Functions of a complex variable

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Selections of theorems from Andrew Lin's class notes from `MIT 18.112 Functions of complex variable`. Note you have yet to complete this course!(You are about 50% done)

Contents

1 Cauchy Integral formula

Definition 1

Prime counting function is denoted by $\pi(x)$, which counts the number of prime numbers p that are smaller than the specified $x \in \mathbb{R}^+$.

Definition 2

A set Ω is **connected** if we cannot write it as disjoint union $\Omega_1 \cup \Omega_2$ of open sets. Ω is **path-connected** if any two points $x, y \in \Omega$ can be connected by a path that lies inside Ω.(We'll call open connected sets **regions**)

Definition 3

Let Ω be an open subset of $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is **holomorphic at z** if the derivateive exists at z, and f is **holomorphic on Ω** if f is holomorphic at every $z \in \Omega$

Fact 4 (Differentiation of Vector Valued Functions)

We know that if $\lim_{x\to a} f = A$ then

$$\lim_{x\to a} \operatorname{Re} f = \operatorname{Re} A$$

and

$$\lim_{x\to a} \operatorname{Im} f = \operatorname{Im} A$$

Theorem 5 (Cauchy-Riemann equation)
Let Ω be an open set and let $f : \Omega \to \mathbb{C}$ be a function where $z = x + iy$ and $f(z) = u + iv$. Then f is holomorphic on Ω if and only if u and v are differentiable satisyfing

$$\tfrac{\partial u}{\partial x} = \tfrac{\partial v}{\partial y} \text{ and } \tfrac{\partial u}{\partial y} = -\tfrac{\partial v}{\partial x}$$

Proof. First consider

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)}{h}$$

We need to prove $f'(z)$ exists which is to say we must prove this limit exists. Just like how $\limsup = \liminf = \lim$ if the limit exists, it should not matter how h approaches zero, be it from positive or from negative. Therefore analagously it should not matter if h goes to zero from the complex or real side. Specifically that means if we had $h = ik$. So $k \to 0 \Rightarrow h \to 0$. Similarly if we had $h = k$. So $k \to 0 \Rightarrow h \to 0$. Then we must have the following for $f'(z)$ to exist

$$\begin{aligned} f'(z) &= \lim_{k\to 0} \frac{f(x+iy+k) - f(x+iy)}{k} && (1) \\ &= \lim_{k\to 0} \frac{f(x+iy+ik) - f(x+iy)}{ik} && (2) \end{aligned}$$

For (1) where $h = k$ this is equivalent to holding the complex part constant then differentiating with respect to real part only

$$\lim_{k\to 0} \frac{f(v(x+k,y)) - f(v(x,y))}{k}$$

hence

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

For (2) where $h = ik$ this is equivalent to holding the real part constant then differentiating with respect to complex part only

$$\lim_{k\to 0} \frac{f(v(x,y+k)) - f(v(x,y))}{k}$$

but comparing with (2) you see that you need to factorize out $\frac{1}{i}$ first, hence we have

$$\frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Because (1) and (2) are equal, their real and complex parts must be equal hence the conclusion follows

Corollary 6
A function f is holomorphic if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0$$

Proof. By multivariable chainrule we have

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} && (1) \\ \frac{\partial f}{\partial \overline{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} && (2) \end{aligned}$$

However we also have

$$x = \frac{z + \overline{z}}{2} \tag{3}$$

$$y = \frac{z - \overline{z}}{2i} \tag{4}$$

therefore

$$\frac{\partial x}{\partial z} = \frac{1}{2} \tag{5}$$

$$\frac{\partial x}{\partial \overline{z}} = \frac{1}{2} \tag{6}$$

$$\frac{\partial y}{\partial z} = \frac{1}{2i} \tag{7}$$

$$\frac{\partial y}{\partial \overline{z}} = -\frac{1}{2i} \tag{8}$$

Therefore sub (5) and (7) into (1) we have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)$$

Sub (6) and (8) into (2) we have

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) = 0$$

equals zero because recall from above we know

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$$

Theorem 7 (Hadamard formula)

The **radius of convergence** R such that

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} \in [0, \infty]$$

Then

$$\sum_{n\geq 0} a_n z^n$$

is convergent whenever $|z| < R$ and divergent whenever $|z| > R$

Proof. recall `rudin`. Every power series of f must have such a radius of convergence

Theorem 8

Define $f = \sum_{n\geq 0} a_n z^n$ for $z \in D_0(R)$(the open disk of convergence). Then f is holomorphic and

$$f'(z) = \sum_{n\geq 0} n a_n z^{n-1}$$

In addition the radius of convergence of $f'(z)$ is the same as that of $f(z)$

Proof. Consider a change of variables $n_0 = n - 1$ to get the desired form

$$f'(z) = \sum_{n_0 \geq 0} n_0 a_{n_0} z^{n_0 - 1} = \sum_{(n+1)\geq 0} (n+1) a_{n+1} z^n$$

so we have by root test

$$\limsup |(n+1)a_{n+1}|^{\frac{1}{n}} = \limsup |(n)a_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}}$$

Because recall from `rudin` that if a sequence converges to a x then so will all subsequences therefore translation invariant thus the first inequality. Also recall too that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1, n > 0$$

Hence why the second equality follows.Therefore both radius of convergences are the same and equal 1. f' is holomorphic because a power series can be differentiated term by term. See lemma below

Fact 9 (Quick Recap)

It can be proven by induction that

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}$$

Let $b = 1$ to get geometric series. This will be used in the lemma below

Lemma 10

a infinite power series is differenitable

Proof. Let for some N

$$f(z) = \sum a_n z^n = \lim_{n\to\infty} s_n(z) = s_n(z) + R_n(z)$$

where the partial sum $s_n(z)$ and its derivative

$$s_n(z) = \sum_{i=0}^{n} a_i z^i \qquad s_n'(z) = \sum_{i=1}^{n} i a_i z^{i-1}$$

because we know $(f+g)' = f' + g'$ so finite term and term differentiation makes sense. The remainder $R_n(z)$ is

$$R_n(z) = \sum_{i=n}^{\infty} a_i z^i$$

We now suppose that because we want to show such term by term differention works even as $n \to \infty$ so we define

$$f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \lim_{n\to\infty} s_n'(z)$$

