Combinatory Analysis

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Contents

1 Pigeonhole principle

Definition 1

Some notational conventions used in these notes

1. $\mathbb{N}_0$: This denotes the set of non-negative integers, i.e., $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. It includes all natural numbers plus zero.
2. $[n]$: In combinatorics, this notation often represents the set of the first n positive integers. Specifically, $[n] = \{1, 2, \ldots, n\}$. For example, $[5] = \{1, 2, 3, 4, 5\}$.
3. $[\![k+1, n+1]\!]$: This is an alternate notation, especially common in some mathematical texts, for a closed interval of integers between $k+1$ and $n+1$. It represents the set of integers from $k+1$ to $n+1$, inclusive. Formally, $[\![k+1, n+1]\!] = \{k+1, k+2, \ldots, n+1\}$.
4. $\binom{[n]}{k}$ is the set of all subsets of $[n]$ that contain exactly k elements. For example, if $n = 4$ and $k = 2$, then:
$$\binom{[4]}{2} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}.$$
5. $\left|\binom{[n]}{k}\right|$ denotes the cardinality (i.e., the number of elements) of the set $\binom{[n]}{k}$, which is the number of k-element subsets of $[n]$. This number is given by the binomial coefficient $\binom{n}{k}$.

   Thus, the equality $\binom{n}{k} = \left|\binom{[n]}{k}\right|$ states that the binomial coefficient $\binom{n}{k}$ counts the number of k-element subsets of the set $[n]$.

Theorem 2 (Pigeonhole principle)

Suppose that we place n pigeons into m holes. If $n > m$ then there must be a hole containing at least two pigeons.

Proof. By contradiction suppose that no hole contains more than one pigeon. Then let $n_i, i = 1 \ldots, n, n_i \in \{0, 1\}$ be the number of pigeons in each hole and hence we have $n = \sum_{i=1}^{m} n_i < m$ which is a contradiction.

Example 3

Given nine lattice points in the space(i.e elements of $\mathbb{Z}^3$) we can always choose a pair of points whose midpoint is also a lattice point.

Proof. Consider the map $f(a, b, c) = (a(\bmod 2), b(\bmod 2), c(\bmod 2))$ This map has clearly $\left|\{0,1\}^2\right| = 8$ possibilities so by **pigeonhole principle** we may find choose 2 distinct points x, y out of the 9 choices of lattice points such that $f(x, y) = 0$ implying that each coordinate of $x + y$ is even hence their midpoint is also in $\mathbb{Z}^3$

Theorem 4 (Generalized Pigeonhole principle)
Suppose that we place n pigeon into m holes. If $n > m$ then there must be a hole containing at least $\lceil n/m \rceil$ pigeons.

Solution. Again let n_i be the number f pigeons in the i-th hole. By contradiction suppose all $n = n_i < \lceil n/m \rceil$. Then we have

$$n = \sum_{i=1}^{m} n_i \leq m\left(\lceil \frac{n}{m} \rceil - 1\right) < m\frac{n}{m} = n$$

so we obviously have a contradiction since $n < n$ is impossible.

Example 5
Suppose that we have 13 points inside a regular hexagon of side 1. Then we can always choose a circle of radius $\sqrt{3}/3$ containing 3 of the 13 given points

Solution. First split the hexagon into 6 equilateral triangles by using its 3 diagonals. Then by **pigeonhole principle** every such triangle must contain 3 equilateral points. Then the circumcircle(a circle touching all the vertices of a triangle or polygon) of this triangle

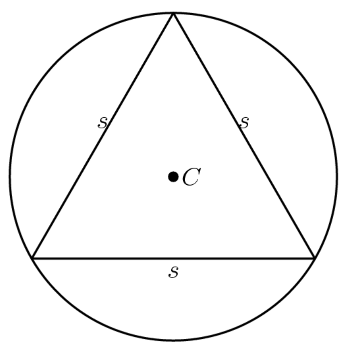


Figure 1: Circumcircle triangle

has exactly a radius of $(1)/\cos(30) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Definition 6 (Principle of strong induction)
Let P be a predicate on nonnegative integers. If

- $P(0)$ is true and
- for all $n \in \mathbb{N}$, $P(0), \ldots, P(n)$ *together*(not just $P(n)$ like in normal induction) imply $P(n+1)$

Then $P(m)$ is true for all $m \in \mathbb{N}$

2 binomial and multinomial theorems

Theorem 7 (Binomial)

For any $n \in N_0$ the following identity holds

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof. Observe that the summands of the form $x^k y^{n-k}$ for $k \in [\![0, n]\!]$ are obtained after expanding $(x+y)(x+y)\dots(x+y)$, choosing for each fixed k all possible ways to get it. That is for each of the n brackets we choose between x or y to get multiply in our expansion. Thus we get $\binom{n}{k}$ ways. $\square$

From which we may obtain interesting identities like

- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ by letting $x = y = 1$ above
- $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 2^n$ by letting $-x = y = 1$ above

Proposition 8 (Vandermonde Identity)

For any $m, n, k \in \mathbb{N}_0$ the following identity holds

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$$

Proof. This is similar to the binomial theorem. This is how many teams of k you can consisting of members from m and n. So for each k it represents one possible composition of members from m and n. Then the $\binom{m}{i}\binom{n}{k-i}$ counts the number of ways we can get this particular composition. $\square$

You might also question is the Vandermode identity just the generalization of the binomial theorem? Why the difference in expression? To investigate this first extract the n number of x(s) and y(s) from the brackets in the binomial expansion and put them together in a set defined below

$$[[x, y]]$$

After choosing k number of x(s) from the above there is only 1 way to pick the remaining $n-k$ number of y(s), that is from the brackets in which you did not pick x. Hence a fixed way just corresponds to multiplication by "1" so we get number of ways= $\binom{n}{k} \times 1$. Now consider the case of $\binom{m+n}{k}$ in the vandermode identity. Now let x, y denote a member of m and n respectively and again we put them together in

$$[[x, y]]$$

So after picking k number of x(s) from m unlike the case for the binomial theorem, there is no fixed way of picking y(s) from n. i.e the event of picking from m is independent of the event of picking from n. From the "conditional probability" standpoint you could see the coeffients of the binomial theorem as an application of bayes theorem

$$\frac{\binom{n}{k}\binom{n}{n-k}}{\binom{n}{k}} = \frac{\binom{n}{k}\binom{n}{k}}{\binom{n}{k}} = \binom{n}{k}$$

By **lattice** path we mean a grid that can be traversed by vertical and horizontal steps

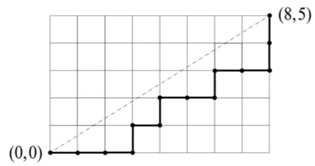


Figure 2: A 13-Lattice Path

Every path taken to reach (i, j) from the origin consists of i and j total horizontal and vertical steps respectively. Hence $\binom{i+j}{j} = \binom{i+j}{i}$ counts the number of possible such paths.Again like the binomial theorem, there is only 1 way to pick of vertical paths after picking horizontal paths and vice versa. These events are dependent.

Proposition 9

prove

$$i\binom{n}{i} = n\binom{n-1}{i-1}$$

Proof. Recall the definition of the binomial coefficient:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

Similarly,

$$\binom{n-1}{i-1} = \frac{(n-1)!}{(i-1)!(n-i)!}.$$

Consider the left-hand side:

$$i\binom{n}{i} = i \cdot \frac{n!}{i!(n-i)!}.$$

Simplify by canceling i in the numerator and denominator:

$$i\binom{n}{i} = \frac{i \cdot n!}{i \cdot i! \cdot (n-i)!} = \frac{n!}{(i-1)!(n-i)!}.$$

Now consider the right-hand side:

$$n\binom{n-1}{i-1} = n \cdot \frac{(n-1)!}{(i-1)!(n-i)!}.$$

Proposition 10

For any $n, k \in \mathbb{N}_0$ the following identities hold

1. $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$
2. $\binom{n+1}{k+1} = \sum_{j=k}^{n} \binom{j}{k}$

Proof. Consider that the left hand side of (1) counts all the lattice paths ending at $(n-k, k+1)$. The number of lattice paths ending at $(n-k, k+1)$ whose last step is the vertical equals the number of lattice paths ending at $(n-k, k)$ hence there are $\binom{n}{k}$ of them. Similary the number of lattice paths ending at $(n-k, k+1)$ whose last step is the horizontal is given by $\binom{n}{k+1}$. Then adding these 2 cases up gives the identity

For (2) consider the left hand side counts the number of $(k+1)$-subsets of $[n+1]$. On the other hand for each $i \in [\![k+1, n+1]\!]$ the number of $(k+1)$ subsets of $[n+1]$ whose maximum element is $i+1$ is obtained in $\binom{i}{k}$ ways, which is to first choose $i+1$ and then complete the $(k+1)$ subset by choosing k elements from $[i]$. Since every

$(k+1)$-subset of $[n+1]$ has a maximum in the set $[\![k+1, n-1]\!]$, the right hand side also counts the number of $(k+1)$-subsets of $[n+1]$

Remark 11. *Notice the proves behind these theorems in general follow the common theme of "counting compositions"*

Our next goal is to generalize the binomial theorem further. First let us generalize the binomial coefficients. For n identically shaped given objects and k colors labelled by $1, 2 \ldots, k$ suppose that there are a_i number of objects of color i for every $i \in [k]$. Then we let $\binom{n}{a_1, \ldots, a_k}$ be the number of ways of linearly arranging the n given objects.

Proposition 12 (Multinomial Theorem)

Let $a_1, \ldots, a_k$ be non-negative integers and set $n = a_1 + \ldots + a_k$. Then

$$\binom{n}{a_1, \ldots, a_k} = \prod_{j=1}^{k} \binom{n - \sum_{i=1}^{j-1} a_j}{a_j} = \frac{n!}{a_1! a_2! \ldots a_k!}$$

Proof. Suppose that we have n identically shaped given objects of k colors with a_i of color i for each $i \in [k]$. We can linearly arrange these objects as follows: out of n given positions choose a_1 in $\binom{n}{a_1}$ ways to place the objects of color 1 then out of the remaining $n - a_1$ positions choose a_2 in $\binom{n-a_1}{a_2}$ ways and so on. Then multiplying these cases togehter yields the first equality. The second equality follows by cancellation. Consider each term in the product

$$\binom{n - \sum_{a_j}^{j-1}}{a_j} = \frac{(n - \sum_{i=1}^{j-1} a_i)!}{(a_j)!(n - \sum_{i=1}^{j-1} a_i - a_j)!} = \frac{(\sum_i^k (a_i) - \sum_{i=1}^{j-1} a_i)!}{(a_j)!(\sum_i^k (a_i) - \sum_{i=1}^{j-1} a_i - a_j)!} = \frac{(\sum_{i=j}^k (a_i))!}{(a_j)!(\sum_{i=j+1}^k (a_i))!}$$

Notice when $j = 1$, we have $\sum_j^k (a_i) = n$. Then all higher terms the $\sum_j^k (a_i)$ in the numerator will be cancelled by $\sum_{j+1}^k (a_i)$ in the denominator of the previous term in the product. And in the last term in the product the denominator is $(\sum_{i=k}^k (a_i) - a_k)! = (0)! = 1$. So the result follows.

Corollary 13

For any $n \in N_0$ we have

$$(x_1 + \ldots x_k)^n = \sum_{(a_1 + \ldots a_k) = n} \binom{n}{a_1, \ldots, a_k} x_1^{a_1} \ldots x_k^{a_k}$$

Where $\sum_{(a_1 + \ldots a_k) = n}$ means the number of permutations of $a_1, \ldots, a_n$ whose sum is n.

Proof. See that for a given composition of powers $a_1, \ldots, a_n$ corresponding to $x_1, \ldots, x_n$ whose sum is n(since every term in the expansion will have total sum of powers to be n). In that case our problem write here is simply solved by direction application of multinomial theorem then summing all distinct cases i.e. permutations of $a_1, \ldots, a_n$ whose sum is n

Theorem 14

If $f : A \to B$ is a bijective function between finite sets then $|A| = |B|$.

Example 15

Let S be a set consisting of n elements namely $S = \{s_1, \ldots, s_n\}$. Let π be a permuation of the functions of S. That is $\pi : [n] \to [n]$ where $\pi(i)$ denotes the position of s_i given a linear arrangement.

The permutation function is bijective. It is injective since distinct elements occupy distinct positions(by definition of set, it contains only distinct elements). So it follows that if $p(i) = p(j)$ then $i = j$. The permutation function is surjective because every position is occupied.

Proposition 16

Let 2^S denote the power set. Then $\left|2^S\right| = 2^{|S|}$

Proof. Suppose we have a set of n elements say $A = \{a_1, a_2 \ldots, a_n\}$. Suppose a possible subset of pow(A) is $\{a_1, a_5, a_7\}$. Define a sequence x_n such that $x_i = 1$ if a_n is in this subset and 0 otherwise for $1 < i < n$. Then this sequece corresponding to this subset will have w elements all 0s except the 1st,5th and 7th element which are 1s.In this way, all possible permutations of this binary sequence will correspond to all possible selections of a_i that can form a subset of A. Thus $f : \text{pow}(A) \to \{1, 0\}^n$ is bijective and their size is 2^n

Definition 17 (Multiset)

A **multiset** is a set with repetitions allowed. More formally for a set S, the multiset on S is a pair (S, f) where $f : S \to \mathbb{N}_0$. The number $f(s)$ is called the **multiplicity** of s which specifies the number of times s is repeated in the given multiset. The **cardinality** or **size** of (S, f) is defined by $k = \sum_{s \in S} f(s)$. In this case (S, f) is said to be a **k-multiset** on S

Definition 18

If $S = \{s_1, \ldots, s_n\}$ we often write $\left\{s_1^{f(s_1)}, \ldots, s_n^{f(s_n)}\right\}$ instead of (S, f). For example

$$1, 2, 2, 4, 4 = \left\{1, 2^2, 3^0, 4^2\right\} \in \left(\!\!\binom{[4]}{5}\!\!\right)$$

is a 5-muliset on the set [4]. We let $\left(\!\binom{S}{k}\!\right)$ denote the set of all k multisets on S and we let $\left(\!\binom{n}{k}\!\right)$ denote the size of $\left(\!\binom{[n]}{k}\!\right)$

Fact 19

The key difference between $\left(\!\binom{[n]}{k}\!\right)$ and $\binom{[n]}{k}$ lies in whether or not repetition of elements is allowed:

1. $\binom{[n]}{k}$: This denotes the set of all k-element subsets of $[n] = \{1, 2, \ldots, n\}$. In a subset, each element can appear at most once (no repetition is allowed), and the order of elements does not matter.The size of this set is given by the binomial coefficient $\binom{n}{k}$, which counts the number of ways to choose k distinct elements from n elements.

   Example: For $n = 3$ and $k = 2$, $\binom{[3]}{2} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and $\binom{3}{2} = 3$.

2. $\left(\!\binom{[n]}{k}\!\right)$: This denotes the set of all k-multisets of $[n] = \{1, 2, \ldots, n\}$. A multiset allows repetition of elements; each element can appear multiple times, and the order of elements still does not matter.The size of this set is given by $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$, which counts the number of ways to choose k elements from n options with repetition allowed(see below theorem for more)

   Example: For $n = 3$ and $k = 2$, $\left(\!\binom{[3]}{2}\!\right)$ includes multisets like $\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 2\}, \{2, 3\}, \{3, 3\}$, and $\left(\!\binom{3}{2}\!\right) = \binom{4}{2} = 6$.

Theorem 20

Prove the final equality

$$\left|\left(\!\!\binom{[n]}{k}\!\!\right)\right| = \left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$$

Proof. For a k-multiset $A = \{a_1, \ldots, a_n\}$ on $[n]$ where we assume that $a_1 \leq \ldots \leq a_k$. Define

$$f(A) = \{a_1, a_2 + 1, \ldots, a_k + k - 1\}$$

Since we are adding numbers in ascending order and that $a_1 \leq \ldots \leq a_k$, the new order of numbers still retain their ascending order and in fact have greater differences between them. That is if we consider subset $B = \{b_1, \ldots, b_k\}$ of $[n+k-1]$ with $b_1 < \ldots < b_k$(notice now *strictly* increasing) we see that $f(A) = B$. It is clear that any element of B can made using elements of A so this map is surjective. It is also clearly injective given it is a translation function $x \mapsto x + c$. Hence f is a bijective map. In which case we know

$$\left|\left(\!\!\binom{[n]}{k}\!\!\right)\right| = \left|\binom{[n+k-1]}{k}\right|$$

where the theorem follows as desired

Example 21

Suppse we want to place k identical balls into n different(distinguishable) boxes. After labeling the boxes by $b_1, \ldots, b_n$ each placement can be identified with a k-multiset on $[n]$ as follows. The number of balls in box b_i specify the mulitplicity of i in the k-multiset. Then by the above thoerem we know the total number of configurations is $\left(\!\binom{n}{k}\!\right)$

Just think of it as each ball can be assigned a box b_i(that is the meaning of on $[n]$ which refers to the index of the boxes in this case). If k balls are assigned to ball b_i then as mentioned the number of balls in b_i is indeed the multiplicity.

