(do Carmo)Differential Geoemtry

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Selected theorems from chapters 1-5 of `do Carmo differential geometry` and Evan Chen's `MIT 18.905 Differential Geometry` lecture notes. The other appendix sections are mostly from `MIT 18.101` 2018 course notes which were not included in the `OCW` version.

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1 readings

- 1-1 introduction **done**
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2 Prerequisites

2.1 Appendix: ODEs

Example 1 (Linear systems)

Now, suppose $\mathbf{f}$ is a linear function, i.e., $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ where A is a constant $m \times n$ matrix. Then for any $\mathbf{x}$, we have:

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

This results in:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \sum_{j=1}^{n} a_{1,j}x_j \\ \sum_{j=1}^{n} a_{2,j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{m,j}x_j \end{bmatrix}$$

The i-th component of $\mathbf{f}(\mathbf{x})$ is:

$$f_i(\mathbf{x}) = \sum_{j=1}^{n} a_{i,j}x_j$$

To find the partial derivative of f_i with respect to x_j, we calculate:

$$\frac{\partial f_i(\mathbf{x})}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{n} a_{i,k}x_k\right)$$

Since $a_{i,k}$ are constants (because A is a constant matrix), this simplifies to:

$$\frac{\partial f_i(\mathbf{x})}{\partial x_j} = a_{i,j}$$

Fact 2

Linear ODEs

$$a_1 y^{(n)} + a_2 y^{(n-1)} \dots a_1 y' + a_0 y = c$$

are just special cases of linear systems Where we have

$$\mathbf{y}' = A\mathbf{y} + g$$

where $g = 0$ if homogenous i.e $c = 0$. To see how so consider what we call an **autonomous system**

$$\begin{aligned} y' &= \mathcal{L}y \\ y'' &= \mathcal{L}y' \\ &\vdots \\ y^{(n-1)} &= \mathcal{L}y^{n-2} \\ y^{(n)} &= F(t, y_1, \dots, y_n) \end{aligned}$$

where $\mathcal{L}$ is just the derivative or equivalently

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, \dots, y_n) \end{aligned}$$

where

$$y_1 = y,\ y_2 = y',\ y_3 = y'' \dots, y_n = y^{(n-1)}$$

which is just a special case of the general form of a linear map given by

$$\begin{aligned} y_1' &= f_1(t, y_1, \dots, y_n) = \sum_{j=1}^{n} a_{1,j} y_j \\ y_2' &= f_2(t, y_1, \dots, y_n) = \sum_{j=1}^{n} a_{2,j} y_j \\ &\vdots \\ y_n' &= f_n(t, y_1, \dots, y_n) = \sum_{j=1}^{n} a_{n,j} y_j \end{aligned}$$

recall above $\mathbf{f}(x) = \mathbf{y}' = \mathbf{Ay}$

Example 3

To see why $\mathcal{L}$ is linear let y be expressed as some linear combination of a linear function x of varying degrees

$$y^{(n)} = a_n x^{(n)} + a_2 y^{(n-1)} \ldots a_1 x' + a_0 x$$

then

$$y^{(n+1)} = a_n x^{(n+1)} + a_2 x^{(n)} \ldots a_1 x'' + a_0 x'$$

well this is just

$$y^{(n+1)} = \sum_{j=1}^{n} a_i y^{(i)}$$

The key idea is any differential can be expressed as a linear combination of other differentials. But you clearly can't do that if say for example it contained $(x^{(n)})^2$ in which case its differential is a product of 2 differentials $2(x^{(n)})(x^{(n-1)})$ which you can't put into matrix form clearly.Though i am curious what can be put into matrix form and solved. And what can't. Hopefully i would find out in `non-linear dynamics and chaos` module.

For this section we denote the following

Definition 4

Let $x \in \mathbb{R}^n$ The **euclidean norm** of x is

$$||x|| = \sqrt{x_1^2 + \ldots x_n^2}$$

The **sup norm** of x is

$$|x| = \sup_i |x_i|$$

In fact they are related by

$$|x - y| \leq ||x - y|| \leq \sqrt{n}\,|x - y|$$

since

$$\sup_i |x_i - y_i| = \sqrt{(x_k - x_k)^2} = |x - y| \leq ||x - y|| = \sqrt{(x_1 - y_1)^2 + \ldots (x_n - y_n)^2}$$

where k is the corresponding i from the sup and that

$$\sqrt{n}\,|x - y| = \sqrt{n\,|x - y|^2}$$

And that the sup norm is $||x||_1$ while euclidean norm is $||x||_2$ in precise terminology.

Problem 5 (Theory of ODEs)

Let U be an open subsetof $\mathbb{R}^n$ and $g_i U \to \mathbb{R}, i = 1, \ldots, n$ be C^1 functions. Describe an $n \times n$ ODE system by

$$\frac{dx_i}{dt} g_i(x_1(t), \ldots, x_n(t)), i = 1, \ldots, n \tag{1}$$

where $x_i(t)$ are C^1 functions on an open interval I of the real line. We will call $x(t) = x_1(t), \ldots, x_n(t)$ an **integral curve** of 1. We can write 1 as

$$\frac{dx}{dt} = g(x(t)) \tag{2}$$

where $g = (g_1, \ldots, g_n)$. We would like to find out if

- *Existence*: Let t_0 be a point on the interval I and x_0 a point on U. Does there exist an integral curve $x(t)$ with the prescribed initial data $x(t_0) = x_0$?
- *Uniqueness*: If $x(t)$ and $y(t)$ are the integral curves and $x(t_0) = x_0 = y_0 = y(t_0)$ then does $x(t) = y(t)$ for all $t \in I$?

The objective of this section is to answer these 2 questions.

Lemma 6

Let U be open in $\mathbb{R}^n$ and $F : U \to \mathbb{R}^k$ be a $\mathcal{C}^1$ mapping. Also assume that U is convex. Suppose that $|Df(a)| \leq c$ for all $A \in U$. Then for all $x, y \in U$

$$|f(x) - f(y)| \leq nc\,|x - y|$$

Proof. Consider any $x, y \in U$. The mean value theoerem says for eery i there exists a point c on the line joining x to y such that

$$f_i(x) - f_i(y) = \sum_j^n \frac{\partial f_i}{\partial x_j}(d_i)(x_j - y_j)$$

for some $d_i \in [x_j, y_j]$ It is mean *value* theorem that's why we must restrict f to a single dimension because "values" of $f(x)$ are actually in $\mathbb{R}^k$. Hence it follows that

$$\begin{aligned} |f_i(x) - f_i(y)| &\leq \sum_j^n c\,|x_j - y_j| \\ &\leq nc\,|x - y| \end{aligned}$$

Because this is true for any i, taking the sup on the left side wont change anything. □

From 4 we know that

$$||g(x) - g(y)|| \leq n\,|g(x) - g(y)|$$

and that $|x - y| \leq ||x - y||$. Therefore we have

$$n^2C\,|x - y| \leq n^2C\,||x - y||$$

which upon combining with

$$n\,|g(x) - g(y)| \leq n^2C\,|x - y|$$

we may obtain

$$||g(x) - g(y)|| \leq L\,||y - x|| \tag{3}$$

where $L = n^2 C$.

Theorem 7

Let $x(t)$ and $y(t)$ be two solutions of 2 and let t_0 be a point on the interval I. Then for all $t \in I$

$$||x(t) - y(t)|| \leq e^{L|t-t_0|} ||x(t_0) - y(t_0)|| \tag{4}$$

Remark 8. *Note*

1. *This result says that in particular if $x(t_0) = y(t_0)$ then $x(t) = y(t)$ for all t, hence proving* uniqueness *as desired*
2. *Let I_1 be the bounded subinterval of I. Then from 4 one can deduce the following theorem*

Theorem 9

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $||x(t_0) - y(t_0)||$ is less than δ then $||x(t) - y(t)||$ is less than ε on the interval $t \in I_1$

Remark 10. *In other words our solution $x(t)$ depends continuously on the initial data*

Proof. Let $p_n \to p$. To prove $x(t)$ is continous on intial data to IVPs we must show that $\lim_{n\to\infty} x(p_n, t) = x(p, t)$. We will denote x_p and x_{p_n} to the respective solutions that satisfy the IVPs $x_p(t_0) = p$ and $x_{p_n}(t_0) = p_n$.

$$||x_{p_n}(t) - x_p(t)|| \leq e^{L|t-t_0|} ||x_{p_n}(t_0) - x_p(t_0)||$$

but since $p_n \to p$ and that norms are uniformly continuous, we can bring the limit in, in shich the RHS goes to zero by definiteness of norms. Hence $x_{p_n} \to x_p$ uniformly(since it is in the norm)

□

Before we prove 7 we need the following lemma

Lemma 11

Let $\sigma : I \to \mathbb{R}$ be a C^1 function. Suppose

$$\left|\frac{d\sigma}{dt}\right| \leq 2L\sigma(t)$$

on the interval I. Let t_0 be a fixed point on this interval. Then for all $t \in I$ we have

$$\sigma(t) \leq e^{2L|t-t_0|}\sigma(t_0)$$

Proof. First assume that $t > t_0$. Differentiating $\sigma(t)e^{-2Lt}$ we get

$$\frac{d}{dt}\sigma(t)e^{-2Lt} = \frac{d\sigma}{dt}e^{-2Lt} - 2L\sigma(t)e^{-2Lt} = \left(\frac{d\sigma}{dt} - 2L\sigma(t)\right)e^{-2Lt}$$

by product rule. But from the above conditions we have

$$-4L\sigma(t) \leq \frac{d\sigma}{dt} - 2L\sigma(t) \leq 0$$

implies the RHS is less than or equal to zero so $\sigma(t)e^{-2Lt}$ is decreasing. That is to say we have

$$\sigma(t_0)e^{-2Lt_0} \geq \sigma(t)e^{-2Lt}$$

where we get

$$\underbrace{\sigma\left(t_0\right)}\, e^{-2L(t-t_0)} \geq \underbrace{\sigma(t)}$$

Notice we are now close to our desired relation! We somehow just need to swap the bracketed terms above.