So we must show $f'(z) = f_1(z)$ So we to show the below goes to zero as $z \to z_0$

$$\begin{aligned}
\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) &= \left(\frac{s_n(z) - s_n(z_0)}{z - z_0} - s_n'(z_0)\right) + (s_n'(z_0) - f_1(z_0)) + \left(\frac{R_n(z) - R_n(z_0)}{z - z_0}\right) && (1)\\
&= \varepsilon/3 + \varepsilon/3 + \sum_{k=n}^{\infty} a_k \left(z^{k-1} + z^{k-2}z_0 + \ldots + zz_0^{k-2} + z_0^{k-1}\right) && (2)\\
&= \varepsilon && (3)
\end{aligned}$$

For (2), the first term corresponds to our defintion so we can surely take $z \to z_0$ to get that. Same for the second term which is what we defined.For (3), consdier for $|z| < R$, on our radius of convergence we may find p such that $|z| < p < r < R$ by archimedian property. Knowing that $1/p > 1/r > 1/R = \limsup |a_n|^{\frac{1}{n}}$ Then recall by basic analysis there exists n_0 such that

$$|a_n| < \frac{1}{r^n}$$

for $n \geq n_0$. Moreover since $|z| \leq p$ we can simplify the 3rd time under such conditions to

$$\sum_{k=n}^{\infty} a_k \left(z^{k-1} + z^{k-2}z_0 + \ldots + zz_0^{k-2} + z_0^{k-1}\right) = \sum_{k=n}^{\infty} \frac{k}{p}\left(\frac{p}{r}\right)^k$$

This is convergent as by ratio test we have letting $x_k = k\frac{p^{k-1}}{r^k}$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{p}{r}\frac{k+1}{k} = \lim_{k\to\infty}\frac{p}{r}\left(1+\frac{1}{k}\right) = \frac{p}{r} < 1$$

Therefore the conclusion follows ☐

Corollary 11

Power series are infinitely differentiable

We can clearly repeat the above theorems. Consider

$$f'(z) = \sum na_nz^{n-1} = \sum b_nz^n$$

a_n is just some constant that is dependent on n so we can just replace it with some b_n and the power can be changed with changed to n because the limit is unchanged. Now clearly the radius of convergence is still the same. As we have proven the radius of convergence for $f'(z)$ is still R and that $\limsup |nb_n|^{\frac{1}{n}} = \limsup |b_n|^{\frac{1}{n}}$ once again. Then everything else follows once more so that we can prove $f''(x)$. By induction this works for all degrees of derivatives.

Definition 12

A function on an open set $\Omega \subseteq \mathbb{C}$ is called **analytic** if for every $z_0 \in \Omega$ there exists a power series $g(z) = \sum_{|n|\geq 0} a_n(z-z_0)^n$ with radius of convergence $R > 0$ such that $f(z) = g(z)$ for all $z \in D_{z_0}(R)$

☐

Recall for real valued functions, the existence of its derivative $f'(x)$ does not necessarily imply that its higher order derivatives like $f''(x)$ should exist. For example

$$f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}, \quad f'(0) = \text{undefined}$$

However for holomorphic functions(complex valued functions g whose derivative g' exists), then its higher order derivatives automatically exists! That is to say holomorphic function and analytic function are *bijective* conditions. We will prove in the coming sections

Definition 13

Fix a function $f : \Omega \to \mathbb{C}$(not necessarily holomorphic). A function $F : \Omega \to \mathbb{C}$ is called a **primitive** of f if it is holomorphic and $F' = f$

Definition 14

A **parametrized smooth curve** γ on Ω is defined by the function $z : [a, b] \to \Omega \subseteq \mathbb{C}$ were $a, b \in \mathbb{R}$(we have a finite interval), such that z is **smooth**: both the real and imaginary parts are infinitely differentiable functions and $z'(t) \neq 0$ for all $t \in [a, b]$. A curve is **closed** if $z(a) = z(b)$

Definition 15

A **piecewise smooth curve** is a cure where we can divide $[a, b]$ into a finite number of pieces so that z is smooth on each piece

Definition 16

The integral along a curve is defined as

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$$

where γ is being represented by the function $z : [a, b] \to \Omega$

Proposition 17

If a function f has a primitive F(is holomorphic meaning its derivative exists and $F' = f$)

$$\int_\gamma f(z)dz = F(z(b)) - f(z(a))$$

Proof. Just a literal restatement of fundamental theorem of calculus

Corollary 18

If a primitive F exists for a function f then

$$\int_\gamma f = 0$$

for any closed curve γ

Corollary 19

Let Ω be connected if $f' = 0$ then f is constant.

Proof. By definition f is a primitive of f' so

$$\int_\gamma f'(z)dz = f(B) - f(A) = 0$$

Definition 20

A curve is **simple** if thhere are no self-intersections: z is injective on the open interval (a, b). A **simple closed curve** is closed an simple

Definition 21

Let the **lenght** of a parameterized curve γ on $[a, b]$ be defined as

$$\text{lenghth}(\gamma) = \int_b^a |z'(t)|\, dt$$

just recall `multivariable calculus`

Proposition 22 (ML inequality)

$$\left| \int_\gamma f(z)dz \right| \leq \sup_{z\in\gamma} |f(z)| \cdot \text{lenght}(\gamma)$$

Proof. The LHS

$$\left| \int_\gamma f(x)dz \right| = \left| \int_a^b f(z(t))z'dt \right| \leq \left| \int_a^b |\sup(f(z))|\, z'(t)dt \right|$$

where the inequality follows by monotonicty. Since $|\sup f(z(t))|$ is just a constant we can pull it out of the integral and out relation follows.

