MIT 2.005 Solid Mechanics 1(2006)

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No fun

Contents

1 Loading and support conditions

Definition 1

The surface forces that develop *at the supports* or points of contacts between bodies are called **reactions**

For example the **normal reaction** force you learnt in high school physics.

Free body diagrams

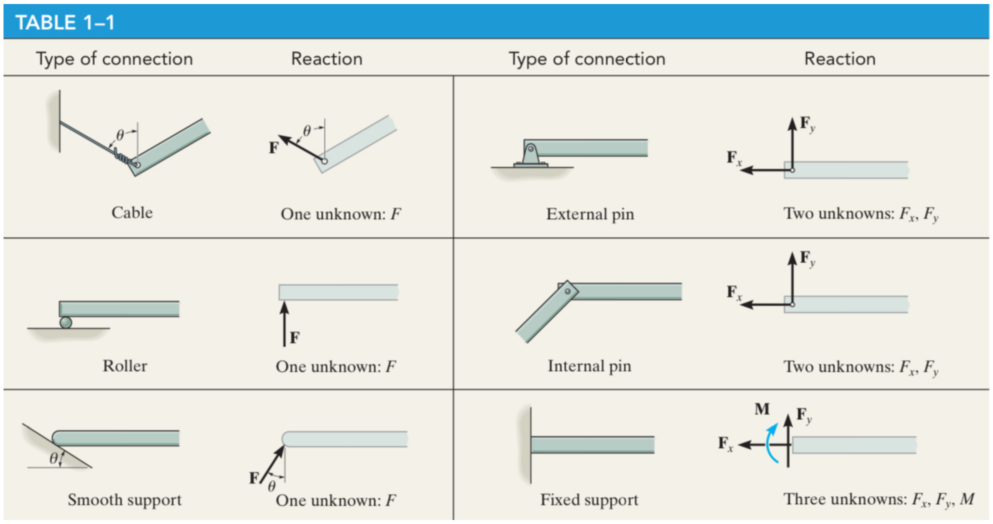


Figure 1: the supports include the cable,roller,pin etc. The member is the the beam thing. The forces by the supports on the member

If the support prevents translation in a given direction then a force must be developed on the member in opposite direction. Likewise if a rotation is prevented, a couple moment must be exerted on the member.

Example 2

For the fixed support where both translation(horizontal and vertical) and rotation is not allowed we have to account for the 3 unknown reactions F_x, F_y, M on the member at the point of contact with the wall when writing equations for equilibrium

Example 3

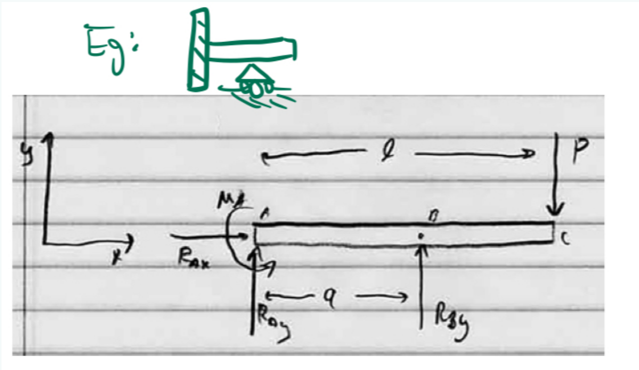
For the roller where only vertical translation is not allowed only the unknown we must account for is F the normal force applied by the roller on the member.There is no couple moment that can be developed as the member is free to rotate about the roller

Definition 4

statically determinate is a situation in which the equations of equlibrium determine the forces and moments that support the structure

Example 5

Consider



which has equations of equilibrium

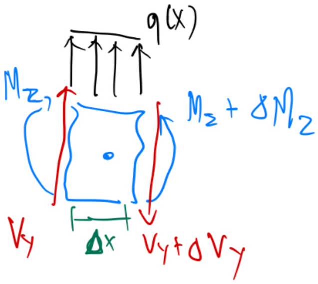
$$\sum F_x = 0$$
$$R_{A_x} = 0$$
$$\sum F_y = -0$$
$$R_{PA_x} + R_{B_y} - P = 0$$
$$\sum M_B = 0$$
$$-aR_{A_y} + M_A - (\ell - a)P = 0$$

However this has 4 unknowns and 3 equations so we cannot get a unique solution

We say this is **statically indeterminate**

2 forces and moments transmitted by slender members

Consider this small cut(so we approximate the load there $q(x)$ to be uniform over δx)



Then solving FBD we have

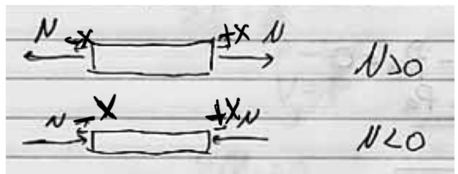
$$\sum F_y = 0 \qquad\qquad \sum M_0 = 0$$

$$V_y - (V_y + \delta V_y) + q\delta x = 0 \qquad\qquad -M_z + M_z + \delta M_z - V_y\left(\frac{\delta x}{2}\right) - (V_y + \delta V_y)\left(\frac{\delta x}{2}\right) = 0$$

$$\delta V_y = q\delta x \qquad\qquad V_y = \frac{\delta M_z}{\delta x} + \frac{\delta x}{2}$$

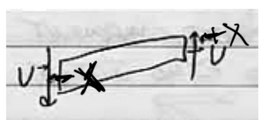
$$\frac{dV_y}{dx} = q, \quad \delta x \to 0 \qquad\qquad V_y = \frac{dM_z}{dx}, \quad \delta x \to 0$$

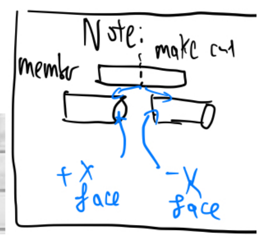
$N > 0$
$N < 0$



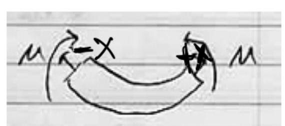
$V > 0$

$V < 0$





$M > 0$



$M < 0$

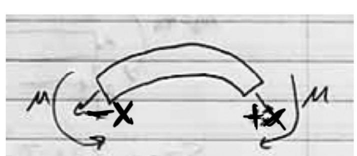


Figure 2: Sign conventions: Internal forces

Steps to finding internal forces and moments

1. Find reactions at supports
2. make appropriate cuts for each member
3. find internal forces and moments on each
4. plot result for each

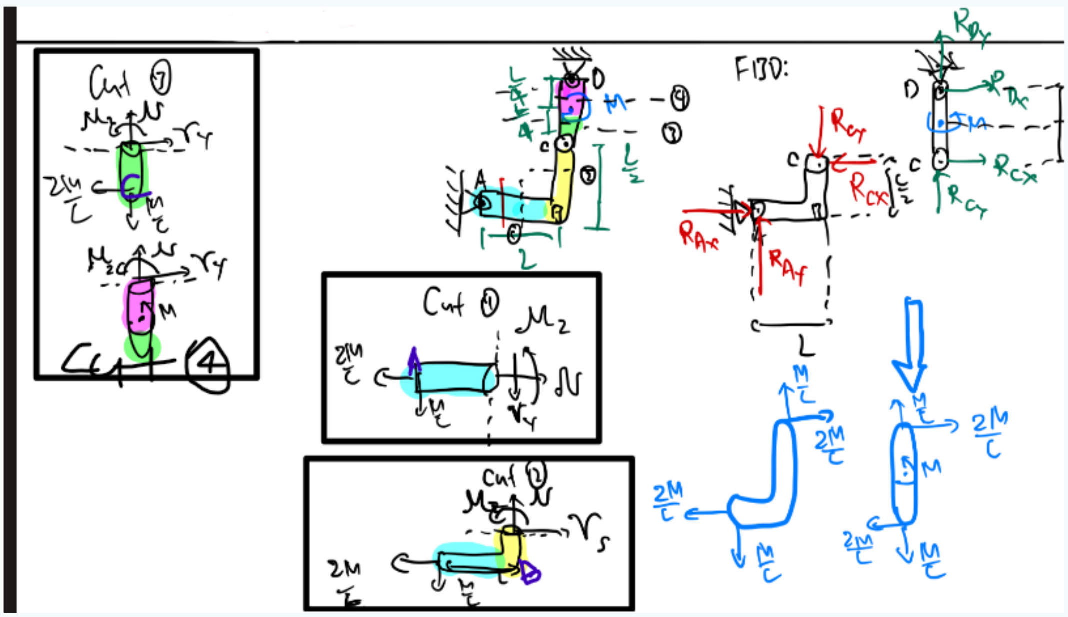
Fact 6 (2 force member)

If forces are only applied at 2 points then

1. forces are equal and opposite
2. forces are aligned with the vector that connects the two points of application of forces

Example 7 (method of sections)

Consider



Step 1 is to find the FBD of all members. Next we draw the reaction forces at the supports(the green and red texts) solving equations of equlibrium we have for the whole system(canceling out the R_{C_x}, R_{C_y})

$$\sum F_x = 0$$
$$R_{A_x} + R_{D_x} = 0$$
$$\sum F_y = 0$$
$$R_{A_y} + R_{D_y} = 0$$
$$\sum M_A = 0$$
$$M - R_{D_x} L + R_{D_y} L = 0$$

Remark 8. *Notice how M contributed to M_A*

Now consider the system only consisting of the red text.Notice that is it *2 force member* system and therefore we know the resultant force of R_{A_x}, R_{A_y} and R_{C_x}, R_{C_y} are each aligned to the direction of the line that connects them. That means

$$\frac{R_{A_y}}{R_{A_x}} = \frac{\frac{L}{2}}{L} = \frac{1}{2}$$

and from previously we also know

$$\frac{R_{A_y}}{R_{D_y}} = -1$$

$$\frac{R_{A_x}}{R_{D_x}} = -1$$

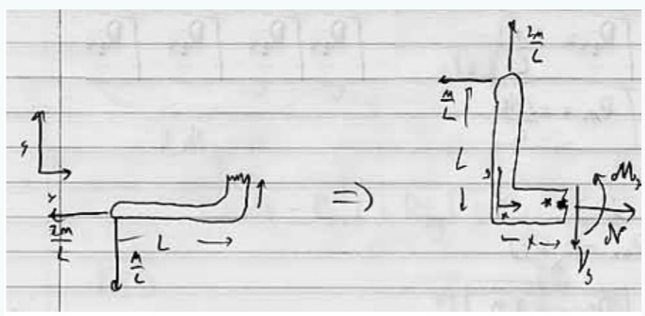
hence

$$\frac{R_{A_x}}{R_{A_y}} = \frac{R_{D_x}}{R_{D_y}} = 2$$

Now solving for equations of equilibrium for the red text labelled system will yield the solutions as indicated by the diagram in blue. Then solving equations of equilibrium for each of the 4 cuts as shown and plotting the graph of $\mathcal{M}_z, \mathcal{N}, \mathcal{V}_y$ against x

Example 9

For example for cut 2 we have



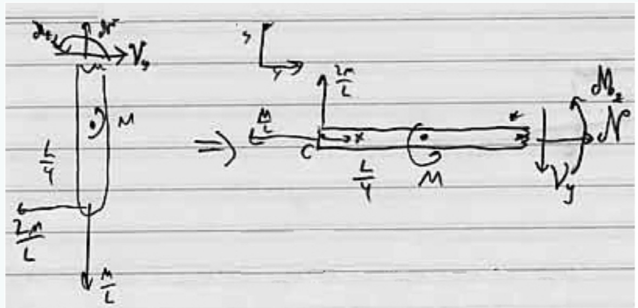
$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_* = 0$$

$$-\frac{M}{L} + N = 0 \qquad \frac{2M}{L} - V_y = 0 \qquad (L)\frac{M}{L} - \frac{2M}{L}x + M_z = 0$$

$$N = \frac{N}{M} \qquad V_y = \frac{2M}{L} \qquad M_z = -M\left(1 - \frac{2x}{L}\right)$$

