prove V is a Banach space. There exists some N_k for every k

$$||v_n - v_m|| < 2^{-k}$$

for $n, m \geq N_k$. Then define

$$n_k = \sum_{j=1}^{k} N_j$$

thus $\{n_k\}$ is an increasing sequence of integers and that $n_k \geq N_k$ for all k Then we can define

$$||v_{n_k+1} - v_{n_k}|| < 2^{-k}$$

Where clearly $\{||v_{n_k+1} - v_{n_k}||\}_k$ is absolutely summable since $\sum_{k=1}^{\infty} 2^{-k} = 1$ Then because every absolutely summable series is summable in our assumption we have

$$\sum_{k=1}^{\infty}(v_{n_k+1} - v_{n_k}) < \varepsilon$$

Thus the sequence of partial sums each defined by $\sum_{k=1}^{m}(v_{n_k+1} - v_{n_k}) = v_{n_{m+1}} - v_{n_1}$ converges. Given that adding a constant to every term does not change convergence(the limit of constants is still equal the constant) we have that $\{v_{n_{m+1}}\}$ converges. Because we have found a convergent subsequence of the cauchy sequence $\{v_n\}$ by 16, $\{v_n\}$ converges too.

Theorem 18

Let V, W be vector spaces. A linear operator $T : V \to W$ is continuous if and only if there exists $C > 0$ such that all $v \in V$, $||Tv||_W \leq C\,||v||_V$

Definition 19 (Bounded Linear Operator)
We call linear operators with either of these 2 properties a **bounded linear operator**. Meaning to say bounded subsets of V are always mapped to bounded subsets of W. The set of bounded linear operators from V to W is denoted by $\mathcal{B}(V, W)$

Definition 20 (operator norm)
The **operator norm** of an operator $T \in \mathcal{B}(V, w)$ is defined by

$$||T|| = \sup_{||v||=1, v \in V} ||Tv||$$

The operator norm of bounded linear operator is basically

$$||T|| = \sup ||Tv|| \leq C\,||v|| = C$$

since bounded means for every v we have $||Tv|| \leq C\,||v||$ and $||v|| = 1$. Thus C can be any upperbound of the pnorms of all maps of unit vectors under the linear operator. So the operator norm is lowest possible value of C. By definition of suprenum we also clearly have the result

$$||Tv|| \leq ||T||$$

for all v with $||v|| = 1$. Let us extend this beyond unit vectors. Consider $||v|| = m$

$$\left|\left|\frac{v}{||v||}\right|\right| = \left|\left|\frac{v}{m}\right|\right| = \frac{1}{m}\,||v|| = 1$$

Thus $\left|\left|\frac{v}{||v||}\right|\right|$ can be seen as unit vector $\hat{v}$ such that $||\hat{v}|| = 1$ Using the homogenity of norms in C_∞ again(recall that Tv is continuous map since we are working with linear operators), we have

$$\left|\left|T\left(\frac{v}{||v||}\right)\right|\right| \leq ||T|| \Rightarrow ||Tv|| \leq ||T||\,||v||$$

Proposition 21
The operator norm is indeed a norm, thus $\mathcal{B}(V, W)$ is a normed space

Proof. (homogenity)

$$||\lambda T|| = \sup_{||v||=1} ||\lambda Tv|| = \sup_{||v||=1} |\lambda|\,||Tv|| = |\lambda|\,||T||$$

. (triangle inquality)

$$||(S+T)v|| = ||Sv + Tv|| \leq ||Sv|| + ||Tv|| \leq ||S|| + ||T||$$

Where the 1st equality follow from linearity and the second follow by the triangle inequality C_∞. Then take sup on the LHS and we are done.

Theorem 22
If V is a normed vector space and W is a Banach space, then $\mathcal{B}(V, W)$ is a Banach space

We consider an absolutely summable sequence of bounded linear operators

$$\sum_{n=1}^{\infty} ||T_n|| = C < \infty$$

and attempt to show that they are summable as well meaning there exists some T where

$$\sum_{n=1}^{\infty} T_n = T$$

First we attempt to make use of the fact that W is banach. Naturally that means should relate it to $T_n v \in W$. Using the inquality from above we have,

$$\sum_{n=1}^{m} ||T_n v|| \leq \sum_{n=1}^{m} ||T_n|| \, ||v|| \leq ||v|| \sum_{n=1}^{\infty} ||T_n|| = C \, ||v||$$

Since the these partial sums $\sum_{n=1}^{m} ||T_n v||$ are bounded, they are absolutely summable. Since W is a Bananch space they are summable too so we can define T here:

$$Tv = \lim_{m \to \infty} \sum_{n=1}^{m} T_n v$$

This is because this statement already implies,

$$T = \sum_{n=1}^{\infty} T_n$$

Since every v is mapped to the same output on both operators so they must be the same

$$T_1 a = T_2 a, \forall a \quad \to \quad T_1 = T_2$$

So now that we have possible candidate of T we aim to prove it is indeed in $\mathcal{B}(V, W)$. Firstly it is a linear operator due to linearity of limits which can be seen from

$$T(\lambda_1 v_1 + \lambda_2 v_2) = \lim_{m \to \infty} \sum_{n=1}^{m} T_n(\lambda_1 v_1 + \lambda_2 v_2)$$

It is obvious that it is bounded since every $\sum_{n=1}^{m} ||T_n v||$ is bounded, hence so will its limit(as usual think of the limit as a $\limsup$ or $\liminf$ to see why). Hence $\mathcal{B}(V, W)$ is a Banach Space

Definition 23

Let V be a normed vector space over $\mathbb{K}$. Then $V' = \mathcal{B}(V, \mathbb{K})$ is the **dual space** of V. An element of the dual space is called a **functional**.

Note that V' Banach Space by the previous theoerem.

Definition 24

Let V be a vector space. A subset $W \subseteq V$ is a **subspace** of V if for all $w_1, w_2 \in W$ and $\lambda_1, \lambda_2 \in \mathbb{K}$ we have

$$\lambda_1 w_1 + \lambda_2 w_2 \in W$$

(that is closed under linear combinations)

Proposition 25
A subspace W of a Banach space V is Banach(with norm inherited from V) if and only W is a *closed* subset of V(with respect to the metric induced by the norm)

From the forward direction if subspace W is Banach, then every convergent sequence in W which are hence also cauchy, must converge to a point in W. That implies W must be closed.

From the backward direction, any cauchy sequence in W is also a cauchy sequence in V. Thus since V is Banach, it must converge to a point in V. But because W is closed, it must be in W which is a subset of V. □

Earlier we defined the **seminorm**. We now aim to define a "norm-equivalent" on subspaces like in 2. We first note that V/E is pronounced as "V mod W".

Theorem 26
Let $||.||$ be a **seminorm** on a vector space V. If we define $E = \{v \in V : ||v|| = 0\}$, then E is a subspace of V, and the function on V/E is defined by

$$||v + E||_{v/E} = ||v||$$

for any $v + E \in V/E$

2 Baire Category and Uniform Boundedness theorem

Theorem 27 (Baire Category Theorem)
Let M be a complete metric space and let $\{C_n\}$ be a collection of closed subsets such that $M = \bigcup_{n=1}^{\infty} C_n$. Then at least one of the C_n must contains an interior point

Proof. By contradiction, suppose M is a complete metric space but none of the C_n contains interior points. Hence we aim to show this implies M is not complete and thus reach a contradiction from there. Naturally we aim to construct a cauchy sequence based on our current assumptions and show that it converges outside of the union of all C_n, that is not in M

Theorem 28 (Uniform Boundedness Theorem)
Let B be a Banach space, and let $\{T_n\}$ be a sequence in $\mathcal{B}(B, V)$. Then for all $b \in B$ we have $\sup_n ||T_n b|| < \infty$. Then $\sup_n ||T_n|| < \infty$ (Hint: Use Baire Category Theorem)

Remark 29. *this theorem is also called the* **Banach–Steinhaus theorem**

Proof. First we break down what the respective propositions mean

$$\forall b, \sup_n ||T_n b|| < \infty \quad \Leftrightarrow \quad \sup_b \sup_n ||T_n b|| < \infty \tag{1}$$
$$\sup_n ||T_n|| < \infty \quad \Leftrightarrow \quad \sup_n \sup_b ||T_n b|| < \infty \tag{2}$$

(1) means bounded for every b, all n - pointwise bounded(works for all b)
(2) means bounded for every n, all b - uniformly bounded(n that works for all b)

Hence we now we aim construct the set of b from the definition of (1) and see how we can get it to imply (2). We note that the linear operator being bounded is continuous. That is

$$||T_m b|| = \lim_{n\to\infty} ||T_m b_n||$$

and we can interchange

$$C_k = \left\{ b \in B : ||b|| \leq 1, \sup_n ||T_n b|| \leq k \right\}$$

3 Open mapping and Closed Graph theorem

Theorem 30 (Open Mapping Theorem)
Let B_1, B_2 be two Banach spaces and let $T \in \mathcal{B}(B_1, B_2)$ be a surjective linear operator. Then T is an **open map**, meaning that for all open subsets $U \subset B_1, T(U)$ is open in B_2

Proof. First we aim to prove that the $T(B(0,1)) \in B_2$ contains an open ball in B_2 that is centered at 0. We now attempt to bring in **baire's theorem** by claiming that

