MIT 2.071 Mechanics of Solid Materials

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Continuum mechanics and basically solid mechanics at the graduate level in general. Material sourced from professor `Rohan Abeyaratne`'s cousenotes volume I,II,III

Contents

1 matrix algebra and indicial notation

Problem 1
Show that any matrix $[A]$ can be additively decomposed into the sum of a symmetric matrix and a skew symmetric matrix

Solution. Define symmetric matrix $[S]$ and skew symmetric $[W]$ as follows

$$S_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) \quad \text{and} \quad W_{ij} = \frac{1}{2}(A_{ij} - A_{ji})$$

By this definition

$$S_{ij} = S_{ji} \quad \text{and} \quad W_{ij} = -W_{ji}$$

which shows that indeed S and W are symmetric and skew symmetric respectively. Upon addition we clearly see

$$A_{ij} = S_{ij} + W_{ij}$$

as desired

2 vectors and linear transformations

Definition 2
Let $\mathbb{E}_3$ be the 3D euclidean space Given two vectors $a, b \in \mathbb{E}_3$, their **tensor product** is the linear transformation

$$(a \otimes b)x = (x \cdot b)a$$

for all $x \in \mathbb{E}_3$

Problem 3

The components of a linear transformation A in an orthonormal basis $\{e_1, e_2, e_3\}$ are the unique real numbers A_{ij} defined by

$$Ae_j = \sum_{i=1}^{3} A_{ij} e_i$$

Show that the linear transformation A can be represented as

$$A = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} (e_i \otimes e_j)$$

Proof. done before in either `algebra I` or `tensor analysis`. But you can refer to professor `Rohan Abeyaratne`'s cousenotes and retex it out here for revision if you want

Problem 4

Let R be a rotation transformation that rotates vectors in $\mathbb{E}^3$(3D euclidean space) through an angle $\theta, 0 < \theta < \pi$ about an axis e(in the sense of the right-hand rule). Show that R can be represented as

$$R = e \otimes e + (e_1 \otimes e_1 + e_2 \otimes e_2) \cos\theta - (e_1 \otimes e_2 - e_2 \otimes e_1) \sin\theta$$

where e_1 and e_2 are any two mutually orthogonal vectors such that $\{e_1, e_2, e\}$ forms a right handed orthonormal basis for $\mathbb{E}_3$

Remark 5. *Recall from* `Algebra I` *that rotation matrices are proper orthogonal(that is it has determinant 1) and that orthogonal matrices in general have determinant* ± 1 *and they are unitary*

Solution. Firstly we know that our rotational transformation R must satisfy

1. Because a rotation preserves lengths:
$$|Rx| = |x|$$
2. From (1) and that rotation only changes the angle between Rx and x :
$$Rx \cdot x = |x|^2 \cos\theta$$
3. Since R rotates vectors about e its projection on e must remain unchanged
$$Rx \cdot e = x \cdot e$$
4. From (3) this also means
$$Re = e$$
5. Finally since the rotation should be orientation preserving
$$(x \times Rx) \cdot e > 0$$
since

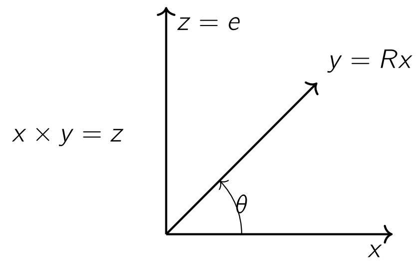


Figure 1: Right handed coordinate system

Where $\theta > 0$ corresponds to an anticlockwise rotation. Therefore considering

$$Re_1 = R_{11}e_1 + R_{21}e_2 + R_{31}e$$
$$Re_2 = R_{12}e_1 + R_{22}e_2 + R_{32}e$$
$$Re = R_{13}e_1 + R_{23}e_2 + R_{33}e$$

From (4) we know that

$$Re = R_{13}e_1 + R_{23}e_2 + R_{33}e = e$$

This implies

$$\boxed{R_{13} = R_{23} = 0, R_{33} = 1}$$

From (3) we know that

$$R_{31} = Re_1 \cdot e = e_1 \cdot e = 0 \quad \text{and} \quad R_{32} = Re_2 \cdot e = e_2 \cdot e = 0$$

Since e_1, e_2, e are orthonormal. Therefore

$$\boxed{R_{31} = R_{32} = 0}$$

From (2) we know that

$$R_{11} = Re_1 \cdot e_1 = |e_1|^2 \cos\theta = \cos\theta \quad \text{and} \quad R_{22} = Re_2 \cdot e_2 = |e_2|^2 \cos\theta = \cos\theta$$

Again since e_1, e_2, e are orthonormal so these basis vectors are of unit length. Therefore

$$\boxed{R_{11} = R_{22} = \cos\theta}$$

Putting everything we have so far together we have

$$Re_1 = \cos\theta e_1 + R_{21}e_2$$
$$Re_2 = R_{12}e_1 + \cos\theta e_2$$
$$Re = e$$

From (5) we know that

$$(e_1 \times Re_1) \cdot e = (R_{21}) > 0$$

and

$$(e_2 \times Re_2) \cdot e = -(R_{12}) > 0$$

But from (1) we know that

$$|Re_1| = |e_1| = 1 \quad \text{and}\, |Re_2| = |e_1| = 1$$

Therefore it follows that

$$\boxed{R_{21} = \sin\theta \quad \text{and} \quad R_{12} = -\sin\theta}$$

Hence altogether we have

$$\begin{aligned} Re_1 &= \cos\theta e_1 + \sin\theta e_2 \\ Re_2 &= -\sin\theta e_1 + \cos\theta e_2 \\ Re &= e \end{aligned}$$

as desired

Corollary 6

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark 7. *Note that because orientation and volume preserving, the determinant of all of the above is* 1

Proof. It follows directly from above. Recall that each column represents Re_1, Re_2, R_e etc. Then to ensure right-handed the column and row orders must follow the following permutation order

$$\begin{tikzcd} e_1 \arrow[r] & e_2 \arrow[r] & e \arrow[ll, bend right] \end{tikzcd}$$

$$\begin{tikzcd} Re_1 \arrow[r] & Re_2 \arrow[r] & Re \arrow[ll, bend left] \end{tikzcd}$$

Hence:

- let x be the rotation axis. Then $[e, Re_1, Re_2]$ corresponds to the column order while $[e, e_1, e_2]^T$ corresponds the row order.
- let y be the rotation axis. Then $[Re_2, e, Re_1]$ corresponds to the column order while $[e_2, e, e_1]^T$ corresponds the row order.
- let z be the rotation axis. Then $[Re_1, Re_2, e]$ corresponds to the column order while $[e_1, e_2, e]^T$ corresponds the row order.

In practice just memorize the columns of the R_z matrix then permuate the rows and columns for the rest accordingly.

Problem 8

If F is a nonsingular linear transformation show that $F^T F$ is symmetric and positive definite

Proof. done before in either `algebra I` or `tensor analysis`. But you can refer to professor `Rohan Abeyaratne`'s cousenotes and retex it out here for revision if you want

Problem 9

Consider a symmetric positive definite linear transformation S. Show that it has unique symmetric positive definite square root i.e there is a unique symmetric positive linear transformation T for which $T^2 = S$

Solution. Since S is symmetric and positive definite it has three real positive eigenvalues $\sigma_1, \sigma_2, \sigma_3$ with corresponding eigenvectors s_1, s_2, s_3 which may be taken to be orthonormal(recall **spectral theorem** for hermitian/symmetric matrices see `Artin Algebra`) Hence we know that

$$S = \sum_{i=1}^{3} \sigma_i (s_i \otimes s_i)$$

Now define

$$T = \sum_{i=1}^{3} \sqrt{\sigma_i} (s_i \otimes s_i)$$

Clearly T is symmetric and positive definite as well. Now, we prove the uniqueness of T. Suppose there is another symmetric positive definite linear transformation T' such that $T'^2 = S$. Since both T and T' are symmetric linear transformations, they are diagonalizable in the same orthonormal basis $\{s_1, s_2, s_3\}$ consisting of eigenvectors of S.

Let the eigenvalues of T be $\lambda_1 = \sqrt{\sigma_1}, \lambda_2 = \sqrt{\sigma_2}, \lambda_3 = \sqrt{\sigma_3}$, and let the eigenvalues of T' be $\lambda'_1, \lambda'_2, \lambda'_3$. Since $T'^2 = S$, we have

$$T'^2 = \sum_{i=1}^{3} (\lambda'_i)^2 (s_i \otimes s_i) = \sum_{i=1}^{3} \sigma_i (s_i \otimes s_i)$$

Thus, $(\lambda'_i)^2 = \sigma_i$, implying $\lambda'_i = \pm\sqrt{\sigma_i}$ for each i.

Since T' is positive definite, all $\lambda'_i > 0$, which forces $\lambda'_i = \sqrt{\sigma_i}$ for each i. Therefore, $T' = T$, proving the uniqueness of T.

Thus, the symmetric positive definite square root T of S is unique.

Problem 10 (Polar Decomposition Theorem)

If F is a non singular linear transformation show that there exist a unique positive definite symmetric linear transformation U and a unique orthogonal linear transformation R such that $F = RU$

Solution. It follows from above that $F^T F$ is symmetric and positive definite and that there exists a unique symmetric positive definite U such that

$$U = \sqrt{F^T F}$$

Since U is positive definite it is non singular so U^{-1} exists. Define the linear transformation R through

$$R = FU^{-1}$$

Now we show that R is orthogonal. This follows from

$$R^T R = (FU^{-1})^T (FU^{-1}) = (U^{-1})^T F^T F U^{-1} = U^{-1} U^2 U^{-1} = I$$

where the last equality used the fact that U^{-1} is also symmetric(its just the same as U but diagonal are reciprocals recall `algebra I`). Uniqueness follows if you follow the same procedure, suppose there is another R' where $R' = FU^{-1}$. But then since U^{-1} is unique, for the same F, $R = R'$.

Problem 11 (Polar decomposition theorem extension)
Besides $F = RU$ we also can write $F = VR$ where similarly R is proper orthogonal and V is symmetric and positive definite

Solution. Recall from `Algebra I` that rotation matrix R is unitary meaning $R^{-1} = R^T$(in particular proper orthogonal with determinant 1) and hence given symmetric A , V defined by

$$V = RUR^{-1} = RUR^T \quad \Rightarrow \quad U = R^{-1}VR$$

is symmetric as well. As for V being positive definite as well also recall from `Algebra I` it means we must prove

$$x^T V x > 0$$

for all $x \neq 0$. Then substituting the above into this expression we have

$$x^T (RUR^T) x$$

letting $R^T x = y$ we get altogether,

$$x^T V x = x^T (RUR^T) = y^T U y > 0$$

the inequality follows for all y since U is positive definite

Corollary 12
V and U have the same eigenvalues

Proof. Recall from `algebra I` that $V = RUR^{-1}$ implies they are *similar matrices* and hence have the same eigenvalues.

