MIT 18.155-6 Differential Analysis I-II(2022)

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December 2024

Graduate course on distribution theory. Material sourced from [1] lecture notes

Contents

1 Prologue: motivation and background(1)

1.1 functional spaces

Definition 1

Let M be a metric space and $U \subset V \subset M$ be two sets then

1. we write
$$U \Subset V$$
if U is relatively open subset of V
2. we say that U is compactly contained in V and write
$$U \Subset V$$
if there exists a compact set such that $U \subset K \subset V$

Definition 2

We denote that $1_A : \mathbb{R}^n \to \mathbb{R}^n$ is the indicator function

$$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Definition 3 (C^k space)

For $U \Subset \mathbb{R}^n$ defin the space of continuous functions

$$C^0(U) = \{f : U \to \mathbb{C} | \text{f is continuous}\}$$

and

$$||f||_{C^0} = \sup_{x \in U} |f(x)|$$

i.e the sup norm

Before we proceed we first deffine

Definition 4

The **multindex** in $\mathbb{R}^n$ is a vector $\alpha = (a_1, \ldots, a_n)$ whose entries are non negative integers. It is used like so

$$|a| = a_1 + \ldots + a_n \quad \partial_x^\alpha = \partial_{x_1}^{a_1} \ldots \partial_{x_n}^{a_n}$$

Now we finally complete our discussion for C^k space by defining

Definition 5

A norm on $C_c^k(U)$ is given by

$$||f||_{C^k} = \max_{|a|\leq k} \sup_{x\in U} |\partial_x^a f(x)|$$

that is the maximum taken over all orders of differentials and over $x \in U$

Definition 6

We define the spaces of **locally** L^p functions

$$L^p_{\text{loc}}(U) = \frac{\{f : U \to \mathbb{C} : 1_k f \in L^p(U) \text{ for all compact } K \subset U\}}{\{f : f = 0 \text{ a.e.}\}}$$

and **compactly supported** L^p functions

$$L^p_c(U) = \{f \in L^p(U) | \text{ there exists compact } K \subset U \text{ such that } f = 1_k f \text{ a.e.}\}$$

Note that for the above

$$1_k(x) \cdot f(x) = 1_K f(x) = \begin{cases} f(x) & x \in K \\ 0 & x \notin K \end{cases}$$

for example $L^p(U)$ consists of measurable functions $f : U \to \mathbb{C}$ such that $1_U f \in L^p(\mathbb{R}^n)$. With this understanding see that the last statement means that f vanishes outside the compact set K and thus is consistent with the definition of compact support if you recall. As for the locally L^p statement it can be interpreted as the set of non trivial functions that are L^p integrable on every compact subset of U

Definition 7 (support)

Let $U \Subset \mathbb{R}^n$ and $f : U \to \mathbb{C}$. Define the **support** of f denoted by $\text{supp}\, f$ as the closure of teh set $\{x \in U | f(x) \neq 0\}$. We say that f is **compactly supported** if $\text{supp}\, f$ is a compact subset of U

Proposition 8

Recall from [4] **Holder's Inequality** where we have

$$||fg||_{L^1} \leq ||f||_{L^p} \, ||g||_{L^q}$$

if $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n), \frac{1}{p} + \frac{1}{q} = 1$

Proposition 9

we see that

$$L^p_{\text{loc}}(U) \subset L^r_{\text{loc}}(U), \quad L^p_c(U) \subset L^r_c(U), \quad \forall p \leq r$$

where $p \leq r$

Proof. Recall $L^p \subset L^q$ when $1 \leq q \leq p$ from [4]

Proposition 10

We have

$$C_c^0(U) \subset L^p(U)$$

for all p.

Proof. recall this result from [4] we shows that compactly supported continuous functions are dense in L^p. This naturally implies the case for C_c^k since

$$C_c^k \subset C^0$$

$\square$

Definition 11

From above we know any function $f \in C_c^0(U)$ is uniformly continuous thus it has a **modulus of continuitity**

$$w_f(\varepsilon) = \sup\{|f(x) - f(y)| : x, y \in U, |x - y| \leq \varepsilon\} \to 0 \text{ as } \varepsilon \to 0+$$

See that uniform continuous is necessary. Because then $|f(x) - f(y)|$ being bounded whenever $|x - y| < \varepsilon$ will depend on say x_0. So it is possible that a x, y pair in which none of them are x_0 might result in an unbounded output difference which makes the definition above undefined. Now clearly the suprenum of the function output difference goes to zero as $\varepsilon \to 0$ by definition of continuity.

1.2 Convolution and approximation by smooth functions

Definition 12

Assume that $f, g \in C_c^0(\mathbb{R}^n)$. Define their **convolution** $f * g \in L^\infty(\mathbb{R}^n)$ to be

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x - y)dy, x \in \mathbb{R}^n$$

Theorem 13 (Properties of convolution)

For $f, g \in C_c^0(\mathbb{R}^n)$ we have

1. $f * g \in C_g^0(\mathbb{R}^n)$ and

$$\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g = \{x + y | x \in \operatorname{supp} f, y \in \operatorname{supp} g\}$$

2. $f * (g * h) = (f * g) * h$ and $f * g = g * f$ that is convolution is associateive and commuative
3. if $f \in C_c^0(\mathbb{R}^n)$ and $g \in C_c^1(\mathbb{R}^n)$ then $f * g \in C_c^1(\mathbb{R}^n)$ and

$$\partial_{x_j}(f * g) = f * (\partial_{x_j} g) \quad , \forall a, |a| \leq k$$

4. if $f \in C_c^0(\mathbb{R}^n)$ and $g \in C_c^k(\mathbb{R}^n)$ then $f * g \in C_c^k(\mathbb{R}^n)$ and

$$\partial_x^a(f * g) = f * (\partial_x^a g) \quad , \forall a, |a| \leq k$$

Proof. Consider

1. We first check that $f * g$ is continuous. Let $x, \tilde{x} \in \mathbb{R}^n$. We compute

$$\begin{aligned}
|(f*g)(x) - (f*g)(\tilde{x})| &= \left| \int_{\mathbb{R}^n} f(y)(g(x-y) - g(\tilde{x}-y))dy \right| \\
&\leq \int_{\mathbb{R}^n} |f(y)| \sup_y |g(x-y) - g(\tilde{x}-y)| \, dy \\
&= \sup_y |g(x-y) - g(\tilde{x}-y)| \int_{\mathbb{R}^n} |f(y)| \, dy \\
&= ||f||_{L^1(\mathbb{R}^n)} \sup_y |g(x-y) - g(\tilde{x}-y)|
\end{aligned}$$

where the second line follows by triangle inequality of integrals $\left|\int f\right| \leq \int |f|$ which applies to both lebesgue and Riemann integrals if you recall [2]. Moreover the suprenum makes sense since g is continuous and compactly supported(set of inputs for which $g \neq 0$ is compact and since g continuous must hav maximum must exist on the compact support). Now recalling **??** therefore it has a modulus of continuity w_g we then estimate

$$|(f*g)(x) - (f*g)(\tilde{x})| \leq ||f||_{L^1(\mathbb{R}^n)} \, w_g(|x - \tilde{x}|)$$

However we know by definition of modulus of continuity that $w_g(|x-\tilde{x}|) \to 0, |x - \tilde{x}| \to 0$ so it follows that $f * g$ is continuous.

For the support property we note first that the set $\{x \in \mathbb{R}^n | f * g(x) \neq 0\}$ is contained in $\operatorname{supp} f + \operatorname{supp} g$ since in order for $f * g(x)$ be be nonzero there must exist some $y \in \mathbb{R}^n$ such that $f(y) \neq 0$ and $g(x-y) \neq 0$. You could easily make a similar argument for x too. Therefore we have

$$\operatorname{supp}(f*g)(x) \subset \operatorname{supp} g + \operatorname{supp} f$$

Next the set $\operatorname{supp} f + \operatorname{supp} g$ is compact as it is the image of the compact set $\operatorname{supp} f \times \operatorname{supp} g$ under the map $(x, y) \mapsto x + y$(which is clearly a continuous map). You should recall this result from real analysis [2]. Finally since compact in $\mathbb{R}^n$ we know that it is closed as well. Then recall from real analysis once more than a closed subset of a compact since by recall 7 by definition all support sets are closed.

2. For associativity compute

$$\begin{aligned}
\big(f * (g * h)\big)(x) &= \int_{\mathbb{T}} f(y) \cdot (g*h)(x-y) \, dy \\
&= \int_{\mathbb{T}} f(y) \cdot \left(\int_{\mathbb{T}} g(z) \cdot h(x-y-z) \, dz \right) \, dy \\
&= \iint_{\mathbb{T}\times\mathbb{T}} f(y) \cdot g(z) \cdot h(x-y-z) \, dz \, dy
\end{aligned}$$

similarly

$$\begin{aligned}
\big((f*g)*h\big)(x) &= \int_{\mathbb{T}} (f*g)(u) \cdot h(x-u) \, du \\
&= \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f(v) \cdot g(u-v) \, dv \right) \cdot h(x-u) \, du \\
&= \iint_{\mathbb{T}\times\mathbb{T}} f(v) \cdot g(u-v) \cdot h(x-u) \, dv \, du.
\end{aligned}$$

then now make the change of variables $y = v, z = u - v$ for the former equation and the equivalence should become clear. Commutativity follows similarly by using a change of variables $y \mapsto x - y$.

3. the fact that $f * g$ is compactly supported has already been proved in (1) and since $g \in C_c^1(\mathbb{R}^n) \subset C_c^0(\mathbb{R}^n)$ we also know that $f * (\partial_{x_j} g)$ is continuous also from (1). Thus it remains to show that $\partial_{x_j}(f * g) = f * (\partial_{x_j} g)$. Denoting $e_1, \dots, e_n$ the canonical basis of $\mathbb{R}^n$(the standard basis) we compute for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}/\{0\}$
$$\frac{(f * g)(x + te_j) - f * g(x)}{t} = \int_{\mathbb{R}^n} f(y) \frac{g(x - y + te_j) - g(x - y)}{t} dy$$
From the mean value theorem and the fact that $\partial_{x_j} g$ is uniformly continuous we get if you recall from real analysis,
$$\frac{g(z + te_j) - g(z)}{t} \to \partial_{x_j} g(z), \quad t \to 0 \text{ uniformly in } z \in \mathbb{R}^n$$
recalling from real analysis again that we may pass the limit under the integral if uniformly convergent as above so that we have
$$\lim_{t \to 0} \frac{(f * g)(x + te_j) - f * g(x)}{t} = \int_{\mathbb{R}^n} f(y) \lim_{t \to 0} \frac{g(x - y + te_j) - g(x - y)}{t} dy = f * (\partial_{x_j} g)(x)$$
as $f(y)$ is just a constant with respect to t and this integral still makes sense since all the terms in the integral are compactly supported by definition.But notice the LHS is just $\partial_{x_j}(f * g)(x)$. So in particular we just shown that the deritivative of $f * g$ exists and is continuous as $f * (\partial_{x_j} g)$ is continuous as mentioned earlier. Hence $f * g \in C_c^1(\mathbb{R}^n)$ as desired.

4. this follows from (3) by induction on k. Specifically first let the multindex a be equal $b + e_j$ where b is another multi index. Let $|a| = m + 1$ and so $|b| = m$ and $g \in C^{m+1}, f \in C^0$
$$\partial_x^a (f * g) = \partial_{x_j} \partial_x^b (f * g)$$
but by induction hypothesis
$$\partial_x^b (f * g) = (f * \partial_x^b g)$$
where $f \in C^0$ and $g \in C^m$ implies $f * g \in C^m$. Therefore we have
$$\partial_x^a (f * g) = \partial_{x_j} (f * \partial_x^b g)$$
but $g \in C^{m+1} \Rightarrow \partial_x^b g \in C^1$ because we have proven the base case in (3) we have
$$\partial_x^a (f * g) = (f * \partial_{x_j} \partial_x^b g) = (f * \partial_x^a g)$$
in particular we know that the RHS is continuous by (1). Hence it follows that $(f * g) \in C^{m+1}$ so we have proved our induction step.

Corollary 14

An immediate result of (4) is that
$$f \in C_c^0(\mathbb{R}^n), g \in C_c^\infty(\mathbb{R}^n) \quad \Rightarrow \quad f * g \in C_c^\infty(\mathbb{R}^n)$$
That is convolving a rough function with a smooth one produces a smooth result. That's powerful!

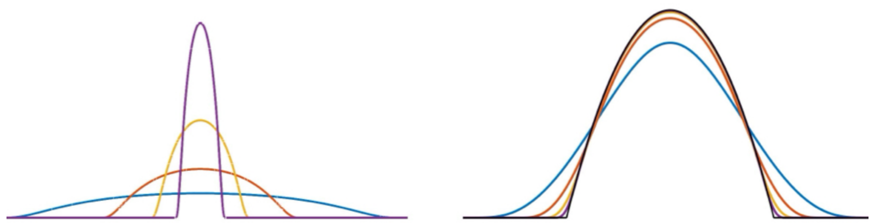

Figure 1: Left: the molifying kernels χ_ε for $n = 1$ and several values of ε. Right: a function f on $\mathbb{R}$ and its successive mollifications f_ε as in 18. We have $f_\varepsilon \to f$ uniformly in x

Example 15

A standard example of a function in $C_c^\infty(\mathbb{R}^n)$ if you recall [3]is the "bump function"

$$f(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

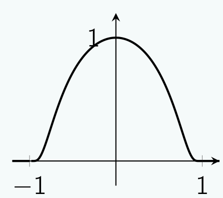

Figure 2: Bump function χ

Example 16 (Extension of a function by zero)

When we *extend a function by zero* we mean for example suppose $f \in C_c^\infty(U)$ where $U \in \mathbb{R}^n$. Then defining

$$\tilde{f}(x) = \begin{cases} f(x) & x \in U \\ 0 & \mathbb{R}^n/U \end{cases}$$

We say that f is now extended to zero to a function $\tilde{f} \in C_c^\infty(\mathbb{R}^n)$. Recall from a similar example in [3] that the zero function is smooth. Moreover the continuity is still preserved given that $f(x)$ has compact support. Meaning that $f(x)$ transitions to 0 in a continuous manner since it is smooth by definition.

Definition 17 (Dense)

Let X be a topological space, and let $A \subseteq X$. The subset A is dense in X if every point of X is either in A or is a limit point of A

Theorem 18

Let $U \Subset \mathbb{R}^n$. Then the space $C_c^\infty(U)$ is dense in the space $C_c^0(U)$ and more precisely for each $f \in C_c^0(U)$ there exists a sequence $f_k \in C_c^\infty(U)$ such that $f_k \to f$ uniformly on U and all the supports $\operatorname{supp} f_k$ are contained in some k independent compact subset of U

Remark 19. *In out context where $C_c^0(U) \subset C_c^\infty(U)$ we say $C_c^0(U)$ is dense in $C_c^\infty(U)$ as every $f \in C_c^0(U)$ can be approximated by $f_k \in C_c^\infty(U)$(limit point) if not a function in $C_c^\infty(U)$ itself.*

Proof. Fix a bump function $\chi \in C_c^\infty(\mathbb{R}^n), \operatorname{supp}\chi \subset B(0,1), \int_{\mathbb{R}^n} \chi(x)dx = 1$(by definition, see 15). For $\varepsilon > 0$ define the rescaling

$$\chi_\varepsilon(x) = \varepsilon^{-n}\chi\left(\frac{x}{\varepsilon}\right), \quad \chi_\varepsilon \in C_c^\infty(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \chi(x)dx = 1$$

Let $U \Subset \mathbb{R}^n$. Take arbitrary $f \in C_c^0(U)$ and extend it by 0 to a function in $C_c^0(\mathbb{R}^n)$ which we still denote by f(recall 16). We then define the **mollifications** of f(see 1 for illustration) as

$$f_\varepsilon = f * \chi_\varepsilon$$

Then we know that

$$f_\varepsilon \in C_c^\infty(\mathbb{R}^n), \operatorname{supp} f_\varepsilon \subset \operatorname{supp} f + B(0,\varepsilon)$$

from 13.We claim that

$$f_\varepsilon \to f \text{ as } \varepsilon \to 0+ \text{ uniformly on } \mathbb{R}^n \tag{1}$$

so our sequence f_k it can be constructed with $\varepsilon = \frac{1}{k}$ which clearly goes to zero for large k. to see this consider

$$\begin{aligned} |f(x) - f_\varepsilon(x)| &= \left| \int_{\mathbb{R}^n} (f(x) - f(x-y))\chi_\varepsilon(y)dy \right| \\ &\leq ||\chi_\varepsilon||_{L^1(\mathbb{R}^n)} \sup_{y \in B(0,\varepsilon)} |f(x) - f(x-y)| \\ &\leq ||\chi||_{L^1(\mathbb{R}^n)}\, w_f(\varepsilon) \end{aligned}$$

this again uses the triangle inequality for integrals as well as the fact that $C_c^0(\mathbb{R}^n)$ functions are uniformly continuous like in a proof for a previous theorem. Note in the first line we used the fact that $\int \chi_\varepsilon = 1$ since

$$\int \varepsilon^{-n}\chi(\frac{x}{\varepsilon})dx = \varepsilon^{-n+1} \underbrace{\int \chi(x)dx}_{=1 \text{ by definition(see above)}} = 1$$

Therefore $\int f(x)\chi(y)dy = f(x)\int \chi(y)dy = f(x)$ which is how we brought $f(x)$ into the integral as above. Like before we know that $w_p(\varepsilon) \to 0, \varepsilon \to 0$ therefore we have shown uniform convergence as desired(uniform because the last line is clearly independent of x). ☐

Proposition 20

Any open set $U \Subset \mathbb{R}^n$ can be **exhausted** by compact sets

$$U = \bigcup_{j=1}^{\infty} K_j \quad \text{where } K_j \Subset U, K_j \Subset K_{j+1}$$

Proof. Recall this from [3].