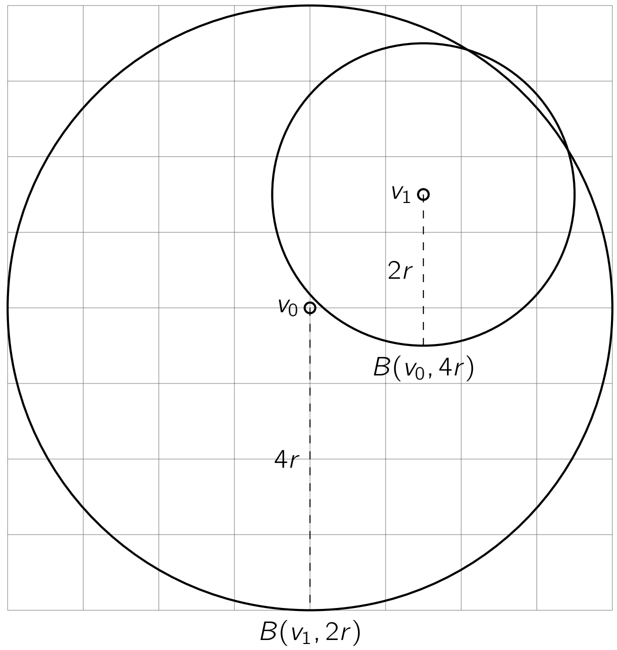
$$B_2 = \bigcup_{n=1}^{\infty} \overline{T(B(0,n))}$$

which is possible because by **surjectivity** everything in B_1 is mapped onto B_2 and that every element in B_1 must exist in one these balls $B(0,n)$ because any element has a finite distance away from 0. Such a form is chosen because we can apply linearity to linear operators knowing that

$$T(B(0,n)) = nT(B(0,1))$$

By Baire's theorem, there must exist some $n_0 \in \mathbb{N}$ where $\overline{T(B(0,n_0))}$ contains an open ball. Suppose this ball centered at some $v_0 \in B_2$. By linearity argument we can find an arbituary $r > 0$ such that

$$B(v_0, 4r) \subset \overline{T(B(0,1))}$$



An to ensure this point is contained just in the closure $\overline{T(B(0,1))}$ but $T(B(0,1))$ itself. First define some $Tu_1 = v_1$. We know v_0 could either be a limit point or an ordinary point which is definitely in the actual set. In the latter case we could simply let $v_1 = v_0$ which would satisfy the above. Otherwise as a limit point we can definitely find a point arbituary close to it that satisfies the above. Now that we have found our open ball, let us return to our objective and prove that it contains a ball $B(0, r)$ centered at 0 in B_2. Naturally this ball is defined to be the set $\{v \in B_2| \, ||v|| < r\}$ We know that

$$(v_1 + 2v) \subset B(v_1, 2r) \subset \overline{T(B(0,1))}$$

To see why consider that $B(v_1, 2r)$ is simply the set of vectors which differ in distance from v_1 by $2r$. Then $||(v_1 + 2v) - (v_1)|| = ||2v|| = 2r$ With this, it remains to show that $v \subset \overline{T(B(0,1))}$ to complete our objective.

$$\begin{aligned} v &= -v_1 + \frac{1}{2}(v_1 + 2v) \\ &\subset -T(\frac{u_1}{2}) + \overline{T(B(0, \frac{1}{2}))} \end{aligned}$$

by linearity we have and knowing that $||u_1|| < 1$ since $u_1 \in B(0, 1)$.

$$\begin{aligned} &= \overline{T\left(-\frac{u_1}{2} + B\left(0, \frac{1}{2}\right)\right)} \\ &\subset \overline{T\left(B\left(0,1\right)\right)} \end{aligned}$$

This result follows when we consider the sum of $||.||$ in this new set of elements defined in here

$$\left\{-\frac{u_1}{2} + \left(B\left(0,1\right)\right)\right\}$$

a We see such a sum is at most 1 clearly.

$$\max(||u_1/2|| + ||x \in B\left(0,1\right)||) = 1$$

So now we can define again by linearity

$$B(0, r2^{-n}) \subset \overline{T(B(0, 2^{-n}))}$$

We can immediately see from this that T is continuous...hence bounded . Consider that it obviously is in the form

$$\lim_{n\to\infty} v_n = v \to \lim_{n\to\infty} f(v_n) = f(v)$$

The conlusion follows by simple translation arguments.

From above we know that there exists some δ such that

$$B(0,\delta) \subset T(B(0,1))$$

Now let $U = B(b_1, \varepsilon)$ be open set in B_1. Let $b_2 = Tb_1$.

$$B(b_2, \varepsilon\delta) = b_2 + \varepsilon B(0,\delta) \subset b_2 + \varepsilon T(B(0,1)) = T(b_1) + \varepsilon T(B(0,1)) = T(b_1, \varepsilon)$$

□

Corollary 31

If B_1, B_2 are two Banach spaces and $T \in \mathcal{B}(B_1, B_2)$ is a bijective map the T^{-1} is in $\mathcal{B}(B_2, B_1)$

Theorem 32 (Closed Graph Theorem)

Let B_1, B_2 be two Banach spaces and let $T : B_1 \to B_2$ be a(not necessarily bounded) linear operator. Then $T \in \mathcal{B}(B_1, B_2)$ if and only if the graph of T defined as

$$\Gamma(T) = \{(u, Tu) : u \in B_1\}$$

is closed in $B_1 \times B_2$

4 L^p space theory

Definition 33

Let $f : E \to \mathbb{C}$ be a measurable function. Then for any $1 \le p < \infty$ we define the ***L^p* norm**

$$||f||_{L^p(E)} = \left(\int_E |f|^p\right)^{\frac{1}{p}}$$

And we define the ***L^∞*** or **essential suprenum** of f as

$$||f||_{L^\infty(E)} = \inf\{M > 0 : m(\{x \in E : |f(x)| > M\}) = 0\}$$

You can see they differ from our last discussion on ℓ^p and ℓ^∞ norms in that $\sum$ is now replaced with $\int$(lebesgue integral). Also the bottom condition is another way of saying

$$||f||_{L^\infty(E)} = \inf\{M > 0 : |f(x)| \le M \text{ a.e.}\}$$

Which in other words is saying these functions must be lebesgue integrable just like the finite case.

Definition 34 (Lebesgue Space L^p)

For any $1 \leq p \leq \infty$ we define the ***L^p* space** to be

$$L^p(E) = \left\{f : E \to \mathbb{C} | \text{f measureable and } ||f||_p < \infty\right\}$$

Theorem 35

L^p space is a vector space

Let $f, g \in L^p$ then we have by triangle inequality

$$\begin{aligned} |f(x) + g(x)|^p &\leq (|f(x)| + |g(x)|)^p \\ &\leq (2\max(|f(x)|, |g(x)|)^p \\ &\leq 2^p (|f(x)|^p + |g(x)|^p) \\ &< \infty \end{aligned}$$

This shows $f + g \in L^p$ too. Therefore from the above we also can tell that it is closed under scalar addition and multiplication. Clearly the space contains the zero function too so therefore there exists an additive inverse too in the L^p space.

Proposition 36

If $f : E \to \mathbb{C}$ is measurable then $|f(x)| \leq ||f||_{L^\infty(E)}$ almost everywhere on E. Also if $E = [a, b]$ is a closed and bounded interval and $f \in C([a, b])$ then $||f||_{L^\infty} = ||f||_\infty$(which is the usual sup norm we defined earlier for the space of bounded continuous functions C^∞)

Proof. The first part follows from the definition of infinum. The second follows from the the definition of suprenum, which is the *lowest* upperbound.

Lemma 37 (Young's Inequality)

Suppose $1 < p < \infty$. Let

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{1}$$

Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2}$$

for all $a \geq 0$ and $b \geq 0$

Proof. First fix $b > 0$ and define the function

$$f(a) = \frac{a^p}{p} + \frac{b^q}{q} - ab \tag{3}$$

From (1) we have

$$1 = \frac{1}{p} + \frac{1}{q} \tag{4}$$
$$= \frac{p+q}{pq} \tag{5}$$
$$p = q(p-1) \tag{6}$$
$$q = \frac{p}{p-1} \tag{7}$$

Sub (6) into (3) to simplify it and begin the "graphing process" using derivatives:

$$f(a) = \frac{a^p}{p} + \frac{b^{p/(p-1)}}{p/(p-1)} - ab \tag{8}$$
$$f'(a) = a^{p-1} - b \tag{9}$$
$$\tag{10}$$

then we find the turning point a_0

$$f'(a_0) = 0 \tag{11}$$
$$a_0 = b^{1/(p-1)} \tag{12}$$

since $a, b \geq 0$ and that a^{p-1} increases as a increases

$$f'(a) < 0, a \in (0, a_0) \tag{13}$$

It is clear from (9) that

$$f(a_0) = \frac{(b^{1/(p-1)})^p}{p} + \frac{b^{p/(p-1)}}{p/(p-1)} - (b^{1/(p-1)})b \tag{14}$$
$$= \frac{b^{p/(p-1)}}{p} + \frac{b^{p/(p-1)}}{p/(p-1)} - (b^{p/(p-1)}) \tag{15}$$
$$= b^{p/(p-1)} \left(\frac{1}{p} + \frac{p-1}{p} - \frac{p}{p} \right) \tag{16}$$
$$= 0 \tag{17}$$

Therefore our calculus-based graphical method shows our function decreases to 0 from $(0 \to a_0)$ then increases above 0 for $(a_0 \to \infty)$. Therefore for $a \geq 0$, we have $f(a) \geq 0$ hence from (3) we have:

$$0 \leq \frac{a^p}{p} + \frac{b^q}{q} - ab \tag{18}$$
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{(QED)} \tag{19}$$

□

Theorem 38 (Holders inequality for L^p spaces)
If $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g : E \to \mathbb{C}$ are measurable functions, then

$$\int_E |fg| \leq ||f||_{L^p(E)} \, ||g||_{L^q(E)}$$

Proof. Letting $a = |f(x)|, b = |g(x)|$ where both ≥ 0 clearly we have by young's inequality that

$$|f(x)|\,|g(x)| = |f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}$$

Taking the lebesgue integral of both sides which if you recall does not affect the sign of the inequality of integrable functions we have

$$\int |f(x)g(x)| = ||f(x)g(x)||_1 = \frac{\int |f(x)|^p}{p} + \frac{\int |g(x)|^q}{q}$$

suppose $||f(x)||_p = ||g(x)||_q = 1$ then we have

$$\left(\int |f|^p\right)^{\frac{1}{p}} = 1$$
$$\left(\int |f|^p\right) = 1$$

same for $g(x)$. Therefore we have

$$\int |f(x)g(x)| = ||f(x)g(x)||_1 \leq \frac{1}{p} + \frac{1}{q} = 1$$

by our assumption in the proposition. Just like 20 we extend this to beyond unit vectors so assume that is we actually had where $||f(x)||, ||g(x)|| \geq 1$ so applying the above again we have

$$\left|\left|\frac{f(x)}{||f(x)||_p}\frac{g(x)}{||g(x)||_q}\right|\right|_1 \leq 1$$

by homogenity of the norm we can do

$$||f(x)g(x)||_1 \leq ||f(x)||_p \, ||g(x)||_q$$

as desired

Theorem 39

Prove that $L^p \subset L^q$ when $1 \leq q \leq p$

Proof. Let $f \in L^a(\mathbb{R}^n)$ for some $1 \leq a < \infty$. We want to show that $f \in L^b(\mathbb{R}^n)$ for $b > a$.