3 components of vectors and tensors: cartesian tensors

We discuss an important propert known as principal scalar invariants. Suppose S is symmetric, then recall from above(or below i think kinematics deformation section) that in **principal basis** of eigenvectors it takes the form

$$[S] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Definition 13

Next we define what we call the 3 **principal scalar invariants**

$$I_1(A) = \operatorname{tr} A$$
$$I_2(A) = \frac{1}{2}[(trA)^2 - tr(A^2)]$$
$$I_3(A) = \det A$$

Problem 14

Suppose that a, b, c are any three linearly independent vectors and that F be an arbitrary non-singular linear transformation. Show that

$$(Fa \times Fb) \cdot Fc = \det F(a \times b) \cdot c$$

Problem 15

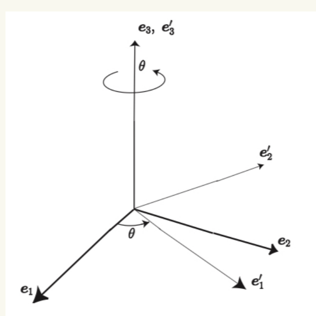
Determine the relationship between components v_i and v_i' of a vector v in two bases

Problem 16

Determine the relationship bewteen the components A_{ij} and A_{ij}' of a linear transformation A in two bases

Problem 17

Suppose that the basis $\{e_1', e_2', e_3'\}$ is obtained by the basis $\{e_1, e_2, e_3\}$ through an angle θ about the unit vector e_3. Write out the transformation rule for 2 tensor explcitly in this case



Solution. In view of the relationship between the two bases it follows that the base change matrix which is the rotation matrix R is given by

$$Re_1 = \cos\theta e_1 + \sin\theta e_2$$
$$Re_2 = -\sin\theta e_1 + \cos\theta e_2$$
$$Re = e$$

The matrix $[Q]$ which relates the two bases is defined by $Q_{ij} = e_i' \cdot e_j$(the standard basis is orthgonal), and so it follows that

$$[Q] = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Substituting this $[Q]$ into $[A'] = [Q][A][Q]^T$(because again rotation matrices are unitary) and multiplying out the matrices leads to the 9 equations

$$\begin{aligned}
A'_{11} &= \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + \frac{A_{12}+A_{21}}{2}\sin 2\theta, \\
A'_{12} &= \frac{A_{12}-A_{21}}{2} + \frac{A_{12}-A_{21}}{2}\cos 2\theta - \frac{A_{11}-A_{22}}{2}\sin 2\theta, \\
A'_{21} &= -\frac{A_{12}-A_{21}}{2} - \frac{A_{12}-A_{21}}{2}\cos 2\theta - \frac{A_{11}-A_{22}}{2}\sin 2\theta, \\
A'_{22} &= \frac{A_{11}+A_{22}}{2} - \frac{A_{11}-A_{22}}{2}\cos 2\theta - \frac{A_{12}+A_{21}}{2}\sin 2\theta, \\
A'_{13} &= A_{13}\cos\theta + A_{23}\sin\theta, \quad A'_{31} = A_{31}\cos\theta + A_{32}\sin\theta, \\
A'_{23} &= A_{23}\cos\theta - A_{13}\sin\theta, \quad A'_{32} = A_{32}\cos\theta - A_{31}\sin\theta, \\
A'_{33} &= A_{33}.
\end{aligned}$$

In the special case when $[A]$ is symmetric, and in addition $A_{13} = A_{23} = 0$(which also means $A_{31} = A_{32} = 0$ since symmetric) , these nine equations simplify to

$$\begin{aligned}
A'_{11} &= \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + A_{12}\sin 2\theta, \\
A'_{22} &= \frac{A_{11}+A_{22}}{2} - \frac{A_{11}-A_{22}}{2}\cos 2\theta - A_{12}\sin 2\theta, \\
A'_{12} &= -\frac{A_{11}-A_{22}}{2}\sin 2\theta + A_{12}\cos 2\theta,
\end{aligned}$$

together with $A'_{13} = A'_{23} = 0$ and $A'_{33} = A_{33}$. These are the well-known equations underlying the **Mohr's circle** for transforming 2-tensors in two-dimensions. This is because we can let A be plane stress tensor

$$[Q] = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

where $\tau_{xy} = \tau_{yx}$(the reason cauchy plane stress tensor being symmetric will be discussed later below)

Problem 18

If W is a skew symmetric tensor show that there is a vector such that $Wx = w \times x$ for all $x \in \mathbb{E}$

Remark 19. *We have used a more elementary proof in your* `tensor analysis` *notes. But we use a more advanced proof here so that can directly express our axial vector in tensorial components*

Solution. Let W_{ij} be the components of W of some basis and let w be a vector whose components in this basis is defined by

$$w_i = -\frac{1}{2}\varepsilon_{ijk}W_{jk}$$

We claim that w is precisely the vector that satisfies this. Multiply both sides of the preceding equation by ε_{ipq}. Recall from `tensor analysis` that $\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$. Hence we have

$$\varepsilon_{ipq}w_i = -\frac{1}{2}(\delta_{jp}\delta_{kq})$$

Problem 20

If A is a tensor such that

$$Ax \cdot x = 0 \quad \forall x$$

show that A is necessarily skew symmetric

Problem 21

For any linear transformation A show that $\det(A - \mu) = \det(Q^T AQ - \mu I)$ for all orthgonal linear transformatoins Q and all scalars μ

Remark 22. *The concepts covered in this problem should be a revision to you having covered it in* `Algebra I`*. This is simply a special case because we are able to use orthogonal matrices which are matrices so we can interchange the usual inverse with transpose on the base change matirx*

Solution. This follows from

$$\det(Q^T AQ - \mu I) = \det(Q^T AQ - \mu Q^T Q = \det(Q^T(A - \mu I)Q) = \det Q^T \det(A - \mu I)\det Q = \det(A - \mu I)$$

Therefore the eigenvalues of $Q^T AQ$ are the same as as those of Q. Then it follows that since the determinant is the product of eigenvalues while trace is the sum,

$$\det(Q^T AQ) = \det A \quad \text{and} \quad \operatorname{tr}(Q^T AQ) = \operatorname{tr} A$$

Remark 23. *Note to be precise we have been considering* **linear operators** *not "transformations" since we are considering linear maps involving the same vector space. If not our base change matrices will look like $A' = Q^{-1}AP$ not $A' = P^{-1}AP$(recall* `Algebra I`*)*

Problem 24

Define a scalar-valued function $\phi(A; e_1, e_2, e_3)$ for all linear transformations A(not necessarily orthonormal) bases $\{e_1, e_2, e_3\}$ by

$$\phi(A; e_1, e_2, e_3) = \frac{Ae_1 \cdot (e_2 \times e_3) + e_1 \cdot (Ae_2 \times e_3) + e_1 \cdot (e_2 \times Ae_3)}{e_1 \cdot (e_2 \times e_3)}$$

Show that its value is independet of basis and that $\phi(A)$ is **scalar invariant** of A

Remark 25. *In fact this corresponds to $I_1(A)$ of the 3 principal scalar invariants defined earlier. In fact you can even see this intuitively. Notice that $Ae_i = \sum_j A_{ij}e_j$ then move the scalar A_{ij} out and realize only $e_1 \cdot (e_2 \times e_3)$ terms will be non zero.*

Proof. We want to evaluate $\phi(A; e_1, e_2, e_3)$ in the general basis $\{e_1, e_2, e_3\}$. Let's consider each term of the numerator separately.

Express Ae_1 in the general basis as:

$$Ae_1 = A_{11}e_1 + A_{21}e_2 + A_{31}e_3.$$

Which is essentially the first column. The dot product with $e_2 \times e_3$ gives:

$$Ae_1 \cdot (e_2 \times e_3) = A_{11}e_1 \cdot (e_2 \times e_3) + A_{21}e_2 \cdot (e_2 \times e_3) + A_{31}e_3 \cdot (e_2 \times e_3).$$

Using the fact that $e_2 \cdot (e_2 \times e_3) = 0$ and $e_3 \cdot (e_2 \times e_3) = 0$, we get:

$$Ae_1 \cdot (e_2 \times e_3) = A_{11}e_1 \cdot (e_2 \times e_3).$$

Express Ae_2 in the general basis as:

$$Ae_2 = A_{12}e_1 + A_{22}e_2 + A_{32}e_3.$$

Then:

$$e_1 \cdot (Ae_2 \times e_3) = A_{12}e_1 \cdot (e_1 \times e_3) + A_{22}e_1 \cdot (e_2 \times e_3) + A_{32}e_1 \cdot (e_3 \times e_3).$$

Since $e_1 \cdot (e_1 \times e_3) = 0$ and $e_1 \cdot (e_3 \times e_3) = 0$, we get:

$$e_1 \cdot (Ae_2 \times e_3) = A_{22}e_1 \cdot (e_2 \times e_3).$$

Similarly, express Ae_3 as:

$$Ae_3 = A_{13}e_1 + A_{23}e_2 + A_{33}e_3.$$

Then:

$$e_1 \cdot (e_2 \times Ae_3) = A_{13}e_1 \cdot (e_2 \times e_1) + A_{23}e_1 \cdot (e_2 \times e_2) + A_{33}e_1 \cdot (e_2 \times e_3).$$

Since $e_1 \cdot (e_2 \times e_1) = 0$ and $e_1 \cdot (e_2 \times e_2) = 0$, we get:

$$e_1 \cdot (e_2 \times Ae_3) = A_{33}e_1 \cdot (e_2 \times e_3).$$

Now, summing up the three terms in the numerator of $\phi(A; e_1, e_2, e_3)$, we get:

$$\text{Numerator} = (A_{11} + A_{22} + A_{33})e_1 \cdot (e_2 \times e_3).$$

The denominator is:

$$\text{Denominator} = e_1 \cdot (e_2 \times e_3).$$

Thus, we have:

$$\phi(A; e_1, e_2, e_3) = \frac{(A_{11} + A_{22} + A_{33})e_1 \cdot (e_2 \times e_3)}{e_1 \cdot (e_2 \times e_3)} = A_{11} + A_{22} + A_{33}.$$

This is exactly the trace of A, since $A_{11} + A_{22} + A_{33}$ represents the sum of the diagonal elements of the matrix representation of A.

We have shown that for any (not necessarily orthonormal) basis $\{e_1, e_2, e_3\}$, the scalar function $\phi(A; e_1, e_2, e_3)$ equals the trace of A. Therefore:

$$\phi(A; e_1, e_2, e_3) = \mathrm{tr}(A)$$

for any basis.

Problem 26 (Jacobi Formula)

Let $\mathbf{F(t)}$ be a one parameter family of non singular 2 tensors that depends smoothly on the paramter t. Calculate

$$\frac{d}{dt} \det \mathbf{F}(t)$$

Solution. recall from previously that

$$(\mathbf{F}(t)\mathbf{a} \times \mathbf{F}(t)\mathbf{b}) \cdot \mathbf{F}(t)\mathbf{c} = \det \mathbf{F}(t)\,(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

for any three linearly-independent constant vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. Differentiating this with respect to t gives

$$(\dot{\mathbf{F}}(t)\mathbf{a} \times \mathbf{F}(t)\mathbf{b}) \cdot \mathbf{F}(t)\mathbf{c} + (\mathbf{F}(t)\mathbf{a} \times \dot{\mathbf{F}}(t)\mathbf{b}) \cdot \mathbf{F}(t)\mathbf{c} + (\mathbf{F}(t)\mathbf{a} \times \mathbf{F}(t)\mathbf{b}) \cdot \dot{\mathbf{F}}(t)\mathbf{c} = \frac{d}{dt} \det \mathbf{F}(t)\,(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

where we have set $\dot{\mathbf{F}}(t) = \frac{dF}{dt}$. Suppressing the argument t and replacing $\dot{\mathbf{F}}$ by $\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{F}$ allows this to be written as

$$(\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{F}\mathbf{a} \times \mathbf{F}\mathbf{b}) \cdot \mathbf{F}\mathbf{c} + (\mathbf{F}\mathbf{a} \times \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{F}\mathbf{b}) \cdot \mathbf{F}\mathbf{c} + (\mathbf{F}\mathbf{a} \times \mathbf{F}\mathbf{b}) \cdot \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{F}\mathbf{c} = \left(\frac{d}{dt} \det \mathbf{F}\right) (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Using the result from the previous problem with $\mathbf{A} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ simplifies this to

$$\mathrm{trace}(\dot{\mathbf{F}}\mathbf{F}^{-1})(\mathbf{F}\mathbf{a} \times \mathbf{F}\mathbf{b}) \cdot \mathbf{F}\mathbf{c} = \left(\frac{d}{dt} \det \mathbf{F}\right) (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

then we get

$$\mathrm{trace}(\dot{\mathbf{F}}\mathbf{F}^{-1}) \det \mathbf{F}\,(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \left(\frac{d}{dt} \det \mathbf{F}\right) (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

or

$$\frac{d}{dt} \det \mathbf{F} = \mathrm{trace}(\dot{\mathbf{F}}\mathbf{F}^{-1}) \det \mathbf{F}.$$

