MIT 6.003 Signals and Systems

Ian Poon

October 2024

Selected theorems from `Alan V Oppenheimer`'s Signals and Systems book

Contents

1 Signals and systems(1)

1.1 continuous time and discrete time signals(1.1)

Definition 1
In **continuous time signals** the independent variable is continuous and thus these signals are defined for a continuum of values

Definition 2
In **discrete time signals** they are defined only at discrete times and consequently the independent variable takes on only a discrete set o values.

Definition 3

To distinguish between continuous and discrete time signal we will use the symbol t t denote the continuous time independent variable and n to denote the discrete time independent variable. In addition for continuous time sigans we will enclose the independent variable in parentheses (.) while for discrete time signals we will use brackets [.] to enclose the independent variable.

Example 4

Continuous time signal

$$x(t)$$

Discrete time signal

$$x[n]$$

Consider an example calculation with continuous time signals

Example 5

If $v(t)$ and $i(t)$ are respectively the voltage and current across a resistor with Resistance R then the instaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

The **total energy** expended over time interval $t_1 \leq t \leq t_2$ is

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

Definition 6

The **total energy** over the time interval $t_1 \leq t \leq t_2$ for a continuous time signal $x(t)$ is

$$\int_{t_2}^{t_1} |x(t)|^2 \, dt$$

Definition 7

Similarly the **total energy** in a discrete time signal $x[n]$ over the time interval $n_1 \leq n \leq n_2$ is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

For the above 2 definitions simply take the limit to infinity as infinite time interval. As for time averaged power simply divide by the time interval like so.

Definition 8

We define the **time-averaged power** over an infinite time interval as

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt$$

and

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{+N} |x[n]|^2$$

Definition 9

We can indentify 3 types of signals

1. signals with finite energy($E_\infty < \infty$) In which case such a signal mut have zero aergae power. For example in the continuous case we must have
$$P_\infty = \lim_{T\to\infty} \frac{E}{2T} = 0$$
2. signals with finite averge power($P_\infty < \infty$)
3. signals with neither of the above

1.2 transformation of the independent variable(1.2)

Example 10

3 basic examples of transforming the independent variable of a signal include

1. **time shift** where
$$x[n] \to x[n-n_0] \quad x(t) \to x(t-t_0)$$
2. **time reversal** where
$$x[n] \to x[-n] \quad x(t) \to x(-t)$$
3. **time scaling**
$$x[n] \to x[kn] \quad x(t) \to x(kt)$$

Another important class of signals are known as **periodic signals**

Definition 11

a **periodic continuous** time signal $x(t)$ takes the form

$$x(t) = x(t+T)$$

we say that $x(t)$ is periodic with period T

Definition 12

However we also that if periodic the continuous time signal $x(t)$ also has the orm $x(t) = x(t + mT)$ for all t and any integer m. That is to say $x(t)$ is also periodic with period $2T, 3T, 4T, \ldots$. The **fundamental period** T_0 of $x(t)$ is the smallest positive value of T for which 11 holds

Definition 13

A **periodic discrete** time signal takes the form

$$x[n] = [n + N]$$

where again there is a **fundamental period** N_0 which is the smallest period N for which the discrete periodic time signal is periodic.

Another class of signals are even and odd signals

Definition 14

A continuous time signal is even if

$$x(-t) = x(t)$$

while a discrete time signal is even if

$$x[-n] = x[n]$$

Definition 15

A continuous time signal is odd if

$$x(-t) = -x(t)$$

while a discrete time signal is odd if

$$x[-n] = -x[n]$$

1.3 exponential and sinusoidal signals(1.3)

Another class of signals are known as exponential and sinusoidal signals

Definition 16

The continuous time **complex exponential signal** is of the form

$$x(t) = Ce^{at}$$

where C and a are in general complex numbers

Fact 17

A second important class of complex exponentials is obtained by contraining a to be purely imaginary. In which case we have

$$x(t) = e^{jw_0 t}$$

An important property of this signal is that it is periodic.

$$e^{jw_0 t} = e^{jw_0(t+T)} = e^{jw_0 t}\underbrace{e^{jw_0 T}}_{=1}$$

by periodicity. The fundamental period T_0 of $x(t)$ is then

$$T_0 = \frac{2\pi}{|w_0|}$$

where w_0 is the **fundamental frequency**(we will learn more below). Thus the signals $e^{jw_0 t}$ and $e^{-jw_0 t}$ have the same fundamental period

Fact 18

A signal closely related to the periodic complex exponential is the **sinusoidal signal**

$$x(t) = A\cos(w_0 t + \phi)$$

Again it is obviously periodic

It is closely related well because

$$A\cos(w_0 t + \phi) = A\,\mathrm{Re}\left\{e^{j(w_0+\phi)}\right\}$$

$$A\sin(w_0 t + \phi) = A\,\mathrm{Im}\left\{e^{j(w_0+\phi)}\right\}$$

Periodic signals - in particular the complex periodic exponential signal and the sinousoidal signal as discussed above provide important examples of signals with *infinite total energy* but *finite average power*.