Indeed, suppose now $t < t_0$. Let I_1 be the interval $s \in I_1 \Leftrightarrow -s \in I$ and let $\sigma_1 : I_1 \to \mathbb{R}$ be the function $\sigma_1(s) = \sigma(-s)$. Then

$$\frac{d\sigma_1}{ds}(s) = -\frac{d\sigma}{ds}(-s) \leq 2L\sigma(-s) = 2L\sigma_1(s)$$

where the inequality in the center again follows due to the condition $\sigma(t_0)e^{-2Lt_0} \geq \sigma(t)e^{-2Lt}$. Thus with $s_0 = -t_0$ we have

$$\sigma_1(s) \leq e^{2L(s-s_0)}\sigma_1(s_0)$$

for $s > s_0$. Thus substituting $-t$ for s we have

$$\sigma(t) \leq e^{2L|t-t_0|\sigma(t_0)}$$

for $t < t_0$ as desired. □

Finally we have everything required to prove 4.Let

$$\sigma(t) = ||x(t) - y(t)||^2 = (x(t) - y(t)) \cdot (x(t) - y(t))$$

Then

$$\frac{d\sigma}{dt} = 2\left(\frac{dx}{dt} - \frac{dy}{dt}\right) \cdot (x(t) - y(t)) = 2(g(x(t)) - g(y(t))) \cdot (x(t) - y(t))$$

by product rule on dot products recall `Gibbs vector analysis`. So

$$\left|\frac{d\sigma}{dt}\right| \leq 2\,||g(x(t)) - g(y(t))||\,||x(t) - y(t)|| \leq 2L\,||x(t) - y(t)||\,||x(t) - y(t)|| \leq 2L\sigma(t)$$

where the second inequality follows from 3 while the first follows from **schwarz inequality**(refer to `functional analysis` under hilbert spaces). Now apply the previous lemma and we have successfully proven uniqueness.

2.2 Appendix: Picard theorem

Previously we have proven uniqueness of solutions to ODEs. This section is a continuation of the previous which is to answer the second question on *existence* of solutions to ODEs.

Definition 12

A real valued function $f : \mathbb{R} \to \mathbb{R}$ is **Lipschitz continuous** if there exists a poistive rael ocnstant K such that for all real x_1, x_2

$$|f(x_1) - f(x_2)| \leq K\,|x_1 - x_2|$$

Fact 13 (Taylor vs Picard Approximation)

Consider $f : D \to \mathbb{R}^n$ that is continuous(at least C^0) in t and lipschitz continuous in y. Then there exists some $\varepsilon > 0$ such that

$$y'(t) = f(y(t)), \quad y(t_0) = y_0$$

has a unique solution $y(t)$ on the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$. In which case it can be calculated by picard interation

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\, ds$$

and that it uniformly converges on that interval

$$\lim_{n\to\infty} y_n(t) = y(t)$$

As for Taylor series its just a way of approximating C^n functions via repeated use of mean value theorem.

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2 + \frac{y^{(3)}(t_0)}{3!}(t - t_0)^3 + \cdots$$

As long as C^n, meaning derivatives of f up till order n exists we can always express it as a taylor series. However it is not necessarily convergent, meaning not analytic since Taylor series is in the form of a power series.

You could in a sense see that picard approximation gets each term by integration while taylor gets each successive term by differentiation. And for each of them to converge they require certain conditions to do so. In fact both of them agree if y is analytic. This is because by definition, that implies y can be expressed by the taylor series. Now $f(y(t))$ is simply a function that relates $y(t)$ to its differential $y'(t)$. So after successive integrations as carried out in Picard iteration you should get back the taylor series. And obviously the conditions for analytic(radius of convergence) already implies lipschitz continuous so the criterion for picard iteration is satisfied as well. Let us consider some such examples of analytic functions

Example 14 (e^t)

Consider the ODE:

$$y'(t) = y, \quad y(0) = 1$$

Using Picard Iteration we have

1. Start with $y_0(t) = 1$.
2. Iteration 1: $y_1(t) = 1 + \int_0^t y_0(s)\, ds = 1 + \int_0^t 1\, ds = 1 + t$.
3. Iteration 2: $y_2(t) = 1 + \int_0^t y_1(s)\, ds = 1 + \int_0^t (1 + s)\, ds = 1 + t + \frac{t^2}{2}$.
4. Iteration 3: $y_3(t) = 1 + \int_0^t y_2(s)\, ds = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}$.

This is building the Taylor series for e^t.We know the solution is $y(t) = e^t$. Expanding e^t as a Taylor series:

$$y(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

Both methods give the same result, demonstrating their underlying similarity.

And of course this applies to the other trigonometric functions too like sin which are also analytic.

For this section let U be an open subset of $\mathbb{R}^n$ and $g : U \to \mathbb{R}^n$ be a C^1 map. Given a point $x_0 \in U$ we will let $B_\varepsilon(x_0)$ be the closed rectagle: $|x - x_0| < \varepsilon$. For ε sufficiently small this rectangle is contained in U. Let

$M = \sup \{|g(x)|, x \in B_\varepsilon(x_0)\}$(this is possible as g is continuous and it is on a compact set B_ε) and let

$$0 < T < \frac{\varepsilon}{M}$$

First we will prove

Theorem 15
There exists an integral curve $x(t), \quad -T \leq t \leq T$ of 2 with $x(t) \in B_\varepsilon(x_0)$ and $x(0) = x_0$.

Proof. The proof will be by a procedure known as **picard iteration**. Given a C^1 curve $x(t), \quad -T \leq t \leq T$ on the rectangle $B_\varepsilon(x_0)$ we will define its *picard iterate* to be the curve

$$\tilde{x}(t) = x_0 + \int_0^t g(x(s))ds$$

Then by monotonicity of reinmann integral and the bound M on $g(x)$ we have

$$|\tilde{x}(t) - x_0| \leq \int_0^t |g(x(s))| \, ds \leq M \, |t| \leq \varepsilon$$

for $|t| \leq T$ so this curve is well defined on the interval $-T \leq t \leq T$ and also contained in $B_\varepsilon(x_0)$. We now define by *induction* a sequence of curves $x_k(t), \quad -T \leq t \leq T$ by letting $x_0(t)$ be the constant curve $x(t) = x_0$ and letting

$$x_k(t) = x_0 + \int_0^t g(x_{k-1}(s))ds, \quad -T \leq t \leq T$$

That is, $x_k(t)$ is the *picard iterate* of $x_{k-1}(t)$. To get convergence first get some estimates for $|x_k(t) - x_{k-1}(t)|$. Let

$$C = \sup_{ij} \left| \frac{\partial g_i}{\partial x_j} \right|, \quad x \in B_\varepsilon(x_0)$$

which cleary exists as this is a continos function on a compact interval. Then from previously we use 3

$$|g(x) - g(y)| \leq L \, |x - y|$$

That is we have now simplified our expression and "gotten rid" of g and expressed all in terms of x_k. Finally we want to claim that

$$|x_k(t) - x_{k-1}(t)| \leq \frac{ML^{k-1} \, |t|^k}{k!}$$

which we can clearly see the infinite sum converges to the exponential function. We will prove that this is indeed possible by induction. For the base case we have

$$x_1(t) = x_0 + \int_0^t g(x_0)ds = x_0 + tg(x_0)$$

so

$$|x_1(t) - x_0(t)| = |x_1(t) - x_0(t)| \leq |t| \, M$$

Great now lets prove the induction step, that case $k-1$ implies case k. We see that

$$|x_k(t) - x_{k-1}(t)| = \int_0^t (g(x_{k-1}(s) - g(x_{k-2}(s)))ds$$

so

$$|x_k(t) - x_{k-1}(t)| \leq \int_0^t |g(x_{k-1}(s) - g(x_{k-2}(s)| \, ds \leq L \int_0^t |x_{k-1}(s) - x_{k-2}(s)|$$

hence by induction hypothesis(that is assuming case $k-1$ is true) we have

$$|x_k(t) - x_{k-1}(t)| \leq ML^{k-1} \int_0^t \frac{s^{k-1}}{k-1} ds \leq \frac{ML^{k-1}t^k}{k!}$$

whose infinite sum clearly *uniformly* converges to some exponential function since recall $e = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and that we have some N where $k \geq N$ onwards the sum converges and this applies to all t in the interval so the convergence is uniform. where we removed the absolute value because we are considering $\int_0^t$. Then it follows naturally from

$$x_k(t) = \left(\sum_{i=1}^{k} x_i(t) - x_{i-1}(t) \right) + x_0$$

that we have the uniform continuous limit

$$x(t) = \lim_{k \to \infty} x_k(t)$$

on $-T \leq t \leq T$. We will now show that this uniform convergence limit is indeed the solution. Finally recall from above we had

$$x_k(t) = x_0 + \int_0^t g(x_{k-1}(s))ds, \quad -T \leq t \leq T$$

Now take the limits on both sides, noting that you may move the limit inside the integral because we have uniform convergence recall `rudin`. We may also move into g as well as g is continuous which recall implies $\lim_{n\to\infty} g(x_n) = g(\lim_{n\to\infty} x_n)$ by definition. So althogether we have

$$x(t) = x_0 + \int_0^t g(x(s))ds$$

First it is clear that $x(0) = x_0$ so we have satisfied the IVP. Next recall `Rudin` that since $x(t)$ is continuous and the second term on the RHS is the anti-derivative of a continuous function and hence is C^1, thus $x(t) \in C^1$ so we may differenitate both sides to get

$$\frac{dx}{dt} = g(x(t))$$

as desired. □

Remark 16. *It can easily be seen that if g is C^k then $x(t)$ must be a C^{k+1} map.*

Fact 17

In summary we have proven that if $0 < T < \frac{\varepsilon}{M}$ there exists a solution, $x(t)$ to

$$\frac{dx}{dt} = g(x(t)), \quad x(0) = x_0$$

for $-T \leq t \leq T$ where $x(t) \in B_\varepsilon(x_0)$, $M = \sup\{|g(x)|, x \in B_\varepsilon(x_0)\}$ and $B_\varepsilon(x_0)$ be the closed rectangle: $|x - x_0| < \varepsilon$ contained in U.

That is to say there exist some closed subset of U in which we can define an integral curve that satisfies an certain IVP.

Fact 18

The technique we used to prove uniqess and existence is known as the **Picard–Lindelöf** theorem. However there is another theorem known as the **Peano existence**. To compare, essentially picard assumes more so concludes more

theorem	requirements	proves
Picard–Lindelöf	lipchitz continuous	existence and uniqueness
Peano	continous	local existence only

We consider a more "global" interpretation of this.