3 Permutation inversions and Q-binomials

Definition 22

Let S_n denote the set consisting of all permutations of $[n]$. The **inversion table** of a permutation $w \in S_n$ is an n-tuple $I(w) = (a_1, \ldots, a_n)$ where a_i denotes the number of elements j in w to the left of i with $j > i$.

Observe that $0 \leq a_i \leq n - i$ for every $i \in [n]$. Forget the permutation first, how many possible elements of $[n]$ are bigger than i? The answer is clearly $n - i$. So there are $n - i$ possible candidates that are bigger than you. In the worst case they are all on your left. In the best case for example imagine you are 1 and you are in the set $1, 2, 3, 4$. You are on the extreme left, no possible candidate bigger than you can be in your left. Hence $0 \leq a_i \leq n - i$ ranging from the best to worst case.

Example 23

Let $n = 5$ and consider the permutation $w = (5, 3, 4, 1, 2)$.

- For $i = 1$: There are no elements greater than 1 to its left, so $a_1 = 0$.
- For $i = 2$: There is 1 element greater than 2 to its left, which is 1, so $a_2 = 1$.
- For $i = 3$: There is 1 element greater than 3 to its left, which is 4, so $a_3 = 1$.
- For $i = 4$: There are 2 elements greater than 4 to its left, which are 5 and 1, so $a_4 = 2$.
- For $i = 5$: There are 3 elements greater than 5 to its left, which are 3, 4, and 1, so $a_5 = 3$.

Thus, the inversion table is $I(w) = (0, 1, 1, 2, 3)$.

Proposition 24

For each $n \in \mathbb{N}$ the map $I : S_n \to [\![0, n-1]\!] \times [\![0, n-2]\!] \times \ldots \times [\![0, 0]\!]$ where $I(w)$ is the inversion table of w is a bijection.

Proof. Set $T_n = [\![0, n-1]\!] \times [\![0, n-2]\!] \times \ldots \times [\![0, 0]\!]$. Form permutation S_n step by step adding element $(n-i)$ in each step in the order $i = 1, \ldots, n$. Now consider for each step before insertion of $(n-i)$ notice that there are exactly $n-i$ elements in the permutation all bigger than $n-i$(well because we were adding in descending order of element size).Now we consider the various values a_{n-i} can take and see if there exists a valid position in the permutation. However given that

(case: 1) $a_{n-i} = n-i$: the obvious position is the extreme right

(case: 2) $a_{n-i} = 0$: the obvious position is the extreme left

(case: 3) $0 < a_{n-i} < n-i$: since the all elements are bigger than $(n-i)$ just slot it in the a_{n-i} position and you will be guaranteed a_{n-i} elements bigger than $(n-i)$ to its left.

Hence our map is by contruction a surjection. Moreover it is by construction an injection too since for every element in T_n, we are putting distinct elments at distinct positions of an element in S_n that is every $(a_1, \ldots, a_n)$ identifies a unique $w \in S_n$.

Remark 25. *En element $t \in T_n$ can have repeating values eg. $(0, 0, 0, 0) \in T_4$ is valid.(the result of n cartesian product of sets is an n-tuple which by definition allows for repetitions)But this is not the case $w \in S_n$ which is a set and hence by definition cannot have repetitions.*

Example 26

Let's walk through this process with an example where $n = 4$ and the inversion table is $(2, 1, 0, 0)$.

1. Begin with an initial arrangement containing only the largest element: $[4]$.
2. Next, we need to insert 3 into the current arrangement $[4]$. According to the inversion table, $a_3 = 0$, meaning there are 0 elements greater than 3 to its left. So, 3 must be placed before 4. - Updated arrangement: $[3, 4]$.
3. Now, insert 2. According to the inversion table, $a_2 = 1$, which means there should be exactly 1 element greater than 2 to its left. Thus, 2 is placed after 3 but before 4. - Updated arrangement: $[3, 2, 4]$.
4. Finally, insert 1. According to the inversion table, $a_1 = 2$, which means there should be exactly 2 elements greater than 1 to its left. So, 1 is placed after 3 and 2 but before 4. - Final arrangement: $[3, 2, 1, 4]$.

Thus, the constructed permutation $w = (3, 2, 1, 4)$ matches the inversion table $(2, 1, 0, 0)$.

Definition 27

An **inversion** of $w = w_1 w_2 \dots w_n \in S$ is pair (w_i, w_j) such that $i < j$ but $w_i > w_j$. The number of inversions of a permutation w is denoted by $\text{inv}(w)$.

Observe that if $I(w) = (a_1, \dots, a_n)$ is the inversion table of w then $\text{inv}(w) = a_1 + \dots + a_n$. Consider in our construction process after addition of $(n - i)$ at a_{n-i}, this value of a_{n-i} will be unaffected as all subsequent values added are smaller so even if they do end up on the left of $(n - i)$ its count will not be affected. Hence each a_{n-i} sums up inversion pairs that contain $(n - i)$ and they will not be double counted by $(n - i)_{\text{previous}}$ because their corresponding values of $(a_{n-i})_{\text{previous}}$ are calculated when $(n - i)$ was not present and will stay constant with remaining addition of elements.

Proposition 28

The identity

$$\sum_{w \in S_n} q^{\text{inv}(w)} = (1 + q)(1 + q + q^2) \dots (1 + q + \dots + q^{n-1})$$

holds for every $n \in \mathbb{N}$

Proof. Set $T_n = [0, n - 1] \times [0, n - 2] \times \dots \times [0, 0]$. Since the assigment $w \mapsto I(w)$ induces a bijection $S_n \to T_n$ and that $\text{inv}(w) = a_1 + \dots + a_n$ for every $w \in S_n$ with $I(w) = (a_1, \dots, a_n)$ it follows that

$$\begin{aligned} \sum_{w \in S_n} q^{\text{inv}(w)} &= \sum_{(a_1, \dots, a_m) \in T_n} q^{a_1 + \dots + a_n} = \sum_{a_1=0}^{n-1} \sum_{a_2=0}^{n-2} \dots \sum_{a_n=0}^{0} q^{a_1} q^{a_2} \dots q^{a_n} \\ &= \sum_{a_1=0}^{n-1} q^{a_1} \sum_{a_2=0}^{n-2} q^{a_2} \dots \sum_{a_n=0}^{0} q^{a_n} = \prod_{k=0}^{n-1} (1 + q + \dots + q^k) \end{aligned}$$

Definition 29

We define the **q-analogs**

$$(n)_q = 1 + q + \ldots + q^{n-1} \quad \text{and} \quad (n)_q! = (1_q)(2)_q \ldots (n)_q$$

of n and $n!$ respectively. By convention $(0)_q = 0$ and $(0)_q! = 1$

Notice that $(n)_1 = n$ and $(n)_1! = n!$. Observe also that

$$(n)_q = \sum_{j=0}^{n-1} q^j = \frac{q^n - 1}{q - 1} \quad \text{and} \quad (n)_q! = \prod_{k=1}^{n} \frac{q^k - 1}{q - 1}$$

where the term on right is just the factorial of the term on the left hence the product.

Definition 30

Similarly we define the **q-analogs for binomial coefficients**

$$\binom{n}{k}_q = \frac{(n)_q!}{(k)_q!(n-k)_q!}$$

4 compositions

Definition 31

A k-tuple $(a_1, \ldots, a_k)$ in $\mathbb{N}_0^k$ is **weak composition** of n into k parts if it is a solution to $x_1 + \ldots + x_k = n$. If in addition each a_i is positive then $(a_1, \ldots, a_k)$ is aclled a **composition** of n into k parts

Proposition 32

For $n, k \in \mathbb{N}_0$ the following hold

1. There are $\binom{n+k-1}{n}$ weak k-compositions of n
2. There are $\binom{n-1}{k-1}$ k-compositions of n

Proof. For (1) it is a problem of dividing n identical balls into k different boxes. In that case we just do $\frac{(n+k-1)!}{n!(k-1)!}$ basically permuting $k-1$ and n identical so order does not matter(thats why we divide by $(k-1)!$ and $n!$ respectively in denominator) walls between and balls(note it possible to have a no balls between a pair of walls here. That basically corresponds to an $x_i = 0$. Hence we have $\binom{n+k-1}{n}$. For (2) consider that a k-tiple $(a_1, \ldots, a_n)$ is a k-composition of n if and only if $(a_1 - 1, \ldots, a_n - 1)$ is a weak-composition of $n-k$. In which case by (1) we have $\binom{(n-k)+k-1}{n-k} = \binom{n-1}{k-1}$ as desired.

Remark 33. *You could see (2) using the same wall analogy. Consider*

$$B_1|B_2|B_3|B_4$$

well clearly to get non zero boxes there only n-1 options from which you can choose your k-1 walls from. Note to partition something into n sets, you need n-1 walls. Quite intuitive.

Corollary 34

For every $n \in \mathbb{N}$ there are 2^{n-1} compositions of n.

Proof. Just sum up all possible cases of k-compositions, knowing that k can only be at most n,

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$$

where the last equality follows from 7.

5 set partitions

Definition 35

A **partition** of a set S such that $S = \bigcup_{j=1}^{k} B_j$. For each $j \in [\![1, k]\!]$ the set B_j is called a **block** of a partition π and we write $|\pi| = k$ when π consists of k blocks. In addition $S(n, k)$ denotes the number of partitions of $[n]$ having k blocks and it is called a **stirling number of the second kind**

Example 36

The set [4] has seven partitions into two nonempty blocks namely

$$\{1, 2, 3\}\,\{4\}\,; \quad \{1, 2, 4\}\,\{3\}\,; \quad \{1, 3, 4\}\,\{2\}\,; \quad \{2, 3, 4\}\,\{1\}\,;$$

$$\{1, 2\}\,\{3, 4\}\,; \quad \{1, 3\}\,\{2, 4\}\,; \quad \{1, 4\}\,\{2, 4\}$$

Example 37

Consider

1. $S(n, 2) = 2^{n-1} - 1$
2. $S(n, n) = S(n, 1) = 1$
3. $S(0, 0) = 1$
4. $S(n, n-1) = \binom{n}{2}$

(1) follows since each element in $[n]$ can be assigned to either the first or second block so we have 2^n possibilities. But because we cannot have cases where all elements are in the first or in the second group in which we case we have $2^n - 2$.Finally we need to account for symmetry so dividing by 2 to eliminate symmetry we have $2^{n-1} - 1$ as desired(see below for more). The last one follows when we consider that in all such permutations at least 1 element must be of size 2. Explicitly they all look like

$$\{x_1 x_2\}\,\{x_3\} \ldots \{x_n\}$$

where order doesn't matter. So our problem is reduced to choosing which item will be size 2. The rest are obvious.

Fact 38

Normally we eliminate similar groups where order doesn't matter in permutations $x_1x_2x_3\dots x_k$

$$\frac{n!}{a_1!a_2!\dots a_k!}$$

where $a_1 + a_2 \dots + a_k = n$. That is if a_j is the number of items in a group of similar objects say $y_i^j, i = 1, \dots a_j$. Then permutations like

$$y_2^j * * y_1^j \quad \text{and} \quad y_1^j * * y_2^j$$

and

$$* \; y_2^j \; y_1^j \; * \quad \text{and} \quad * \; y_1^j \; y_2^j \; *$$

where $a_j = 2$ and $n = 4$ in this case. See that every permutation will contain $a_j = 2$ number of y_i^j. And in each positional permutation there are $a_j!$ of such repeated permutations of similar objects. So dividing the total number of permutations by $a_j!$ to eliminate all similar groups sequentially makes sense. In our case our every "positional permutation" is the partition (left or right) each element is assigned. corresponds to for example

$$\{1, 2\}\,\{3, 4\} \Leftrightarrow \{3, 4\}\,\{1, 2\}$$

and

$$\{1, 2, 3\}\,\{4\} \Leftrightarrow \{4\}\,\{1, 2, 3\}$$

so in our case we remove these repeated symmetrical groups by dividing by 2 from all permutations works. Alternatively you may see each "positional permutation" as a binary number like 01001 where say 1 is right and 0 is left. Then for every such binary number we know its complement, in our example 10110 exists and we have to account for its repeated count by dividing by 2.

Theorem 39

For any $n, k \in \mathbb{N}$ with $k \leq n$ the following identity holds

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

Proof. We are basically considering when n is a block itself

$$[n]\,\underbrace{[*]\dots[*]}_{\text{k-1 blocks}}$$

and n is part of a non-empty block containing other elements

$$\underbrace{[*]\dots[*]}_{\text{k blocks, choose one of them to contain n}}$$

Hence summing up these 2 cases we get our desired theorem.

Proposition 40

FOr every $n, k \in \mathbb{N}$ the number of surjective functions $f : [n] \to [k]$ is $S(n, k)k!$

Proof. The situation is essentially: taking n-indexed elements and mapping it to k indexed elements. How many surjective functions can do this? That is

- $f_1 : [n] \to [k]$
- $f_2 : [n] \to [k]$
- $\vdots$
- $f_q : [n] \to [k]$ what is q?

First our function essentially must map $[n]$(n elements) into k groups so they can be mapped to $[k]$(k-elements). In which case we have $S(n,k)$. Now the number of ways to map these k-groups of elements to k indexes is clearly $k!$. Therefore we have $k!S(n,k)$ as desired

Corollary 41

For all non-negative integers n

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k$$

where $(x)_k = x(x-1)\dots(x-k+1)$

Proof. First assume $x \in \mathbb{N}$. See this as we trying to count the number of surjective functions from $[s]$ to $[x]$ where $|[x]| = k$. So clearly we have

$$x^n = S(n,k)\binom{x}{k}k! = \sum_{k=0}^{n} S(n,k)(x)_k$$

Definition 42

For $n \in \mathbb{N}$ the total number of partitions of $[n]$ is denoted by $B(n)$ and called a **Bell number**. In which case it is clear that

$$B(n) = \sum_{k=1}^{n} S(n,k)$$

since we just simply summing all cases of number of partitions which are clearly distinct independent cases

Theorem 43

For every $n \in \mathbb{N}_0$ the following identity holds

$$B(n+1) = \sum_{j=0}^{n} \binom{n}{j} B(j)$$

Proof. By definition of bell number the LHS counts the set of of partitions of $[n+1]$. In which case we may do

$$B(n+1) = \sum_{s=1}^{n+1} \binom{n}{s-1} B(n+1-s) = \sum_{j=0}^{n} \binom{n}{n-j} = \sum_{j=0}^{n} \binom{n}{j} B(j)$$

where the second equality follows by considering the block containing $\{n+1\}$ and enumerating through all it possible sizes. We also enumerated through all its possible members too via $\binom{n}{s-1}$(number of ways to choose other $s-1$ members not $n-1$ in this block of size s). Then finally $B(n+1-s)$ counts the number of ways to partition the remaining $n+1-s$ elements. Clearly all distinct cases(all cases the block containing $\{n+1\}$ will differ in size and members) so summing them up works. The 3rd equality follows by change of variable $j = n+1-s$ and the last is obvious.

6 Integer Partitions

Definition 44

Let $N, a_1, \ldots, a_k$ be positive integers with $a_1 \geq \ldots \geq a_k$ and $n = a_1 + \ldots + a_k$. When $(a_1, \ldots, a_k)$ is a **partition** of n we often write $(a_1, \ldots, a_k) \vdash n$. The number of partitions of n into k parts is denoted by $p_k(n)$ while the number of partitions of n(that is for any possible k or $\sum_{k=1}^{n} p_k(n)$) by $p(n)$

Note that $p_k(0) = 0$ for every $k \in \mathbb{N}$ and by convention $p_0(0) = 1$ and $p(0) = 1$. We also see that $p_1(n) = p_n(n) = 1$ for every $n \in \mathbb{N}$.

Example 45

Let (a_1, a_2) be a partition of n into two parts. Observe that any value $a_2 \in [\lfloor \frac{n}{2} \rfloor]$ is permissible and determines the partition (a_1, a_2) . On the other hand if $a_2 > \lfloor \frac{n}{2} \rfloor$ we then obtain

$$n = a_1 + a_2 \geq (\left\lfloor \frac{n}{2} \right\rfloor + 2) + (\left\lfloor \frac{n}{2} \right\rfloor + 1) = 2\left\lfloor \frac{n}{2} \right\rfloor + 3 > n$$

which is a contradiction

I'm not sure what is this too. If it comes up again ill try to understand it

Example 46

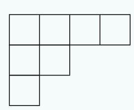
The positive integer 5 has 7 partitions. Indeed they are

$$(5); \quad (4,1); \quad (3,2); \quad (3,1,1); \quad (2,2,1); \quad (2,1,1,1); \quad (1,1,1,1,1)$$

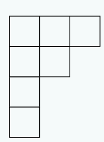
Therefore $p(5) = 7$

Example 47

Ferrer or young diagram of $(4,2,1) \vdash 7$



and its **conjugate partition** $(3,2,1,1)$ of $(4,2,1)$



It is obtained essentially by swapping row i with column i

Definition 48

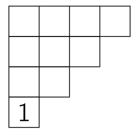
A partition is called **self conjugate** if it equals its conjugate which happens precisely when the Ferrer diagram is symmetric with respect to its main diagonal line.