Theorem 21 (Partitions of Unity on a compact neighborhood)
Let $U_1, \ldots, U_m \Subset \mathbb{R}^n$ and $K \subset U_1 \cap \ldots \cup U_m$ be a compact set. Then there exists functions

$$\chi_j \in C_c^\infty(U_j), j = 1, \ldots, m \quad \chi_j \geq 0, \chi_1 + \ldots + \chi_m \leq 1$$

where $\chi_1 + \ldots + \chi_m = 1$ in a neighborhood K. That is $(1 - \chi_1 - \ldots - \chi_m) \cap K = \emptyset$.

Proof. You should be able to see this is just a theorem regarding the existence of a **partition of unity** which you covered before in [3].

Lemma 22 (fundamental lemma of the calculus of variations(FLCV))
Let $U \Subset \mathbb{R}^n, f \in L^1_{\text{loc}}(U)$ and assume that

$$\int_U f(x)\varphi(x)dx = 0 \quad \text{for all } \varphi \in C_c^\infty(U)$$

then $f(x) = 0$ for almost every $x \in U$

Proof. now by 18 we know that there exists $\varphi_k \in C_c^\infty(U)$ which converges uniformly to φ. Because the convergence is uniform we may pass it under the integral where we see

$$\lim_{k\to\infty} \int_U f(x)\varphi_k(x)dx = \int_U f(x) \lim_{k\to\infty} \varphi_k(x)dx = \int_U f(x)\varphi(x)dx = 0$$

This immediately implies that $\int_U f\varphi_k(x)dx = 0$ for all k as well(recall definition of limit if not refer to [5] where i remembered encountering something similar there in the proofs.) We now claim that the set of $A = \{U : f(x) \neq 0\}$ is a set of positive measure by contradiction i.e $m(A) > 0$.

Without loss of generality, suppose $f(x) > 0$ on A. (If $f(x)$ takes both positive and negative values, split $f(x)$ into $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$, and handle the positive and negative parts separately.)

Since $|A| > 0$, by definition of Lebesgue measure, there exists a compact subset $K \subset A$ such that $|K| > 0$ and $f(x) > 0$ on K. (The existence of such a compact set of a result of **inner regularity** of the lebesgue measure. Again recall [5])

Define a simple test function φ based on the set K:

$$\varphi(x) = \begin{cases} 1, & x \in K, \\ 0, & x \notin K. \end{cases}$$

Note that φ may not smooth, but we will approximate it by smooth functions.

Since φ is compactly supported (its support is K, which is compact), we can approximate it by a sequence of smooth functions $\varphi_k \in C_c^\infty(U)$ such that:

$$\varphi_k(x) \to \varphi(x) \quad \text{pointwise and in } L^1(U).$$

For example, this can be done by convolving φ with a smooth mollifier. Specifically:

$$\varphi_k(x) = (\varphi * \eta_k)(x),$$

where η_k is a standard mollifier supported in a shrinking neighborhood of radius $1/k$. These φ_k satisfy:

1. $\varphi_k(x) \to \varphi(x)$ pointwise,

2. $\|\varphi_k - \varphi\|_{L^1(U)} \to 0$, and

3. $\varphi_k \in C_c^\infty(U)$.

you should know this from 18

Using the assumption that $\int_U f(x)\psi(x)\,dx = 0$ for all $\psi \in C_c^\infty(U)$, we have:

$$\int_U f(x)\varphi_k(x)\,dx = 0 \quad \text{for all } k.$$

Taking the limit as $k \to \infty$, and using the dominated convergence theorem (since $f \in L^1_{\text{loc}}(U)$ and $|\varphi_k(x)| \leq |\varphi(x)|$. recall [5]. Moreover C^0 is dense in L^p for $1 < p < \infty$ recall [4]), we get:

$$\int_U f(x)\varphi(x)\,dx = \int_K f(x) \cdot 1\,dx = \int_K f(x)\,dx.$$

Since $f(x) > 0$ on K and $m(K) > 0$, it follows that:

$$\int_K f(x)\,dx > 0.$$

(recall again [5] that if K has measure zero this lebesgue integral will be zero) But this contradicts the assumption that $\int_U f(x)\varphi(x)\,dx = 0$ for all $\varphi \in C_c^\infty(U)$. Thus, our original assumption that $f(x) \neq 0$ on a set of positive measure must be false. Therefore the set A must be one of measure zero which is the very meaning of $f(x) = 0$ almost everywhere.

Remark 23. *Alternatively there is a proof using an extended version of the* **riesz represenation theorem** *in relation to measure theory. You should look at Andrew Lin's Stanford Graduate Real Analysis math 205a notes for more. Essentially we can see the above integral as an inner product then conclude f must be unique. Note you can't directly use it with our current understanding because the space of smooth compactly supported functions apparently isn't hilbert!*

Theorem 24

Let $U \Subset \mathbb{R}^n$. Assume that $f \in C^1(U)$ and $g \in C_c^1(U)$. Then we have for all j

$$\int_U (\partial_{x_j} f(x)) g(x) dx = -\int_U f(x)(\partial_{x_j} g(x)) dx$$

Proof. we will show that

$$\int_U \partial_{x_j} h(x) dx = 0 \quad \forall h \in C_c^1(U)$$

then the theorem follows when we let $h = fg$. See the equivalance by product rule differentiation! So prove this line first extend h by zero to a function in $C_c^\infty(\mathbb{R}^n)$ which we still denote by h(recall 16). Note that this is acceptable afterall by definition h vanishes outside its compact support anyway so we have not changed the value of the integral. For notational convenience assume that $j = 1$ and write $x = (x_1, x')$ where $x' \in \mathbb{R}^n$. Now by **fubini theorem** we have

$$\int_{\mathbb{R}^n} \partial_{x_1} h(x) dx = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \partial_{x_1} h(x_1, x') dx_1 dx' = 0$$

since $\int_{\mathbb{R}} \partial_{x_1} \varphi(x_1) dx_1 = 0$ for any $\varphi \in C_c^1(\mathbb{R})$ by the **fundamental theorem of calculus**. This is because

$$\int_{\mathbb{R}} \partial_{x_1} h(x_1, x') dx_1 = h(a, x') - h(b, x') = 0 - 0 = 0$$

where a, b are the limit points of out integration which are sufficiently far away from the compact set within U as we have extended all the way to $\mathbb{R}^n$ in our integration area.

Remark 25. *If U is bounded with a smooth boundary then the Divergence theorem for the vector field $h(x)e_j$ where e_j is the jth coordinate vector on $\mathbb{R}^n$ gives*

$$\int_{\mathbb{R}^n} \partial_{x_j} h(x)dx = \int_{\partial U} h(x)n_j(x)dS(x)$$

instead of $\int_{\mathbb{R}^n} \partial_{x_j} h(x)dx = 0$ where $n_j(x)$ is the jth coordinate of the outward unit normal vector to ∂U at x and dS is the area masure on ∂U. This in turn gives the intergration of parts identity for $f, g \in C^1(\overline{U})$

$$\int_U (\partial_{x_j} f(x))g(x)dx = \int_{\partial U} f(x)g(x)n_j(x)dS(x) - \int_U f(x)(\partial_{x_j} g(x))dx$$

2 Basics of distribution theory(2)

2.1 Definition of distributions

Definition 26

Let $U \Subset \mathbb{R}^n$ and assume that

$$u : C_c^\infty(U) \to \mathbb{C}$$

is a linear functional. We say that u is a **distribution** on U if for each compact set $K \subset U$ there exists constants C, N such that

$$|u(\varphi)| \leq C\,||\varphi||_{C^N} \text{ for all } \varphi \in C_c^\infty(U) \text{ such that } \operatorname{supp}\varphi \subset K$$

We denote the set of all distributions on U by

$$\mathcal{D}'(U)$$

In other words for *any* compact set in U, all φ whose compact support lies in K must satisfy the bounds stated above.

Proposition 27

$\mathcal{D}'(U)$ is a vector space

Proposition 28

Let $U \Subset \mathbb{R}^n$ and $f \in L^1_{\text{Loc}}(U)$. Define the linear functional

$$\tilde{f} : C_c^\infty(U) \to \mathbb{C}, \tilde{f}(\varphi) = \int_U f(x)\varphi(x)dx \text{ for all } \varphi \in C_c^\infty(U)$$

Then $\tilde{f}$ is a distribution in $\mathcal{D}'(U)$ and the map $f \mapsto \tilde{f}$ is linear and injective

Proof. it is a distribution because

$$\left|\tilde{f}(\varphi)\right| \leq ||1_K f||_{L^1} \cdot ||\varphi||_{C^0}$$

by the triangle inequality for integrals and the definitions of $L^1_{\text{Loc}}(U)$ and the sup norm $||\bullet||_{C^0}$ in the previous section. Linearity of the map $f \mapsto \tilde{f}$ is immediate by observation. Injectivity follows from the **FLCV** proven previously which implies that the nullspace is zero looking at $f(x)$ as the input to our functional.

Definition 29

We introduce the following important notations to be used for the preceding discussions

- for $f \in L^1_{\text{loc}}(U)$ we identify the function f with the distribution $\tilde{f}$ from 28
- for $f \in L^1_{\text{loc}}(U)$ and $\varphi \in C_c^\infty(U)$ we define the pairing

$$(f, \varphi) = \int_U f(x)\varphi(x)dx$$

- for $u \in \mathcal{D}'(U)$ and $\varphi \in C_c^\infty(U)$ we define the pairing

$$(u, \varphi) = u(\varphi)$$

Fact 30

Essentially distributions is a map from functions to $\mathbb{C}$. Say u is a distribution. Then $(u, f) = u(f) \to \mathbb{C}$. If u is a function itself we can write it as the "inner product over all of (x)" with its argument f. If not its just $u(f)$. In general u is not necessarily a function even if it can take in an argument $u(x)$ like a function $f(x)$ would. For example $\int \delta_x f(x)dx$ is undefined. See more below on dirac delta functions.

Definition 31 (Dirac Delta Function)

Let $U \Subset \mathbb{R}^n$ and $y \in U$ be a point. Define the distribution $\delta_y \in \mathcal{D}'(U)$ by

$$(\delta_y, \varphi) = \varphi(y) \quad \forall \varphi \in C_c^\infty(U)$$

Example 32 (Dirac Delta not a function)

labeleg31 A standard example of a distribution which is not a function is given by the *dirac delta function*. To see how this is true assume by contradiction that $\delta_y = f$ for some $f \in L^1_{\text{loc}}(U)$. Take $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(0) = 1$ and consider the test function $\varphi_\varepsilon(x) = \chi((x - y)/\varepsilon)$ which lies in $C_c^\infty(U)$ for sufficiently small $\varepsilon > 0$. Then $(f, \varphi_\varepsilon) \to 0$ as $\varepsilon \to 0+$ by the *dominated convergence theorem*. However we know that

$$(\delta_y, \varphi_\varepsilon) = \varphi_\varepsilon(y) = \chi((y - y)/\varepsilon) = \chi(0) = 1$$

therefore we can't have a well defined function as 1 input 2 outputs.

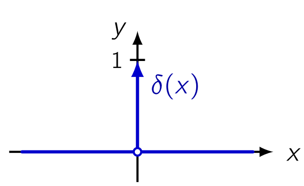

Figure 3: dirac delta "function"

Another way to see that dirac delta "function" isn't a function is to consider its original definition. Essentialy it is just a straight line to $y + \infty$ (at $x = 0$ in our case). Basic high school techniques tell you that by vertical line test, you immediately see there is more than one(in fact infinite ouputs y) for a single input $x = 0$. Clearly this ain't a well defined function.

Fact 33

This exemplifies one of the key motivations for distribution theory, which allows us to model the dirac delta like a "function". Later you will learn that it we can indeed model the dirac delta function as a "distributuional derivative" of the heaveside function. Moreover in later parts of the course you wil learn that the dirac delta "function" belongs to general class of objects known as the "tempered distributions" which has certain nice properties. For example we could even define fourier transforms on them etc.

2.2 Distributions and convergence

Fact 34

Recall [4] that if X is a **Banach Space** then a functional $u : X \to \mathbb{C}$ is bounded if and only if it is continuous. We now give an analagous statement for distributions.

Definition 35

Let $U \Subset \mathbb{R}^n$ and assume $\varphi_k \in C_c^\infty(U)$ is a sequence and $\varphi \in C_c^\infty(U)$ We say that

$$\varphi_k \to \varphi \text{ as } k \to \infty \in C_c^\infty(U)$$

if the following two conditions hold

1. there exists compact $K \subset U$ such that $\operatorname{supp} \varphi_k \subset K$ for all k, and
2. we have $||\varphi_k - \varphi||_{C^N}$ as $k \to \infty$ for all N

Proposition 36

Let $u : C_c^\infty(U) \to \mathbb{C}$ be a linear functional. Then the following are equivalent

1. u is a distribution that is it satisfies the norm bounds in 26
2. (sequential continuitity)for each sequence $\varphi_k \in C_c^\infty(U)$ if $\varphi_k \to 0$ in $C_c^\infty(U)$ then $(u, \varphi_k) \to 0$

Remark 37. *(2) can be seen as, as the argument goes to zero so must the output of the distribtuion*

Proof. To show that (1) $\to$ (2) Assume that $\varphi_k \to 0$ in $C_c^\infty(U)$. Then in particular there exists $K \Subset U$ such that $\operatorname{supp} \varphi_k \subset K$ for all K. The norm in 26 implies there exists C, N such that for all k

$$|(u, \varphi_k)| \leq C\, ||\varphi_k||_{C^n}$$

but the RHS goes to zero as $k \to \infty$ so $(u, \varphi_k) \to 0$ by the continuity of norms as required.

to show that (2) $\to$ (1) we argue by contradiction. Assume that u does not satisfy the norm bounds in 26. That is there exists $K \Subset U$ such that for any choice of C, N there exists $\varphi \in C_c^\infty(U)$ such that $\operatorname{supp} \varphi K$ and $|(u, \varphi)| \geq C\, ||\varphi||_{C^n}$. Choosing $C = N = k$ and dividing φ by $u(\varphi)$ we construct a sequence

$$\varphi_k(x) = \frac{\varphi}{u(\varphi)} \quad \to \quad (u, \varphi_k) = \frac{(u, \varphi)}{u(\varphi)} = 1$$

so we have

$$\varphi_k \in C_c^\infty(U). \quad \operatorname{supp}\varphi_k \subset K, \quad (u,\varphi_k) = 1, \quad ||\varphi_k||_{C^k} \geq \frac{1}{k}$$

the terms in blue follow from the terms in blue above. The sequence φ_k converges to 0 in $C_c^\infty(U)$ since for all $k \geq m$ we have $||\varphi_k||_{C^m} \leq ||\varphi_k||_{C^k} \leq \frac{1}{k}$. Now take $k \to \infty$ to see it. Thus u does not satisfy the sequential continuity property (2) as $(u,\phi_k) = 1$ for all k by definition □

Definition 38 (Weak boundedness)

Let $U \Subset \mathbb{R}^n$, $u^k \in \mathcal{D}'(U)$ be a sequence and $u \in \mathcal{D}'(U)$. We say that

$$u_k \to k \text{ as } k \to \infty \text{ in } \mathcal{D}'(U)$$

if we have

$$(u_k,\varphi) \to (u,\varphi) \text{ as } k \to \infty \text{ for all } \varphi \in C_c^\infty(U)$$

Proposition 39

If $u_k, u \in L^1_{\text{loc}}(U)$ satisfy $u_k(x) \to u(x)$ for almost every $x \in U$ and there exists $g \in L^1_{\text{loc}}$ such that $|u_k(x)| \leq g(x)$ for all k, then $u_k \to u$ in $\mathcal{D}'$

Proof. this follows immediately from the **dominated convergence theorem**

Proposition 40

Define functions $u_k \in L^1_{\text{loc}}(\mathbb{R})$ by

$$u_k(x) = k1_{[-1/k,1/k]}(x)$$

i.e compactly supported clearly on this interval then $u_k \to 2\delta_0$ in $\mathcal{D}'(\mathbb{R})$

Proof. take arbitrary $\varphi \in C_c^\infty(\mathbb{R})$ then

$$(u_k,\varphi) = k\int_{-1/k}^{1/k} \varphi(x)dx \xrightarrow{k\to\infty} 2\varphi(0) = (2\delta_0,\varphi)$$

by the continuitity of φ.