Example 19

For example consider the periodic exponential signal. Calculating we have

$$E_{\text{period}} = \int_0^{T_0} \left|e^{jw_0 t}\right|^2 dt = \int_0^{T_0} 1 \cdot dt = T_0$$

This follows because the magnitude of the exponential $\left|e^{jw_0 t}\right| = 1$ obviously for any valid t in the interval.Then we have

$$P_{\text{period}} = \frac{1}{T_0} E_{\text{period}} = 1$$

In which case the complex periodoic exponential signal as finite average power equal to

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|e^{jw_0 t}\right|^2 dt = 1$$

So clearly the time averaged power over an interval is finite but because it is not equal zero clearly the signal has infinite power.

Definition 20

We will often find useful to define sets of **harmonically related** complex exponentials - that is sets of peroidic exponentials all of which are periodic with common period T_0

Recall that a necessary condition for a complex exponential e^{jwt} to be periodic is that

$$e^{jwT_0} = 1$$

which implies

$$wT_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

so we define

$$w_0 = \frac{2\pi}{T_0}$$

in which case we have

$$\underbrace{\left(k\frac{2\pi}{T_0}\right)}_{w=kw_0} T_0 = 2\pi k$$

Definition 21

We define the kth harmonic $\phi_k(t)$ by

$$\phi_k(t) = e^{jkw_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

Proposition 22

See that for $k = 0, \phi_k(t)$ is a constant while for any other value of k, $\phi_k(t)$ is period with fundamental frequency $|k|\, w_0$ and fundamental period

$$\frac{2\pi}{|k|\, w_0} = \frac{T_0}{|k|}$$

Remark 23. *This does not violate the fact that they all still have the same period T_0. But that does not mean they cannot have different fundamental periods.*

Proof. this follows since recall the fundamental period is the smallest t in which ϕ_k is periodic and we know that happens if and only if

$$1 = e^{jkw_0 t}$$

but we know $w_0 T_0 = 2\pi$. Hence it follows that for a fixed k, the smallest t such that $w_o t$ is a multiple of 2π corresponds to a period of $\frac{T_0}{|k|}$ as desired. In which case, the corresponding fundamental frequency is calculated as usual by

$$\frac{2\pi}{T_0/\, |k|} = |k|\, w_0$$

since by definition $w_0 = \frac{2\pi}{T_0}$ Graphically the effect of different fundamental frequencies and periods appear as follows

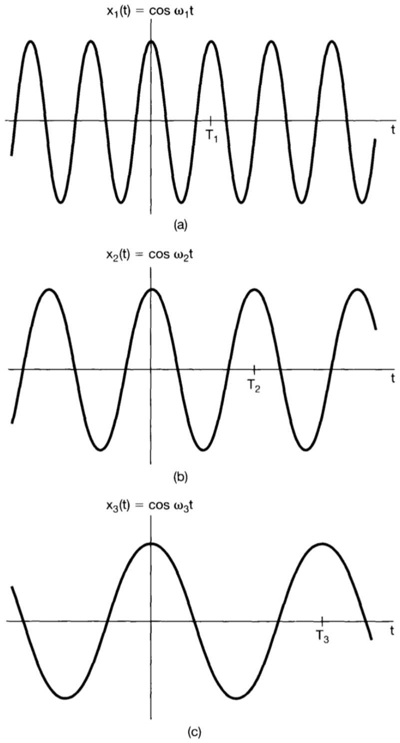


Figure 1.21 Relationship between the fundamental frequency and period for continuous-time sinusoidal signals; here, $\omega_1 > \omega_2 > \omega_3$, which implies that $T_1 < T_2 < T_3$.

To repeat again although all having different fundamental frequencies they are still periodic about T_0. Moreover their average power are all clearly still 1(refer to 19) no matter the harmonic number.

Example 24

The most general case of a complex exponential signals can be expressed and interpreted in terms of two cases that we have examined so far: the real exponential and periodic complex exponential Specifically if we had the form

$$C = |C| e^{j\theta}$$

and

$$a = r + jw_0$$

then

$$Ce^{at} = |C| e^{j\theta} e^{(r+jw_0)t} = |C| e^{rt} e^{j(w_0 t+\theta)}$$

Essentially we now have a product of the real exponential and periodic complex exponential which is basically just a decaying signal with the same graph as the familiar damped oscillations.

Let us now investigate the equivalent for discrete time signals.

Definition 25
As in continuous time an important discrete time signal is the **complex exponential signal**/sequence defined by

$$x[n] = Ca^n$$

where C and a are in general complex numbers. Alternatively we could express this as

$$x[n] = Ce^{\beta n}$$

where $a = e^{\beta}$.

Again we constrict to purely imaginary to get

$$x[n] = e^{jw_0 n}$$

and there is also the closely related sinusoidal signal

$$x[n] = A\cos(w_0 n + \phi)$$

also related by euler relations. Also similary we have the general complex form given by

$$C = |C|\, e^{j\theta}$$

and

$$a = |a|\, e^{jw_0}$$

then

$$Ca^n = |C|\,|a|^n \cos(w_0 n + \theta) + j\,|C|\,|a|^n \sin(w_0 n + \theta) = |C|\,|a|^n\, e^{j\theta} e^{jnw_0} = |C|\,|a|^n\, e^{j(\theta + nw_0)}$$