Proposition 49

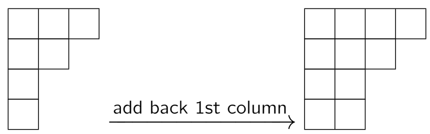
For every $n \in \mathbb{N}$ and $k \in [n]$ the follwoing recurrence indentity holds

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k)$$

Proof. Consider that the LHS counts the set P of partitions of n into k parts. The first term on the RHS counts the the set of partitions of n into k parts whose last part is length 1.



So clearly this is in bijection with the set of all partitions of $n-1$ into $k-1$ parts. Note that we draw young diagrams by default in decreasing order of row lenghth. So in other words this condition means there exist(s) an $a_i = 1$. (reserve one guy where $a_i = 1$, vary the others). Now the second part corrresponds to the set of partitions of n into k parts whose last part is at least length 2. Recall above in other words all $a_i \geq 2$. Then we have say



which removes all rows with length 1. We then repartition this into The end result is only rows corresponding to those at least of length 2. This is clearly in bijection with the set $p_k(n-k)$.(That is, fix 1 column so any partition formed(which has at least length 1 rows) will now have at least length 2 rows. □

Let $q(n)$ denote the number of partitions of n whose parts have size at least 2

Proposition 50

For each $n \in \mathbb{N}$ with $n \geq 2$ the following statements hold

1. $q(n)$ equals the number of partitions of n whose largest two parts are equal
2. $q(n) = p(n) - p(n-1)$

Proof. (1) follows by...to be continued

Let $p(j, k, n)$ be the number of partitions of n into at most k parts whose largest part is at most j.

Proposition 51

For all $j, k \in \mathbb{N}$ the followig identity holds:

$$\sum_{n \geq 0} p(j, k, n) q^n = \binom{j+k}{j}_q$$

Proof. I'm not sure what is this too. If it comes up again ill try to understand it

7 Permutations

Lemma 52

Let $p : [n] \to [n]$ be a permutation and let $x \in [n]$. Then there exists a positive integer $1 \leq i \leq n$ so that $p^i(x) = x$ where

$$p^i(x) = \underbrace{p \circ p \ldots \circ p(x)}_{\text{i times}}$$

Proof. By the pigeonhole principle knowing that p produces a different element each time and that there are only n distinct values, we are gauranteed to get a repeat value within $0 \leq i < j \leq n$.

Example 53

Find the disjoint cycles of 321564

Proof. The permutation $p = 321564$ represents the mapping:

$$1 \to 3, \quad 2 \to 2, \quad 3 \to 1, \quad 4 \to 5, \quad 5 \to 6, \quad 6 \to 4.$$

Starting with 1, we get the cycle: $1 \to 3 \to 1$, which gives (31). For 2, since $2 \to 2$, we have the cycle (2). Starting with 4, we have $4 \to 5 \to 6 \to 4$, giving the cycle (564). Therefore, the cycles of p are (31), (2), and (564). □

Remark 54. *Essentially the notation means that what ever number in this sequence of numbers corresponds to the image of the permutation map of its positional index as shown above. This is what we call the* **one-line notation** *of permutations*

$$f(1)f(2)\ldots f(n)$$

Definition 55

For a permutation $\pi : [n] \to [n]$ and $k \in [n]$ we call $(k, \pi(k), \ldots, \pi^{s-1}(s))$ a **cycle** of π of length s provided that $k, \pi(k), \ldots, \pi^{s-1}(k)$ are pairwise distinct and $\pi^s(k) = k$

Example 56

One-line notation vs canonical cycle form

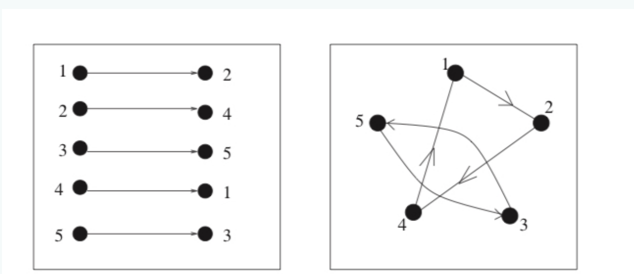


Figure 3: Two ways to look at $24513 = (412)(53)$

Definition 57

Let $\pi \in S_n$. If for every $i \in [n]$ the disjoint cycle decomposition of π has precisely a_i cycles of lenght i then $(a_1, \ldots, a_n)$ is called the cycle type of π

Example 58

The cycle type of permutation π in $783295146 = (1,7)(2,8,4)(3)(5,9,6)$ is $(1,1,2,0,0,0,0,0,0)$

See that there is 1 cycle of length 1. Hence $a_1 = 1$. There is 1 cycle of length 2 so $a_2 = 1$. There is 2 cycles of length 3 so $a_3 = 2$. Finally the number of cycles for other higher lenghts is zero clearly.

Theorem 59

Let $a_1, a_2, \dots, a_n$ be nonnegative integers so that the equality $\sum_{i=1}^{n} i \cdot a_i = n$ holds. Then the number of n-permutations with a_i cycles of length i where $i \in [n]$ is

$$\frac{n!}{a_1!a_2! \dots a_n! \cdot 1^{a_1}2^{a_2} \dots n^{a_n}}$$

Proof. Consider for example $(1,2,0,1,0,0,0,0,0)$ and look at brackets of length 2. So first start with

$$(-)(--)(--)(----)$$

now we fill these positions with elements from $[n] = [9]$(in other words assigning elements from $[9]$ their respective cycles) knowing that there will be 9! permutations in total. Some of them will indeed be distinct permutations like

$$(1)(23)(45)(6789) \quad (2)(13)(45)(6789)$$

Since different elements in cycles. However there will be redundant ones as you might have suspected like

$$(1)(23)(45)(6789) \quad (1)(32)(54)(7896)$$

which are basically the same cycles but with different starting points or

$$(1)(23)(45)(6789) \quad (1)(45)(23)(6789)$$

different orders of brackets These are precisely the cases of redundant permutation we need to eliminate. Specifically we have for instance

$$(-) \quad \underbrace{\overbrace{(--)}^{\text{cycle start points in each bracket rotates i=2 times}} \qquad (--)}_{a_2!=2! \text{ bracket position permutations}} \quad (----)$$

Therefore $a_i!$ essentially counts the number of times brackets permute while i^{a_i} accounts for rotations in each bracket. Hence the theorem follows ☐

Recall that for every $n \in \mathbb{N}$ and $k \in [n]$ the stirling number of the second kind $S(n,k)$ counts the set of partitions of $[n]$ into k blocks. Now we introduce the first kind which essentially counts number of ways to decompose into k cycles instead of partition into k blocks.

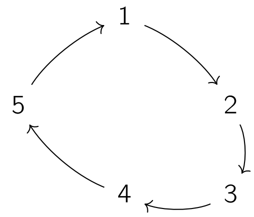
Definition 60

For $n, k \in \mathbb{N}$ we let $c(n,k)$ denote the number of permutations of S_n whose disjoint cycle decompositions consists of k cycles and we call $c(n,k)$ a **signless stirling number of the first kind**

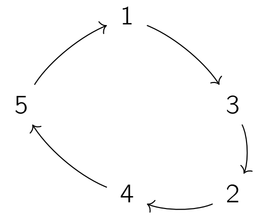
See that by convention $c(0,0) = 1$ and $c(0,k) = 0$. Moreover $c(n,n) = 1$ since obviously this must correspond to the identity permutation. These special/default values are similar to integer partitions, stirling numbers of second kind as you would recall. However unlike them, notice that $c(n,1) = (n-1)!$. This is because in one partition say

$$(1,2,3,4,5) \quad \text{and} \quad (1,3,2,4,5)$$

are equivalent. But in one cycle this same example means



and



which are clearly different cycles. Moreover notice that the number of cycles is equal to $\frac{n!}{n}$ where recall we divide by n to account for the rotation in start points and hence we have $(n-1)!$ as desired.

Proposition 61
For $n \in \mathbb{N}$ and $k \in [n]$ the following recurrence identity holds

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$$

Proof. Similar to how we have done previously we break this into n being a cycle of length 1 by itself and n as part of a cycle of at least length 2. In that case the former case is clearly represented by $c(n-1,k-1)$(reserve 1 cycle and element n then vary others). Now for the latter case, after doing $c(n-1,k)$(decomposing into k cycles without n), then say we have $n-1$ elements in k cycles as in $(---)(--)(--)$. Then the possible positions to add n are

$$({}_\uparrow -{}_\uparrow -{}_\uparrow -)({}_\uparrow -{}_\uparrow -)({}_\uparrow -{}_\uparrow -)$$

which is just $n-1$. All of which result in distinct cycles. So summing up all distinct cases we get $(n-1)c(n-1,k)$ and the proposition follows as desired $\square$

Proposition 62
For each $n \in \mathbb{N}_0$ the following identity holds

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\dots(x+n-1)$$

Proof. Set $F_n(x) = x(x+1)\dots(x+n-1)$ for every $n \in \mathbb{N}_0$. Since $F_n(x)$ is a polynomial of degree n with non-negative integer coefficients we could write $F_n(x) = \sum_{k=0}^{n} d(n,k)x^k$ where $d(n,k) \in \mathbb{N}_0$ for every $k \in [\![0,n]\!]$. Also $F_0(x) = 1$(the product of no factors is 1 by convention like how $x^0 = 1$). So $d(0,0) = 1$ and $d(0,k) = 1$(only non zero coefficient is the constant). Moreover $d(n,n) = 1$ since in the expansion of $x(x+1)\dots(x+n-1)$ clearly the coefficient of the highest degree of x is 1. We will now investigate other values

$$
\begin{aligned}
F_n(x) = \sum_{k=0}^{n} d(n,k)x^k &= x(x+1)\dots(x+n-1) = (x+n-1)F_{n-1}(x) \\
&= (\underbrace{x} + \underbrace{n-1})\sum_{k=0}^{n-1} d(n-1,k)x^k \\
&= \sum_{k=0}^{n-1} d(n-1,k)\underbrace{x^{k+1}} + \sum_{k=0}^{n-1}\underbrace{(n-1)}\, d(n-1,l)x^k \\
&= \sum_{k=1}^{n} d(n-1,k-1)x^k + \sum_{k=0}^{n-1}(n-1)d(n-1,k)x^k \\
&= \sum_{k=1}^{n-1}(d(n-1,k-1)+(n-1)d(n-1,k))x^k + d(n-1,n-1)x^n + (n-1)d(n-1,0)
\end{aligned}
$$

Comparing x^k terms we immediately see that

$$d(n,k) = d(n-1,k-1) + (n-1)d(n-1,k)$$

for $k = 1, \dots, n-1$. And again recall from above that for $k = n$ $d(n,n) = 1$. However this is exactly the same initial values(recall $c(n,n) = 1$) and recurrence relation satisfied for $c(n,k)$ and so $d(n,k) = c(n,k)$. So the theorem follows

Definition 63

For $n, k \in \mathbb{N}_0$ we call $(-1)^{n-k}c(n,k)$ a **stirling number of the first kind** and we denote it by $s(n,k)$)

Corollary 64

For each $n \in \mathbb{N}$ the following polynomial identity holds

$$\sum_{k=0}^{n} s(n,k)x^k = (x)_n$$

Proof. Consider

$$\sum_{k=0}^{n} s(n,k)x^k = (-1)^n \sum_{k=0}^{n} c(n,k)(-x)^k = (-1)^n \prod_{j-0}^{n-1}(-x+j) = \prod_{j=0}^{n-1}(x-j) = (x)_n$$

which follows from the definitions of the terms we learnt above.

Definition 65

We say that $i \in [n-1]$ is a **descent** of $w = w_1w_2\dots w_n \in S_n$ if $w_i > w_{i+1}$. In addition we set

$$D(w) = \{i \in [n-1] | w_i > w_{i+1}\} \quad \text{and} \quad \mathrm{des}(w) = |D(w)|$$

Definition 66

For $n \in \mathbb{N}$ we let $A(n,k)$ denote the number of permutations in S_n and $A(0,k) = 0$ if $k \neq 0$ and we call $A(n,k)$ an **Eulerian number**

Proposition 67
For all $n \in \mathbb{N}$ and $k \in \mathbb{N}$ the following recurrence identity holds

$$A(n,k) = (n-k_1)A(n-1,k-1) + kA(n-1,k)$$

Proof. to be continued...I wanna do sieve already

Fact 68
As a recap we have covered 3 different structures. Given $[n], k$ you should be able to recognize the variables below involved for each respective structure

1. Composition(number of ways to sum to n). Types of variables involved:
$$\binom{n+k-1}{n}, \quad \binom{n-1}{k-1}$$
2. Partitions(number of ways to break $[n]$ into k blocks).Types of variables involved:
$$S(n,k), \quad B(n,k), \quad p(n), \quad p_k(n)$$
3. Permutations(number of arrangements of $[n]$).Types of variables involved:
$$s(n,k), \quad c(n,k), \quad A(n,k), \quad D(w)$$

Note that in compositions order matters while in permutations it doesn't. For example

$$c_1 + c_2 = n$$

when $c_1 = 1, c_2 = 2$ not the same composition as $c_1 = 2, c_2 = 1$ vs

$$B_1 + B_2 = n$$

when $B_1 = \{1\}, B_2 = \{2\}$ is the same partition as $B_1 = \{2\}, B_2 = \{1\}$ Hence in practice we don't number B_i to avoid confusion we just write

$$\{1\}\{2\} = \{2\}\{1\}$$

8 The Sieve method

The following result is what we call the **sieve method** or **principle of inclusion and exclusion**. We will see how it is able to help us sum up the number of ways to get various structures on a set.

Theorem 69
If $A_1, \ldots, A_n$ are finite sets then
$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right|$$

Proof. Suppose that $A_1, \ldots, A_n \subseteq U$ for some universal set U. For each $A \subseteq U$ consider the characteristic function $f_A : U \to \{\pm 1\}$(1 if in A and 0 otherwise). In addition define $F : U \to \{\pm 1\}$ by

$$F(x) = \prod_{k-1}^{n} (1 - f_{A_k}(x))$$

This clearly is a characteristic function for $A / \bigcup_{k=1}^{n}$ because it only returns 1 if x is not in $\bigcup_{k=1}^{n}$ and 0 otherwise.

$$\begin{aligned}
\left| U / \bigcup_{k=1}^{n} A_k \right| &= \sum_{x \in U} F(x) = \sum_{x \in U} \prod_{k=1}^{n} (1 - f_{A_k}(x)) \\
&= \sum_{x \in U} \sum_{I \subseteq [n]} (-1)^{|I|} \prod_{i \in I} f_{A_i}(x) \\
&= \sum_{I \subseteq [n]} (-1)^{|I|} \sum_{x \in U} \prod_{i \in I} f_{A_i}(x) \\
&= \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|
\end{aligned}$$

The 2nd equality follows by binomial expansion consider

$$(1 - f_{A_1})(1 - f_{A_1}) \ldots (1 - f_{A_n})$$

In each expansion, during multiplication you could choose to multiply 1 or $-f_{A_i}$ in each bracket. So we let I correspond to the i of $-fA_i$ chosen to be multiplied in the expansion. Expanding this we have

$$\sum_{I \subseteq [n]} (-1)^{|I|} \prod_{i \in I} f_{A_i}(x)$$

where I takes values like for example $\{1, 3, 4, 5\}, \emptyset, \{1, 2\} \subseteq [n]$ etc. Where the empty set means no $-fA_i$ was chosen to be multiplied, so the expansion is just 1. In the 3rd equality after bringing $\sum_{x \in U}$ in, it is obvious to see that $\prod_{i \in I} f_{A_i}(x)$ is the characteristic function for $\bigcap_{i \in I} A_i$. Now finally consider

$$\left| U / \bigcup_{k=1}^{n} A_k \right| + \left| \bigcup_{k=1}^{n} A_k \right| = \left| \bigcup_{k=1}^{n} A_k \cup U \right| = |U|$$

in our case since $\bigcup_{k=1}^{n} A_k \subseteq U$

$$|U| - \left| \bigcup_{k=1}^{n} A_k \right| = \left| U / \bigcup_{k=1}^{n} A_k \right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right| = |U| + \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|$$

where the last equality follows because the intersection of no subsets/empty subsets of the universal set is just the universal set itself. Why? Well there was no intersection criteria to be considered so every $x \in U$ is valid.(Search this up its called the nullary intersection its so fucking stupid). That basically corresponds to the scenario when $I = \emptyset$. Now finally minus $|U|$ from both sides and flip polarity the theorem follows. $\square$

Definition 70

A permutation $\pi \in S_n$ is called a **derrangement** if $\pi(i) \neq i$ for any $i \in [n]$. For every $n \in \mathbb{N}$ we let $D(n)$ denote the number of derrangements in S_n.