Since $f \in L^a(\mathbb{R}^n)$, it follows that:

$$\int_{\mathbb{R}^n} |f(x)|^a \, dx < \infty.$$

To prove that $f \in L^b(\mathbb{R}^n)$, we need to show that:

$$\int_{\mathbb{R}^n} |f(x)|^b \, dx < \infty.$$

We begin by writing $|f(x)|^b$ as:

$$|f(x)|^b = |f(x)|^a \cdot |f(x)|^{b-a}.$$

Now, we apply Hölder's inequality to the product $g(x) = |f(x)|^a$ and $h(x) = |f(x)|^{b-a}$. Hölder's inequality states that for functions $g \in L^p(E)$ and $h \in L^q(E)$, where $\frac{1}{p} + \frac{1}{q} = 1$, we have:

$$\int_E |g(x)h(x)| \, dx \leq \left(\int_E |g(x)|^p \, dx\right)^{1/p} \left(\int_E |h(x)|^q \, dx\right)^{1/q}.$$

In our case, we apply Hölder's inequality with $p = \frac{a}{b}$ and $q = \frac{b}{b-a}$, where $\frac{1}{p} + \frac{1}{q} = 1$. We then get:

$$\int_{\mathbb{R}^n} |f(x)|^b\,dx = \int_{\mathbb{R}^n} |f(x)|^a \cdot |f(x)|^{b-a}\,dx \le \left(\int_{\mathbb{R}^n} |f(x)|^a\,dx\right)^{\frac{b}{a}} \left(\int_{\mathbb{R}^n} |f(x)|^b\,dx\right)^{\frac{b-a}{b}}.$$

Now, since $f \in L^a(\mathbb{R}^n)$, we know the first term on the right-hand side is finite:

$$\int_{\mathbb{R}^n} |f(x)|^a\,dx < \infty.$$

Therefore, we can conclude that upon rearrangment

$$\left(\int_{\mathbb{R}^n} |f(x)|^b\right)^{\frac{a}{b}} dx \le \left(\int_{\mathbb{R}^n} |f(x)|^a\,dx\right)^{\frac{b}{a}} < \infty$$

Thus, we have shown that $f \in L^b(\mathbb{R}^n)$, and therefore the inclusion $L^a(\mathbb{R}^n) \subset L^b(\mathbb{R}^n)$ holds for $1 \le a < b < \infty$. ☐

Theorem 40 (Minkowski's inequality for L^p spaces)
If $1 \le p \le \infty$ and $f, g : E \to \mathbb{C}$ are two measurable functions then $||f+g||_{L^p(E)} \le ||f||_{L^p(E)} + ||g||_{L^p(E)}$

Proof. Using the triangle inequality

$$|f+g|^p \le |f+g|^{p-1}(|f|+|g|)$$

integrating both sides, we get:

$$\int |f+g|^p\,d\mu \le \int |f+g|^{p-1}(|f|+|g|)\,d\mu$$

This can be split into two integrals:

$$\int |f+g|^p\,d\mu \le \int |f+g|^{p-1}|f|\,d\mu + \int |f+g|^{p-1}|g|\,d\mu$$

We apply Hölder's inequality to each of the integrals on the right side. Note that p and $\frac{p}{p-1} = q$ are conjugate exponents:

$$\int |f+g|^{p-1}|f|\,d\mu \le \left(\int |f+g|^{(\frac{p}{q})^q}\,d\mu\right)^{(1/q)} \left(\int |f|^p\,d\mu\right)^{1/p} = \left(\int |f+g|^p\,d\mu\right)^{(p-1)/p} \left(\int |f|^p\,d\mu\right)^{1/p}$$

Similarly,

$$\int |f+g|^{p-1}|g|\,d\mu \le \left(\int |f+g|^p\,d\mu\right)^{(p-1)/p} \left(\int |g|^p\,d\mu\right)^{1/p}$$

We combine the inequalities to get:

$$\int |f+g|^p\,d\mu \le \left(\int |f+g|^p\,d\mu\right)^{(p-1)/p} \left(\int |f|^p\,d\mu\right)^{1/p} + \left(\int |f+g|^p\,d\mu\right)^{(p-1)/p} \left(\int |g|^p\,d\mu\right)^{1/p}$$

Factor out $\left(\int |f+g|^p\,d\mu\right)^{(p-1)/p}$:

$$\int |f+g|^p\,d\mu \le \left(\int |f+g|^p\,d\mu\right)^{(p-1)/p} \left(\left(\int |f|^p\,d\mu\right)^{1/p} + \left(\int |g|^p\,d\mu\right)^{1/p}\right)$$

We now simplify and solve for the p-norm. Let $A = \left(\int |f+g|^p\,d\mu\right)^{1/p}$:

$$A^p \le A^{p-1}\left(\|f\|_p + \|g\|_p\right)$$

Divide both sides by A^{p-1}:

$$A \leq \|f\|_p + \|g\|_p$$

Thus,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Corollary 41

Our ***L^p* norm** defined earlier is indeed a norm

Proof. Just use properties of lebesgue integrals which we have proven before

(Homogenity)

$$||cf||_{L^p(E)} = \left(\int_E |cf|^p\right)^{\frac{1}{p}} = \left(|c|^p \int_E |f|^p\right)^{\frac{1}{p}} = |c| \left(\int_E |f|^p\right)^{\frac{1}{p}}$$

(Definiteness)

$$||0||_{L^p(E)} = \left(\int_E |0|^p\right)^{\frac{1}{p}} = 0$$

We have proven triangle inquality with **minkowski's inequality** above.

Theorem 42 (Riesz-Fischer)

For all $1 \leq p \leq \infty$, $L^p(E)$ is a banach space

Proof. We will use the fact that a normed space is banach if and only if every absolutely summable series is summable. Suppose we have

$$\sum_k ||f_k|| p = M < \infty$$

...to be continued

Theorem 43 (Riesz Fischer)

For all $1 \leq p \leq \infty$, ℓ^p is a banach space

5 Hilbert Spaces

We now move on to hilbert spaces.

Fact 44

You will learn that

- Banach spaces are *complete normed vector spaces*
- Hilbert spaces are *complete inner product spaces*

We have the inclusion

$$\text{Hilbert Spaces} \subset \text{Banach Spaces}$$

because the inner product in hilbert spaces defines a normed vector space too via the inner product.

Example 45
Recall that you have recently just proved **Riesz Fischer** theorem which shows that L^p for $1 \leq p \leq \infty$ are *banach spaces*. Althought they are all complete normed vector spaces, only the L^2 space is also hilbert since it is the only one that defines the billinear inner product as a norm. Specifically by

$$\langle f(x), g(x) \rangle = \int |f(x)g(x)|^2 dx$$

see 67.

Example 46
Other examples of key hilbert spaces include the **sobolev spaces**(see [1]) and certain spaces **holomrphic** functions(see [2])

Definition 47 (pre-hilbert space)
A **pre-hilbert space** H is a vector space over $\mathbb{C}$ with a **hermitian inner product** which is a map $H \times H \to \mathbb{C}$ (recall `artin algebra I`). That satisfies the following:

1. for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $v_1, v_2, w \in H$ we have

$$\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle$$

2. for all $v, w \in H$ we have $\langle v, w \rangle = \overline{\langle w, v \rangle}$
3. for all $v \in H$ we have $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$

Points (1) and (2) show:

$$\langle v, \lambda w \rangle = \overline{\langle \lambda w, v \rangle} = \overline{\lambda}\ \overline{\langle w, v \rangle} = \overline{\lambda} \langle v, w \rangle$$

Nothing new here. Recall that 1 variable linear while the other conjugate linear. 0 is the only vector that is orthogonal to all vectors so $\langle v, v \rangle = 0$. Also recall that $\langle v, v \rangle$ is real number. We also see that the form on the pre-hilbert space is positive definite.

Definition 48
Let H be a pre-hilbert space. Then for any $v \in H$ we define

$$||v|| = \langle v, v \rangle^{\frac{1}{2}}$$

Theorem 49 (homogenity of the norm)
Suppose V is an inner product space $f \in V$ and $a \in \mathbb{K}$ then

$$||af|| = |a|\,||f||$$

Proof. Consider

$$||af||^2 = \langle af, af \rangle = a\overline{a}\langle f, f \rangle = |a|^2\,||f||^2$$

Taking square roots gives the desired expression

Lemma 50 (Pythagorean Theorem)
Suppose f and g are orthogonal elements of an inner product space. Then

$$||f+g||^2 = ||f||^2 + ||g||^2$$

Proof. we have

$$\begin{aligned} ||f+g||^2 &= \langle f+g, f+g\rangle \\ &= \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle \\ &= ||f||^2 + ||g||^2 \end{aligned}$$

because the middle 2 terms are zero due to orthogonality.

Theorem 51 (Orthogonal Decomposition)
Suppose f and g are elements of an inner product space with $g \neq 0$. Suppose we have $f = cg + h$ Then we have that

$$\langle h, g\rangle = 0 \text{ if and only if } f = \tfrac{\langle f,g\rangle}{||g||^2} g + h$$

Proof. Suppose $\langle h, g\rangle = 0$

$$\langle f, g\rangle = \langle cg + h, g\rangle = c\,||g||^2$$

thus we have $c = \frac{\langle f,g\rangle}{||g||^2}$ and therefore the conclusion follows.(relate this to orthogonal projection as covered in `artin algebra I`). From the other direction we have

$$\langle h, g\rangle = \langle f - \frac{\langle f, g\rangle}{||g||^2}, g\rangle = \langle f, g\rangle - \frac{\langle f, g\rangle}{||g||^2}, \langle g, g\rangle = 0$$

Theorem 52 (Cauchy-Schwartz inequality)
Let H be a pre-hilbert space. For all $u, v \in H$ we have

$$|\langle u, v\rangle| \leq ||u||\,||v||$$

Proof. Consider the orthoganal decomposition $f = \frac{\langle f,g\rangle}{||g||^2} g + h$ where h is orthognal to g. Then by pythagorean theorem we have,

$$||f||^2 = \left|\left|\frac{\langle f, g\rangle}{||g||^2} g\right|\right|^2 + ||h||^2$$

since $\frac{g}{||g||} = 1$ and that $\langle f, g\rangle$ is some complex scalar. Applying homogentiy of norms we have

$$||f||^2 = \left|\left|\frac{\langle f, g\rangle}{||g||}\right|\right|^2 + ||h||^2 = \frac{|\langle f, g\rangle|^2}{||g||^2} + ||h||^2 \leq \frac{|\langle f, g\rangle|^2}{||g||^2}$$