Example 10

For example for cut 4 we have



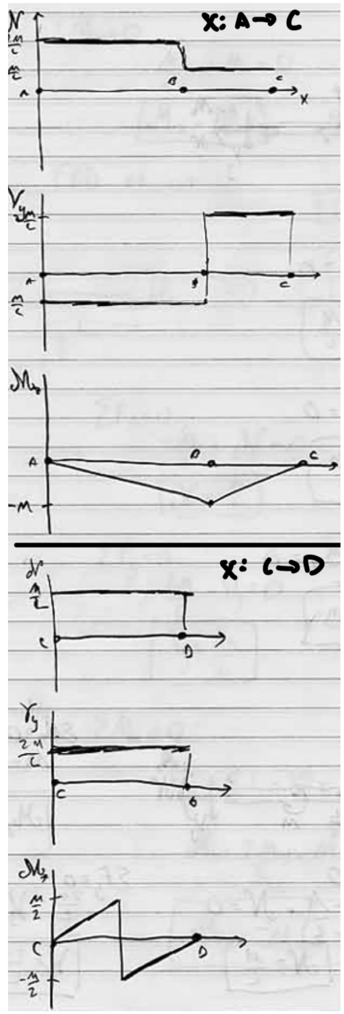
$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_* = 0$$

$$-\frac{M}{L} + N = 0 \qquad \frac{2M}{L} - V_y = 0 \qquad M - \frac{2M}{L}x + M_z = 0$$

$$N = \frac{N}{M} \qquad V_y = \frac{2M}{L} \qquad M_z = -M\left(1 - \frac{2x}{L}\right)$$

Remark 11. *Note that cut 1 and 2 were analyzed on the same component they were both on while cut 3 and 4 was analyzed on another separate component in which they were both on*

each starting from $a \to d$ yields



Steps to determine if statically indeterminate

1. How many uncontrained degrees of freedom? For example we can form one equation with $\sum F_y = 0, \sum M_z = 0, \sum F_x = 0$ each for a total of 3 degrees of freedom. Clearly this differs from support type to type
2. how many unkowns? For example F_A, F_B, F_C are 3 unknowns

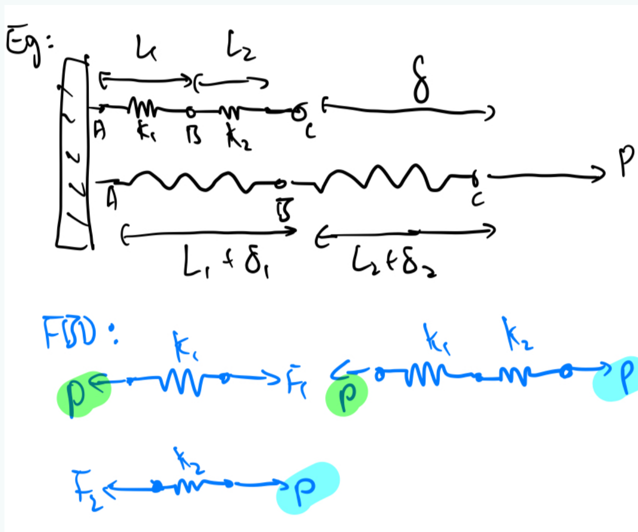
3 Force Deformation and compatibility relationships

Definition 12

Note that $u(x)$ is a measure of how much a member moves while δ is a measure of how much the member stretches

Example 13

Consider



In this case $u_x = \delta$. Note for this system we may also add force deformation comptability relationships like so

Equations of equilibrium

$$F_1 = P, \quad F_2 = P$$

Compatability

$$\delta_1 + \delta_2 = \delta$$

Force deformation

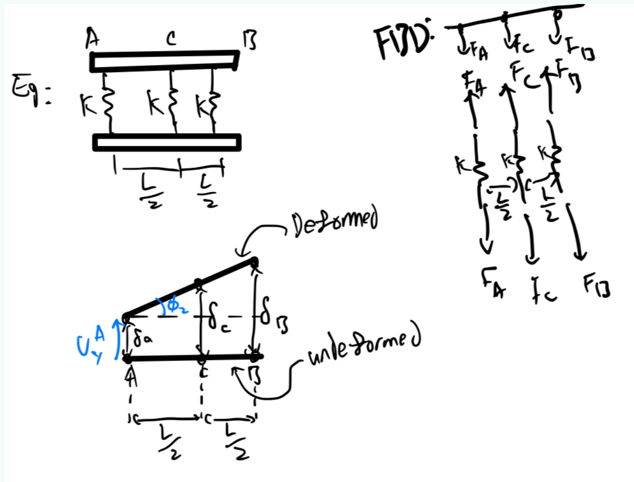
$$F_1 = k_1\delta_1, \quad F_2 = k_2\delta_2$$

and together we have

$$\delta = P \underbrace{\left(\frac{k_1 + k_2}{k_1 k_2}\right)}_{k_{\text{eff}}}$$

Example 14

Now consider



Notice that this system is **statically indeterminate** since the degrees of freedom ($\sum F_y = 0, \sum M_z = 0$) is 2 but we have three unknowns F_A, F_B, F_C(additionally we also want to solve for $u(x)$).Hence we add force deformation relationships

$$F_A = k\delta_A \quad F_B = k\delta_B \quad F_A = k\delta_C \quad F_A = k\delta_C$$

which are basically **hooke's law** equations. Then we add compatibility relationships so they can be applicable to our system

$$\begin{aligned} \delta_A &= u_Y^A \\ \tan\varphi &= \frac{\delta_B - \delta_A}{L} \\ \tan\varphi &= \frac{\delta_C - \delta_A}{L/2} \end{aligned}$$

to increase the number of independent equations to solve. Namely using small angle approximation $\tan\varphi_z^A = \frac{\sin\varphi_z^A}{\cos\varphi_z^A} \approx \varphi_z^A$ upon substitution of compatability equations to our force deformation equations

$$\begin{aligned} F_A &= ku_y^A \\ F_B &= k(u_y^A + L\varphi_z^A) \\ F_C &= k(u_y^A + \frac{L}{2}\varphi_z^A) \end{aligned}$$

and then into our equations of equilibrium obtain

$$\varphi_z^A = \frac{-P}{Lk}\left(1 - \frac{2a}{L}\right)$$

and

$$u_z^A = -\frac{P}{2k}\left(1 - \frac{2a}{L}\right)$$

see that $\tan\phi \approx \phi_z^A$ is the gradient of the straight line while $u(0) = u_y^A$ is the y intercept. Therefore we obtain

$$u(x) = mx + c = \frac{P}{3k} + \frac{P}{2k}\left(1 - \frac{2a}{L}\right) + \left(-\frac{P}{2k}\left(1 - \frac{2a}{L}\right)\right)x$$

4 Uniaxial

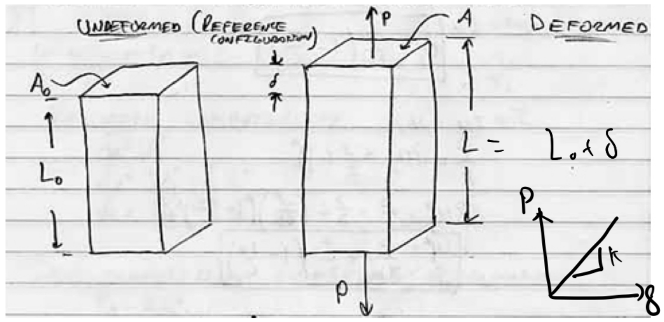


Figure 3: K is a constant dependent on material and geometry

Note that P here refers to force

Definition 15

The **engineering stress** is defined by

$$\sigma = \frac{P}{A_0}$$

while the **engineering strain** is defined by

$$\varepsilon = \frac{\delta}{L_0}$$

Remark 16. *See that stress(σ) has the same units as pressure while strain(ε) being a ratio of lengths is dimensionless*

If we took a stress-strain plot instead of pressure-change in lenghth plot instead. That is $\sigma = \frac{P}{A_0}$ against $\varepsilon = \frac{\delta}{L_0}$ instead of P against δ as shown in the diagram above, we will clearly again get linear plot. The gradient of this straight line graph is denoted by E which is known as the **young's modulus**. Putting the relationships together we have

$$\boxed{\sigma = E\varepsilon}$$

Then using the definition of σ, ε from above we have

$$\frac{P}{A_0} = \frac{E\delta}{L_0}$$

so

$$P = \frac{EA_0}{L_0}\delta$$

recall from the diagram above that k is the gradient of the straight line plot(a consequence of hooke's law, linear elasticity) of P against δ so

$$k = \frac{EA_0}{L_0}$$

for uniaxial loading.So in summary, consistent with hooke law we have $P \propto \delta$

$$\boxed{P = k\delta}$$

Now consider

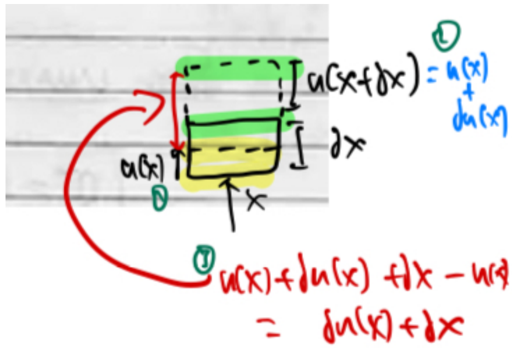


Figure 4: deformation from box(solid) to box(dashed)

To see this recall 12 that $u(x)$ is the measure of how much an item moves. Therefore the displacement of the bottom face(the change in the bottom dashed line from the solid bottom line aka the yellow higlighted parts) is $u(x)$. While the difference green highlighted part is clearly $u(x + dx)$ as it similarly represents the displacement of the top face. Then the red component is clearly calculated as shown. Now recall that δ represents the chance in lenghth so we have

$$\delta = (du(x) + dx) - dx$$

and so

$$\varepsilon = \frac{\delta}{L} = \frac{du(x)}{dx}$$

where L represents the inital lenghth(recall definition of engineering strain).

Example 17

Consider this scenario.As usual now find $F_A, F_B, F_C, u_B(x)$, using force-deformation comptability relationships if necessary.

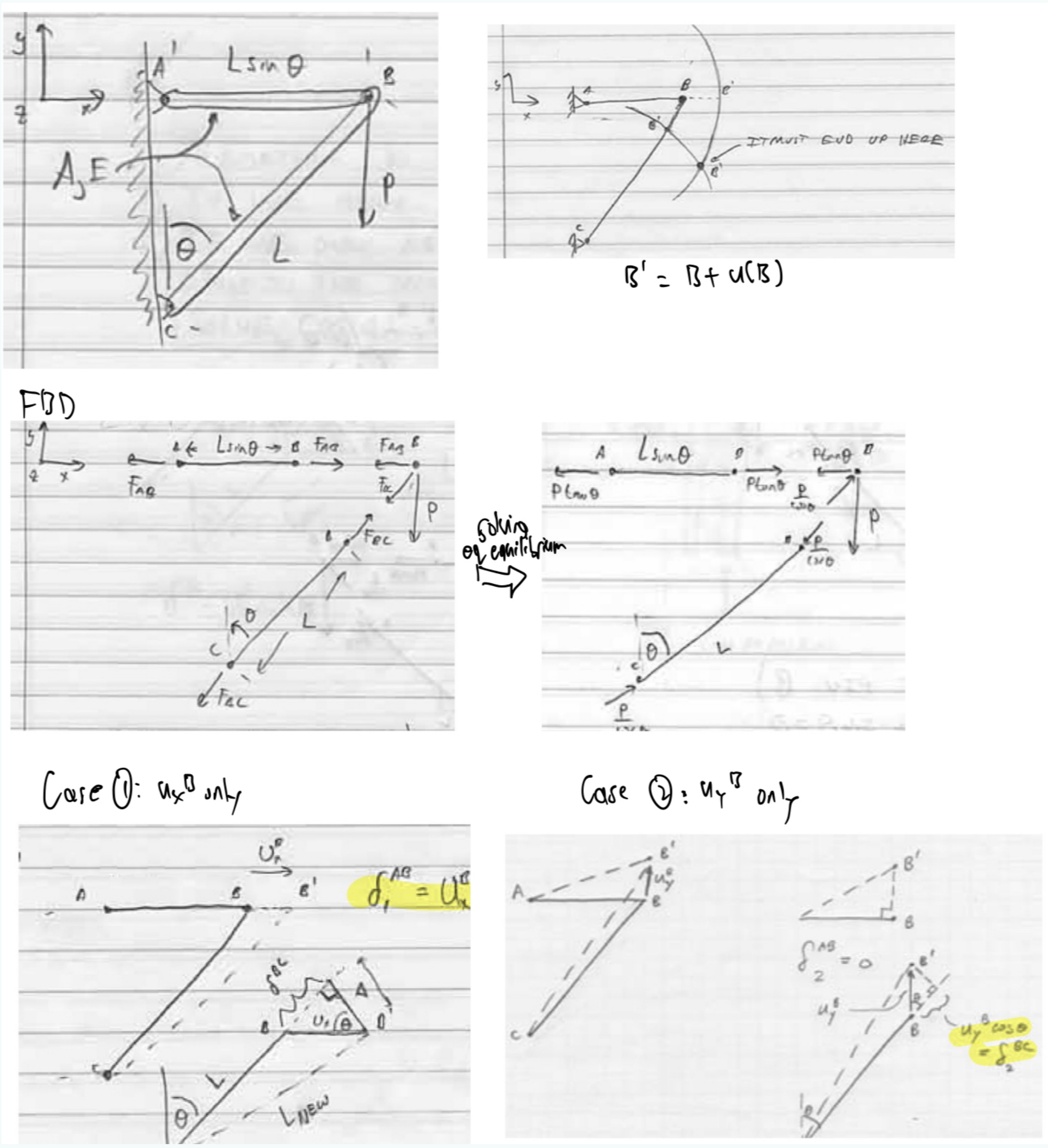


Figure 5: Assume trusses that deform(if not this would budge)