4 Calculus of Vector and Tensor Fields

Fact 27

Notation matters ugh. Consider a scalar field $\phi(x)$ and vector v

$$\phi_{,i} = \frac{\partial \phi}{\partial x_i}, \quad \phi_{,ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}, \quad v_{i,j} = \frac{\partial v_i}{\partial x_j}$$

Example 28

$$(\mathrm{grad}\,\phi)_i = \phi_{,i}$$

so that

$$\mathrm{grad}\,\phi = \phi_{,i} e_i$$

Example 29

$$(\mathrm{grad}\,v)_{ij} = v_{i,j}$$

so that

$$(\mathrm{grad}\,v) = v_{i,j} e_i \otimes e_j$$

Problem 30

Let $\phi(x)$, $u(x)$, $A(x)$ be scalar,vector and 2 tensor fields respectively. Prove

1. $\mathrm{div}(\phi u) = u \cdot \mathrm{grad}\,\phi + \phi\,\mathrm{div}\,u$
2. $\mathrm{grad}(\phi u) = u \otimes \mathrm{grad}\,\phi + \phi\,\mathrm{grad}\,u$
3. $\mathrm{div}(\phi A) = A\,\mathrm{grad}\,\phi + \phi\,\mathrm{div}\,A$

5 Orthogonal Curvilinear Coordinates

Remark 31. *We denote local curvilinear coordinate system and all components and quantities associated with t by similar symbols with "hats" over them*

Example 32

Fixed cartesian coordinate system symbols:

$$x_i, \mathbf{e}_i, f(x_1, x_2, x_3)$$

Local curvilinear coordinate system symbols:

$$\hat{x}_i, \hat{\mathbf{e}}_i, \hat{f}(\hat{x}_1, \hat{x}_2, \hat{x}_3)$$

We introduce curvilinear cooridinates $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ through a triplet of scalar mappings

$$x_i = x_i(\hat{x}_1, \hat{x}_2, \hat{x}_3)$$

Each curvilinear coordinate $\hat{x}_i$ belongs to some linear interval $\mathcal{L}_i$ and $\hat{R} = \mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3$. We assume further this mapping is one to one and sufficiently smooth in the interior in the interior of $\mathcal{R}$ so that by inverse function theorem there exists a local neighbourhood where the inverse mapping

$$\hat{x}_i = \hat{x}_i(x_1, x_2, x_3)$$

Definition 33

The **jacobian** matrix $[J]$ of has elements

$$J_{ij} = \frac{\partial x_i}{\partial \hat{x}_j}$$

Putting the above together the position vector denoted by

$$\mathbf{x} = \mathbf{x}(\hat{x}_1, \hat{x}_2, \hat{x}_3)$$

has the normalized tangent space basis vectors

$$\hat{\mathbf{e}}_i = \frac{1}{|\partial \mathbf{x}/\partial \hat{x}_i|} \frac{\partial \mathbf{x}}{\partial \hat{x}_i}$$

Remark 34. *Correctly speaking, we are taking our local curvilinear coordinates to have the basis of the* **tangent bundle***(recall riemannian manifolds) in* $\mathbb{R}^3$

Definition 35

We denote g_{ij} to be the metric coefficients

$$g_{ij} = \frac{\partial \mathbf{x}}{\partial \hat{x}_i} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_j} = \frac{\partial x_k}{\partial \hat{x}_i} \frac{\partial x_k}{\partial \hat{x}_j}$$

Remark 36. *The metric matrix is not the same as the jacobian matrix. Rather it is the matrix of form.*

Where the extreme RHS using summation convention means enumerating over all k.(recall how the jacobian looks like, notice that the components of $\mathbf{x}$ go to the output i.e the rows. Now dot product returns a scalar which sums the product of same components) Also notice that $g_{ij} = 0$ if $i \neq j$ since we have restricted ourselves to *orthogonal curvilinear systems* specifically cylinderical and spherical(recall how in `vector analysis` we have explicitly calculated the metric coefficients and found that only the diagonal are non-zero). Explicitly this means

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0 \quad \text{or} \quad \frac{\partial \mathbf{x}}{\partial \hat{x}_i} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_j} = 0 \quad \text{or} \quad \frac{\partial x_k}{\partial \hat{x}_i} \frac{\partial x_k}{\partial \hat{x}_j} = 0$$

Remark 37. *Not all curvilinear systems are orthgonal, meaning there are off-diagonal elements that are non-zero. In those cases we cannot apply the following analysis that follows after this because in what follows you will see that we have equated the elements* g_{ij} *to resemble a kronecker delta function* δ_{ij} *which only makes sense if orthogonal obviously.*

Remark 38. $\frac{\partial \mathbf{x}}{\partial \hat{x}_i}$ *returns a vector while* $\frac{\partial x_k}{\partial \hat{x}_i}$ *returns a scalar. Well just look the numerator which decides the output and* 41 *below*

Therefore this is say in our discussion of *orthogonal curvilinear systems* specifically cylinderical and spherical our metric coefficients take the form

$$[g] = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$$

and that

$$h_i = \sqrt{g_{ii}} = \sqrt{\frac{\partial x_k}{\partial \hat{x}_i}\frac{\partial x_k}{\partial \hat{x}_i}} = \sqrt{\left(\frac{\partial x_1}{\partial \hat{x}_i}\right)^2 + \left(\frac{\partial x_2}{\partial \hat{x}_i}\right)^2 + \left(\frac{\partial x_3}{\partial \hat{x}_i}\right)^2}$$

Definition 39

We denote the scale moduli to be the magnitude

$$h_{\underline{i}} = |\partial \mathbf{x} / \partial \hat{x}_i|$$

combining our existing results we may therefore rewrite the above expressions as

$$g_{ij} = h_{\underline{i}} h_{\underline{j}} \delta_{ij}$$

and

$$\hat{\mathbf{e}}_i = \frac{1}{h_{\underline{i}}}\frac{\partial \mathbf{x}}{\partial \hat{x}_i}$$

Proposition 40

$$\frac{\partial x_j}{\partial \hat{x}_m} = g_{km}\frac{\partial \hat{x}_k}{\partial x_j} = h_{\underline{m}}^2 \frac{\partial \hat{x}_m}{\partial x_j}$$

Proof. from chain rule we know

$$\frac{\partial x_j}{\partial \hat{x}_k}\frac{\partial \hat{x}_k}{\partial x_j} = \delta_{ij}$$

Multiplying this by $\frac{\partial x_i}{\partial \hat{x}_m}$ we have

$$\frac{\partial x_i}{\partial \hat{x}_m}\frac{\partial x_i}{\partial \hat{x}_k}\frac{\partial \hat{x}_k}{\partial x_j}$$

then noticing the first 2 terms we have

$$g_{km}\frac{\partial \hat{x}_k}{\partial x_j} = h_{\underline{k}} h_{\underline{m}} \delta_{km}\frac{\partial \hat{x}_k}{\partial x_j}$$

in view of the kronecker delta function this is clearly equal to

$$h_{\underline{m}}^2 \frac{\partial \hat{x}_m}{\partial x_j}$$

where you replace all k by m essentially as they are the only non zero terms

From the LHS we also know

$$\frac{\partial x_j}{\partial \hat{x}_k}\frac{\partial \hat{x}_k}{\partial x_i} = \delta_{ji} = \delta_{ij}$$

Multiplying the LHS by $\frac{\partial x_i}{\partial \hat{x}_m}$ we have

$$\frac{\partial x_j}{\partial \hat{x}_k}\frac{\partial \hat{x}_k}{\partial x_i}\frac{\partial x_i}{\partial \hat{x}_m} = \frac{\partial x_j}{\partial \hat{x}_m}$$

then the proposition follows

Lemma 41

$$\frac{\partial \mathbf{x}}{\partial x_i} = \mathbf{e}_i$$

Proof. Recall that $x_i = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ is a coordinate function(which is a scalar field). Notice, using summation convention to express the position vector

$$\mathbf{x} = x_i \mathbf{e}_i$$

Recall differentiation of vector valued functions

$$\frac{\partial \mathbf{x}}{\partial x_i} = \frac{d}{dx_i} x_1 \mathbf{e}_1 + \ldots \frac{d}{dx_i} x_i \mathbf{e}_i \ldots + \frac{d}{dx_i} x_n \mathbf{e}_n$$

clearly the since the other $x_j, j \neq i$ are just treated as constants the lemma follows

Remark 42. *now you finally know the motivation behind the "wierd" vector field*

$$\frac{\partial}{\partial x_i}(p) = \mathbf{e}_i$$

well that is because p here is a point which you may treat as a "position vector" as above

Proposition 43

$$\hat{\mathbf{e}}_i \cdot \mathbf{e}_j = \frac{1}{h_{\underline{i}}} \frac{\partial \mathbf{x}}{\partial \hat{x}_i} \cdot \frac{\partial \mathbf{x}}{\partial x_j} = \frac{1}{h_{\underline{i}}} \frac{\partial x_j}{\partial \hat{x}_i}$$

Proof. Since $\frac{\partial \mathbf{x}}{\partial x_j} = \mathbf{e}_j$ then we have

$$\hat{\mathbf{e}}_i \cdot \mathbf{e}_j = \frac{1}{h_{\underline{i}}} \frac{\partial \mathbf{x}}{\partial \hat{x}_i} \cdot \frac{\partial \mathbf{x}}{\partial x_j} = \frac{1}{h_{\underline{i}}} \frac{\partial x_k}{\partial \hat{x}_i} \frac{\partial x_k}{\partial x_j} = \frac{1}{h_{\underline{i}}} \frac{\partial x_k}{\partial \hat{x}_i} \delta_{kj} = \frac{1}{h_{\underline{i}}} \frac{\partial x_j}{\partial \hat{x}_i}$$

since $\frac{\partial x_k}{\partial x_j} = \delta_{kj}$ so clearly the proposition follows

alternatively notice that

$$\hat{\mathbf{e}}_i \cdot \mathbf{e}_j = \frac{1}{h_{\underline{i}}} \frac{\partial \mathbf{x}}{\partial \hat{x}_i} \cdot \mathbf{e}_j = \frac{1}{h_{\underline{i}}} \frac{\partial x_j}{\partial \hat{x}_i}$$

because recall $Df \cdot e_i$ in munkres, this essentially selects the ith component of the output so we have the last equality ☐

Now compare our result with 40 we see that...