Proposition 71
For each $n \in \mathbb{N}$ the following identity holds

$$D(n) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

Proof. For each $k \in [n]$ set $A_k = \{\pi \in S_n | \pi(k) = k\}$. Observe that $|A_k| = (n-1)!$. This is because every $\pi \in S_n$ in this set takes the form say for example $n = 7$ and $k = 4$,

$$\pi = - - - k - --$$

The remaining other positions to permute corresponds to $(n-1)!$ clearly. For every k subset I of $[n]$ the set $\bigcap_{i \in I} A_i$ has size $(n-k)!$. The reason why follows the same logic as above. Any element of $\bigcap_{i \in I} A_i$ has now $|I|$ places fixed. That is for example for some $|I| = 3$ and $n = 7$ then elements in $\bigcap_{i \in I} A_i$ could look like

$$i_1 - -i_2 \; i_3 - -$$

Again clearly permutating the remaining elements follows $(n-k)!$ as desired.Hence in light of the sieve method we have

$$\begin{aligned}
\left|\bigcup_{k=1}^{n} A_k\right| &= \sum_{\emptyset I \subseteq [n]} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right| \\
&= \sum_{k=1}^{n} \sum_{I \subseteq [n]:|I|=k} (-1)^{k+1} \left|\bigcap_{i \in I} A_i\right| \\
&= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)!
\end{aligned}$$

Where we have $\binom{n}{k}$ due to $\sum_{I \subseteq [n]:|I|=k}$. Now finally we may calculate $D(n)$ by

$$D(n) = n! - \left|\bigcup_{k=1}^{n} A_k\right| = n! - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)! = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

since $(n-k)!\binom{n}{k} = (n-k)!\frac{n!}{(n-k)!k!}$ and that $(-1)^0 \frac{n!}{0!} = n!$. The first equality is evidently all permutations minus the union of all non derangements.

We will conclude this section by establishing a non-recurrent formula for the Stirling numbers of the second kind.

Proposition 72
For every $n \in \mathbb{N}$ and $k \in [n]$ the following identities hold

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

Proof. Recall from previous lectures that $S(n,k)k!$ is the number of surjectve functions $[n] \to [k]$. We will now use the **sieve method** to count the number of surjective functions this time. For each $j \in [k]$ set

$$A_j : \{f : [n] \to [k] | j \notin f([n])\}$$

Observe that a function is clearly not surjective if and only if it belong s to $\bigcup_{j=1}^{k} A_j$. Then it follows from the sieve method that

$$\left|\bigcup_{j\in[k]} A_j\right| = \sum_{\emptyset\neq I\subseteq[k]} (-1)^{|I|+1} \left|\bigcup_{i\in I} A_i\right| = \sum_{j=1}^{k} (-1)^{j+1}\binom{k}{j}(k-j)^n$$

The last equality follows from the fact that for all functions in $\bigcap_{i\in I} A_i$, the set I is not in their image. We let the size of $|I| = j$. In that case, for every $x \in [n]$, there are $k-j$ outputs a function $f(x)$ can have. Functions must only have one output per input to be well defined so we assign each function to one of the $k-j$ outputs. Therefore for all $x \in [n]$, we will know that we may create $(k-j)^n$ possible functions. We have $\binom{k}{j}$ since that is precisely the number of I sets of lenght j that can be constructed from $[k]$. We loop through all sizes starting from 1 because it is assumed $I \neq \emptyset$ so length zero is impossible. Now finally consider number of possible functions minus number of non surjective functions = number of surjective functions.

$$S(n,k)k! = k^n - \left|\bigcup_{j\in[k]} A_j\right| = k^n - \sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}(k-j)^n = \sum_{0}^{k}(-1)^j\binom{k}{j}(k-j)^n$$

for the last equality see for yourself that $(-1)^{0+1}\binom{k}{0}(k-0)^n = -k^n$. Now combine the sums and account for the polarity and the desired formula follows.

9 Ordinary Generating functions

Note that previously we only had binomial theorem for $r \in \mathbb{Z}^+$. What if r was not nonnegative integer? Lets investigate

Theorem 73 (Generalized Binomial Theorem)
For any $r \in \mathbb{R}$ we have

$$(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n$$

where

$$\binom{r}{n} = \frac{r(r-1)\dots r-n+1}{n!}$$

Proof. Set $f(x) = (1+x)^r$. For each $n \in \mathbb{N}_0$ we see that $f^{(n)} = (r)_n(1+x)^{r-n}$ and so $\frac{f^{(n)}}{n!} = \binom{r}{n}$. Recall $(r)_n = r(r-1)\dots(r-n+1)$ Then by the maclaurin formula of $f(x)$ the theorem follows □

Example 74
Let us find the macluarin series of $(1-x)^{-m}$ when $m \in \mathbb{N}$. First note that each $n \in \mathbb{N}_0$

Solution. Consider

$$\binom{-m}{n} = \frac{1}{n!}\prod_{i=0}^{n-1}(-m-i) = \frac{(-1)^n}{n!}m(m+1)\dots(m+n-1)$$
$$= (-1)^n\frac{(m+n-1)!}{n!(m-1)!} = (-1)^n\binom{m+n-1}{m-1}$$

In light of the above thoerem we have

$$(1+x)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} x^n = \sum_{n=0}^{\infty} (-1)^n \binom{m+n-1}{m-1} x^n = \sum_{n=0}^{\infty} (-x)^n$$

Now sub $x = -x$ which we then get

$$(1-x)^{-m} = \sum_{n=0}^{\infty} \binom{m+n-1}{m-1} x^n$$

Definition 75

We can associate any sequence $(a_n)_{n\geq 0}$ to the formal power series $\sum_{n=0}^{\infty} a_n x^n$ then we say that this power series is the the (ordinary) **generating function** of $(a_n)_{n\geq 0}$

If $\sum_{n=0}^{\infty} a_n$ converges to a function $F(x)$ in some neighbourhood of 0 we also call $F(x)$ the (ordinary) **generating function** of $(a_n)_{n\geq 0}$

In the example above we say $(1-x)^{-m}$ is the generating function of the series $\left\{\binom{m+n-1}{m-1}\right\}_{n\geq 0}$

Example 76

The formula

$$a_{n+1} = 3a_n + 1$$

is called a **reccurence relation** while the formula

$$a_{n+1} = \frac{3^{n+1}-1}{2}$$

is called a **explicit formula** since it provides a way to calculate a_{n+1} directly without calculating other elements

The *explicit formula* is also called the **closed formula** because it does not contain the $\sum$ or $\prod$ sign. More precisely a closed formula can contain the sum or product of a *fixed* number of parts . For example

$$f(n) = \sum_{i=1}^{n} i$$

is not closed but

$$f(n) = (n+1)n/2$$

is

Example 77 (Fibonacci Sequence)

Find the explicit formula for fibonacci $F_{n+1} = F_n + F_{n-1}$ series with generating functions

Solution. Let $F(x) = \sum_{n=0}^{\infty} F_n x^n$ be the generating function of the Fibonacci sequence(recall definition generating function above). Now lets try to find a **closed formula** so we can solve this(well you can only compare fixed variables...). Since $F_0 = 0$. Hence $F(x) = \sum_{n=0}^{\infty} F_n x^n = \sum_{n=1}^{\infty} F_n$. Since $F_1 = 1$, then

$$F(x) - x = \sum_{n=2}^{\infty} F_n x^n$$

in which case shifting n by 1 we have

$$F(x) = \sum_{n=1}^{\infty} F_{n+1} x^{n+1}$$

Then

$$F(x) - x = \sum_{n=1}^{\infty} F_{n+1} x^{n+1} = \sum_{n=1}^{\infty} (F_n + F_{n+1}) x^{n+1} = x \sum_{n=1}^{\infty} F_n x^n + x^2 \sum_{n=1}^{\infty} F_{n-1} x^{n-1} = xF(x) + x^2 F(x)$$

which is closed as $F(x)$ is the same fixed sum used on both sides of the equation. So right now we seek to find a solution to this given the initial conditions $F_0 = 0, F_1 = 1$. This looks like an ODE IVP doesn't it? Lets try to find a series solution for this(like fourier series for periodic functions. And geometric series for well geometric functions and taylor/maclaurin for smooth differentiable functions.(Both of whom if convergent then analytic), so many series can solve so many problems). So intuitively we can try to use to geometric formula by first converting our equation above to partial fractions.

$$F(x) = -\frac{x}{x^2 + x - 1} = -\left(\frac{A}{x - a} + \frac{B}{x - \beta}\right)$$

In which case by cover up rule we have

$$F(x) = \frac{1}{a - \beta}(1 - \frac{x}{a})^{-1} + \frac{1}{\beta - a}(1 - \frac{x}{\beta})^{-1}$$

Then applying geometric series we have

$$= \sum_{n=0}^{\infty} \left(\frac{a^{-n}}{a - \beta} + \frac{\beta^{-n}}{\beta - a}\right) x^n$$

By elementary calculus we obtain the roots of $x^2 + x - 1$ to be $a = \frac{-1+\sqrt{5}}{2}$ and $\beta = \frac{-1-\sqrt{5}}{2}$ and so subbing everything in we have

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Remark 78. *Notice that solving linear first order equations that are solvable with power series method gives rise to recurrence series too.In a sense you could see solving recurrence relations as precisely using a power series method to solve some IVP. The*

Proposition 79

Let $F(x)$ and $G(x)$ be the generating functions of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. Then the following statements hold

1. $F'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$
2. $F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$
3. $F(x)G(x) = \sum_{n=0}^{\infty} c_n x^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Proof. Done before see rudin or complex analysis(the section on power series)

Definition 80

Fix $k \in \mathbb{N}$ and for every $n \in \mathbb{N}$ let $p_{\leq k}(n)$ be the number of partition of n which each part having size at most k.

We will now proceed to find the generating function of the sequence $\{p_{\leq n}(n)\}_{n\in\mathbb{N}_0}$

Proposition 81

The following identity holds

$$\sum_{n=0}^{\infty} p_{\leq k}(n)x^n = \prod_{i=1}^{k} \frac{1}{1-x^i}, \quad \forall k \in \mathbb{N}$$
$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

Proof. For each $n \in \mathbb{N}_0$ set $R_n = \{(c_1, \ldots, c_k) \in \mathbb{N}_0 | c_1 + 2c_2 + \ldots + kc_k = n\}$. Observe that R_n is in bijection with the set $P_{\leq k}(n)$ consisting of all partitions of n with large parts at most k.(See that in ic_i where i is the partition size and c_i is number of partitions of size i). Then consider

$$\begin{aligned}\prod_{i=1}^{k} \frac{1}{1-x^i} &= \prod_{i=1}^{k}(1 + x^i + x^{2i} + x^{3i} \ldots)\\ &= \sum_{n\in\mathbb{N}_0} \sum_{(c_1,\ldots,c_k)\in R_n} x^{c_1+2c_2+\ldots+kc_k} = \sum_{n\in\mathbb{N}_0} \sum_{\lambda\in P_{\leq k}(n)} x^{|\lambda|}\\ &= \sum_{n=0}^{\infty} |P_{\leq k}(n)|\, x^n = \sum_{n=0}^{\infty} p_{\leq k}(n)x^n\end{aligned}$$

this easily follows when you see $c_1 + 2c_2 + \ldots + kc_k = n$. Essentially we are summing up all powers of x^n(to infinity as the term $\sum_{n\in\mathbb{N}_0}$ implies). And the coefficient is simply the number of ways you can do that.

For the latter identity juts consider the same thing but now no restiction on k so replace $p_{\leq k}(n)$ with $p(n)$

$$\begin{aligned}\prod_{i=1}^{k} \frac{1}{1-x^i} &= \prod_{i=1}^{k}(1 + x^i + x^{2i} + x^{3i} \ldots)\\ &= \sum_{n\in\mathbb{N}_0} \sum_{(c_1,\ldots,c_k)\in R_n} x^{c_1+2c_2+\ldots+kc_k} = \sum_{n\in\mathbb{N}_0} \sum_{\lambda\in P(n)} x^{|\lambda|}\\ &= \sum_{n=0}^{\infty} |P(n)|\, x^n = \sum_{n=0}^{\infty} p(n)x^n\end{aligned}$$

□

Definition 82

Let $F(x) = \sum_{n\geq 0} f_n x^n$ be a **formal** power series and let G be a formal power series with constant term 0. Then we define

$$F(G(x)) = \sum_{n\geq 0} f_n(G(x))^n = f_0 + f_1 G(x) + f_2(G(x))^2 + \ldots$$

Note that the formal power series of a function is not necessarily the taylor series of a function. A formal power series is defined by the coefficients themselves. Not by a function like taylor which then defines its coefficients.

Theorem 83 (Product Formula)
For each $n \in \mathbb{N}_0$ let a_n be the number of ways to build certain a-structure on an n-set and let b_n be the number of ways to build a certain β-structure on an n-set. Assume that $a_0 = 0$ and $b_0 = 1$. Now let c_n be the number of ways to split $[n]$ into 2 sets $S = \{1, 2, \ldots, i\}$ and $T = \{i+1, i+2, \ldots, n\}$(the intervals S, T are allowed to be empty) and then build an a-structure on S and then build a β-structure on T. If $A(x)$, $B(x)$ and $C(x)$ denote the generating functions of $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ respectively. Then $C(x) = A(x)B(x)$

Proof. There are a_i ways to build a structure of β on T and a_i of a on S. This is true for all i as long as $0 \leq i \leq n$. Therefore $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ so the theorem is true considering the products of power series.

Theorem 84
For each $n \in \mathbb{N}_0$ let a_n be the number of ways to build certain a-structure on an n-set on an n-set. Assume that $a_0 = 0$. Now let h_n be the number of ways to split $[n]$ into an *unspecified* number of nonempty disjoint sub-intervals then build a structure of the given kind on each of these intervals. Assume that $h_0 = 1$ and let $A(x)$ and $H(x)$ denote the generating functions of $(a_n)_{n\geq 0}$ and $(h_n)_{n\geq 0}$ respectively. Then $H(x) = \frac{1}{1-A(x)}$

Proof. Consider
$$H(x) = 1 + \sum_{k\geq 1} A(x)^k = \sum_{k\geq 0} A(x)^k = \frac{1}{1 - A(x)}$$
In the first equality we have 1 because $h_0 = 1$. From the **product formula** we know $A(x)^k$ is the number of ways to split $[n]$ into exactly k intervals then build a structure of the given kind on each interval. Now because $a_0 = 0$, none of the power series $A(x)^k$ has a non-zero constant term. This shows that we are assured that $H(x)$ is in the form above, with the only constant term being 1.

Remark 85. *We require non-empty here because now we have an unspecified number.There are infinite ways to split* $[n]$ *into empty intervals. In the previous case we specified only 2 possible intervals, empty or not. But here there are no restrictions.*

Definition 86
Let $A(x)$ be the generating function of a sequence $(a_n)_{n\geq 0}$ satisfying $a_0 = 0$ and let $G(x)$ be the generating function of a sequence $(b_n)_{n\geq 0}$. Then the **composition** of $B(x)$ with $A(x)$ is

$$B(A(x)) = \sum_{n=0}^{\infty} b_n A(x)^n$$

Theorem 87 (The Compositional formula)
For each $n \in \mathbb{N}_0$ let a_n be the number of ways to build certain a-structure on an n-set and let b_n be the number of ways to build a certain β-structure on an n-set. Assume that $a_0 = 0$ and $b_0 = 1$. Now let f_n be the number of ways to split $[n]$ into a number of nonempty sub-intervals build an a-structure on each sub-interval and then build a β-structure on the set consisting of all such sub-intervals. Assume that $f_0 = 1$ and let $A(x)$, $B(x)$ and $F(x)$ denote the generating functions of $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(f_n)_{n\geq 0}$ respectively. Then $F(x) = B(A(X))$

Proof. Consider

$$F(x) = 1 + \sum_{n=1}^{\infty} f_n x^n = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} b_k \sum_{(c_1,\ldots,c_k)\in C(n)} a_{c_1} \ldots a_{c_k} \right) x^n$$
$$= 1 + \sum_{k=1}^{\infty} b_k \sum_{n=1}^{\infty} \left(\sum_{c_1,\ldots,c_k \in C(n)} a_{c_1} \ldots a_{c_k} \right) x^n = 1 + \sum_{k=1}^{\infty} b_k A(x)^k$$
$$= B(A(x))$$

10 Catalan Numbers

Definition 88

Let C_n be the number of strings of balanced parentheses of lenght $2n$ where $C_0 = 1$. We call them the **catalan numbers**

Example 89

Consider $2n = 4$ the possible strings are

$$()() \quad (())$$

Definition 90

The double factorial is not the same as applying the factorial twice. It means for even n

$$n!! = \prod_{k=1}^{\lceil \frac{n}{2} \rceil - 1} (2k) = n(n-2)(n-4)\ldots 4 \cdot 2$$

for odd n

$$n!! = \prod_{k=1}^{\lceil \frac{n}{2} \rceil - 1} (2k-1) = n(n-2)(n-4)\ldots 3 \cdot 1$$

Theorem 91

The *explicit* formula for the catalan number C_n where $C_0 = 0$ is

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

for every $n \in \mathbb{N}$

Proof. Notice that every string of balanced parenthese of lenght $2n$ has the form $P_k P_{n-k}$ for some $k \in [\![1, n]\!]$ where P_k is the string of balanced parenthesis of length $2k$. This is because there must exist a day other than the first day when the paranthesis is balanced. For example given $2n = 4$ It could be either at the end like

$$P_2 P_0 = (())$$

in which case $k=2$ or it could be in

$$P_1P_1 = ()()$$

in which case $k=1$. But notice that for any P_k above the number of balanced strings is the same as P_{k-1} since you minus the start and end bracket the inside must be a balanced string. So expressing C_0C_n as C_0C_{n-1}, that is we change one of the indexes in each pair of C_i(pick the bigger one by convention to ensure non-zero string lenghts) we have

$$C_n = C_0C_{n-1} + \ldots + C_{n-1}C_0 = \sum_{k=0}^{n-1} C_kC_{(n-1)-k}$$

The let $C(x)$ be the generating function of the sequence $(C_n)_{n\geq 0}$ that is $\sum_{n=0}^{\infty} C_nx^n$ using the recurrence rleation above we have

$$C(x) - 1 = \sum_{n=1}^{\infty} C_nx^n = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} C_kC_{(n-1)-k}\right)x^n = x\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} C_kC_{n-k}\right)x^k = xC(x)^2$$

where the 3rd equality follows by a shift of n by 1 and the 1 in the extreme LHS is due to $C_0 = 1$ which we minus from the sum.Now we have

$$xC(x)^2 - C(x) + 1 = 0$$

and solving this quadratic equation we have

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

Then by the generalized binomial theorem it can be shown that

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = 1 + \sum_{n=1}^{\infty} \frac{1}{n+1}\binom{2n}{n}x^n$$

and so $C_n = \frac{1}{n+1}\binom{2n}{n}$ as desired

11 Exponential Generating Functions

There are many sequences that grow too quickly and therefore their generating functions cannot be expressed n closed form.For some of such sequences it is convenient to use exponential generating functions instead of ordinary generating functions.