2.3 Localization

We now discuss how the space $\mathcal{D}'(U)$ depends on the open set U

Fact 41

recall the restriction operator

$$L^1_{\text{loc}}(U) \to L^1_{\text{loc}}(V), f \mapsto f|_V$$

we now define the analagous statement for distribtutions

Definition 42

Let $V \Subset U \Subset \mathbb{R}^n$. For $u \in \mathcal{D}'(U)$ define its **restriction** $u|_V \in \mathcal{D}'(V)$ as follows

$$(u|_V, \varphi) = (u, \varphi) \quad \forall \varphi \in C_c^\infty(V)$$

Here $C_c^\infty(V)$ is considered a subset of $C_c^\infty(U)$ as follows:
for $\varphi \in C_c^\infty(V)$ we extend it by zero to produce an element of $C_c^\infty(U)$(via 16)

Proposition 43

The restriction map $r_{V,U} : \mathcal{D}'(U) \to \mathcal{D}'(V)$ from 42 is linear and satisfies

$$r_{U,U} = I$$
$$r_{W,V} r_{V,U} = r_{W,U} \quad \forall W \Subset V \Subset U \Subset \mathbb{R}^n$$

Proof. obvious by observation

Theorem 44 (Sheaf Property of distributions)

Assume $\mathcal{J}$ is an arbitrary set and

$$U_j \Subset \mathbb{R}^n \quad \text{for } j \in \mathcal{J}, \quad U = \bigcup_{j \in \mathcal{J}} U_j$$

assume next that we are given $u_j \in \mathcal{D}'(U_j), j \in \mathcal{J}$ satisfying the comptability conditions

$$u_j|_{U_j \cap U_\ell} = u_\ell|_{U_j \cap U_\ell} \quad \forall j, \ell \in \mathcal{J}$$

then there exists unique $u \in \mathcal{D}'(U)$ such that

$$u|_{U_j} = u_j \quad \forall j \in \mathcal{J}$$

Proof. we first show uniqueness of u, which can be reformulated as follows

$$u \in \mathcal{D}'(U), u|_{U_j} = 0 \quad \forall j \in \mathcal{J} \Rightarrow u = 0$$

this is because if there were two distributions $u, v \in \mathcal{D}'(U)$ such that $u|_{U_j} = u_j$ and $v|_{U_j} = u_j$ then there difference $w = u - v$ satisfies

$$w|_{U_j} = 0, \quad \forall j$$

therefore if we show the only possible difference is 0 there must be only be 1 unique distribution that satisfies this. So now using 42 to express this equivalently in terms of restrictions we have

$$(u, \psi) = 0 \quad \forall j \in \mathcal{J}, \psi \in C_c^\infty(U_j) \Rightarrow (u, \varphi) = 0 \quad \forall \varphi \in C_c^\infty(U) \tag{1}$$

Take arbitrary $\varphi \in C_c^\infty(U)$. We can decompose it as

$$\varphi = \sum_{j \in \mathcal{J}} \varphi_j, \quad \varphi_j \in C_c^\infty(U_j)$$

where only finitely many of φ_j are nonzero. This is because $\operatorname{supp} \varphi$ is compact. Hence given that is covered by the open sets U_j we will know there exists a finite set $\mathcal{J}' \subset J$ such that $\operatorname{supp} \varphi \subset \bigcup_{j \in \mathcal{J}'} U_j$. Nowing by 21 we take a

partition of unity

$$\chi_j \in C_c^\infty(U_j), j \in \mathcal{J}', \sum_{j\in\mathcal{J}'} \chi_j = 1 \quad \text{on } \operatorname{supp}\varphi$$

then multiplying the above by φ we get our desired decomposition if we put $\varphi_j = \chi_j\varphi$ which is clearly in $C_c^\infty(U_j)$ so we have $\varphi_j = 0$ if $j \notin \mathcal{J}'$. That is we have the decomposition

$$\sum_{i\in\mathcal{J}'} \chi_i\varphi = \sum_{i\in\mathcal{J}'} \varphi_i = \varphi$$

Then we get upon pairing with u

$$(u, \varphi) = \sum_{j\in\mathcal{J}'} (u, \varphi_j)$$

It is then clear to see that if (1) holds, then $(u, \varphi_j) = 0$ for all j which gives $(u, \varphi) = 0$ as desired

It remains to show that given $u_j \in \mathcal{D}'(U_j)$ satisfying the compatbility conditions there exists $u \in \mathcal{D}'(U)$ satisfying

$$u|_{U_j} = u_j \quad \forall j \in \mathcal{J}$$

as required in the theorem statement. To define u we need to specify (u, φ) for each $\varphi C_c^\infty(U)$. Take such φ and decompose like earlier

$$\varphi = \sum_{j\in\mathcal{J}} \chi_j\varphi, \quad C_c^\infty(U_j)$$

again only finitely many χ_j are non zero. So we then put

$$(u, \varphi) = \sum_{j\in\mathcal{J}} (u_j, \chi_j\varphi)$$

the rest of the proof follows in the following steps

- we claim the value of (u, φ) does not depdend on the choice of decomposition. Indeed if we had

  $$\varphi = \sum_{j'\in\mathcal{J}} \tilde{\chi}_{j'}\varphi, \quad \tilde{\chi}_{j'} \in C_c^\infty(U_{j'})$$

  we write

  $$\begin{aligned}\sum_{j\in\mathcal{J}} (u_j, \chi_j\varphi) &= \sum_{j,j'\in\mathcal{J}} (u_j, \chi_j\tilde{\chi}_{j'}\varphi) \\ &= \sum_{j,j'\in\mathcal{J}} (u_{j'}, \chi_j\tilde{\chi}_{j'}\varphi) \\ &= \sum_{j'\in\mathcal{J}} (u_{j'}, \tilde{\chi}_{j'}\varphi)\end{aligned}$$

  Gives the required indepedence. Here the first equality above follows by the fact that $\chi_j\varphi = \sum_{j'\in\mathcal{J}} \chi_j\tilde{\chi}_{j'}\varphi$.In the second equality we use the compatability conditions where we have $\chi_j\tilde{\chi}_{j'}\varphi \in C_c^\infty(U_j \cap U_{j'})$ and the restrictions u_j and u_j' to $U_j \cap U_{j'}$ are wqual. Finally the last equality lisimilar to the first equality follows by

  $$\sum_{j\in\mathcal{J}} \chi_j\tilde{\chi}_{j'}\varphi$$

- we claim that the map $\varphi \mapsto (u, \varphi)$ is linear. Indeed take any $\varphi^{(1)}, \varphi^{(2)} \in C_c^\infty(U)$ and $a_1, a_2 \in \mathbb{C}$. Take a partition of unity as before

$$\chi_j \in C_c^\infty(U_j), \quad \sum_{j\in\mathcal{J}} \chi_j = 1 \quad \text{on } \operatorname{supp}\varphi^{(1)} \cup \operatorname{supp}\varphi^{(2)}$$

then again we have the decomposition $\varphi = \sum_{j'\in\mathcal{J}} \tilde{\chi}_{j'}\varphi, \quad \tilde{\chi}_{j'} \in C_c^\infty(U_{j'})$. Now we consider their linear combination where we have

$$\begin{aligned}
(u, a_1\varphi^{(1)} + a_2\varphi^{(2)}) &= \sum_{j\in\mathcal{J}} (u_j, \chi_j(a_1\varphi^{(1)} + a_2\varphi^{(2)})) \\
&= a_1 \sum_{j\in\mathcal{J}} (u_j, \chi_j\varphi^{(2)}) + a_1 \sum_{j\in\mathcal{J}} (u_j, \chi_j\varphi^{(2)}) \\
&= a_1(u, \varphi^{(1)}) + a_2(u, \varphi^{(2)})
\end{aligned}$$

which clearly shows linearity

- we claim that the linear map u satisfies the bounds 26 and thsu a distribution in $\mathcal{D}'(U)$. Indeed take any $K \Subset U$. Again fix a partition of unity

  $$\chi_j \in C_c^\infty(U_j), \sum_{j\in\mathcal{J}'} \chi_j = 1 \in K$$

  where $\mathcal{J}' \subset \mathcal{J}$ is a finite set. Since we have finitely many such distributions u_j if we take $j \in \mathcal{J}'$, by definition of being a distribution we may find C, N such that

  $$|(u_j, \psi)| \leq C\, ||\psi||_{C^n} \quad \forall j \in \mathcal{J}', \psi \in C_c^\infty(U_j), \operatorname{supp}\psi \subset \operatorname{supp}\chi_j$$

  for each $\varphi \in C_c^\infty(U)$ with $\operatorname{supp}\varphi \subset K$.So now upon taking triangle inequality on $(u, \varphi) = \sum_{j\in\mathcal{J}'} (u, \varphi_j)$ which to remind again is a finite sum,the following will now make sense

  $$\begin{aligned}
  |(u, \varphi)| &\leq \sum_{j\in\mathcal{J}'} |(u_j, \chi_j\varphi)| \\
  &\leq C \sum_{j\in\mathcal{J}'} ||\chi_j\varphi||_{C^n} \\
  &\leq C'\, ||\varphi||_{C^n}
  \end{aligned}$$

  This immediately defines the required bound of the linear map u and so by definition proves is indeed a distribution as desired.

- Finally we claim that we have $u|_{U_j} = u_j$ for all $j \in \mathcal{J}$ that is $(u, \varphi) = (u_j, \varphi)$ for each $\varphi \in C_c^\infty(U_j)$.Again taking a partition of unity $(u, \varphi) = \sum_{j\in\mathcal{J}} (u_j, \chi_j\varphi)$ but this time only on the compact set $\operatorname{supp}\varphi$ is only U_j not $U = \bigcup U_j$ like in the other earlier proofs. So now we have only 1 term in the summand where $\chi_j = 1$ on $\operatorname{supp}\varphi$ and $\chi_{j'} = 0$ for all $j' \neq j$. Hence

  $$(u, \varphi) = (u_j, \chi_j\varphi) = (u_j, \varphi)$$

3 operations with distributions(3)

3.1 differentiation

Definition 45

The deritivative of a distribution is defined to satisfy

1. we are looking for linear operator $\tilde{\partial}_{x_j} : \mathcal{D}'(U) \to \mathcal{D}'(U)$ where $U \Subset \mathbb{R}^n$
2. the operator should agree with the classical partial derivative on well defined functions. That is $f \in C^1 \quad \Rightarrow \tilde{\partial}_{x_j} f = \partial_{x_j} f$. Here as we know $C^1(U) \subset L^1_{\text{loc}}(U)$(recall 10) is embedded into $\mathcal{D}'(U)$ similar to how we did in 28
3. The operator should also be (sequentially) continuous that is if $u_k \in \mathcal{D}'(U)$ is a sequence converging to 0 in $\mathcal{D}'$ then $\tilde{\partial}_{x_j} u_k \to 0$ in $\mathcal{D}'(U)$ as well

do define such an operator we do so by **duality**:

1. First let us take a well defined function $f \in C^1(U)$ and a test function $\varphi \in C_c^\infty(U)$. Using integration by parts as in 24 we see that
$$(\partial_{x_j} f, \varphi) = \int_U (\partial_{x_j} f)\varphi dx = -\int_U f(\partial_{x_j}\varphi)dx = -(f, \partial_{x_j}\varphi)$$
2. similarly for distributions, we may also take (1) as the definition of $\tilde{\partial}_{x_j}$, more precisely if $u \in \mathcal{D}'(U)$ and $\varphi \in C_c^\infty(U)$ then we define
$$(\tilde{\partial}_{x_j} u, \varphi) = -(u, \partial_{x_j}\varphi)$$
Here we use that $\partial_{x_j}\varphi \in C_c^\infty(U)$
3. It is direct to see that (2) defines a linear functinoal $\tilde{\partial}_{x_j} u : C_c^\infty(U) \to \mathbb{C}$. We now show that this functinoal satisfies the bound in 26 and thus gives a distribution $\tilde{\partial}_{x_j} u \in \mathcal{D}'(U)$. Fix an arbitrary $K \Subset U$. Since u is a distribtuion we it does the satisfy the bound where there exists C, N such that
$$|(u, \varphi)| \leq C\,||\psi||_{C^n} \quad \forall \psi \in C_c^\infty(U) \text{ such that } \operatorname{supp}\psi \subset K$$
If $\varphi \in C_c^\infty(U)$ and $\operatorname{supp}\varphi \subset K$ then we apply the above bound with $\varphi = \partial_{x_j}\varphi$ to get
$$\underbrace{\left|(\tilde{\partial}_{x_j} u, \varphi)\right| = \left|(u, \partial_{x_j}\varphi)\right|}_{\text{from (2)}} \leq C\,\left|\left|\partial_{x_j}\varphi\right|\right|_{C^n} \leq C\,||\varphi||_{C^{N+1}}$$
4. From (1) we see that if $f \in C^1(U)$ then $\tilde{\partial}_{x_j} f = \partial_{x_j} f$. Moreover the operator $\tilde{\partial}_{x_j}$ is sequentialy continuous in $\mathcal{D}'(U)$. Indeed if $u_k \in \mathcal{D}'(U)$ converges to zero in $\mathcal{D}'(U)$ then for each $\varphi \in C_c^\infty(U)$ we have
$$(\tilde{\partial}_{x_j} u_k, \varphi) = -(u_k, \partial_{x_j}\varphi) \to 0$$
That is u_k becomes the zero distribution so for any functional $\partial_{x_j}\varphi$ it takes in the output is zero and thus $\tilde{\partial}_{x_j} u_k \to 0$ in $\mathcal{D}'(U)$ as well.

Fact 46
We have hence constructed our desired $\tilde{\partial}_{x_j}$ that satisfies properties (1)-(3) above. By a slight abuse of notation we will henceforth forget the tilde for the rest of the notes and write

$$\partial_{x_j} u = \tilde{\partial}_{x_j} u \quad \forall \mathcal{D}'(U)$$

We remark that we still have $\partial_{x_j}\partial_{x_\ell} = \partial_{x_\ell}\partial_{x_j}$ so we can define $\partial_x^a : \mathcal{D}'(U) \to \mathcal{D}(U)$ for any multindex a.

Having defined derivatives of destributions we look at a few examples with $U = \mathbb{R}$.

Example 47 (absolute function)
$u(x) = |x|$. To compute $u' = \partial_x u \in \mathcal{D}'(\mathbb{R})$ take arbitrary $\varphi \in C_c^\infty(\mathbb{R})$ and write

$$\begin{aligned}(u', \varphi) = -(u, \varphi') &= -\int_{\mathbb{R}} |x|\, \varphi'(x)dx \\ &= \int_{-\infty}^{0} x\varphi'(x)dx - \int_{0}^{\infty} x\varphi'(x)dx \\ &= -\int_{0}^{-\infty} \varphi(x)dx + \int_{0}^{\infty} \varphi(x)dx\end{aligned}$$

where in the last equality we used integration by parts with the boundary terms being zero(due to compact support). Notice that our result is equivalent to

$$(u', \varphi) = \int \operatorname{sgn}(x)\varphi(x)dx$$

This shows that $\partial_x |x|$ is given by the locally integrable function $\operatorname{sgn} x$

$$\partial_x |x| = \operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Example 48 (Heavside function)
The heaveside function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Take arbitrary $\varphi \in C_c^\infty(\mathbb{R})$ and compute

$$(H', \varphi) = -(H, \varphi') = -\int_0^\infty \varphi'(x)dx = \varphi(0)$$

where the last equality follows by the fundamental theorem of calculus.Notice that

$$(H', \varphi) = (\delta_0, \varphi) = \varphi(0)$$

that is to say the distributional derivative of the heavside function is the dirac delta "function"(recall 16)

$$H'(x) = \delta_0(x)$$

We are now ready to solve our first differential equation $u' = 0$.

Proposition 49

Assume that $U \subset \mathbb{R}$ is an open inerval, $u \in \mathcal{D}'(U)$ and $u' = 0$. Then u is a constant function.