Example 23

Compute

$$\int_0^{2\pi} \cos^n x dx$$

Solution. Instead of doing this by *integration by parts* we can integrate using a complex variable. Use the parameterization $z = e^{ix}$. Notice that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = \frac{z}{2} + \frac{1}{2z}$$

So if we want $\cos^n x$ we can consider where C is the unit circle

$$\int_C \left(\frac{z}{2} + \frac{1}{2z}\right)^n \frac{dz}{iz} = \frac{1}{2^n i} \int_C \left(z + \frac{1}{z}\right)^n \frac{dz}{z} = \frac{1}{2^n} \int_C \left(z + \frac{1}{z}\right)^n \frac{dz}{z} = \frac{1}{2^n} \int_C \sum_{k=0}^n \binom{n}{k} z^{n-k} \left(\frac{1}{z}\right)^k \frac{dz}{z}$$

which is equal using binomial expansion,

$$\frac{1}{2^n} \int_C \sum_{k=0}^n \binom{n}{k} z^{n-2k-1} dz$$

where we have used the change of variable

$$z = e^{ix} \quad \Rightarrow \quad dz = ie^{ix} dx$$

However recall that z^n is periodic

$$\int_C z^m dz = \int_0^{2\pi} (e^{i\theta})^m i e^{i\theta} d\theta = \int_0^{2\pi} (i e^{i\theta(m+1)}) d\theta$$

reverting back to our original variable observe that

$$\int_0^{2\pi} i e^{i(m+1)x} dx = \begin{cases} \int_0^{2\pi} 1 dx = 2\pi i & m = -1 \\ \frac{e^{i(m+1)2\pi} - e^{in0}}{i(m+1)} = 0 & m \neq -1 \end{cases}$$

Therefore $n-2k=0$ must be satisfied for $n-2k-1=-1$. This would be impossible if n is odd and for even the only case where this is true is when $k=\frac{n}{2}$.Therefore the only non-zero term in our binomial expansion is

$$\frac{1}{2^n i}\int_C \binom{n}{n/2} z^{-1} dz = \binom{n}{n/2}\frac{2\pi i}{2^n i} = \binom{n}{n/2}\frac{\pi}{2^{n-1}}$$

$\square$

Problem 24 (Gaussian Integral)

Compute

$$\int_{-\infty}^{\infty} e^{-ax^2} dx$$

Solution. Let

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx$$

Then

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-ax^2}\right)$$

switching to polar coordinates we have

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-ar^2}\cdot r dr d\theta$$

since

$$\det Da = \det\begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{bmatrix} = r$$

Therefore we have by change of variables theorem

$$\int_0^{2\pi}\int_0^{\infty} e^{-ar^2}\cdot r dr d\theta = \int_0^{2\pi}\left(\frac{1}{2a}\right) d\theta = \frac{\pi}{a}$$

Therefore taking square roots we have

$$\sqrt{\frac{\pi}{a}}$$

$\square$

Corollary 25

the function $\frac{1}{z}$ does not have a primitive on Ω

Proof. .Recall $\int_0^{2\pi} z^{-1} = 2i\pi \neq 0$, so there exists a closed curve where this integral is not zero. Therefore our corollary is simply the contrapositive of the fact that primitive implies $\oint_\gamma f = 0$ for all closed curves γ . $\square$

We have stated the implications of f having a primitive. So this begs the question, when does f have a primitive? We make a new claim, that if $f : \Omega \to \mathbb{C}$ is a holomorphic function and Ω is "nice" then it has a primitive and consequently we have $\int_\gamma f dz = 0$ for any closed curve γ

Theorem 26 (Goursat simple case)

If f is holomorphic and $\gamma = T$ is any triangle such that the interior of T is contained in Ω(so our domain can't have a hole) then

$$\int_T f(z)dz = 0$$

Proof. consider that any triangle by construction can be contained in any circle in this manner. You might want to study `topology` in future to have more insight into this

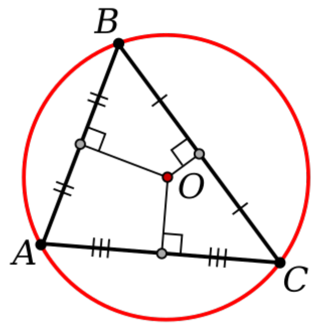

Figure 1: inscribed circle

Hence there exists a close curve to connect any 2 points on triangle vertexes. This will make sense after we study **Cauchy integral formula**. For now we just consider the following. Construct a sequence of triangles by bisecting each side of the triangle and connecting the midpoints(like above but connecting with each other not to some center point among the 3 midpoints).

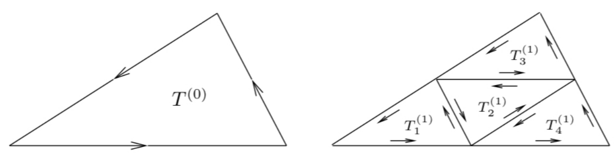

Figure 2: Goursat Triangle

$$\int_{T^{(0)}} f(z)dz = \int_{T_1^{(1)}} f(z)dz + \int_{T_2^{(1)}} f(z)dz + \int_{T_3^{(1)}} f(z)dz + \int_{T_4^{(1)}} f(z)dz$$

Therefore by triangle inequality and taking the integral that gives the greates magtidue among the 4 terms on the RHS we have

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq 4 \sup_{j\in\{1,2,3,4\}} \left|\int_{T_j^{(1)}} f(z)dz\right|$$

Continuing this process we have for any *arbitrary* n

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)dz\right|$$

Essentially we decomposed the integral recursively into sub integrals of the largest magnitudes. Moreover we know that by construction the perimeter of every any triangle is half of that of the original. We can clearly see this by considering the $T_1^{(1)}, T_3^{(1)}, T_4^{(1)}$ above. See that each of them have 2 sides that scaled by eactly half but they are all similar triangles with respect to $T^{(0)}$ hence their perimeters are halved. Given the middle triangle T_3^1 shares 1 side each from $T_1^{(1)}, T_3^{(1)}, T_4^{(1)}$, it too will also have its perimiter halved. Therefore we have the relationship

$$p^{(n)} = 2^{-n}p^{(0)} \quad \text{and} \quad d^{(n)} = 2^{-n}d^{(0)}$$

Where $p^{(i)}, p^{(i)}$ denotes the perimeter and diameter(height) of $T^{(i)}$. Since $f(z)$ is holomorphic at z_0 we can express it as

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \phi(z)(z - z_0)$$

where $\phi(z) \to 0$ as $z \to z_0$. So we have

$$\int_{T^{(n)}} f(z)dz = \int_{T^n} \underbrace{(f(z_0) + f'(z_0)(z - z_0))}_{=0} + \int_{T^{(n)}} \phi(z)(z - z_0)dz$$

Because we know that the constant $f(z_0)$ and $f'(z_0)$ has a primitive by 19 . Then by the **ML inequality** we have

$$\left| \int_{T^{(n)}} f(z)dz \right| \leq \varepsilon_n d^n p^n$$

Where is d clearly the sup of $(z - z_0)$ while p is the sup of the total length integrated and $\varepsilon_n = \sup_{z \in T^{(n)}} |\phi(z)| \to 0$ as $n \to \infty$ since $z \to z_0$ as our triangles get smaller. Then by our previous inequality we have

$$\left| \int_{T^{(0)}} f(z)dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)dz \right| \leq e_n d^{(0)} p^{(0)}$$

it is clear that the RHS goes to zero as $n \to \infty$. Equality follows because magntidudes cant be less than zero. Then finally

$$\left| \int_{T^{(0)}} f(z)dz \right| = 0 \quad \to \quad \int_{T^{(0)}} f(z)dz = 0$$

by monotoncity. □

Corollary 27

This applies when $\gamma = T$ is a rectangle

Proof. Consider that any rectangles can be composed of 2 triangles. We will see an example below.