Definition 92

Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. The **exponential generating function** of $(a_n)_{n\geq 0}$ is the formal series $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$

Example 93

Exponential generating function whose coefficients are 1(s) is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

and when coefficients is the sequence $(n!)_{n\geq 0}$ it is

$$\sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Example 94

An interesting of this special case is that is used to solve ordinary differential equations. To this end we denote the exponential generating function sequence $EG_n = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x$.Now we consider the fibonacci sequence again which is to solve $f_{n+2} = f_{n+1} + f_n$. Now notice that $EG_{n-i} = EG_n^{(i)}$. Therefore using EG to solve for the fibonacci series we see that

$$EG_{n+2} = EG_{n+1} + EG_n \quad \Leftrightarrow \quad EG_{n+2} = EG'_{n+2} + EG''_{n+2}$$

are equivalent problems. Now you see how ODE solutions are motivated!

Lemma 95

Let $\{a_i\}$ and $\{b_k\}$ be two sequences and let $A(x)$ and $B(x)$ be their exponential generating functions. Define $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$. Then just like for ordinary normal generating functions we also have

$$A(x)B(x) = C(x)$$

In other words the coefficient of $\frac{x^n}{n!}$ in $A(x)B(x)$ is $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$

Proof. Consider

$$a_i \frac{x^i}{i!} \cdot b_j \frac{x^j}{j!} = a_i b_j \cdot \frac{x^{i+j}}{i!j!} \cdot \frac{(i+j)!}{(i+j)!} = a_i b_j \cdot \frac{x^{i+j}}{(i+j)!} \cdot \binom{i+j}{i}$$

Remark 96. *Contrast with $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ which is the coefficient of $A(x)B(x)$ is they were ordinary generating fuctions*

Proposition 97 (Product Formula: Exponential Version)

For each $n \in \mathbb{N}_0$ let a_n be the number of ways to build certain a-structure on an n-set and let b_n be the number of ways to build a certain β-structure on an n-set. Assume that $a_0 = 0$ and $b_0 = 1$. Now let f_n be the number of ways to split $[n]$ into 2 sets S and T and then build an a-structure on S and then build a β-structure on T. If $A(x)$, $B(x)$ and $F(x)$ denote the generating functions of $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$ and $(f_n)_{n\geq 0}$ respectively. Then $F(x) = A(x)B(x)$

Proof. Observe that for every $n \in \mathbb{N}_0$ we have

$$f_n = \sum_{S\subseteq [n]} a_{|S|} b_{n-|S|} = \sum_{k=0}^{n} \sum_{S\subseteq [n]:|S|=k} a_k b_{n-k} = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

Therefore

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{x^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) x^n = A(x)B(x)$$

Example 98

Let us find an explicit forumla for the exponential generating function $B(x)$ of the sequence $(B(n))_{n\geq 0}$ where $B(n)$ denotes the n-th Bell number. Recall that

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k)$$

Solution. First notice that

$$B'(x) = \sum_{n=1}^{\infty} B(n) \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} B(n+1) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} B(k) \frac{x^n}{n!} = B(x)e^x$$

Where the 3rd equality follos by a shift in n and the 3rd equality is subbing in $B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k)$ into our expression. Finally the last equality follows since

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} B(k) 1^{n-k} \right) \frac{x^n}{n!} = B(x)e^x$$

since

$$\sum_{n=0}^{\infty} (1) \frac{x^n}{n!} = e^x \quad \text{and} \quad \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} = B(x)$$

Then we obtain $e^x = (\ln B(x))' = \frac{B'(x)}{B(x)}$ then by fundamental theorem of calculus we know

$$\ln B(x) = e^x + C$$

but knowing $B(0) = 1$ we have $C = -1$ so

$$B(x) = e^{e^x - 1}$$

□

We will now generalize the product formula like so

Proposition 99

Fix $k \in \mathbb{N}_{\geq 2}$ and for each $i \in [k]$ let $a_n^{(i)}$ be the number of ways to build a certain a_i structure on an n set. Let f_n be the number of ways to subdivide $[n]$ into k disjoint *subsets* namely $[n] = T_1 \cup \ldots \cup T_k$ and then build an a_i structure on T_i for every $i \in [k]$. Let $A_i(x)$ be the exponential generating function of $(a_n^{(i)})_{n\geq 0}$ for every $i \in [k]$ and let $F(x)$ be the exponential generating function $(f_n)_{n\geq 0}$. Then $F(x) = A_1(x)A_2(x)\ldots A_k(x)$

Proof. Consider

$$f_n = \sum_{(T_1,\ldots,T_k)} a_{|T_1|}^{(1)} \ldots a_{|T_k|}^{(k)} = \sum_{(t_1,\ldots,t_k)} \binom{n}{t_1,\ldots,t_k} a_{t_1}^{(1)} \ldots a_{t_k}^{(k)}$$

where $\sum_{(T_1,\ldots,T_k)}$ represents the number of ways to subdivide $[n]$ into k disjoint *partitions*. And then multiply by the number of ways to build an a_i structure on each set T_i as represented by $a^{(1)}_{|T_1|}\ldots a^{(k)}_{|T_k|}$ for each $(T_1,\ldots,T_k)$. For each $(T_1,\ldots,T_k)$ let $t_i = |T_i|$. Then notice that $\sum_{(T_1,\ldots,T_k)} = \sum_{(t_1,\ldots,t_k)} \binom{n}{t_1,\ldots,t_k}$ where $\sum_{(t_1,\ldots,t_k)}$ represents the number of *compositions* $t_1+\ldots+t_k = n$. This make sense as essentially we are looping over size then for each sized set permute the elements.Also recall from **multinomial theorem** that since

$$\binom{n}{t_1,\ldots,t_k} = \frac{n!}{t_1!\ldots t_k!}$$

subbing our expression for the coefficients of f_n into our generating function $F(x)$ we see that

$$F(x) = \sum_{n=0}^{n} f_n \frac{x^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{(t_1,\ldots,t_k)}\binom{n}{t_1,\ldots,t_k} a^{(1)}_{t_1}\ldots a^{(k)}_{t_k}\right)\frac{x^n}{n!} = \prod_{i=1}^{k} A_1(x)\ldots A_k(x)$$

as desired. The last equality follows by induction from the previou theorem □

Theorem 100 (Compositional formula: Exponential Version)
Let a_n be the number of ways to build a certain a-structure on an n-set and assume $a_0 = 0$. Let b_n be the number of ways to build a certain β structure on an n-set and assume $b_0 = 1$. Let f_n be the number of ways to partition $[n]$ build an a-structure on each block of the partition and then build a β structure on the set of all blocks. Assume $f_0 = 1$ and let $F(x)$ be the exponential generating function $(f_n)_{n\geq 0}$. Then $F(x) = B(A(x))$. N

Proof. Consider

$$\begin{aligned} f_n &= \sum_{k=1}^{n} b_k \sum_{\{T_1,\ldots,T_k\}\in\prod_n} a_{|T_1|}\ldots a_{|T_k|} = \sum_{k=1}^{n}\frac{b_k}{k!}\sum_{(T_1,\ldots,T_k)} a_{|T_1|}\ldots a_{|T_k|} \\ &= \sum_{k=1}^{n}\frac{b_k}{k!}\sum_{(t_1,\ldots,t_k)}\binom{n}{t_1,\ldots,t_k} a_{t_1}\ldots a_{t_k} \end{aligned}$$

The first equality is simple, we are summing through the number of partitions as represented by k. Given each k partition $[n]$ accordingly and build a structure on each the k blocks. Then given the resulting k blocks build a β structure on them And then for these k subintervals build β structure so multiply all by b_k Notice that f_n is not the number of ways to subdivide $[n]$ into k disjoint subsets in which case *order matters*(see that $T_1 \cup \ldots T_k$ are numbered sets! However for partition it does not. As such we divide b_k by $k!$ in the second equality. Finally as usual sub f_n into our generating function $F(x)$ and see that

$$\begin{aligned} F(x) &= 1 + \sum_{n=1}^{\infty} f_n \frac{x^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{b_k}{k!} \sum_{(t_1,\ldots,t_k)} \binom{n}{t_1,\ldots,t_k} a_{t_1} \ldots a_{t_k} \right) \frac{x^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \frac{b_k}{k!} \left(\sum_{k=1}^{n} \sum_{(t_1,\ldots,t_k)} \binom{n}{t_1,\ldots,t_k} a_{t_1} \ldots a_{t_k} \right) \frac{x^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \frac{b_k}{k!} \left(\sum_{(t_1,\ldots,t_k)} \frac{a_{t_1}}{t_1!} \cdots \frac{a_{t_k}}{t_k!} \right) x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{b_k}{k!} A(x)^k \\ &= B(A(x)) \end{aligned}$$

as desired. Recall from above that $A(x)^k$ is the number of ways to divide into k intervals and apply an a structure on each on interval. Just that our coefficients now have $t_i!$ in the denominator because we are using exponential generating functions

12 Graph Theory Introduction and Eulerian Trails

In the interest of time i will do psets for the above after graph theory...

Definition 101

A **simple graph** is a pair (V, E) where V is a finite nonempty set and E is a set consisting of 2-subsets of V. The elements of V are called **vertices** while the elements of E are called **edges**

In other words the edges are basically *pairs* of elements in V We often denote an edge $\{v, w\}$ as vw or wv.

Remark 102. *We will define what is meant by "unique" in the coming lectures*

Example 103

A unique graph having n vertices where any two distinct vertices are adjacent is called **complete**.

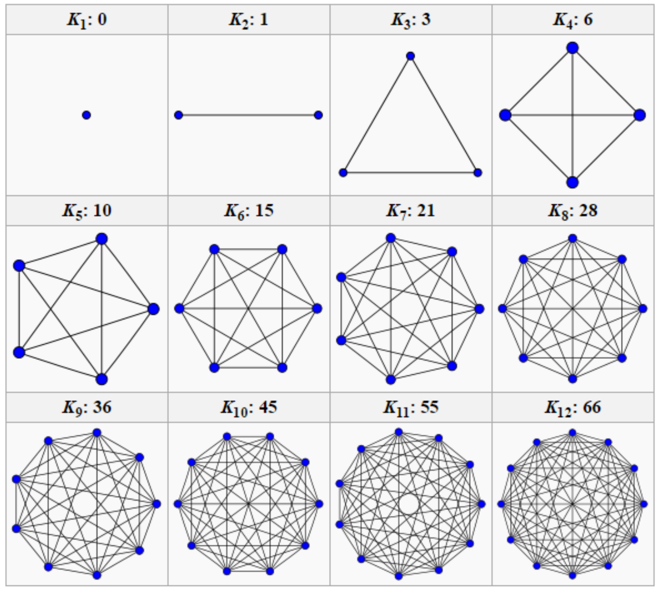


Figure 4: Some Complete graphs

Notice that K_n has precisely $\binom{n}{2}$ edges. Also notice that every vertex has degree $n-1$ since every vertex must be connect to the other $n-1$ vertices.

Example 104

A unique graph having n vertices with no edges is called **trivial**

Proposition 105

Every graph contains an even number of vertices of odd degree

Proof. Let $G = (V, E)$ be a graph. It suffices to observe that every edge of G contributes 1 towards the degree of each of the two vertices it is connected to. Therefore

$$\sum_{v \in V} \deg v = 2|E|$$

Since the sum of all degrees of vertices of G is even there must be an even number of vertices of G with odd degrees

Definition 106

A sequence of edges of the form $v_1v_2, v_2v_3, \ldots, v_\ell v_{\ell+1}$ where $v_1, \ldots, v_{\ell+1} \in V(G)$ is called a **walk** of length ℓ and if $v_iv_{i+1} \neq v_jv_{j+1}$ for any distinct $i, j \in [\ell]$ then we call $v_1v_2, v_2v_3, \ldots, v_\ell v_{\ell+1}$ a **trail** of length ℓ.

Example 107 (Trail)
A trail is a walk where no edge is repeated. Here's an example with 4 vertices and 3 distinct edges v_1v_2, v_2v_3, v_3v_4.

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow v_4$$

but on the other hand

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow v_4 \longrightarrow v_2 \longrightarrow v_3$$

is a walk as $\{v_2, v_3\}$ is repeated
Both of these could be just part of the graph not the whole graph

Definition 108
A a trail satisfies $v_{\ell+1} = v_1$ then we call it a **closed trail** or **circuit**

Example 109 (Closed Trail (Circuit))
A closed trail (circuit) is a trail that starts and ends at the same vertex. Here's an example:

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow v_1$$

This could be just part of the graph not the whole graph

Definition 110
A trail that uses all the edges of the graph is called an **eulerian trail**. It if it's closed then **eulerian circuit**

Example 111 (Eulerian Trail)
An Eulerian trail is a trail that uses all the edges of the graph. Here's a simple graph with an Eulerian trail:

$$\begin{tikzcd} v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[r] & v_4 \arrow[ll, bend left] \end{tikzcd}$$

This is the whole graph. This trail visits each edge exactly once but doesn't necessarily return to the starting point, so it is an Eulerian trail but not a circuit.

Example 112 (Eulerian Circuit)
An Eulerian circuit is an Eulerian trail that is also closed. Here's an example:

$$\begin{tikzcd} v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[r] & v_4 \arrow[lll, bend left] \end{tikzcd}$$

This is the whole graph. This example has an Eulerian circuit since all edges are used exactly once, and the trail returns to the starting vertex.

Definition 113

If a trail $v_1v_2, v_2v_3, \ldots, v_\ell v_{\ell+1}$ satisfies that $v_i \neq v_j$ for any $i \neq j$ then it is called a **path**. A **subgraph** of G is a graph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$. A subgraph of G is called an **induced subgraph** provided that for any $v, w \in V$ with $v \neq w$ if $vw \in E$ then $vw \in E'$

Example 114 (Path)

A path is a trail where no vertices are repeated. To see the distinction with trails consider:

Trail(vertex v_1 is visited twice):

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow v_1 \longrightarrow v_4$$

Path(each vertex is visited exactly once):

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow v_4$$

but in both cases by definition of trail we have distinct edges as you can see

Definition 115

We say the graph G is **connected** if any two distinct vertices of G can be connected by a path. That is, for any $v, w \in V$ with $v \neq w$ there exists a path $v_1v_2, v_2v_3, \ldots, v_\ell v_{\ell+1}$ such that $v_1 = v$ and $v_{\ell+1} = w$. A graph that is not connected is **disconnected**

Example 116 (Connected Graph)

A connected graph means there is a path between any two vertices. Here's an example of a simple connected graph:

$$\begin{tikzcd}
v_1 \arrow[r] & v_2 \arrow[r] & v_3 \\
& v_4 \arrow[u] &
\end{tikzcd}$$

This graph is connected because you can find a path between any two vertices.

Example 117 (Disconnected Graph)

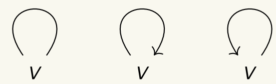
A disconnected graph means there are at least two vertices with no path between them. Here's an example:

$$\begin{tikzcd}
v_1 \arrow[r] & v_2 \\
v_3 \arrow[r] & v_4
\end{tikzcd}$$

In this graph, v_1 and v_2 are disconnected from v_3 and v_4.

Fact 118

In a undirected graph degree of a self loop is considered as 2 just to avoid contradiction in proving sum of degree theorem. More accurately since undirected, the following 2 "directed" self loops are equivalent when considering undirected graphs



hence the degree of a self loop is 2

Theorem 119

A connected graph G has an eulerian circuit if and only if every vertex of G has even degree

Proof. Let $G = (V, E)$ be a connected graph. If G has only one vertex the statement follows trivially(self loops contribute 2 to the degree). So assume $|V| \geq 2$. Since G is connected, $|E| \geq 1$

For the direct implication suppose that $C = v_1, v_2 \dots v_2 v_3 \dots v_\ell v_1$ is an Eulerian ciruit of G.That is $v_1 = v_{\ell+1}$ Clearly from C we see that every v_i has an even degree. Since every v_i can be associated with a pair of edges like so $v_{i-1}v_i, v_i v_{i+1}$. If it is visited multiple time in C then there will be multiple pairs of edges like this, each contributing to 2 to the degree of v_i. To illustrate this, since circuit every element in it can be visited again after traversing in a loop. Then it must be that every vertex has even degree(see diagram below)

For the converse suppose that every vertex G has an even degree. Proceed by induction on the number n of edges. The implication trivially holds when $n = 0$ or $n = 3$ so suppose that $|E| = n > 3$ and also that the statement holds for any connected graph with less than n edges. Choose a vertex $v_1 \in V$ and start travelling from v_1 through consecutive edges of G without repeating any of them until encountering a vertex from which we cannot continue travelling because all the edges connected to this vertex have already been used. This will give a trail $T = v_1 v_2, v_2, v_3 \dots v_\ell, v_{\ell+1}$. If $v_\ell \in V/\{v_1\}$ then $v_{\ell+1}$ would have been connected to an odd number of edges of T and ggiven $\deg v_{\ell+1}$ is even, we would have been able to continue travelling(see illustration below). Thus $v_{\ell+1} = v_1$. We have proved that G contains a closed trail.