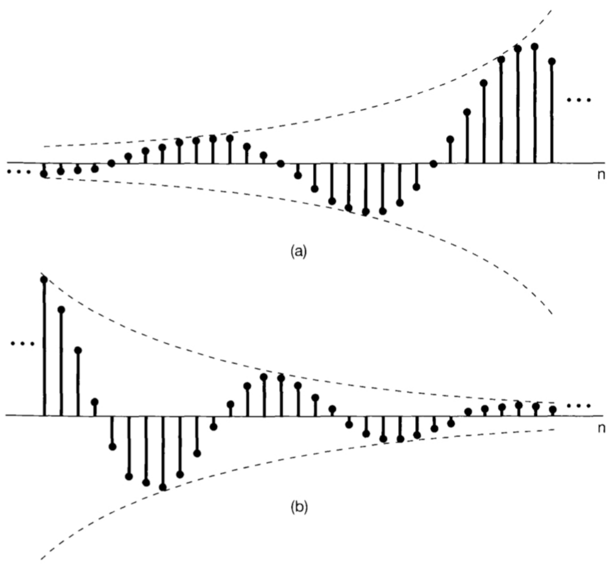


Figure 1.26 (a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

While there are many similarities between continuous time and discrete time signals there also a number of important differences. One of them involve the exponential signal $e^{jw_0 n}$

Fact 26

Consider the discrete time complex exponential with frequency $w_0 + 2\pi$ where we have

$$e^{j(w_0+2\pi)n} = e^{j2\pi n}e^{jw_0 n} = e^{jw_0 n}$$

but you cant say they same for continuous

$$e^{j(w_0+2\pi)t} = e^{j2\pi t}e^{jw_0 t} \neq e^{jw_0 n}$$

we only have equality when t is a whole number.

It helps to consider

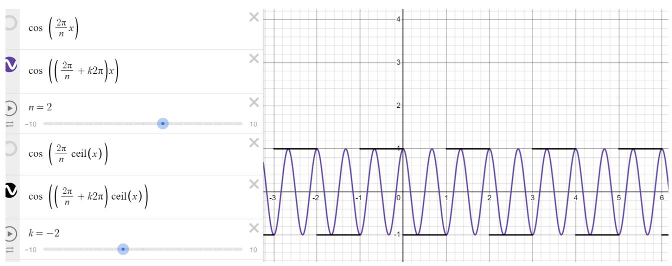


Figure 1: discrete vs continuous complex exponential

See when n and t are superimposed on the x axis the adding of multiples of 2π to w_0 clearly changes the signal as oscillations get more and more frequent. However in the discrete case these changes don't show as whatever happens between integers is not reflected. Afterall, discrete can be seen as continuous but filter out all non integer inputs.

As a consequence of 26 we have that $e^{jw_0 n}$ does not have contniually increasing rate of oscillation like the continuous case, instead as we increase w_0 from 0 we obtain signals that oscillate more and more rapidly until we reach $w_0 = \pi$. As we continue to increase w_0 we decrease the rate of oscillation until we reach $w_0 = 2\pi$ which produces the same constant sequence as $w_0 = 0$.

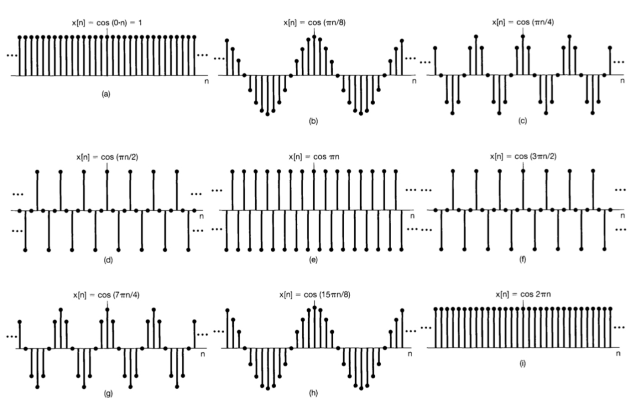


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

Note that in particular for $w_0 = \pi$ or any other odd multiple of π we have

$$e^{j\pi n} - (e^{j\pi})^n = (-1)^n$$

With our calculations we can also determine the fundamental period and frequency of discrete time complex exponential wher we define the fundmental frequence of a discrte time periodic signal as we did in continuous time.

Fact 27

Consider the signal e^{jwn}. For such a signal to be periodic it must satisfy

$$e^{jw(n+N)} = e^{jwn}$$

or equivalently

$$e^{jwN} = 1$$

so that we have

$$wN = 2\pi m, \quad m = 0, \pm 1, \ldots$$

Again we define N to be the fundamental period and the fundamental frequency to be $w_0 = 2\pi/N$. So we have

$$\underbrace{\left(m\frac{2\pi}{N}\right)}_{w=mw_0} N = 2\pi m$$

Definition 28

As in continuous time we are now able to consider sets of harmonically related periodic exponential for discrete time with a common period N which are at frequencies multiples of $2\pi/N$. That is

$$\phi_k[n] e^{jk(2pi/N)n}, \quad k = 0, \pm 1, \ldots$$

with corresponding fundamental period and frequency

$$\frac{N}{|k|} \quad \text{and} \quad |k|\, w_0$$

respectively for each k

However unlike the continuous time case not all of harmonically related complex exponentials are distinct due to 26. Specifically

$$\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n} = e^{jk(2\pi/N)n} e^{j2\pi n} = \phi_k[n]$$

So therefore there are only N distinct periodic exponentials for any given set of k for a common period N.