Let us define a matrix of the form relating the inner product of $\mathbf{e}_j$ with $\hat{\mathbf{e}}_i$ which we denote to $[Q]$. Well clearly by definition of euclidean dot product it will be symmetrical.From the above we have

$$Q_{ij} = \hat{\mathbf{e}}_i \cdot \mathbf{e}_j = \frac{1}{h_{\underline{i}}} \frac{\partial x_j}{\partial \hat{x}_i}$$

5.1 inverse partial deriatives

5.2 transformation of components in local basis of curivlinear coordinates

For the first derivative from 39 we have

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial \hat{x}_j} \cdot \hat{\mathbf{e}}_k = \left\{ -\frac{1}{h_{\underline{i}}^2} \frac{\partial h_i}{\partial \hat{x}_j} \frac{\partial \mathbf{x}}{\partial \hat{x}_i} + \frac{1}{h_{\underline{i}}} \frac{\partial^2 \mathbf{x}}{\partial \hat{x}_i \partial \hat{x}_j} \right\} \cdot \underbrace{\frac{1}{h_{\underline{k}}} \frac{\partial \mathbf{x}}{\partial \hat{x}_k}}_{\hat{\mathbf{e}}_k}$$

while the term in the curly brackets follows from product rule of differentiation. ☐

Corollary 44

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial \hat{x}_j} \cdot \hat{\mathbf{e}}_k = -\frac{\delta_{ik}}{h_{\underline{i}}} \frac{\partial h_i}{\partial \hat{x}_j} + \frac{1}{h_{\underline{i}} h_{\underline{k}}} \frac{\partial^2 \mathbf{x}}{\partial \hat{x}_i \partial \hat{x}_k} \cdot \frac{\partial \mathbf{x}}{\partial x_k}$$

Proof. Notice that dot product is distributive so applying it to the second term we see it how it follows. As for the firt term see that

$$\begin{aligned} -\frac{1}{h_{\underline{i}}^2} \frac{\partial h_i}{\partial \hat{x}_j} \frac{\partial \mathbf{x}}{\partial \hat{x}_i} \cdot \frac{1}{h_{\underline{k}}} \frac{\partial \mathbf{x}}{\partial \hat{x}_k} &= -\frac{1}{h_{\underline{i}}^2} \frac{1}{h_{\underline{k}}} \frac{\partial h_i}{\partial \hat{x}_j} \frac{\partial \mathbf{x}}{\partial \hat{x}_i} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_k} \\ &= -\frac{1}{h_{\underline{i}}^2} \frac{1}{h_{\underline{k}}} \frac{\partial h_i}{\partial \hat{x}_j} \delta_{ik} h_{\underline{i}} h_{\underline{k}} \\ &= -\frac{\delta_{ik}}{h_{\underline{i}}} \frac{\partial h_i}{\partial \hat{x}_j} \end{aligned}$$

as desired □

Now for the second dierivative first differentiate

$$\frac{\partial \mathbf{x}}{\partial \hat{x}_i} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_k} = g_{ij} = \delta_{ij} h_{\underline{i}} h_{\underline{j}}$$

with respect to $\hat{x}_k$ where we then obtain by multivariate chain rule on dot products,

$$\frac{\partial^2 \mathbf{x}}{\partial \hat{x}_i \partial \hat{x}_k} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_j} + \frac{\partial^2 \mathbf{x}}{\partial \hat{x}_j \partial \hat{x}_k} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_i} = \delta_{ij} \frac{\partial}{\partial \hat{x}_k} (h_{\underline{i}} h_{\underline{j}})$$

which is basically product rule on the LHS and noticing that δ_{ij} is a constant with respect to our variable on the RHS. Now permutnig (i, j, k) we have swapping i,k

$$\frac{\partial^2 \mathbf{x}}{\partial \hat{x}_k \partial \hat{x}_i} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_j} + \frac{\partial^2 \mathbf{x}}{\partial \hat{x}_j \partial \hat{x}_i} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_k} = \delta_{kj} \frac{\partial}{\partial \hat{x}_i} (h_{\underline{k}} h_{\underline{j}})$$

and swapping j,k

$$\frac{\partial^2 \mathbf{x}}{\partial \hat{x}_i \partial \hat{x}_j} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_k} + \frac{\partial^2 \mathbf{x}}{\partial \hat{x}_k \partial \hat{x}_j} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_i} = \delta_{ik} \frac{\partial}{\partial \hat{x}_j} (h_{\underline{i}} h_{\underline{k}})$$

too. These 3 equations be solved simulataneously to obtain

$$\frac{\partial^2 \mathbf{x}}{\partial \hat{x}_i \partial \hat{x}_j} \cdot \frac{\partial \mathbf{x}}{\partial \hat{x}_k} = \frac{1}{2} \left\{ \delta_{jk} \frac{\partial}{\partial \hat{x}_i} (h_{\underline{j}} h_{\underline{k}}) + \delta_{ki} \frac{\partial}{\partial \hat{x}_j} (h_{\underline{k}} h_{\underline{i}}) - \delta_{ij} \frac{\partial}{\partial \hat{x}_k} (h_{\underline{i}} h_{\underline{j}}) \right\}$$

Now sub this result into 44 we obtain

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial \hat{x}_j} \cdot \hat{\mathbf{e}}_k = -\frac{\delta_{ik}}{h_{\underline{i}}} \frac{\partial h_i}{\partial \hat{x}_j} + \frac{1}{h_{\underline{i}} h_{\underline{k}}} \frac{1}{2} \left\{ \delta_{jk} \frac{\partial}{\partial \hat{x}_i} (h_{\underline{j}} h_{\underline{k}}) + \delta_{ki} \frac{\partial}{\partial \hat{x}_j} (h_{\underline{k}} h_{\underline{i}}) - \delta_{ij} \frac{\partial}{\partial \hat{x}_k} (h_{\underline{i}} h_{\underline{j}}) \right\}$$

Now see that

5.3 gradient of a scalar field

Definition 45

Let $\phi(\mathbf{x})$ be a scalar valued function and let $v(\mathbf{x})$ denote its gradient

The components of $\mathbf{v}$ in the two bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. Well by definition of gradient of a scalar field we want

$$\hat{v}_k = \frac{\partial \hat{\phi}}{\partial \hat{\mathbf{e}}_i} = \frac{1}{}$$

5.4 gradient of a vector field

5.5 divergence of a vector field

5.6 laplacian of a scalar field

5.7 curl of a vector field

6 some preliminary notions

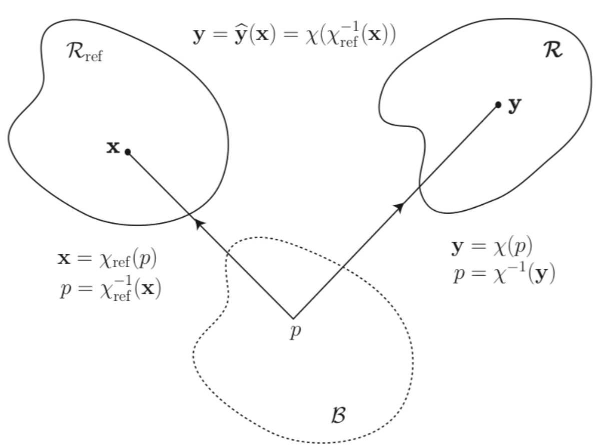
Definition 46

Note the following terms

- A **body** $\mathcal{B}$ is a collection of elements which can be put into one to one correspsnedence with some region $\mathcal{R}$ of a euclidean point space
- the set of all points in space corresponding to the locations of all particles is the **region** $\mathcal{R}$ occupied by the body in that configuration
- an element $p \in \mathcal{B}$ is called a **particle**
- A mapping χ that takes particles $p \in \mathcal{B}$ into their geometric locations $y \in \mathcal{R}$ in a 3D euclidean space is called a **configuration**

Definition 47

Let χ_{ref} and χ be two configurations of body $\mathcal{B}$ and let $\mathcal{R}_{\text{ref}}$ and $\mathcal{R}$ denote the regions occupied by body $\mathcal{B}$ in these two configurations.



The mapping $\hat{y}(x) : \mathcal{R}_{\text{ref}} \to \mathcal{R}$ defined by

$$\hat{y}(x) = \chi(\chi_{\text{ref}}^{-1}(x)), \quad x \in \mathcal{R}_{\text{ref}}, y \in \mathcal{R}$$

is called a **deformation** of the body from the reference configuaration χ_{ref}

The temperature θ of particle p in the configuration χ is given by

$$\theta = \theta_*(p)$$

where $p \in \mathcal{B}$. This basically deals with abstract particles and is known as the **material description**. And

$$\theta = \overline{\theta}(y) = \theta_*(\chi^{-1}(y))$$

for $y \in \mathcal{R}$ which basically now deals with the positions of the particles in the deformed configuration and is called the **eulerian** or **spatial** description.

If a reference configuration has been inroduced we can again rewrite the temperature of a particle but this time in terms of $x = \chi_{\text{ref}}(p)$ where we denote

$$\theta = \hat{\theta}(x) = \theta_*(\chi_{\text{ref}}^{-1}(x))$$

This is what we call the **Lagrangian** or **referential** form. Note that we could also have

$$\hat{\theta}(x) = \overline{\theta}(\hat{y}(x)) = \theta_*(p) \tag{1}$$

where recall $\hat{y}(x) = \chi(\chi_{\text{ref}}^{-1})$ is the deformation of the body from the reference configuration χ_{ref}

Definition 48

We define the **langarian spatial gradient** and **eulerian spatial gradient** respectivley by

$$\text{Grad}\theta = \nabla\hat{\theta}(x) \quad \text{and} \quad \text{grad}\,\theta = \nabla\overline{\theta}(y)$$

where the former has components $\frac{\partial\hat{\theta}}{\partial x_i}(x)$ and the latter $\frac{\partial\overline{\theta}}{\partial y_i}(y)$

We similarly define Curl vs curl and Div vs div. Now to relate them,differentiating both sides of 1 we have

$$\frac{\partial\hat{\theta}}{\partial x_i} = \frac{\partial\overline{\theta}}{\partial y_j}\frac{\partial\hat{y}_j}{\partial x_i} = \frac{\partial\overline{\theta}}{\partial y_j}F_{ji}$$

Hence we have for scalar field θ

$$\text{Grad}\theta = \mathbf{F}^T\,\text{grad}\,\theta$$

Similarly if $\mathbf{w}$ is any vector field we have

$$\text{Grad}\mathbf{w} = (\text{grad}\,\mathbf{w})\mathbf{F}$$

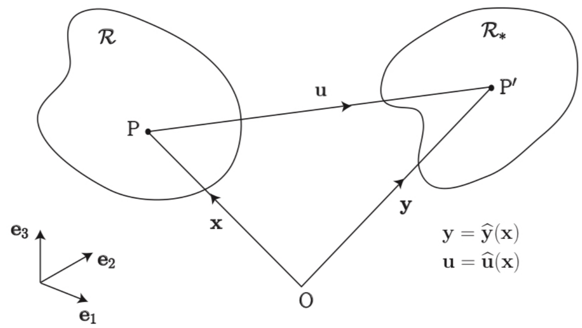
and for any tensor field $\mathbf{T}$ that

$$\text{Div}\mathbf{T} = J\,\text{div}(J^{-1}\mathbf{F}\mathbf{T})$$

where $J = \det\mathbf{F}$ and equivalently we have

7 kinematics: deformation

Consider now



Definition 49

The **displacement** vector field $\hat{u}(x)$ is defined on $\mathcal{R}_0$ by

$$u = y - x$$

which in terms of $x \in \mathcal{R}$ is

$$\hat{u}(x) = \hat{y}(x) - x$$

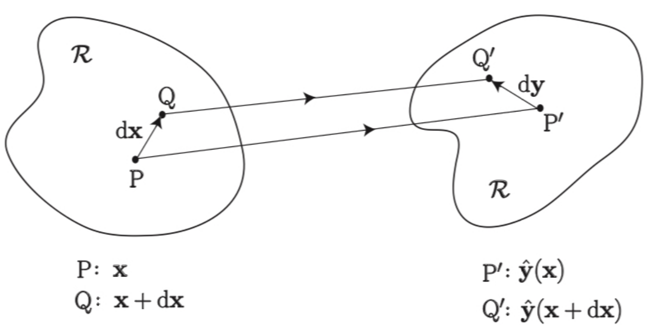
where $\hat{y} : \mathcal{R} \to \mathcal{R}_*$

Unless explicIty stated otherwise we will assume

$$\hat{y} \in C^2(\mathcal{R}_0), \quad \overline{x} \in C^2(\mathcal{R})$$

7.1 Deformation gradient tensor: deformation in the neighborhood of a particle

Consider now the deformation of a small neighborhood of a generic particle.