Among all such closed trails let C be one with maximum number of eges and we claim that C is an eulerian circuit. Suppose by way of contradiction that this is not the case.

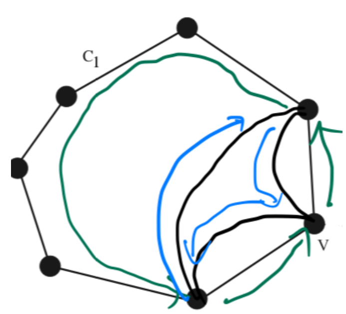


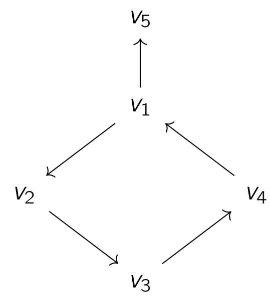
Figure 5: Eulerian Circuit: All vertex with even degree

Consider the subgraph $G' = (V, E')$ of G where $E' = E/C$. Since C is not an eulerian circuit E' is nonempty. Take $e \in E'$ and let H be the connected component of G' containing e. Since H is connected and $|E'| < |E|$ it follows from the induction hypothesis that H has an eulerian circuit C'. Observe that one of the edges of C must be connected to a vertex x in H as otherwise G will be disconnected. Now notice that the closed trail that results from concatenating C and C' via x has more edges than C. This however contradicts the maximality of C. To illustrate this see that the green + blue circuit has more edges than the green or blue circuit alone.

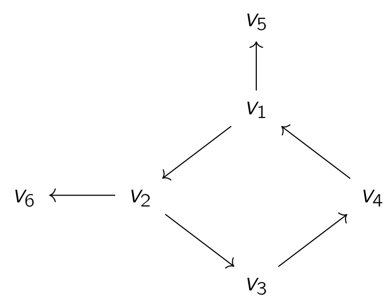
Corollary 120

A connected graph has an eulerian *trail* if and only if it contains at most two vertices with odd degrees

Proof. First consider any eulerian circuit which we know there is all vertices have even degrees. So we want to add a vertex in so that we have some vertexes with odd degrees



but as seen clearly this will always contribute to 2 vertices with odd vertices. You cannot induce an odd degree on any vertex given a eulerian circuit using self loops as it contribute degree 2. Then it is clear to see if we have say



You may only choose 1 *trail*, one that ends at v_6 or one that ends at v_5. Clearly neither are eulerian.

Definition 121

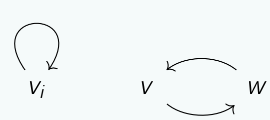
A (simple) **directed graph** is a pair (V, E) where V is a finite set and E is a subset of $V \times V$ satisfying the following two conditions

1. $(v, v) \notin E$ for any $v \in V$ and
2. $|\{(v, w)(w, v)\} \cap E| \leq 1$ for all $v, w \in V$

This means no self loops and directed loops.

Example 122

These are two cases that are not allowed in a directed graph



Definition 123

We define the in-degree of v denoted by indeg v by the number of edges incident and directed to v. Similarly we define the out-degree of v denoted by outdeg v. We say that G is **balanced** provided that indeg v=outdeg v for all $v \in V$

Definition 124

We define eulerian trails,circuits and paths for directed graphs in the same way we did for undirected graphs. We say that a directed graph G is **strongly connected** if for any two distinct vertices v and w of G we can find a directed path from v to w

Theorem 125

A directed graph has an eulerian circuit if and only if it is a balanced strongly connected graph.

Proof. A closed eulerian trail W leaves each vertex as many times as it enters that vertex so G must be balanced. Similarly if W provides a trail from any vertex to any vertex so G is strongly connected

13 Hamilton Cycles and Paths

Definition 126

Let G be a graph. A **cycle** in G is a closed trail that only repeats the first and last vertices. A **hamilton cycle** in G is a cycle that visits all the vertices of G.

Similarly a **hamiltonian path** is a path that visits all vertices.

Definition 127

A simple graph that has a hamilton cycle is called a **hamiltonian graph**

Not every graph is Hamiltonian for instance t is clear that a disconnected graph cannot contain any Hamilton cycle/path. There are also connected graphs that are not Hamiltonian. Such as a vertex of degree one.

Example 128

The complete graph is hamiltonian if and only if $n \geq 3$

Refer to 103. See that $n = 1$ is one isolated vertex while $n = 2$ is just a pair of vertices each with degree 1.

Example 129

Consider

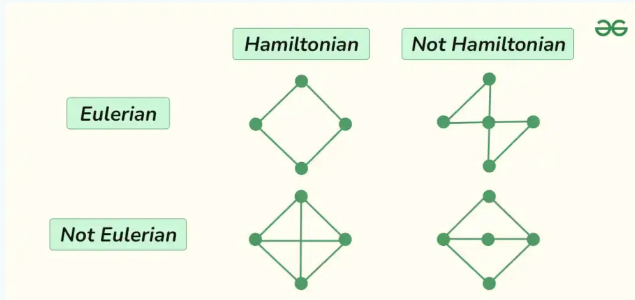


Figure 6: Hamiltonian vs Eulerian Examples

See the bottom left, there exists a cycle that visits all vertices but there no cycle that visits all edges. On the other hand you can do neither using cycles in the bottom right.

Lemma 130

Let n be the number of vertices of a graph G.If all vertices in G have a degree of at least $\frac{n}{2}$ then G is connected

Proof. By contradiction suppose G is disconnected but satisfies the condition above. Disconnected means there exists a way to partition G into two components say each containing a and b vertices each. In which case every vertex in the former and latter component will have degree of at most $a-1$ and $b-1$ each. Then by our contradiction hypothesis we have

$$a - 1 \geq \frac{n}{2}$$
$$b - 1 \geq \frac{n}{2}$$

However adding these 2 inequalities we have

$$a - 1 + b - 1 \geq n$$

which implies

$$n - 2 \geq n$$

which is a contradiction

Proposition 131

Fix $n \in \mathbb{N}$ with $n \geq 3$ and let $G = (V, E)$ be a simple graph $|V| \geq n$. If $\deg v \geq \frac{n}{2}$ for all $v \in V$ then G is hamiltonian

Proof. By contradiction suppose there exists a graph satisfying the hypothesis of the proposition that is not hamiltonian. Now add more and more edges to G until adding one anywhere will create a hamiltonian cycle. Let this new graph be $G' = (V, E')$. This means G' has a hamiltonian path given that one more closes the path and turns it into a cycle. Now pick two distinct elements $v, w \in V$ that are not adjacent. Let the hamiltonian path be $v_1 v_2 \dots v_n$ with $v_1 = v$ and $v_n = w$(that is all elements are represented here). Now consider

$$X = \{i \in [\![2, n-1]\!] | v_i w \in E'\} \quad \text{and} \quad Y = \{i \in [\![2, n-1]\!] | v_{i+1} v \in E'\}$$

Clearly $|X| = \deg w$ and $|Y| = \deg v$. Thus

$$|X \cap Y| = |X| + |Y| - |X \cup Y| = \deg w + \deg v - |X \cup Y| \geq n - |X \cup Y| \geq 2$$

since $[\![2, n-1]\!]$ is of size $n-2$. Since $n \geq 3$ and $|X \cup Y| \leq n-2$ we have that $X \cup Y$ is nonempty. In that case we have

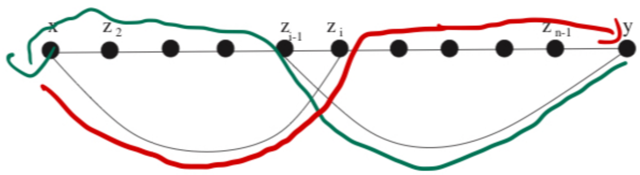


Figure 7: $v_1 v_2 \dots v_j v_n v_{n-1} v_{j+1} v_1$ is a hamiltonian cycle

Oh no contradiction! $\square$

We now consider hamiltonian cycles in directed graphs.

Definition 132

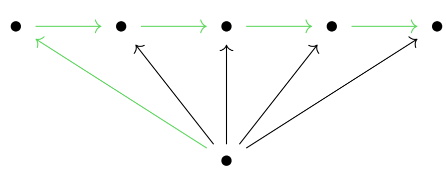
A directed graph is called a **tournament** if there is a directed edge between any two vertices. Observe that a directed grraph (V, E) is a tournament if and only if it contains $\binom{n}{2}$ edges where $n = |V|$

Proposition 133

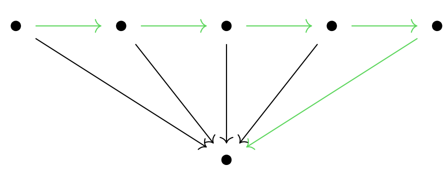
Every tournament has a hamiltonian path

Proof. We will do induction on the number of vertices our tournament. It is clear that a tournament with one or two vertices has a hamiltonian path. So we got our bases cases covered. Now for the induction step, Let $h_1 h_2 \dots h_{n-1}$ be a hamiltonian path. We will show that if tournament, then adding any new v in, there will also still exist a hamiltonian path .Consider the only 3 cases:

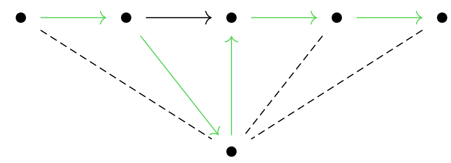
Case 1(all away from the new vertex)



Case 2(all to new vertex)



Case 3(there exists an edge $h_j V$ and $V h_{j+1}$)



Hence by induction our proposition follows $\square$

14 Intro to Trees

Definition 134

A simple connected graph is called a **tree** if it does not contain any cycle

Definition 135

A connected simple graph G is said to be **minimally connected** if any graph obtained from G by deleting an edge is disconnected.

Proposition 136

A graph is a tree if and only if it is minimally connected

Proof. Suppose graph G is a tree but is not minimally connected. That means that removing some edge between v, w, G is still connected. That is there must exist a path $v_1 \dots v_\ell$ connecting v, w. That implies that there was a cycle in G which is a contradiction

Suppose graph G is minimally connected but is not a tree. This means it contains a cycle. However if you remove an edge in the cycle, clearly G is still connected so contradiction again.

Corollary 137

A connected graph H is a tree if and only if for each pair of vertices (x, y) there is exactly one path joining x and y

Proof. Suppose connected graph H is a tree but there is more than one path joining some x and y. Then removing one path between x and y the graph is still connected with violates the above condition on minimal connectedness.

For the converse suppose there is only one path joining every x and y but G is a tree(no cycles).Clearly the removal of one results in an edge disconnected because without cycles there will be no other way other path other than the direct edge between xy that connects them.

Proposition 138

If G is a tree then $|E(G)| = |V(G)| - 1$

Proof. We will agure that for every G this equality hols by induction on $|V(G)|$. When $|V(G)| = 1$ this is true trivially if you consider that single isolated point with no edges. With the base case now let us add a node into our tree G of

n vertices. In order for G' to remain a tree every new edge can only be used to connect an existing edge to our new edge as connecting any 2 existing edges will result in a cycle(recall there is exactly one path between edges so adding an edge not already there will clearly result in a cycle). Therefore for $|V(G)| = n+1$, we have $|E(G)| = n$. So the proposition follows.

Definition 139

A **forest** is a graph whose connected components are trees.

Definition 140

If G is not connected then let k be the smallest integer so that G can be obtained as the union of k connected graphs. Then we say that G has **k connected components**

Proposition 141

Let G be a forest with exacly k connected components. Then $|E(G)| = |V(G)| - k$

Proof. By 138 each connected component exceeds the number of vertices exceeds that of edges by 1 so the proposition follows

Lemma 142

Let T be a tree on n vertices where $n \geq 2$. Then T has at least two vertices whose degree is 1. They are known as the **leaves**

Proof. We claim that for any path p of maximum length in T, then the endpoints of p must be the leaves. Indeed if any one of them weren't leaves, then they can be extended by one of the edges adjacent to a not part of p. Well, there must be maximum path in a connected tree end so the lemma follows.

Theorem 143

All trees on n vertices have $n-1$ edges. Conversely all connected graph on n vertices with exactly $n-1$ edges are trees

Proof. Using induction on n, if $n = 1$ this tatement is trivially true as a 1 vertex cycle free graph has no edges. With the base case settled let us prove the induction step. Let T be a tree on $n+1$ vertices. Find a leaf ℓ of T which we know exists by the previous lemma. Then delete ℓ and its only edge e adjacent to it from T to get a new tree T'. This new tree has T' has n vertices so by induction hypothesis it has $n-1$ edges.That directly meant that T with $n+1$ vertices had n edges, proving the induction step.

15 Cayley's theorem

Theorem 144

For each $n \in \mathbb{N}$ the number of trees on $[n]$ is n^{n-2}

Proof. Fix a positive integer n and let t_n denote the number of trees on $[n]$. Consider the set $\mathcal{J}_n$ consisting of trees on $[n]$ such that T in $\mathcal{J}_n$ has a distinguished pair of vertices $(b, e) \in V \times V$(it may be that $b = e$.)

We will prove that the set of functions $f : [n] \to [n]$ is bijective with the set of doubly rooted trees on $[n]$. First construct

$$C = \{c \in [n] | f^m(c) = c \text{ for somme m} \in \mathbb{N}\}$$

we write $C = \{c_1, \ldots, c_k\}$ in ascending order of c_1 and connect $d_i = f(c_i)$ in a straight line. Then for every element v in $[n]/C$ we connect v to $f(v)$. To see this consider

Example 145

Consider $f : [8] \to [8]$ where

$$f(1) = 3, f(2) = 4, f(3) = 1, f(4) = 5, f(5) = 5, f(6) = 7, f(7) = 8, f(8) = 6$$

then

$$\begin{tikzcd}
1 \arrow[r, bend right] & 3 \arrow[l, bend right] & & 6 \arrow[d] & \\
 & & & 7 \arrow[r] & 8 \arrow[ul] \\
2 \arrow[r] & 4 \arrow[dl] & & & \\
5 \arrow[loop below] & & & &
\end{tikzcd}$$

So the function f creates the cycles $(13), (5), (678)$ then the construction as described in the theorem above turns this unique function into a corresponding unique double rooted tree

$$\begin{tikzcd}
3_{\text{start}} \arrow[r, dash] & 1 \arrow[r, dash] & 5 \arrow[r, dash] \arrow[d, dash] & 7 \arrow[r, dash] & 8 \arrow[r, dash] & 6_{\text{end}} \\
 & & 4 \arrow[d, dash] & & & \\
 & & 2 & & &
\end{tikzcd}$$

As you can see it essentially forces all cycle forming elements into a single straight path with no cycles.In other words it is basically (13 $\cancel{)}$ $\cancel{(}$5 $\cancel{)}$ $\cancel{(}$678) stopping the self loop and linking the last element to the first element of the other cycle instead. This is because the connecting process without this is

$$v \to f(v) \to f \circ f(v) \to f \circ f \circ f(v) \ldots$$

Clearly the elements in C and their images which are responsible for the cycles and they are all belong to some permutation groups.See the cycle notation for the example above for reference. Also notice that $f(C) \in C$.So they are self contained. Then connects the rest normally according to $v \mapsto f(v)$. Clearly for every unique function, it produces a unique tree hence injective. As for surjectivity, every doubly rooted tree can be created by defining functions $f : [n] \to [n]$ as demonstrated above so surjective. Since the number of trees by definition is independent of the choice of root. Since the number of functions $f : [n] \to [n]$ is n^n which is injective with the number of trees on $[n]$ each with a start and end root, we divide by n^2 which is the number of ways such pairs of start and end roots can permuate obtaining n^{n-2} as desired

Corollary 146

For all positie integers n, the number of rooted forerest on $[n]$ is $(n+1)^{n-1}$

Proof. Take a rooted forest on $[n]$ and ... to be continued

16 Spanning trees and kruskal algorithm

Definition 147

Let G be a graph. A subgraph T of G is called a **spanning tree** of G provided that T is a tree and $V(T) = V(G)$

Proposition 148

Every connected simple graph has a spanning tree

Proof. Let G be a connected simple graph. Then the set $\mathcal{G}$ consists of all connected sub graphs G' of G with $V(G') = V(G)$ is nonempty since $G \in \mathcal{G}$. Among all the graphs of $\mathcal{G}$ let T be the one minimizing the number of edges. (Recall the 136) that doing this to simple connected graph results in minimal connected graph which is a tree. Since we are only removing edges not vertices $V(T) = V(G)$. We conclude that T is a spanning tree of G. ☐

Definition 149

Let G be a connected simple graph and let $w : E(G) \to \mathbb{R}_{>0}$ be a map. For every subgraph G' of G define the **weight** of G' to be

$$w(G') = \sum_{e \in E(G')} w(e)$$

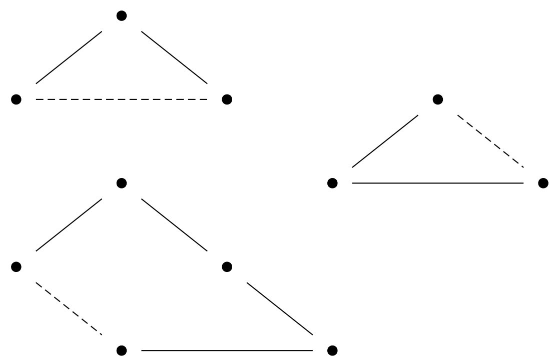
Fact 150 (Kruskal Algorithm)

Let G be a connnected graph nd $w : E(G) \to \mathbb{R}_{>0}$ be a function.