Theorem 28

Any function f that is holomorphic on an open disk has a primitive

Proof. Define

$$F(z) = \int_\gamma f(z)dz$$

on an open disk centred at z_0.

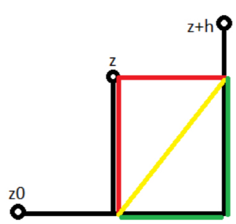

Figure 3: Disk Contour

where γ is unique path joining z_0 to z using only horizontal and vertical paths. Our objective is to show $F(z)$ is holomorphic and $F(z + h) - F(z)$ is related by

$$F(z + h) - F(z) = f(z)h + \phi(z, h)h$$

where $\phi(z,h) \to 0$ as $h \to 0$ so that we have $F'(z) = f(z)$ by definition. First consider

$$\int_{\text{red+green CCW}} f(z)dz = \int_{\text{red+yellow CCW}} f(z)dz + \int_{\text{yellow+green CCW}} f(z)dz = 0 + 0 = 0$$

where the 2nd equality follows by **Goursat Theorem** for triangles from earlier. Therefore the integral over the red and green paths are the same since this implies

$$\int_{\text{red+yellow CCW}} f(z)dz = \int_{\text{yellow+green CW}} f(z)dz$$

Now that we proven that F is well defined we can let η to be the path between z and $z+h$ using only vertical and horizontal components.

$$F(z+h) - F(z) = \int_{\eta} f(w)dw$$

Since f is holomorphic like previous we can make the linear approximation

$$f(w) = f(z) + f'(z)(w-z) + \phi(w)(w-z)$$

where $\phi \to 0$ as $w - z \to 0$.So we have

$$F(z+h) - F(z) = \int_{\eta} \left(f(z) + f'(z)(w-z) + \phi(w)(w-z)\right) dw$$

but doing simple calculations we see that the 2nd term on the RHS integrates to $f'(z)\frac{h^2}{2}$ while the last term on the RHS integrates to an even smaller term since its the remainder function. Therefore we do indeed have

$$F(z+h) - F(z) = f(z)h + \phi(z,h)h$$

with the necessary conditions for the theorem to hold □

Corollary 29

If f is holomorphic on an open disk Ω then $\int_{\gamma} f(z)dz = 0$ for any closed curve $\gamma \in \Omega$.

This is a direct implication from the fact that if f has a primitive, then by the fundamental theorem of calculus $\int_{\gamma} f = 0$ for any closed curve $\gamma \in \Omega$ as mentioned earlier □

Note we did not even require the region where f is holomorphic, Ω to be a disk. As you can observe all we required is a well defined closed area in which you can join connect any pair of interior points by horizontal and vertical lines. In other words we just need some closed connected space(doesn't have to be simply connected/no holes). In which any f defined inside this space has a primitive and any closed curve in the interior of this space will integrate to zero.We will consider an application of this below.

Theorem 30 (Cauchy Integral formula)

Let D be a closed disk. Let C be a circle oriented counterclockwise and forms the boundary of D. Given a holomorphic function $f : D \to \mathbb{C}$ for any $\zeta \in \operatorname{Int} D$

$$f(\zeta) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \zeta} dz$$

where we integrate once counterclockwise around the circle C enclosing D

Proof. Recall from earlier we know that

$$\frac{1}{2\pi i}\int_C \frac{dz}{z} = 1$$

Now consider an abirtuary small subdisk C_ε(oriented clockwise) that contains ζ(which should exist since Int D is open). Then notice that

$$0 = \int_{\gamma_1} \frac{f(z)}{z-\zeta}dz + \int_{\gamma_2} \frac{f(z)}{z-\zeta}dz = \int_C \frac{f(z)}{z-\zeta}dz - \int_{C_\varepsilon} \frac{f(z)}{z-\zeta}dz$$

where we integrate over C_ε clockwise and the equality with zero follows from the previous theorem that a holomorphic function on an open disk has a primitive. This means for any closed curve like γ_1 and γ_2 their integrals should be zero. Therefore this implies

$$\int_{C_\varepsilon} \frac{f(z)}{z-\zeta}dz = \int_C \frac{f(z)}{z-\zeta}dz$$

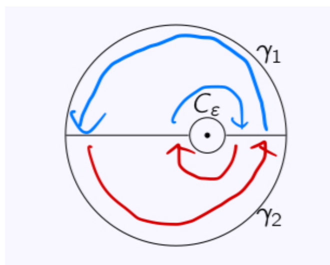

Figure 4: Toy Contour

As usual because f is holomorphic on D there exists

$$f(z) = f(\zeta) + (z-\zeta)g(z)$$

where $g(z)$ is continuous(recall differentiability implies continuity) and bounded(since on small disk). Hence

$$\int_{C_\varepsilon} \frac{f(z)}{z-\zeta}dz = f(\zeta)\int_{C_\varepsilon} \frac{1}{z-\zeta}dz + \int_{C_\varepsilon} g(z)dz$$

the second term disappear as $\varepsilon \to 0$(which we can do because ε is arbitrary. Notice that by change of variables we have

$$u = z-\zeta \quad \to \quad du = dz \quad \to \quad \int_{C_\varepsilon} \frac{1}{u}dz = 2i\pi$$

Therefore

$$f(\zeta)\int_{C_\varepsilon} \frac{1}{z-\zeta}dz = f(\zeta)2\pi i$$

and the theorem clearly follows upon comparison we see that

$$\int_{C_\varepsilon} \frac{f(z)}{z-\zeta}dz = \int_C \frac{f(z)}{z-\zeta}dz = 2\pi i f(\zeta)$$

and dividing both sides by 2π yields the result $\square$

Let us prove something even more remarkable: we can recover all higher order derivatives from the value of f on a single circle!