Proof. The statement $u' = 0$ in distributions in equivalent to

$$(u, \psi') = 0 \quad \forall \psi \in C_c^\infty(U)$$

We now rewrite this as follows

$$(u, \varphi) = 0 \quad \forall \varphi \text{ in the space } \nu = \{\psi' | \psi \in C_c^\infty(U)\}$$

We claim that the space ν has codimension 1 inside $C_c^\infty(U)$ that is it can charaterized by

$$\nu = \left\{\varphi \in C_c^\infty(U) | \int_U \varphi(x)dx = 0\right\}$$

To check this we show that each $\varphi \in C_c^\infty(U)$ which integrates to 0 can be written as some $\psi \in C_c^\infty(U)$ which can be done by putting $\psi(x) = \int_a^x \varphi(t)dt$ where $a \in U$ lies to the left of $\operatorname{supp} \varphi$. Quite straightforward to see using basic calculus. Now fix $\chi_0 \in C_c^\infty(U)$ such that $\int_U \chi_0(x) = 1$.(the bump function 15). Then for each $\varphi \in C_c^\infty(U)$ we have

$$\varphi - (1, \varphi)\chi_0 \in \nu \text{ where } (1, \varphi) = \int_U \varphi(x)dx$$

then by the above we have for all $\varphi \in C_c^\infty(U)$

$$(u, \varphi) = (1, \varphi)(u, \chi_0) = ((u, \chi_0)1, \varphi)$$

that is $u = (u, \chi_0)1$ is a constant function $\square$

3.2 mutliplication by smooth functions

We now seek the extend to our operations to distributions involving the multiplication operator

$$f \in L^1_{\text{loc}}(U) \mapsto af$$

where $a \in C_c^\infty(U)$ is given. For each $f \in L^1_{\text{loc}}(U)$ and a test function $\varphi \in C_c^\infty(U)$ we have

$$(af, \varphi) = \int_U a(x)f(x)\varphi(x)dx = (f, a\varphi)$$

Definition 50

Thus we define for $u \in \mathcal{D}'(U)$ and $a \in C^\infty(U)$ the product $au \in \mathcal{D}'(U)$ as follows

$$(au, \varphi) = (u, a\varphi) \quad \forall \varphi \in C_c^\infty(U)$$

arguing similar to how we did for 45 we should be able to see that au is indeed a distribution and tha map $u \mapsto au$ is sequentially continuous.

Proposition 51

Lebniz Rule for distributions Assume that $u \in \mathcal{D}'(U)$ and $a \in C^\infty(U)$. Then

$$\partial_{x_j}(au) = (\partial_{x_j} a)u + a(\partial_{x_j} u)$$

Note the abuse of notation where ∂_{x_j} refers to the distributional deritivative as noted earlier.

Proof. By direct computation letting $\varphi \in C_c^\infty(U)$ we then have

$$\begin{aligned}
(\partial_{x_j}(au), \varphi) &= -(au, \partial_{x_j}\varphi) = -(u, a(\partial_{x_j}\varphi)) \\
((\partial_{x_j} a)u, \varphi) &= (u, (\partial_{x_j} u)\varphi) \\
(a(\partial_{x_j} u), \varphi) &= (\partial_{x_j} u, a\varphi) = -(u, \partial_{x_j}(a\varphi))
\end{aligned}$$

which gives our proposition as desired since $\partial_{x_j}(a\varphi) = (\partial_{x_j} a)\varphi + a(\partial_{x_j}\varphi)$ where we simply applied the familiar product rule since $a\varphi$ is a product of differentiable well defined functions. Therefore this immediately implies

$$((\partial_{x_j} a)u, \varphi) = (\partial_{x_j}(au), \varphi) + (a(\partial_{x_j} u), \varphi)$$

which directly implies our proposition

4 distributions and support(4)

4.1 Support of distribution

Like how we defined a support for well defined functions as in 7 we seek to do so analagously for distributions. To that end we define

Definition 52

Let $U \Subset \mathbb{R}^n$ and $u \in \mathcal{D}'(U)$. We say a point $x \in U$ does *not* lie in $\operatorname{supp} u$ if there exists $V \Subset U$ containing x and such that $u|_V = 0$. This defines the subset

$$\operatorname{supp} u \subset U$$

Similar to the case for differentiation notice that for $f \in C^(U)$ 7 and 52 agree, that is they both give the same set $\operatorname{supp} f$.

Proposition 53

Let $u \in \mathcal{D}'(U)$. Then

$$u|_{U/\operatorname{supp} u} = 0$$

That is if $\varphi \in C_c^\infty(U)$ and $\operatorname{supp} u \cap \operatorname{supp}\varphi = \emptyset$ then $(u, \varphi) = 0$.

Proof. For each $x \in U/\operatorname{supp} u$ there exists $V_x \Subset U/\operatorname{supp} u$ containing x and such that $u|_{V_x} = 0$. The sets V_x cover $U/\operatorname{supp} u$ so by the uniqueness part of 44 applied to $u|_{U/\operatorname{supp} u}$ we see that $u|_{U/\operatorname{supp} u} = 0$.

4.2 Distributions with compact support

So far cleary we have defined $\mathcal{D}'(U)$ as the dual to the space $C_c^\infty(U)$. We now consider the dual to the space $C^\infty(U)$(which likewise the space of smooth functions but however not necessarily compactly supported). We denote such a dual space by $\mathcal{E}'(U)$.

Definition 54

For each $K \Subset U$ and $\varphi \in C^\infty(U)$ define the **seminorm**

$$||\varphi||_{C^N(U,K)} = \max_{|a|\leq N} \sup_{x\in K} |\partial_x^a \varphi(x)|$$

Notice the nesscity of defining for all compact sets K. This is because only if a set is compact will the maximum of a continuous function exist on it. Previously this was not required in the C_c^∞ case because they are compactly supported by definition we can specifiy the norm on the whole of U knowing that terms not on the compact support vanish anyway.So by definition this includes all compact sets on U as well.

Definition 55

Let $\varphi_k \in C^\infty(U)$ be a sequence and $\varphi \in C^\infty(U)$. We say that $\varphi_k \to \varphi$ in $C^\infty(U)$ if

$$||\varphi_k - \varphi||_{C^k(U,K)} \to 0 \quad \text{as } k \to \infty \quad \forall K \Subset U \text{ and } N$$

Essentially such a definition requires uniform convergence of all derivatives on *every compact set*

Let us compare this definition with that of the C_c^∞ case 35

- (Similarity)As a reminder here uniform here since we took the sup over K in the definition of seminorm. That should convergence occur there must had been a k_0 where all $k > k_0$ converges. Such k_0 then clearly applies to *all* k. This was also true for the $C_c^\infty(U)$ case too clearly.
- (Difference)our definition here is more broad in a sense that the definition for the $C_c^\infty(U)$ case is a subset of this(as it should given that $C_c^\infty(U) \subset C^\infty(U)$).The previous case specified not just convergence on every compact set specifically one of them must be the case where supp φ_k is supported in it for all k.

Definition 56

Let $u : C^\infty(U) \to \mathbb{C}$ be a linear functional. We say that u lies in $\mathcal{E}'(U)$ if it is **sequentially continuous** namely for each sequence φ_k converging to 0 in $C^\infty(U)$ we have $u(\varphi_k) \to 0$.

As in the case of $\mathcal{D}'(U)$ we use the notation $(u, \varphi) = u(\varphi)$ when $u \in \mathcal{E}'(U)$ and $\varphi \in C^\infty(U)$.

Definition 57

The functional $u : C^\infty(U) \to \mathbb{C}$ can be restricted to $C_c^\infty(U)$ which yields the distribution in $D'(U)$ by 36 and since $\varphi_k \to 0$ in $C_0^\infty(U)$ this implies $\varphi_k \to 0$ in $C^\infty(U)$ as well. These are achieved via the operator

$$\iota : \mathcal{E}'(U) \to \mathcal{D}'(U), \quad (\iota(u), \varphi) = (u, \varphi) \quad \forall u \in \mathcal{E}(U)', \varphi \in C_c^\infty(U)$$

The next theorem shows that ι is injective and its range is exactly the space of distributions in $\mathcal{D}'(U)$ with compact support. That is ι gives the following identification

$$\boxed{\mathcal{E}'(U) \simeq \{u \in \mathcal{D}'(U) | \operatorname{supp} u \Subset U\}}$$

That is to say $\mathcal{E}'(U) \subset \mathcal{D}'(U)$!

Theorem 58

consider some relationships between $\mathcal{E}'(U)$ and $\mathcal{D}'(U)$.

1. Assume that $\mathcal{E}'(U)$ and $\iota(u) = 0$. Then $u = 0$.
2. Assume that $u \in \mathcal{E}'(U)$. Then $\operatorname{supp} \iota(u) \Subset U$
3. Assume that $v \in \mathcal{D}'(U)$ and $\operatorname{supp} v \Subset U$. Then there exists $u \in \mathcal{E}'(U)$ such that $\iota(u) = v$

Proof. Consider

1. Take arbitrary $\varphi \in C^\infty(U)$. Then there exists a sequence $\varphi_k \in C_c^\infty(U)$ which coverges to φ in $C^\infty(U)$. Indeed using 20 take a sequnce of compact subsets exhausting U:

   $$U = \bigcup_{k=1}^{\infty} K_k, K_k \Subset U$$

   take cutoff functions(many ways to do this,bump functions are one way)

   $$\chi_k \in C_c^\infty(U), \qquad \underbrace{\operatorname{supp}(1 - \chi_k) \cap K_k = \emptyset}_{\text{recall this is another way 1 on } K_k \text{ 0 otherwise}}$$

   and put $\varphi_k = \chi_k \varphi \in C_c^\infty(U)$. Then $\varphi_k \to \varphi$ in $C^\infty(U)$ since for any arbitrary compact set $K \Subset U$ there exists k_0 such that for all $k \geq k_0$ we have $K \subset K_k$(again recall how exhaustions are defined 20) and thus $||\varphi_k - \varphi||_{C^n(U,K)} = 0$. Because on K we have $\varphi_k = \varphi$ by definition of χ_k.Recall the definitions above if you can't see this! Now since $\iota(u) = 0$ we have $(u, \varphi_k) = 0$ for all k recall 4.2(restriction is now a zero distribution). Passing to the limit we see that $(u, \varphi) = 0$(by sequential continuitity 4.2) as well which shows that $u = 0$

2. we argue by contradiction. Assume that $\operatorname{supp} \iota(u)$ is not compactly contained in U. Take a sequence K_k(exhaustions) like before then we have $\operatorname{supp} \iota(u) \not\subset K_k$ for each k.(no compact subset can contain it, means it cant be contained in any of the exhaustions which cover the whole space). So now we may construct a test function $\varphi_k \in C_c^\infty(U/K_k)$ such that $(u, \varphi_k) = 1$ in a similar manner to 40. Clearly $\phi_k \to 0$ in $C^\infty(U)$ as in the limit its exhaustions will cover U. So we have uniform convergence to 0 on any compact subset of U thus we drop the c subscript. However like above, applying sequential continuitity we know that then $(u, \varphi_k) = 1$ which contradicts its definition just earlier.

3. Fix a cutoff $\chi \in C_c^\infty(U)$ such that $\operatorname{supp}(1 - \chi) \cap \operatorname{supp} v = \emptyset$. For $\varphi \in C^\infty(U)$ define

   $$(u, \varphi) = (v, \chi\varphi)$$

   It is easy to check this defines $u \in \mathcal{E}'(U)$.You can directly see from the definition that v is the restricted distribution of u as desired. Indeed if $\varphi_k \to 0$ in C^∞ then $\chi\varphi_k \to 0$ in $C_c^\infty(U)$. Moreover if $\varphi \in C_c^\infty(U)$ then by 53 applied to v and $(1 - \chi)\varphi$ we have

   $$(v, \varphi) - (u, \varphi) = (v, (1 - \chi)\varphi) = 0$$

this shows that $\iota(u) = v$.

Recall in 16 we showed how we can extend $C_c^\infty(U) \to C_c^\infty(V)$ where $U \subset V$ via an extension by zero. We now show analagously for $\mathcal{E}'(U)$

Problem 59

Let $V \Subset U \Subset \mathbb{R}^n$ and $v \in \mathcal{E}'(V)$. Then there exists a *unique* $u \in \mathcal{E}'(U)$ such that $u|_V = v$ and $\operatorname{supp} u \subset V$. In fact we have $\operatorname{supp} u = \operatorname{supp} v$.

Proof. First we prove existence. By definition of $\mathcal{E}'(V)$, the distribution v acts on test functions $\varphi \in \mathcal{D}(V)$ (compactly supported smooth functions in V). Consider the extension u of v to U by defining u as:

$$u(\psi) = v(\psi|_V), \quad \forall \psi \in \mathcal{D}(U),$$

where $\psi|_V$ denotes the restriction of ψ to V. This is well-defined by 16

To verify that $u \in \mathcal{E}'(U)$, note that u is linear and continuous. Linearity is clear from the linearity of v. For continuity, let $\psi_k \to 0$ in $\mathcal{D}(U)$. Then $\psi_k|_V \to 0$ in $\mathcal{D}(V)$, and thus $v(\psi_k|_V) \to 0$ since v is continuous. Hence $u(\psi_k) \to 0$, proving sequential continuity.

Since ψ is supported in U, $u(\psi) = v(\psi|_V)$ implies that $\operatorname{supp} u \subset V$, because v does not depend on values of ψ outside V.

Now we prove uniqueness Suppose there exist two distributions $u_1, u_2 \in \mathcal{E}'(U)$ such that $u_1|_V = v$, $u_2|_V = v$, and $\operatorname{supp} u_1, \operatorname{supp} u_2 \subset V$. Then for any $\psi \in \mathcal{D}(U)$, we have

$$(u_1 - u_2)(\psi) = (u_1 - u_2)(\psi|_V).$$

Since $u_1|_V = u_2|_V = v$, it follows that $(u_1 - u_2)(\psi) = 0$ for all $\psi \in \mathcal{D}(U)$. Therefore, $u_1 = u_2$, proving uniqueness.

Finally, since u is defined by the action of v and $\operatorname{supp} u \subset V$, it follows that $\operatorname{supp} u = \operatorname{supp} v$, completing the proof. □

4.3 Frechet metric and Banach-Steinhaus for distribution

Definition 60 (Metric on $C^\infty(U)$)

Let $U \Subset \mathbb{R}^n$. Using 21 we take the exhaustions of U

$$U = \bigcup_{j=1}^{\infty} K_j, \quad K_j \Subset U, \quad K_j \Subset K_{j+1}$$

For $N \in \mathbb{N}$ define the nth **seminnorm** $||\bullet||_N$ on $C^\infty(U)$ as in 55

$$||\varphi||_N = ||\varphi||_{C^n(U, K_N)}$$

Problem 61

Let $\varphi_k \in C^\infty(U)$ be a sequence and $\varphi \in C^\infty(U)$. Prove that then $\varphi_k \to \varphi$ in $C^\infty(U)$ if and only if $||\varphi_k - \varphi||_N \to 0$ as $k \to \infty$ for each N. In particular the set of seminorms $||\bullet||_N$ makes $C^\infty(U)$ a complete space

Solution. left as an exercise. Hint: prove this in the same you have as in [4]

Problem 62

Assume that $\varphi_k \in C^\infty(U)$ is a *cauchy sequece* in the following sense

$$\sup_{k,\ell \geq r} ||\varphi_k - \varphi_\ell||_N \to 0 \quad \text{as r} \to \infty$$

Then there exists $\varphi \in C^\infty(U)$ such that $\varphi_k \to \varphi$ in $C^\infty(U)$

Solution. left as an exercise. Hint: prove this in the same you have as in [4]

Let us try to formulate 4.2 in terms of norm bounds like we have for $\mathcal{D}'(U)$

Proposition 63

Let $u : C^\infty(U) \to \mathbb{C}$ be a linear map. Then $u \in \mathcal{E}'(U)$ if and only if there exists $K \Subset U$ and constants C, N such that

$$|(u, \varphi)| \leq C\, ||\varphi||_{C^N(U,K)} \quad \forall \varphi \in C^\infty(U)$$

Proof. The proof here is very similar to the case for $\mathcal{D}'(U)$ done earlier. First,if this holds then it is immediate that $\varphi \to 0$ in $C^\infty(U)$ iplies that $(u, \varphi_k) \to 0$ and thus $u \in \mathcal{E}'(U)$ by sequential continuitity.

For the other direction suppose now that $u \in \mathcal{E}'(U)$. By contradiction suppose the proposition doesn't hold so for each N there exist

$$\varphi_N \in C^\infty(U), \quad (u, \varphi_N) = 1, \quad ||\varphi_N|| \leq \frac{1}{N}$$

Then $\varphi_N \to 0$ as $N \to \infty$ in $C^\infty(U)$ which contradicts the sequential continuitity of $u : C^\infty(U) \to \mathbb{C}$. □

Definition 64 (Frechet Metric)

We define the a metric on $C^\infty(U)$ via the following. For $\varphi, \psi \in C^\infty(U)$ put

$$d_{C^\infty}(\varphi, \psi) = \sum_{N=1}^{\infty} 2^{-N} \frac{||\varphi - \psi||_N}{1 + ||\varphi - \psi||_N}$$

Problem 65

Show that the metric d_{C^∞} satisfies

1. d_{C^∞} is indeed a well defined metric on $C^\infty(U)$
2. for a sequence $\varphi_k \in C^\infty(U)$ we have $\varphi_k \to \varphi$ in $C^\infty(U)$ if and only if $d_{C^\infty}(\varphi_k, \varphi) \to 0$
3. the metric space $(C^\infty(U), d_{C^\infty})$ is complete

Recall Banach Steinhaus also known as uniform boundedness theorem from [4]. We now define the analagous for distributions.