1. Assume that the edges of G are initially unmarked
2. Let S be the subset of $E(G)$ consisting of all unmarked edges that do not create any cycles with the marked edges
3. If S is not empty
   - take $e \in S$ with $w(e) = \min\{w(s) | s \in S\}$
   - mark e
   - return to step (2)
4. Set T to be the sub graph of G whose edges are the marked edges

Proposition 151
Let F and F' be two forests with $V(F) = V(F')$. If $|E(F)| < |E(F')|$ then there is an edge $e \in E(F')/E(F)$ such that the graph we obtain from F by addin the edge e is still a forest

Proof. Suppose by contradiction that if we add any edge $e \in E(F')/E(F)$ to F we produce a cycle then it must be that every degree in $E(F')$ connected two vertices in the same cycle. For example



Where the dotted lines are all the possible addition areas. So it follows that the number of connected components of F' must be at least equal to that of F which we denote as k' and k respectively.Therefore

$$|E(F')| = |V(F')| - k' \leq |V(F)| - k = |E(F)|$$

which is a contradiction. $\square$

Theorem 152
et G and $w : E(G) \to \mathbb{R}_{>0}$ be as above. If T is a spanning tree obtained from successive iterations of steps (1) and (2) above, then $\sum_{e \in E(T')} w(e)$ for any spanning tree T' of G.

Proof. Set $n = |V(G)|$. Suppose by way of contradiction that there exists a spanning tree T' of G such that $\sum_{e \in E(T')} w(e) < \sum_{e ß E(T)} w(e)$. Let $e_1, \dots, e_{n-1}$ and $e'_1, \dots, e'_{n-1}$ be the edges of T and T' labeled in increasing order of weight, that is

$$w(e_1) \leq w(e_2) \leq \dots \leq w(e_{n-1}) \text{ and } w(e'_1) \leq w(e'_2) \leq \dots \leq w(e'_{n-1})$$

Let j be the minimum index in $[\![1, n-1]\!]$ such that

$$\sum_{i=1}^{j} e'_i < \sum_{i=1}^{j} e_i$$

The minimality of j gaurantees that right at $j-1$ this sum is still not lesser. We need just one more to be so we have

$$\sum_{i=1}^{j-1} e'_i \geq \sum_{i=1}^{j-1} e_i$$

given that T, T' are spanning trees of G, $V(T) = V(T')$. In this entire operation this will always stay true as we are only adding edges in, not affecting the vertices at all. Consider the subforests F and F' of G determined by the edges $e_1, e_2, \dots, e_{j-1}$ and $e'_1, e'_2, \dots, e'_j$. That is, $|E(F)| = j - 1 < j = |E(F')|$. Well its a sub forest because you cant guarantee the subset of edges of any tree is still just 1 single tree. It could be broken into connected components of trees. Now by the previous lemma we then know there exists an index $k \in [\![1, j]\!]$ such that $e'_k \notin \{e_1, \ldots, e_{j-1}\}$ and that the subgraph of G obtained by adding e'_k to F is still a forest. However that means

$$\begin{aligned} w(e'_k) \leq w(e'_j) < w(e'_j) + \left(\sum_{i=1}^{j} w(e_i) - \sum_{i=1}^{j} w(e'_i)\right) \\ = w(e_j) + \left(\sum_{i=1}^{j-1} w(e_i) - \sum_{i=1}^{j-1} w(e'_i)\right) \\ \leq w(e_j) \end{aligned}$$

the 2nd equality follows because $\left(\sum_{i=1}^{j} w(e_i) - \sum_{i=1}^{j} w(e'_i)\right)$ so adding it to the bigger side does not change the inequality sign. The 4th equality follows if you consider adding $-w(e'_j)$ to both sides to it and the inequality before it. So we have

$$w(e'_j) - w(e'_j) + \left(\sum_{i=1}^{j} w(e_i) - \sum_{i=1}^{j} w(e'_i)\right) = w(e_j) - w(e'_j) + \left(\sum_{i=1}^{j-1} w(e_i) - \sum_{i=1}^{j-1} w(e'_i)\right)$$

which is equivalent to

$$w(e'_j) - w(e'_j) + \left(\sum_{i=1}^{j} w(e_i) - \sum_{i=1}^{j} w(e'_i)\right) = \left(\sum_{i=1}^{j} w(e_i) - \sum_{i=1}^{j} w(e'_i)\right)$$

Now finally the last equality is since $\left(\sum_{i=1}^{j-1} w(e_i) - \sum_{i=1}^{j-1} w(e'_i)\right) \leq 0$. We now have a contradiction because the steps outlined above clearly says that $w(e_j)$ is the lowest weighted edge you can pick that doesn't create a cycle when joined with the existing edges but we have just shown $w(e_k)$ is an even lower choice.

17 Adjacency matrices and the matrix tree theorem

Definition 153

Let G be a multigraph with $V(G) = [n]$. Then the **adjacency matrix** A of G is defined as follows

- if G is undirected then A_{jk} is the number of edges between j and k and
- if G is directed then A_{jk} is the number of edges from j to k

It is clear that the adjacency matrix of any undirected multigraph is symmetric.

Proposition 154

Let G be a graph(directed or undirected) on $[n]$ with adjacency matrix A. For any $j, k \in [n]$ and $\ell \in \mathbb{N}$ there are A^{ℓ}_{jk} walks of lenght ℓ from j to k

Proof. I know this see your `6.042 Math for CS` notes

Proposition 155

Let G be a simple graph on $[n]$, and let A be the adajency matrix of G. Then G is connected if and only if all the entries of $(I_n + A)^{n-1}$ are positive.

Proof. We know that G is connected if and only if any two distinct vertices j and k of G are connected by a path of length at most $n-1$. This happens if and only if $A^{\ell}_{jk} > 0$ for some $\ell \in [n-1]$. Then the proof follows from

$$(I_n + A)^{n-1} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} A^{\ell}$$

well clearly the binomial theorem applies here too, you can see for yourself hence the coefficients $\binom{n-1}{\ell}$. Now since we summing up the different lenght paths adj matrices, the entries will positive if and only there is at least one such lenght path which is non-zero. (Btw adj matrices never have negative entries for obvious reasons) $\square$

Definition 156

Let G be a directed graph. We say that the **underlying graph** of G is the graph we obtain from G by ignoring the orientation of the edges. A **spanning tree of a directed graph** G is a subgraph T sch that the underlying graph of T is a spanning tree of the underlying graph of G

Definition 157

Let G be a directed graph with $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. The **incidence matrix** of G is the $n \times m$ matrix with

$$A_{ij} = \begin{cases} 1 & \text{edge } e_j \text{ starts at } v_i \\ -1 & \text{edge } e_j \text{ ends at } v_i \\ 0 & \text{edge } e_j \text{ not connected to } v_i \end{cases}$$

Theorem 158

Let G be a connected directed graph(without loops) and let A be the incidence matrix of G. If A_0 is the matrix obtained from A by removing the last row, then $\det(A_0 A_0^T)$ is the number of spanning trees of G

Proof. Let B be an $(n-1)\times(n-1)$ submatrix of A_0 which recall is an $(n-1)\times m$ matrix. This must exist if not G isn't even connected which is not the case. It is obvious to be connected you there must at least be $n-1$ edges, that is $m \geq n-1$. Recall that $m = n-1$ corresponds to the minimally connected graph/tree where every pair of distinct vertices has exactly 1 path. For example when all vertices are arranged linearly on a single path like so

$$v_1 \xrightarrow{e_1} v_2 \cdots\cdots\cdots v_{n-1} \xrightarrow{e_{n-1}} n$$

Let G' be the the subgraph of G whose edges correspond to the columns of B, that is it has $n-1$ edges with $V(G) = V(G')$. Now this makes sense if you are using B to count spanning subtrees of G because $V(G) = V(G') = n$. So each subtree must contain $n-1$ edges(recall theorems above). Note that the removal of the last row corresponding to a vertex v_n does not affect whether G' is a spanning tree or not meaning to say we may still assume the conditions "$V(G) = V(G')$" even when it appears vertex is "removed". Explicitly the removal of last row results in some columns of B having missing tails or heads(because $V(G) = V(G')$ so for the set of n-1 edges chosen some must connect v_n) and this does not affect our count.We will explain more below

We claim that $|\det B| = 1$ if and only if the subgraph G' is a spanning tree of G and $\det B = 0$ otherwise. We prove by induction on n. Consider the following cases

1. If G' is a spanning tree of G then should be vertex $v_i (i \neq n)$ of degree one in G'(this is precisely the leaf of the tree which must exists by 142 and because there are 2 of them even if the removed vertex v_n was one of them you should be able to identifying another from B) . Also we need $i \neq n$ since recall here that v_n corresponds to the row removed from A to get A_0 so you won't find it in B and hence not in G'. Then we know that there is exactly that row i of B corresponding to v_i has exactly one element that is either 1 or -1(with the rest all zeroes). Then expanding $|\det B| = |\det B'|$ where B' is obtained from removing the column and row of this element ± 1. In other words this is precisely removing the leaf and its only adjacent edge from G'.If G' were indeed a tree, the result of this removal and all subsequent removals should still be trees. In which we case we can inductively reason that $|\det B| = 1$ if G' where a tree
2. Now consider the case where G' has no vertices of degree one(in which case it has no leaves and can't be tree). We aim to show that this means $\det B = 0$. Since G' cant be a spanning tree it is possible for it to have a vertex of degree zero(which is basically an isolated vertex,meaning G' isn't even connected). In that case B has a zero row so $\det B = 0$. Now in any case every column of B contains one 1 and one -1 and the rest zeroes(since every edge by definition has one head and tail). But because by assumption there are no zero rows or rows with exactly one -1 or 1 it follows that every row must also have exactly one -1 and 1. In which case the linear sum of rows is zero. Hence the rows of B ar linearly dependent and $\det B = 0$

Having proving that the theorem follows from

$$\det A_0 A_0^T = \sum (\det B)^2$$

which is basically cauchy binet formula if you recall states

$$\det AB = \sum_{S \in \binom{[n]}{m}} \det(A_{[m],S}) \det(B_{S,[m]})$$

We see that our formula runs over all $(n-1) \times (n-1)$ sub-matrices B of A_0. Now these counts all spanning trees G' of B because the result proves that given this set of $n-1$ edges, no cycles were made with $V(G)/\{v_n\}$. That implies when removed row data of v_n is added back, the result is still a tree.This would not work if we "removed"(more accurately hidden) 3 rows instead. In which case adding back to the matrix with the same given set of edges, we are guaranteed that it does not cause cycles with the preexisting edges, but not with the 3 new edges(because you need minimally 3 elements to form a cycle) □

Now let us extend this to undirected graphs too.

Theorem 159 (Matrix Tree Theorem)
Let U be a simple undirected graph with $V(U) = \{v_1, \ldots, v_m\}$. Let L be the $(m-1) \times (m-1)$ matrix defined by

$$L_{ij} = \begin{cases} \deg v_i & i = j \\ -1 & i \neq j \text{ and } v_i v_j \in E(U) \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in [m-1]$. Then the number of spanning trees of U is $\det L$.

Proof. Let G be the directed graph that we obtain from U by replacing each edge of U by two arrows, one in each direction.We claim that $A_0A_0^T = 2L$. Set $M = A_0A_0^T$ and observe that

$$M_{ij} = A_{i1}A_{j1} + A_{i2}A_{j2} + \dots A_{in}A_{jn}$$

Notice that when $i = j$, then every edge that starts or ends at v_i contributes to 1 to this inner product. So we have

$$M_{ii} = \text{indeg}_G v_i + \text{outdeg}_G v_i = 2L_{ii}$$

we multiply L_{ii} by 2 because we must add in the other directions as mentioned above. Similarly if $i \neq j$ then every edge that starts at v_i and ends at v_j and every edge that starts at v_j and ends at v_i contributes to -1 to this inner product. Thus

$$M_{ij} = -2 = 2L_{ij}$$

if $v_iv_j \in E(U)$. Otherwise no edges connecting v_i and v_j in U or in G and so $M_{ij} = 0 = 2L_{ij}$. Hence $A_0A_0^T = 2L$. Then by linear algebra properties of determinants

$$2^{m-1} \det L = \det 2L = \det A_0A_0^T$$

see that $2L$ means each and every $m-1$ rows/columns of L are scaled by 2 so applying the rules of determinants which is to $\times 2$,$m-1$ times to $\det L$.

18 Bipartite Graphs

Definition 160

Let G be a simple graph. We say that G is **bipartite** if $V(G) = X \cup Y$ for some disjoint sets of vertices X and Y such that every edge of G connects a vertex of X with a vertex of Y.

X and Y are known as **parts** of a bipartite graph G and they are called **color classes**(this terminology is justified when we generalize bipartite graphs in our discussion of graph coloring)

Theorem 161

For a simple connected graph G the following conditions are equivalent

(a) G is bipartite

(b) Does not contain an odd length cycle

Proof. For $(a) \Rightarrow (b)$: Assume that G is bipartite on the parts X and Y. Suppose by contradiction that G has a cycle of odd length namely $C = v_1v_2 \dots v_{2n+1}v_1$. And that $v_1 \in X$. Then easily we see

$$v_1 \in X \rightarrow v_2 \in Y \rightarrow v_3 \in X \rightarrow v_4 \in Y \dots \underbrace{v_{2n+1} \in X \rightarrow v_1 \in X}$$

where we know ever $2k, k \in [\![0, n]\!]$ is in Y. But $v_{2n+1}, v_1 \in X$ is a contradiction.

For $(b) \Rightarrow (a)$: You can clearly construct a bipartite graph using an even length cycle say defined by an even path lenght function to decide if in X or Y.If no cycles at all you can very easily separate into 2 disjoint subsets with no contradictions too.

Corollary 162

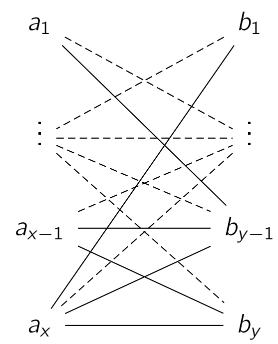
Every forest is bipartite

Proof. No cycles

Proposition 163

Let G be a simple bipartite graph on n vertices. Then $|E(G)| \leq \frac{n^2}{4}$ if n is even and $|E(G)| \leq \frac{n^2-1}{4}$ if n is odd

Proof. Suppose that G is bipartite on the parts X and Y and set $x = |X|$ and $y = |Y|$. It is clear that $|E(G)| \leq xy = x(n-x)$ Well, that is just the maximum number of edges you can form



Just enumerate through a_i knowing it each has y choices of b_i to connect to. Hence we have

$$M_n = \max\left\{xn - x^2 | x \in [\![0, n]\!]\right\}$$

However $xn - x^2$ is a concave parabola with vertex at $x = \frac{n}{2}$. So if $n = 2k$ for some $k \in \mathbb{N}_0$ then

$$M_n = M_{2k} = \frac{2k}{2}\left(2k - \frac{2k}{2}\right) = k^2 = \frac{n^2}{4}$$

On the other hand if $n = 2k+1$ for some $k \in \mathbb{N}_0$ then

$$M_n = M_{2k+1} = \lceil\frac{2k+1}{2}\rceil\left(2k+1-\lceil\frac{2k}{2}\rceil\right) = k(k+1) = \frac{n-1}{2}\frac{n+1}{2} = \frac{n^2-1}{4}$$

as desired

19 Matching and Hall's Theorem

Definition 164

Let G be a simple graph and let S be a subset of $E(G)$. If no two edges in S form a path then we say that S is a **matching** of G. A matching S of G is called a **perfect matching** if every vertex of G is covered by an edge of S

For a graph G and a subset T of $V(G)$ we let $N_G(T)$ denote the vertices of G that are adjacent to some vertex in T that is

$$N_G(T) = \{v \in V(G) | vw \in E(G) \text{ for some } w \in T\}$$

You could see this a neighbourhood of w.