Theorem 31 (Regularity Theorem)
If f is holomorphic then f is *infinitely* complex differentiable(in other words $f', f'', f''' \ldots$ are all also holomorphic). In addition for any $z \in D$ and $n \geq 0$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

Proof. This will follow after considering the following theorem.

Theorem 32
All holomorphic functions are analytic. That is recalling the general form of a power series(in fact it is also the same form as taylor series)

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = \sum_{n \geq 0} a_n (z - z_0)^n$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$ and z_0 is the center of the radius of convergence for all $z \in \Omega$

Proof. Let D be the open disk of radius R centered at z_0. By **Cauchy theorem** for all $z \in D$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

We know the power series expansion

$$\frac{1}{1 - z} = \sum_{n \geq 0} z^n$$

holds for all $|z| < 1$ Therefore in order to try to get a form similar to that of the power series we

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n \geq 0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n$$

for all $|z| < |\zeta|$ which is true because ζ is the boundary of the disk of radius R while z is in the interior. Notice we have defined our radius of convergence. Therefore we may write

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_C \sum_{n \geq 0}^{\infty} f(\zeta) \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta = \frac{1}{2\pi i} \sum_{n \geq 0}^{\infty} (z - z_0)^n \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

where we can swap the order of the sum and integral as the series converges *uniformly* in C(we will prove below).Therefore f can represented as an infinite power series with a radius of convergence.Hence recalling from earlier discussions on properties of such power series,we know it is infinitely differentiable. Therefore comparing coefficients we see that

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

so

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

Proposition 33

Let C be compact. If

$$f(\zeta) = \sum_{n \geq 0}^{\infty} A_n(\zeta)$$

converges *uniformly*, meaning that for any ε, there exists a fixed N such that for all $\zeta \in C$ we have

$$\left| \sum_{n=M}^{M} A_n(\zeta) \right| < \varepsilon$$

then we can swap the sum and integral

$$\int_C \sum_{n \geq 0}^{\infty} A_n(\zeta) = \sum_{n \geq 0}^{\infty} \int_C A_n(\zeta)$$

Proof. Consider

$$\left| \sum_{n=N}^{M} \frac{z^n}{\zeta^{n+1}} f(\zeta) \right| \leq \sum_{n=N}^{M} \frac{|z|^n}{|\zeta|^{n+1}} f(\zeta) \leq \sum_{n=N}^{M} \frac{|z|^n}{R^{n+1}} |\sup_C f(\zeta)|$$

the sup exists as C is compact f holomorphic(differentiability implies continuity). Then it is obviou we have uniform convergence by definition of sup, we have an N that works for all x for each ε. □

To appreaciate the power of regularity theorem let us contrast with real valued functions. Recall we once remarked that a differentiable real valued function is not necessarily smooth unlike for complex functions. Now we also illustrate via the following example that smooth real valued functions are not necessarily analytic unlike complex functions. See 39 for a summary of such comparisons.

Example 34 (Smooth non-analytic real valued function)

Consider the real valued function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

although continuous and infinitely differentiable but all derivatives are equal to zero so f not equal to its power series centered $x = 0$. See this from

$$a_n = f^{(n)}(0)/n! = 0$$

for all n so for the power series centered at zero we have

$$\text{power series:} f(x) = \sum a_n x^n = 0$$

However this is not true because for the same $x \neq 0$ argument

$$f(x) \neq 0 \quad \text{but} \quad \text{power series:} f(x) = 0$$

so they are not equal

Definition 35

A holomorphic function f is **entire** if it is holomorphic on all of $\mathbb{C}$.

Theorem 36 (Liouville)

If f is entire and bounded then it is constant

Proof. The goal is clearly to get $f'(z) = 0$ for all z. By **regularity theorem** we have

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

Since f is bounded we have

$$|f'(z)| \leq \frac{1}{2\pi} \int_C \left| \frac{f(\zeta)}{(\zeta - z)^2} \right|$$

and if C is a circle centered at z with radius R(this is a valid assumption, afterall when entire any point could be the center of any arbituary disk) by **ML inequality** this can be bounded by

$$\frac{1}{2\pi} \sup_{\zeta \in C} |f(\zeta)| \frac{1}{R^2} 2\pi R = \frac{\sup_{\zeta \in C} |f(\zeta)|}{R}$$

since we want entire we will take $R \to \infty$ to the circle "covers" the whole of $\mathbb{C}$. In which case clearly $|f'(x)| \leq 0$ and equality follows obviously.

Theorem 37 (Cauchy Inequality)

For all non-negative integers n we have

$$\left| f^{(n)}(z) \right| \leq \sup_{\zeta \in C} \frac{n!}{2\pi} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} \cdot 2\pi R = \frac{n!}{R^n} \sup_{\zeta \in C} |f(\zeta)|$$

for where C is a circle of radius R centered at z

Proof. Follows very easily by applying ML inequality to regularity theorem. In fact we have already derived and used it in the previous theorem.

Corollary 38

If $f : \mathbb{C} \to \mathbb{C}$ is entire then the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

is convergent for all $z \in \mathbb{C}$

Proof. This is a direct application of the above theorems now centering the disk at 0 instead. Since entire, in fact you can center at anywhere and this will still hold in which case will get the same form as we did in when proving liouville and Cauchy inequality.

$$\left| f^{(n)}(z) \right| \leq \frac{n!}{R^n} \sup_{\zeta \in C} |f(\zeta)|, R \to \infty$$

where z and 0 in our corollary are analgous to ζ and z that we have here. Anyway back to proving, it is obvious that in the corollary, the only non-zero term will be the constant term as all derivatives are zero. □

Fact 39

As a summary thus far we have proven in the following order

$$
\begin{tikzcd}
f \text{ holomorphic on } \Omega \arrow[d, "F'{=}f\text{,F holomorphic via Goursat Triangle}"] \\
f \text{ has a primitive on } \Omega \arrow[d, "\text{fundamental theorem of calculus}"] \\
\oint_\gamma f = 0, \gamma \in \Omega \arrow[d, "\text{fails for real valued functions since } e^{2x\pi}\notin\text{periodic}"] \\
\text{Cauchy integral formula} \arrow[d, "\text{re-express as geometric series show analytic}"] \\
\text{choice of C} \rightarrow \text{radius of convergence for power series} \arrow[d, "\text{recall properties of power series from earlier: smoothness implied}"] \\
\text{regularity theorem: formula for computing } f^{(n)}(x) \arrow[d, "\text{cauchy inequality via ML inequality}"] \arrow[uuuuu, bend left, "C^\infty \supset C^1"] \\
|f^{(n)}(z)| \text{bounded on disk centered at z and } \rightarrow 0 \text{ if entire} \arrow[d, "\text{Liouville Theorem showing only constant term} \neq 0"] \\
\text{convergence power series if f entire}
\end{tikzcd}
$$