Proposition 19 (Extension to the Flow Theorem)

The theorem on flows generalizes our current result as follows:

1. Existence of Solutions Near x_0: The theorem guarantees that not only does a solution exist for the initial point x_0, but solutions also exist for initial points p in a small neighborhood V around x_0. In other words, if you slightly perturb the initial condition x_0 to some nearby point p, you still have a well-defined solution $x(p,t)$ to the ODE:

$$\frac{dx}{dt} = g(x(t)), \quad x(p,0) = p.$$

2. Smoothness of the Dependence on Initial Conditions: The map $F(p,t) = x(p,t)$ that takes an initial point $p \in V$ and time $t \in (-\epsilon, \epsilon)$ to the corresponding point on the integral curve is smooth (C^k). This means the solutions vary smoothly with respect to both initial conditions and time.

3. Flow as a Family of Maps: For each fixed time t, the flow map f_t is defined by $f_t(p) = x(p,t)$, which describes how each point p evolves after time t. The family of these maps f_t (as t varies) describes how points in V flow under the influence of the vector field g.

We will now prove the existence of "flows"

$$f_r(p+te_j) = \int_0^t \frac{d}{ds} f_r(p+se)j)ds = \int_0^t \frac{\partial f_r}{\partial p_j}(p+se_j)$$

for small t. Since $\frac{\partial f_r}{\partial p_j}$ and $f_r(p)$ converges uniformly to $h_j(p)$ and $f(p)$ respectively. We have

$$f(p+te_j) = \int_0^t h_j(p+e_j)ds$$

hence

$$\frac{\partial f}{\partial p_j} = h_j(p)$$

Lemma 20

For every $p \in B_{\frac{\varepsilon}{2}}(x_0)$ there exists a solution

$$x_p(t), \quad -\frac{T}{2} \leq t \leq \frac{T}{2}$$

of 2 with $x_p(0) = p$ and $x(t) \in B_\varepsilon(x_0)$

Remark 21. *We are now well defined IVPs defined at points in a half-sized neighbourhood around.*

Now let us generalize this result.

Theorem 22
Let W be a compact subset of U. Then for some $T > 0$ there exists for every $q \in W$ a solution $x_q(t)$, $\quad -T \leq t \leq T$ of 2 with $x_q(0) = q$

Remark 23. *Note, to solve the equation we just need to solve the IVPs. We don't need the solution to be defined on all of $t \in I$. So the T here does not contradict anything.*

2.3 Appendix: Flows

Following our discussion with regard to the domain in which there exists a solution to ODEs, we aim to extend this discussion "globally". That is we aim to investigate the equivalent for *vector fields on manifolds* and the *flows* they generate. This section will first discuss flows.

Let U be an open subset of $\mathbb{R}^n$ and $g : U \to \mathbb{R}^n$ be a C^1 function. We have shown in the previou section that for a given point $p_0 \in U$ there exists an $\varepsilon > 0$ a neighbourhood V of p_0 in U such that for every $p \in V$ where we define an IVP, there exists a solution(integral curve $x(p,t)$) with $x(p,0) = p$ for $-\varepsilon < t < \varepsilon$. Moreover we also know that $x(p,t)$ is continuous in *both* p and t. However this time we want to show that there exists a *family* of curves $x(p,t)$ such that when g is $C^{k+1}, k > 1$ then this map is a C^k map of $V \times (-\varepsilon, \varepsilon)$ into U. So far we know for each p the corresponding $x(p,t)$ which we denote as the curve satisfying the IVP $x(p,t_0) = p$ by $x_p(t)$, that if $g \in C^k$ then $x_p \in C^{k+1}$ in t but this is still not global, in a sense we still don't know how smooth x with respect to *both* (p,t) is, other than it is at least C^0 in (p,t).

Well you could let $\frac{dx}{dt} = g(x(p,t))$ be part of a linear system of one order ODEs. In that case we could express our system in terms of other possible orders of derivatives not just $\frac{dx}{dt}$ and we be assured there still exists a continuous unique solution to it from picard theorem above. For example consider the below where we also express derivatives $\frac{\partial}{\partial p_j}\frac{dx}{dt} = h(p,t)$ in our linear system(you would begin to realize why so when you study tangent bundles and geodesics)

Fix $1 \leq j \leq n$ and let

$$y(p,t) = \frac{\partial x}{\partial p_j}(p,t) = \left(\frac{\partial x_1}{\partial p_j}, \dots, \frac{\partial x_n}{\partial p_j}\right)$$

Then

$$\begin{aligned}
\frac{d}{dt}y(p,t) &= \frac{d}{dt}\frac{\partial}{\partial p_j}x(p,t) \\
&= \frac{\partial}{\partial p_j}\frac{d}{dt}x(p,t) \\
&= \frac{\partial}{\partial p_j}g(x(p,t)) \\
&= \sum_k \frac{\partial g}{\partial x_k}(x(p,t))\frac{\partial x_k}{\partial p_j} \\
&= h(x(p,t), y(p,t))
\end{aligned}$$

where

$$h(x,y,t) = \sum_k \frac{\partial g}{\partial x_k}(x,t) y_k$$

This gives us for each j a solution $(x(p,t), y(p,t))$ of the $(2n \times 2n)$ first order system of ODEs

$$\frac{dx}{dt}(p,t) = g(x(p,t)) \tag{5}$$
$$\frac{dy}{dt}(p,t) = h(x(p,t), y(p,t)) \tag{6}$$

with initial data

$$x(p,0) = p, \quad y(p,0) = \frac{\partial x}{\partial p_j}(p,0) = \frac{\partial p}{\partial p_j} = e_j$$

where the second term follows from the definition of $y(p,t)$ above and that $x(p,0) = p$.

Lemma 24

Starting $x_0(p,t) = p$ and $y_0(p,t) = e_j$ we generate interatively a sequence using picard iteration

$$(x_r(p,t), y_r(p,t)), \quad r = 1, 2, 3 \ldots$$

Then to prove existence with picard theorem with aim to prove the induction step at stage $r-1$

$$y_{r-1}(p,t) = \frac{\partial}{\partial p_j} x_{r-1}(p,t)$$

then at stage r

$$y_r(p,t) = \frac{\partial}{\partial p_j} x_r(p,t)$$

Proof. By picard iteration we first have

$$x_r(p,t) = p + \int_0^t g(x_{r-1}(p,s))ds$$

$$y_r(p,t) = e_j + \int_0^t h(x_{r-1}(p,s), y_{r-1}(p,s))ds$$

Thus

$$\begin{aligned}
\frac{\partial}{\partial p_j} x_j(p,t) &= e_j + \int_0^t Dg(x_{r-1}(p,s)) \frac{\partial}{\partial p_j} x_{r-1}(p,s)ds \\
&= e_j + \int_0^t Dg(x_{r-1}(p,s)) \frac{\partial}{\partial p_j} x_{r-1}(p,s)ds \\
&= e_j + \int_0^t Dg(x_{r-1}(p,s)) y_{r-1}(p,s)ds \\
&= e_j + \int_0^t h(x_{r-1}(p,s), y_{r-1}(p,s))ds \\
&= y_r(p,s)
\end{aligned}$$

Theorem 25

Assume $g : U \to \mathbb{R}^n$ is C^{k+1}. Then given a point $p_0 \in U$ there exists a neighbourhood V of p_0 in U and an $\varepsilon > 0$ such that

(a) for every $p \in V$ there exists an integral curve $x(p, t), \quad -\varepsilon < t < \varepsilon$ of 2 with $x(p, 0) = 0$ and

(b) the map

$$(p, t) \in V \times (-\varepsilon, \varepsilon) \to U, \quad (p, t) \to x(p, t)$$

is a C^k map

We denote this map by F, that is $F(p, t) = x(p, t)$ and for each $t \in (-\varepsilon, \varepsilon)$ let

$$f_t : V \to U$$

be the map $f_t(p) = F(p, t)$. The family of maps each corresponding to a p denoted by $f_t, \quad -\varepsilon < t < \varepsilon$ is called the **flow** generated by the system of equations 2

Remark 26. *recall*

$$(p, t) \in \underbrace{V}_{\text{corresponds to an IVP}} \times \underbrace{(-\varepsilon, \varepsilon)}_{\text{the range of the integral curve of the solution}} \to U, \quad (p, t) \to x(p, t)$$

Basically there is a unique curve passing through each point in this neighbourhood and these curves are continuous in both p and t

Proof. Base case. We have shown that if g is C^2 in x then h is C^1 in (x, y). Note this is the lowest case because we need both g and h to be at least C^1 to their respective variables to use picard theorem

$$\frac{dx}{dt}(p, t) = g(x(p, t)) \tag{7}$$
$$\frac{dy}{dt}(p, t) = h(x(p, t), y(p, t)) \tag{8}$$

with initial data

$$x(p, 0) = p, \quad y(p, 0) = \frac{\partial x}{\partial p_j}(p, 0) = \frac{\partial p}{\partial p_j} = e_j$$

Now recall since $y(p, t) = \frac{\partial x}{\partial p_j}(p, t)$ we can infer that x is C^1 in (p, t) since its derivative with respective to p as shown is just y which we know is continuous in p, t. On the other hand its derivative $\frac{\partial x}{\partial t}$ is simply g which we know is continuous in x and that x is continuous in p, t so $\frac{\partial x}{\partial t} = g \circ x$ is continuous in p, t. Therefore we now have in the base case: $g \in C^2$ implies $x \in C^1$. Now lets prove the inductive step. Our inductive for case k is $g \in C^{k+1}$ implies $x \in C^k$.

Suppose $g \in C^{k+1}$. But we know that $g \in C^{k+1}$ implies g^k so hence we already know by induction hypothesis that $x \in C^{k-1}$. Then considering $\frac{dx}{dt} = g(t, p)$ and $\frac{\partial x}{\partial p_j} = y(t, p)$ again, we know that there exists a higher derivative of x that is continuous. Hence x is at the very least 1 order higher than it currently is, that is $x \in C^k$ as desired.

2.4 Appendix: Vector fields

Example 27 (Constant vector field)
Given a fixed vector $v \in \mathbb{R}^n$ the function

$$p \in \mathbb{R}^n \to (p, v)$$

is a vector field

Example 28
In particular let $e_i, i = 1, \ldots, n$ be the standard basis vectors of $\mathbb{R}^n$. If $v = e_i$ we will denote the constant vector field

$$p \in \mathbb{R}^n \to (p, e_i)$$

by $\frac{\partial}{\partial x_i}$

Example 29 (Tangent Vector field)
$(p, e_i), i = 1, \ldots, n$ are the basis of $T_p\mathbb{R}^n$ and that we may write

$$v = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}$$

where $g_i : U \to \mathbb{R}$ is the function $p \to g_i(p)$. Recall we say v is a C^k vector field if the g_i's are in $C^k(U)$. Also recall the **lie differentiation** operation on vector fields where

$$Df(p)v = L_v f(p)$$

and

$$L_v f = \sum g_i \frac{\partial}{\partial x_i} f$$

we covered a bit of these in our `18.101` notes.