Multiplying both sides by $||g||^2$ and taking square roots gives the desired conclusion

Theorem 53 (triangle inqeuality)
Suppose f and g are elements of an inner product space then

$$||f+g|| \leq ||f|| + ||g||$$

Proof. Consider

$$\begin{aligned}
||f+g||^2 &= \langle f+g, f+g\rangle && (1)\\
&= \langle f,f\rangle + \langle g,g\rangle + \langle f,g\rangle + \langle g,f\rangle && (2)\\
&= \langle f,f\rangle + \langle g,g\rangle + \langle f,g\rangle + \overline{\langle f,g\rangle} && (3)\\
&= ||f||^2 + ||g||^2 + 2\,\mathrm{Re}\langle f,g\rangle && (4)\\
&\leq ||f||^2 + ||g||^2 + 2|f,g| && (5)\\
&= ||f||^2 + ||g||^2 + 2\,||f||\,||g|| && (6)\\
&= (||f|| + ||g||)^2 && (7)
\end{aligned}$$

where (6) follows from *Cauchy inequality*

Fact 54
Notice that

$$\begin{aligned}
||f+ig||^2 &= \langle f+ig, f+ig\rangle\\
&= \langle f,f\rangle + \langle ig,ig\rangle + \langle f,ig\rangle + \langle ig,f\rangle\\
&= \langle f,f\rangle + (-i^2)\langle g,g\rangle - i\langle f,g\rangle + i\overline{\langle f,g\rangle}\\
&= ||f||^2 + ||g||^2 + i(\langle f,g\rangle - \overline{\langle f,g\rangle})\\
&= ||f||^2 + ||g||^2 + 2\,\mathrm{Im}\langle f,g\rangle
\end{aligned}$$

Theorem 55
If H is a pre-hilbert space then $||.||$ is indeed a norm on H

Proof. we have already proven homogenity and triangle inequality. For definiteness consider

$$||v|| = 0 \leftrightarrow \langle v,v\rangle^{\frac{1}{2}} \leftrightarrow v = 0$$

from our definitions above.

Theorem 56 (Continuity of inner product)
If $u_n \to u$ and $v_n \to v$ in a pre-hilbert space equipped with norm $||.||$ then $\langle u_n, v_n\rangle \to \langle u, v\rangle$

Proof. Consider

$$
\begin{aligned}
|\langle u_n, v_n\rangle - \langle u, v\rangle| &= |\langle u_n, v_n\rangle - \langle u, v_n\rangle + \langle u, v_n\rangle - \langle u, v\rangle| &(1)\\
&= |\langle u_n - u, v_n\rangle + \langle u, v_n - v\rangle| &(2)\\
&\leq |\langle u_n - u, v_n\rangle| + |\langle u, v_n - v\rangle| &(3)\\
&\leq ||u_n - u||\,||v_n|| + ||u||\,||v_n - v|| &(4)\\
&= 0, n \to \infty &(5)
\end{aligned}
$$

Where (3) is consequence of trianlge inquality while (4) is consequence of cauchy inequality. (5) follows by the continuity of norms. Therefore we have proven continuity of inner products as desired

Proposition 57 (Parallelogram law)
Let H be a pre-hilbert space. Then for any $u, v \in H$ we have

$$||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$$

in addition if H is a normed vector satisying this equality thn H is a pre-hilbert space. In other words this is a bijective proposition.

Proof. Let H be a pre-Hilbert space with norm $||u|| = \sqrt{\langle u, u\rangle}$. Expanding $||u+v||^2$ and $||u-v||^2$, we have recalling how we expanded in 53

$$
\begin{aligned}
||u+v||^2 &= \langle u+v, u+v\rangle\\
&= \langle u+v, u\rangle + \langle u+v, u\rangle\\
&= \langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle\\
&= \langle u, u\rangle + \langle v, u\rangle + \overline{\langle v, u\rangle} + \langle v, v\rangle \quad \text{by definition(see above)}\\
&= \langle u, u\rangle + 2\,\mathrm{Re}(\langle u, v\rangle) + \langle v, v\rangle
\end{aligned}
$$

similarly

$$||u-v||^2 = \langle u-v, u-v\rangle = \langle u, u\rangle - 2\,\mathrm{Re}(\langle u, v\rangle) + \langle v, v\rangle.$$

Adding these yields:

$$||u+v||^2 + ||u-v||^2 = 2\langle u, u\rangle + 2\langle v, v\rangle = 2(||u||^2 + ||v||^2).$$

For the converse, assume the norm satisfies the parallelogram law. Define an inner product via the polarization identity:

$$\langle u, v\rangle = \frac{1}{4}(||u+v||^2 - ||u-v||^2).$$

This inner product satisfies linearity, conjugate symmetry, and positive-definiteness, making H a pre-Hilbert space. ☐

Theorem 58 (Bessel)
Let $\{e_n\}$ be a countable orthonormal subset of pre-hilbert space H. Then for all $u \in H$ we have

$$\sum_n |\langle u, e_n\rangle|^2 \leq ||u||^2$$

Proof. The general "naive" idea of the proof is to have (length of u)^2 - (length of projection of u onto this orthonormal basis)$^2 \geq 0$ like pythagorean theorem. In other words, the lenghth of projection onto an orthoganal basis is bounded by the lenght of u itself. First consider that

$$\left|\left|\sum_{n=1}^{N}\langle u, e_n\rangle e_n\right|\right|^2 = \left\langle \sum_{n}\langle u, e_n\rangle e_n, \sum_{m}\langle u, e_m\rangle e_m \right\rangle \tag{1}$$

since linear in 1st variable and conjugate linear in the 2nd (1) is equal

$$= \sum_{n,m}\langle u, e_n\rangle\overline{\langle u, e_m\rangle}\langle e_n, e_m\rangle = \sum_{n=1}^{N}|\langle u, e_n\rangle|^2$$

as due to orthonormal property the only nonzero terms are when $n = m$ in which case they are equal one. Next consider that

$$\left\langle u, \sum_{n=1}^{N}\langle u, e_n\rangle \right\rangle = \sum_{n=1}^{N}\overline{\langle u, e_n\rangle}\langle u, e_n\rangle = \sum_{n=1}^{N}|\langle u, e_n\rangle|^2 \tag{2}$$

Notice that

$$\left|\left|\sum_{n=1}^{N}\langle u, e_n\rangle e_n\right|\right|^2 = \left\langle u, \sum_{n=1}^{N}\langle u, e_n\rangle \right\rangle = \sum_{n=1}^{N}|\langle u, e_n\rangle|^2$$

You can see they are all the "length of projection". The middle is yhe "dot product" of u and the projection which will clearly ignore the orthogonal part of u to also get the length of projection.

Now considering the projection of u into the orthonognal space(simply minus all the projected parts) we have

$$0 \leq \left|\left|u - \sum_{n=1}^{N}\langle u, e_n\rangle e_n\right|\right|$$

like our proof for triangle inquality we can do

$$0 \leq \left|\left|u - \sum_{n=1}^{N}\langle u, e_n\rangle e_n\right|\right| \leq ||u||^2 + \left|\left|\sum_{n=1}^{N}\langle u, e_n\rangle e_n\right|\right|^2 - 2\,\mathrm{Re}\left\langle u, \sum_{n=1}^{N}\langle u, e_n\rangle e_n \right\rangle$$

Now we appply our results from (1) and (2) to get

$$0 \leq ||u||^2 + \sum_{n=1}^{N}|\langle u, e_n\rangle|^2 - 2\sum_{n=1}^{N}|\langle u, e_n\rangle|^2 = ||u||^2 - \sum_{n=1}^{N}|u, e_n|^2$$

So we have

$$\sum_{n=1}^{N}|u, e_n|^2 \leq ||u||^2 \quad \Rightarrow \quad \lim_{N\to\infty}\sum_{n=1}^{N}|u, e_n|^2 \leq ||u||^2$$

Definition 59 (Separable)
A set is **separable** if it contains a countable **dense** subset. Recall that a set A is dense in metric space X if and only if its closure is equal X that is

$$\overline{A} = X$$

Example 60

$\mathbb{R}^k$ is separable because $\mathbb{Q}^k$ is countable dense subset of $\mathbb{R}^k$. That is to say every point of $\mathbb{R}^k$ is either a point of $\mathbb{Q}^k$ or a limit point of $\mathbb{Q}^k$

Proposition 61

$C([a, b])$ is dense in $L^p([a, b])$ for $1 \leq p < \infty$

Proof. refer to link in notes...

Remark 62. *Because we have proven that $C([a, b])$ is dense in L^p and that we know the latter is a Banach space which is complete, we can say that L^p is the* **completion** *of $C([a, b])$. Note that C is not be confused with C^∞. The latter is continuous on all degrees of differention of the function while the former is just the function itself(aka C^0).C^∞ is Banach as we have proven earlier.*

Definition 63 (maximal)

An orthonormal subset $\{e_i\}$ of a pre-hilbert space H is **maximal** if and only if the only vector $u \in H$ satisfying $\langle u, e_i \rangle = 0$ for all i is $u = 0$

Theorem 64

Every nontrivial **separable** pre-hilbert space H has a countable maximal orthonormal subset

Proof. Since H contains a countable dense subset we may define $\{v_j\}$ to be a countable(finite or countably infinite) subset dense subset of H. Simply craft an orthnormal basis from $\{v_i\}$ using **Gram-Schidt** such that our orthogonal basis $(e_1, \dots e_{m(n)})$ spans $(v_1, \dots, v_n)$ for all n. After doing so note by density property, every limit point of this subset must be in H. Suppose there exists $u \in H$ such that $\langle u, e_\ell \rangle = 0$ for all ℓ. Then there exists a subsequence $\left\{v_{j(k)}\right\}$ such that

$$\lim_{k \to \infty} v_{j(k)} \to u \in H$$

because u is either a limit point of this subset or a point in this subset. For the former if its not in the subset then it has to be a limit point in H. For the latter consider x in this subset. Then we can just pick constant sequence $\{x\}$ in the dense which obviously converges to x. In both cases such a subsequence exists Now writing each $v_{j(k)}$ as a span of $\left\{e_1, \dots, e_{m(j(k))}\right\}$

$$\left|\left|v_{j(k)}\right|\right|^2 = \sum_{\ell=1}^{m(j(k))} \left|\langle v_{j(k)}, e_\ell \rangle\right|^2$$

Where recall above this is the "lenght of projection". Therefore

$$\sum_{\ell=1}^{m(j(k))} \left|\langle v_{j(k)}, e_\ell \rangle\right|^2 = \sum_{\ell=1}^{m(j(k))} \left|\langle v_{j(k)} - u, e_\ell \rangle\right|^2 \leq \left|\left|v_{j(k)} - u\right|\right|^2$$

The first equality follows because $\langle u, e_\ell \rangle = 0$ for all ℓ so adding it into the inner product doesn't change anything. The rest of the terms in the RHS are due to **Bessel Inequality**. Since $\lim_{k \to \infty} \left|\left|v_{j(k)} - u\right|\right| = 0$ we have

$$||u|| = \lim_{k \to \infty} \left|\left|v_{j(k)}\right|\right|^2 = 0$$

Therefore $u = 0$ as desired by the definiteness property of norms and continuity of inner products.