Theorem 66 (Banach Steinhaus for $\mathcal{E}'(U)$)
Let $U \Subset \mathbb{R}^n$ and assume that a sequence of copactly supported distributions $u_k \in \mathcal{E}'(U)$ is weakly bounded in the following sense:

for each $\varphi \in C^\infty(U)$ there exists C_φ such that for all k we have

$$\begin{aligned} &\operatorname{supp} u_k \subset K \\ &|(u_k, \varphi)| \leq C\, ||\varphi||_{C^N(U,K)} \text{ for all } \varphi \in C^\infty(U) \end{aligned}$$

Proof. For $L \in \mathbb{N}$ define the subset of $C^\infty(U)$ and $\varphi \in A_L$ for all m. For each k we have $u_k \in \mathcal{E}'(U)$ so $(u_k, \varphi_m) \to (u_k, \varphi)$ as $m \to \infty$. Thus $|(u_k, \varphi_m)| \leq L$ implies that $|(u_k, \varphi)| \leq L$ which shows that $\varphi \in A_L$. By the weak bound as defined above we have

$$C^\infty(U) = \bigcup_{L \geq 1} A_L$$

Then by the **Baire Category theorem** for complete metric spaces(recall [4]) we can fix L such that the interior A_L is nonempty that is A_L contains a metric ball:

$$B_{d_{C^\infty}}(\psi, \varepsilon) \subset A_L \quad \text{for some } \psi \in C^\infty(U), \varepsilon > 0$$

from the above we get

$$B_{d_{C^\infty}}(0, \varepsilon) \subset A_{2L}$$

Indeed take arbitrary $\varphi \in B_{d_{C^\infty}}(0, \varepsilon)$. Then both $\psi + \varphi$ and ψ lie in $B_{d_{C^\infty}}(\psi, \varepsilon)$ which is contained in A_L thus $\varphi \in A_{2L}$...to be continued

5 Convolution I(6)

5.1 Convolution of a distribution and a smooth function

Recall that we have defined the convolution earlier for two well defined functions. Now we seek to extend this to distributions.

Definition 67
For a function $\varphi \in C^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ define the function

$$\varphi(x, \bullet) \in C^\infty(\mathbb{R}^n), \quad \varphi(x - \bullet)(y) = \varphi(x - y)$$

Then the convolution can be written in terms of the pairing $(\bullet, \bullet)$ as

$$f * \varphi(x) = (f, \varphi(x - \bullet))$$

You can see $(f, \varphi(x - \bullet))$ as for a fixed x we take the inner product over y that is $(f * y)(x) = \int f(y)\varphi(x - y)dy$ which is literally the definition of the convolution.

Definition 68

Assume $u \in \mathcal{D}'(\mathbb{R}^n), \varphi \in C^\infty(\mathbb{R}^n)$ and either u or φ is compactly supported. Define the function $u * \varphi : \mathbb{R}^n \to \mathbb{C}$ by

$$u * \varphi(x) = (u, \varphi(x - \bullet)), \quad x \in \mathbb{R}^n$$

Fact 69

As we know a convolution between 2 functions always defines a function. But notice the definition above makes sense even though we convolved a function with distribution(which likewise defines a function too clearly) because by definition above the output is $\mathbb{C}$ and the input is $x \in \mathbb{R}^n$.

let us see an example of how this works. recall δ_0 is not a function but a distribution. Let's see how its is convolved to get a function.

Example 70 (Convolution with a delta function)

Show that for any $\varphi \in C^\infty(\mathbb{R}^n)$ we have

$$\delta_0 * \varphi = \varphi$$

Solution. Let $x \in \mathbb{R}^n$ then

$$\delta_0 * \varphi(x) = (\delta_0, \varphi(x - \bullet)) = \varphi(x - \bullet)(0) = \varphi(x)$$

$\square$

We now want to show that our convolution of a distribution with a smooth function is smooth(analagous to how we did for a continuous function and a smooth function earlier). And that this result applies to both $\mathcal{E}'(U)$ and $\mathcal{D}'(U)$.

Proposition 71

Assume that $U \Subset \mathbb{R}^n, V \Subset \mathbb{R}^m$ and

$$v \in \mathcal{E}'(V), \psi \in C^\infty(U \times V) \tag{1}$$

For $x \in U$ define $\psi(x, \bullet) \in C^\infty(V)$ by $\psi(x, \bullet)(y) = \psi(x, y)$ then the function

$$f(x) = (v, \psi(x, \bullet)), \quad x \in U \tag{2}$$

lies in $C^\infty(U)$ with derivatives given by $\partial_x^a f(x) = (v, \partial_x^a \psi(x, \bullet))$

Assume that (1) holds. Now recall 63 there exists $K_V \Subset V$ and constants C, N such that

$$|(v, \varphi)| \leq C \, ||\varphi||_{C^N(V, K_V)} \quad \forall \varphi \in C^\infty(V)$$

Fix $x \in U$ and estimate $\tilde{x} \in U$ close to x.

$$|f(x) - f(\tilde{x})| = |(v, \psi(x, \bullet) - \psi(\tilde{x}, \bullet))| \leq C \, ||\psi(x, \bullet) - \psi(\tilde{x}, \bullet)||_{C^N(V, K_V)}$$

since $\psi \in C^\infty(U \times V)$ then we have for $\tilde{x} \to x$

$$||\psi(x, \bullet) - \psi(\tilde{x}, \bullet)||_{C^N(V, K_V)} = \max_{|\beta| \leq N} \sup_{y \in K_V} \left|\left|\partial_y^\beta \psi(x, y) - \partial_y^\beta \psi(\tilde{x}, y)\right|\right| \to 0$$

by the continuity of norms. This immediately shows continuitity.

We now show that f is differentiable(in the classical sense) that is

$$\partial_{x_j} f(x) = (v, \partial_{x_j}\psi(x, \bullet))$$

assuming that (1) holds we again similarly fix $x \in U$ and estimate for small $t \in \mathbb{R}$

$$\begin{aligned} \left|f(x+te_j) - f(x) - t(v, \partial_{x_j}\psi(x,\bullet))\right| &= \left|v, \psi(x+te_j, \bullet) - \psi(x,\bullet) - t\partial_{x_j}\psi(x,\bullet)\right| \\ &\leq C\left|\left|\psi(x+te_j,\bullet) - \psi(x,\bullet) - t\partial_{x_j}\psi(x,\bullet)\right|\right|_{C^N(V,K_V)} \end{aligned}$$

since $\psi \in C^\infty$ we have as $t \to 0$

$$\left|\left|\psi(x+te_j,\bullet) - \psi(x,\bullet) - t\partial_{x_j}\psi(x,\bullet)\right|\right|_{C^N(V,K_V)} = o(t)$$

where we used big o notation here. This shows that

$$\lim_{t\to 0} \frac{f(x+te_j) - f(t)}{t} = (v, \partial_{x_j}\psi(x,\bullet))$$

as desired □

As promised we now consider the case for $\mathcal{D}'(U)$ which recall 57 is a superset of $\mathcal{E}'(U)$.

Corollary 72

The same conclusion above holds if instead we assume that $v \in \mathcal{D}'(V), \psi \in C^\infty(U \times V)$ and the restriction of the projection $\pi_x : U \times V \to U$ to $\operatorname{supp}\psi$ is proper, namely for each compact subset $K_U \subset U$ the preimage $\pi_x^{-1}(K_U) \cap \operatorname{supp}\psi$ is compact

Remark 73. *Notice instead of $\mathcal{E}'(V)$ we are dealing with $\mathcal{D}'(V)$ now. The restriction of the projection means we have*

$$\pi_x|_{\operatorname{supp}\psi} : \{U \times V\} \cap \operatorname{supp}\psi \to \{u \in (u,v) \in U \times V | \psi(u,v) \neq 0\} \to U$$

. Note that the restriction does not change the value of u. If simply drops the v for every (u,v) pair. Also recall the definition of proper from [3]

Proof. Assuming this alternative condition holds, we fix $x \in U$ and put $K_U = B(x,\varepsilon)$ where $\varepsilon > 0$ is small enough so that $K_U \Subset U$.Let this be a compact set in the range of the restriction. Since $\pi_x|_{\operatorname{supp}\psi}$ is proper there exists $K_V \Subset V$ such that $\operatorname{supp}\psi(\tilde{x},\bullet) \subset K_V$ for all $\tilde{x} \in (K_U)$. This refers to the compact set in the domain:

$$\operatorname{supp}\psi(\tilde{x},\bullet) = \{v \in V | \psi(\tilde{x}, v) \neq 0\} \subset K_V \Subset V$$

Hence $\psi(\tilde{x},\bullet) \in C_c^\infty(V)$ now so we use 26, applying the bound for v with this set K_V we similarly get like above a C, N such that for all $\tilde{x} \in (K_U)$

$$|f(x) - f(\tilde{x})| \leq C\,||\psi(x,\bullet) - \psi(\tilde{x},\bullet)||_{C^N} \to 0 \quad \tilde{x} \to x$$

giving again continuitity of f. □

Remark 74. *Notice that the use of restriction operator is to restrict $\varphi \in C^\infty(U \times V)$ to a compactly supported case as required to prove for the $\mathcal{D}'(U)$ which takes in only compactly supported smooth functions.*

With this we now prove

Theorem 75
Assume that $u \in \mathcal{D}'(\mathbb{R}^n), \varphi \in C^\infty(\mathbb{R}^n)$ and either u or φ is compactly supported. Then $u * \varphi \in C^\infty(\mathbb{R}^n)$ and

$$\partial_x^a(u * \varphi) = u * (\partial_x^a \varphi) = (\partial_x^a u) * \varphi$$

Proof. Define $\psi \in C^\infty(\mathbb{R}^{2n})$ by the formula

$$\psi(x, y) = \varphi(x - y)$$

then we may rewrite 68 as

$$u * \varphi(x) = (u, \psi(x, \bullet))$$

If $u \in \mathcal{E}'(\mathbb{R}^n)$ then applying 71 we have

$$\partial_x^a(u * \varphi) = (u, \partial_x^a \psi(x, \bullet)) = u * \partial_x^a \varphi$$

as desired. Having proven the u is compactly supported case(recall 57) we move on to if instead $\varphi \in C_c^\infty(\mathbb{R}^n)$ then the projection $\pi_x|_{\operatorname{supp}\psi}$ is proper. Indeed if $K \subset \mathbb{R}^n$ is compact then we have

$$\pi_x^{-1}(K) \cap \operatorname{supp}\psi = \left\{(x, y) \in \mathbb{R}^{2n} | x \in K, x - y \in \operatorname{supp}\varphi\right\}$$

which is a compact set as it is the image of $K \times \operatorname{supp}\varphi$ by a continuous map. Finally the last equality in theorem follows by 45 the definition of distributive differentiation where we have

$$u * (\partial_x^a \varphi)(x) = (u, (\partial_x^a \varphi)(x - \bullet)) = (-1)^{|a|}(u, \partial_y^a(\varphi(x - \bullet)))$$

which is equal to

$$(\partial_y^a u, \varphi(x - \bullet)) = (\partial_x^a u) * \varphi(x)$$

□

Problem 76
Consider

1. if $u \in \mathcal{D}'(U), \varphi \in C^\infty(\mathbb{R}^n)$ and either u or φ is compactly supported then

$$\sup(u * \varphi) \subset \operatorname{supp} u + \operatorname{supp}\varphi$$

   In particular if both u and φ are compactly supported then so is their convolution

2. if $u \in \mathcal{D}'(U)$ and $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$ then

$$(u * \varphi) * \psi = u * (\varphi * \psi)$$

Proof. We will prove each part of the problem separately.

For (1) Let $u \in \mathcal{D}'(U)$, $\varphi \in C^\infty(\mathbb{R}^n)$, and assume that either u or φ is compactly supported(recall 57). Recall that the convolution $u * \varphi$ is defined by:

$$(u * \varphi)(x) = \langle u, \varphi(x - \cdot)\rangle.$$

For $x \in \mathbb{R}^n$, the function $\varphi(x - \cdot)$ has its support shifted by x, i.e., $\operatorname{supp}\varphi(x - \cdot) = x - \operatorname{supp}\varphi$. Hence, $(u * \varphi)(x) \neq 0$

only if the shifted support of φ intersects $\operatorname{supp} u$, that is:

$$x \in \operatorname{supp} u + \operatorname{supp} \varphi.$$

Therefore, $\operatorname{supp}(u * \varphi) \subset \operatorname{supp} u + \operatorname{supp} \varphi$.

In the particular case where both u and φ are compactly supported, $\operatorname{supp} u$ and $\operatorname{supp} \varphi$ are bounded sets. Their Minkowski sum $\operatorname{supp} u + \operatorname{supp} \varphi$ is also bounded, so $u * \varphi$ has compact support.

For (2) Let $u \in \mathcal{D}'(U)$ and $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$. By the definition of convolution, we have:

$$(u * \varphi)(x) = \langle u, \varphi(x - \cdot) \rangle.$$

Convolving $u * \varphi$ with ψ yields:

$$((u * \varphi) * \psi)(x) = \int_{\mathbb{R}^n} (u * \varphi)(y) \psi(x - y) \, dy.$$

Substituting the expression for $u * \varphi$, we get:

$$((u * \varphi) * \psi)(x) = \int_{\mathbb{R}^n} \langle u, \varphi(y - \cdot) \rangle \psi(x - y) \, dy.$$

Interchanging the order of integration (justified by the compact support of φ and ψ), this becomes:

$$((u * \varphi) * \psi)(x) = \langle u, \int_{\mathbb{R}^n} \varphi(y - \cdot) \psi(x - y) \, dy \rangle.$$

The inner integral defines the convolution $\varphi * \psi$:

$$(\varphi * \psi)(x - \cdot) = \int_{\mathbb{R}^n} \varphi(y - \cdot) \psi(x - y) \, dy.$$

Substituting back, we find:

$$((u * \varphi) * \psi)(x) = \langle u, (\varphi * \psi)(x - \cdot) \rangle.$$

By the definition of convolution, this is precisely:

$$u * (\varphi * \psi)(x).$$

Hence, $(u * \varphi) * \psi = u * (\varphi * \psi)$, completing the proof. $\square$

5.2 Approximation of distributions by smooth functions

The main goal of this section is to show that the space $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$ and also $\mathcal{E}'(U)$ by a similar argument. We first consider the case of $\mathbb{R}^n$. Fixing a bump function similar to how we have done so previously we have

$$\chi \in C_c^\infty(\mathbb{R}^n) \quad \operatorname{supp} \chi \subset B(0,1) \quad \int_{\mathbb{R}^n} \chi(x) dx = 1$$

and define the rescaling for $\varepsilon > 0$

$$\chi_\varepsilon(x) = \varepsilon^{-n} \chi\left(\frac{x}{\varepsilon}\right) \in C_c^\infty(\mathbb{R}^n)$$

now consider

Theorem 77 (Density of distribution: $\mathbb{R}^n$ case)
Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and define for $\varepsilon > 0$

$$u_\varepsilon = u * \chi_\varepsilon$$

which lies in $C^\infty(\mathbb{R}^n)$ by 75. Then

$$u_\varepsilon \to u \text{ as } \varepsilon \to 0+ \text{ in } \mathcal{D}'(\mathbb{R}^n)$$

before we proceed consider

Lemma 78
Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and u_ε be defined as in the above, $\varphi \in C_c^\infty(\mathbb{R}^n)$ and define $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ by

$$\varphi_\varepsilon(y) = \int_{\mathbb{R}^n} \chi_\varepsilon(x-y)\varphi(x)dx = \chi_\varepsilon * \varphi$$

then we have

$$(u_\varepsilon, \varphi) = (u, \varphi_\varepsilon)$$

Remark 79. *In otherwords passing functions convolved with bump function as arguments to a distribution is equivalent to convolving the distribution with bump function then passing functions in as arguments*

Proof. If $u_\varepsilon \in L^1_{\text{loc}}(\mathbb{R}^n)$ then the lemma follows by Fubini' theorem

$$(u_\varepsilon, \varphi) = \int_{\mathbb{R}^{2n}} u(y)\chi_\varepsilon(x-y)\varphi(x)dxdy = (u, \varphi_\varepsilon)$$

However we need to also handle the case when u is a distribution so first denote

$$\varphi_\varepsilon = \int_{\mathbb{R}^n} \chi_\varepsilon(x - \bullet)\varphi(x)dx$$