1.4 Unit Impulse and unit step functions(1.4)

Definition 29

One of the simplest discrete time signals is the **unit impulse** defined as

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Definition 30

Another one of the simplest discrte time signals is the **unit step** defined as

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

There is close relationship between the discrete time unit impulse and unit step.

Fact 31

In particular we have the **first difference** of the discrete time step

$$\delta[n] = u[n] - u[n-1]$$

To see this consider that the only case where this gets 1 is when $n = 0$. If not we will have $1 - 1 = 0$ or $0 - 0 = 0$. Refer to the graphs below Conversely the discrete time unit step is the **running sum** of the unit sample. That is

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \in \mathbb{R}$$

To see this consider the functions below. This will get 1 the $n \geq 0$ and 0 otherwise.

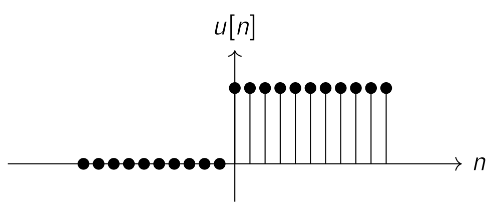


Figure 2: Unit Step function

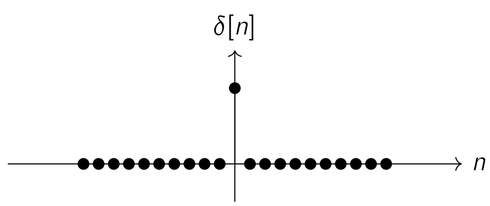


Figure 3: Unit Impulse function

Moreover chaning the variable of summation from m to $k = n - m$ in the 2nd equation of 31 we have

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

which follows when you consider that $m = n - k$. At the lower limit where $m = -\infty$ we have $k = n - (-\infty) = \infty$ and when $m = n$ at the upper limit we have $n - k = n \Rightarrow k = 0$. Which is just equal to

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

Fact 32

Another observation is that we have

$$x[n]\delta[n] = x[0]\delta[n]$$

This quite obvious think about what happens when a value is nonzero and and when the value is equal zero. More generally

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

Follows very simply from above, just consider two cases only when $n = n_0$ will get zero if not won't otherwise.

Let us consider the continues time version of the unit step and impulse functions.

Definition 33

The continuous time **unit step** function $u(t)$ is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Note the discontinuity at $t = 0$

Definition 34

The continuous time **unit impulse** function $\delta(t)$ is defined as

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ 1 & t = 0 \end{cases}$$

it looks similar to 3

Fact 35

Analagous to above the continuous time unit step is the **running integral** of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

From this we also see that the continuous time unit impulse can be thought of as the **first derivative** of the continuous time unit step

$$\delta(t) = \frac{du(t)}{dt}$$

However as you might have questioned this is not well defined because of the discontinuity at $t = 0$. So we consider the unit step $u_\Delta(t)$ where $u(t)$ is the limit as $\Delta \to 0$. Hence we redefine the derivative like so

$$\delta_\Delta(t) = \frac{u_\Delta(t)}{dt}$$

This continuous approximation specifically looks like this

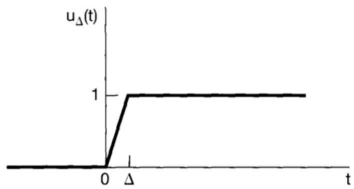


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

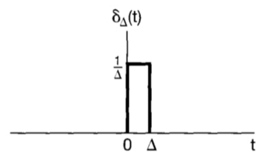


Figure 1.34 Derivative of $u_\Delta(t)$.

See how this makes sesne. As $\Delta \to 0$ so does $\delta_\Delta(t)$ making $\delta(t)$ is limit as well. That is

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

which as you recall should look similar to 3 We give a few more example calculations

Example 36

Consider the scaled impulse $k\delta(t)$ notice that from 35

$$\int_{-\infty}^{t} k\delta(\tau)d\tau = ku(t)$$

Also from 35 if we change our variable of integration from τ to $\sigma = t - \tau$ we have from elementary calculus

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma)$$

or equivalently

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)d\sigma$$

As with the discrete time impulse the continuous time impulse has very important sampling property.

Fact 37

Consider

$$x_1(t) = x(t)\delta_\Delta(t)$$

By construction $x_1(t)$ is zero outside the interval $0 \leq t \leq \Delta$. So for Δ sufficiently small so that $x(t)$ is approximately consant over this interval

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

and taking $\Delta \to 0$ we have

$$x(t)\delta(t) = x(0)\delta(t)$$

And just like the discrete case we also can have

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

1.5 continuos and discrete time systems(1.5)

Definition 38

A **continuous time system** is a system in which *continuous time input signals* are applied and result in continuous time output signals

Definition 39

Similarly a **discrete time system** is a system that transforms discrete time inputs into discrete time outputs

$$x(t) \longrightarrow \boxed{\text{continuous time system}} \longrightarrow y(t)$$

$$x[n] \longrightarrow \boxed{\text{discrete time system}} \longrightarrow y[n]$$

Definition 40

A system can be made of various subsystems so it makes sense to define system interconnections. Consider a few important ones

- series(cascade) interconnections

$$\text{input} \longrightarrow \boxed{\text{system 1}} \longrightarrow \boxed{\text{system 2}} \longrightarrow \text{output}$$