Definition 50

The **deformation gradient** $\mathbf{F}(x) = \mathrm{Grad}(\mathbf{y}(x))$ is a 2 tensor field and its components $F_{ij}(x) = \frac{\partial y_i(x)}{\partial x_j}$

Notice that $d\mathbf{y} = \mathbf{y}(\mathbf{x} + d\mathbf{x}) - \mathbf{y}(\mathbf{x})$ Thus $\mathbf{F}$ carries an infinitesimal undeformed material fiber $d\mathbf{x}$ into its location $d\mathbf{y}$ in the deformed configuration.Formally we write

$$dy_i = F_{ij} dx_j \quad \text{or} \quad d\mathbf{y} = \mathbf{F} d\mathbf{x}$$

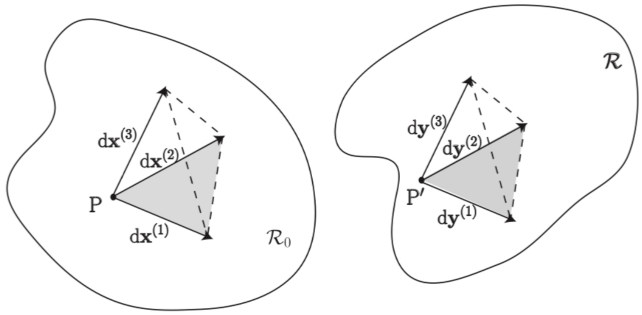


Figure 2: Orientation preserving deformation

Notice that

$$(d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)}) \cdot d\mathbf{y}^{(3)} = (\mathbf{F}d\mathbf{x}^{(1)} \times \mathbf{F}d\mathbf{x}^{(2)}) \cdot \mathbf{F}d\mathbf{x}^{(3)} = (\det \mathbf{F})(d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) \cdot d\mathbf{x}^{(3)}$$

. To see this consider that

$$\det[ka, kb, kc] = \det k \det[a, b, c] = \det k[a, b, c] = (ka \times kb) \cdot kc$$

Consequently orientation is preserved if and only if

$$J = \det F > 0$$

In this set of notes we will consider *only orientation preserving deformations*

Formally we may write since $d\mathbf{y} = \mathbf{y}(\mathbf{x} + d\mathbf{x}) - \mathbf{y}(\mathbf{x})$,

$$y(x + dx) = y(x) + F dx$$

Suppose that the deformation gradient tensor is constant on the entire region $\mathcal{R}_0$. Then we may integrate $dy = F dx$ both sides to obtain the following

Definition 51

A deformation $y(x)$ is said to be **homogeonus** if the deformation gradient tensor is contant on the entire region $\mathcal{R}_0$ so it can be denoted by

$$y(x) = Fx + b$$

where F is a constant tensor while b is a constant vector.

Finally consider the scenario of the **simple shear**

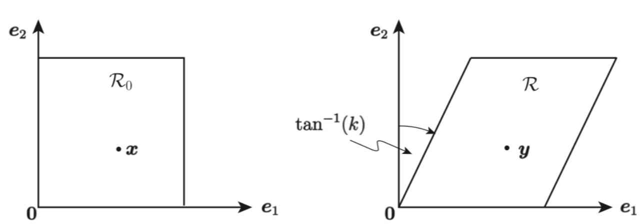
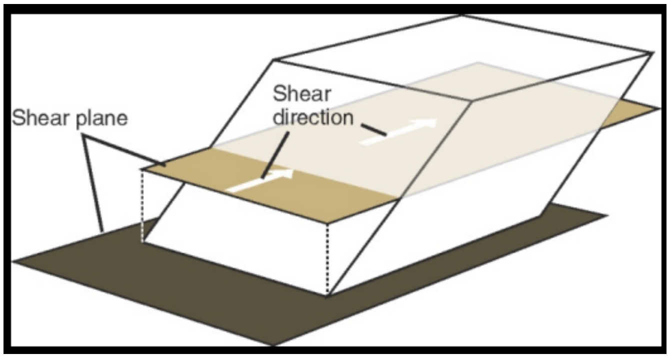


Figure 3: simple shear

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where $x_2 = c$ is the **shearing(or glide) plane** while x_1 direction as the **shearing direction** and k is the **amount of shear**



7.2 transformation of length,orientation,angle,volume and area

So far we consider the deformation gradient tensor $F(x)$ and how it characterizes all geometric changes in the neighbourhood of the particle x. We now examine the infinitemesmal material fiber(recall meaning dx), infitenesmmianl material surface and material region.

Fact 52
The change in length is related to the notion of fibre stretch(or strain), the change in angle is related to the notion of shear strain and the change in volume is related to the notion of volumetric(or dilational) strain.

Suppose that we are given a material fiber that has lenght ds_y and orientation $\mathbf{n}_0$ in the reference configuration $d\mathbf{x} = (ds_x)\mathbf{n}_0$. We want now to calculate its length and orientation $\mathbf{n}$ then $d\mathbf{y} = (ds_y)\mathbf{n}_0$

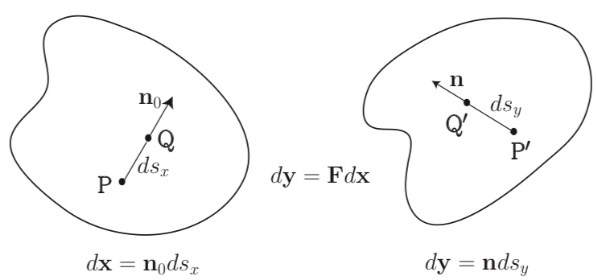


Figure 4: An infinitesimal material fiber

Now since $dy = F dx$ it follows that

$$dy = (ds_y)n = (ds_x)F n_0$$

Thus the deformed lenght of the fiber is

$$ds_y = |dy| = |F dx| = ds_x \, |F n_0|$$

note that clearly we are using unit directional vectors n_0 and n. See that the new orientation can be found from

$$\mathbf{n} = \frac{ds_x}{ds_y}\mathbf{F}\mathbf{n}_0 = \frac{ds_x}{ds_x\,|\mathbf{F}\mathbf{n}_0|} = \frac{\mathbf{F}\mathbf{n}_0}{|\mathbf{F}\mathbf{n}_0|}$$

Definition 53

The **stretch ratio** λ at the particle x in the direction n_0 is defined as the ratio

$$\lambda = ds_y/ds_x$$

and so

$$\lambda = |Fn_0|$$

Let us now explore changes angles

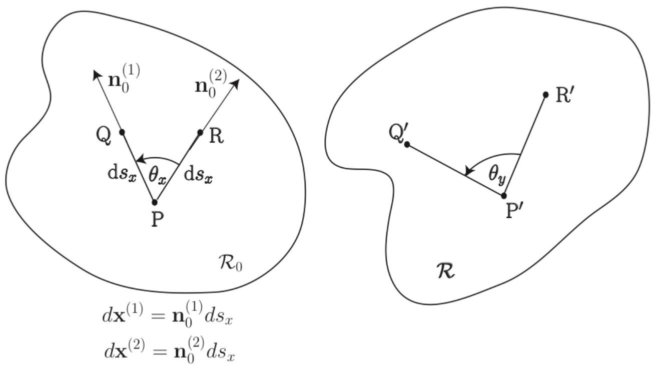
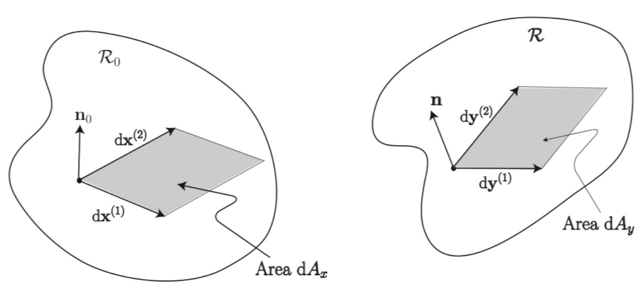


Figure 5: Two infinitesimal material fibres

Note that $Fdx^{(1)} \cdot Fdx^{(1)} = \left|dx^{(1)}\right|\left|dx^{(1)}\right|\cos\theta_y$ while $\cos\theta_x = n_0^{(1)} \cdot n_0^{(2)}$ we can write

$$\cos\theta_y = \frac{\mathbf{F}d\mathbf{x}^{(1)}}{\left|\mathbf{F}d\mathbf{x}^{(1)}\right|} \cdot \frac{\mathbf{F}d\mathbf{x}^{(2)}}{\left|\mathbf{F}d\mathbf{x}^{(2)}\right|} = \frac{\mathbf{F}\mathbf{n}_0^{(1)} \cdot \mathbf{F}\mathbf{n}_0^{(2)}}{\left|\mathbf{F}\mathbf{n}_0^{(1)}\right|\left|\mathbf{F}\mathbf{n}_0^{(2)}\right|}$$

As for **change in area** consider



From the definition of vector product we know that

$$d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} = dA_x\mathbf{n}_0$$

$$d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)} = dA_y \mathbf{n}$$

Since $d\mathbf{x}^{(i)} = \mathbf{F} d\mathbf{y}^{(i)}$ we may rewrite the above as

$$\mathbf{F} d\mathbf{x}^{(1)} \times \mathbf{F} d\mathbf{x}^{(2)} = dA_y \mathbf{n}$$

Then(refer to the lemma below) we have

$$dA_y \mathbf{n} = dA_x J \mathbf{F}^{-T} \mathbf{n}_o$$

and

$$dA_y = dA_y J \left| \mathbf{F}^{-T} \mathbf{n}_0 \right|$$

where we find that

$$\mathbf{n} = \frac{\mathbf{F}^{-T} \mathbf{n}_0}{\left| \mathbf{F}^{-T} \mathbf{n}_0 \right|}$$

Lemma 54

Let $\alpha_0 = |\mathbf{a} \times \mathbf{b}|$ and $\mathbf{n}_0 = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$. Let the new area under deformation be defined by $\alpha = |\mathbf{Fa} \times \mathbf{Fb}|$ and $\mathbf{n} = \frac{\mathbf{Fa} \times \mathbf{Fb}}{|\mathbf{Fa} \times \mathbf{Fb}|}$ Then

$$\mathbf{n} = \frac{\mathbf{F}^{-T} \mathbf{n}_0}{|\mathbf{F}^{-T} \mathbf{n}_0|}.$$

Proof.

$$\text{Therefore} \quad \alpha_0 \mathbf{n}_0 = \mathbf{a} \times \mathbf{b}, \quad \text{and} \quad \alpha \mathbf{n} = \mathbf{Fa} \times \mathbf{Fb}.$$

But

$$(\mathbf{Fa} \times \mathbf{Fb})_s = e_{sij} (\mathbf{Fa})_i (\mathbf{Fb})_j = e_{sij} F_{ip} a_p F_{jq} b_q.$$

Also recall the identity $e_{pqr} \det[\mathbf{F}] = e_{ijk} F_{ip} F_{jq} F_{kr}$ from your tensor analysis notes. Multiplying both sides of this identity by F_{rs}^{-1} lea

$$e_{pqr} \det[\mathbf{F}] F_{rs}^{-1} = e_{ijk} F_{ip} F_{jq} F_{kr} F_{rs}^{-1} = e_{ijs} F_{ip} F_{jq} \delta_{ks} = e_{ijs} F_{ip} F_{jq} = e_{sij} F_{ip} F_{jq}.$$

Substituting gives

$$(\mathbf{Fa} \times \mathbf{Fb})_s = \det[\mathbf{F}] e_{pqr} F_{rs}^{-1} a_p b_q = \det[\mathbf{F}] e_{rpq} a_p b_q F_{rs}^{-1} = \det[\mathbf{F}] \mathbf{F} (\mathbf{a} \times \mathbf{b})_r F_{sr}^{-T} = \det[\mathbf{F}] \mathbf{F} \left(\mathbf{F}^{-T} (\mathbf{a} \times \mathbf{b}) \right)_s$$

and

$$\alpha \mathbf{n} = \alpha_0 \det \mathbf{F} (\mathbf{F}^{-T} \mathbf{n}_0).$$

This describes how (vectorial) areas are mapped by the transformation $\mathbf{F}$. Taking the norm of this vector equation gives

$$\frac{\alpha}{\alpha_0} = |\det \mathbf{F}| |\mathbf{F}^{-T} \mathbf{n}_0|;$$

and substituting this result into the preceding equation gives

$$\mathbf{n} = \frac{\mathbf{F}^{-T} \mathbf{n}_0}{|\mathbf{F}^{-T} \mathbf{n}_0|}.$$

Fact 55

where the second equality follows from the scalar ds_x on the numerator and denominator cancelling Let the decrease in angle $\gamma = \theta_x - \theta_y$ be the **shear** associated with $\gamma = \gamma(n_0^{(1)}, n_0^{(2)})$

7.3 rigid deformation

we now consider the special case of a *rigid deformation*.