Definition 65

A **hilbert space** is a pre-hilbert space that is complete with respect to the norm $||.|| = \langle ., . \rangle^{\frac{1}{2}}$

Example 66

The space of n-tuples of complex numbers $\mathbb{C}^n$ with inner product $\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$ is a (finite dimensionsal) **hilbert space**

Example 67

The space $\ell^2 = \left\{ a : \sum_n |a_n|^2 < \infty \right\}$ is a **hilbert space** where we define

$$\langle a, b \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k} = \sum_{k=1}^{\infty} |a_k b_k|^2$$

Consider that $\langle a, a \rangle^{\frac{1}{2}}$ coincides with the ℓ^2 norm $||a||_{\ell^2}$ which we have proven is bananch, complete and hence Hilbert(since used an inner product as the norm too)!

Example 68

Let $E \subset \mathbb{R}$ be measurable. Then $L^2(E)$, the space of measurable functions $f : E \to \mathbb{C}$ with $\int_E |f|^2 < \infty$ is a hilbert space with inner product

$$\langle f, g \rangle = \int_E \overline{f} g = \int_E |fg|^2$$

Consider that $\langle f, f \rangle^{\frac{1}{2}}$ coincides with the L^2 norm $||f||_{L^2}$ which we have proven is bananch, complete and hence Hilbert!

Example 69

The functions $f_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ form an orthonormal subset of $L^2([-\pi, \pi])$ which is a hilbert space

see below for proof

Definition 70

Let H be a hilbert space. An **orthonormal basis** of H is a countale maximal orthnormal subset $\{e_n\}$ of H.

Theorem 71 (fourier bessel series)

Let $\{e_n\}$ be an orthonormal basis in a hilbert space H. Then for all $u \in H$ we have convergence of the **fourier-bessel series**

$$\lim_{m \to \infty} \sum_{n=1}^{m} \langle u, e_n \rangle e_n = \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n = u$$

Proof. By **Bessel Inequality** we know that

$$\left|\left| \sum_n \langle u, e_n \rangle e_n \right|\right|^2 = \sum_n |\langle u, e_n \rangle|^2 \leq ||u||^2$$

Therefore the partial sums for the above must be a cauchy sequence because every convergent sequence is cauchy. Because by defintion a Hilbert Space is **complete** so every cauchy sequence must be convergent in H. That is there must exist

$$u' = \lim_{m\to\infty} \sum_{n=1}^{m} \langle u, e_n \rangle e_n \in H$$

Therefore it now remains to show that $u' = u$.Now consider

$$\langle u - u', e_\ell \rangle = \lim_{n\to\infty} \left\langle u - \sum_{n=1}^{m} \langle u, e_n \rangle e_n, e_\ell \right\rangle$$

by continuity of inner products. Then by linearity we can get this to be equal

$$\lim_{n\to\infty} \left(\langle u, e_\ell \rangle - \sum_{n=1}^{m} \langle u, e_n \rangle \langle e_n, e_\ell \rangle \right)$$

where we get

$$\langle u - u', e_\ell \rangle = 0$$

However by **maximality** the only possible solution that satisfy all ℓ is when $u - u' = 0$

Corollary 72

If a hilbert space H has an orthonormal basis then H is separable

Proof. First we contruct a countable subset of H defined by

$$S = \bigcup_{m\in\mathbb{N}} \left\{ \sum_{n=1}^{m} s_n e_n | s_i \in \mathbb{Q} + i\mathbb{Q} \right\}$$

So essentially this is the set of all approximations by fourier bessel series with rational coefficients only. Recall that rational numbers are countable and a countable union of countable sets if countable. We now need to prove that S is dense in H. That is every $x \in H$ is a limit point of S. First Let $x \in H$ be represented by a fourier bessel series

$$x = \sum_i c_i e_i \tag{1}$$

where $c_i = \langle x, e_i \rangle \in \mathbb{R}$ We also that by bessel inquality there exists N such that where get a cauchy sequence as follows

$$\left|\left| \sum_{N+1}^{\infty} c_i e_i \right|\right| \leq \frac{\varepsilon}{2}$$

But recall that $\mathbb{C}$ is dense in $\mathbb{R}$ hence we can find for every c_i an element of S that is arbituary close to it which we then take to be s_i because every c_i is a limit point of S. We do this for $i : 1 \to N$

$$|c_i - s_i| \leq \frac{\varepsilon}{2^{i+1}} \tag{2}$$

We now define $x_n = \sum_i^N s_i e_i \in S$. Therefore for any $x \in H$ we have

$$\begin{aligned}
||x - x_n|| &= \left|\left|\sum_i^{\infty} c_i e_i - \sum_i^N s_i e_i\right|\right| && (3)\\
&= \left|\left|\sum_{N+1}^{\infty} c_i e_i + \sum_i^N (c_i - s_i) e_i\right|\right| && (4)\\
&\leq \left|\left|\sum_{N+1}^{\infty} c_i e_i\right|\right| + \left|\left|\sum_i^N (c_i - s_i) e_i\right|\right| && (5)\\
&\leq \frac{\varepsilon}{2} + \sum_i^N |(c_i - s_i)| && (6)\\
&\leq \varepsilon && (7)
\end{aligned}$$

Where (6) is consequence of triangle inequality and applying homogeneity of norms. This shows we can find an element in S arbiturary close to every element in H so S is dense in H

Theorem 73 (Parseval's Identity)
Let H be a Hilbert space, and let $\{e_n\}$ be a countable orthonormal basis of H. Then for all $u \in H$

$$\sum_n |\langle u, e_n \rangle|^2 = ||u||^2$$

Proof. Since we have

$$u = \sum_n \langle u, e_n \rangle e_n$$

then

$$||u||^2 = \langle u, u \rangle = \lim_{m \to \infty} \left\langle \sum_{n=1}^m \langle u, e_n \rangle e_n, \sum_{\ell=1}^m \langle u, e_\ell \rangle e_\ell \right\rangle$$

and then moving constants out again like usual we have this equal to

$$\lim_{m \to \infty} \sum_{n,\ell}^m \langle u, e_n \rangle \overline{\langle u, e_\ell \rangle} \langle e_n, e_\ell \rangle = \lim_{m \to \infty} \sum_{n=1}^m |\langle u, e_n \rangle|^2$$

by orthonormality.

Theorem 74
If H is an infinite dimensional separable Hilbert space, then H is isometrically isomorphic to ℓ^2. In other words, there exists a bijective bounded linear operator $T : H \to \ell^2$ so that for all $u, v \in H$, $||Tu||_{\ell^2} = ||u||_H$ and $\langle Tu, Tv \rangle_{\ell^2} = \langle u, v \rangle_H$

Proof. Suppose we define our map to ℓ^2 by

$$Tu = \{\langle u, e_n \rangle\}_n$$

which is essentially the sequence of coefficients that show up in our expansion by orthnormal basis. By our result in fourier besel series we know(see 53)

$$\left(\sum_{n=1}^{\infty} |\langle u, e_n \rangle|^2\right)^{\frac{1}{2}} = \left|\left|\sum_{n=1}^{\infty} \langle u, e_n \rangle e_n\right|\right| = ||u||$$

This immediately shows the equivalence of norms of ℓ^2 and H, so we have $u, v \in H$, $||Tu||_{\ell^2} = ||u||_H$ as desired. This also clearly implies the linear operator is bounded. It is linear in u clearly due to linearity in the first variable of the inner product in hilbert space. Finally it is injective because $Tu = 0$ implies $\langle u, e_j \rangle = 0$ for all j and by maximality that can only be when $u = 0$. It is surjective because every $Tu = \{c_j\}$ is mapped from some $u = \sum_{j=1}^{\infty} c_j e_j$

6 Fourier Series

Proposition 75

The subset of functions $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_n$ is an orthonormal subset of $L^2([-\pi, \pi])$

(Note: Recall we are referring to an L^p space here)

Proof. Recall from 68 for how we define a norm on L^2, we have

$$\langle e^{inx}, e^{imx} \rangle = \int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}}dx = \int_{-\pi}^{\pi} e^{i(n-m)x}dx$$

However the function e^{inx} is periodic which means it repeats every 2π interval in x. Therefore

$$\int_y^x e^{i(n-m)x}dx = \begin{cases} \int_y^x 1dx = 2\pi & n = m \\ \frac{e^{i(n-m)y} - e^{i(n-m)x}}{i(n-m)} = 0 & n \neq m \end{cases}$$

if the difference between x and y is 2π then clearly the integral is zero. Since our integral is an interval $x - y = \pi - (-\pi) = 2\pi$ then clearly $e^{i(n-m)y} - e^{i(n-m)x} = 0$

$$\left\langle \frac{e^{inx}}{\sqrt{2\pi}}, \frac{e^{imx}}{\sqrt{2\pi}} \right\rangle = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Fact 76

In general

$$\int_y^x e^{inx}dx = \begin{cases} \int_y^x 1dx = 2\pi & n = 0 \\ \frac{e^{iny} - e^{inx}}{in} = 0 & n \neq 0 \end{cases}$$

where $|x - y| = 2\pi$

Definition 77
For a function $f \in L^2([-\pi.\pi])$ the **fourier coefficient** $\hat{f}(n)$ of f is given by

$$\hat{f} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$$

and the Nth **partial fourier sum** is

$$S_n f(x) = \sum_{|n|\leq N} \hat{f}(n)e^{inx} = \sum_{|n|\leq N}\left\langle f, \frac{e^{inx}}{\sqrt{2\pi}}\right\rangle \frac{e^{inx}}{\sqrt{2\pi}}$$