- parallel interconnections

$$\begin{tikzcd}
 & \bullet \arrow[r] & \boxed{\text{system 1}} \arrow[r] & \bullet \arrow[d] & \\
\text{intput} \arrow[r] & \bullet \arrow[u] \arrow[d] & & \oplus \arrow[r] & \text{output} \\
 & \bullet \arrow[r] & \boxed{\text{system 2}} \arrow[r] & \bullet \arrow[u] &
\end{tikzcd}$$

- feedback interconnection

$$\begin{tikzcd}
\text{input} \arrow[r] & \oplus \arrow[r] & \boxed{\text{system 1}} \arrow[r] & \bullet \arrow[r] \arrow[d] & \text{output} \\
 & \bullet \arrow[u] & \boxed{\text{system 2}} \arrow[l] & \bullet \arrow[l] &
\end{tikzcd}$$

1.6 basic system properties(1.6)

Definition 41

A system is said to be **memoryless** if the output for each value of the independent variable at a given time is dependent only on the input at that same time.

Example 42

A simple example of a memorless sysem is the **identity system**

$$y(t) = x(t)$$

Example 43

An example of a discrete time system with memory is an **accumulator**

$$y[n] = \sum_{-\infty}^{n} x[k]$$

and another example will be the **delay**

$$y[n] = x[n-1]$$

note the outputs in these examples depend on past outputs

Definition 44

A system is said to be **invertible** if distinct inputs lead to distinct outputs

Example 45

Suppose an invertible continuous time system is

$$y(t) = 2x(t)$$

then its **inverse system** is

$$w(t) = \frac{1}{2}y(t)$$

Definition 46

a system is **causal** if the output at any time depends only on values of the input at the present time and in the past.

such a system is also described as **nonanticipative** because it does anticipate future outputs

Example 47

The following system is not causal

$$y(t) = x(t+1)$$

Definition 48

Informally a **stable** system is one in which small inputs lead to responses that do not diverge. Formally its one where bounded inputs must result in bounded outputs(BIBO stability)

Example 49

Suppose $|x(t)| \leq M$ The following is unstable

$$|tx(t)| \leq M|t|$$

but t is unbounded.The following are stable

$$\left|e^{x(t)}\right| \leq e^M \quad \text{and} \quad |x(t)| \leq M$$

Definition 50

a system is **time invariant** if the behavior and the characteristics of the system are fixed over time

Example 51

The following is not time invariant

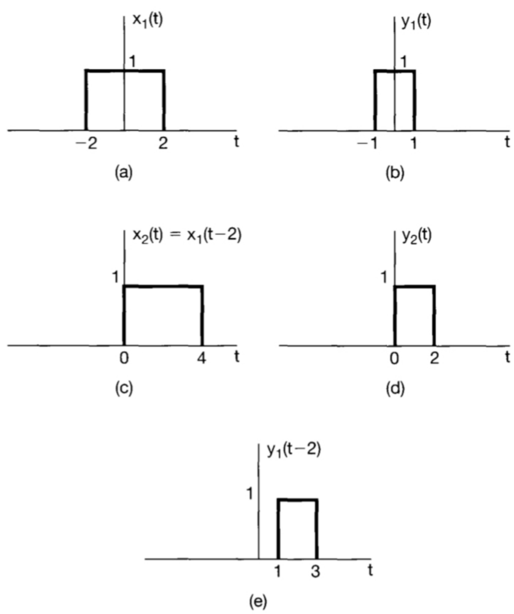


Figure 1.47 (a) The input $x_1(t)$ to the system in Example 1.16; (b) the output $y_1(t)$ corresponding to $x_1(t)$; (c) the shifted input $x_2(t) = x_1(t-2)$; (d) the output $y_2(t)$ corresponding to $x_2(t)$; (e) the shifted signal $y_1(t-2)$. Note that $y_2(t) \neq y_1(t-2)$, showing that the system is not time invariant.

Figure 4: follows quite obvious since the order of translation and scaling matters! Recall high school graphs...

Example 52

The continuous time system defined by

$$y(t) = \sin[x(t)]$$

is time invariant since denoting

$$y_1(t) = y(t - t_0) = \sin[x(t - t_0)]$$

and

$$y_2(t) = \sin[x(t - t_0)]$$

we see that $y_1(t) = y_2(t)$

Example 53

The discrete time system defined by

$$y[n] = nx[n]$$

is not time invariant since denoting

$$y_1[n] = y[n - n_0] = (n - n_0)x[n - n_0]$$

and

$$y_2[n] = nx[n - n_0]$$

we see that $y_1[n] \neq y_2[n]$

Fact 54

You should be able to see from the examples that a direct time shift of the output must match the the corresponding output due to a the same time shift in input to be considered time invariant. In other words we are saying

$$y = f(x) \text{ and not } y = f(t, x)$$

where y, x are time varying signals. In other words f itself is independent of t and only dependent on x. So if not time invariant we may something like

$$y_{\text{new}} = f(t - \tau, x_{\text{new}}) \neq f(t, x_{\text{new}})$$

where we assume the time varying signals $y_{\text{new}}, x_{\text{new}}$ are shifted by τ that is considering the signal output at t/its component,

$$y_{\text{old}}(t - \tau) = y(t)$$

equivalently for x and the discrete case. That is componentwise if time invariant before shifting

$$x[t] \mapsto y[t]$$

and after shifting

$$x_{\text{new}}[t] = x[t - \tau] \mapsto y[t - \tau] = y_{\text{new}}[t]$$

that is the same components map to the same components no matter when they come in the signal.