Definition 56

A deformation is said to be **rigid** if the distance between all pair of particles is preserved under the deformation. That is

$$|z - x| = |y(z) - y(x)|$$

Notice that this implies

$$|y(z) - y(x)|^2 = [y_i(z) - y_i(x)][y_i(z) - y_i(x)] = (z_i - x_i)(z_i - x_i)$$

for all $x, z \in \mathcal{R}_0$. That implies that if we take its derivative with respect to x_j we get

$$-2F_{ij}(x)(y_i(z) - y_i(x)) = -2(z_j - x_j)$$

for all $x, z \in \mathcal{R}_0$ where recall that $F_{ij} = \frac{\partial y_i(x)}{\partial x_j}$ which are the components of the deformation gradient tensor. Since this holds for all z we may take its derivative with respect to z_k

$$-2\frac{\partial}{\partial z_k} F_{ij}(x)(y_i(z) - y_i(x)) = -2\frac{\partial}{\partial z_k}(z_j - x_j)$$

which is clearly equal

$$F_{ij}(x)F_{ik}(z) = \delta_{jk}$$

Note that the above clearly represents a dot product. Hence we have

$$F^T(x)F(z) = I$$

but we know that this holds for all $x, z \in \mathcal{R}_0$ including $z = x$ hence

$$F^T(x)F(x) = I$$

So we conclude that $F(x)$ is indeed an orthonormal tensor at each x. Moreover the deformation gradient tensor F satisfies $\det F > 0$ since recall 50 we only consider orientation preserving deformations. Therefore it follows that $\det F = 1$ since $\det^2 F = \det F^T \det F = \det I = 1$. Hence we conclude that $F(x)$ must represent a **rotation** since it has been shown to be proper orthgonal by definition(recall 4 the conditions for a rotation).Moreover multiplying $F^T(x)F(z) = I$ both sides by $F(x)$, since $F(x)$ is orthogonal(meaning $F^T F = I$ recall `algebra I`) we have

$$F(z) = F(x)$$

at all $x, z \in \mathcal{R}_0$. That implies $F(x)$ is a constant tensor and the deformation $y(x)$ is homogenous which implies by 51 that we may write

$$y(x) = Qx + b$$

where Q is the constant rotation tensor and b is constant vector.

7.4 decomposition of deformation gradient tensor into a rotation and a stretch

As mentioned repeatedly above the deformation tensor $F(x)$ completel characterizes the deformation in the vinicinity of particle x. Part of this deformation is rotation, the rest is a "distortion" or "strain" if you recall. The question is now, so which part of F is the rotation and which part is the strain? The answer lies in **polar decomposition theorem** recall 10. According to this theorem every nonsingular tensor F with positive determinant(as we assumed from the start if you recall) can be uniquely written as the product of a proper orthgonal tensor R and a symmetric positive definite tensor U as

$$F = RU$$

where $U^2 = F^T F$ We claim that R represents the rotational part of F while U represents the part that is not a rotation.

Explicitly this means

$$[U] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \lambda_i > 0$$

where $\lambda_1, \lambda_2, \lambda_3$ are three real eigenvalues corresponding to a triplet of orthonormal eigenvectors r_1, r_2, r_3 We say $\{r_1, r_2, r_3\}$ forms a **principal basis**. The components of $d\mathbf{x}$ in this principal basis are

$$\{dx\} = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \quad \text{then} \quad [U]\{dx\} = \begin{bmatrix} \lambda_1 dx_1 \\ \lambda_2 dx_2 \\ \lambda_3 dx_3 \end{bmatrix}$$

The tensor U is called the **right stretch tensor** However we also know from 11 that the polar decomposition theorem admits another form:

$$F = VR$$

where V is a symmetric positive definite tensor while R is a proper orthogonal tensor and again we claim that R represents the rotation part while V represents the rest. Hence immediately we also see after eliminating R,

$$R = V^{-1}F = FU^{-1} \quad \Rightarrow \quad V^2 = (F\sqrt{F^T F}F^{-1})(F\sqrt{F^T F}F^{-1}) = F(F^T F)F^{-1} = FF^T$$

so

$$V = \sqrt{FF^T}$$

Notice that these are valid claims because a rotation does not change length angle area and volume so it should be independent of the rotation matrix. Indeed subbing $F = RU$ into the equations we had above expressing change in lenght,angle,volume and area respectively, noticing that $R \cdot R = R^T R = I$ we see that R cancels out in all 4 equations confirming this act

$$ds_y = ds_x \sqrt{U^2 \mathbf{n}_0 \cdot \mathbf{n}_0} = ds_x |U\mathbf{n}_0| = ds_x |RU\mathbf{n}_0| = ds_x |F\mathbf{n}_0|, \tag{2}$$

$$\cos\theta_y = \frac{U^2 \mathbf{n}_0^{(1)} \cdot \mathbf{n}_0^{(2)}}{\sqrt{U^2 \mathbf{n}_0^{(1)} \cdot \mathbf{n}_0^{(1)}}\sqrt{U^2 \mathbf{n}_0^{(2)} \cdot \mathbf{n}_0^{(2)}}}, \tag{3}$$

$$dV_y = dV_x \det U, \tag{4}$$

$$dA_y = dA_x(\det U)|U^{-1}\mathbf{n}_0|. \tag{5}$$

Then how about for change of orientation? Subbing $F = RU$ we have

$$\mathbf{n} = \frac{\mathbf{F}\mathbf{n}_0}{|\mathbf{F}\mathbf{n}_0|} = \mathbf{R}\frac{\mathbf{U}\mathbf{n}_0}{|\mathbf{U}\mathbf{n}_0|}$$

noting that $|\mathbf{R}\mathbf{U}\mathbf{n}_0| = |\mathbf{U}\mathbf{n}|_0$ again since rotation does not change lengths. Note that similary if you sub in $F = VR$ instead you will get the same relations but in terms of $ds_x, \cos\theta_x, dV_x, dA_x$ instead.

Fact 57

We call the **left stretch tensor** V **eulerian stretch tensor** since it is in terms of the *reference configuration*(x) while we call the **right stretch tensor** U **eulerian stretch tensor** since it is in terms of the *deformed configuration*(y)

7.5 strain

Definition 58

We define the **langarian strain** to be all related to the stretch U in a one to one manner

Example 59 (Examples langarian strain)

The green strain and the generalized green strain and the hencky strain defined by

$$\frac{1}{2}(U^2 - I), \quad \frac{1}{m}(U^m - I), \quad \ln U$$

respectively

Note the principal directions of each of these strain tensors are the same as those of U and the associated princpal strains are

$$\frac{1}{2}(\lambda_i^2 - 1), \quad \frac{1}{m}(\lambda_i^m - 1), \quad \ln \lambda_i$$

respectively

Definition 60

The **eulerian strain** is related too the stretch V in a one t one manner.

Example 61

The almansi strain, generalized almansi strain the logarithmic strain defined by

$$\frac{1}{2}(I - V^{-2}), \quad \frac{1}{m}(V^m - I), \quad \ln V$$

respectively

The preceding examples may be unified and generalized as follows

Definition 62

Let $e(\cdot)$ be any scalar valued function that is defined on $(0, \infty)$ such that

(a) $e(1) = 0$

(b) $e'(1) = 1$

(c) $e'(\lambda) > 0$ for all $\lambda > 0$

This means we can define the langarian strain tensor $E(U)$ to be the tensor with eigenvectors r_i and corresponding eigenvalues $e(\lambda_i)$ as

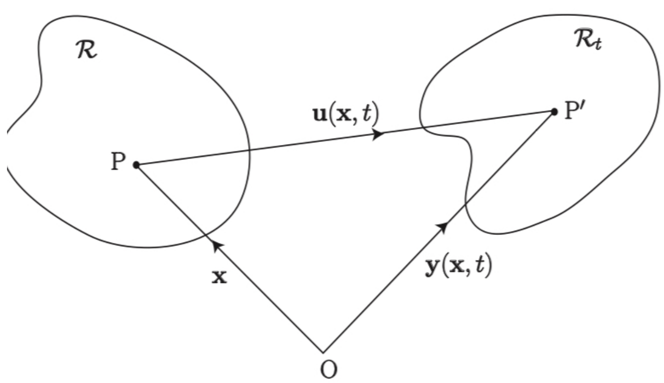
$$E = \sum_{i=1}^{3} e(\lambda_i)(r_i \otimes r_i)$$

Recall that U is symmetric positive definite. Therefore we see that all langarian strain tensors are symmetric(since I only affects the diagonal and the matrix product of symmetric matrices are still symmetric as row i equal column i)

8 kinematics motion

8.1 motion

In the previous chapter we exmamined the single configuration of a body. Here we consider time dpendent sequnce of configurations. Consider now the time dependent deformation



With respect to a fixed reference configuration a **motion** can be characterised by

$$y = y(x, t), x \in \mathcal{R}_0, y \in \mathcal{R}_t$$

The **initial configuration** is denoted by

$$x = y(x, t_0), F(x, t_0) = I, \quad \forall x \in \mathcal{R}_0$$

Definition 63
The **velocity** and **acceleration** of a particle is given by

$$v(x,t) = \frac{\partial y}{\partial t}(x,t) = \dot{y} \quad \text{and} \quad a(x,t) = \frac{\partial v}{\partial t}(x,t) = \dot{v}$$

respectively

Recall the quantities discussed earlier, in particular the deformation gradient tensor $F(x,t)$, Jacobian determinant $J(x,t)$ and propert orthogonal rotation tensor $R(x,t)$ and the symmetric positive definite stretch tensor $U(x,t), V(x,t)$ where we have

$$\begin{aligned} F(x,t) &= \text{grad}\, y(x,t) \\ J(x,t) &= \det F(x,t) > 0 \\ F(x,t) &= R(x,t)U(x,t) = V(x,t)R(x,t) \end{aligned}$$

The principal stretches $\lambda_1(x,t), \lambda_2(x,t), \lambda_3(x,t)$ are the common eigenvalues of $U(x,t)$ and $V(x,t)$. The corresponding eigenvectors $r_1(x,t), r_2(x,t) r_3(x,t)$ and $\ell_1(x,t), \ell_2(x,t), \ell)3(x,t)$ are the principal directions of the Lagrangian and eulerian stretch tensors and so on.

8.2 rigid motions

Recall that a rigid motion is characterized by

$$y = Q(t) + b(t)$$

where this time we are expressing the time dependence on t. Then differentiating with respect to t yields the velocity feild in langarian form(in terms of x)

$$v(x,t) = \dot{Q}(t)x + \dot{b}(t)$$

To convert to eulerian form we get from above that

$$Q^{-1}(y - b(t)) = Q^T y - Q^T b = x$$

where the second equality follows because rotational matrices are unitary $Q^{-1} = Q^T$ if you recall. Then subbing it our expression for v yields

$$v(y,t) = \Omega(t)y + c(t)$$

where $\Omega = \dot{Q}Q^T$ and $c = \dot{b} - \dot{Q}Q^T b$. However notice that $Q(t)$ being an orthgonal tensor at each time t for means

$$Q(t)Q^T(t) = I$$

for all t. Therefore since it is constant its derivative is zero so by product rule we have

$$\dot{Q}Q^T + Q\dot{Q}^T = 0$$

this means we have

$$\Omega = -\Omega^T$$

Since $\Omega(t)$ is skew symmetric recall from `tensor analysis` that it can then be realized as a croos product of some vector w that is

$$\Omega p = w \times p$$

for all vectors p. THen it follows that the velocity field can be written equivalentlly as

$$v(y,t) = w(t) \times y + c(t)$$

where $w(t)$ is called the angular velocity of the rigid motion(recall classical mechanics...this looks familiar). Then notice that

$$\Omega = \operatorname{grad} v \quad \text{and} \quad \mathbf{w} = \frac{1}{2} \operatorname{curl} v$$

reminder that we are using eulerian(in terms of y, the deformed configuration) vector identities clearly if not we will have used "Grad" and "Curl". To understand this consider the lemma below.