Paring both sides with the distribution u and putting the pairing inside the integral we get

$$(u, \varphi_\varepsilon) = \int_{\mathbb{R}^n} (u, \chi_\varepsilon(x-\bullet))\varphi(x)dx = \int_{\mathbb{R}^n} u_\varepsilon(x)\varphi(x)dx$$

which gives our lemma as desired. To see how this works the above is possible we need to show that pairing with u can be put under the integral sign as we did. A way to do so is use **Riemann sums**. Namely for $\delta > 0$ define the Riemann sum for the integral above via

$$\mathcal{R}_\delta = \delta^n \sum_{x \in \delta\mathbb{Z}^n} \chi_\varepsilon(x - \bullet)\varphi(x) \in C_c^\infty(\mathbb{R}^n)$$

we claim that

$$\mathcal{R}_\delta \to \varphi_\varepsilon \text{ as } \delta \to 0+ \text{ in } C_c^\infty(\mathbb{R}^n)$$

Indeed the support condition is immediate since φ and χ_ε are compactly supported. Next observe that indeed the convergence result is true, following in the same way that usual reinmann sums to the integral.Well they are clearly continuous functions on a closed and bounded interval(by definition compact for $\mathbb{R}^n$). You don't have to consider the rest that vanish. Therefore this is reinmann integrable. Note that δ^n reflects the infinitesimal change in all n components of $x \in \mathbb{R}^n$.Since $u \in \mathcal{D}'(\mathbb{R}^n)$ by the sequential continuity property we know that

$$\lim_{\delta \to 0+} (u, \mathcal{R}_\delta - \varphi_\varepsilon) = 0$$

applying linearity of u this is equivalent of saying $(u, \mathcal{R}_\delta) \to (u, \varphi_\varepsilon)$ as $\delta \to 0+$. Now consider

$$(u, \mathcal{R}_\delta) = \delta^n \sum_{x \in \delta\mathbb{Z}^n} (u, \chi_\varepsilon(x - \bullet))\varphi(x) = \delta^n \sum_{x \in \delta\mathbb{Z}^n} u_\varepsilon(x)\varphi(x)$$

where the ability to bring the sum and δ^n once again follows by linearity of u. But once again notice this precisely another riemann sum this time for the function $u_\varepsilon\varphi$. Recall that u_ε is a function and a smooth one as well. So we may once again take this to be rienmann integral and know that as $\delta \to 0+$ we have

$$\int_{\mathbb{R}^n} u_\varepsilon(x)\varphi(x)dx = (u_\varepsilon, \varphi)$$

as desired. □

We are now ready to prove 77. First fix $\varphi \in C_c^\infty(\mathbb{R}^n)$. We need to show that

$$(u_\varepsilon, \varphi) \to (u, \varphi) \text{ as } \varepsilon \to 0$$

By 78 we have $(u_\varepsilon, \varphi) = (u, \varphi_\varepsilon)$. Since u is a distribution by it suffices to show that

$$\varphi_\varepsilon \to \varphi \text{ in } C_c^\infty(\mathbb{R}^n) \text{ as } \varepsilon \to 0+$$

which as we have seen 1, it does indeed. □

We now generalize our result to open sets in general and define

Theorem 80 ([Density of distribution: General Open set case)
Let $U \Subset \mathbb{R}^n$ and $u \in \mathcal{D}'(U)$. Then there exists a sequence

$$f_k \in C_c^\infty(U), f_k \to u \text{ in } \mathcal{D}'(U)$$

Proof. once again we take the exhaustion of U

$$U = \bigcup_{k=1}^\infty K_k, K_k \Subset K_{k+1}$$

and the cut off function

$$\psi_k \in C_c^\infty(U), \operatorname{supp}(1 - \psi_k) \cap K_k = \emptyset$$

and fix a number $\varepsilon_k > 0$ small enough so that

$$\operatorname{supp} \psi_k + B(0, \varepsilon_k) \Subset U$$

we also require that $\varepsilon_k \to 0$ as $k \to 0$ for example we could let $\varepsilon_k = \frac{1}{k}$. Let χ_ε be the scaled bump function from previously and put

$$f_k(\psi_k u) * \chi_{\varepsilon_k}$$

Here $\psi_k u \in \mathcal{D}'(U) \in \mathcal{E}'(U)$ is extended by zero to an alement of $\mathcal{E}'(\mathbb{R}^n)$ via 59. Then by 75 we know that convolving a distribution with a smooth function leads to a smooth function so we know $f_k \in C^\infty(\mathbb{R}^n)$. By 76 we know that it is supported in $\operatorname{supp} \varphi_k + B(0, \varepsilon_k) \Subset U$. Therefore in fact $f_k \subset C_c^\infty(U)$. We now claim $f_k \to u$ in $\mathcal{D}'(U)$ as tasked. Take arbitary $\varphi \in C_c^\infty(U)$ and extend it to 0 by 16. We need to show that

$$(f_k, \varphi) \to (u, \varphi) \text{ as } k \to \infty$$

By 78 we know that

$$(f_k, \varphi) = (u, \psi_k \varphi_{\varepsilon_k})$$

where $\varphi_{\varepsilon_k} \in C_c^\infty(\mathbb{R}^n)$. Since $u \in \mathcal{D}'(U)$ therefore by sequential continuitty we only need to show that

$$\psi_k \varphi_{\varepsilon_k} \to \varphi \text{ in } C_c^\infty(U)$$

as $k \to \infty$ which would imply $f_k \to u$ as desired. As we did for the $\mathbb{R}^n$ case this is indeed true as in 1

6 tensor products and distributional kernels(7)

6.1 Tensor product of distributions

Definition 81

Let $U \Subset \mathbb{R}^n, V \Subset \mathbb{R}^m$. We use the letter x to denote a point in U and the letter y to denote a point in V. If $f \in L^1_{\text{loc}}(U), g \in L^1_{\text{loc}}(V)$ then we define their **tensor product** $f \otimes g \in L^1_{\text{loc}}(U \otimes V)$ as follows

$$(f \otimes g)(x, y) = f(x)g(y)$$

we now seek to extend this to distributions as usual.To that end letting $\varphi \in C_c^\infty(U), \psi \in C_c^\infty(V)$ we have

$$(f \otimes g, \varphi \otimes \psi) = \int_{U \times V} f(x)g(y)\varphi(x)\psi(y)dxdy = (f, \varphi)(g, \psi)$$

Theorem 82

Let $U \Subset \mathbb{R}^n, V \Subset \mathbb{R}^m, u \in \mathcal{D}'(U), v \in \mathcal{D}'(V)$ then there exists unique $w \in \mathcal{D}'(U \times V)$ such that

$$(w, \varphi \otimes \psi) = (u, \varphi)(v, \psi)$$

for all $\varphi \in C_c^\infty(U), \psi \in C_c^\infty(V)$. We call w the **tensor product** of u and v and denote $u \otimes v = w$

Proof. We first show the existence of w. If f, g are locally integrable functions then for any $\beta \in C_c^\infty(U \times V)$ by *fubini theorem* we have

$$(f \otimes g, \beta) = \int_U f(x) \left(\int_V g(y)\beta(x, y)dy \right) dx$$

In similar fashion writing in terms of pairings and now letting $u \in \mathcal{D}'(U), v \in \mathcal{D}'(V)$. Take any $\beta \in C_c^\infty(U \times V)$ and consider the function

$$(v(y), \beta(x, y)) = (v, \beta(x, \bullet)) \tag{1}$$

where $\beta(x, \bullet) \in C_c^\infty(V)$ is defined by $\beta(x, \bullet)(y) = \beta(x, y)$. Instantly you recognize that (1) defined above is convolution between a smooth function and a distribution so by 75 we know it lies in $C^\infty(U)$. It is also compactly supported given that β is compactly supported so may further infer that it lies in $C_c^\infty(U)$. Pairing it with u we define w like so

$$(w, \beta) = (u(x), (v(y), \beta(x, y))) \quad \text{for all } \beta \in C_c^\infty(U \times V)$$

We now want to show that w is a distribution. Define arbitary $K \Subset U \times V$ which implies we have $K \subset K_U \times K_V$ for some $K_U \Subset U, K_V \Subset V$. Using the norm bounds for $\mathcal{D}'(E)$ on u, v we see that

$$|(u,\varphi)| \leq C\,||\varphi||_{C^N}\,\forall\varphi\in C_c^\infty(U), \operatorname{supp}\varphi\subset K_U$$
$$|(v,\psi)| \leq C\,||\psi||_{C^N}\,\forall\psi\in C_c^\infty(V), \operatorname{supp}\psi\subset K_V$$

Therefore for arbitary $\beta\in C_c^\infty(U\times V)$ such that $\operatorname{supp}\beta\subset K$ we estimate

$$\begin{aligned}|(w,\beta)| &\leq C\,||v,\beta(x,\bullet)||_{C_x^N(U)}\\ &= C\max_{|a|\leq N}\sup_{x\in U}|(v(y),\partial_x^a\beta(x,y))|\\ &\leq C^2\max_{|a|\leq N}\sup_{x\in U}||\partial_x^a\beta(x,y)||_{C_y^N(V)} \leq C^2\,||\beta||_{C^{2N}(U\times V)}\end{aligned}$$

The 2nd line uses the formula for x derivatives from 75. since the norm bound is satisfied w is indeed a distribution by definition. Finally existence follows by letting $\beta = \varphi\times\psi$ then $(v(y),\beta(x,y)) = \varphi(x)(v,\psi)$ which implies

$$(w,\beta) = (u\otimes v,\beta) = (u,\varphi)(v,\psi)$$

as desired

We now show uniqueness. That is equivalent to showing

$$(w,\varphi\otimes\psi) = 0 \text{ for all} \varphi\in C_c^\infty(U), \psi\in C_c^\infty(V) \quad\Rightarrow\quad w=0$$

the reasoning is similar to our proof in 44. Essentially we need the zero difference to onl correspond to the case of $w-w$ showing uniqnuess. Alternatively you could see this as "injectivity". To this end we approximate the distribution with smooth compactly supported functions via 80 which can be used for general open sets like our $U\times V\in\mathbb{R}^{n+m}$.Taking K_k^U, K_k^V to be the families of compact sets exhausting U, V respectively we put

$$\theta_k = \theta_k^U\otimes\theta_k^V\in C_c^\infty(U\times V)$$

which are the respective cut off functions i.e $\theta_k^U\in C_c^\infty(U)$ satisfies $\operatorname{supp}(1-\theta_k^U)\cap K_k^U=\emptyset$ and analagous for θ_k^V. Next define smooth bump functions

$$\chi = \chi^U\otimes\chi^V, \chi^U\in C_c^\infty(B_{\mathbb{R}^n}(0,\frac{1}{2})), \chi^V\in C_c^\infty(B_{\mathbb{R}^m}(0,\frac{1}{2})), \int_{\mathbb{R}^n}\chi^U = 1, \int_{\mathbb{R}^m}\chi^V = 1$$

where $\chi_\varepsilon^U\in C_c^\infty(\mathbb{R}^n), \chi_\varepsilon^V\in C_c^\infty(\mathbb{R}^m), \chi_\varepsilon\in C_c^\infty(\mathbb{R}^{n+m})$. So then we have the approximation

$$(\theta_k w)*\chi_{\varepsilon_k}\to w \text{ in } \mathcal{D}'(U\times V)$$

But expanding this out we notice that for each $(x,y)\in U\times V$ we have

$$(\theta_k w)*\chi_{\varepsilon_k}(x,y) = (\theta_k w, \chi_{\varepsilon_k}((x,y)-\bullet)) = (w,\varphi_{k,x}\otimes\psi_{k,y})$$

where $\varphi_{k,x}(\tilde{x}) = \theta_k^U(\tilde{x})\chi_{\varepsilon_k}^U(x-\tilde{x}), \quad \psi_{k,y}(\tilde{y}) = \theta_k^V(\tilde{y})\chi_{\varepsilon_k}^V(y-\tilde{y})$.by 78

Therefore as $k\to\infty$ we get $(w,\varphi\otimes\psi)=0$ by sequential continuity. But from the RHS of the above since $(\theta_k w)*\chi_{\varepsilon_k}\to w$ as $k\to\infty$ we also conclude $w=0$ as desired. ☐

6.2 Distributional Kernels

Definition 83

Let $U \Subset \mathbb{R}^n, V \Subset \mathbb{R}^m$. For any function $\mathcal{K}(x,y) \in L^1_{\text{loc}}(U \times V)$ we can define the integral operator $A : L^\infty_c(V) \to L^1_{\text{loc}}(U)$ by

$$A\varphi(x) = \int_V \mathcal{K}(x,y)\varphi(y)dy, \varphi \in L^\infty_c(V), x \in U$$

Definition 84

If instead $\mathcal{K}$ is a distribution we define for all $\psi \in C^\infty_c(V)$ and $\psi \in C^\infty_c(U)$ by fubini theorem

$$(A\varphi, \psi) = \int_{U\times V} \mathcal{K}(x,y)\psi(x)\varphi(y)dxdy = (\mathcal{K}, \psi \otimes \varphi)$$

Definition 85

Let $\mathcal{K} \in \mathcal{D}'(U \times V)$. Then using the defintion above and defining the linear operator here $A : C^\infty_C(V) \to \mathcal{D}'(U)$ as follows

$$(A\varphi, \psi) = (\mathcal{K}, \psi \otimes \varphi) \text{ for all} \varphi \in C^\infty_c(V), \psi \in C^\infty_c(U)$$

We call $\mathcal{K}$ the distributional kernel or the **Schwartz kernel** of A.

Proposition 86

$A\varphi$ is indeed a distribution

Proof. For all $\varphi \in C^\infty_c(V)$ take any sequence $\psi_k \to 0$ in $C^\infty_c(U)$. Then $\psi_k \otimes \varphi \to 0$ in $C^\infty_c(U \otimes V)$. Then from this definition

$$(A\varphi, \psi_k) = (\mathcal{K}, \psi_k \otimes \varphi)$$

by the fact that $\mathcal{K}$ is defined to be a distribution we have by sequential continuity the proposition in the limit of k. □

In particular this proposition says that A is a *sequentially continuous* operator. By that we mean

Definition 87

A linear operator $A : C^\infty_c(V) \to \mathcal{D}'(U)$ that satisfies

$$\varphi_k \to 0 \text{ in } C^\infty_c V \quad \Rightarrow \quad A\varphi_k \to 0 \text{ in } \mathcal{D}'(U)$$

is said to be **sequentially continuous**

The next statement shows that *every* sequentiall continuous operator $A : C^\infty_c(V) \to \mathcal{D}'(U)$ can indeed be charaterized by 85 for a unqiue choice of kernel $\mathcal{K}$. In other words we immediatley have the identification

$$\text{Operators } C^\infty_c(V) \to \mathcal{D}'(U) \simeq \text{Distributions in } \mathcal{D}'(U \times V)$$

...to be continued

6.3 the transpose of an opeator and defining operators by duality

Definition 88
Let $U \Subset \mathbb{R}^n, V \Subset \mathbb{R}^m$ and $A : C_c^\infty(V) \to \mathcal{D}'(U)$ be a sequentially continuous linear operator. Define its transpose $A^t : C_c^\infty(U) \to \mathcal{D}'(V)$ by the formula

$$(A^t\psi, \varphi) = (A\varphi, \psi) \text{ for all } \varphi \in C_c^\infty(V), \psi \in C_c^\infty(U)$$

7 pullbacks by smooth maps

Definition 89
For $f \in C^\infty(V)$ define the pullback of f by φ as

$$\Phi f = f \circ \Phi \in C^\infty(U)$$

This gives a linear sequential contiuous operator

$$\Phi* : L^1_{\text{loc}}(V) \to L^1_{\text{loc}}(U)$$

provided that Φ satisfies: for each $K \Subset U$ there exists a constant C_k so that

$$\text{vol}(K \subset \Phi^{-1}(\Omega)) \leq C_k \text{vol}(\Omega)$$

for all measurable $\Omega \subset V$.

Proposition 90
A pullback is contravariant that is for two C^∞ maps we have

$$U \underset{\Phi_2}{\longrightarrow} V \underset{\Phi_1}{\longrightarrow} W$$

then the pullback $\Phi_1 \circ \Phi_2 : U \to W$ satisfies

$$(\Phi_1 \circ \Phi_2)* = \Phi_2 * \Phi*_1$$

Proof. recall this from [3]

Definition 91
Let

8 Fourier Transform I(11)

8.1 Fourier Transform on Schwartz functions

Definition 92 (Fourier Transform on L^1)

Let $f \in L^1(\mathbb{R}^n)$. Define the Fourier Transform

$$\hat{f} = \mathcal{F}(f) \in L^\infty(\mathbb{R}^n)$$

by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

Proposition 93

It is immediate to see that$\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is a bounded linear opeator too because

$$||\hat{f}||_{L^\infty(\mathbb{R}^n)} \leq ||f||_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n)$ which by definition implies the RHS$< \infty$

Proof. The Fourier transform $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is bounded and linear:

For linearity consider For any $f, g \in L^1(\mathbb{R}^n)$ and scalars $a, b \in \mathbb{C}$,

$$\mathcal{F}(af + bg)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} (af(x) + bg(x))\, dx.$$

Using the linearity of the integral, this becomes

$$\mathcal{F}(af + bg)(\xi) = a \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx + b \int_{\mathbb{R}^n} e^{-ix\cdot\xi} g(x)\, dx = a\hat{f}(\xi) + b\hat{g}(\xi).$$

Thus, $\mathcal{F}$ is linear.

