# MIT 18.03 Differential Equations Introduction

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Very basic first year engineering math content essentially. Largely generated by ChatGPT in 2025 because your original notes from 2023 in logseq are just random screenshots of the same thing anyway. Also added are refreshers from high school trigonometry,calculus,complex numbers and more. Not to worry all my other notes are hand typed for purpose of understanding and practice.

## Contents



# 1 Method of Undetermined Coefficients for Linear Constant Coefficient ODEs

The *method of undetermined coefficients* is a systematic way to find a particular solution for linear constant coefficient ODEs when the non-homogeneous term is of a specific form (e.g., exponential, polynomial, sine, cosine, etc.).

## General Form of the Equation

A linear constant coefficient ODE looks like this:

$$ay'' + by' + cy = f(x),$$

where:

- $a$, $b$, and $c$ are constants,
- $f(x)$ is a non-homogeneous term (the forcing function).

The solution consists of:

1. The **general solution to the homogeneous equation** $y_h$, and
2. A **particular solution** $y_p$ to the inhomogeneous equation.

The method of undetermined coefficients helps find $y_p$.

## Steps for the Method of Undetermined Coefficients

### 1. Solve the Homogeneous Equation

Write the homogeneous part:

$$ay'' + by' + cy = 0.$$

Find the roots of the characteristic equation:

$$ar^2 + br + c = 0.$$

The form of $y_h$ depends on the roots:

1. **Distinct real roots:** $r_1, r_2$:
$$y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$
2. **Repeated real roots:** $r_1 = r_2 = r$:
$$y_h = (C_1 + C_2 x)e^{rx}.$$
3. **Complex roots:** $r = \alpha \pm \beta i$:
$$y_h = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right).$$

### 2. Guess the Form of $y_p$

For the non-homogeneous term $f(x)$, guess the form of $y_p(x)$. The guess depends on the type of $f(x)$:

| **Type of $f(x)$** | **Guess for $y_p(x)$** |
|---|---|
| $f(x) = ke^{mx}$ | $y_p = Ae^{mx}$ |
| $f(x) = kx^n$ | $y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0$ |
| $f(x) = k\cos(mx)$ | $y_p = A\cos(mx) + B\sin(mx)$ |
| $f(x) = k\sin(mx)$ | $y_p = A\cos(mx) + B\sin(mx)$ |
| $f(x) = kx^n e^{mx}$ | $y_p = e^{mx}(A_n x^n + \cdots + A_0)$ |

### 3. Adjust for Overlap with $y_h$

If your guess for $y_p(x)$ overlaps with terms in $y_h(x)$, multiply your guess by $x^m$, where $m$ is the smallest positive integer that eliminates the overlap.

### 4. Plug $y_p(x)$ into the ODE

Substitute $y_p(x)$ into the original equation:

$$ay'' + by' + cy = f(x),$$

to determine the coefficients (e.g., $A, B, C, \ldots$).

### 5. Write the General Solution

Combine the homogeneous solution $y_h$ with the particular solution $y_p$:

$$y(x) = y_h(x) + y_p(x).$$

## Example: Solve $y'' - 3y' + 2y = e^x$

1. **Solve the Homogeneous Equation:** The homogeneous equation is:

   $$y'' - 3y' + 2y = 0.$$

   The characteristic equation is:

   $$r^2 - 3r + 2 = 0.$$

   Factorise:

   $$(r-1)(r-2) = 0.$$

   Roots: $r = 1, r = 2$.

   The homogeneous solution is:

   $$y_h = C_1 e^x + C_2 e^{2x}.$$

2. **Guess the Form of $y_p$:** The non-homogeneous term is $f(x) = e^x$, so guess:

   $$y_p = Ae^x.$$

3. **Check for Overlap:** The term $e^x$ already appears in $y_h$. To remove the overlap, multiply the guess by $x$:

   $$y_p = Axe^x.$$

4. **Plug $y_p$ into the ODE:** Compute the derivatives:

   $$y_p' = Ae^x + Axe^x, \quad y_p'' = 2Ae^x + Axe^x.$$

   Substitute into the original equation:

   $$y_p'' - 3y_p' + 2y_p = e^x.$$

Expand:

$$(2Ae^x + Axe^x) - 3(Ae^x + Axe^x) + 2(Axe^x) = e^x.$$

Simplify:

$$(-A)e^x + (0)xe^x = e^x.$$

Equate coefficients of $e^x$:

$$-A = 1 \implies A = -1.$$

Thus:

$$y_p = -xe^x.$$

5. **General Solution:** The general solution is:

$$y(x) = y_h + y_p = C_1e^x + C_2e^{2x} - xe^x.$$

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# 2 Derivation for Repeated Roots

To derive the solution of a second-order linear differential equation with **repeated roots**, let's start with the general equation:

$$ay'' + by' + cy = 0,$$

where $a$, $b$, and $c$ are constants.