Solving for FBD we obtain

$$F_{AB} = P \tan \theta$$

$$F_{BC} = \frac{P}{\cos \theta}$$

Then using the force deformation relationships($P = k\delta$ and $k = \frac{EA}{L}$) as we found above we have

$$\delta_{AB} = \frac{F_{AB}}{k_{AB}} \qquad k_{AB} = \frac{AE}{L\sin\theta}$$

$$\delta_{BC} = \frac{F_{BC}}{k_{BC}} \qquad k_{BC} = \frac{AE}{L}$$

In case 1 where we have only u_x^B see that

$$\delta_1^{AB} = U_x^B \qquad \delta_1^{BC} = u_x^B \sin\theta$$

where we denote the change in lenght for case 1 by δ_1. Similary δ_2 for case 2. We can similary solve for case 2 where there is only u_y^B where we get

$$u_y^B \cos\theta = \delta_2^{BC} \qquad \delta_2^{AB} = 0$$

5 multi axial stress and strain

Recall that in uniaxial stress we had the following. However in the general case, $d\vec{F}$ is not necessarily parallel to $\hat{n}$

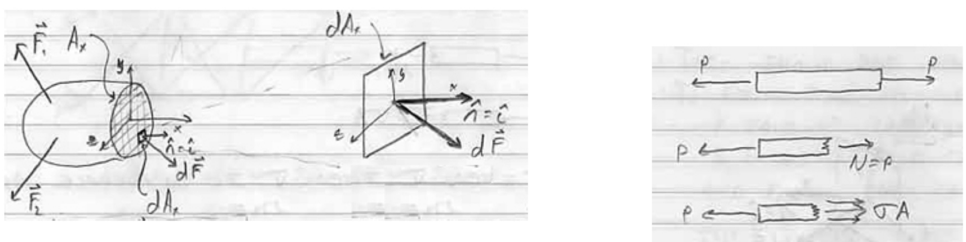


Figure 6: uniaxial vs general case

So it makes sense to define

Definition 18

the **traction** $\vec{t}$ is the force per unit area at a point

$$\vec{t}^{(i)} = \frac{dF_x^{(i)}}{dA_x}\hat{i} + \frac{dF_y^{(i)}}{dA_x}\hat{j} + \frac{dF_z^{(i)}}{dA_x}\hat{k}$$

and

Definition 19

Note that

$$\sigma_{xx} = \sigma_x x = \frac{dF_x^{(i)}}{dA_x} \text{ (Normal Stress)}$$
$$\sigma_{xy} = \sigma_x y = \frac{dF_y^{(i)}}{dA_x} \text{ (Shear Stress)}$$
$$\sigma_{xz} = \sigma_x z = \frac{dF_z^{(i)}}{dA_x} \text{ (Shear Stress)}$$

so we may rewrite the traction $\vec{t}^{(i)}$ as

$$\vec{t}^{(i)} = \sigma_{xx}\hat{i} + \sigma_{xy}\hat{j} + \sigma_{xz}\hat{k}$$

We may define similary for the othe faces as shown in the figure below

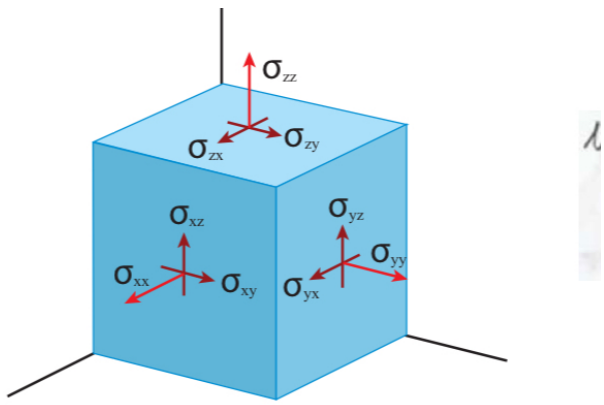


Figure by MIT OCW.

Matrix form of stress tensor

$$[\sigma\ ] = \left| \begin{matrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{matrix} \right| .$$

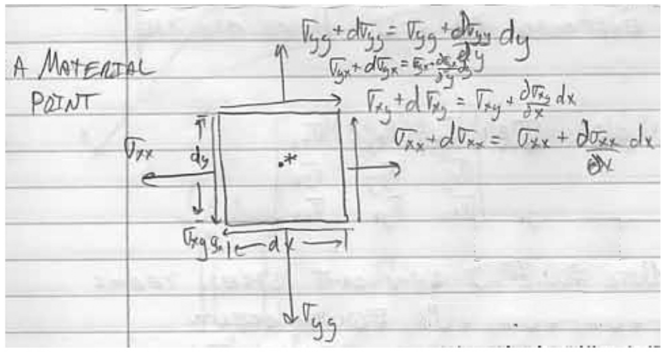
the diagonal terms are normal stresses while off diagonal terms are shear stresses.

Proposition 20

to obey the equations of equlibrium the stress tensor σ is symmetric

Remark 21. *We will show the proof for 2D, for 3D please refer to your* `2.071 Mechanics of solid materials` *notes.*

Proof. Consider



To see this consider the x face. At the intial x coordinate(the left face) the stress on it is σ_{xx}. Then dx dispacement later the stress(on the right face) is then naturally equal to $\sigma_{xx} + d\sigma_{xx}$. As for she shear forces on the x face we have $\sigma_{yx} + \sigma_{yx}$ on the right side too after displacement dx. The same logic applies to other faces. Now solving for equations of equlibrium we find that for $\sum F_x = 0$

$$-\sigma_{xx}dydz + (\sigma x + \frac{\partial \sigma_{xx}}{\partial x}dx)dydz - \sigma_{yx}dxdz + (\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}dy)dxyz = 0$$

where the red and blue part is $d\sigma_{xx}$ and $d\sigma_{yx}$ respectively so we have

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

and similary for $\sum F_y = 0$ we can obtain

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$$

Nowing $\sum M = 0$ about the centre of the block we have

$$(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x}dx)\frac{dx}{2}dydz + \sigma_{xy}dydz\frac{dx}{2} - \sigma_{yx}dxdx\frac{dy}{2} - (\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}dy)dxdz\frac{dy}{2} = 0$$

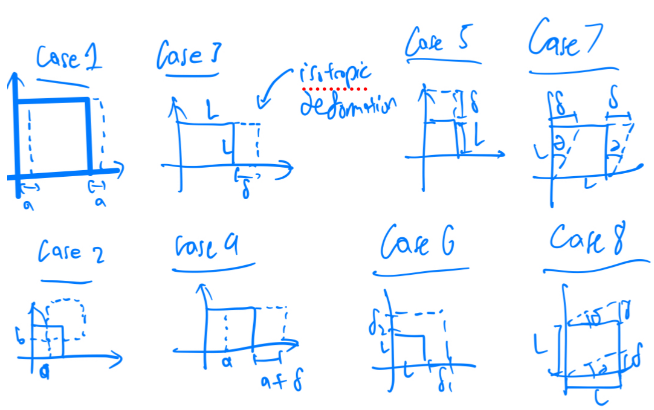
where we have

$$\sigma_{xy} - \sigma_{yx} + \frac{\partial \sigma_{xy}}{\partial x}\frac{dx}{2} - \frac{\partial \sigma_{yx}}{\partial y}\frac{dy}{2} = 0$$

but as $dx, dy \to 0$ the last equation yields

$$\sigma_{xy} = \sigma_{yx}$$

Now consider



(Case 1) $\vec{u}(x,y,z) = a\hat{i}$

(Case 2) $\vec{u}(x,y,z) = a\hat{i} + b\hat{j}$

(Case 3) $\vec{u}(x,y,z) = \frac{\delta}{L}x\hat{i}$

$$\frac{\partial u_x}{\partial x} = \frac{\delta}{L}$$

(Case 4) $\vec{u}(x,y,z) = (a + \frac{\delta}{L}x)\hat{i}$

$$\frac{\partial u_x}{\partial x} = \frac{\delta}{L}$$

(Case 5) $\vec{u}(x,y,z) = \frac{\delta}{L}y\hat{j}$

(Case 6) $\vec{u}(x,y,z) = \frac{\delta_1}{L}x\hat{i} + \frac{\delta_2}{L}y\hat{j}$

(Case 7) $\vec{u}(x,y,z) = \frac{\delta}{L}y\hat{i}$

$$\frac{\partial u_x}{\partial y} = \frac{\delta}{L} = \tan\theta \approx \theta$$

(Case 8) $\vec{u}(x,y,z) = \frac{\delta}{L}x\hat{j}$

$$\frac{\partial u_y}{\partial x} = \frac{\delta}{L} = \tan\theta \approx \theta$$

Definition 22

The **shear modulus** is defined for example in this case

$$\sigma_{xy} = G\gamma_{xy}$$

where the shear modulus as the units Pa(pressure) as seen clearly below

Strain tensor

Consider the case of both case 7 and case 8 occurring

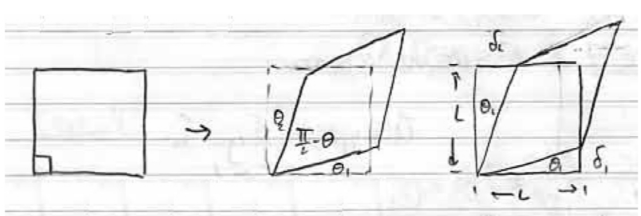


Figure 7: notice that we can apply case 7 then 8 independently like 17

So we see that

$$\gamma_{xy} = \theta_1 + \theta_2 = \frac{\delta_1}{L} + \frac{\delta_2}{L}$$
$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Now define

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Remark 23. *Notice that* $\varepsilon_{yx} = \varepsilon_{xy} = \frac{\gamma_{xy}}{2}$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

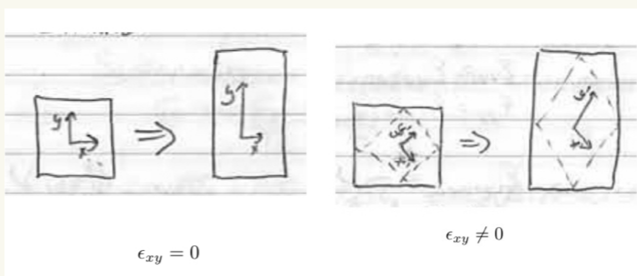
and the strain tensor

$$[\varepsilon] = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

Hence by definition the strain tensor is also symmetric as in 23

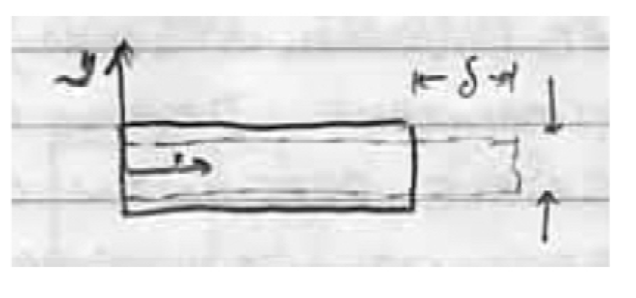
Fact 24

Note that the orientation of axis matters



Example 25

Consider



where we have

$$\varepsilon_{xx} = \frac{\delta}{L}; \varepsilon_{yy} = \text{some negative value}$$

in this case

$$\varepsilon_{xx} = \varepsilon_{\text{axial}} \quad \varepsilon_{yy} = \varepsilon_{\text{lateral}}$$

then we say the ν is the **poisson ratio** which is denoted by

$$\varepsilon_{\text{lateral}} = -\nu\varepsilon_{\text{axial}}$$

where clearly poisson ratio is unitless as the name suggests

Definition 26

Isotropic means material properties are the same in all orientations

elastic means deformation is removed when load is released(deformation is fully recovarable)

Definition 27

For **linear isotropic elastic** members we may use what we call **equations of linear isotropic elasticity** defined by