For boundedness consider The operator $\mathcal{F}$ is bounded because the L^∞-norm of $\hat{f}$ is controlled by the L^1-norm of f. This is explained in the next part.

As for the bound on $\|\hat{f}\|_{L^\infty}$: The inequality

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$$

follows from the definition of $\hat{f}$ and the properties of integrals. Specifically, for any $\xi \in \mathbb{R}^n$,

$$|\hat{f}(\xi)| = \left| \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx \right|.$$

Using the triangle inequality for integrals, we get

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} |e^{-ix\cdot\xi}| \cdot |f(x)|\, dx.$$

Since $|e^{-ix\cdot\xi}| = 1$ (as it is a complex exponential), this simplifies to

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_{L^1(\mathbb{R}^n)}.$$

Taking the supremum over all $\xi \in \mathbb{R}^n$, we get

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} = \sup_{\xi\in\mathbb{R}^n} |\hat{f}(\xi)| \leq \|f\|_{L^1(\mathbb{R}^n)}.$$

This inequality confirms that $\mathcal{F}$ maps $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ continuously with operator norm ≤ 1. Thus, $\mathcal{F}$ is bounded.

$\square$

Also $\hat{f}$ is a continuous function which we prove below.

Proposition 94

Assume that $f \in L^1(\mathbb{R}^n)$ then $\hat{f} \in C^0(\mathbb{R}^n)$.

Proof. we have for any $\xi \in \mathbb{R}^n$

$$\hat{f}(\eta) = \int_{\mathbb{R}^n} e^{-ix\cdot\eta} f(x)dx \to \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx = \hat{f}(\xi)$$

as $\eta \to \xi$ by the dominated convergence theorem. Since $\left|e^{-ix\cdot\eta}f(x)\right| = |f(x)|, f \in L^1(\mathbb{R}^n)$ and $e^{-ix\cdot\eta} \to e^{-ix\cdot\eta}$ as $\eta \to \eta$ for all $x \in \mathbb{R}^n$. This immediately shows continuitity.

Definition 95 (Schwartz Functions)

We say that $\varphi \in C^\infty(\mathbb{R}^n)$ is a **Schwartz fuction** if for all multindices α, β we have

$$\sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial_x^\beta \varphi(x)\right| < \infty$$

Denote by $\mathscr{S}(\mathbb{R}^n)$ the space of all shwartz functions on $\mathbb{R}^n$. For a sequence $\varphi_j \in \mathscr{S}(\mathbb{R}^n)$ we say it converges to $\varphi \in \mathscr{S}(\mathbb{R}^n)$ in $\mathscr{S}(\mathbb{R}^n)$ if for all α, β we have

$$\sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial_x^\beta (\varphi_j - \varphi)\right| \to 0 \quad \text{as } j \to \infty$$

We think of $\mathscr{S}(\mathbb{R}^n)$ as a space of test functions(arguments to distributions) which is well suited to study the fourier transform. We sometimes call this the space of rapidly decreasing functions.

Corollary 96

we have $\mathscr{S}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$

Proof. quite obvious. the above definition applies to any α, β so that includes

$$\sup_{x\in\mathbb{R}^n} |\varphi(x)| = ||\varphi||_{L^\infty(\mathbb{R}^n)} < \infty$$

where $\alpha = \beta = 0$.

Definition 97

A family of **seminorms** on $\mathscr{S}(\mathbb{R}^n)$ is given by

$$||\varphi||_{N,M} = \max_{|a|\leq N, |\beta|\leq M} \sup_{\mathbb{R}^n} \left|x^a \partial_x^\beta \varphi\right|, \quad N, M \in \mathbb{N}_0$$

Where essentially N, M are sums of powers and orders of partial differentiaon over each of component of x and φ respectively.

by the definitions above it is clear to see that the multiplication operator x_i and the differentation operator ∂_x^β are sequentially continuous. Moreover we also see $\varphi_n \to 0$ in $\mathscr{S}(\mathbb{R}^n)$ if and only if $||\varphi_n||_{N,M} \to 0$ for all N, M. Moreover it should be obvious that $||\varphi||_{N,M+1} < \infty$ for any N, M should $\varphi \in \mathscr{S}(\mathbb{R}^n)$. In fact we have

Proposition 98

There exists constant C such that for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have

$$||x_j\varphi||_{N,M} \leq C\,||\varphi||_{N+1,M} \quad \text{and} \quad \left|\left|\partial_{x_j}\varphi\right|\right|_{N,M} \leq C\,||\varphi||_{N,M+1}$$

. This should be very obvious from the above definition but explicity you may consider the following proof

Proof. To prove $||x_j\varphi||_{N,M} \leq C\,||\varphi||_{N+1,M}$:

By the definition of the seminorm $||\cdot||_{N,M}$, we have

$$||x_j\varphi||_{N,M} = \max_{|\alpha|\leq N, |\beta|\leq M} \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial_x^\beta (x_j\varphi(x))\right|.$$

Using the product rule,

$$\partial_x^\beta(x_j\varphi(x)) = x_j\partial_x^\beta\varphi(x) + \beta_j\partial_x^{\beta-e_j}\varphi(x),$$

where e_j is the multi-index corresponding to the unit vector in the j-th direction (if $\beta_j = 0$, the second term vanishes). Substituting this into the seminorm, we get

$$||x_j\varphi||_{N,M} = \max_{|\alpha|\leq N, |\beta|\leq M} \sup_{x\in\mathbb{R}^n} \left|x^\alpha\left(x_j\partial_x^\beta\varphi(x) + \beta_j\partial_x^{\beta-e_j}\varphi(x)\right)\right|.$$

Using the triangle inequality,

$$||x_j\varphi||_{N,M} \leq \max_{|\alpha|\leq N, |\beta|\leq M} \sup_{x\in\mathbb{R}^n} \left(\left|x^{\alpha+e_j}\partial_x^\beta\varphi(x)\right| + \left|\beta_j x^\alpha\partial_x^{\beta-e_j}\varphi(x)\right|\right).$$

Notice that the term $x^{\alpha+e_j}$ meant that the sum of powers of each component of x increases by 1. For the second term, multiplication by β_j simply scales each x component by a constant. Its powers and differentation orders is unchanged. Now By definition, the seminorm $||\varphi||_{N+1,M}$ accounts for terms with $|\alpha| \leq N+1$ and $|\beta| \leq M$, so there exists a constant C (depending on j) such that

$$||x_j\varphi||_{N,M} \leq C\,||\varphi||_{N+1,M}.$$

To prove $\left|\left|\partial_{x_j}\varphi\right|\right|_{N,M} \leq C\,||\varphi||_{N,M+1}$:

Again, by the definition of the seminorm, we have

$$\left|\left|\partial_{x_j}\varphi\right|\right|_{N,M} = \max_{|\alpha|\leq N, |\beta|\leq M} \sup_{x\in\mathbb{R}^n} \left|x^\alpha\partial_x^\beta(\partial_{x_j}\varphi(x))\right|.$$

By the commutativity of partial derivatives,

$$\partial_x^\beta(\partial_{x_j}\varphi(x)) = \partial_x^{\beta+e_j}\varphi(x),$$

where e_j is the multi-index corresponding to the unit vector in the j-th direction. Substituting this into the seminorm, we get

$$\left|\left|\partial_{x_j}\varphi\right|\right|_{N,M} = \max_{|\alpha|\leq N, |\beta|\leq M} \sup_{x\in\mathbb{R}^n} \left|x^\alpha\partial_x^{\beta+e_j}\varphi(x)\right|.$$

Immediately you see that the sum of orders of partial differentials of each component of x increased by 1. Therefore again by defition $||\varphi||_{N,M+1}$ handles sum of orders of differentiation increased by 1. That is, Since $|\beta + e_j| \leq M+1$

when $|\beta| \leq M$, it follows that

$$\left|\left|\partial_{x_j}\varphi\right|\right|_{N,M} \leq C \left|\left|\varphi\right|\right|_{N,M+1}.$$

$\square$

Now this implies $C_c^\infty(\mathbb{R}^n) \in \mathscr{S}(\mathbb{R}^n)$ and that the differential and multiplication operators are closed in $\mathscr{S}(\mathbb{R}^n)$. This is because to remind once again if in $\mathscr{S}(\mathbb{R}^n)$ we have $||\varphi||_{N,M} < \infty$ for any N, M by defintion!!

Problem 99

Prove that we have the inclusions

$$C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$$

That is convergence of sequences in $C_C^\infty(\mathbb{R}^n)$ is the stronger than in $\mathscr{S}(\mathbb{R}^n)$ which in turn is strongger than in $C^\infty(\mathbb{R}^n)$. Also prove that the space $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ too.

Remark 100. *We have already proven inclusion previously but you may want to consider a more direct and compact alternative proof below.*

Proof. **1. Inclusions:**

(i) Inclusion $C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$:

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. By definition, $\varphi \in C^\infty(\mathbb{R}^n)$ and has compact support, meaning there exists a bounded set $K \subset \mathbb{R}^n$ such that $\varphi(x) = 0$ for all $x \notin K$.

For any multi-indices α and β, the quantity $x^\alpha \partial_x^\beta \varphi(x)$ is zero outside K, since $\varphi(x)$ vanishes there. Inside K, $\varphi(x)$ and all its derivatives are smooth and bounded because K is compact. Hence,

$$\sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial_x^\beta \varphi(x)\right| < \infty,$$

showing that $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Thus, $C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$.

(ii) Inclusion $\mathscr{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$:

Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. By definition, $\varphi \in C^\infty(\mathbb{R}^n)$, since all partial derivatives of φ exist and are continuous. Moreover, the Schwartz condition implies rapid decay of $\varphi(x)$ and its derivatives at infinity, but this does not affect smoothness. Hence, $\mathscr{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$.

2. Convergence Strength:

(i) Convergence in $C_c^\infty(\mathbb{R}^n)$ implies convergence in $\mathscr{S}(\mathbb{R}^n)$:

If $\varphi_j \to \varphi$ in $C_c^\infty(\mathbb{R}^n)$, then there exists a compact set K such that $\mathrm{supp}(\varphi_j) \subset K$ for all j, and $\partial_x^\beta \varphi_j \to \partial_x^\beta \varphi$ uniformly on $\mathbb{R}^n$ for all multi-indices β. This implies that for any multi-indices α, β,

$$\sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial_x^\beta (\varphi_j - \varphi)\right| \to 0,$$

because outside K, the terms are zero, and inside K, convergence is uniform. Hence, $\varphi_j \to \varphi$ in $\mathscr{S}(\mathbb{R}^n)$.

(ii) Convergence in $\mathscr{S}(\mathbb{R}^n)$ implies convergence in $C^\infty(\mathbb{R}^n)$:

If $\varphi_j \to \varphi$ in $\mathscr{S}(\mathbb{R}^n)$, then for all α, β,

$$\sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial_x^\beta (\varphi_j - \varphi)\right| \to 0.$$

In particular, taking $\alpha = 0$, we see that $\partial_x^\beta \varphi_j \to \partial_x^\beta \varphi$ uniformly on $\mathbb{R}^n$. This is precisely the condition for convergence in $C^\infty(\mathbb{R}^n)$.

Thus, convergence in $C_c^\infty(\mathbb{R}^n)$ is stronger than in $\mathscr{S}(\mathbb{R}^n)$, and convergence in $\mathscr{S}(\mathbb{R}^n)$ is stronger than in $C^\infty(\mathbb{R}^n)$.

3. Density of $C_c^\infty(\mathbb{R}^n)$ in $\mathscr{S}(\mathbb{R}^n)$:

Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Define a sequence $\varphi_j \in C_c^\infty(\mathbb{R}^n)$ by multiplying φ with a smooth cutoff function $\chi_j(x)$ as follows:

$$\chi_j(x) = \begin{cases} 1, & \text{if } |x| \leq j, \\ 0, & \text{if } |x| > j+1, \end{cases}$$

with a smooth transition in $j \leq |x| \leq j+1$. The sequence is given by $\varphi_j(x) = \chi_j(x)\varphi(x)$.

- Each φ_j has compact support, so $\varphi_j \in C_c^\infty(\mathbb{R}^n)$.
- For any multi-indices α and β, we have

$$x^\alpha \partial_x^\beta(\varphi - \varphi_j) = x^\alpha \partial_x^\beta((1-\chi_j)\varphi).$$

Since $(1-\chi_j)(x) \to 0$ pointwise as $j \to \infty$ and $(1-\chi_j)(x)\varphi(x)$ is supported in $|x| \geq j$, the supremum over all x tends to 0 as $j \to \infty$. Hence, $\varphi_j \to \varphi$ in $\mathscr{S}(\mathbb{R}^n)$.

Therefore, $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$. □

Proposition 101

The schwartz space $\mathscr{S}(\mathbb{R}^n)$ is contained in $L^1(\mathbb{R}^n)$. In fact if $\varphi \in \mathscr{S}(\mathbb{R}^n)$ then

$$|\varphi(x)| \leq C_n \,||\varphi||_{n+1,0}\,(1+|x|)^{-n-1}$$

for constant C_n depending only on n so

$$||\varphi||_{L^1(\mathbb{R}^n)} \leq C_n\,||\varphi||_{n+1,0}$$

Proof. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. By the definition of the Schwartz space, for all multi-indices α and β, we have

$$\sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial_x^\beta \varphi(x)\right| < \infty.$$

The word "all" multindices implies we may choose $\beta = 0$, where we have

$$\left|x^{n+1}\varphi(x)\right| = \left|x^{n+1}\right| |\varphi(x)| \leq C_{n+1},$$

where $C_{n+1} = ||\varphi||_{n+1,0}$, we divide both sides by $(1+|x|)^{n+1}$, which is a positive quantity for all $x \in \mathbb{R}^n$. The division gives:

$$\frac{\left|x^{n+1}\varphi(x)\right|}{(1+|x|)^{n+1}} \leq \frac{C_{n+1}}{(1+|x|)^{n+1}}.$$

On the left-hand side, observe that:

$$\frac{\left|x^{n+1}\varphi(x)\right|}{(1+|x|)^{n+1}} = |\varphi(x)| \cdot \frac{|x|^{n+1}}{(1+|x|)^{n+1}}.$$

Since $(1+|x|)^{n+1}$ dominates $|x|^{n+1}$ for all $x \in \mathbb{R}^n$, we have:

$$\frac{|x|^{n+1}}{(1+|x|)^{n+1}} \leq 1.$$

Substituting this back, we find:

$$|\varphi(x)| \cdot \frac{|x|^{n+1}}{(1+|x|)^{n+1}} \leq |\varphi(x)|\,.$$

Therefore, the original inequality simplifies to:

$$|\varphi(x)| \leq \frac{C_{n+1}}{(1+|x|)^{n+1}}.$$

$$|\varphi(x)| \leq C_{n+1}(1+|x|)^{-n-1},$$

where $C_{n+1} = ||\varphi||_{n+1,0}$ depends only on the seminorm of φ. This shows that $\varphi(x)$ decays at least as fast as $(1+|x|)^{-n-1}$, which ensures integrability of $\varphi(x)$ on $\mathbb{R}^n$.

To estimate $||\varphi||_{L^1(\mathbb{R}^n)}$, we compute:

$$||\varphi||_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\varphi(x)|\, dx \leq \int_{\mathbb{R}^n} C_{n+1}(1+|x|)^{-n-1}\, dx.$$

The integral $\int_{\mathbb{R}^n}(1+|x|)^{-n-1}dx$ converges because $(1+|x|)^{-n-1}$ decays sufficiently fast as $|x| \to \infty$. Denoting this integral by C_n, we have:

$$||\varphi||_{L^1(\mathbb{R}^n)} \leq C_n\, ||\varphi||_{n+1,0}\,,$$

where C_n depends only on n. From this it immediately implies that if in $\mathscr{S}(\mathbb{R}^n)$ the RHS is less than infinity which immediately implies $||\varphi||_{L^1(\mathbb{R}^n)} < \infty$ as desired $\square$

Corollary 102

we have that $\mathscr{S}(\mathbb{R}^n) \subset L^p$ for $1 \leq p \leq \infty$

Proof. we have already proven the L^∞ and L^1 case above. But recall

$$L^1 \subset L^2 \subset \ldots L^n$$

from [4].So the corollary follows obviously.