Lemma 64

Given that $\mathbf{v} = \mathbf{w} \times \mathbf{r}$ where $\mathbf{r}$ is the position vector and $\mathbf{w}$ is constant

$$\mathbf{w} = \frac{1}{2} \operatorname{curl} \mathbf{v}$$

Proof. We start with the vector equation:

$$v_i = (\boldsymbol{w} \times \boldsymbol{r})_i$$

Using the definition of the cross product in terms of the Levi-Civita symbol, this becomes:

$$v_i = \epsilon_{ijk} w_j r_k$$

Here, ϵ_{ijk} is the Levi-Civita symbol, w_j are the components of the vector $\boldsymbol{w}$, and r_k are the components of the position vector $\boldsymbol{r}$. In Cartesian coordinates, we can express $r_k = x_k$, where x_k are the spatial coordinates.

Thus, the equation becomes:

$$v_i = \epsilon_{ijk} w_j x_k$$

Next, we compute the curl of $\boldsymbol{v}$ in index notation. The curl of a vector field is given by:

$$(\nabla \times \boldsymbol{v})_i = \epsilon_{ijk} \partial_j v_k$$

Substitute $v_k = \epsilon_{klm} w_l x_m$ from the previous step:

$$(\nabla \times \boldsymbol{v})_i = \epsilon_{ijk} \partial_j (\epsilon_{klm} w_l x_m)$$

Apply the product rule of differentiation: where we have assumed that w is constant

$$\partial_j (\epsilon_{klm} w_l x_m) = \epsilon_{klm} w_l \partial_j x_m$$

Since $\partial_j x_m = \delta_{jm}$, the Kronecker delta simplifies the derivative:

$$\partial_j (\epsilon_{klm} w_l x_m) = \epsilon_{klm} w_l \delta_{jm} = \epsilon_{klj} w_l$$

Thus, we have:

$$(\nabla \times \boldsymbol{v})_i = \epsilon_{ijk}\epsilon_{klj}w_l$$

We now use the identity for the product of two Levi-Civita symbols:

$$\epsilon_{ijk}\epsilon_{klj} = \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl}$$

Since $\delta_{jj} = 3$, we get:

$$\epsilon_{ijk}\epsilon_{klj} = 3\delta_{il} - \delta_{il}$$

Thus, simplifying:

$$\epsilon_{ijk}\epsilon_{klj} = 2\delta_{il}$$

Substituting this back into our expression for $\nabla \times \boldsymbol{v}$, we obtain:

$$(\nabla \times \boldsymbol{v})_i = 2w_i$$

Therefore, the vector form of this result is:

$$\nabla \times \boldsymbol{v} = 2\boldsymbol{w}$$

Finally, solve for $\boldsymbol{w}$:

$$\boldsymbol{w} = \frac{1}{2}\nabla \times \boldsymbol{v}$$

Thus, we have proven the desired result:

$$\boxed{\boldsymbol{w} = \frac{1}{2}\nabla \times \boldsymbol{v}}$$

8.3 velocity gradient, stretching and spin tensors

Definition 65

The **velocity gradient tensor** is defined as the (spatial) gradient of the velocity field

$$L(y, t) = \text{grad}\, v(y, t) = \frac{\partial v}{\partial y}(y, t)$$

So in term of componeents in an orthonormal basis we have

$$L_{ij} = \frac{\partial v_i}{\partial y_j}$$

Then we have

$$\text{tr}\, \mathbf{L} = \text{div}\, \mathbf{v}$$

This is because $\text{div}\, \mathbf{v} = \mathcal{C}_1^2\, \text{grad}\, \mathbf{v}$(self contraction of gradient) and because $\text{grad}\, \mathbf{v}$ is a 2 tensor clearly such a contraction produces the trace. If you can't see this please refer back to `tensor analysis`. Now differentiating the deformation tensor we have

$$\dot{\mathbf{F}} = \frac{\partial}{\partial t}\mathrm{Grad}\mathbf{y}(\mathbf{x}, t) = \mathrm{Grad}(\frac{\partial \mathbf{y}}{\partial t}(\mathbf{x{,}t})) = \mathrm{Grad}(\mathbf{v}) = (\mathrm{grad}\,\mathbf{v})\mathbf{F} = \mathbf{LF}$$

The 2nd last inequality follows by considering the elements

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_i}{\partial y_k}\frac{\partial y_k}{\partial x_j}$$

see that summation of indices on the RHS precisely corresponds to matrix multiplication in that order. So we now know

$$\boxed{\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}} \tag{6}$$

Definition 66

Now let us break this into its symmetric and skew symmetric parts(recall the 1) where we have

$$D = \frac{1}{2}(L + L^T) \quad \text{and} \quad W = \frac{1}{2}(L - L^T)$$

We call D the **stretching tensor** and W the **spin tensor**

Then we have

Proposition 67

$$\mathrm{tr}\, D = \mathrm{tr}\, L, \quad \mathrm{tr}\, W = 0$$

Proof. Consider

$$\mathrm{tr}\, D = \frac{1}{2}(\mathrm{tr}\, L + \mathrm{tr}\, L^T) = \frac{1}{2}(\mathrm{tr}\, L + \mathrm{tr}\, L) = \mathrm{tr}\, L$$

Similar reasoning for W. □

and since W is skew symmetric there exists a unique axial vector w such that $Wp = w \times p$ so we have

$$w = \frac{1}{2}\,\mathrm{curl}\, v$$

Definition 68

Then the **vortocity** ω is defined as

$$\omega = 2w = \mathrm{curl}\, v$$

8.4 rate of change of length,orientation and volume

Let $d\mathbf{x}$ be a material fiber through the particle $\mathbf{x}$ in the reference configuarion and let $d\mathbf{y}$ be its image at time $t/$ Then $d\mathbf{y} = \mathbf{F}(\mathbf{x}, y)d\mathbf{x}$. Differentiating with respect to t and using 6 we have

$$(d\mathbf{y})^{\cdot} = \dot{F}d\mathbf{x} = (\mathbf{LF})d\mathbf{x} = \mathbf{L}d\mathbf{y}$$

Now let

$$d\mathbf{y} = \mathbf{n}ds$$

given that $ds^2 = d\mathbf{y} \cdot d\mathbf{y}$ and using the above relation gets upon differentiating both sides by product rule

$$ds(ds)^{\cdot} = d\mathbf{y} \cdot (d\mathbf{y})^{\cdot} = \mathbf{L}d\mathbf{y} \cdot d\mathbf{y} = \mathbf{Ln} \cdot \mathbf{n}ds^2$$

which simplifies further to $(ds)^{\cdot}/ds = \mathbf{Ln} \cdot \mathbf{n}$. Becaus $\mathbf{Wn} \cdot \mathbf{n} = 0$ since $\mathbf{W}$ is skew symmetric we get knowing that $\mathbf{L} = \mathbf{D} + \mathbf{W}$,

$$\frac{(ds)^{\cdot}}{ds} = \mathbf{Dn} \cdot \mathbf{n} = \mathbf{Ln} \cdot \mathbf{n}$$

we consequently conclude that the **stetching tensor** $\mathbf{D}$ characterizes the rate of change in length.

As for rate of change in orienation we know from the above 2 relations we have

$$(d\mathbf{y})^{\cdot} = (\mathbf{n}ds)^{\cdot} = \mathbf{L}d\mathbf{y} = \mathbf{Ln}ds = \dot{\mathbf{n}}ds + \mathbf{n}(ds)^{\cdot}$$

so

$$\dot{\mathbf{n}} = \frac{1}{ds}(\mathbf{Ln}ds - \mathbf{n}(ds)^{\cdot})$$

which gets

$$\dot{\mathbf{n}} = \mathbf{Ln} - (\mathbf{n} \cdot \mathbf{Ln})\mathbf{n}$$

finally using $\mathbf{L} = \mathbf{D} + \mathbf{W}$ gets

$$\dot{\mathbf{n}} = \mathbf{Wn} + \mathbf{Dn} - (\mathbf{Dn} \cdot \mathbf{n})\mathbf{n}$$

8.5 rate of change in volume

Consider a material volume dV_x in the reference configuration and let dV_y be the volume occupied by this same collection of particles in the current configuration. Now recall from 2 that

$$dV_y = JdV_x$$

where $J = \det \mathbf{F}$. Using **jacobi formula** recall 26 we have

$$\frac{d}{dt}(\det \mathbf{F}) = (\det \mathbf{F}) \operatorname{tr} \left(\frac{d\mathbf{F}}{dt} \mathbf{F}^{-1} \right)$$

So combining these 2 observations we have

$$\dot{J} = \det \mathbf{F} \operatorname{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1}) = J \operatorname{tr} \mathbf{L}$$

but recall that $\operatorname{div} \mathbf{v} = \operatorname{tr} \mathbf{D} = \operatorname{tr} \mathbf{L}$ so we have

$$\frac{(dV_y)^{\cdot}}{dV_y} = \frac{\dot{J}dV_x}{JdV_x} = \operatorname{div} \mathbf{v}$$

since dV_x is the material volume at t_0 which is just a constant

8.6 rate of change in area and orientation

Proposition 69
Consider a material surface which at time t has area dA_y and unit normal $\mathbf{n}$. Then the rate of change of area and the rate of rotation of the unit normal are given by

$$(dA_y)^{\cdot} = (\operatorname{tr}\mathbf{L} - \mathbf{n}\cdot\mathbf{Ln})dA_y$$

and

$$\dot{\mathbf{n}} = (\mathbf{n}\cdot\mathbf{Ln})\mathbf{n} - \mathbf{L}^T\mathbf{n}$$

Proof. We begin by recalling that the deformation gradient $\mathbf{F}$ and its inverse transpose $\mathbf{F}^{-T}$ relate the initial and current areas. The Jacobian $J = \det\mathbf{F}$ relates the change in volume during deformation.

1. **Time derivative of dA_y**:

$$dA_y = dA_x J|\mathbf{F}^{-T}\mathbf{n}_0|$$

Taking the time derivative:

$$(dA_y)^{\cdot} = dA_x\left(\dot{J}|\mathbf{F}^{-T}\mathbf{n}_0| + J\frac{d}{dt}|\mathbf{F}^{-T}\mathbf{n}_0|\right)$$

The rate of change of J is related to the velocity gradient tensor $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ via:

$$\dot{J} = J\operatorname{tr}(\mathbf{L})$$

(recall the derivation for volume above) Therefore, the first term becomes:

$$dA_x\dot{J}|\mathbf{F}^{-T}\mathbf{n}_0| = \frac{dA_y}{\mathbf{J}\left|\mathbf{F}^{-T}\mathbf{n}_0\right|}\mathbf{J}\operatorname{tr}(\mathbf{L})\left|\mathbf{F}^{-T}\mathbf{n}_0\right| = dA_y\operatorname{tr}(\mathbf{L})$$

2. **Rate of change of $|\mathbf{F}^{-T}\mathbf{n}_0|$**: The second term involves the time derivative of $|\mathbf{F}^{-T}\mathbf{n}_0|$. Let:

$$a = |\mathbf{F}^{-T}\mathbf{n}_0| = \sqrt{\mathbf{F}^{-T}\mathbf{n}_0\cdot\mathbf{F}^{-T}\mathbf{n}_0}$$

Taking the time derivative of a, we use the chain rule:

$$\frac{d}{dt}|\mathbf{F}^{-T}\mathbf{n}_0| = \frac{1}{2|\mathbf{F}^{-T}\mathbf{n}_0|}\frac{d}{dt}\left(\mathbf{F}^{-T}\mathbf{n}_0\cdot\mathbf{F}^{-T}\mathbf{n}_0\right)$$

The time derivative of $\mathbf{F}^{-T}\mathbf{n}_0$ is:

$$\frac{d}{dt}(\mathbf{F}^{-T}\mathbf{n}_0) = -\mathbf{F}^{-T}\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{n}_0 = -\mathbf{F}^{-T}\mathbf{L}^T\mathbf{n}$$

Therefore:

$$\frac{d}{dt}\left(\mathbf{F}^{-T}\mathbf{n}_0\cdot\mathbf{F}^{-T}\mathbf{n}_0\right) = -2\mathbf{F}^{-T}\mathbf{n}_0\cdot\mathbf{F}^{-T}\mathbf{L}^T\mathbf{n}$$

Substituting this back into the expression for the rate of change of $|\mathbf{F}^{-T}\mathbf{n}_0|$:

$$\frac{d}{dt}|\mathbf{F}^{-T}\mathbf{n}_0| = -\mathbf{n}\cdot\mathbf{Ln}$$

Hence the second term becomes:

$$dA_x J\frac{d}{dt}|\mathbf{F}^{-T}\mathbf{n}_0| = -dA_y\mathbf{n}\cdot\mathbf{Ln}$$

3. **Combining terms**: Thus, the total rate of change of area is:

$$(dA_y)^{\cdot} = dA_y\left(\operatorname{tr}(\mathbf{L}) - \mathbf{n}\cdot\mathbf{Ln}\right)$$

For the **second equation** recall we are also given that $\mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{n}_0}{|\mathbf{F}^{-T}\mathbf{n}_0|}$.