Now we move on to some applications

Theorem 40 (Fundmanetal Theorem of Algebra)

Let $P(z) = a_0 + a_1 z + \dots a_n z^n$ be a polynomial of degree n(so all $a_l \in \mathbb{C}$ and $a_n \neq 0$). Then P has n roots, counting multiplicity. This means we can write

$$P(z) = a_n \prod_{i=1}^{n} (z - z_i)$$

for some (not necssarily distinct) $z_i \in \mathbb{C}$

Proof. It suffices to show that $P(z)$ has where $n \geq 1$ has at least 1 root in which case we can write by long division

$$P = (z - z_1)Q(z)$$

where $Q(z)$ is a polynomial of degree $n-1$ and proceed by induction. First assume for the sake of contradiction $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then can let $f(z) = \frac{1}{P(z)}$ and know that it is well defined. Because $P(z)$ takes the form of a power series, it is holomorphic. Therefore so is $f(z)$ as know from chain rule.

$$\frac{d}{dz}\frac{1}{P(z)} = -\frac{1}{(P(z))^2}P'(z)$$

which is clearly well defined since again $P(z) \neq 0, \forall z \in \mathbb{C}$ and $P'(z)$ exists since holomorphic. Because this is true for all $z \in \mathbb{C}$ which its domain, $f(z)$ is entire. However by the previous theorem that implies f is constant but that is a contradiction clearly.

Theorem 41
Let Ω be a connected open region and say that $f : \Omega \to \mathbb{C}$ is holomorphic. Say that we have a sequence of points $\{z_n\}$ such that $f(z_n) = 0$ and $\lim_n z_n = z \in \Omega$. Then $f = 0$ everywhere in Ω(that is including $f(z)$ of course)

In other words if the set of roots $f^{-1}(0)$ contains a limit point in Ω then f is identically zero and the set $f^{-1}(0)$ is closed. Because for any arbiturary sequence $\{z_n\} \subset f^{-1}(0)$ we will know $\lim_{n\to\infty} z_n = z \in f^{-1}(0)$.

Proof. First let $\lim_{n\to\infty} z_n = z_\infty, f(z_n) = 0$ and assume for the sake of contradiction $f(z_\infty) \neq 0$. Without loss of generality since we only need 1 case to prove contradiction, let $z_\infty = 0$. Then expressing $f(z)$ as a power series centered at $z_\infty = 0$ we have

$$f(z) = \sum_{n\geq 0}^{\infty} a_n z^n$$

However this implies there exists a neighbourhood $z \neq 0$ around z_∞(as defined by the radius of convergence) in which $f(z) \neq 0$. In other words there exists a neighbourhood of z_∞ that does not include z_n Hence this immediately contradicts the definition of a limit point. Therefore we know $f^{-1}(0)$ is a closed set. Now we need to prove $f = 0$ everywhere on Ω. That is to say let $f^{-1}(0) = U$ then $U = \Omega$. Because Ω is connected it cannot be decomposed into disjoint sets $U \cup (\Omega - U)$ unless one of those is empty. Given that U is non-empty as it contains $\{z_n\}$ by assumption and z too from the previous part, we know that $f = 0$ everywhere in Ω

Corollary 42
If f, g are holomorphic functions on a connected open region Ω and $f = g$ on any disk D then $f = g$ on Ω

Proof. Let $h = f - g : \Omega \to \mathbb{C}$ which is clearly holomorphic. Then for any disk D, there exists certainly sequence $\{z_n\} \in D$ where $h(z_n) = 0$. Therefore due to connectivity of open region Ω we have $h = 0$ on Ω. So the corollary follows.

Definition 43
Lef $f : \Omega \to \mathbb{C}$ and $F : \Omega' \to \mathbb{C}$ be holomorphic, where $\Omega \subseteq \Omega'$. If $f = F$ on all of Ω then F is an **analytic continuation** of f.

Example 44
We know that

$$f(z) = 1 + z + z^2 + \ldots + z^n$$

is convergent for all $|z| < 1$ by **hammard formula**. We know that this agrees with $F(z) = \frac{1}{1-z}$ which is holomorphic on $\mathbb{C}/\{1\}$. Therefore since $\{z \in \mathbb{C}; |z| < 1\} \subseteq \mathbb{C}/\{1\}$ and $f = F$ on all of $\{z \in \mathbb{C}; |z| < 1\}$ we can conclude that F is an **analytic continuitation** of f.

Theorem 45 (Morera)
If complex valed function f is continuous on an open disk D and $\int_T f(z)dz = 0$ for any triangle $T \subseteq D$ then f is holomorphic on D.

Proof. Just refer to 39.

Theorem 46

If $\{f_n\}$ is a sequence of holomorphic functions on Ω such that $f_n \to f$ uniformly on every compact subset of Ω then the limit $f(z) = \lim_{n\to\infty} f_n(z)$ is holomorphic on Ω

Proof. By **Morera** theorem it suffices to show $f(z)$ is holomorphic in a neighbourhood Naround every satisfies

$$\int_T f(z)dz = 0$$

for all $T \subset N$ then applying uniform convergence we have

$$\left| \int_T (f - f_n)(z)dz \right| \le \int_T |(f - f_n)(z)| \le \varepsilon \cdot \text{lenght}(T)$$

then taking $n \to \infty$ gets the desired conclusion

Definition 47 (Riemann zeta function)

The **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is defined for $s \in \Omega = \{s \in C : \text{Re}(s) > 1\}$

Let $s = x + iy$ and consider that

$$\left| \frac{1}{n^x} \cdot \frac{1}{n^{iy}} \right| = \left| \frac{1}{n^x} \right| \left| \frac{1}{e^{i\ln(n)y}} \right| = \left| \frac{1}{n^x} \right|$$

so therefore convergence of the Reinmann zeta function only depends on the real part. The condition $x > 1$ follows from recall `rudin`.