Definition 30
A C^1 curve $\gamma : (a, b) \to U$ is an integral curve of v if for all $a < t < b$ and $p = \gamma(t)$

$$\left(p, \frac{d\gamma}{dt}(t)\right) = v(p)$$

That is, if v is a vector field given by 29 and $g : U \to \mathbb{R}^n$ is the function $(g_1, \ldots, g_n)$ the condition for $\gamma(t)$ to be an integral curve of v is that it satisfy the system of ODEs

$$\frac{d\gamma}{dt}(t) = g(\gamma(t)) \tag{9}$$

From ODE theory in the previous 3 sections we can infer

Theorem 31 (Existence)
Given a point $p_0 \in U$ and $a \in \mathbb{R}$ there exists an interval $I = (a - T, a + T)$, a neighbourhood U_0 of p_0 in U and for every $p \in U_0$ an *integral curve* $\gamma_p : I \to U$ with $\gamma_p(a) = p$

Theorem 32 (Uniqueness)
Let $\gamma_i : I_i \to U, i = 1, 2$ be the itnegral curves. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$ thenn $\gamma_1 \equiv \gamma_2$ on $I_1 \cap I_2$ and the curve $\gamma : I_1 \cup I_2$ defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in I_1 \\ \gamma_2(t) & t \in I_2 \end{cases}$$

is an integral curve

The second half of this section will discuss properties of vector fields required to extend the notion of "vector field" to manifolds.

Let U and W be open subsets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and let $f : U \to W$ be a C^{k+1} map. If v is a C^k vector field on U and w is a C^k vector field on W we will say that v and w are "f-related" if for all $p \in U$ and $q = f(p)$

$$df_p(v_p) = w_q$$

writing

$$v = \sum_{n=1}^{\infty} v_i \frac{\partial}{\partial x_i}, \quad v_i \in C^k(U)$$

and

$$w = \sum_{j=1}^{m} w_j \frac{\partial}{\partial y_j}, \quad w_j \in C^k(V)$$

3 curves

3.1 Regular Curves & Arc Length

Definition 33
A **parameterized differentiable curve** is a differentiable map $a : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ on the real line $\mathbb{R}$ into $\mathbb{R}^3$. That is to say

$$a(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3, t \in I$$

The variable t is called the **parameter** of the curve. **Differentiable** meaning that the functions $x(t), y(t), z(t)$ are differentiable. So we can define the tangent vector

$$(x'(t), y'(t), z'(t)) = a'(t) \in \mathbb{R}^3$$

The image of set $a(I) \subset \mathbb{R}^3$ is called the **trace** of a.

Example 34

Consider the trace of parametrized curve in $\mathbb{R}^3$ below

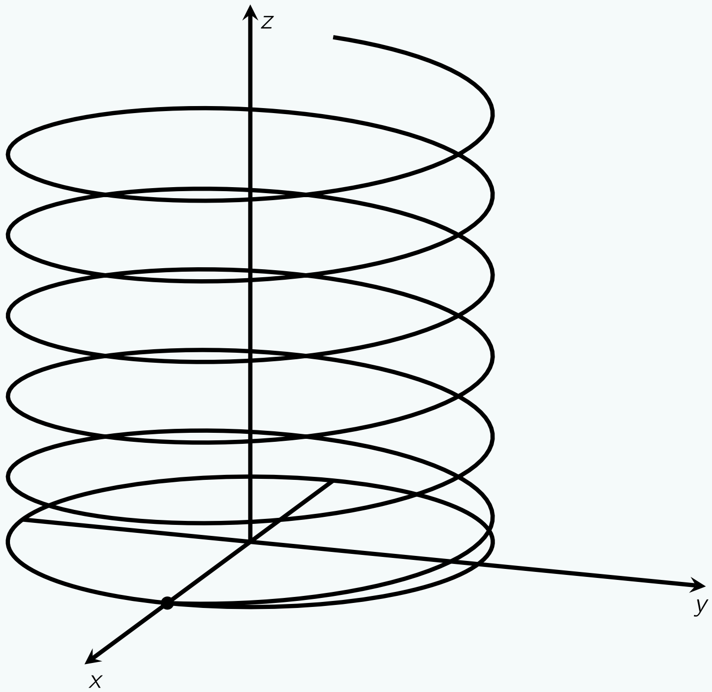

Figure 1: Trace of $a(t) = (a\cos t, a\sin t, bt)$, $t \in \mathbb{R}$

Definition 35

A parameterized differentiable curve $a : I \to \mathbb{R}^3$ is said to be **regular** if $a'(t) \neq 0$ for all $t \in I$

Definition 36

according to **big o notation** if there exists constants M, k such that

$$f(x) \leq Mg(x)$$

for $x > k$ we write

$$f(x) = O(g(x))$$

Proposition 37

For a polynomial f of degree n then $f = O(x^n)$

Proof. Refer to `MIT Math for CS 6.042` □

Consider a partition

$$P : a = t_0 < t_1 < \ldots < t_n = b$$

and consider

$$\ell(a, P) = \sum_i ||a(t_i) - a(t_{i-1})||$$

and the **mesh** of P is defined as the $\max_i(t_i - t_{i-1}) \to 0$

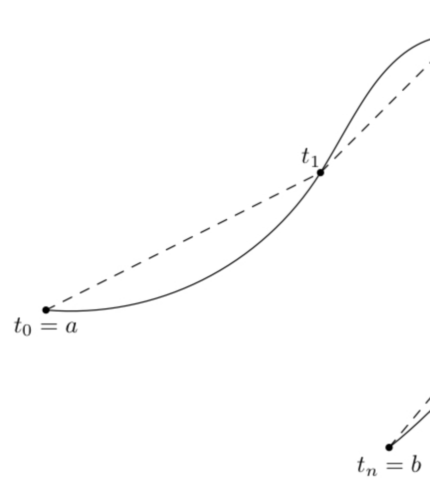

Figure 2: Riemann Arc Integral

Proposition 38

As the mesh of the partition approahces zero

$$\ell(a, P) \to \int_a^b ||a'(t)||\, dt$$

where $||.||$ denotes the standard *euclidean norm*

Proof. Because $[a, b]$ is compact and that since a is smooth, the derivative of any order of a is continuous and thus is bounded. So we may let $||a''(t)|| < C$. Hence for any $t > t_{i-1}$ we may write

$$||a'(t) - a'(t_{i-1})|| = \left|\left|\int_{t_{i-1}}^t a''(r)dr\right|\right| \leq \int_{t_{i-1}}^t ||a''(r)||\, dr \leq C[t - t_{i-1}]$$

where the second equality follows by the fundamental theorem of calculus. Therefore we have

$$\int_{t_{i-1}}^{t_i} ||a'(t) - a'(t_{i-1})||\, dt = O([t_i - t_{i-1}]^2)$$

where we have used big O notation. Notice that $\int_{t_{i-1}}^{t_i}[t - c]dt = (t_i - t_{i-1})((t_i + t_{i-1})/2 - t_{i-1}) = (t_i - t_{i-1})^2/2$. So we may do

$$\begin{aligned} a(t_i) - a(t_{i-1}) &= \int_{t_{i-1}}^{t_i} [a'(t_{i-1}) + a'(t) - a'(t_{i-1})]dt \\ &= (t_i - t_{i-1})a'(t_{i-1}) + \int_{t_{i-1}}^{t_i} [a'(t) - a'(t_{i-1})]dt \end{aligned}$$

This implies

$$||a(t_i) - a(t_{i-1})|| = (t_i - t_{i-1})\, ||a'(t_{i-1})|| + O([t_i - t_{i-1}]^2)$$

since $t_i - t_{i-1} > 0$. Hence

$$\int_{t_{i-1}}^{t_i} ||a'(t)||\, dt = \int_{t_{i-1}}^{t_i} ||a'(t) - a'(t_{i-1} + a'(t_{i-1})||\, dt$$
$$\leq \int_{t_{i-1}}^{t_i} ||a'(t)||\, dt = \int_{t_{i-1}}^{t_i} ||a'(t) - a'(t_{i-1}||\, dt + (t_i - t_{i-1})\, ||a'(t_{i-1})||$$
$$= ||a(t_i) - a(t_{i-1})|| + O([t_i - t_{i-1}]^2)$$

Summing up and taking the mesh of the partition $\to 0$ the conclusion follows

Definition 39

$$(u \wedge v) \cdot w = \det(u, v, w)$$

where vectors $u, w, v \in \mathbb{R}^3$ then

This is essentially the triple scalar product and the $\wedge$ here refers to the cross product. Refer to `Gibbs vector analysis` for more.

3.2 the local theory of curves paramterized by arc length

Definition 40 (Curvature)
Let $a : I \to \mathbb{R}^3$ be a curve parameterized by arc lenght $s \in I$. The number $||a''(s)|| = k(s)$ is called the **curvature** of a at s.

note that $K(s) \equiv 0$ if and only if a is a straight line. The curvature remains unchanged by change of orientation(due to the absolute value $||a''(s)||$ in its definition)

Proposition 41

$$||a'(s)|| = 1$$

implies

$$a''(s) \perp a'(s)$$

Proof. If

$$||a'(s)|| = a'(s) \cdot a'(s) = 1$$

then we know $a'(s) \neq 0$(positive definiteness of dot product). Differentiating both sides with respect to s we have

$$a'(s) \cdot a''(s) + a''(s) \cdot a'(s) = 2(a'(s) \cdot a''(s)) = 0$$

by symmetry of dot product Hence $a''(s) \perp a'(s)$.

Definition 42
We define the unit **normal vector** to be

$$n(s) = \frac{a''(s)}{||a''(s)||} \quad \text{or} \quad n(s)k(s) = a''(s)$$

At this point we define $t(s)$ to be the unit **tangent vector** $a'(s)$ which is also of unit length by definition. The plane spanned by $a'(s)$ and $a''(s)$ is called the **osculating plane**. If $k(s) = 0$ at some point we say s is a singular point of order 1.