Recall the following example from `MIT CS 6.042`

Example 165

Consider the following bipartite graph G where the set of men M make up $L(G)$ while the set of women W make up $R(G)$

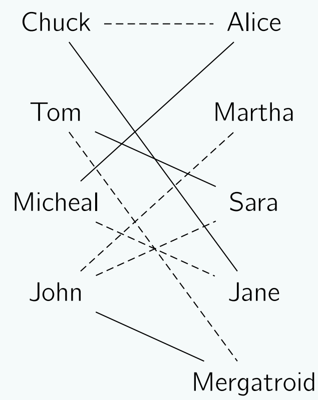


Figure 8: A graph where an edge between a man and woman denotes that the man likes the woman

- The bolded edges represent one possible matching
- This matching *covers* the set $V(G)$ − Mergatroid.
- This is not a *perfect* matching
- Notice that every subset of men likes at least as many women

Theorem 166 (Hall)

For a bipartite graph G on the parts X and Y the following conditions are equivalent

(a) There is a perfect matching of X into Y

(b) For each $T \subseteq X$ the inequality $|T| \leq |N_G(T)|$ holds

Proof. For $(a) \Rightarrow (b)$: Let S be a perfect matching of X into Y. As S is a perfect matching, for every $x \in X$ there exists a unique $y_x \in Y$ such that $xy_x \in S$. Define the map $f : X \to Y$ by $f(x) = y_x$. Since S is a matching the function f is injective. Therefre for any $T \subseteq X$ we see that $|T| = |f(T)| \leq |N_G(T)|$ because $f(T) \subseteq N_G(T)$

For $(b) \Rightarrow (a)$: Conversely suppose that $|T| \leq |N_G(T)|$ for each $T \subseteq X$. We will prove that there exists a perfect matching of X into Y by induction on $n = |X|$. When $n = 1$ this proposition is trivially true. Now assume that every bipartite graph on the parts X' and Y' with $|X'| < |X|$ and satisfying condition (b) has a perfect matching of X' into Y'. Then to prove the induction step we need to show that this implies there is a perfect matching of X into Y. We split the rest of the proof into two cases, one to handle subsets $|T| = |N_G(T)|$ and one to handle subsets $|T| < |N_G(T)|$.

(Case: 1) For every nonempty *proper* subset T of X, that is $T \subset X$ the strict inquality $|T| < |N_G(T)|$ holds. Take $x \in X$ and $y \in N_G(\{x\})$. Let G' be the bipartite graph we obtain by removing x and y(and the edges incident to them) from G. In other words we have successfully matched x with y so we kicked them out of the graph to help find matches for the other guys. Now for every subset A of $X/\{x\}$ we see that

$$|N_{G'}(A)| \geq |N_G(A)| - 1 \geq |A|$$

where the last inquality holds because A is a strict subset of $X(N_G(A) > |A|)$. Hence with the conditions that $|N_{G'}(A) \geq |A||$ for every subset A of $X/\{x\}$ and that $|X/\{x\}| < |X|$, by induction hypothesis there exists a perfect matching S' in G' of $X/\{x\}$ into $Y/\{y\}$. It is clear that $S' \cup \{xy\}$ is a perfect matching in G of X into Y(recall we have successfully matched them earlier)

(Case: 2) There exists a nonempty subset A of X such that $|A| = |N_G(A)|$. Let G_1 be the subgraph of G induced by the set of vertices $A \cup N_G(A)$ and G_2 be the result of removal of G_1(yes including all vertices $A \cup N_G(A)$ and its edges) from G. It is clear $G_1(A, N_G(A))$ and $G_2 = (X/A, Y/N_G(A))$ are bipartite graphs.

- To show that G_1 satisfies (b) take $T \subseteq A$. It follows by the way G_1 was constructed that $N_{G_1}(T) = N_G(T)$. Well any subset of A cant possibly like guys in $Y/N_G(A)$. Hence $|N_{G_1}(T)| = |N_G(T)| \geq |T|$. Then G_1 satisfies condition (b)
- To show that G_2 also satisfies (b) , take $T' \subseteq X/A$ and observe that

$$N_G(T \cup A) = N_G(A) \cup N_{G_2}(T')$$

because the union on the right-hand side is disjoint. Well $N_{G_2}(T')$ is restricted to only the possible matches $Y/N_G(A)$. Hence since

$$N_G(T' \cup A) \geq |T' \cup A| \text{ and } |N_G(A)| = |A|$$

Recall in our first sentence we have supposed that $|T| \leq |N_G(T)|$ for each $T \subseteq X$. Clearly $T' \cup A$ is a subset of X so this applies to it too. In which case combining the above we have

$$|N_{G_2}(T')| = |N_G(T' \cup A)| - |N_G(A)| \geq |T' \cup A| - |A| = (|T'| + |A|) - |A| = |T'|$$

Therefore G_2 also satisifies (b).

Since $|A| < |X|$(recall A is a non empty subset of X) and $|X/A| < |X|$ and having proven both G_1 and G_2 satisfy (b) our induction hypothesis guarantees the existence of a perfect matching S_1 in G_1 of A into $N_G(A)$ and a perfect matching S_2 in G_2 of X/A into $Y/N_G(A)$. Then it follows from the construction of G_1 and G_2 that $S_1 \cup S_2$ is a perfect matching in G of X into Y which concludes our proof.

$\square$

Definition 167

In a graph G a matching M is called **maximal** if we cabbit extend M by addinga new edge to it. A matching N is called a maximum if no mathcing of G contain more edges than N

Example 168

Consider

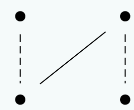


Figure 9: A maximal but not maximum matching

Definition 169

Let G be a graph and let M be a matching of G. A path $P = v_1, v_2 \dots v_\ell$ is called **M-alternating** provided that $v_{i-1}v_i \in M$ if and only if $v_i v_{i+1}$

20 Chromatic Numbers and Polynomials

Definition 170

For $k \in \mathbb{N}$ a **proper k-coloring** of a simple graph G is a (coloring) function $f : V(G) \to [k]$ such that no two adjacent vertices of G have the same image under f.

Essentially every map of f goes to a different color. No two adjacent vertices can have the same color. You can immediately see this is the generalization of the bipartite definition which corresponds a proper 2-colorable graph.

Definition 171

We say that a simple graph G is **k-colorable** if it admits a proper *k-coloring*

It is clear that if G is $k - colorable$ then it is *m-colorable* for every $m \geq k$. Well just combine colors into one to get that.

Definition 172

The **chromatic number** of a simple graph G denoted by $\chi(G)$ is the minimum among all $k \in \mathbb{N}$ such that G is k-colorable

Example 173

For $n \in \mathbb{N}$ let G be a graph satisfying $|V(G)| = n$ and $|E(G)| = 0$. Sinnce G does not have any adjacency relation we easily see that $\chi(G) = 1$. In this case a proper 1-coloring is given by the constand function $f : V(G) \to \{1\}$

Example 174

Suppose that G is a bipartite graph on the parts X and Y. If $E(G) = 0$ then it is just the case of the previous example. But otherwise we easily see that $\chi(G) = 2$. Hence we can color vertices in X black and Y white. which yield the proper 2-coloring $f : V(G) \to \{1, 2\}$ corresponding to the 2 colors.

Well this easily applies to trees as well. Recall a tree is bipartite as well. So every tree is 2-colorable. Another example will be the complete bipartite graphs(recall from above, it's the one with $m \times n$ edges) are also 2-colorable.

Example 175

Let us find the chromatic number of C_n for each $n \in \mathbb{N}$ with $n \geq 3$. If n is even then C_n is bipartite and so $\chi(C_n) = 2$. Suppose then that n is odd. Since C_n is not bipartite, $\chi(C_n) \geq 3$. Now write $n = 2k + 1$ for some $k \in \mathbb{N}$ and set $V(C_n) = \{v_1, v_2, \ldots, v_{2k+1}\}$ such that $v_1 v_2 \ldots v_{2k+1} v_1$ is the only cycle of C_n. To exhibit a proper 3-coloring of G color the vertices in $\{v_{2j-1} | j \in [k]\}$ blue, the vertices in $\{v_{2j} | j \in [k]\}$ green and the vertex 2_{2k+1} red. Hence $\chi(C_n) = 3$ whe n is odd

Basically same logic as bipartite. Just handle the remaining one element in the odd cycle by a separate color. In a sense this is a "tripartite" situation.

Example 176

For $n \in \mathbb{N}$ let K_n be the complete graph on $[n]$. Since any two distinct vertices of K_n are adjacent in order to have a proper coloring of K_n no two vertex can have the same color. From this observation it follows immediately that $\chi(K_n) = n$

Proposition 177

Let G be a simple graph with labelled vertices. For every $k \in \mathbb{N}$ let p_k denote the unmber of proper k-coloring of G. Then there exists a unique polyomial $p(x) \in \mathbb{Q}[x]$ such that $p(k) = p_k$ for every $k \in \mathbb{N}$

Proof. Set $n = |V(G)|$. For each $m \in \mathbb{N}$ let c_m denote the number of proper m-coloring of G that use all the m colors. Its is clear that $c_m = 0$ when $m > n$...to be continued

21 Brooks' theorem

Definition 178

For a simple graph G we let $\Delta(G)$ denote the maximum of all degrees of the vertices of G. A simple graph G is called **k-regular** if any two vertices of G have the same degree. That is $\deg v = \Delta(G)$ for every $v \in V(G)$

Example 179

A path graph P is regular if and only if its lenght is 1 in which case $\Delta(P)$. TI see this consider that if a path graph P has length at least 2 then it contains one vertex of at least degree 2

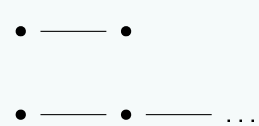


Figure 10: Path graph length 1 and length ≥ 2 respectively

Example 180

For every $n \geq 3$ the cycle graph C_n is 2-regular and so $\Delta(G) = 2$

Example 181

The complete graph K_n is the $(n-1)$ regular graph with n vertices. In this case $\Delta(K_n) = n - 1$

(recall above that every vertex in complete graph connected to $n - 1$ vertices)

The main objective of this section is to prove **Brooks' Theorem** which gives an upper bound for the chromatic number of simple connected graphs with two exceptions.

Definition 182

In simple graph G a **cut-vertex** $v \in V(G)$ is a vertex satisfying that $G/\{v\}$ has more connected components than G does.

Fact 183

Observe that $\chi(C_{2n+1}) = 3 = \Delta(C_{2n+1}) + 1$ and $\chi(K_n) = n = \Delta(K_n) + 1$. That is to say that $\chi(G) > \Delta(G)$ if G is an odd length cycle or complete graph.

The following theorem will prove that these are exceptions.Specifically, every other simple connected graph satisfies $\chi(G) \leq \Delta(G)$.

Theorem 184 (Brook)

Let G be a simple connected graph. If G is neither complete nor can odd cycle then $\chi(G) \leq \Delta(G)$

Proof. Set $k = \Delta(G)$.When $k = 2$, clearly since connected, G can only be either a path or a cycle. And specifically an even cycle because we assumed G is not an odd cycle. In that case $\chi(G) = \Delta(G) = 2$. So we now assume $k \geq 3$.

(Case 1) The graph G is not k-regular. This implies there exists a vertex $v_n \in V(G)$ with $\deg v_n \leq k - 1$. Since G is connected we can take a spanning tree T of G and label the vertices of G by $v_n, v_{n-1}, \ldots, v_2, v_1$, coloring first the vertices which are closer to v_n in T.(That is to say these labels are in decreasing order of distance from v_n as we travel the v_n rooted tree T by levels. Thus for any $j \in [n-1]$ the vertex j has at most $k - 1$ adjacent vertices to its left in the sequence $v_1, v_2, \ldots, v_n$ because v_j has at least one adjacent vertex in the subsequence $v_{j+1}, \ldots, v_n$ More precisely, the number of adjacent vertices(both bold and dotted lines considered) those that correspond to a level below have at most $k - 1$ since there is at least 1 adjacent corresponding to a level above given that the maximum degree of any vertex is k. Hence in every step, going from v_1 to v_n, we only need to pick a color for v_i not the same as the at most $k - 1$ adjacent vertices 1 level below.We don't need to consider the level above as it will considered when you enumerate to it. For example looking at 3, we just need to pick a color different from 1, no need to worry about 4 and 6. When we get to say 4, it will then pick a color different from 3 and 1. Hence it total there will only be k different colors for every element.

(Case 2) The graph G is k-regular

(Case 2.1) The graph G contains a cut vertex. Consider the subgraphs induced by $G'_i = G/G_i$(delete all other $G_j, j \neq i$ both edges and vertices except the cut vertex. Then we have the exact same scenario each time as case 1

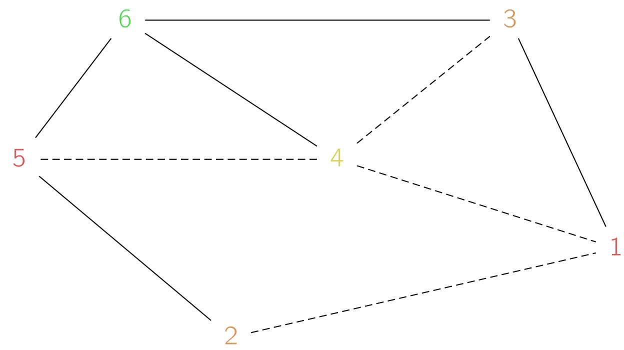


Figure 11: spanning tree in non 4-regular graph rooted at v_6

since each $\deg_{G'_j} v \leq k-1$. Now we may fix a color of v for all cases and find a coloring for the rest of the G_is.

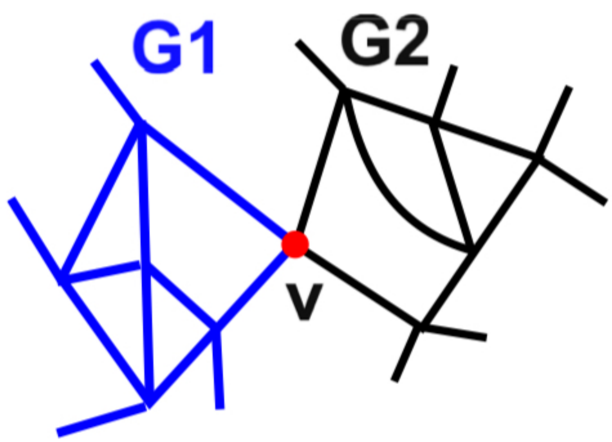


Figure 12: 4-regular Cut vertex graph

(Case 2.2) Now consider $G' = G\{v\}$.We split into two cases again.

(Case 2.2.1) The graph G' does not contain a cut vertex. Because G is k=regular and not complete v_1 cannot be adjacent to the rest of the vertices.So there must exist a vertex that is of at least path lenght 2 away from v_1. Well if there exsits such an element,then obviously there must exist a vertex of path lenght exactly 2 away from v_1(the intermediate elements in the path)

$$v_1 \to v_n \to v_2 \to \dots$$

We denote this exactly path lenghth 2 vertex to be v_2. That means there some element which we denote to be v_n in between them/adjacent to both of v_1, v_2 respectively. Now take a spanning tree rooted at v_n and color like in case 1. This is feasible even though v_n has k adjacent vertices because v_1 and v_2 can be given the same color since they are not adjacent from each other as gaurateed above. In which case v_n similar to case 1 can also choose a color different from $k-1$ possible colors from its adjacent vertices.

(Case 2.2.2) The graph G' does contain a cut vertex. This means if we delete 2 random vertices say v_1, v_2 then $G/\{v_1, v_2\}$ is disconnected. Essentially we have

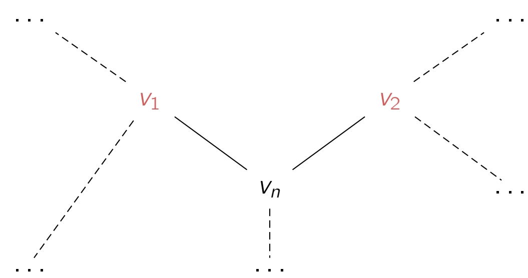


Figure 13: G' no cut vertex

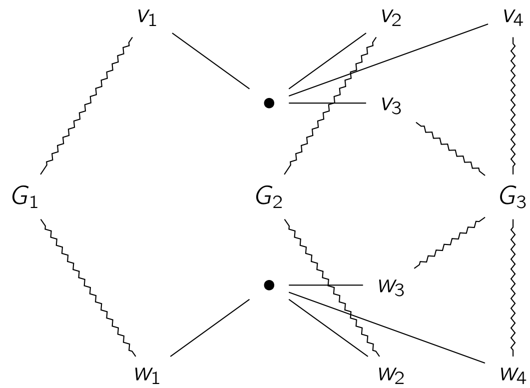


Figure 14: 2 Cut vertex

Notice that the jagged lines indicates $k-1$ adjacent vertices(there's already one between v_i/w_i with the cut vertex and every vertex has k adjacent vertices). By case 1 you may fix a color on say the top vertex then color the rest of the graph that doesn't include w_i and the bottom cut vertex.

Hence by induction any k-regular G no matter how many cut vertices or having no cut vertices are have their chromatic number bounded by their maximum degree as desired.

22 Intro to Lattices and Posets

Definition 185

A pair $(P, \leq)$ where P is a nonempty set and $\leq$ is a relation on P is called a **partially ordered set** or **poset** provided

1. Reflexivity
2. Anti-symmetry
3. Transitivity