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \qquad \varepsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$
$$\varepsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] \qquad \varepsilon_{xz} = \frac{1}{2G}\sigma_{xz}$$
$$\varepsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] \qquad \varepsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$

where

$$G = \frac{E}{2(1+\nu)}$$

We denote that G is the **shear modulus** while ν is the **poisson ratio**

Remark 28. *note I will do derivation after advanced statistical thermodynamics in physics if not you will never have the full picture apparently*

additionally we also have

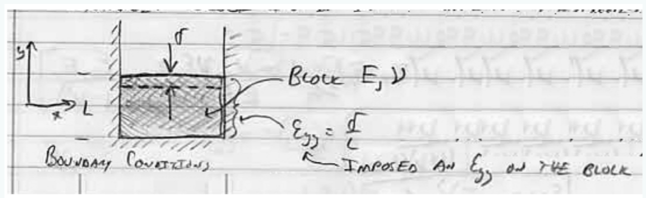
Definition 29

These

$$\gamma_{xy} = \frac{1}{G}\sigma_{xy} \qquad \varepsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$
$$\gamma_{xz} = \frac{1}{G}\sigma_{xz} \qquad \varepsilon_{xz} = \frac{1}{2G}\sigma_{xz}$$
$$\gamma_{yz} = \frac{1}{G}\sigma_{yz} \qquad \varepsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$

Example 30

Consider



Find all the missing components below. Note that ε_{xx}

Solution. The boundary conditions are such that

$$\varepsilon_{xx} = 0 \qquad \sigma_{xx} = ?$$
$$\varepsilon_{yy} = \text{given} \qquad \sigma_{yy} = ?$$
$$\varepsilon_{zz} = ? \qquad \sigma_{zz} = 0$$

using the equations as well the Poisson ratio relationship

$$\varepsilon_{xx} = \nu\varepsilon_{yy}$$

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} - \sigma_{zz})\right)$$

to obtain

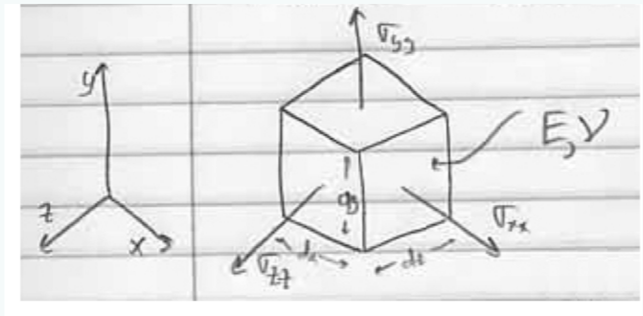
$$\sigma_{yy} = \frac{\varepsilon_{yy}E}{(1-\nu^2)}$$

and

$$\varepsilon_{zz} = \frac{-\nu}{1-\nu}\varepsilon_{yy}$$

Example 31

Consider



Find the change in volume as well as all the missing components below

Solution. In this case

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$$

the initial volume is

$$V_i = dxdydz$$

and the final volume

$$V_f = (1+\varepsilon_{xx})dx(1+\varepsilon_{yy})dy(1+\varepsilon_{zz})dz$$

we may find the change of volume as

$$\Delta V = (1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})(V_i - V_i)$$

assuming small strains so higher order strains like below are zero

$$\varepsilon^3, \varepsilon^2 \approx 0$$

so our equation simplies to

$$(1+\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{yy})V_i - V_i$$

now using the constitutive relation

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{xx}+\sigma_{zz})] = \frac{1}{E}[-p - \nu(-p-p)] = \frac{-1(1-2\nu)}{E}p$$

see that we will also get

$$\varepsilon_{yy} = \frac{-1(1-2\nu)}{E}p$$

$$\varepsilon_{zz} = \frac{-1(1-2\nu)}{E}p$$

therefor subsituting these relations back into our equation for ΔV we will find that

$$\frac{-p}{\Delta V/V_i} = \frac{E}{3(1-2\nu)} = k$$

where k is called the **bulk modulus**

6 thermal strain

Definition 32

thermal strain

$$\varepsilon_{xx}^T = a\Delta T$$
$$\varepsilon_{yy}^T = \Delta T$$
$$\varepsilon_{zz}^T = \Delta T$$

where T stands for thermal

Definition 33

the **total strain** is given by

$$\varepsilon_{xx} = \varepsilon_{xx}^E + \varepsilon_{xx}^T$$

where E stands for elastic. The same appplies for yy,zz

So we rewrite our constitutive relation as

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] + a\Delta T$$

same for yy,zz

Example 34

We now return back to our block on a frictionless channel example.This time instead of imposing a strain ε_{yy} we now subject the block to an increased $T, \Delta T > 0$.(In other words subject to a thermal strain instead) Again find the missing components

Solution. so now our boundary conditions are The boundary conditions are such that

$$\begin{aligned} \varepsilon_{xx} &= 0 & \sigma_{xx} &=? \\ \varepsilon_{yy} &=? & \sigma_{yy} &= 0 \\ \varepsilon_{zz} &=? & \sigma_{zz} &= 0 \end{aligned}$$

Now using our constitutive relations

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] + a\Delta T$$

on subbing in boundary conditions we find that

$$\sigma_{xx} = -a\Delta T E$$

solving for the rest similarly we find that

$$\varepsilon_{yy} = \frac{-\nu}{E}(-a\Delta T E) + a\Delta T$$

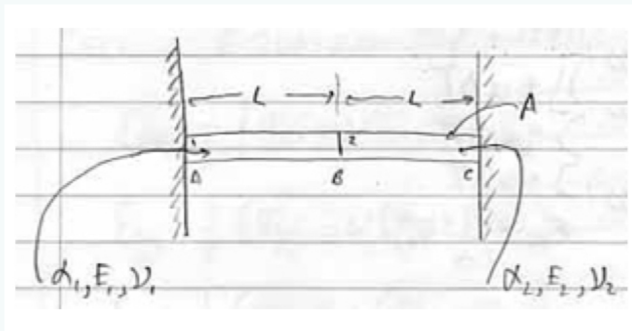
$$\varepsilon_{zz} = \frac{-\nu}{E}(-a\Delta T E) + a\Delta T$$

which simplifies to

$$\varepsilon_{yy} = \varepsilon_{zz} = (\nu + 1)a\Delta T$$

Example 35

Now consider a bar made of two component materials with respective a, E, V



Now subject the bar to increased $T, \Delta T > 0$. Find the displacement $\vec{u}$ of B

Solution. first consider the FBD of the entire rod. Suppose we have two arrows F_A, F_B both pointing horizontally right acting on A and B respectively. Now at equilbrium we have

$$\sum F_x = 0, \quad F_A = -F_C$$

now adding in the constitutive relationships we have

$$\varepsilon_{xx_1} = \frac{1}{E_1}(\sigma_{xx_1} - \nu_1(\sigma_{yy_1} + \sigma_{zz_1})) + a_1\Delta T$$

but because there are no forces acting in the y or z direction we have that the yy,zz stress terms disappear so we have

$$\varepsilon_{xx_1} = \frac{1}{E_1}(\sigma_{xx_1}) + a_1\Delta T$$

similarly for the other side we have

$$\varepsilon_{xx_2} = \frac{1}{E_2}(\sigma_{xx_2}) + a_2\Delta T$$

now add in the compatibility relationships we define u_x^B to be the displacement from point B in the x direction letting right be positive so we have

$$\delta_1 = u_x^B$$
$$\delta_2 = -u_x^B$$

(that is you can see as the 1st component extending while the the second component shortening proporationately while the total length stays the same) so

$$\delta_1 = -\delta_2$$

. Recall that for a uniaxially loaded bar we can then write

$$\delta_1 = \varepsilon_{xx_1} L$$

$$\delta_2 = \varepsilon_{xx_2} L$$

so we have $\varepsilon_{xx_1} = \varepsilon_{xx_2}$. A;so by definition of stress we have

$$F_A = -\sigma_{xx_1} A$$

$$F_B = \sigma_{xx_1} A$$

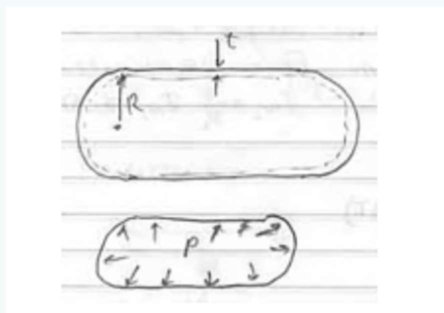
therefore considering the FBD equilibrium condition on the left component where we have $F_A = F_B$ we have that

$$\sigma_{xx_1} = -\sigma_{xx_2}$$

Finally subbing all this relations back into our original constitutional relationships derived above we

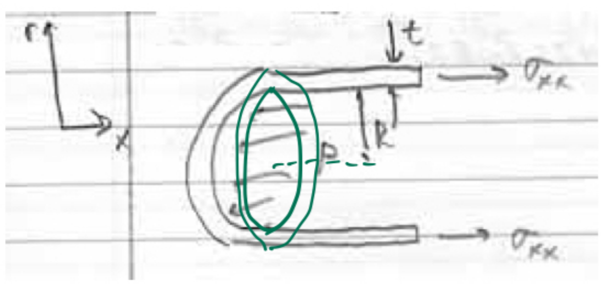
Example 36

Consider



$t << R \Leftarrow$ Thin Walled Vessel

Solution. Consider

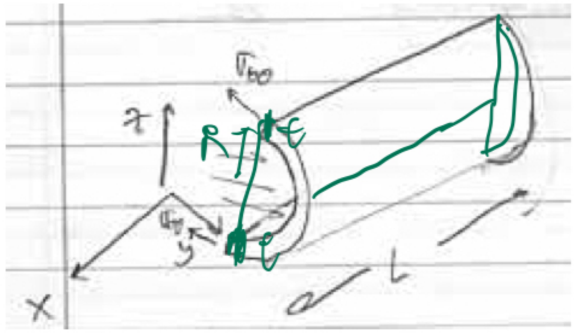


solving FBD we have

$$\sum F_x = 0$$

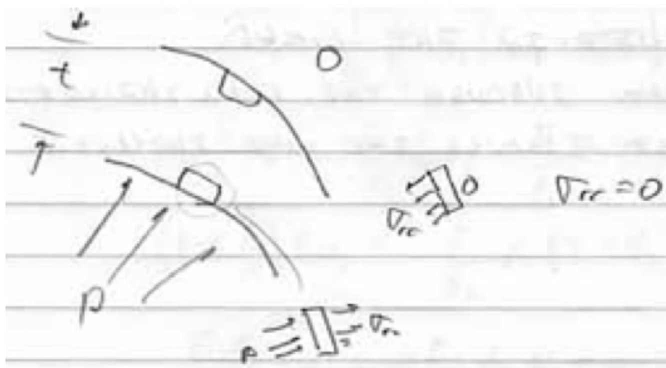
$$\sigma_{xx}(2\pi R t) - P(\pi R^2) = 0$$

$$\sigma_{xx} = \frac{PR}{2t}$$



solving FBD we have

$$\sum F_y = 0$$
$$\sigma_{\theta\theta} L t(2) - P(2R)L = 0$$
$$\sigma_{\theta\theta} = \frac{PR}{t}$$



solving FBD we have

$$\sigma_{rr} = -P$$

since uniform this applies throughout the thickness as shown. Now since $R >> t$ so $R/t >> 1$ and so $P << \frac{PR}{t}$. Therefore we approximate $\sigma_{rr} = 0$ relative to the axial and radial stresses. That is our stress tensor now looks like so

$$[\sigma] = \begin{pmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{xx} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{Rp}{t} & 0 \\ 0 & 0 & \frac{pR}{2t} \end{pmatrix}$$

which is essentially simplifies to a plane stress situation.To find the strains simply use the constitutive relationships and the fact that

$$\varepsilon_{\theta\theta} = \frac{\Delta R}{R}$$
$$\varepsilon_{xx} = \frac{\Delta L}{L}$$
$$\varepsilon_{rr} = \frac{\Delta t}{t}$$

7 stress transformation

Recall from your `MIT 2.071 Continuum Mech` notes that you using tensors(or rather just base change matrices) we have

$$A'_{11} = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + A_{12}\sin 2\theta,$$
$$A'_{22} = \frac{A_{11}+A_{22}}{2} - \frac{A_{11}-A_{22}}{2}\cos 2\theta - A_{12}\sin 2\theta,$$
$$A'_{12} = -\frac{A_{11}-A_{22}}{2}\sin 2\theta + A_{12}\cos 2\theta,$$

where A is the stress tensor/matrix and 1 and 2 represent x and y.