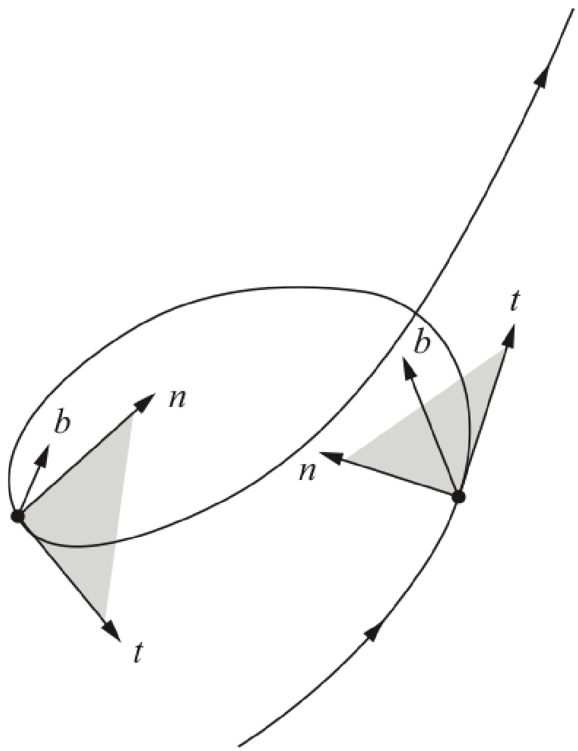

Figure 3: Osculating Plane

Definition 43
The unit **binormal vector** $b(s)$ is defined by $t(s) \times n(s)$

we know that $t(s) \perp n(s)$ by definition. Therefore $|t(s)|\,|n(s)| \sin 90 = 1$ so $b(s)$ is a unit vector. Therefore $|b'(s)|$ measures the rate of change of neighbouring osculating planes(as in the change in the directin of the unit normals that define them).

Proposition 44
In $\mathbb{R}^3$ we have that $b'(s)$ is parallel to $n(s)$

Proof. Notice that

$$b'(s) = \underbrace{t'(s) \wedge n(s)}_{=0} + t(s) \wedge n'(s)$$

Since $t'(s) = a''(s)$ but it is in the same direction as $n(s) = \frac{a''(s)}{||a''(s)||}$ so cross product of parallel vectors is zero. Now we know by 41 that $t(s) \perp t'(s) \Rightarrow t \perp n(s)$ and $n(s) \perp n'(s)$. Therefore n is a vector $\perp$ to both t and n'. Since $\mathbb{R}^3$ it must be that $b'(s) \parallel n(s)$

Definition 45 (Torison)
Let $a : I \to \mathbb{R}^3$ be a curve parameterized by arc length s such that $a''(s) \neq 0, s \in I$. The number $\tau(s)$ defined by $b'(s) = \tau(s)n(s)$ is called the **torison** of a at s.

Unlike the curvature the torison may be positive or negative(which has a geometric interpretation as we will see later)

Fact 46
Geometrically you could see torison as a result of "twists" and curvature as a result of "turns"

Proposition 47
The curve a is a plane curve if and only if has zero torison(meaning $\tau(s) = 0$ everywhere)

Proof. From 3 we know that the osculating plane agrees withe plane of the curve at all points(direction wise so $b' = 0$ and assume $k \neq 0$). For the converse consider

$$\tau(s) \Leftrightarrow b'(s) = 0 \Leftrightarrow b(s) = b_0$$

for some fixed/constant vector b_0. Then we have

$$\frac{d}{ds}[a(s) \cdot b_0] = a'(s) \cdot b_0 = t(s) \cdot b_0 = 0$$

since $b(s) \perp t(s)$ by definition. Again this means the osculating plane agrees with plane of the curve at all points s.

Theorem 48 (Frenet Formulas)
Let $a : I \to \mathbb{R}^3$ be parameterized by arc length. So far we associated three orthogonal unit vectors

$$\begin{aligned} t(s) &= a'(s) \\ n(s) &= \frac{a''(s)}{||a''(s)||} \\ b(s) &= t(s) \times n(s) \end{aligned}$$

Then we always have

$$t'(s) = k(s)n(s) \tag{1}$$
$$n'(s) = -k(s)t(s) - \tau(s)b(s) \tag{2}$$
$$b'(s) = \tau(s)n(s) \tag{3}$$

Proof. (1) and (3) follow by defnition. (3) follows by taking $b = t \times n \to n = b \times t$ then differeitate both sides applying product rule. That is

$$\begin{aligned} n'(s) &= (b(s) \wedge t(s))' \\ &= b'(s) \wedge t(s) + b(s) \wedge t'(s) \\ &= \tau(s)n(s) \wedge t(s) + b(s) \wedge k(s)n(s) \\ &= -\tau(s)b(s) - k(s)t(s) \end{aligned}$$

Thast line follows since $b = t \times n \to -b = n \times t$ and similar logic for the other term.

Lemma 49

The torison τ of a curve is given by

$$\tau(s) = -\frac{a'(s) \wedge a''(s) \cdot a'''(s)}{||k(s)||^2}$$

Definition 50

A **translation** by a vector v in $\mathbb{R}^3$ is a map $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $Au = u + v$

Definition 51

A linear map $p : \mathbb{R}^3 \to \mathbb{R}^3$ is an **orthgonal transformation** if $pu \cdot pv = u \cdot v$ for all vectors $u, v \in \mathbb{R}^3$

Definition 52

A **rigid motion** in $\mathbb{R}^3$ is the result of composing a translation with an orthogonal tranformation with positive determinant; that is if A is a rigid motion then there exists an orthgonal map p of $\mathbb{R}^3$ with $\det p > 0$ and a vector c such that $Av = pv + c$.

Lemma 53

The norm of a vector and the angle between vectors $0 \leq \theta \leq \pi$ are invariant under orthogonal transformations with positive determinant

Proof. Let $p : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation with positive determinant. Choose an arbitrary v in $\mathbb{R}^3$. Recall `artin algebra` sine orthogonal transformation matrix we then we have

$$||pv||^2 = pu \cdot pv = v \cdot v = ||v||^2$$

Similary for the angle

$$\cos\overline{\theta} = \frac{pu \cdot pv}{||pu||\,||pv||} = \frac{u \cdot v}{||u||\,||v||} = \cos\theta$$

Lemma 54

The vector product of two vectors is invariant under orthogonal transformations with positive determinant.Note that because we are assuming rotations so in our case determinant is equal one.

Proof. Let $p : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation with positive determinant. Choose an arbitrary u, v in $\mathbb{R}^3$ and let θ be the angle between them. Because $\det p \neq 0$ so the map p is onto. That is for every vector $w \in \mathbb{R}^3$ there exists $w = p\overline{w}$ where vector $\overline{w} \in \mathbb{R}^3$. Therefore consider the triple scalar product

$$(pu \wedge pv) \cdot w = \det(pu, pv, p\overline{w}) = \det(p)\det(u, v, \overline{w}) = \det(p)(u \wedge v) \cdot \overline{w}$$

On the other hand since p is an orthogonal transformation we have

$$(u \wedge v) \cdot \overline{w} = p(u \wedge v) \cdot p\overline{w} = p(u \wedge v) \cdot w$$

So on comparison with the above we have, multiping our latter equation by $\det(p)$ on all sides we see that

$$\det(p)(u \wedge v) \cdot \overline{w} = (pu \wedge pv) \cdot w = \det(p)p(u \wedge v) \cdot w$$

where the middle term is the 1st term on our first equation and the last term is the last term in our second equation. Then knowing that $\det(p) = 1$ it can be implied

$$(pu \wedge pv) = p(u \wedge v)$$

as desired

Theorem 55

The arc length, curature and the torison of a parameterized curve are invariant under rigid motion

Proof. For arc length consider

$$\int_a^b \left|\frac{da}{dt}\right| dt = \int_a^b \left|\frac{d(M \circ a)}{dt}\right| dt$$

where M represents the rigid motion. Firstly translation does not affect the value of derivative since

$$\frac{d(M \circ a)}{dt} = \lim_{t \to 0} \frac{(a(x+t)+c) - (a(x)+c)}{t} = \lim_{t \to 0} \frac{a(x+t) - a(x)}{t} = \frac{da}{dt}$$

Moreover we have proven than orthogonal transformations preserve the norm the arc lenghth is invariant. To see this consider the next steps for proving invariane of curvature, the logic is the same(recall `vector analysis gibbs` for the below)

$$k_A(s) = ||(Aa(s))''|| = ||Aa''(s)|| = ||a''(s)|| = k(s)$$

Now finally for torison recall

$$b'(s) = t(s) \wedge n'(s) = \tau(s)n(s)$$

Now consider the case when all terms have been acted upon by A

$$Ab'(s) = At(s) \wedge An'(s) = \tau_A(s)An(s)$$

but recall from our above lemma that

$$(At(s) \wedge An'(s)) = A(t(s) \wedge n'(s)) = A(\tau(s)n(s)) = \tau(s)An(s)$$

so it must be that $\tau_A(s) = \tau(s)$ as desired

Theorem 56 (Fundamental Theorem of the local theory of curves)
Given differentiable functions $k(s) > 0$ and $\tau(s)$ on an interval $I = (a, b)$ there exists a regular parameterized curve $a : I \to \mathbb{R}^3$ with that curvature and torsion.

Moreover this curve is unique up to a rigid motion in the sense that if a and $\overline{a}$ both satisfy the condition then there is an orthogonal linear map $p : \mathbb{R}^3 \to \mathbb{R}^3$ with $\det p > 0$ such that $\overline{a} = p \circ a + c$. Intuitively, this means "uniqueness up to rotation and translation"

Proof. The existence part we will return after ODE theory. For the uniquenss part of the fundamental theorem, consider two curves $a(s)$ and $\overline{a}(s)$ satisfying the conditions(same curvature and torison). The big idea in our following proof is construct such a rigid motion that when applied to one of the curves then $a(s) = \overline{s}$ on all s. Hence proving they only differ by a rigid motion as desired. First, at an arbituary point say $a(s_0)$, its is categorized by its position and direction of $t(s_0), t(s_0), b(s)$. So it is possible to perform rigid motion to transalate and rotate $\overline{a}$ such that $a(s_0) = \overline{a}(s_0)$. That is to say $n(s_0) = \overline{n}(s_0)$,$b(s_0) = \overline{b}(s_0)$,$t(s_0) = \overline{t}(s_0)$. So now we consider the frenet equations in general for all s after such this rigid motion as been applied . We could still use the condition that curvature,torison,arc lenght is the same as reflected in the equations below even after rigid motion(recall the previous lemma).

$$\begin{aligned} t'(s) &= k(s)n(s) \\ n'(s) &= -k(s)t(s) - \tau(s)b(s) \\ b'(s) &= \tau(s)n(s) \end{aligned}$$

and

$$\begin{aligned} \overline{t}'(s) &= k(s)\overline{n}(s) \\ \overline{n}'(s) &= -k(s)\overline{t}(s) - \tau(s)b(s) \\ \overline{b}'(s) &= \tau(s)\overline{n}(s) \end{aligned}$$