Theorem 48

Let holomorphic functions $f_n \to f$ converge uniformly on compact set C. Then $f_n' \to f'$ also converges uniformly on C

Proof. Just use cauchy integral formula

$$f_n' = \int_C \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta$$

then apply uniform convergence which allows you to move the limit in. Therefore

$$f = \lim_{n\to\infty} f_n' = \int_C \lim_{n\to\infty} \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta = \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

The choice of $n$is clearly independent of x here so we have uniform convergence $f_n' \to f'$ as desired.

2 Zeroes and Poles

Lemma 49

Let $f \neq 0$ be holomorphic on Ω and let $z_0 \in \Omega$. Then there exists an open disk D centered around z_0 and a unique function g and an integer $n \geq 0$ such that for all $z \in D$

$$f(z) = (z - z_0)^n \cdot g(z)$$

where g is holomorphic and $g(z) \neq 0$ on the whole disk

Remark 50. *$f \neq 0$ means f is not the zero function/vanishes identically on D. It does not mean $f(z) \neq 0$ for some $z \in D$ is not allowed.*

Proof. Since f holomorphic on Ω from our previous theorems we certainly can find an open disk D centered around $z_0 \in \Omega$ which defines the radius of convergence for our power series representation of f on D

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = (z - z_0)^{n_0} \left(\sum_{n=0}^{\infty} a_{(n_0+n)} (z - z_0)^n \right)$$

where n_0 is the smallest n where $a_n \neq 0$ which should exists because $f \neq 0$. The term in the latter bracket on the RHS is g. Clearly $g(z) \neq 0$ on all of D since

$$\left(\sum_{n=0}^{\infty} a_{(n_0+n)} (z - z_0)^n \right) = a_{n_0} + a_{n_0+1}(z - z_0)^1 + a_{n_0+2}(z - z_0)^2 \ldots$$

and again $a_n \neq 0$. It is clearly holomorphic too as it is analytic. As for uniquess suppose we have

$$(z - z_0)^{n_0} g_0(z) = (z - z_0)^{n_1} g_1(z)$$

where $n_0 > n_1$ and so $g_1 \neq g_0$ however that implies

$$g_0(z) = (z - z_0)^{n_0 - n_1} g_1(z)$$

but then $g_0(z_0) = 0$ which is a contradiction.

Definition 51

The **multiplicity** of a root z_0 is the value of n_0 in the unique representation

$$P(z) = (z - z_0)^{n_0} Q(z)$$

where $Q(z_0) \neq 0$

Definition 52

A **deleted neighbourhood** of z_0 is defined as

$$\{z : 0 < |z - z_0| < r\}$$

which is some disk centered around z_0 minus z_0 with radius $r > 0$

Definition 53

If $f(z)$ is holomorphic on a deleted neighbourhood of z_0. then we call z_0 an **isolated singularity** of f. Such a singularity is called a **pole** if $\frac{1}{f}$ is holomorphic in a neighbourhood z_0. The **order** of the pole is the multiplicity of z_0 at $\frac{1}{f}$

Theorem 54

If f has pole of order n at z_0 then we write

$$f(z) = (z - z_0)^{-n} h(z)$$

where h is holomorphic near z_0 and $h(z_0) \neq 0$

Proof. By definition $\frac{1}{f(z)}$ is holomorphic in a neighbourhood of z_0. Then from the previous theorem there exists

$$\frac{1}{f(z)} = (z - z_0)^n g(z)$$

so the result follows with $h(z) = \frac{1}{g(z)}$ which is well defined as $g(z) \neq 0$ on all of the Disk centered at z_0

This allows us to write the more explicit form know as the **laurant series** as follows

Corollary 55

If f has a pole of order n at z_0 then

$$f(z) = \left(\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z - z_0} \right) + G(z)$$

where G is holomorphic near z_0

proof By the previous theorem we can directly write knowing that clearly h is of order $n - 1$ and holomorphic

$$f(z) = (-z_o)^{-n} h(z) = (z - z_0)^{-n}(A_0 + A_1(z - z_0) + \ldots) = \frac{A_0}{(z - z_0)^n} + \ldots + \frac{A_{n-1}}{z - z_0} + (A_n + \ldots)$$

but is power series

Definition 56

In the above expression for f,

$$\left(\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z - z_0} \right)$$

is known as the **principle part** of f while a_{-1} is known as the **residue** of f at z_0 denoted as $\text{res}_{z_0}(f)$. If f has a **simple pole** at z_0 we can write

$$f = \frac{a_{-1}}{z - z_0} + G(z)$$

Example 57

If

$$f(z) = \left(\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z-z_0}\right)$$

To calculate $\text{res}_{z_0}(f)$ we can consider

$$(z-z_0)^n f = a_{-n} + a_{-n+1}(z-z_0) + \ldots + a_{-1}(z-z_0)^{n-1} + (z-z_0)^n G(z)$$

so it follows that we take the $n-1$ th derivative and let $z \to z_0$ which yields

$$a_{-1} = \text{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)]$$

Example 58

In the case of a simple pole we just need to do

$$a_{-1} = \text{res}_{z_0} = \lim_{z \to z_0} [(z-z_0) f(z)]$$

which is quite easy to see why

Theorem 59 (Residue Theorem)

If we have a holomorphic function f in an open set containing a curve C and its interior, except for a pole at z_0 then integrating once counterclockwise around C

$$\int_C f(z)dz = 2\pi i \, \text{res}_{z_0}(f)$$

Proof. Just perform integration over C term by term on

$$f(z) = \left(\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z-z_0}\right) + G(z)$$

From above we **cauchy integral formula** we know that

$$\int_C \frac{a_{-k}}{(z-z_0)^k} = \int_{C^\epsilon} \frac{a_{-k}}{(z-z_0)^k}$$

where $k \neq 1$ and C_ε is the smallest circle enclosing the pole. We know that closed integrals in holomorphic regions will integrate to zero by 30. Therefore we have

$$\int_C \frac{a_{-k}}{(z-z_0)^k} = \int_{C_\varepsilon} \frac{a_{-k}}{(z-z_0)^k} = 0$$

where $k \neq 1$ and

$$\int_C G(z) = \int_{C_\varepsilon} G(z) = 0$$

since G is holmorphic near z_0 by assumption. However for $k = 1$ where we have

$$\frac{a_{-1}}{z-z_0}$$

recall 25 this not a holomorphic function! This why we have been letting $k \neq 1$ above. In such a case we have if you recall

$$\int_C \frac{a_{-1}}{z-z_0}dz = \int_{C_\varepsilon} \frac{a_{-1}}{z-z_0}dz = 2\pi i a_{-1}$$

which will be our only non-zero term in $\int_C f(z)dz$ Therefore

$$\int_C f(z)dz = 2\pi i a_{-1} = 2\pi i \operatorname{res}_{z_0}(f)$$

□

Now you can see why we have defined a_{-1} to be our residue because it is associated with a term that doesn't integrate to zero for a closed curve in a holomorphic region of f. In which case our non-zero term will correspond to the integral over C_ε that is the smallest circle enclosing z_0.