The **fourier series** of f is the formal series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$$

Proposition 78
For all $f \in L^2([-\pi,\pi])$ and all $N \in \mathbb{Z}_{\geq 0}$ we have $S_n f(x) = \int_{-\pi}^{\pi} D_N(x-t)f(t)dt$ where

$$D_n(x) = \begin{cases} \frac{2N+1}{2\pi} & x = 0 \\ \frac{\sin((N+\frac{1}{2})x)}{2\pi\sin\frac{x}{2}} & x \neq 0 \end{cases}$$

We call D_n the **Dirichlet Kernel**

Proof. Consider that, we now aim to show that the term in the parenthesis is indeed $D_n(x-t)$

$$S_n f(x) = \sum_{|n|\leq N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt\right)e^{inx} = \int_{-\pi}^{\pi} f(t)\left(\frac{1}{2\pi}\sum_{|n|\leq N} e^{in(x-t)}\right)dt$$

We can interchange sums and integral bcause we are using lebesgue intergral, recall its properties. Let $x = x - t$ then define

$$D_n(x) = \frac{1}{2\pi}\sum_{|n|\leq N} e^{inx} = \frac{1}{2\pi}e^{-iNx}\sum_{n=0}^{2N} e^{inx}$$

which is clearly equivalent as both sum the range $-N \leq n \leq N$. For the case $e^{ix} \neq 1$ which is when $x \neq 0$ we can apply geometric series to find

$$D_n(x) = \frac{1}{2\pi}e^{-iNx}\frac{1-e^{i(2N+1)x}}{1-e^{ix}} = \frac{1}{2\pi}\frac{e^{i(N+1/2)x} - e^{-i(N+1/2)x}}{e^{ix/2} - e^{-ix/2}} = \frac{1}{2\pi}\frac{2i\sin((N+1)x)}{2i\sin(\frac{1}{2})}$$

as desired and that recall $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$
If we now have the case for $e^{-ix} = 1$ which is when $x = 0$ then obviously

$$D_n(x) = \frac{2N+1}{2\pi}$$

$\square$

Definition 79

Let $f \in L^2([-\pi,\pi])$. The Nth **Cesaro-Fourier mean** of f is

$$\sigma_n f(x) = \frac{1}{N+1}\sum_{k=1}^{\infty} S_k f(x)$$

Proposition 80

For all $f \in L^2([-\pi,\pi])$ we have

$$\sigma_N f(x) = \int_{-\pi}^{\pi} K_n(x-t)f(t)dt$$

and

$$K_N(x) = \begin{cases} \frac{N+1}{2\pi} \\ \frac{1}{2\pi(N+1)}\left(\frac{\sin(\frac{N+1}{2}x}{\sin\frac{x}{2}}\right)^2 \end{cases}$$

The function $K_N(x)$ is called the **Fejer kernel** and it satisfies

1. $K_N(x) \geq 0$ and $K_N(x) = K_N(-x)$ for all x
2. K_N is periodic with period 2π
3. $\int_{-\infty}^{\infty} K_N(t)dt = 1$
4. for any $\delta \in (0,\pi)$ and for all $\delta \leq |x| \leq \pi$ we have $|K_n(x)| \leq \frac{1}{2\pi(N+1)\sin^2\frac{\delta}{2}}$

Theorem 81 (Ferger)

Let $f \in C([-\pi,\pi])$ be 2π periodic (so $f(-\pi) = f(\pi)$. Then $\sigma_N f \to f$ uniformly on $[-\pi,\pi]$

7 Minimizers,Orthogonal Complements and Riesz Representation

Theorem 82 (Existence Length Miminizer)

Let C be a nonempty closed subset of a Hilbert space H which is **convex**, meaning that for all $v_1, v_2 \in C$ we have $tv_1 + (1-t)v_2 \in C$ for all $t \in [0,1]$. Then there exists a unique element $v \in C$ with $||v|| = \inf_{u\in C}||u||$. This is called a **length minimizer**

Remark 83. *The condition that C is closed is required. This is clear because the minimum norm would not be achieved because its on the boundary(limit point,infinum). Now convexity simply implies* the line segment between any two eleements of C is conatained in C.*This condition is also essential in this theorem. Suppose we took C to be the complement of an open disk. Then clearly the minimum norm will exist on the whole boundary so we do not have uniqueness.*

Proof. Let $d = \inf_{u\in C}||u||$. Knowing that norms by definition are bounded below and C is closed we know that it exists. Therefore there exists a sequence $\{u_n\}$ where $||u_n|| \to d$ so we can write

$$2\,||u_n||^2 < 2d^2 + \frac{\varepsilon^2}{2}$$

for $n \geq N$. then by **parallelogram law**(recall above) we also can write

$$||u_m - u_n||^2 = 2\,||u_m|| + 2\,||u_n||^2 - 4\left|\left|\frac{u_n + u_m}{2}\right|\right|^2$$

and we know that $\frac{u_n+u_m}{2}$ lies on the line segment between u_n and u_m where $t = \frac{1}{2}$ and we know that it exists in C by **convexity**. Therefore

$$\left|\left|\frac{u_n + u_m}{2}\right|\right|^2 \geq d^2$$

so we have

$$||u_m - u_n||^2 \leq 2\,||u_m||^2 - 2d^2 + 2\,||u_n||^2 - 2d^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2$$

so $\{u_n\}$ is cauchy and because hilbert spaces are complete we know that it converges to some $v \in C$. Due to the continuity of norms we therefore can write

$$||v|| = \lim_{n\to\infty} ||u_n|| = d$$

To prove for uniqueness is very much to same as how we do so for sequences using the distance metric in $\mathbb{R}^n$. Consider the parallelogram law

$$||v - \overline{v}||^2 = 2\,||v||^2 + 2\,||\overline{v}||^2 - 4\left|\left|\frac{v + \overline{v}}{2}\right|\right|^2 \leq 2d^2 + 2d^2 - 4d^2 = 0$$

Theorem 84

Let H be a Hilbert space, and let $W \subset H$ be a subspace. Then the **orthogonal complement**

$$W^\perp = \{u \in H : \langle u, w\rangle = 0 \quad \forall w \in W\}$$

is a closed linear subspace of H. Furthermre if W is closed then $H = W \oplus W^\perp$; in other words, for all $u \in H$ we can write $u = w + w^\perp$ for some unique $w \in W$ and $w^\perp \in W^\perp$

Proof. Showing that $W^\perp$ is a subspace is clear. If $\langle u_1, w\rangle = 0$ and $\langle u_2, w\rangle = 0$ for all $w \in W$, then any linear combination

$$a\langle u_1, w\rangle + b\langle u_2, w\rangle = \langle au_1 + bu_2, w\rangle = 0 + 0 = 0$$

To show that $W^\perp$ is closed consider any arbituary sequence $\{u_n\}$ in $W^\perp$ that converges to some $u \in H$ then

$$\langle u, w\rangle = \lim_{n\to\infty} \langle u_n, w\rangle = \lim_{n\to\infty} 0 = 0$$

by the continuity of inner prducts.. Finally to show that $H = W \oplus W^\perp$ if W is closed first define some arbituary $u \in H/W$ and define

$$C = u + W = \{u + w : w \in W\}$$

where this set is convex since

$$t(u + w_1) + (1 - t)(u + w_2) = u + (tw_1 + (1 - t)w_2)$$

since W is subspace so it is closed under linear combinations so the bracketed term o the RHS is in W. So it is clear the RHS in C as desired. Now consider any arbiturary sequence $u + w_n$ in C that coverges to some $v \in H$ just like how we proved $W^\perp$ is closed earlier we see that

$$u + w_n \to v \quad \Rightarrow \quad w_n \to v - u = w$$

since W is closed we know that $w = v - u \in W$. But notice that $v = u + w$ is exactly the defines the elements in C. So having proved that C is closed and convex we can the apply **existence of length minimizer theorem** from previously where we know that there exsts some unique $v \in C$ where

$$||v|| = \inf_{c \in C} ||c|| = \int_{w \in W} ||u + w||$$

So we lets use what we call a **variational argument**(in physics this is the **euler langrange equations** consider recall 53

$$f(t) = ||v + tw||^2 = ||v||^2 + t^2 \, ||w||^2 + 2t \operatorname{Re}\langle v, w\rangle$$
$$g(t) = ||v + itw||^2 = ||v||^2 + t^2 \, ||w||^2 + 2t \operatorname{Im}\langle v, w\rangle$$

because we know the minimum occurs exactly at $v \in C$ we can infer that

$$f'(0) = 0 \quad \text{and} \quad g'(0) = 0$$

where we can see that

$$\operatorname{Re}\langle v, w\rangle = \operatorname{Im}\langle v, w\rangle$$

therefore $\langle v, w\rangle = 0$ since this is true for all $w \in W$ we conclude that $v \in W^\perp$. Therefore if we show that for every u written in the form $u = w + w^\perp$(recall `Axler Linear Algebra`) *uniquely* then $H = W \oplus W^\perp$ as desired. Suppose by contradiction there exist

$$u = w_1 + w_1^\perp = w_2 + w_2^\perp$$

but that implies

$$w_1 - w_2 = w_2^\perp - w_1^\perp$$

where we know the LHS and RHS in W and $W^\perp$ respectively. But $W \cap W^\perp = \{0\}$ so it must be that $w_1 = w_2$.

Corollary 85

If $W \subset H$ is a subspace then $(W^\perp)^\perp$ is the closure $\overline{W}$ of W. In particular if W is closed then $(W^\perp)^\perp = W$

Proof. We know that $W^\perp$ is closed. Therefore by the previous theorem we know that

$$H = (W^\perp) \oplus (W^\perp)^\perp$$

and if W is closed we have

$$H = (W^\perp) \oplus W$$

it is evident to see that $W = (W^\perp)^\perp$ if closed.

Note that as we have used the lenght minimizer theorem which requires completeness as you might suspect the hilbert condition(basically complete inner product spaces) is essential. Consider the following example where the Inner Product Space is Not Complete (Not Hilbert) and Decomposition Fails

Example 86
Consider the space of polynomials $P([0,1])$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

This space is not complete with respect to the norm induced by the inner product because the space of polynomials is not closed under limits of Cauchy sequences. For instance, the sequence of polynomials $f_n(x) = x^n$ converges pointwise to the function $f(x) = 0$ on the interval $(0,1)$ and $f(1) = 1$, which is not a polynomial. Thus, $P([0,1])$ is not closed. Let W be the subspace of polynomials of degree at most 1, i.e.,

$$W = \operatorname{span}\{1, x\}.$$

The orthogonal complement $W^\perp$ would be the set of functions orthogonal to all constant and linear polynomials under the inner product. However, since the space of polynomials is not complete, there are functions that are orthogonal to every polynomial in W but do not belong to $W^\perp$ (since $W^\perp$ would need to be contained within the space of polynomials, which isn't closed).