Clearly the goal is show

$$\frac{d}{ds}\{|t - \overline{t}|^2 + |n - \overline{n}|^2 + |b - \overline{b}|^2\} = 0$$

after which we can conlude the difference between t, n, s and $\overline{t}, \overline{n}, \overline{s}$ remains constant over s. And since we already know that at $s = s_0$ the difference is zero, then the difference is hence zero throughout the whole curve so $a(s) = \overline{a}(s)$ as desired.Indeed,

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\{|t - \overline{t}|^2 + |n - \overline{n}|^2 + |b - \overline{b}|^2\} &= \langle t - \overline{t}, t' - \overline{t}' \rangle + \langle b - \overline{b}, b' - \overline{b}' \rangle + \langle n - \overline{n}, n' - \overline{n}' \rangle \\ &= k\langle t - \overline{t}, n - \overline{n} \rangle + \tau\langle b - \overline{b}, n - \overline{n} \rangle - k\langle n - \overline{n}, t - \overline{t} \rangle \\ &= 0 \end{aligned}$$

Definition 57
Assume that $a : I \to \mathbb{R}^2$ so we have a tangent vector $t(s)$. Then we can give a unique normal vector $n(s)$ such that (t, n) forms a positive basis. We define the **signed curvature** according to

$$t'(s) = k(s)n(s)$$

Definition 58

Given a *plane curve* $a : I \to \mathbb{R}^2$ we define the **evolute** of a to be the curve

$$\beta(s) = \alpha(s) + \frac{1}{k(s)} n(s)$$

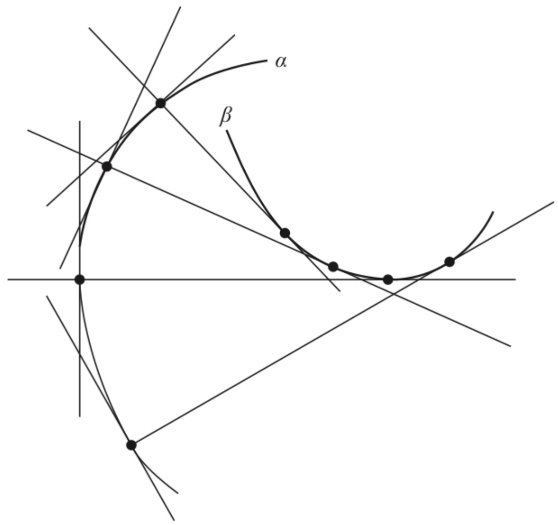

Figure 4: $\beta(s)$, the evolute of curve $a(s)$

Proposition 59

Let $a(s)$ be a curve parameterized by arc lenght and β be its evolute. Then

(a) tangent at s of the evolute of a is the normal to curve a at s

(b) Let s_1 and s_2 be points and consider the intersection p of their normal vectors to a. As $s_1, s_2 \to s$, $p \to \beta(s)$. In other words the intersection of the normals of any 2 points on curve $a(s)$ will converge onto a point on the evolute as these 2 points get arbitrarily close on curve a

We call $\beta(s)$ the **center of curvature** of $a(s)$

Remark 60. *$n(s) = (-y'(s), x'(s))$ is a good choice because $n(s) = \frac{a''(s)}{||a''(s)||}$ where $a''(s) = (x''(s), y''(s))$. Instead of explicIty going to find $||a''(s)||$ then scalar dividing $a''(s)$ use the fact that $t(s) \cdot n(s) = 0$ as they are by definition perpendicular to each other. A natural choice will be $n(s) = (-y'(s), x'(s))$ where you can easily tell it is of unit length as desired.*

Proof. For (a) first recall 47 $\tau = 0$ everywhere since a is a plane curve. So $n'(s) = -kt$ by the frenet equations. Then

$$\beta'(s) = a'(s) - \frac{1}{k(s)^2} k'(s) n(s) + \frac{1}{k} n'(s) = t - \frac{k'}{k^2} n + \frac{1}{k}(-kt) = -\frac{k'}{k^2} n(s)$$

which is parallel to $n(s)$ since k and k' are both scalar fields/real valued functions.

For (b) We are given $a(s) = (x(s), y(s))$, $t(s) = (x'(s), y'(s))$, and $n(s) = (-y'(s), x'(s))$, with the relation $t'(s) = k(s)n(s)$.

Step 1: Deriving $k(s)$.

Compute $t'(s)$:

$$t'(s) = (x''(s), y''(s))$$

Since $t'(s) = k(s)n(s)$, we have:

$$(x''(s), y''(s)) = k(s)(-y'(s), x'(s))$$

This gives:

$$x''(s) = -k(s)y'(s), \quad y''(s) = k(s)x'(s)$$

Multiply $x''(s) = -k(s)y'(s)$ by $x'(s)$ and $y''(s) = k(s)x'(s)$ by $y'(s)$:

$$x'(s)x''(s) = -k(s)y'(s)x'(s), \quad y'(s)y''(s) = k(s)x'(s)y'(s)$$

Subtract the two equations:

$$y'(s)y''(s) - x'(s)x''(s) = k(s)$$

Thus, the curvature is:

$$k(s) = x'(s)y''(s) - y'(s)x''(s)$$

Step 2: Computing λ_1.

We are given:

$$p = a(s_1) + \lambda_1 n(s_1) = a(s_2) + \lambda_2 n(s_2)$$

which gives:

$$(x(s_1), y(s_1)) + \lambda_1(-y'(s_1), x'(s_1)) = (x(s_2), y(s_2)) + \lambda_2(-y'(s_2), x'(s_2))$$

Equating components:

$$x(s_1) - \lambda_1 y'(s_1) = x(s_2) - \lambda_2 y'(s_2)$$

$$y(s_1) + \lambda_1 x'(s_1) = y(s_2) + \lambda_2 x'(s_2)$$

Solve for λ_2 from the first equation:

$$\lambda_2 = \frac{x(s_1) - x(s_2) + \lambda_1 y'(s_1)}{y'(s_2)}$$

Substitute into the second equation:

$$y(s_1) + \lambda_1 x'(s_1) = y(s_2) + \frac{x'(s_2)(x(s_1) - x(s_2) + \lambda_1 y'(s_1))}{y'(s_2)}$$

Simplifying:

$$\lambda_1 \left(x'(s_1) - \frac{x'(s_2)y'(s_1)}{y'(s_2)} \right) = \frac{x'(s_2)(x(s_1) - x(s_2))}{y'(s_2)} + y(s_2) - y(s_1)$$

Thus:

$$\lambda_1 = \frac{x'(s_2)(x(s_1) - x(s_2)) + y'(s_2)(y(s_2) - y(s_1))}{y'(s_2)x'(s_1) - x'(s_2)y'(s_1)}$$

Step 3: Taking the limit $s_1, s_2 \to s$.

Expand $x(s_1)$, $x(s_2)$, $y(s_1)$, and $y(s_2)$ using Taylor series around s where we ignored the high power terms like ΔS_1^3. So we have with big O notation,

$$x(s_1) = x(s) + x'(s)\Delta s_1 + \frac{1}{2}x''(s)\Delta s_1^2 + O(\Delta S_1^3), \quad x(s_2) = x(s) + x'(s)\Delta s_2 + \frac{1}{2}x''(s)\Delta s_2^2 + O(\Delta S_2^3)$$

$$y(s_1) = y(s) + y'(s)\Delta s_1 + \frac{1}{2}y''(s)\Delta s_1^2 + O(\Delta S_1^3), \quad y(s_2) = y(s) + y'(s)\Delta s_2 + \frac{1}{2}y''(s)\Delta s_2^2 + O(\Delta S_2^3)$$

where $\Delta s_1 = s - s_1$ and $\Delta s_2 = s - s_2$ so $\Delta s = \Delta s_1 - \Delta s_2$ In the limit $s_1, s_2 \to s$, the numerator becomes:

$$(x'(s)^2 + y'(s)^2)(\Delta s_1 - \Delta s_2)$$

and the denominator becomes:

$$(x'(s)y''(s) - y'(s)x''(s))(\Delta s_1 - \Delta s_2)$$

Thus:

$$\lim_{s_1, s_2 \to s} \lambda_1 = \frac{x'(s)^2 + y'(s)^2}{x'(s)y''(s) - y'(s)x''(s)} = \frac{1}{k(s)}$$

where the numerator is 1 because $t(s)$ is a unit vector. Then we have

$$p = a(s_1) + \lambda_1 n(s_1) \quad \to \quad \beta(s) = \alpha(s) + \frac{1}{k(s)} n(s)$$

as desired $\square$

Theorem 61 (Isoperimetric Inequality)
Let a be a simple closed plane curve of lenght ℓ and let A be the area of the bounded region. Then

$$\ell^2 \geq 4\pi A$$

with equality if and only if a gives a circle.

4 regular surfaces

4.1 inverse images of regular values

Definition 62
A **regular surface** is a subset $S \subseteq \mathbb{R}^3$ such that for every point p of S there is a neighbourhood p which we denote as $V \subseteq \mathbb{R}^3$ which is *diffeomorphic* to an open subset of $\mathbb{R}^2$. By **diffeomoprhic** we mean there is map

$$\overline{x} : \mathbb{R}^2 \supset U \to V \cap S \subset \mathbb{R}^3$$

such that $\overline{x}$ is differentiable and bijective with continuous inverse. Moreover the differential $d\overline{x} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective(full rank). We call $\overline{x}$ a **parameterization** at x and the ordered triple $(x, U, V \cap S)$ a **coordinate chart**

Proposition 63
Let $S \subset \mathbb{R}^3$ and $p \in S$. Then there exists a neighbourhood V of $p \in S$ such that V is the graph of a differetiable function in one of the following forms:

$$x = f(y, z), y = f(z, x), z = f(x, y)$$

Proof. By definition of regular surface, for each p on it, the differential $d\overline{x}_p : \mathbb{R}^2 \to \mathbb{R}^3$ is injective we can take a parameterization at p. So we know one of the jacobian determinants is nonzero by hypothesis.