Definition 55

A **linear system** in continuous or discrete time satisfies

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2)$$

Where a, b are constants. Needless to say equivalently for the discrete(replace the brackets with [ ] and t with n) where $x_1, x_2, f(x_1), f(x_2)$ are time varying signals and $f : \ell^\infty \to \ell^\infty$(map from signal to signal. again you may alternatively use $\mathbb{R}^\infty$ instead)

Remark 56. *It is not necessarily true that $\sum_k y[i]\delta[i-k] = f(\sum_k x[i]\delta[i-k])$.Equivalently treating singals as vectors, $f(x[i]\mathbf{e}_i) = y[i]\mathbf{e}_i$.Clearly you are implying an eigenvalue.This confusion probably arose since the book wrote*

$$x(t) \mapsto y(t)$$

. To resolve this, consider that this is actually the component to component map of $f : x \to y$ which we denote to be $f_n : x[n] \to y[n]$. Thus what we were doing previously was

$$f : (x \cdot \mathbf{e}_i) \cdot \mathbf{e}_i \to y$$

then component to component map in this case $f_n : x[n] \mapsto y[n]$ is not necessarily the same because we are considering a different (filtered)function altogether!

Remark 57. *A system need not be time invariant to be linear. Consider the example from earlier $f(x) = tx$ where x is a time varying signal. See that*

$$k(y[n]) = nk(x[n])$$

where k is some scalar is indeed a linear system but it is not time invariant

Example 58

The system $y[n] = 2x[n] + 3$ is not linear since

$$k(y[n]) \neq 2k(x[n]) + 3, \forall n$$

2 Linear Time Invariant Systems(2)

2.1 Discrete time LTI sytems: the convolution sum(2.1)

Fact 59

It is easy to see that

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

this is known as the **sifting property** of the discrete time unit impulse now considier a linear time invariant system. Let $x[n]$ and $y[n]$ be their arbituary inputs and outputs respectively at time n.We also denote the system as

$$y = f(x)$$

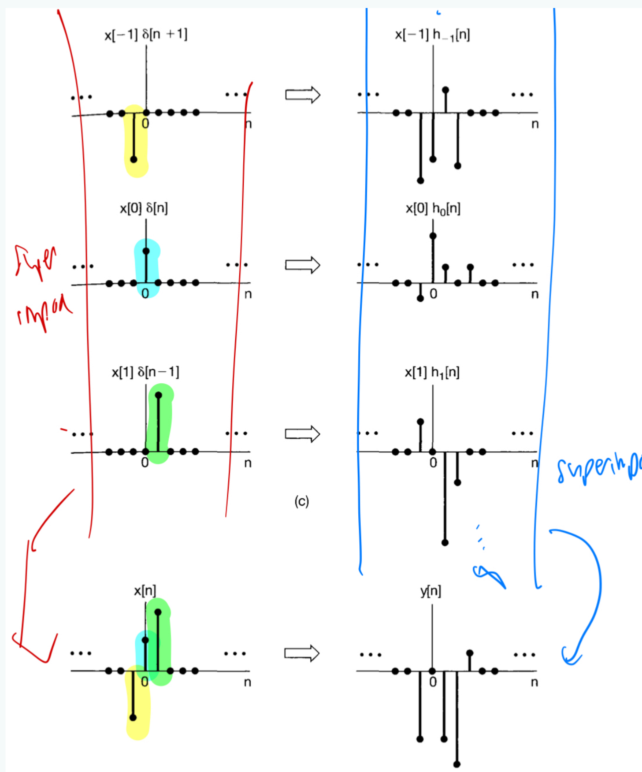
where $y, x \in \ell^\infty$ and $f : \ell^\infty \to \ell^\infty$. Note that

$$\left\{ \sum_{k=-\infty}^{\infty} \delta[n-k] \right\}_n$$

is a basis for ℓ^∞. Note that is perfectly fine to think of signals as an infinite dimensional vector instead

Example 60

Now consider for example



Fact 61 (Involing Linearity)

you can see time very signals x and y as infinite dimensional sequences where $x[n], y[n]$ denote the values at the nth summand of their respective sequences. Then given a linear map $f : \ell^\infty \to \ell^\infty$ as a map between signals

$$y = f(x) = f(\sum_{k=-\infty}^{\infty} x[k])$$

By **linearity** since (recall 59)

$$x = \sum_k x[k] = \sum_k \sum_n x[k]\delta[n-k] = \sum_k x[k] \left(\sum_n \delta[n-k] \right)$$

we have a sum of superimposed signals

$$y = f(x) = \sum_{k=-\infty}^{\infty} x[k]h_k \quad \text{basically every element of } h_k \text{ scaled by } x[k] \in \mathbb{R}$$

where the time varying signal $h_k \in \ell^\infty$ is defined to be

$$h_k = f(\sum_{n=-\infty}^{\infty} \delta[n-k])$$

which is the response of the linear system to the shifted unit impulse $\delta[n-k]$. in other words component wise we have isolating the nth component of signal y, summing all contributions to component n we have