Corollary 60

If f has poles $z_1, \ldots, z_n$ inside a circle γ then

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname{res}_{z_j} f$$

Proof. First note that we are simply applyinng the multikeyhoel verions of 30 and I am using reverse orientations(C should be clockwise) by convention accidentally but it should not affect the results.

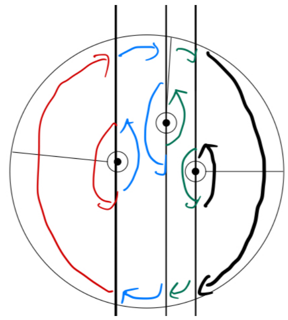

Figure 5: Multiple Poles

Let C be oriented clockwise(the path consisting of all arrow that circle the boundary of our large circle)Let C_{ε_i}(oriented anti clockwise) denote the circle that encloses the smallest pole i. Now similar to 30 just that this time we have multiple keyholes γ_i, each associated with the leftmost side of a pole and taken clockwise we see that

$$\int_C f - \sum_i \int_{C_{\varepsilon_i}} f = \sum_i \int_{\gamma_i} f$$

Then by cauchy integral formula we know the RHS is zero and

$$\int_C f = \sum_i \int_{C_{\varepsilon_i}} f$$

Now calculate applying 54 to each the integral f over C_{ε_i} which encloses a deleted neighbourhood of a pole i. In which case this corollory follows after the residue theorem.

Problem 61

Compute
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

Solution. Consider

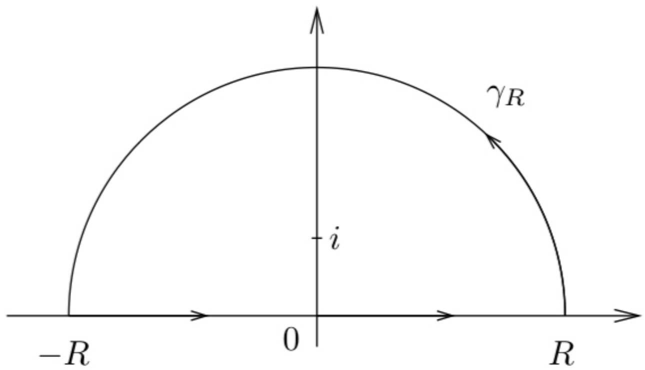

Figure 6: Half Circle Countour

where the y axis is the complex and x axis is the real part of $z \in \mathbb{C}$. Define a function
$$f(z) = \frac{1}{1+z^2}$$
Then we have
$$\int_{C_R^+} f(z)dz = \int_{\gamma_R} f(z)dz + \int_{-R}^{R} f(z)dz = \pi$$
where C_R^+ denotes the half circle above. The equality with π follows from the **residue formula** where we notice
$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$
where clearly $z = \pm i$ are poles as the derivative at those values are clearly undefined. Therefore noticing they are each of multiplicity/order 1 they are simple poles so
$$\lim_{z \to i}((z-i)f) = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}$$
and likewise
$$\lim_{z \to -i}((z+i)f) = \lim_{z \to -i} \frac{1}{z-i} = -\frac{1}{2i}$$
Then considering i is only pole enclosed by C_R^+ we have
$$\int_{C_R^+} f(z)dz = 2\pi i \frac{1}{2i} = \pi$$
Now consider
$$\int_{\gamma_R} f(z)dz \leq \sup_{z \in \gamma_R} f(z) \cdot \text{length}(\gamma_R) = \frac{1}{R^2-1} \cdot \pi R \leq \frac{1}{R^2} \pi R = \frac{\pi}{R}$$

by the **ML** inequality.Then taking $R \to \infty$ we see that

$$\lim_{R\to\infty} \int_{C_R^+} f(z)dz = \underbrace{\lim_{R\to\infty} \int_{\gamma_R} f(z)dz}_{=0} + \lim_{R\to\infty} \int_{-R}^{R} f(z)dz = \pi$$

and we may replace z with x since the horizontal segment $-R$ to R is the real segment so

$$\lim_{R\to\infty} \int_{-R}^{R} f(z)dz = \int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \pi$$

as desired

Corollary 62

In fact we can calculate in the same way:

$$\int_{-\infty}^{\infty} \frac{1}{P(z)}$$

where real valued $P(z) = a_0 + a_1x + a_2x^2 \dots a_nx^n$ is a polynomial of degree $n \geq 2$

Proof. Looking at the above we see that the ML inequality can be bound by

$$\frac{\pi R}{R^n} = \frac{\pi}{R^{n-1}}$$

so as long as $n \geq 2$ taking $R \to \infty$ make the arc integral go to zero. As for evaualting the real horizontal segment we simply need to find all the poles(which are the roots of $P(z)$) with respect to their multiplicities and apply residue formula.

3 other isolated singularities

Definition 63

An isolated singularity z_0 of a function $f : \Omega - \{z_0\} \to \mathbb{C}$ is called **removable** if f exends to a holomoprhic function on Ω(that is we can fill in $f(z_0)$ with some value)

Example 64

We know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and indeed the singularity of $f(z) = \frac{\sin z}{z}$ is removable at $z = 0$

Theorem 65 (Riemann)

A singularity z_0 of $f : \Omega - \{z_0\} \to \mathbb{C}$ is removable if and only if $f(z)$ is bounded in a neighbourhood around z_0(not containing z_0 itself).

4 gamma functions

TODO

1. meromorphic continuation
2. singularities
3. weier strass
4. hamard theorem

Definition 66

The **Gamma Function**

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt = \lim_{\varepsilon \to 0} \int_\varepsilon^{\frac{1}{\varepsilon}} \phi(t) dt$$