Example 64

For a sphere $S^2 = \{(x, y, z)|x^2 + y^+ z^2 = 1\}$ we can take a projection onto the xy, yz, zx planes for a total of six charts(recall injective so we have $\binom{3}{2} = 6$ planes of $\mathbb{R}^3$ in the domain to choose from. For example on the charts. For example one of the charts is

$$\mathbb{R}^2 \supset \{u^2 + v^2 < 1\} \to \mathbb{R}^3 \quad \text{by} \quad (u, v) \mapsto (u, v, \sqrt{1 - u^2 - v^2})$$

First it is clear the map is differentiable. To see full rank consider

$$\frac{\partial(u, v)}{\partial(u, v)} = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

Finally to construct a continuous inverse simply take the projection $(x, y, z) \mapsto (x, y)$(reverse calculate the above). In which case you also find that it is continuous.

Definition 65

Given a differentiable map $F : U \to \mathbb{R}^m$ (where $U \subset \mathbb{R}^n$ is open) a **critical point** $p \in U$ of F is one such that $(dF)_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective. The value of $F(p)$ is called a **critical value**. Values in the image $\mathbb{R}^m$ which are not critical are called **regular values**

Example 66

Let us relate this to a standard result we know in elementary calculus that the derivative of critical/turning points are 0. In the case of $f : \mathbb{R}^n \supset U \to \mathbb{R}$ for every critical point p, we know that df_p is not onto $\mathbb{R}$. That can only mean $df_p = \{0\}$.

Theorem 67

Let $f : \mathbb{R}^3 \supset U \to \mathbb{R}$ be differentiable and c be a regular value. Then the pre-image

$$\{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = c\}$$

is a regular surface

Proof. Recall `Munkres`. Essentially use canonical submersion theorem.

While we know what being differentiable means on real numbers, we would now like to know what differentiable means on manifolds now.

Definition 68

Let $f : V \subset S \to \mathbb{R}$ be a function defined in an open subest V of a regular surface S. Then f is **differentiable** at $p \in V$ if for some parameterization $x : U \subset \mathbb{R}^2 \to S$ the composition

$$U \xrightarrow{x} V \xrightarrow{f} \mathbb{R}$$

is differentiable at $x^{-1}(p)$. In other words $(f \circ x)'(x^{-1}(p))$ exists.

It is clear to see this is independent of choice of parameterization x. Suppose $f \circ x$ is differentiable. Now let another parameterization be $y : W \subset \mathbb{R}^2 \to S$ then we know $h = x \circ y^{-1}$ is also a diffeomorphism recall transition functions in `munkres`. So it follows that $f \circ y = f \circ x \circ h$ is also differentiable.

4.2 tagent plane: the differential of a map

Definition 69

A **tangent vector** to $p \in S$ is the tangent vector $a'(0)$ of a smooth curve $a : (-\varepsilon, \varepsilon) \to S$ with $a(0) = p$.

Proposition 70

Let $x : U \subset \mathbb{R}^2 \to S$ be a parameterization of a regular surface S and let $q \in U$. The vector subspace of dimension 2

$$dx_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincides with the set of tangent vectors to S at $x(q)$.

Proof. Let w be a tangent vector at $x(q)$, that is let $w = a'(0)$ where $a : (-\varepsilon, \varepsilon) \to x(U) \subset S$ is differentiable and $a(0) = x(q) = p$.

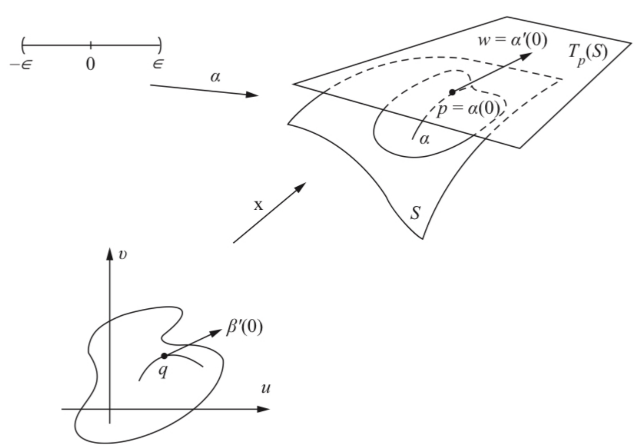

Figure 5: Tangent space

The curve $\beta = x^{-1} \cdot a : (-\varepsilon, \varepsilon) \to U$ is differentiable since it is a composition of differentiable funtions and so it is continuous too. By definition

$$w = \frac{d}{dt} a(0) = \frac{d}{dt}(x \circ \beta(0)) = [Dx(q)][\beta'(0)] = dx_q(\beta'(0))$$

by chain rule

Thus, $w \in dx_q(\mathbb{R}^2)$ as desired $\square$

Having defined maps to tangent space from a plane let us define maps between tangent spaces in $\mathbb{R}^3$. Consider

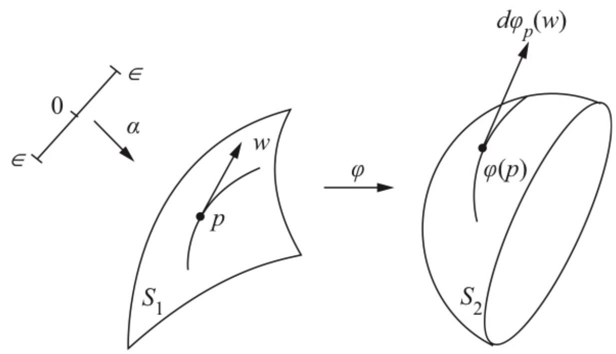

Figure 6: Tangent Space 2

Where S_1 and S_2 are two regular surfaces an $\phi : V \subset S_1 \to S_2$ is a differentiable mapping. The curve $\beta = \phi \circ a$ is such that $\beta(0) = \phi(p)$ therefore $\beta'(0)$ is a vector of $T_{\phi(p)}(S_2)$

Proposition 71

In the disccusion above, given w, the vector $\beta'(0)$ does not depend on the choice of a. Moreover, the map $d\phi_p : T_p(S_1) \to T_{\phi(p)}(S_2)$ defined by $d\phi_p(w) = \beta'(0)$ is linear

Let $x(u, v)$ and $\overline{x}(\overline{u}, \overline{v})$ be the parameterizations in neighbourhoods of p and $\phi(p)$ respectively. Suppose

$$\phi(u, v) = (\phi_1(u, v), \phi_2(u, v))$$

and

$$a(t) = (u(t), v(t)), \quad t \in (-\varepsilon, \varepsilon)$$

then

$$\beta(t) = (\phi_1(u(t), v(t)), \phi_2(u(t), v(t)))$$

so

$$\beta'(0) = \left(\frac{\partial \phi_1}{\partial u} u'(0) + \frac{\partial \phi_1}{\partial v} v'(0), \frac{\partial \phi_2}{\partial u} u'(0) + \frac{\partial \phi_2}{\partial v} v'(0) \right)$$

so

$$\beta'(0) = d\phi_p(w) = \frac{\partial(\phi_1, \phi_2)}{\partial(u, v)} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix} = D\phi_p \cdot w$$

where $\cdot$ here refers to the matrix product. It is clear the jacobian matrix here only depends on ϕ. So it is clear β' only relies only on $a'(t) = w$ not a itself. That is to say if $a'(t) = b'(t) = w$ even if $a \neq b$ the result still holds.

Remark 72. *From the above 2 secenarios you can clearly see how the defitition in* `munkres` *given by $df_q(v) = Df(q) \cdot v$ where the $\cdot$ here refers to the matrix product came about!*

Definition 73

A map $\phi : U \subset S_1 \to S_2$ is a **local diffeomorphism** at $p \in U$ if there exists a neighborhood $p \in V \subset U$ such that ϕ is a diffeomophism of V onto its image in S_2

Example 74

The map $S^1 \to S^2$ given by $z \mapsto z^2$ is a local diffeomorphism but not globally since it is not injective(eg.both 2 and -2 in z maps to $4 \in z^2$). But let $f(z) = z^2$ then $f'(z) = 2z$ which is bijective so by inverse function theorem local diffeomorphism.

Proposition 75

Let S_1 and S_2 be regular surfaces. $\phi : U \subset S_1 \to S_2$ is differentiable. If the differential

$$(d\phi)_p : T_pS_1 \to T_{\phi(p)}S_2$$

is a linear isomorphism(linear map that is bijective), then ϕ is local diffeomorphism at p.

Proof. Consider

$$\begin{tikzcd} S_1 \arrow[r, "\phi"] & S_2 \\ U \arrow[u, leftrightarrow, "x"] \arrow[r] & \overline{U} \arrow[u, leftrightarrow, "\overline{x}"'] \end{tikzcd}$$

As seen we have induced a map $U \to \overline{U}$ whose differential is an isomorphism. Then by inverse function theorem there is a local diffeomorphism between neighbourhood in U and $\overline{U}$ which we can lift to S_1 and S_2.

Definition 76

Let $p \in S$ be a point and let $x : U \to S$ be a local parameterization of S at p. Then we define the **normal** vector to the surface as follows

$$N(p) = \pm \frac{x_1 \times x_2}{|x_1 \times x_2|}$$

The $\pm$ sign corresponds to the choice of orientation which we will talk more about below.

4.3 the first fundamental form: area

Definition 77

Let $S \subset \mathbb{R}^3$ be a regular surface. Then for any $p \in S$ the tangent plane T_pS inherits an inner product from $\mathbb{R}^3$ because it is a subspace of $\mathbb{R}^3$. Thus we have the map

$$I_p : T_pS \to \mathbb{R} \quad \text{by} \quad w \mapsto \langle w, w \rangle = ||w||^2$$

Note that by abuse we use $\langle ., . \rangle$ instead of $\langle ., . \rangle_p$ despite the fact that this form depends on p if it is clear what point we are referring to. The map I_p is called the **first fundamental form** of S at $p \in S$

One can compute this explcitly given local coordinates. Suppose we paramterize $\overline{x} : U \subset \mathbb{R}^2 \to S$ and take $p \in U$. Then if we denote $\mathbb{R}^2 = u\mathbb{R} \oplus v\mathbb{R}$ then T_pS has a basis $\{x_u, x_v\}$ where $x_u = \frac{\partial x}{\partial u}(p)$ and $x_v = \frac{\partial u}{\partial v}(p)$. Now let $w \in T_p(S)$. Let $a : (-\varepsilon, \varepsilon) \to S$ such that $a(t) = (x(t), v(t))$ and

$$w = a'(0) = u'(0)x_u + v'(0)x_v$$

then by expanding we obtain

$$
\begin{aligned}
I_p(w) &= \langle w, w\rangle_p \\
&= \langle u'(0)x_u + v'(0)x_v, u'(0)x_u + v'(0)x_v\rangle \\
&= ||x_u||^2\, u'(0)^2 + 2\langle x_u, x_v\rangle u'(0)v'(0) + ||x_v||^2\, v'(0)^2
\end{aligned}
$$

which we write as

$$= E \cdot u'(0)^2 + 2F \cdot u'(0)v'(0) + G \cdot v'(0)^2$$

where

$$E = \langle x_u, x_u\rangle$$

$$F = \langle x_u, x_v\rangle$$

$$G = \langle x_v, x_v\rangle$$

Example 78

A coordinate system for a plane $P \subset \mathbb{R}^3$ passing through p_0 and containing orthonormal vectors w_1 and w_2 is given by(recall A-level vector calculus):

$$x(u, v) = p_0 + uw_1 + vw_2, \quad (u, v) \in \mathbb{R}^2$$

Then to compute the first fundamental form for an arbitrary point of P we observe that

$$x_u = w_1, x_v = w_2$$

Then since w_1 and w_2 are unit orthogonal vectors, we have

$$E = 1, F = 0, G = 1$$

In other words for any plane in $\mathbb{R}^3$ there exists a paramterization where the first fundamental form gives $E = 1, F = 0, G = 1$ for any point in the plane

The power of the first fundamental form is that we can make metric measurements on S without referring back to the ambient space $\mathbb{R}^n$ that S is in as long as S can be parameterized by $\overline{x} : U \subset \mathbb{R}^2 \to S$ for every $p \in S$. We will some examples and applications below.