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]}$$

Fact 62 (Invoking time invariance)

By assumption of **time invariant** we have that the *responses*

$$x[k]h_k = f(\sum_n x[k]\delta[n-k]) = x_k[h_0]_{\text{shifted by k}}$$

in other words component wise

$$h_0[n-k] = h_k[n]$$

Specifically $[h_0]_{\text{shifted by k}}$ means

$$h_0 = f(\sum_n \delta[n]) \xrightarrow[\text{only true by assumption}]{\text{(output)shift time(n) by k}} \underbrace{f(\sum_n \delta[n-k]) = h_k}_{\text{input time shift}}$$

since clearly $\sum_n \delta[n-k]$ is the k time shifted input signal of $\sum_n \delta[n]$ Recall by definition if time invariant, the time shifted signal reponse/output y_{new} will correspond to the time shifted input x_{new}. Therefore for convencience for LTI systems we now drop the subscript for h and let $h = h_0$. Then for an LTI system component wise we have

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

Definition 63

Symbolically we may write this as the **convolution sum** or the superposition sum

$$y[n] = x[n] * h[n]$$

Example 64

Consider

$$x[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$h[n] = \begin{cases} a^n & 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$

Now find

$$y[n] = \sum_k x[k][n-k]$$

Solution. We know the convolution will only be non zero when the intersection of the non zero ranges for

$$0 \le k \le 4$$

and

$$0 \le n-k \le 6 \quad \Rightarrow \quad n-6 \le k \le n$$

for $x[k]$ and $h[n-k]$ respectively is nonempty since we are summing their products over k for each given n.So you could see the below as a "sliding window inner product" Therefore it now makes sense to split up into cases to since for an inner product over k, the range k to sum over will clearly depend on n

- when $n < 0$ clearly no overlap
- when $0 \le n \le 4$ we have
  - $(n = 0)k : \{-6 \le k \le 0\} \cap \{0 \le k \le 4\} = \{0\}$
  - $(n = 1)k : \{-5 \le k \le 1\} \cap \{0 \le k \le 4\} = \{0, 1\}$
  - $\vdots$
  - $(n = 4)k : \{-2 \le k \le 4\} \cap \{0 \le k \le 4\} = \{0, 1, 2, 3, 4\}$
- when $n = 5$ we have $(n = 5)k : \{-1 \le k \le 5\} \cap \{0 \le k \le 4\} = \{0, 1, 2, 3, 4\}$
- when $6 \le n \le 10$ we have
  - $(n = 6)k : \{0 \le k \le 6\} \cap \{0 \le k \le 4\} = \{0, 1, 2, 3, 4\}$
  - $(n = 7)k : \{1 \le k \le 7\} \cap \{0 \le k \le 4\} = \{1, 2, 3, 4\}$
  - $\vdots$
  - $(n = 10)k : \{4 \le k \le 10\} \cap \{0 \le k \le 4\} = \{4\}$
- when $n > 10$ clearly no overlap

Therefore we have

$$y[n] = \begin{cases} 0 & n < 0 \\ \sum_{k=0}^{n} a^{n-k} & 4 \leq n \leq 6 \\ \sum_{k=0}^{4} a^{n-k} & n = 5 \\ \sum_{k=n-6}^{4} a^{n-k} & 6 \leq n \leq 10 \\ 0 & n > 10 \end{cases}$$

2.2 Continuous time LTI systems: The convulution integral

Definition 65

Define as 1.4

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 \leq t < \Delta \\ 0 & \text{otherwise} \end{cases}$$

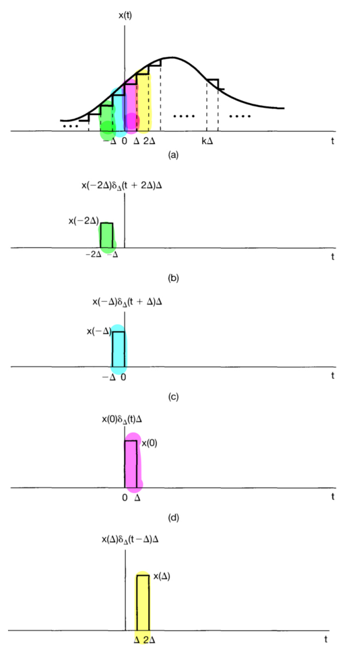


Figure 5: recall the sifting property

so defining

$$\hat{x} = \sum_{k,t=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta = \sum_{k=-\infty}^{\infty} x(k\Delta) \sum_{t=-\infty}^{\infty} \delta_\Delta(t - k\Delta)\Delta$$

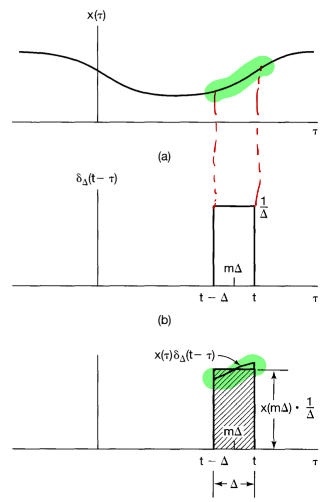
and by sifting property

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

there we have when $\Delta \to 0$(partitions becomes infinitesimal)

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

to see this consider



Now from 65 we see that for $\delta_\Delta(t - m\Delta)$ we have from its definition when $\delta_\Delta(t - m\Delta) = \frac{1}{\Delta}$ the corresponding input range

$$0 \leq t - m\Delta < \Delta \quad \rightarrow \quad t - \Delta < m\Delta \leq t$$

therefore each term has area $x(m\Delta)\delta_\Delta(t - m\Delta)\Delta = x(k\Delta)$. And clearly as $\Delta \to 0$ the approximation becomes closer and closer to the actual area under the convolution over τ for a given t.