Definition 37

The average stress $\overline{\sigma}$ is given by

$$\overline{\sigma} = \frac{\sigma_{xx}+\sigma_{yy}}{2}$$

with this, consider that from the transformation that we have

$$(\sigma_{x'x'} - \overline{\sigma}) = \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta\right)^2$$
$$\sigma_{x'y'}^2 = \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta\right)^2$$

expanding then adding these 2 relations we realize we obtain

$$(\sigma_{x'x'} - \overline{\sigma})^2 + \sigma_{x'y'}^2 = \left(\frac{\sigma_{xx}-\sigma_{yy}}{2} + \sigma_{xy}^2\right)^2$$

this is essentially the equation of a circle which we call the **mohr's circle**

Example 38

Now consider a uniaxial scenario such that $\sigma_0 = \sigma_{xx}$.

$$\boxed{\quad\text{rod}\quad} \to \sigma_0 = \sigma_{xx}$$

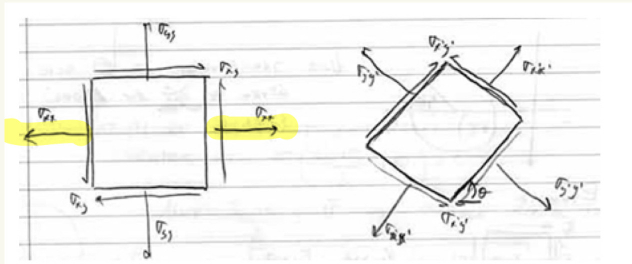
$$\to (+ve \text{ x axis}), \quad \uparrow (+ve \text{ y axis})$$

Then we have the mohr circle equation

$$(\sigma_{x'x'} - \frac{\sigma_0}{2})^2 + \sigma_{x'y'}^2 = \left(\frac{\sigma_0^2}{2}\right)\underbrace{(\cos^2 2\theta + \sin^2 2\theta)}_{=1}$$

Fact 39 (Sign convention of stress element)

Consider the positive sign notation



Draw for $\sigma_{xx} > \sigma_{yy} > 0;\sigma_{xy} > 0$.

See that

- $\sigma_{xx}, \sigma_{yy} > 0$ because they are pointing outwards from their respective sides
- $\sigma_{xy} > 0$ because they are pointing inwards the top right diagonal and bottom left diagonal respectively

Fact 40 (Sign Convention of Mohr's Circle)

The moor circle is drawn such that the

- horizontal axis represents the normal stress σ with positive to the right
- vertical axis represents the shear stress τ with positive downwards

to see why positive downwards consider

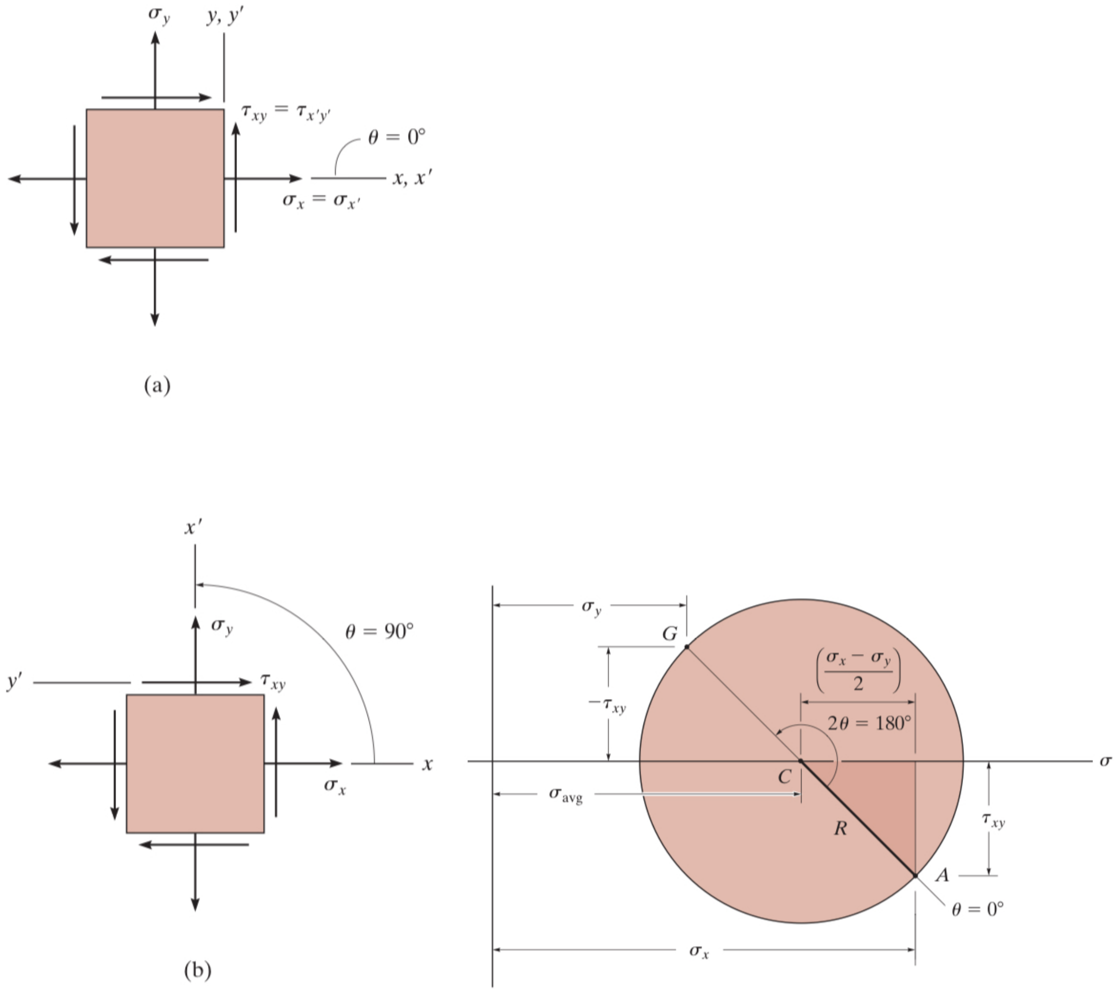


Figure 8: no actually idk its not even stated just take it as some wierd ass convention

Example 41

Consider

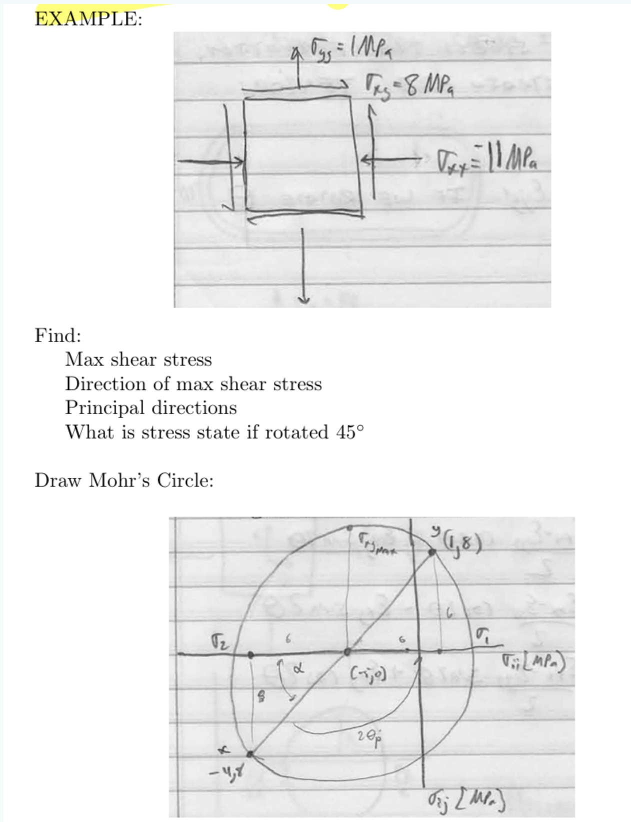


Figure 9: y axis: $\sigma_{x'y'}$(shear);x axis: $\sigma_{x'x'}$(stress)

then we have

$$\overline{\sigma} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{-11 + 1}{2} = -5MPa$$

notice the sign convention for $\sigma_{yy} < 0$ here. Notice

$$R = \sqrt{6^2 + 8^2} = 10MPa$$

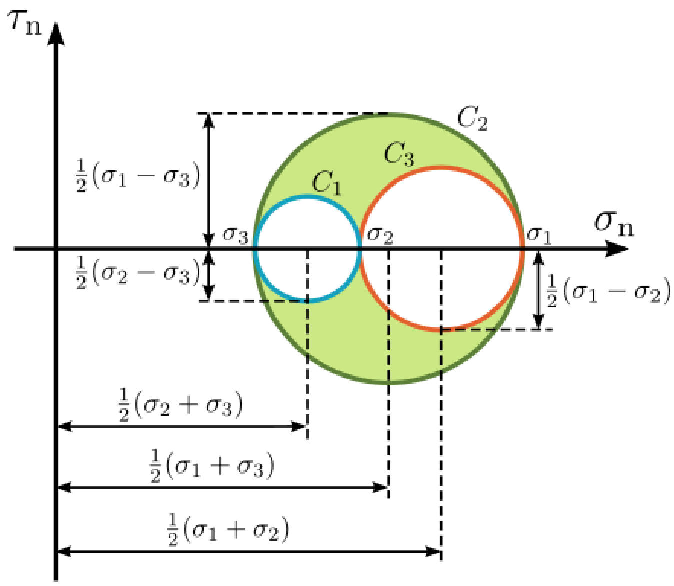
so we easily obtain the max/mind shear stress to be $\pm 10MPa$ respectively. As for the principal stresses it is obvious that we have

$$\sigma_1 = \overline{\sigma} + R = -5 + 10 = 5$$
$$\sigma_2 = \overline{\sigma} + R = -5 - 10 = -15$$

as the maximum and minimum stresses

Example 42

consider now a state of triaxial principal stress assuming $\sigma_1 > \sigma_2 > \sigma_3$ that act outwards on the x,y,z face respectively. Then we have



where the circle centres correspond to

1. x-z plane: $\frac{\sigma_1+\sigma_3}{2}$
2. x-y plane: $\frac{\sigma_1+\sigma_2}{2}$
3. y-z plane: $\frac{\sigma_2+\sigma_3}{2}$

as for radius recall the formula is given by

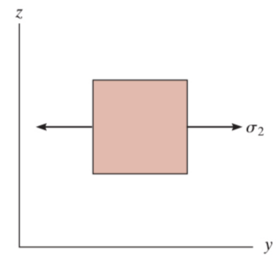
$$\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} + \sigma_{xy}^2\right)$$

but in our case since principal stresses there is not sheer therefore the square term in the bracket disappears and we get what is reflected in the diagram above.

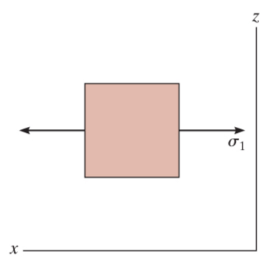
8 failure theory

8.1 tresca

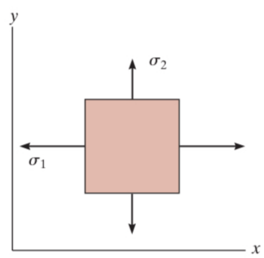
Consider the case when the material is subjected to in plane principal(i.e no shear) stresses σ_1, σ_2. viewing the element in two dimensions we see that



(b)

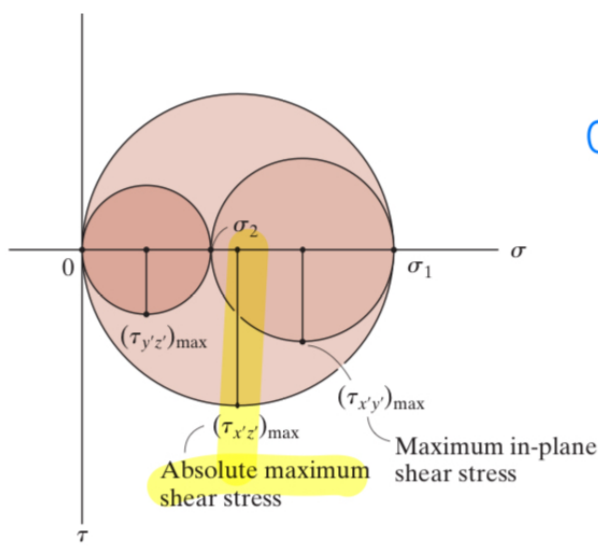


(c)



(d)