## Step 1: Write the Characteristic Equation

The solution of the differential equation is related to the roots of its characteristic equation:

$$ar^2 + br + c = 0.$$

For **repeated roots**, the discriminant ($b^2 - 4ac$) is zero:

$$b^2 - 4ac = 0.$$

In this case, the two roots are equal:

$$r_1 = r_2 = r = \frac{-b}{2a}.$$

Thus, the characteristic equation has a **double root**.

## Step 2: General Solution for Repeated Roots

If the roots are repeated ($r_1 = r_2 = r$), the homogeneous solution cannot simply be:

$$y_h = C_1e^{rx} + C_2e^{rx},$$

because the two terms are identical, and this would only yield one independent solution.

To account for the second independent solution, we use the fact that the second solution must satisfy the differential equation but is linearly independent of $e^{rx}$. It can be shown (using methods from differential equations theory) that the second independent solution is of the form:

$$xe^{rx}.$$

Thus, the general solution for the repeated roots case is:

$$y_h = (C_1 + C_2 x)e^{rx}.$$

### Step 3: Justification for $xe^{rx}$ as a Solution

To see why $xe^{rx}$ is a valid second solution, consider the following reasoning:

1. For a second-order linear ODE, the general solution must involve two linearly independent solutions. For repeated roots, $e^{rx}$ is one solution, but a second linearly independent solution must exist.

2. Using the **reduction of order method**, assume the second solution is of the form:

$$y_2 = v(x)e^{rx},$$

   where $v(x)$ is a function to be determined.

3. Substitute $y_2 = v(x)e^{rx}$ into the homogeneous differential equation. After simplifications, it can be shown that $v(x) = x$.

Thus, the second solution is:

$$y_2 = xe^{rx}.$$

### Step 4: Final General Solution

The final general solution for the differential equation with repeated roots is:

$$y_h = (C_1 + C_2 x)e^{rx},$$

where:

- $r = \frac{-b}{2a}$ is the repeated root of the characteristic equation,
- $C_1$ and $C_2$ are arbitrary constants determined by initial conditions.

## 3 Derivation of the Case for Complex Roots in Constant Coefficient Linear ODEs

Consider the second-order constant coefficient linear differential equation:

$$ay'' + by' + cy = 0,$$

where $a$, $b$, and $c$ are constants.

## Step 1: Write the Characteristic Equation

The characteristic equation associated with this differential equation is:

$$ar^2 + br + c = 0.$$

The roots of this quadratic equation are given by the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

## Step 2: Complex Roots

If the discriminant $b^2 - 4ac < 0$, the roots are complex. Let:

$$b^2 - 4ac = -\Delta, \quad \text{where } \Delta > 0.$$

Then the roots can be written as:

$$r = \frac{-b}{2a} \pm i\frac{\sqrt{\Delta}}{2a}.$$

Define:

$$\alpha = \frac{-b}{2a} \quad \text{(real part of the root)},$$

$$\beta = \frac{\sqrt{\Delta}}{2a} \quad \text{(imaginary part of the root)}.$$

Thus, the roots are:

$$r = \alpha \pm i\beta.$$

## Step 3: General Solution for Complex Roots

For a second-order ODE, the general solution is based on the two linearly independent solutions associated with the roots. For complex roots, the solutions are of the form:

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}.$$

Substitute $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$:

$$y_1 = e^{(\alpha + i\beta)x}, \quad y_2 = e^{(\alpha - i\beta)x}.$$

Using Euler's formula:

$$e^{i\beta x} = \cos(\beta x) + i\sin(\beta x),$$

and:

$$e^{-i\beta x} = \cos(\beta x) - i\sin(\beta x),$$

we can rewrite $y_1$ and $y_2$ as:

$$y_1 = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right),$$

$$y_2 = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right).$$

The real part and imaginary part of these solutions form two linearly independent real solutions:

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x).$$

## Step 4: General Solution

The general solution is a linear combination of these two independent solutions:

$$y(x) = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x),$$

where $C_1$ and $C_2$ are arbitrary constants determined by initial conditions.