To compute the time derivative of $\mathbf{n}$, we apply the product rule:

$$\dot{\mathbf{n}} = \frac{d}{dt}\left(\frac{\mathbf{F}^{-T}\mathbf{n}_0}{|\mathbf{F}^{-T}\mathbf{n}_0|}\right)$$

This gives:

$$\dot{\mathbf{n}} = \frac{1}{|\mathbf{F}^{-T}\mathbf{n}_0|}\frac{d}{dt}(\mathbf{F}^{-T}\mathbf{n}_0) - \frac{\mathbf{F}^{-T}\mathbf{n}_0}{(|\mathbf{F}^{-T}\mathbf{n}_0|)^2}\frac{d}{dt}|\mathbf{F}^{-T}\mathbf{n}_0|$$

Substitute the results from before:

- $\frac{d}{dt}(\mathbf{F}^{-T}\mathbf{n}_0) = -\mathbf{F}^{-T}\mathbf{L}^T\mathbf{n}$ - $\frac{d}{dt}|\mathbf{F}^{-T}\mathbf{n}_0| = -\mathbf{n}\cdot\mathbf{L}\mathbf{n}$

We then have:

$$\dot{\mathbf{n}} = \frac{-\mathbf{F}^{-T}\mathbf{L}^T\mathbf{n}}{|\mathbf{F}^{-T}\mathbf{n}_0|} + \mathbf{n}(\mathbf{n}\cdot\mathbf{L}\mathbf{n})$$

Recognizing that $\mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{n}_0}{|\mathbf{F}^{-T}\mathbf{n}_0|}$, we simplify this to:

$$\dot{\mathbf{n}} = -\mathbf{L}^T\mathbf{n} + (\mathbf{n}\cdot\mathbf{L}\mathbf{n})\mathbf{n}$$

Thus, we have proven that:

$$\dot{\mathbf{n}} = (\mathbf{n}\cdot\mathbf{L}\mathbf{n})\mathbf{n} - \mathbf{L}^T\mathbf{n}$$

This completes the proof of both the rate of change of area and the rate of rotation of the unit normal. $\square$

8.7 current configuration as reference configuration

8.8 transport equations

Theorem 70 (Transport Equations)
Let $\beta(y,t)$ and $b(y,t)$ characterize a scalar and vector property associated with a motion of the body. Let $\mathcal{D}_t, \mathcal{S}_t, \mathcal{C}_t$ be the material subregion, surface, and curve in the time-dependent region $\mathcal{R}_t$ occupied by the a particular fixed set of particles $\mathcal{P}$. Then

1.
$$\frac{d}{dt}\int_{\mathcal{D}_t} \beta dV_y = \int_{\mathcal{D}_t} (\dot{\beta} + \beta \operatorname{div} v) dV_y$$
$$\frac{d}{dt}\int_{\mathcal{D}_t} b dV_y = \int_{\mathcal{D}_t} (\dot{b} + b \operatorname{div} v) dV_y$$

2.
$$\frac{d}{dt}\int_{\mathcal{S}_t} \beta dA_y = \int_{\mathcal{S}_t} [(\dot{\beta} + \beta \operatorname{div} v)n - \beta L^T n] dA_y$$
$$\frac{d}{dt}\int_{\mathcal{S}_t} b dA_y = \int_{\mathcal{S}_t} (\dot{b} + b \operatorname{div} v - Lb) \cdot n dA_y$$

3.
$$\frac{d}{dt}\int_{\mathcal{C}_t} \beta dy = \int_{\mathcal{C}_t} (\dot{\beta} + \beta L) dy$$
$$\frac{d}{dt}\int_{\mathcal{C}_t} b dy = \int_{\mathcal{C}_t} (\dot{b} + L^T b) \cdot dy$$

Proof. The approach is the convert the time dependent $\mathcal{D}_t, \mathcal{S}_t, \mathcal{C}_t$ into their time independent original reference configuration $\mathcal{D}_0, \mathcal{S}_0, \mathcal{C}_0$. Then using the identities we established earlier to prove the proposition for example

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{D}_t} b(y,t) dV_y &= \frac{d}{dt}\int_{\mathcal{D}_0} b(x,t)J(x,t) dV_x \\
&= \int_{\mathcal{D}_0} \frac{\partial}{\partial t}(b(x,t)J(x,t)) dV_x \\
&= \int_{\mathcal{D}_0} \left(\dot{b}J + b\dot{J}\right) dV_x \\
&= \int_{\mathcal{D}_0} \left(\dot{b}J + bJ \operatorname{div} v\right) dV_x \\
&= \int_{\mathcal{D}_0} \left(\dot{b} + b \operatorname{div} v\right) dV_y
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{S}_t} \beta(y,t) n dA_y &= \frac{d}{dt}\int_{\mathcal{S}_0} \beta(x,t) J F^{-T} n_0 dA_x \\
&= \int_{\mathcal{S}_0} \frac{\partial}{\partial t}(\beta J F^{-T}) n_0 dA_x \\
&= \int_{\mathcal{S}_0} \left(\dot{\beta} J F^{-T} + \beta \dot{J} F^{-T} + \beta J (F^{-T})^{\cdot}\right) n_0 dA_x \\
&= \int_{\mathcal{S}_0} (\dot{\beta} J F^{-T} + \beta J (\operatorname{div} v) F^{-T} + \beta J(-F^{-T}(\dot{F})^T F^{-T})) n_0 dA_x \\
&= \int_{\mathcal{S}_t} [(\dot{\beta} + \beta \operatorname{div}) I - \beta L^T] n dA_y
\end{aligned}$$

9 mechanical balance laws and field equations

9.1 conservation of mass

Definition 71

Given any part $\mathcal{P}$ of a body, its mass is a positive scalar valued property whose dimension is independent of lenght and time that we denote by $m(\mathcal{P}, t, \chi)$. In terms of **mass desnity** $p(y, t; \chi)(> O)$

$$m(\mathcal{P}, t; \chi) = \int_{\mathcal{D}_t} p(y, t; \chi) dV_y$$

Theorem 72

The **conservation of mass** states that the mass of any part $\mathcal{P}$ does not depend on the motion or time i.e $m(\mathcal{P}, t; \chi) = m(\mathcal{P})$.

Since $m(\mathcal{P})$ is time independent we differiate the above with respect to t to get the **balance law**

$$\frac{d}{dt} \int_{\mathcal{D}_t} p(y, t; \chi) dV_y = 0$$

where using the **transport equations** from above we

$$\int_{\mathcal{D}_t} (\dot{p} + p \operatorname{div} v) dV_y = 0$$

because this holds for all $\mathcal{D}_t \subset \mathcal{R}_t$ and sice the integrand is continuous we can conclude that

$$\dot{p} + p \operatorname{div} v = 0$$

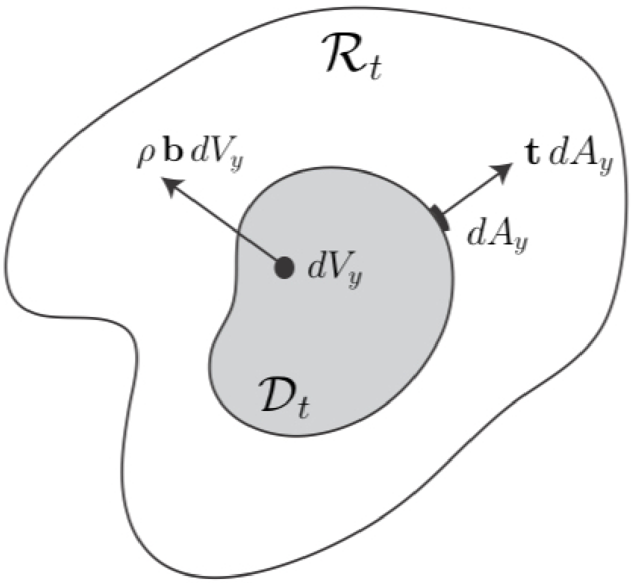
which holds for all $y \in \mathcal{R}_t$ and all t. Now suppose you want to multiply p ny some smooth scalar or vector valued fied ϕ. Then using the transport equations and product rule differentiation we have

$$\frac{d}{dt} \int_{\mathcal{D}_t} p\phi dV_y = \int_{\mathcal{D}_t} [(p\phi)^{\cdot} + p\phi \operatorname{div} v] dV_y = \int_{\mathcal{D}_t} [p\dot{\phi} + (\dot{p} + p \operatorname{div} v)\phi] dV_y$$

so we get

$$\frac{d}{dt} \int_{\mathcal{D}_t} p\phi dV_y = \int_{\mathcal{D}_t} p\dot{\phi} dV_y$$

9.2 force



We assume that there are two types of forces:

1. **body forces** that act at each point in the interior of the region $\mathcal{D}_t$
2. **contract forces**(or tractions) that act at the points on the boundary $\mathcal{D}_t$ and represent forces due to the contact between $\mathcal{P}$ and the rest of the body $\mathcal{B}$ across the surface $\partial\mathcal{D}_t$

Definition 73

Let b denote the **body force per unit mass** distributed over the interior of $\mathcal{D}_t$

- the **resultant body force** is

$$\int_{\mathcal{D}_t} pbdV_y$$

- the **resultant moment about fixed point O** is

$$\int_{\mathcal{D}_t} y \times pbdV_y$$

- the **rate at which the body forces do work** is

$$\int_{\mathcal{D}_t} v \cdot pbdV_y$$

where y is position, v is particle velocity and p is the mass density in the current configuration.

Definition 74

Similary let t represent the traction distributed over the boundary of $\mathcal{D}_t$

- the **resultant contact force per unit area** on $\mathcal{P}$ is

$$\int_{\partial D_t} t dA_y$$

- the **resultant moment of contact forces** about a fixed point O on $\mathcal{P}$ is

$$\int_{\partial D_t} t dA_y$$

- the **rate at which the contact forces do work** of the external forces on $\mathcal{P}$ is

$$\int_{\partial D_t} t \cdot v dA_y$$

Definition 75

Altogether letting t represent a force per unit area we combine above to get

- the **resultant force** on $\mathcal{P}$ is

$$\int_{\mathcal{D}_t} pbdV_y + \int_{\partial D_t} t dA_y$$

- the **resultant moment** on $\mathcal{P}$ is

$$\int_{\mathcal{D}_t} pbdV_y + \int_{\partial D_t} t dA_y$$

- the **total rate of working** of the external forces on $\mathcal{P}$ is

$$\int_{\mathcal{D}_t} v \cdot pbdV_y + \int_{\partial D_t} t \cdot v dA_y$$

applications

We first assume $b = b(y, t)$ and $t = t(y, t, n)$ where n is the unit normal vector of the surface t acts on. The latter assumption is known as **cauchy's hypothesis**