Suppose σ_1, σ_2 have the same sign as above then we have

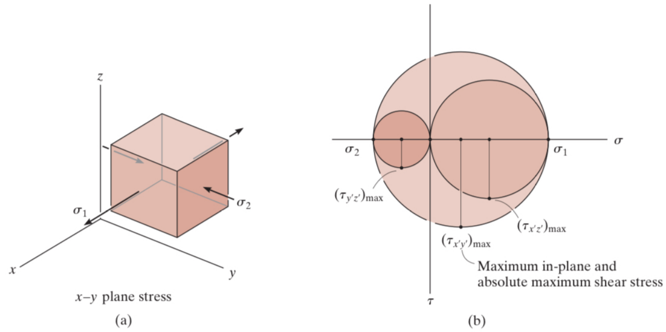


(e)

which means

$$\tau_{\text{abs max}} = \frac{\sigma_1}{2}$$

assuming $\sigma_1 > \sigma_2$ and if different signs



this means

$$\tau_{\text{abs max}} = \frac{\sigma_1 - \sigma_2}{2}$$

again assuming $\sigma_1 > \sigma_2$

Example 43

For example consider a ductile material where planes are aligned by θ with respect to the stress axis. For $\theta = 0$ each element in a plane can be modeled as the uniaxial princpal stress case. Here we have no shear and no stress components in the y or z axis. We just have single positive component in the x axis

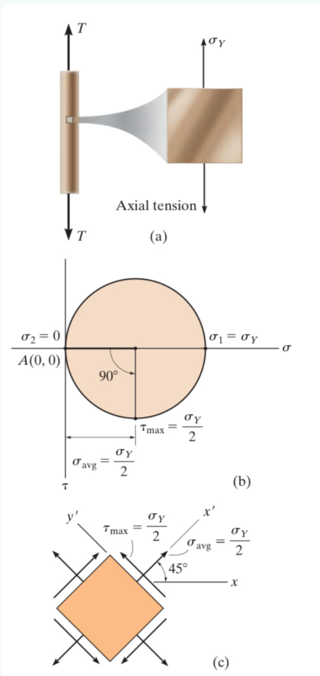


Fig. 10–27

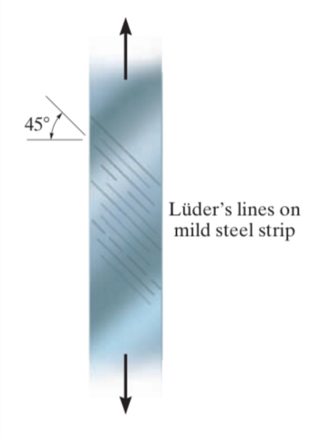


Fig. 10–26

So it is clear the angle in which there is maximum shear will be when the planes are 45 degrees counterclockwise from the principal stress state(as in 10-27(a))

Now as above because relate the maximum sheer stress σ_Y a given pair of principal stresses (σ_1, σ_2) like so:

$$\begin{cases} |\sigma_1| = \sigma_Y, |\sigma_2| = \sigma_Y & \sigma_1, \sigma_2 \text{ have the same signs} \\ |\sigma_1 - \sigma_2| = \sigma_Y & \sigma_1, \sigma_2 \text{ have opposite signs} \end{cases}$$

Graphically we have

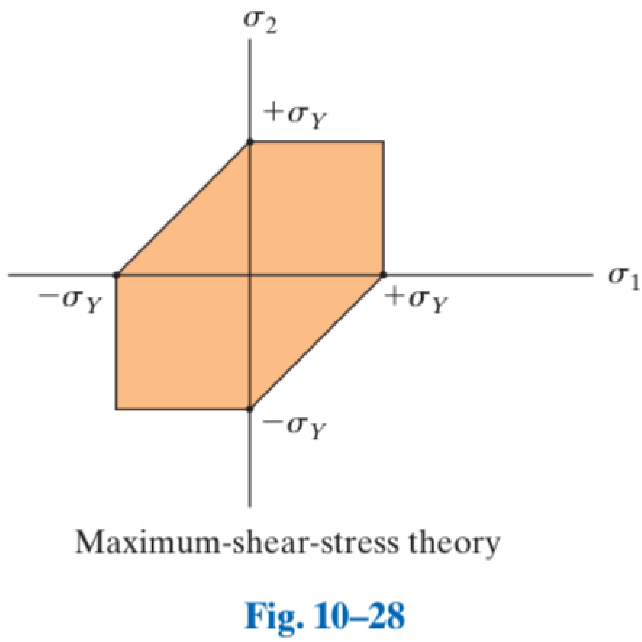


Figure 10: the boundary of the graph below represents σ_Y, the absolute maximum stress values when subjected the various *principal streses* (σ_1, σ_2)

where clearly points outside the orange region will result shear stresses that exceed the maximum allowable sheer stress and hence fail

8.2 maximum distortion energy theory

Recall that we have defined **strain density** as

$$u = \frac{1}{2}\sigma\varepsilon$$

now when subjected to princpal stresses as shown in (a) below we can break this into components

$$u = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

now assuming the material behaves in a linera elastic manner we may use the consitutive relations and rewrite the above as

$$u = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_3\sigma_2)]$$

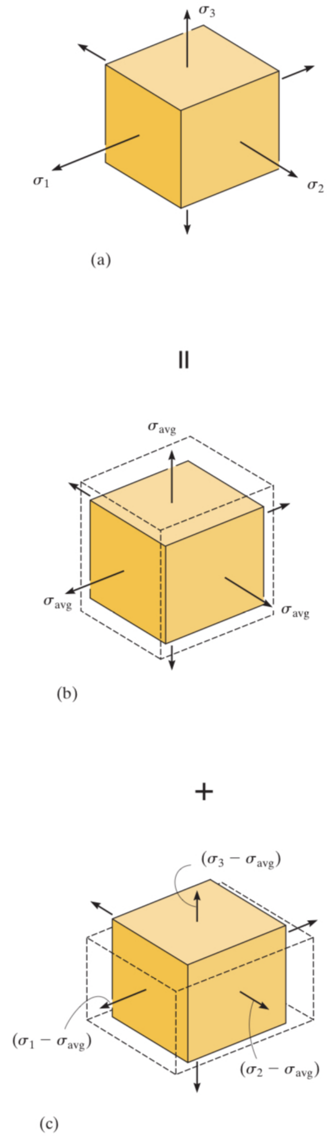


Fig. 10–29

Now associate $\sigma_{\text{avg}} = \frac{\sigma_1+\sigma_2+\sigma_3}{3}$ with the stress that acts on each face. Now because the magntiude of the force on each face is the same the volume change associated will be uniform, that it scales with the original volume, preserving the ratios between its dimensions. Then the remaining $(\sigma_1 - \sigma_{\text{avg}}), (\sigma_2 - \sigma_{\text{avg}}), (\sigma_3 - \sigma_{\text{avg}})$ components of the stress are then associated with the stress components that distort the volume. Therefore we may define

Definition 44

the **distortion energy per unit volume** as

$$u_d = \frac{1}{2}(\sigma_1 - \sigma_{\text{avg}})\varepsilon_1 + \frac{1}{2}(\sigma_1 - \sigma_{\text{avg}})\varepsilon_2 + \frac{1}{2}(\sigma_1 - \sigma_{\text{avg}})\varepsilon_3$$

then subbing σ_{avg} we obtain

$$u_d = \frac{1+\nu}{6E}[(\sigma_1 - \sigma_2^2) + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

Consider the state of plane stress, $\sigma_3 = 0$ then our equation reduces to

$$u_d = \frac{1+\nu}{3E}(\sigma_1^2 - \sigma_1\sigma_3 + \sigma_2^2)$$

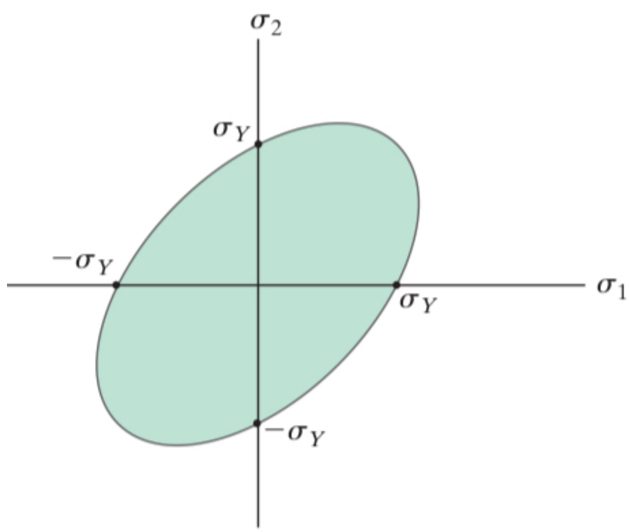
for a uniaxial tension test(i suppose the maximum stress on one axis?) that is $\sigma_1 = \sigma_Y, \sigma_2 = \sigma_3 = 0$ the distortion energy in that case is

$$(\mu_d)_Y = \frac{1+\nu}{3E}(\sigma_Y)^2$$

Theorem 45

By the **maximum distortion energy theorem** says that at all times $u_d \leq (u_d)_Y$

Graphically we have



Maximum-distortion-energy theory

Fig. 10–30

Comparing tresca with distortion theory for the plane stress case

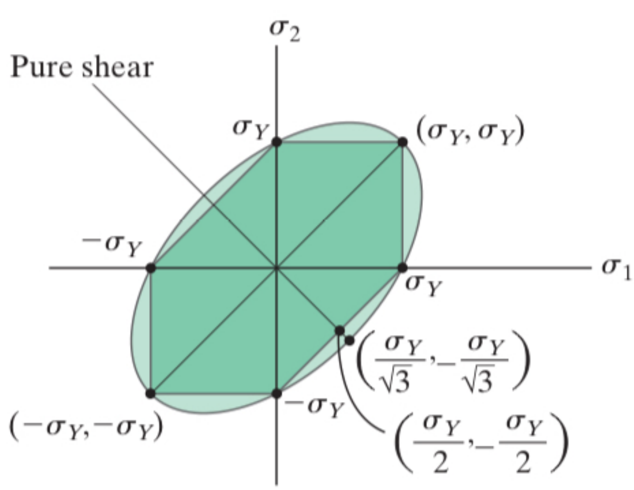
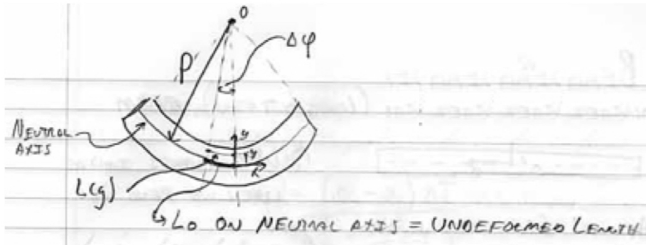
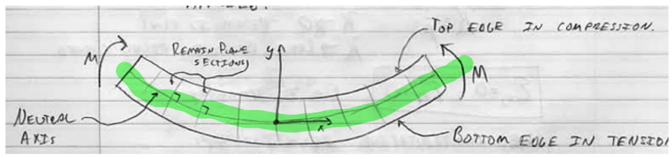


Fig. 10–31

9 Beam Bending

Consider what happense when we bend a slender(long and thin) beam





Notice that one side is stretched while one side is compressed. Hence it follows that in between somewhere there must be plane with no strain which we call the **neutral axis**. Now let the lenght of the neutral axis be L_0. Now consider

$$L_0 = p\Delta\varphi$$
$$L(y) = (p - y)\Delta\varphi$$

where y is the *displacement* from the *neutral axis*. hence we find that

$$\varepsilon_{xx} = \frac{\Delta L}{L_0} = \frac{(p-y)\Delta\varphi - p\Delta\varphi}{p\Delta\varphi} = \frac{-\Delta\varphi y}{p\Delta\varphi} = \frac{-y}{p}$$

notice that when $p \to \infty$ we essentially have a flat beam(think of what happens when infinite radius for a circle segment like above). Conversely when $p \to 0$ it corresponds to a very sharp curve. By assumption of slender, this means the length L is $>>$ than the width or breadth. So should that there be any deformation due to bending in the y or z axis which might change the breadth and width, they are assumed to be very negligible. Therefore we may assume

$$\sigma_{yy} = \sigma_{zz} = \sigma_{yz} = 0$$

now looking at the constitutive relations we hence have

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] = \frac{\sigma_{xx}}{E} = \frac{-y}{p} \\
\varepsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] = \frac{-\nu}{E}\varepsilon_{xx} = \frac{-\nu}{EL}\left(\frac{-y}{p}E\right) = \frac{\nu y}{p} \\
\varepsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] = \frac{\nu y}{p} \\
\varepsilon_{xy} &= \frac{\sigma_{xy}}{2G} = 0 \\
\varepsilon_{yz} &= \frac{\sigma_{yz}}{2G} = 0 \\
\varepsilon_{xz} &= \frac{\sigma_{xz}}{2G} = 0
\end{aligned}$$

to see why the last 3 relations hold true recall their defintions in 22 noting that $\vec{u}_y, \vec{u}_z = 0$. Therefore noticing that

$$\sigma_{xx} = \frac{-y}{p} E \tag{1}$$

and that $y = 0$ at the neutral axis(see the diagrams above) we have

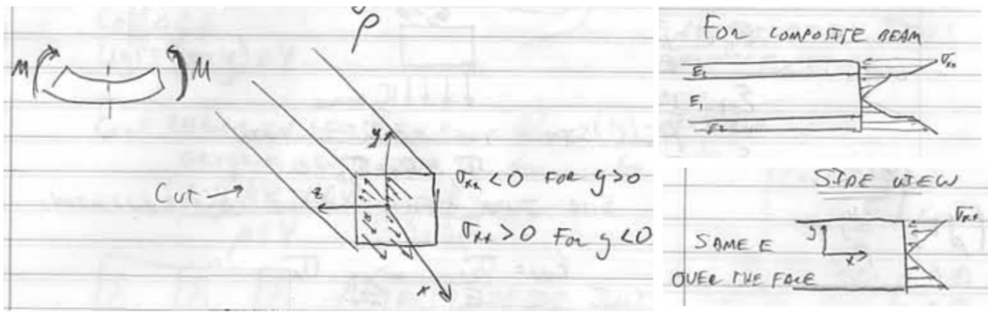


Figure 11: notice the arrow directions of the stress σ_{xx} flip polarity after $y = 0$ on the right image which shows an arbitrary cut element in the beam. This should make sense considering that this generates the moment