## Summary of the Complex Roots Case

If the characteristic equation has complex roots $r = \alpha \pm i\beta$:

- $\alpha = \frac{-b}{2a}$ is the real part of the root.
- $\beta = \frac{\sqrt{4ac-b^2}}{2a}$ is the imaginary part of the root.

The general solution to the ODE is:

$$y(x) = e^{\alpha x}\left(C_1\cos(\beta x) + C_2\sin(\beta x)\right).$$

Here:

- $e^{\alpha x}$ accounts for exponential decay or growth,
- $\cos(\beta x)$ and $\sin(\beta x)$ account for oscillatory behaviour due to the imaginary part.

# 4 Integrating Factors for First-Order Linear Differential Equations

Consider the first-order linear differential equation:

$$\frac{dy}{dt} + p(t)y = g(t). \tag{1}$$

## General Solution Using an Integrating Factor

To solve this equation, we introduce an *integrating factor* that simplifies the equation into an easily integrable form.

### Step 1: Multiply by the Integrating Factor

The integrating factor is defined as:

$$\mu(t) = e^{\int p(t)\,dt}.$$

Multiply both sides of the differential equation by $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t).$$

**Step 2: Simplify the Left-Hand Side**

The left-hand side becomes the derivative of the product $\mu(t)y$:

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t).$$

**Step 3: Integrate Both Sides**

Integrate both sides with respect to $t$:

$$\mu(t)y = \int \mu(t)g(t)\,dt + C,$$

where $C$ is the constant of integration.

**Step 4: Solve for $y$**

Divide through by $\mu(t)$ to isolate $y$:

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right).$$

## Summary of the Solution

The general solution to (1) is:

$$y(t) = \frac{1}{e^{\int p(t)\,dt}}\left(\int e^{\int p(t)\,dt}g(t)\,dt + C\right).$$

## Special Case: Homogeneous Equation

If $g(t) = 0$, the equation becomes:

$$\frac{dy}{dt} + p(t)y = 0.$$

In this case, the solution simplifies to:

$$y(t) = Ce^{-\int p(t)\,dt}.$$

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# 5 Integrating Factors for the General First-Order Differential Equation

Consider the general first-order differential equation in the form:

$$M(x, y)dx + N(x, y)dy = 0.$$

If the equation is not already exact, it can be made exact by multiplying through by an *integrating factor*.

## Exact Equations

An equation is *exact* if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If this condition is satisfied, the solution is found by solving:

$$\Phi(x, y) = C,$$

where $\Phi(x, y)$ satisfies:

$$\frac{\partial \Phi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = N(x, y).$$

## Non-Exact Equations and Integrating Factors

If the equation is not exact, an integrating factor can be used to make it exact. The integrating factor is typically a function of $x$, $y$, or both. Common cases include:

**Case 1: Integrating Factor Depends on $x$**

If the integrating factor $\mu(x)$ depends only on $x$, it satisfies:

$$\mu(x) = e^{\int \left( \frac{1}{N} \frac{\partial M}{\partial y} - \frac{1}{M} \frac{\partial N}{\partial x} \right) dx}.$$

**Case 2: Integrating Factor Depends on $y$**

If the integrating factor $\mu(y)$ depends only on $y$, it satisfies:

$$\mu(y) = e^{\int \left( \frac{1}{M} \frac{\partial N}{\partial x} - \frac{1}{N} \frac{\partial M}{\partial y} \right) dy}.$$

**Case 3: Integrating Factor Depends on Both $x$ and $y$**

In more complicated cases, the integrating factor may depend on both $x$ and $y$. In such situations, identifying the integrating factor requires additional techniques or insights.

## Example: Solving a Non-Exact Equation

Consider:

$$\left(2xy + y^2\right) dx + \left(x^2 + 2xy\right) dy = 0.$$

**Step 1: Check for Exactness**

Compute the partial derivatives:

$$\frac{\partial M}{\partial y} = 2x + 2y, \quad \frac{\partial N}{\partial x} = 2x + 2y.$$

Since:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

the equation is **exact**.

**Step 2: Solve for $\Phi(x, y)$**

Integrate $M(x, y)$ with respect to $x$:

$$\Phi(x, y) = \int \left(2xy + y^2\right) dx = x^2 y + y^2 x + h(y),$$

where $h(y)$ is an arbitrary function of $y$.