Thus, the decomposition $V = W \oplus W^\perp$ fails in this case, as V is not Hilbert and does not support the existence of such a decomposition.

Definition 87
We first denote $f \in H' = \mathcal{B}(H, \mathbb{C})$ which refers to the space of bounded linear operators(and thus continuous) between from the hilbert space H to $\mathbb{C}$.

Tbhis notation should be familiar from 19

Theorem 88 (Riesz Representation Theorem)
Let H be a Hilbert space. Then for all $f \in H'$ there exists a unique $v \in H$ so that $f(u) = \langle u, v\rangle$ for all $u \in H$. In other words every element of the dual(in this case linear maps $f \in \mathcal{L}(H)$) can be realized the inner product with a fixed vector.

Remark 89. *This is a powerful theorem...you will realize why in tensor algebra...*

Proof. To prove uniqueness suppose

$$f(u) = \langle u, v\rangle = \langle u, \overline{v}\rangle = 0$$

then $\langle u, v - \overline{v}\rangle = 0$ for all $u \in H$. This means we may set $u = v - \overline{v}$ and we can easily see that $v - \overline{v} = 0$.

The easiest case is $f = 0$ in which case $v = 0$. Hence we assume otherwise, there exists some $u_1 \in H$ so that $f(u_1) \neq 0$ and we take $u_0 = \frac{u_1}{f(u_1)}$ so that $f(u_0) = 1$ so that $f(u_0) = 1$.(recall we can always normalize linear operators like that). Hence we may define the non empty set

$$C = \{u \in H : f(u) = 1\} = f^{-1}(\{1\})$$

which is closed because f is a continuous function. This because $\{1\}$ only 1 element, so it is closed and the preimage of a closed set by a continuous function is a closed set. We calim that C is convex. Indeed if $u_1, u_2 \in C$ and $t \in [0,1]$ then

$$f(tu_1 + (1-t)u_2) = tf(u_1) + (1-t)f(u_2) = t \cdot 1 + (1-t) \cdot 1 = 1$$

so that $tu_1 + (1-t)u_2$ is also in C. By 82 we have there exists $v_0 \in C$ so that $v_0 = \inf_{u \in C} ||u||$ and we define $v = \frac{v_0}{||v_0||^2}$ noting that $v_0 \neq 0$ by assumption.

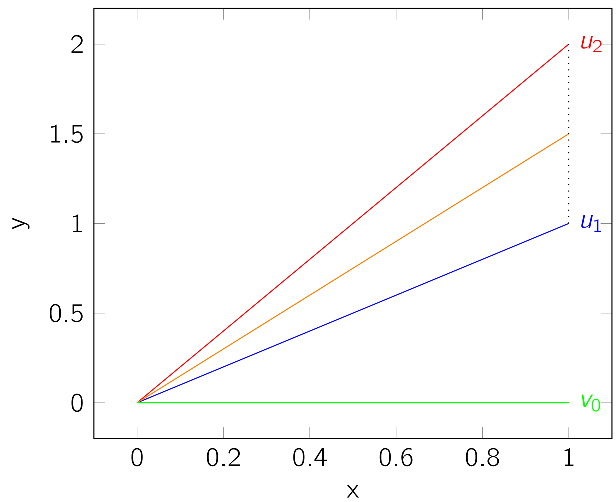


Figure 1: For example, intuitively you may see v_0 like so. The orange line corresponds to a particular $tu_1 + (1-t)u_2$

We claim that this is indeed the v we want. So we check if $f(u) = \langle u, v \rangle$ satifies the proposition.First we let

$$N = f^{-1}(\{0\}) = \{w \in H : f(w) = 0\}$$

be the nullspace of f. Again by the same logic as above, we know this is a closed set. Now notice that

$$\langle u, v \rangle = \frac{1}{||v_0||^2} \langle u, v_0 \rangle = \frac{1}{||v_0||^2} [\langle (u - f(u)v_0), v_0 \rangle + f(u)\langle v_0, v_0 \rangle]$$

By linearity of the inner product. But notice that

$$f(u - f(u)v_0) = f(u) - f(u)f(v_0) = 0$$

by linearity of f, noting that $f(u) \in \mathbb{R}$ so its a scalar that we can bring out. Therefore $u - f(u)v_0 \in N$. Hence by 84 knowing that N is a closed set we realize that

$$u = \underbrace{(u - f(u)v_0)}_{\in N} + \underbrace{f(u)v_0}_{\in N^\perp}$$

since we now know that $H = N \oplus N^\perp$. Since $f(u)$ is but a scalar constant we know that $v_0 \in N^\perp$. Therefore we realize the inner product in blue above is an product between a terms that are orthogonal to each other so it is zero by definition. Therefore we are left with

$$\langle u, v \rangle = f(u) \frac{\langle v_0, v_0 \rangle}{||v_0||^2} = f(u)$$

□

Lemma 90

all norms on a finite-dimensional vector space V are equivalent. This means that for any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V, there exist constants $C_1 > 0$ and $C_2 > 0$ such that:

$$C_1\|v\|_1 \leq \|v\|_2 \leq C_2\|v\|_1 \quad \text{for all } v \in V.$$

Proof. Simply define the norms on closed and bounded finite sets in finite dimensional spaces. Now we know that by heine borel this means this set is compact and because norms are continuous if you recall they must be bounded so say we pick $||.||_2$ then there exists $C_1, C_2 \in \mathbb{R}$ where

$$C_1 \leq \|\overline{x}\|_2 \leq C_2 \quad \text{for all } \overline{x} \in \mathbb{R}^n.$$

but also recall by homogenity of the norm so we can scale it. For example say $\overline{x} = \frac{x}{||x||_1}$ in which case we have

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1 \quad \text{for all } x \in \mathbb{R}^n.$$

Corollary 91

This for any linear map $T : V \to W$ where V, W are finite dimensional, T is continuous

Proof. Recall from above, it is the same as to prove bounded. Specifically, there is a constant $C \geq 0$ such that:

$$\|T(v)\|_W \leq C\|v\|_V \quad \text{for all } v \in V.$$

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for the finite-dimensional space V. Every vector $v \in V$ can be written as:

$$v = \sum_{i=1}^{n} c_i e_i,$$

where $c_i \in \mathbb{R}$ (or $\mathbb{C}$) are the coordinates of v in this basis.

Now, consider the map T. Since T is linear, we have:

$$T(v) = T\left(\sum_{i=1}^{n} c_i e_i\right) = \sum_{i=1}^{n} c_i T(e_i).$$

Let $w_i = T(e_i) \in W$. Then:

$$T(v) = \sum_{i=1}^{n} c_i w_i.$$

Using the triangle inequality for norms:

$$\|T(v)\|_W = \left\|\sum_{i=1}^{n} c_i w_i\right\|_W \leq \sum_{i=1}^{n} |c_i|\|w_i\|_W.$$

Define:

$$M = \max_{1 \leq i \leq n} \|w_i\|_W.$$

which exists as clearly $\{w_i\}_{1 \leq i \leq n}$ is a closed bounded set on a finite space so compact and that norms are continous so together means it is bounded. Then:

$$\|T(v)\|_W \leq M \sum_{i=1}^{n} |c_i|.$$

The above inequality involves the coefficients c_i, which depend on the choice of basis. To connect this to the norm $\|v\|_V$, we use the fact that norms in finite-dimensional spaces are equivalent.

Let $\|v\|_1$ denote the ℓ_1-norm on the coordinates, i.e.,

$$\|v\|_1 = \sum_{i=1}^{n} |c_i|.$$

Since V is finite-dimensional, there exist constants $C_1, C_2 > 0$ such that:

$$C_1 \|v\|_1 \leq \|v\|_V \leq C_2 \|v\|_1.$$

Thus, we can bound $\|T(v)\|_W$ in terms of the norm $\|v\|_V$:

$$\|T(v)\|_W \leq M \|v\|_1 \leq M \frac{\|v\|_V}{C_1}.$$

Hence, T is bounded, with:

$$\|T(v)\|_W \leq \frac{M}{C_1} \|v\|_V.$$

This shows that there is a constant $C = \frac{M}{C_1}$ such that:

$$\|T(v)\|_W \leq C \|v\|_V \quad \text{for all } v \in V.$$

Thus, T is bounded, and consequently, continuous.

8 the adjoint of a bounded linear operator on a hilbert space

Theorem 92 (Duality)

Let H be a hilbert space and let $A : H \to H$ be a bounded linear operator. Then there exists a unique bounded linear operator $A^* : H \to H$ known as the **adjoint** of A satisfying

$$\langle Au, v \rangle = \langle u, A^* v \rangle$$

for all $u, v \in H$. In addition we have that $||A^*|| = ||A||$

Proof. Like how we did for reisz representation theorem to show uniqueness consider 2 potential canditates A_1, A_2 where

$$\langle u, A_1^* v \rangle = \langle u, A_2^* v \rangle$$

then we know that

$$\langle u, (A_1^* v - A_2^* v) \rangle = 0$$

for all u, v but we may let $u = (A_1^* v - A_2^* v)$ in which case by definiteness of norms $A^*1v = A_2^* v$ for all v meaning that A_1^* and A_2^* must be the same operator to begin with.

To show existence, first fix $v \in H$ and define a map $f_v : H \to \mathbb{C}$ by $f_v(u) = \langle Au, v \rangle$. Clearly this a linear map in the argument u. We also claim our linear operator here is continuous. Indeed we can check that if $||u|| = 1$(recall how to proved in 18, any u can be normalized to 1 so it suffices to jsut consider that case) we have

$$|f_v(u)| = |\langle Au, v\rangle| \leq ||Au|| \cdot ||v||$$

by the cauchy Schwartz inequality. Therefore we have indeed $f_v \in H' = \mathcal{B}(H, \mathbb{C})$. Then by the **reisz representation theorem** we can hence find a unique elment which we denote A^*v of H satisfying

$$\langle Au, v\rangle = f_v(u) = \langle u, A^*v\rangle$$

Now we need to show that A^* is indeed a bounded linear operator. For linearity consider

$$\langle u, A^*(\lambda_1 v_1 + \lambda_2 v_2)\rangle = \langle Au\lambda_1 v_1 + \lambda_2 v_2\rangle = \overline{\lambda_1}\langle Au, v_1\rangle + \overline{\lambda_2}\langle Au, v_2\rangle$$

by conjugate linearity of the second variable. This is equal to

$$\overline{\lambda_1}\langle u, A^*v_1\rangle + \overline{\lambda_2}\langle u, A^*v_2\rangle = \langle u, \lambda_1 A^*v_1 + \lambda_2 A^*v_2\rangle$$

this shows that

$$\overline{\lambda_1}\langle A^*v_1\rangle + \overline{\lambda_2}\langle u, A^*v_2 = \lambda_1 A^*v_1 + \lambda_2 A^*v_2\rangle$$

which immediately shows linearity.