Remark 66. *Such an approximation is valid since continuous function on a closed real bounded interval means reinmann integrable if you recall real analysis*

Now define the convolution integral in similar fashion to how we did so for the convolution sum first define the signal

$$\hat{h}_{k\Delta}(t) = f(\sum_{t=-\infty}^{\infty} \delta_\Delta(t - k\Delta))$$

therefore again by linearity from 2.2 we have

$$\hat{y} = f(\hat{x}) = f(\sum_{k=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}\Delta$$

and so componentwise

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

once again invoking time invariance we have

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}(t - k\Delta)\Delta$$

where $\hat{h}_0 = \hat{h}$ and because f is continuous as $\Delta \to 0$ we have

$$\lim_{\Delta \to 0} f(\hat{x}) = f(x) = y$$

therefore we have

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}(t - k\Delta)\Delta = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

where $\lim_{\Delta \to 0} \hat{h}(t - k\Delta) = h(t - \tau)$ and $d\tau \approx \Delta, \tau \approx k\Delta$ with this we can

Definition 67

Define the **convolution integral** or the superposition integral as

$$y(t) = x(t) * h(t)$$

representing the above as desired

2.3 properties of linear time invariant systems(2.3)

Proposition 68

Convolution properties. We will do them for the continuous case. they hold true for the discrete case as well

1. (Commutative) $x(t) * h(t) = h(t) * x(t)$
2. (Distributive) $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$
3. (Associative) $x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$

Proof. See the first by change of variables. The rest are also very trivial. To be continued...

2.4 Causal LTI systems described by differential and difference equations(2.4)

Note the following block diagram notation for discrete time systems

- adding elements

$$\begin{tikzcd} & x_2[n] \arrow[d] & \\ x_1[n] \arrow[r] & \oplus \arrow[r] & x_1[n] + x_2[n] \end{tikzcd}$$

- scaling elements

$$x[n] \xrightarrow[a]{} ax[n]$$

- unit delay

$$x[n] \longrightarrow \boxed{D} \longrightarrow x[n-1]$$

Example 69

Consider a block diagram

$$\begin{tikzcd} x[n] \arrow[r, "b"] & \oplus \arrow[r] & \bullet \arrow[r] \arrow[d] & y[n] \\ & y[n-1] \arrow[u] & \boxed{D} \arrow[l, "-a"'] & \end{tikzcd}$$

This essentially describes the recursive equation

$$y[n] = -ay[n-1] + bx[n]$$

Now consider the following block diagram terminology for continuous time systems

- adding elements

$$\begin{tikzcd} & x_2(t) \arrow[d] & \\ x_1(t) \arrow[r] & \oplus \arrow[r] & x_1(t)+x_2(t) \end{tikzcd}$$

- scaling elements

$$x(t) \xrightarrow[a]{} ax(t)$$

- unit delay

$$x(t) \longrightarrow \boxed{D} \longrightarrow \frac{dx(t)}{t}$$

Example 70

Consider a block diagram

$$\begin{tikzcd} x(t) \arrow[r, "b/a"] & \oplus \arrow[r] & \bullet \arrow[r] \arrow[d] & y(t) \\ & \frac{dy(t)}{dt} \arrow[u] & \boxed{D} \arrow[l, "-1/a"'] & \end{tikzcd}$$

This essentially describes the recursive equation

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$

Next consider what is known as the integrator for block diagrams

$$x[t] \longrightarrow \boxed{\int} \longrightarrow \int_{-\infty}^{t} x(\tau)d\tau$$

Example 71

Consider

$$\begin{tikzcd} x[t] \arrow[r, "b"] & \oplus \arrow[r] & \boxed{\int} \arrow[r] & y[t] \arrow[ll, bend left, "-a"] \end{tikzcd}$$

which esssentiall represents

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$

2.5 singularity functions(2.5)

recall 1.4 that if $\delta \to 0$ $\delta_\Delta(t)$ acts like an impulse. to be continued...

3 fourier series representation of periodic signals(3)

Definition 72

A signal for which the system output is a (possibly complex) constant times the input is referred to as an eigenfunction of the system and the amplitude factor is referred to as the system's eigenvalue

Proposition 73

Complex exponentials are eigenfunctions of LTI systems.(under certain assumptions of course)

Proof. Consider the convolution integral

$$y(t) = \int h(\tau)x(t-\tau)d\tau$$

so if the complex exponential were an eigenfunction, express x as the complex exponential we have

$$y(t) = \int h(\tau)e^{s(s-\tau)}d\tau = e^{st}\int h(\tau)e^{-s\tau}d\tau$$

and assuming the the integral on the RHS converges we will have

$$y(t) = H(s)e^{st}$$

where $H(s)$ is a complex constant whose value depends on s...to be continued

4 the continuous time fourier transform(4)