Corollary 103

We also see that defining the fourier transform $\hat{\varphi} \in L^\infty(\mathbb{R}^n)$ we have that

$$||\hat{\varphi}||_{L^\infty(\mathbb{R}^n)} \leq C_n\, ||\varphi||_{n+1,0}$$

Proof. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. The Fourier transform of φ is defined as:

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\, dx.$$

Using the bound from the proposition:

$$|\varphi(x)| \leq C_n\, ||\varphi||_{n+1,0}\, (1+|x|)^{-n-1},$$

we estimate:

$$|\hat{\varphi}(\xi)| = \left| \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\, dx \right| \leq \int_{\mathbb{R}^n} |\varphi(x)|\, dx.$$

From the proposition, we know that:

$$\int_{\mathbb{R}^n} |\varphi(x)| \; dx \leq C_n \, ||\varphi||_{n+1,0} \, .$$

Since ξ is arbitrary taking the suprenum we have

$$||\hat{\varphi}||_{L^\infty(\mathbb{R}^n)} = \sup_{\xi \in \mathbb{R}^n} |\hat{\varphi}(\xi)| \leq C_n \, ||\varphi||_{n+1,0} \, .$$

This proves the corollary. □

Theorem 104

For each $\varphi \in \mathscr{S}(\mathbb{R}^n), \mathcal{F}(\varphi) = \hat{\varphi}$ also lies in $\mathscr{S}(\mathbb{R}^n)$. Moreover the operator $\mathcal{F} : \ell(\mathbb{R}^n) \to \ell(\mathbb{R}^n)$ is sequentially continuous

before proving this we first introduce the following

Definition 105

We define the modified differentiation operators

$$D_{x_j} := -i\partial_{x_j}$$

For multindex a we have

$$D_x^a = D_{x_1}^{a_1} \dots D_{x_n}^{a_n} = (-i)^{|a|}\partial_x^a$$

Proposition 106

Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then

$$\widehat{D_{x_j}\varphi}(\xi) = \xi_j \hat{\varphi}(\xi) \tag{1}$$

$$\widehat{x_j\varphi}(\xi) = -D_{\xi_j}\hat{\varphi}(\xi) \tag{2}$$

To show (1) we integrate by parts

$$\begin{aligned} \widehat{D_{x_j}\varphi}(\xi) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} D_{x_j}\varphi(x)dx = -\int_{\mathbb{R}^n} (D_{x_j}e^{-ix\cdot\xi})\varphi(x)dx \\ &= \int_{\mathbb{R}^n} \xi_j e^{-ix\cdot\xi}\varphi(x)dx = \xi_j\hat{\varphi}(\xi) \end{aligned}$$

To justify why we treated the boundary terms as zero consider that we integrating on a ball $B(0, R)$ and let $R \to \infty$. However φ is rapidly decreasing so the integral vanishes there. To show (2) we do

$$-D_{\xi_j}\hat{\varphi} = \int_{\mathbb{R}^n} (-D_{\xi_j}e^{-ix\cdot\xi})\varphi(x)dx = \int_{\mathbb{R}^n} x_j e^{ix\cdot\xi}\varphi(x)dx = \widehat{x_j\varphi(\xi)}$$

to justify integration under the integral as above denote e_j to be the jth standard basis vector on $\mathbb{R}^n$ and write for any $\xi \in \mathbb{R}^n$ and $t/\{0\}$

$$\frac{\hat{\varphi}(\xi + te_j) - \hat{\varphi}(\xi)}{t} = \int_{\mathbb{R}^n} \frac{e^{-ix\cdot(\xi+te_j) - e^{ix\cdot\xi}}}{t}\varphi(x)dx$$

applying the inequality $\left|e^{ia}-1\right| \leq |a|$ with $a = -tx_j$ we see that

$$\left|\frac{e^{-ix\cdot(\xi+te_j)-e^{ix\cdot\xi}}}{t}\varphi(x)\right| = \left|\frac{e^{-itx_j}-1}{t}\varphi(x)\right| \leq |x_j\varphi(x)|$$

since φ is a schwartz function from 101 we have $x_j\varphi \in L^1(\mathbb{R}^n)$ so we may pass the limit $t \to 0$ under the integral by dominated convergence theorem. $\square$

We are now ready to prove 104. Assume that $\varphi \in \mathscr{S}(\mathbb{R}^n)$.Since $\mathscr{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, we may use 94 to find that $\hat{\varphi} \in C^0(\mathbb{R}^n)$.However consider

$$\widehat{x_j\varphi}(\xi) = -D_{\xi_j}\hat{\varphi}(\xi)$$

from 106. The LHS is continuous since is just a continuous C^0 function scaled by x_j. The RHS is the differential of $\hat{\varphi}$. This instantly shows $\hat{\varphi} \in C^1(\mathbb{R}^n)$. However by induction we can clearly extend this to C^∞ knowing that multiplication by x_j of the continuous function is still continuous so in fact we may prove any order of differentiation is continuous. For example

$$\widehat{x_\ell x_j\varphi}(\xi) = D_{\xi_\ell}D_{\xi_j}\hat{\varphi}(\xi)$$

Now consider an arbitary pair of multindices α, β,

$$\xi^\alpha D_\xi^\beta\hat{\varphi} = (-1)^{|\beta|}\widehat{D_x^\alpha x^\beta\varphi}$$

this relation should make sense from 106. Since x_j, ∂_{x_j} map $\mathscr{S}(\mathbb{R}^n)$ to itself and $\mathscr{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, from 93 we see that $\widehat{D_x^\alpha x^\beta\varphi}$ must be a bounded function(in the L^∞ norm). This means we have

$$\sup_{\xi\in\mathbb{R}^n}\left|\xi^\alpha D_\xi^\beta\hat{\varphi}\right| < \infty$$

but notice that our choice of α, β was arbitary. This is precisely the definition of being $\mathscr{S}(\mathbb{R}^n)$!!! $\square$

Proposition 107

Let $f, g \in L^1(\mathbb{R}^n)$. Then

$$(\hat{f}, g) = (f, \hat{g})$$

Proof. by fubini theorem oth sides are equal to

$$\int_{\mathbb{R}^{2n}} e^{-ix\cdot\xi}f(x)g(\xi)dxd\xi$$

Proposition 108

Let $f, g \in L^1(\mathbb{R}^n)$ then

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

Proof. by fubini theorem and change of variables $x = y + z$ we have

$$\begin{aligned}\widehat{f * g}(\xi) &= \int_{\mathbb{R}^{2n}} e^{-ix\cdot\xi}f(y)g(x-y)dxdy \\ &= \int_{\mathbb{R}^{2n}} e^{-iy\cdot\xi}e^{-iz\cdot\xi}f(y)g(z) = \hat{f}(\xi)\hat{g}(\xi)\end{aligned}$$

Proposition 109

Assume that $f \in L^1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^m)$ then

$$\widehat{f \otimes g}(\xi, \eta) = \hat{f} \otimes \hat{g}(\xi, \eta)$$

Remark 110. *You may see $f \otimes g = h \in :^1 (\mathbb{R}^{n+m})$ and let $\zeta = \xi + \eta$(decomposed as components here essentially). Then $\hat{h}(\zeta) = \hat{h}(\xi + \eta)$ then applying componentwise the following wil make sense*

Proof. Recall the definition of the tensor product:

$$(f \otimes g)(x, y) = f(x)g(y),$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. The Fourier transform of $f \otimes g$ is:

$$\widehat{f \otimes g}(\xi, \eta) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} e^{-i(x \cdot \xi + y \cdot \eta)} f(x)g(y)\, dx\, dy.$$

Separating the integrals, we have:

$$\widehat{f \otimes g}(\xi, \eta) = \left(\int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x)\, dx \right) \left(\int_{\mathbb{R}^m} e^{-iy \cdot \eta} g(y)\, dy \right).$$

By definition of the Fourier transform:

$$\widehat{f \otimes g}(\xi, \eta) = \hat{f}(\xi)\hat{g}(\eta).$$

Recognizing this as the tensor product:

$$\widehat{f \otimes g} = \hat{f} \otimes \hat{g}.$$

This concludes the proof. $\square$

Proposition 111

Assume that $A : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map and $f \in L^1(\mathbb{R}^n)$. Then

$$\widehat{A * f}(\xi) = |\det A|^{-1} \hat{f}(A^{-T}\xi)$$

where A^{-T} denotes the inverse transpose of A

Remark 112. *Please review pullbacks before returing here. the following is found from MSE. But it should be quite clear using change of var theorem learnt in [3]*

Proof. For any $\phi \in \mathcal{S}(\mathbb{R}^n)$ the definitions provide

$$\widehat{T^*u}(\phi) = \int_{\mathbb{R}^n} T^*u(x)\hat{\phi}(x)\, dx = \int_{\mathbb{R}^n} u(Tx)\hat{\phi}(x)\, dx.$$

Now change variables: with $y = Tx$ you get $dy = |\det T| dx$ so that

$$\int_{\mathbb{R}^n} u(Tx)\hat{\phi}(x)\, dx = |\det T|^{-1} \int_{\mathbb{R}^n} u(y)\hat{\phi}(T^{-1}y)\, dy.$$

Time to investigate the Fourier transform. Observe

$$\hat{\phi}(T^{-1}y) = \int_{\mathbb{R}^n} e^{-i\langle z, T^{-1}y \rangle} \phi(z)\, dz$$

and $\langle z, T^{-1}y\rangle = \langle (T^{-1})^t z, y\rangle = \langle (T^t)^{-1}z, y\rangle$, where the last equality uses the fact that the matrix representing T is real.

Now combine the formulas. I will be a bit cavalier with the use of Fubini's theorem trusting that you can make it precise.

$$\begin{aligned}\widehat{T^*u}(\phi) &= |\det T|^{-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y)e^{-i\langle (T^t)^{-1}z,y\rangle}\phi(z)\,dzdy \\ &= |\det T|^{-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle (T^t)^{-1}z,y\rangle}u(y)\,dy\phi(z)\,dz.\end{aligned}$$

As long as $u \in L^1(\mathbb{R}^n)$ (which doesn't seem to be implied by the hypotheses, but whatever) the inner integral equals $\hat{u}((T^t)^{-1}z)$ so that

$$\widehat{T^*u}(\phi) = |\det T|^{-1}\int_{\mathbb{R}^n}\hat{u}((T^t)^{-1}z)\phi(z)\,dz = |\det T|^{-1}\int_{\mathbb{R}^n}((T^t)^{-1})^*\hat{u}(z)\phi(z)\,dz$$

where the last expression is $|\det T|^{-1}((T^t)^{-1})^*\hat{u}(\phi)$. Finally disregard ϕ to obtain

$$\widehat{T^*u} = |\det T|^{-1}((T^t)^{-1})^*\hat{u}.$$

Proposition 113

Let $f \in L^1(\mathbb{R}^n)$ and $\overline{f}$ be the complex conjugate of f i.e $\overline{f}(x) = \overline{f(x)}$ then

$$\mathcal{F}(\overline{f})(\xi) = \overline{(\mathcal{F}f)(-\xi)}$$

Proof. By definition of the Fourier transform:

$$\mathcal{F}(\overline{f})(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\overline{f(x)}\,dx.$$

but this is equivalent to

$$\mathcal{F}(\overline{f})(\xi) = \overline{\int_{\mathbb{R}^n} e^{ix\cdot\xi}f(x)\,dx}.$$

The term inside is the Fourier transform evaluated at $-\xi$:

$$\mathcal{F}(\overline{f})(\xi) = \overline{(\mathcal{F}f)(-\xi)}.$$

This concludes the proof. □

Definition 114

The **Gaussian function** is defined by

$$G(x) = e^{-\frac{|x|^2}{2}}$$

Proposition 115

The Gaussian function $G(x)$ satisfies

$$\hat{G}(\xi) = (2\pi)^{\frac{n}{2}}e^{\frac{-|\xi|^2}{2}}$$

that is $\hat{G} = (2\pi)^{\frac{n}{2}}G$

Proof. It suffices to show the case for $n = 1$ since if G_n is the gaussian for $\mathbb{R}^n$ then $G_{n+m} = G_n \otimes G_m$ so the formula for $\hat{G}_{n+m}$ follows by 109. The Fourier transform of $G(x)$ is defined by

$$\hat{G}(\xi) = \int_{\mathbb{R}} e^{-ix\cdot\xi} G(x)\, dx,$$

where $G(x) = e^{-\frac{x^2}{2}}$. Substituting, we have

$$\hat{G}(\xi) = \int_{\mathbb{R}} e^{-ix\cdot\xi} e^{-\frac{|x|^2}{2}}\, dx.$$

Combine the terms in the exponent:

$$-ix\cdot\xi - \frac{x^2}{2} = -\frac{1}{2}\left(x^2 + 2ix\cdot\xi\right).$$

Completing the square, we write

$$|x|^2 + 2ix\cdot\xi = x + i\xi^2 - \xi^2.$$

Substituting, the exponent becomes

$$-\frac{1}{2}x + i\xi^2 + \frac{\xi^2}{2}.$$

The integral now reads:

$$\hat{G}(\xi) = e^{\frac{\xi^2}{2}} \int_{\mathbb{R}} e^{-\frac{1}{2}x + i\xi^2}\, dx.$$

Using the invariance of the Gaussian integral under shifts in the complex plane(recall [6]), this simplifies to:

$$\int_{\mathbb{R}} e^{-\frac{1}{2}x + i\xi^2}\, dx = \int_{\mathbb{R}} e^{-\frac{sx^2}{2}}\, dx = (2\pi)^{\frac{n}{2}}.$$

Therefore,

$$\hat{G}(\xi) = (2\pi)^{\frac{n}{2}} e^{-\frac{\xi^2}{2}}.$$

This concludes the proof. □

Theorem 116 (Fourier inversion formula)
Assume that $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then we have for all $x \in \mathbb{R}^n$

$$\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{\varphi}(\xi) d\xi$$

Remark 117. *It follows that the operator $\mathcal{F} : \mathscr{S}(\mathbb{R}^n) \to \mathbb{R}$ is invertible and its inverse is given by the formula*

$$\mathcal{F}^{-1}\psi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot xi} \hat{\varphi}(\xi) d\xi$$

Proof. We rewrite the RHS of our integral above as

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(\int_{\mathbb{R}^n} e^{iy\cdot\xi} \varphi(y) dy \right) d\xi$$

However we can't just apply fubini theorem gien that $e^{i(x-y)\cdot\xi}\varphi(y)$ is not integrable on $\mathbb{R}^{2n}$ since it is complex exponential. To fix this usse we regularize the integral using the gaussian G defined earlier. Since $\hat{\varphi} \in \mathbb{R} \subset L^1(\mathbb{R}^n)$

and $G(0) = 1$ by the dominated convergence theorem we have

$$\begin{aligned}\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^n} e^{ix\cdot\xi}G(\varepsilon\xi)\hat{\varphi}(\xi)d\xi &= \lim_{\varepsilon\to 0+}(2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}G(\varepsilon\xi)\varphi(y)dyd\xi \\ &= \lim_{\varepsilon\to 0+}(2\pi\varepsilon)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i(x-y)\cdot\eta}{\varepsilon}}G(\eta)\varphi(y)dyd\eta \\ &= \lim_{\varepsilon\to 0+}(2\pi\varepsilon)^{-n}\int_{\mathbb{R}^n}\widehat{G}\left(\frac{y-x}{\varepsilon}\right)\varphi(y)dy \\ &= \lim_{\varepsilon\to 0+}(2\pi)^{-n}\int_{\mathbb{R}^n}\widehat{G}(w)\varphi(x+\varepsilon w)dw\end{aligned}$$

The first line follows by fubini theorem since ow $G \in L^1(\mathbb{R}^n)$. The second line used the change of bariables $\xi = \eta/\varepsilon$. THe third line uses fubini theorem again to integate out η and in the last line we made the change of variables $y = x + \varepsilon w$. Since $\hat{G} \in L^1(\mathbb{R}^n)$ by the dominated convergence theorem and 115, we bring the limit under the integral where we obtain

$$(2\pi)^{-n}\left(\int_{\mathbb{R}^n}(2\pi)^{\frac{n}{2}}e^{-\frac{|w|^2}{2}}dw\right)\varphi(x) = \varphi(x)(2\pi)^{\frac{n}{2}}(2\pi)^{-\frac{n}{2}} = \varphi(x)$$

as desired □

Corollary 118

Assume that $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$. Then

$$\widehat{\varphi\psi} = (2\pi)^{-n}\hat{\varphi} * \hat{\psi}$$

Proof. From previously knowing that $\mathcal{F}$ is invertible we have that

$$\mathcal{F}^{-1}(f * g) = (2\pi)^n\mathcal{F}^{-1}(f)\mathcal{F}^{-1}(g)$$

□

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