Now clearly considering its FBD, for equilibrium we must have

$$\sum F_x = 0$$
$$\int_A \sigma_{xx} dA = 0$$

and for moment equilibrium we have

$$\sum M_z = 0$$
$$M = -\int_A \sigma_{xx} y dA$$
$$M = \int_A \frac{Ey^2}{p} dA$$

where to emphasize M is a moment about axis z

Fact 46

Consider the special case where E is constant through the cross section in which case we have

$$M = E\int_A \frac{y^2}{p}dA = \frac{1}{p}EI$$

where the moment of inertia I(or more precisely here the 2nd moment of area here) is

$$I = \int_A y^2 dA$$

also recall from 1 we have

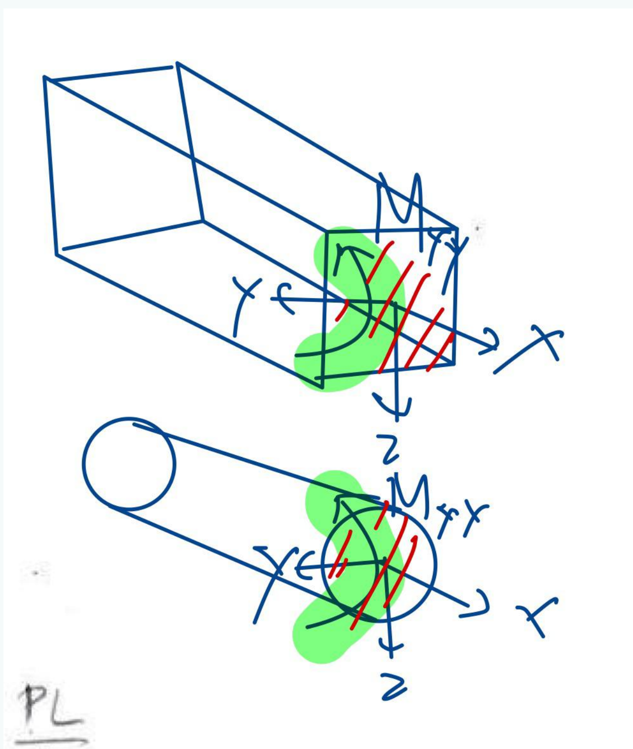
$$\sigma_{xx} = \frac{-Ey}{p}$$

so we have

$$\frac{E}{p} = \frac{M}{I} = \frac{-\sigma_{xx}}{y}$$
$$\sigma_{xx} = \frac{-My}{I}$$

Example 47

Find the second moment of area I for the rectangular and circular member respectively



Let b(// y direction) be the breadth and h(// z direction) be the height of the rectangular face while d be the diameter of the circular face

Solution. For rectangular we have

$$I_{yy} = \int z^2 dA = \int z^2 dy dz = \int_{-h/2}^{h/2} z^2 dz \int_{b/2}^{-b/2} dy = \frac{bh^3}{12}$$

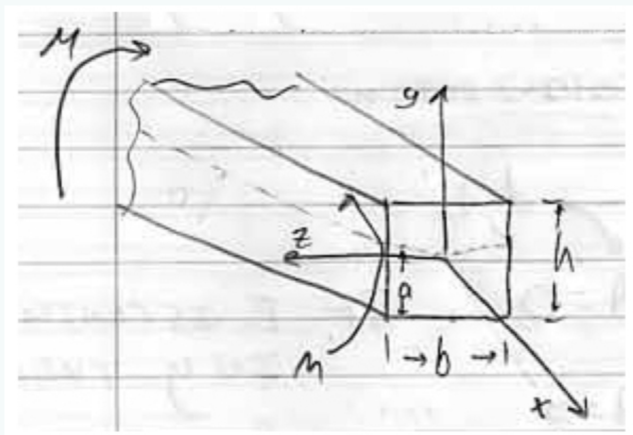
as for circular we have

$$I_{yy} = \int z^2 dA = \int_0^{d/2} r^2 2\pi r dr = \frac{\pi d^2}{64}$$

basically we were integrating over circular rings over the circular surface area.

Example 48

Consider



assume E is constant across the cross-section. Find a the position of the neutral axis

Solution. Recall for force equilibrium we have since E is constant

$$\begin{aligned}
\sum F_x &= 0 \\
\int_A \sigma_{xx} \frac{Ey}{p} dA = \frac{E}{p} \int_A y dA = \frac{E}{p} \int_{\frac{-b}{2}}^{\frac{b}{a}} \int_a^{h-a} y dy dz \\
&= \frac{1}{2} \int_{\frac{-b}{2}}^{\frac{b}{a}} [(h-a)^2 - a^2] = \frac{1}{2}(h^2 - 2ha)(\frac{b}{2} + \frac{b}{2}) dz = \frac{b}{2}(h^2 - 2ha) = 0 \\
h^2 &= 2ha \\
a &= \frac{h}{2}
\end{aligned}$$

Example 49

Consider the case in which $E_1 < E_2$ and are separated at the halfway point. Then we have

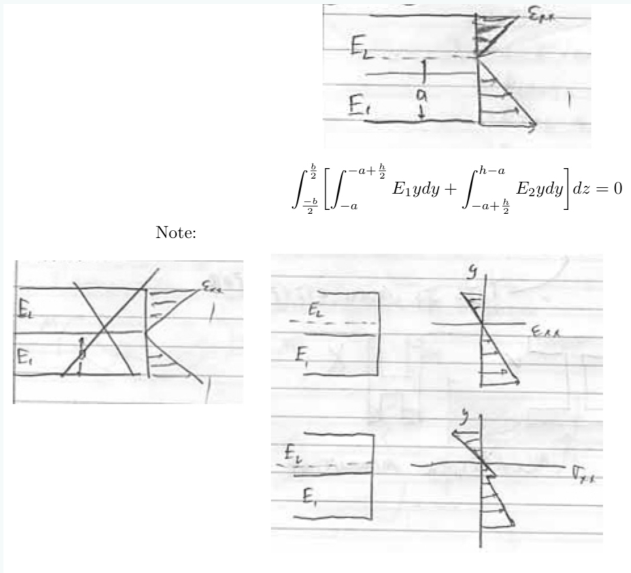


Figure 12: bottom left image: $E_1 < E_2$; bottom right image: $E_2 < E_1$

9.1 unsymetrical bending

Theorem 50

The general formula for the normal stress in unsymmetrical bending is:

$$\sigma_x = -\frac{M_z I_{yy} + M_y I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} y + \frac{M_y I_{zz} + M_z I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} z.$$

Proof. We begin by assuming that the linear-elastic normal stress at any point (y, z) on the cross section is given by:

$$\sigma_x = E\left(-\frac{y}{R_y} + \frac{z}{R_z}\right),$$

where R_y and R_z are the radii of curvature about the y- and z-axes, respectively. Our goal is to find $\frac{1}{R_y}$ and $\frac{1}{R_z}$ in terms of the applied bending moments M_y and M_z and the section properties.

The applied bending moments M_y and M_z are given by the internal moment resultants:

$$M_y = \int_A z\sigma_x \, dA, \quad M_z = -\int_A y\sigma_x \, dA.$$

Moment about the y-axis: Substituting $\sigma_x = E\left(-\frac{y}{R_y} + \frac{z}{R_z}\right)$ into the expression for M_y, we get:

$$M_y = \int_A z\left[E\left(-\frac{y}{R_y} + \frac{z}{R_z}\right)\right] dA.$$

Expanding and separating terms:

$$M_y = E\left(-\frac{1}{R_y}\int_A yz\,dA + \frac{1}{R_z}\int_A z^2\,dA\right).$$

Using the definitions of the second moments of area:

$$I_{yy} = \int_A z^2\,dA, \quad I_{yz} = \int_A yz\,dA,$$

we can write:

$$M_y = E\left(-\frac{I_{yz}}{R_y} + \frac{I_{yy}}{R_z}\right).$$

Moment about the z-axis: Similarly, substituting $\sigma_x = E\left(-\frac{y}{R_y} + \frac{z}{R_z}\right)$ into the expression for M_z, we get:

$$M_z = -\int_A y\left[E\left(-\frac{y}{R_y} + \frac{z}{R_z}\right)\right]dA.$$

Expanding and separating terms:

$$M_z = -E\left(-\frac{1}{R_y}\int_A y^2\,dA + \frac{1}{R_z}\int_A yz\,dA\right).$$

Using the definitions of the second moments of area:

$$I_{zz} = \int_A y^2\,dA, \quad I_{yz} = \int_A yz\,dA,$$

we can write:

$$M_z = E\left(\frac{I_{zz}}{R_y} - \frac{I_{yz}}{R_z}\right).$$

System of equations: We now have the following system of equations:

$$M_y = E\left(-\frac{I_{yz}}{R_y} + \frac{I_{yy}}{R_z}\right),$$
$$M_z = E\left(\frac{I_{zz}}{R_y} - \frac{I_{yz}}{R_z}\right).$$

Dividing through by E, we write:

$$-\frac{I_{yz}}{R_y} + \frac{I_{yy}}{R_z} = \frac{M_y}{E},$$
$$\frac{I_{zz}}{R_y} - \frac{I_{yz}}{R_z} = \frac{M_z}{E}.$$

Letting $X = \frac{1}{R_y}$ and $Y = \frac{1}{R_z}$, the system becomes:

$$-I_{yz}X + I_{yy}Y = \frac{M_y}{E},$$
$$I_{zz}X - I_{yz}Y = \frac{M_z}{E}.$$

Solving the system: The determinant of the coefficient matrix is:

$$\Delta = (-I_{yz})(-I_{yz}) - (I_{yy})(I_{zz}) = I_{yz}^2 - I_{yy}I_{zz}.$$

The solution is given by:

$$X = \frac{I_{yz}M_y + I_{yy}M_z}{E(I_{yy}I_{zz} - I_{yz}^2)}, \quad Y = \frac{I_{yz}M_z + I_{zz}M_y}{E(I_{yy}I_{zz} - I_{yz}^2)}.$$

Substituting back $X = \frac{1}{R_y}$ and $Y = \frac{1}{R_z}$, we have by **Cramer's rule**:

$$\frac{1}{R_y} = \frac{I_{yz}M_y + I_{yy}M_z}{E(I_{yy}I_{zz} - I_{yz}^2)}, \quad \frac{1}{R_z} = \frac{I_{yz}M_z + I_{zz}M_y}{E(I_{yy}I_{zz} - I_{yz}^2)}.$$

Expression for σ_x: Recall that:

$$\sigma_x = E\left(-\frac{y}{R_y} + \frac{z}{R_z}\right).$$

Substitute $\frac{1}{R_y}$ and $\frac{1}{R_z}$ into this expression:

$$\sigma_x = -\frac{M_z I_{yy} + M_y I_{yz}}{I_{yy}I_{zz} - I_{yz}^2}y + \frac{M_y I_{zz} + M_z I_{yz}}{I_{yy}I_{zz} - I_{yz}^2}z.$$