Next we show boundedness, in particular $||A^*|| = ||A||$ as tasked. Once again take the unit norm vector $||v|| = 1$. if $A^*v = 0$ then clearly $||A^*v|| \leq ||A||$ clearly. As for if $A^*v \neq 0$ then

$$||A^*v||^2 = \langle A^*v, A^*v\rangle = \langle AA^*v, v\rangle$$

by definition of adjoint and now by cauchy Schwartz inequality we have this to be

$$\leq ||AA^*v|| \cdot ||v|| = ||AA^*|| \leq ||A|| \cdot ||A^*v||$$

dividing by the nonzero constant(since $A^*v \neq 0$) $||A^*v||$ yields $||A^*v|| \leq ||A||$ as desired. Now taking the sup over all v with $v = 1$ yields $||A^*|| \leq ||A||$. To finish we need to show equality. For all $u, v \in H$ we have by definition

$$\langle A^*u, v\rangle = \overline{\langle v, A^*u} = \overline{\langle Av, u\rangle} = \langle u, Av\rangle$$

so the adjoint of the adjoint A is A itself. Therefore we may write

$$||(A^*)^*|| \leq ||A^*|| \Rightarrow ||A|| \leq ||A^*||$$

and putting the two inequalities above together yields equality as desired.

Definition 93
Let X be a metric space. A subset $K \subset X$ is **compact** if every sequence of elements in K has a subsequence converging to an element of K

You should recall this from `Rudin real analysis`(equivalent defintitions of compact sets using sequences). You should also recall from there too that

Theorem 94
A subset $K \subset \mathbb{R}$(also $\mathbb{R}^n$ and $\mathbb{C}^n$) is compact if and only if K is closed and bounded.

However this not true for any arbitrary metric spaces.

Example 95

Let H be an infinite dimensional Hilbert space. Then the closed ball

$$F = \{u \in H : ||u|| \leq 1\}$$

is a closed and bounded set but it not compact.

This is because we can let $\{e_n\}_{n=1}^{\infty}$ be a countably infinite orthonormal subset of H(it doesn't need to be a bsis) which we can find by Gram-Schmidt so tha all elements e_n are in F but see that(recall 57)

$$||e_n - e_k||^2 = \langle e_n - e_k, e_n - e_k\rangle = ||e_n||^2 + ||e_k||^2 + 2\,\mathrm{Re}\langle e_n, e_k\rangle = 2$$

however this means the distance any two elements of the sequence is 2 so there is no convergent subsequence

Remark 96. *This might be a little shocking but remember how we proved that bounded closed k cells are compact in rudin. We specifically made use of specific properties of euclidean spaces such as the use of L2 norm and that k cells are finite dimensional(since $\mathbb{R}^k$ not $\mathbb{R}^\infty$). Then we used archidemean property of real numbers. But here we are infinite dimensional*

Definition 97

Let H be a Hilbert space. A subset $K \subset H$ has **equi small tails** with respect to a countable orthonormal subset $\{e_n\}$ if for all $\varepsilon > 0$ there is some $n \geq N$ so that for all $v \in K$ we have

$$\sum_{k>N} |\langle v, e_k\rangle|^2 < \varepsilon^2$$

Hey! This looks very similar to the bessel inequality 58! Indeed, we know sequnce for any such v will converge by the bessel inequality. However the catch here is that we need to pick an N that works for all $v \in K$ at the same time. In other words we need uniform convergence.

Theorem 98

Let H be a hilbert space and let $\{v_n\}$ be a convergent sequence with $v_n \to v$. If $\{e_k\}$ is a countable orthonormal subset then $K = \{v_n : n \in \mathbb{N}\} \cup \{v\}$ is compact and K has equi small tails with respect to $\{e_k\}$.

Proof. Let $\varepsilon > 0$ since $v_n \to v$ there is some $M \in \mathbb{N}$ so that $n \geq M$ we have $||v_n - v|| < \frac{\varepsilon}{2}$. We choose N large enough so that for this fixed v

$$\sum |\langle v, e_k\rangle|^2 + \max_{1\leq n\leq M-1} \sum_{k>N} |\langle v_n, e_k\rangle|^2 < \frac{\varepsilon^2}{4}$$

Such an N exists due to the bessel inequality. We claim that this N uniformly bounds our tails. Indeed

$$\sum_{k>N} |\langle v, e_k\rangle| < \frac{\varepsilon^2}{4} < \varepsilon^2$$

and for all $1 \leq n \leq M-1$ we also have

$$\sum_{k>N} |\langle v_n, e_k\rangle|^2 < \frac{\varepsilon^2}{4} < \varepsilon^2$$

because the bessel inequalty applies to all $v_n \in H$. Moreover the maximum is taken over the a finite number of terms($1 \leq n \leq M-1$) for each summand so its bounded. Now we check the condition for $n \geq M$. Bessel inequality tells us that

$$\left(\sum_{k>N} |\langle v_n, e_k\rangle|\right)^{\frac{1}{2}} = \left(\sum_{k>N} |\langle v_n - v, e_k\rangle + \langle v, e_k\rangle|\right)^{\frac{1}{2}}$$

and this is the ℓ^2 norm of the sum of two sequences indexed by k so by the triangle inequality this is bounded by

$$\leq \left(\sum_{k>N} |\langle v, e_k\rangle|\right)^{\frac{1}{2}} + \left(\sum_{k>N} |\langle v_n - v, e_k\rangle|\right)^{\frac{1}{2}}$$

where the first term is at most $\frac{\varepsilon}{2}$ and the scond term is bounded by Bessel's inequality too by $||v_n - v||$. Since we chose N large enough so that norm is less than $\frac{\varepsilon}{2}$ we indeed have that this is bonuded by

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as desired $\square$

9 Compact subsets of a hilbert space and finite rank operators

Theorem 99

Let H be a separable Hilbert space and let $\{e_k\}$ be an orthonormal basis of H. Then a subset $K \subset H$ is compact if and only if K is closed,bounded and has equi small tails with respect to $\{e_k\}$.

Proof. For the forward direction suppose first that K is compact. We know by general metric space theory that K is then closed and compact and we'lll show that K has equi small tails wit respect to $\{e_k\}$ by contradiction. Suppose there exists some ε_0 such that for each natural N ther is some $u_N \in K$ such that

$$\sum_{k \geq N} |\langle u_N, e_k\rangle|^2 \geq \varepsilon_0^2$$

this gives use a sequence $\{u_n\}$ by picking such a u_N for every natural number N. Thus by assumption of compactness there is some subsequence $\{v_m\} = \{u_{n_m}\}$ and some $v \in K$ such that $v_m \to v$(Bolnanzo Weirstrass). But we also kno wthat for all $n \in \mathbb{N}, \sum_{k>n} |\langle v_n, e_k\rangle| \geq \varepsilon_0^2$ because $v_m = u_{n_m}$ is the nrth or late term of the orginal sequence so summing over $k > n_m$ is at most the value we get summing over $k > n$. That means the subset $\{v_n : n \in \mathbb{N}\} \cup \{v\}$ does not have equi small tails which is a contradiction of the previous theorem 98

Fact 100

From here on H will be a Hilbert space and we'll denote $\mathcal{B}(H, H)$ by $\mathcal{B}(H)$

Definition 101

A bounded linear operator $T \in \mathcal{B}(H)$ is a **finite rank operator** if the range of T(a subspace of H) is finite dimensional. We denote this as $T \in \mathcal{R}(H)$

Definition 102
An operator $K \in \mathcal{B}(H)$ is a **compact operator** if $\overline{K(\{u \in H : ||u|| \leq 1\})}$(the closure of the image of the unit ball under k) is compact.

10 Compact Operators and the Spectrum of Bounded Linear Operator s

Example 103
An example of a compact operator include $K : \ell^2 \to \ell^2$ sending $a = (a_1, a_2, a_3, \ldots)$ to $(\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \ldots)$

see that every element of a is smaller so if the norm of a satisfies $||a|| \leq 1$ so will $K(a)$.

Theorem 104
Let H be a separable Hilbert space...to be continued

Proposition 105
The space of invertible linear operators $GL(H) = \{T \in \mathcal{B}(H) : \text{T is invertible}\}$ is an open subset of $\mathcal{B}(H)$

Definition 106
Let $A \in \mathcal{B}(H)$ be a bounded linear operator. The **resolvant set** of A denoted $\operatorname{res}(A)$ is the set $\{\lambda \in \mathbb{C} : A - \lambda I \in GL(H)\}$ and the **spectrum** of A denoted $\operatorname{Spec}(A)$ is the complement $\mathbb{C}/\operatorname{res}(A)$

11 The spectrum of self adjoint operators and the eigenspaces of compact self adjoint operators

12 the spectral theorem for a compact self-adjoint operator

see for proof of spectral theorem for the langrange multiplier part use the fact that we only have 1 gradient so we can use KKT by LICQ(see wikipedia). In other words in this case KKT is a necessary condition(must be fulfilled if our solution is optimal).But from the other direction finding a valid langrange multiplier does not imply that it is it indeed a solution(notice the previous implication was not bidirectional). That is why we have the condition compact which ensures the existence of the minimum. With these the use of langrange multipliers is justified here.

References

[1] Ian Poon. *MIT 18.155-6 Differential Analysis I-II(2022) Workbook*. Dec. 2024. URL: https://github.com/extremefattypunch/MIT-18.155-6-Differential-Analysis-I-II-2022-.

[2] Ian Poon. *MIT18.112Functions of a Complex Variable(Wei Zhang) Workbook*. Sept. 2024. URL: https://github.com/extremefattypunch/MIT18.112Functions-of-a-complex-variable-Wei-Zhang-workbook.