Differentiate $\Phi(x, y)$ with respect to $y$:

$$\frac{\partial \Phi}{\partial y} = x^2 + 2xy + h'(y).$$

Set this equal to $N(x, y)$:

$$x^2 + 2xy + h'(y) = x^2 + 2xy.$$

This gives:

$$h'(y) = 0 \implies h(y) = C,$$

where $C$ is a constant.

Thus, the solution is:

$$\Phi(x, y) = x^2y + y^2x = C.$$

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# 6 Summary of Integrating Factor Methods

## 1. Linear First-Order Equations

For $\frac{dy}{dt} + p(t)y = g(t)$, the integrating factor is:

$$\mu(t) = e^{\int p(t)\,dt}.$$

## 2. General First-Order Equations

For $M(x, y)dx + N(x, y)dy = 0$, if not exact, find an integrating factor $\mu(x, y)$ to make it exact.

### Key Cases for Integrating Factors

- $\mu = \mu(x)$: Depends only on $x$.
- $\mu = \mu(y)$: Depends only on $y$.
- $\mu = \mu(x, y)$: Depends on both $x$ and $y$.

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# 7 Solving Angles Using ASTC

The **ASTC rule** (also known as the quadrant rule) is a mnemonic used to determine the signs of trigonometric functions in different quadrants of the unit circle:

- **A**: All trigonometric functions (sin, cos, tan) are positive in Quadrant I.
- **S**: sin is positive in Quadrant II.
- **T**: tan is positive in Quadrant III.

- **C**: cos is positive in Quadrant IV.

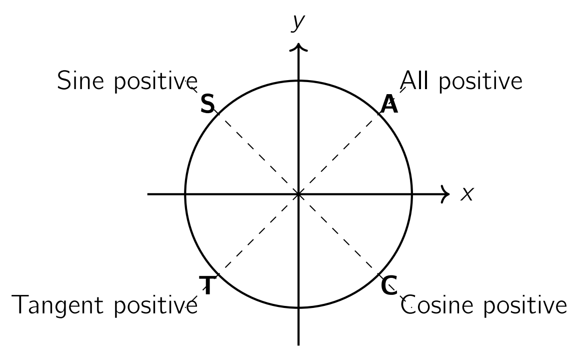


Figure 1: ASTC Diagram: Quadrants and the Unit Circle.

## Example: Solve $\sin(\theta) = -\frac{\sqrt{3}}{2}$ for $0^\circ \leq \theta < 360^\circ$

### Step 1: Determine the Quadrants

Since $\sin(\theta)$ is negative, the angle $\theta$ must lie in quadrants where $\sin(\theta)$ is negative. Using the ASTC rule:

- $\sin(\theta)$ is positive in Quadrants I and II.
- $\sin(\theta)$ is **negative in Quadrants III and IV**.

Thus, $\theta$ lies in Quadrants III and IV.

### Step 2: Find the Reference Angle

The *reference angle*, denoted as $\theta_r$, is the acute angle between the terminal side of $\theta$ and the x-axis.

The reference angle is calculated from the equation:

$$\sin(\theta_r) = \frac{\sqrt{3}}{2}.$$

From standard trigonometric values, we know:

$$\theta_r = 60^\circ.$$

### Step 3: Determine the Angles in the Appropriate Quadrants

The angles in Quadrants III and IV are calculated using the reference angle:

- In Quadrant III: $\theta = 180^\circ + \theta_r$,

$$\theta = 180^\circ + 60^\circ = 240^\circ.$$

- In Quadrant IV: $\theta = 360^\circ - \theta_r$,

$$\theta = 360^\circ - 60^\circ = 300^\circ.$$

**Step 4: Verify the Solution**

To confirm, substitute the values into the original equation:

$$\sin(240^\circ) = -\frac{\sqrt{3}}{2}, \quad \sin(300^\circ) = -\frac{\sqrt{3}}{2}.$$

Both values satisfy the equation.

**Final Answer:**

The solutions to $\sin(\theta) = -\frac{\sqrt{3}}{2}$ are:

$$\boxed{\theta = 240^\circ \quad \text{and} \quad \theta = 300^\circ.}$$

# 8 Essential Trigonometry

## Basic Trigonometric Functions

$$\sin(x), \quad \cos(x), \quad \tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}.$$

## Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1,$$
$$1 + \tan^2(x) = \sec^2(x),$$
$$1 + \cot^2(x) = \csc^2(x).$$

## Angle Sum and Difference Formulas

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b),$$
$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b),$$
$$\tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{1 \mp \tan(a)\tan(b)}.$$