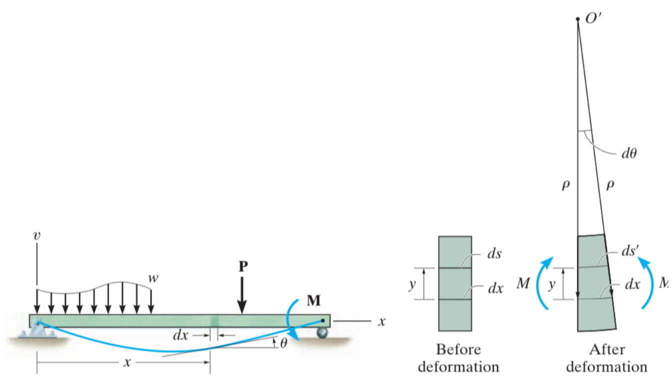
□

10 beam deflection

Definition 51

The **elastic curve** is the curved axis of a beam that has been deflected by a load, while remaining within its elastic limit It is defined by

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2)]^{\frac{3}{2}}}$$



Definition 52

The v axis extends *positive upward* from the x axis. It measures the displacement of the elastic curve. ρ is the radius of the *curvature* of the elastic curve at a point

We would consider an application of this. First we make a few modelling assumptions

1. in the special case when E is constant we have recall 46

$$\boxed{\frac{1}{p} = \frac{M}{EI}} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2)]^{\frac{3}{2}}}$$

also recall that

$$\frac{1}{p} = -\frac{\sigma}{Ey}$$

2. In many applications we have that the slope of the elastic curve is very small therefore the higher term of a small quantity $\left(\frac{dv}{dt}\right)^2$ is estimated to be so our equation simplifies to

$$\frac{d^2v}{dx^2} = \frac{M(x)}{E(x)I(x)}$$

Another implication is that we may then use small angle approximation

$$dv/dx = \tan\theta \approx \theta$$

Now recall from our discussion on moment diagrams early on that $V = dM/dx$ so differentiating we have

$$\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V(x)$$

similarly also recalling that $w = dV/dx$ we have

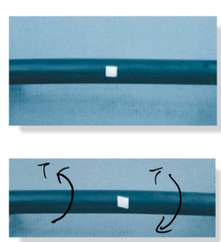
$$\frac{d^2}{dx^2}\left(EI\frac{d^2x}{dx^2}\right) = w(x)$$

3. For most prolems the **flexural rigidity** (EI) will be constant so our results further simplify to

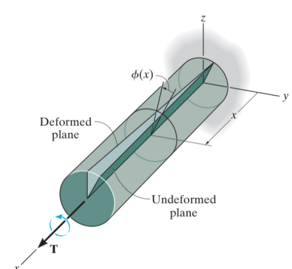
$$EI\frac{d^4v}{dx^4} = w(x) \tag{1}$$
$$EI\frac{d^3v}{dx^3} = V(x) \tag{2}$$
$$EI\frac{d^2v}{dx^2} = M(x) \tag{3}$$

11 torison

Consider a rod twisted on both ends like so(like you are scrunching dry a towel)



Notice the deformation of the rectangular element when this rubber bar is subjected to a torque.



The angle of twist $\phi(x)$ increases as x increases.

Fig. 5–2

Consider the angles

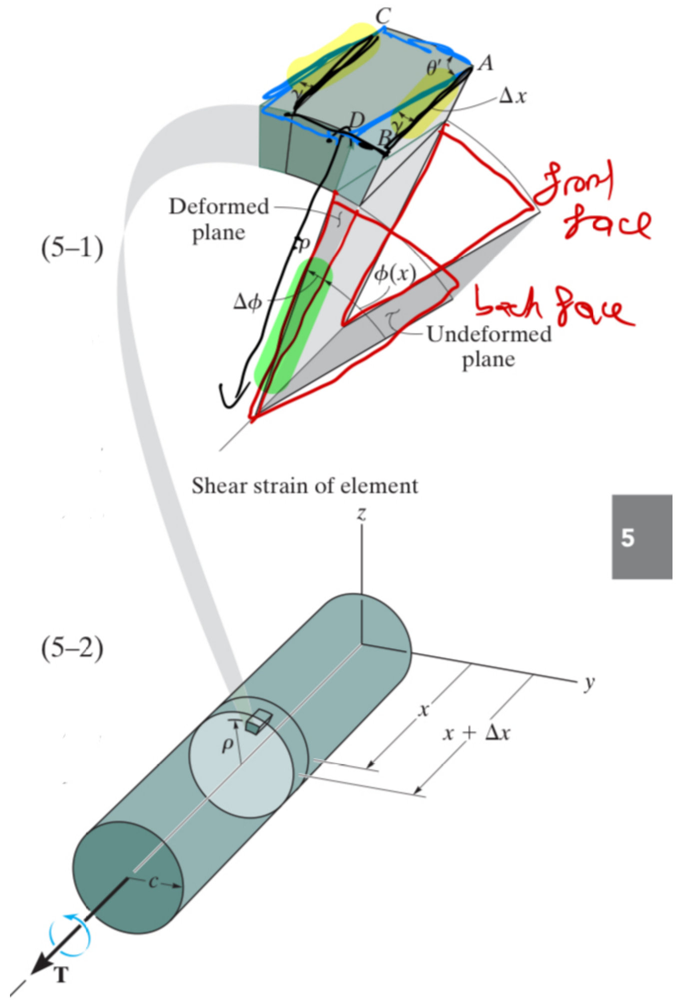


Fig. 5–3

Notice that

$$BD = p\Delta\phi = \Delta x\gamma$$

therefore letting $\Delta\phi \to d\phi$ and $\Delta x \to dx$ we have

$$\gamma = p\frac{d\phi}{dx}$$

Now consider

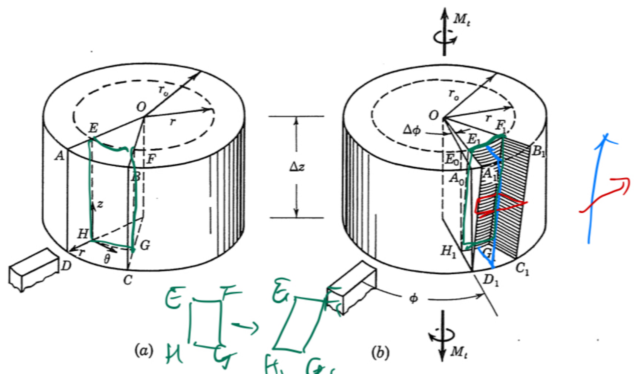


Fig. 6.8 Analysis of deformation of a slice of circular shaft subjected to torsion.

Figure 13: Notice that only the $r\theta$ plane(green) is sheared, the rest(red:$r\theta$ plane;blue:rz plane) are merely translated but there are still the same plane i.e no strain. Also z in this figure refers to x for us

so now also recalling the consitution relations from 22 we have

$$\varepsilon_{\theta x} = \frac{\gamma_{x\theta}}{2} = \frac{R}{2}\frac{d\varphi}{dx}$$

and

$$\varepsilon_{rr} = \varepsilon_{\theta\theta} = \varepsilon_{xx} = \varepsilon_{r\theta} = \varepsilon_{rx} = 0$$

therefore from the consitutive relations we obtain

$$\sigma_{\theta x} = 2\varepsilon_{\theta x} = G\gamma_{x\theta} = Gr\frac{d\varphi}{dx}$$

where G is the shear modulus

Remark 53. *Compare this to our result for beam bending* $\sigma_{xx} = \frac{-Ey}{\rho}$

To solve for equilibrium we have

$$\begin{aligned} \sum M_x &= 0 \\ -M_t + \int_A r dF &= 0 \\ dF &= \sigma_{\theta x} dA = 0 \\ -M_t + \int_A r\sigma_{\theta x} dA &= 0 \\ M_t &= \int_A Gr^2 \frac{d\varphi}{dx} dA \end{aligned}$$

Remark 54. *compare this with our result for beam bending* $M = \frac{1}{\rho}\int_A Ey^2 dA$

Fact 55

Similar to how we did for bending beams we consider the special case in which G is constant

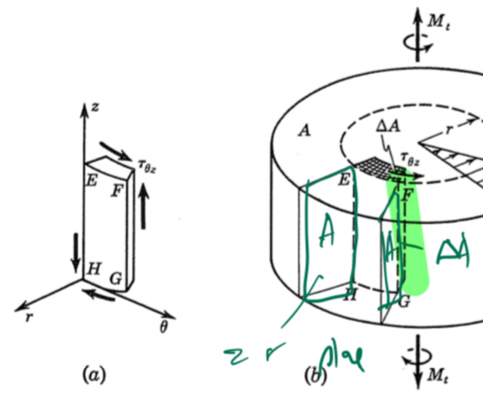


Fig. 6.9 (*a*) Stress components acting on a small element; (*b*) distribution of shearing stress on cross section.

Figure 14: because $\tau_{\theta z}$(in our case, $\sigma_{\theta z}$) does not change in the z(in our case x) nor θ direction by symmetry, $\tau_{\theta z}$ is therefore uniform the z, θ plane corresponding to the r there

this means we can bring both G and $\sigma_{\theta z}$ out of the integral obtaining

$$M_t = \frac{d\varphi}{dx} G \int_A r^2 dA$$

since we know the blue part has no dependence on r and is constant over z, θ(which reminder again is x in our case and refers to height) $J \equiv \int_A r^2 dA$ be the polar moment of inertia. So we obtain

$$M_t = GJ \frac{d\varphi}{dx}$$

Remark 56. *compare this with the case for beam bending where E is constant*

$$M = \frac{EI}{\rho}$$

Let us do some example calculations for J

Example 57

Consider a circular solid shaft then we have

$$J = \int_A r^2 dA = \int_0^{2\pi} \int_0^R r^2 r dr d\theta = \frac{\pi}{2} R^4$$

Example 58

A hollow circular shaft with R_2(inner) and R_1(outer) then we have

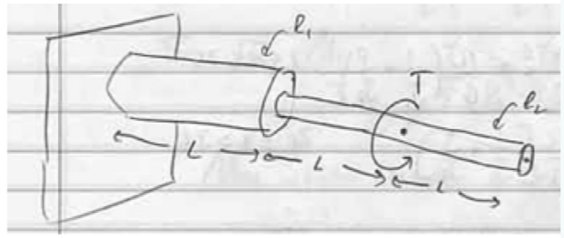
$$J = \frac{\pi}{2} \left(R_2^4 - R_1^4\right)$$

Now lets do some example problems

Example 59

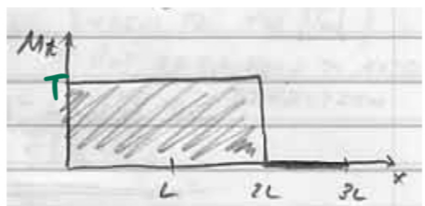
Consider

Example:



What is $\varphi(3L)$?

$$M_t = T, 0 \le x \le 2L$$
$$M_t = 0, 0 \le x \le 3L$$



Solution. From our earlier discussions we know that

$$M_t = GJ\frac{d\varphi}{dx}$$
$$J = \frac{\pi}{2}R_1^4 = J_1, 0 \le x \le L$$
$$J = \frac{\pi}{2}R_2^4 = J_2, L \le x \le 3L$$
$$\frac{d\varphi}{dx} = \frac{T}{GJ_1}, 0 \le x \le L$$
$$\frac{d\varphi}{dx} = \frac{T}{GJ_1}, L \le x \le 2L$$
$$\frac{d\varphi}{dx} = \frac{T}{GJ_1}, 2L \le x \le 3L$$

upon integration yields

$$\varphi(x) = \frac{T}{GJ_1}x + c_1, 0 \le x \le L$$
$$\varphi(x) = \frac{T}{GJ_2}x + c_2, L \le x \le 2L$$
$$\varphi(x) = c_3, 2L \le x \le 3L$$

now apply boundary conditions that is $\phi(0) = 0$. So immediately we have $c_1 = 0$. Now because the angle of twist is a continuous function we must have

$$\phi(L) = \frac{T}{GJ_1}(L) + c_1 = \frac{T}{GJ_2}(L) + c_2 \quad \rightarrow \quad c_2 = \frac{TL}{G}\left(\frac{1}{J_1} - \frac{1}{J_2}\right)$$

for the same reason we must have that

$$\phi(2L) = \frac{T}{GJ_2}(2L) + c_2 = c_3 \quad \rightarrow \quad c_3 = \frac{TL}{G}\left[\frac{1}{J_1} + \frac{1}{J_2}\right]$$

thus putting everything together we have

$$\phi(x) = \begin{cases} \frac{T}{GJ_1} & 0 \leq x \leq L \\ \frac{T}{GJ_2} + \frac{TL}{G}\left(\frac{1}{J_1} - \frac{1}{J_2}\right) & L \leq x \leq 2L \\ \frac{TL}{G}\left[\frac{1}{J_1} + \frac{1}{J_2}\right] & 2L \leq x \leq 3L \end{cases}$$

12 Buckling

we now begin part 2 of MIT 2.005 solid mechanics.