## Double Angle Formulas

$$\sin(2x) = 2\sin(x)\cos(x),$$
$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1,$$
$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}.$$

## Reflections, Shifts, and Periodicity

- **Reflections:**

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \tan(-x) = -\tan(x).$$

- **Shifts:**

$$\sin(x+\pi) = -\sin(x), \quad \cos(x+\pi) = -\cos(x), \quad \tan(x+\pi) = \tan(x).$$

- **Periodicity:**

$$\sin(x+2\pi) = \sin(x), \quad \cos(x+2\pi) = \cos(x), \quad \tan(x+\pi) = \tan(x).$$

### Derivatives and Antiderivatives

- $\frac{d}{dx}\sin(x) = \cos(x), \quad \int \sin(x)\,dx = -\cos(x) + C.$
- $\frac{d}{dx}\cos(x) = -\sin(x), \quad \int \cos(x)\,dx = \sin(x) + C.$
- $\frac{d}{dx}\tan(x) = \sec^2(x), \quad \int \tan(x)\,dx = \ln|\sec(x)| + C.$
- $\frac{d}{dx}\sec(x) = \sec(x)\tan(x), \quad \int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C.$
- $\frac{d}{dx}\cot(x) = -\csc^2(x), \quad \int \cot(x)\,dx = \ln|\sin(x)| + C.$
- $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x), \quad \int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C.$

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# 9 Hyperbolic Functions

### Definitions of Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2},$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \operatorname{sech}(x) = \frac{1}{\cosh(x)},$$
$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}, \quad \operatorname{csch}(x) = \frac{1}{\sinh(x)}.$$

### Identities

$$\cosh^2(x) - \sinh^2(x) = 1,$$
$$1 - \tanh^2(x) = \operatorname{sech}^2(x),$$
$$\coth^2(x) - 1 = \operatorname{csch}^2(x).$$

### Derivatives and Antiderivatives

- $\frac{d}{dx}\sinh(x) = \cosh(x), \quad \int \sinh(x)\,dx = \cosh(x) + C.$
- $\frac{d}{dx}\cosh(x) = \sinh(x), \quad \int \cosh(x)\,dx = \sinh(x) + C.$
- $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x), \quad \int \tanh(x)\,dx = \ln|\cosh(x)| + C.$
- $\frac{d}{dx}\operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x), \quad \int \operatorname{sech}(x)\,dx = \arctan(\sinh(x)) + C.$
- $\frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x), \quad \int \coth(x)\,dx = \ln|\sinh(x)| + C.$

- $\frac{d}{dx}\operatorname{csch}(x) = -\operatorname{csch}(x)\coth(x)$, $\int \operatorname{csch}(x)\,dx = \ln|\coth(x) + \operatorname{csch}(x)| + C$.

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# 10 Complex Numbers

## Basic Definitions

A complex number is written as:

$$z = a + bi,$$

where:

- $a$ is the real part: $\mathrm{Re}(z) = a$,
- $b$ is the imaginary part: $\mathrm{Im}(z) = b$,
- $i = \sqrt{-1}$.

## Polar Form of Complex Numbers

The polar form of $z$ is:

$$z = r(\cos\theta + i\sin\theta),$$

where:

$$r = |z| = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right).$$

## Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus, the polar form can also be written as:

$$z = re^{i\theta}.$$

## De Moivre's Theorem

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right).$$

**Example: Solve $z^3 + 1 = 0$**

Rewrite the equation as:

$$z^3 = -1.$$

Write $-1$ in polar form:

$$-1 = e^{i\pi}.$$

Thus:

$$z = e^{i\pi/3}, \quad e^{i\pi}, \quad e^{i5\pi/3}.$$

These are the three cube roots of $-1$.

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# Integration by Parts: DI Tabulation Method

The **DI Tabulation Method** is a systematic approach to perform repeated integration by parts efficiently. It is especially useful for integrals involving a product of a polynomial and an exponential, trigonometric, or logarithmic function. First we use the LIATE rule to choose which product to integrate in integration by parts.

## LIATE Rule (Integration by Parts)

The LIATE rule provides guidance for choosing $u$ and $dv$ in integration by parts. The acronym stands for:

- **L**: Logarithmic functions ($\ln(x)$),
- **I**: Inverse trigonometric functions ($\arctan(x)$, $\arcsin(x)$, etc.),
- **A**: Algebraic functions ($x^n$),
- **T**: Trigonometric functions ($\sin(x), \cos(x)$),
- **E**: Exponential functions ($e^x$).

When applying integration by parts:

$$\int u\,dv = uv - \int v\,du,$$

choose $u$ as the function highest on the LIATE priority list, and $dv$ as the remaining part.

### Example: Evaluate $\int xe^x\,dx$

Using the LIATE rule:

- $u = x$ (algebraic),
- $dv = e^x\,dx$ (exponential).

Compute $du$ and $v$:

$$du = dx, \quad v = e^x.$$

Apply integration by parts:

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Final result:

$$\int xe^x\,dx = e^x(x-1) + C.$$

Now we show how to use the DI tabulation table in the next example.

## Example: Evaluate $\int x^2e^x\,dx$

### Step 1: Setup the Table

Start by identifying the two parts of the integral:

$$u = x^2 \quad \text{(choose as } u \text{ since it can be differentiated to zero)}, \quad dv = e^x\,dx \quad \text{(choose as } dv \text{ since it is easily integrated)}.$$

Construct the table for derivatives of $u$ and integrals of $dv$:

| **Derivative (D)** | **Integral (I)** |
|---|---|
| $x^2$ | $e^x$ |
| $2x$ | $e^x$ |
| $2$ | $e^x$ |
| $0$ | $e^x$ |

**Step 2: Assign Signs (+ and -)**

Alternate the signs starting with $+$ for the first row:

| **Sign** | **Derivative (D)** | **Integral (I)** |
|---|---|---|
| $+$ | $x^2$ | $e^x$ |
| $-$ | $2x$ | $e^x$ |
| $+$ | $2$ | $e^x$ |
| $-$ | $0$ | $e^x$ |

**Step 3: Multiply Diagonally and Add Terms**

Multiply diagonally across the table ($D$ term $\times$ $I$ term) to construct the solution:

$$\int x^2 e^x \, dx = (+)\,(x^2)(e^x) - (-)\,(2x)(e^x) + (+)\,(2)(e^x).$$

Simplify:

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

**Final Answer:**

$$\boxed{\int x^2 e^x \, dx = e^x(x^2 - 2x + 2) + C.}$$

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## When to Use the DI Tabulation Method

The DI method is most effective when:

- The derivative of $u$ becomes zero after a finite number of steps (e.g., polynomials).
- $dv$ is easy to integrate repeatedly (e.g., $e^x$, $\sin(x)$, $\cos(x)$).

This method avoids performing integration by parts repeatedly by structuring it into a simple tabular process.

# 11 Graphs

trigo and hyperbolic

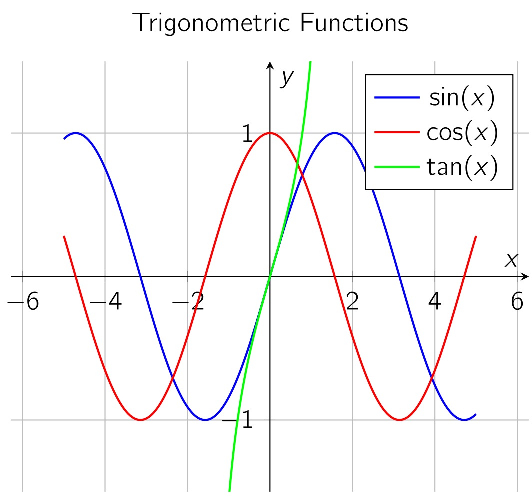


Figure 2: Trigonometric Functions: $\sin(x)$, $\cos(x)$, and $\tan(x)$.

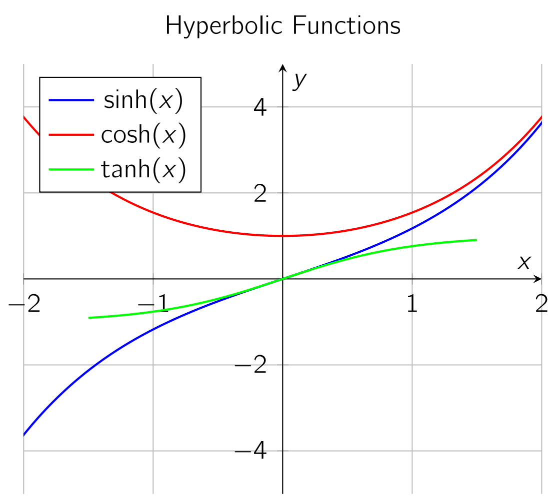


Figure 3: Hyperbolic Functions: $\sinh(x)$, $\cosh(x)$, and $\tanh(x)$.