Definition 79

Let $R \subset S$ be a bounded region of regular surface contained in the coordinate neighborhood of the parameterization $x : U \subset \mathbb{R}^2 \to S$. The positive number

$$\int\int_Q |x_u \wedge x_v|\, dudv = A(R), \quad Q = x^{-1}(R)$$

is called the area of R

The justification is by the "volume function" and change of variables theorem you learnt in `18.101 Analysis on manifolds Munkres`. Observe that $|x_u \wedge x_v|^2 + \langle x_u, x_v\rangle^2 = |x_u|^2 \cdot |x_v|^2$ just recall that our form here assumes $a \cdot b = |a|\,|b| \cos\theta$ and analogously for the cross product which is associated with sin and denoted with $\wedge$ here. So the equality follows by $\sin^2 + \cos^2 = 1$. Now this implies

$$|x_u \wedge x_v| = \sqrt{EG - F^2}$$

Example 80

The arc length can be calculated by

$$\begin{aligned} s(t) = \int_0^t |a'(t)|\, dt &= \int_0^t \sqrt{I(a'(t))} dt \\ &= \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} dT \end{aligned}$$

Example 81

The angle in which two parameterized regular curves $a : I \to S$ and $\beta : I \to S$ intersect at $t = t_0$ is given by

$$\cos\theta = \frac{\langle a'(t_0), \beta'(t_0)\rangle}{|a'(t_0)||\beta'(t_0)|}$$

Just imagine the scenario, not that hard to see why this is equivalent to the angle between tangents of the curves at t_0 so by the 1st fundmanental form we have

$$\cos\theta = \frac{F}{\sqrt{EG}}$$

Example 82

Calculate the surface area of the torus which is parameterized by

$$x(u,v)((a + r\cos u)\cos v, (a + r\cos u)\sin v), r\sin u), 0 < u < 2\pi, 0 < v < 2\pi$$

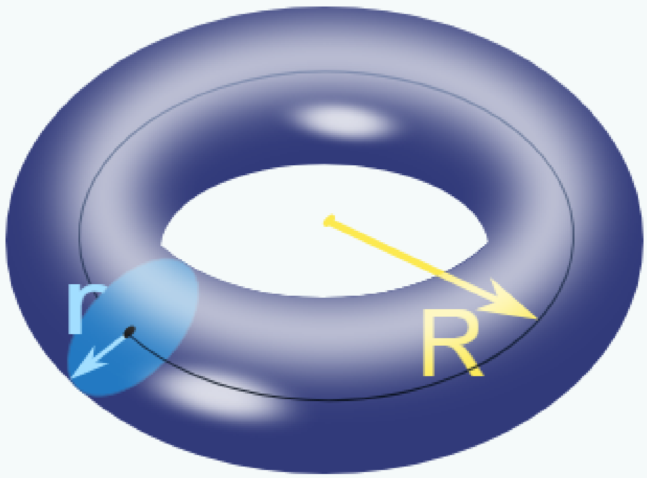

Figure 7: In our case R=r,r=a

Then we have

$$\begin{aligned} A(R) &= \int_0^{2\pi}\int_0^{2\pi}\sqrt{EG - F^2}dudv \\ &= \int_0^{2\pi}\int_0^{2\pi}(r^2\cos u + ra)dudv \\ &= 4\pi^2 ra \end{aligned}$$

5 geometry of a gauss map

5.1 Appendix: Linear Maps and Quadratic Forms

Lemma 83

If the function $Q(x,y) = ax^2 + 2bxy + cy^2$ restricted to the unit circle $x^2 + y^2 = 1$ has a maximum at point $(1,0)$ then $b = 0$

Proof. Parameterize by

$$Q(t) = a\cos^2 t + 2b\cos t\sin t + c\sin^2 t$$

so if Q has maximum at $(1,0)$ it means

$$\left(\frac{dQ}{dt}\right)_{t=0} = 2b = 0$$

So $b = 0$ as desired

Proposition 84
Given a quadratic form for Q in V there exists an orthonormal basis $\{e_1, e_2\}$ of V such that if $v \in V$ is given by $v = xe_1 + ye_2$ then

$$Q(v) = \lambda_1 x^2 + \lambda_2 y^2$$

where λ_1 and λ_2 are the maximum and minimum of Q repectively on the unit circle $|v| = 1$

Since the basis $\{e_1, e_2\}$ is orthonormal and v must be in the unit circle we have

$$|v| = x^2\langle e_1, e_1\rangle + y^2\langle e_2, e_2\rangle = x^2 + y^2 = 1$$

Then by the previous lemma we may associate the symmetric bilinear form B with the a quadratic form like so

$$B(v, v) = Q(v) = Q(x, y) = ax^2 + bxy + cy^2$$

where the RHS is just the general form of a possible quadratic form. by the previous lemma we know $b = 0$ and since $(0, 1)$ is the maximum we can let $\lambda_1 = Q(0, 1) = a$. Now it remains to show $\lambda_2 = Q(0, 1) = c$ is the minimum.Notice that

$$Q(v) = \lambda_1 x^2 + \lambda_2 y^2 \geq \lambda_2(x^2 + y^2) = \lambda_2$$

since $\lambda_2 \leq \lambda_1$ given that λ_1 is asssumemd to be the maximum. This clearly shows that λ_1 is the minium of Q over all v in the unit circle.

Theorem 85
Let $A : V \to V$ be a self-adjoint linear map. Then there exists an orthonormal basis $\{e_1, e_2\}$ of V such that $A(e_1) = \lambda_1 e_1, A(e_2) = \lambda_2 e_2$(that is e_1 and e_2 are eigenvectors and λ_1, λ_2 are eigenvalues of A).

In the basis $\{e_1, e_2\}$ the matrix of A is clearly diagonal, the elements λ_1, λ_2 where $\lambda_1 \geq \lambda_2$ on the diagonal are the maximum and minimum respectively of the quadratic form which we define to be

$$Q(v) = \lambda_1 x^2 + \lambda_2 y^2 = \langle Av, v\rangle$$

where $v \in V$ is given by $v = xe_1 + ye_2$ on the *unit circle* in V.

Proof. For the 1st part on the existence of eigenvalues of A we are basically trying to show

$$A(e_1) = \lambda_1 e_1 \quad \text{and} \quad A(e_2) = \lambda_2 e_2$$

Because A is self adjoint/hermitian operator by definition recall `artin algebra`, the matrix of the linear map satisfies $A = A^*$ when A is with respect to the othonormal basis $\{e_1, e_2\}$ which also implies A must be othogonal under this basis obviously. In that case λ_1 and λ_2 become eigenvalues with respect to the orthognal basis eigenvector e_1 and e_2 respectively as desired.

For the second part if we use the quadratic form

$$Q(v) = \langle Av, v\rangle$$

then clearly since orthonormal basis

$$Q(xe_1 + ye_2) = \lambda_1 x^2 + \lambda_2 y^2$$

as desired. Finally from the above lemma if v is restricted to a unit circle in v we will then know λ_1 and λ_2 are the maximum and minimum values of this form.

Remark 86. *Note that if A is the matrix of an adjoint linear operator, $A = A^*$ is not necessarily true unless orthonormal basis is assumed. In other words the condition for orthonormal basis as in the definition is necessary. See spectral theorem which investigates such properties beyond the orthonormal basis for more.*

5.2 The definition of gauss map and fundamental properties

As we have seen previously given a prametrization $x : U \subset \mathbb{R}^2 \to S$ of a regular surface S at a point $p \in S$ we can choose a unit normal vector at each point $x(U)$ by the rule

$$N(q) = \frac{x_u \wedge x_v}{|x_u \wedge x_v|}(q), \quad q \in x(U)$$

So we may define a differeenitable map $N : x(U) \to \mathbb{R}^3$ that associates to each $q \in x(U)$ a unit normal vector $N(q)$

More generally if $V \subset S$ is an open set in S and $N : V \to \mathbb{R}^3$ is a differentiable map which associates to each $q \in V$ a unit nrmal vector at q we say that N is a **differentiable field of unit normal vectors** on V.

Remark 87. *Not all surfaces can admit a differentiable field of unit normal vectors defined on the whole surface. For example the Mobius strip is one such surface*

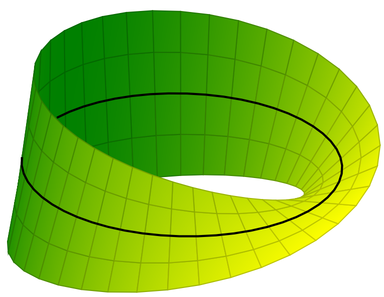

Figure 8: Mobius Strip

> **Definition 88**
> We say a regular surface is **orientable** if it admits a differentiable field of unit normal vectors defined on the whole surface. The coice of such a field N is called an **orientation** of S

We see that the mobius strip is not orientable. Notice that after one turn, the vector field N would come back as $-N$ which is a contradiction to the continuity of N(recall how it is defined).

Example 89

Let S be the cylinder $\left\{(x,y,z)|x^2+y^2=1\right\}$. We claim that $N=(-x,-y,0)$ are the unit normal vectors. Since let $a(t)=(x(t),y(t),z(t))$ be an arbitrary curve on S and its tangent at $a(0)=p$ is $a'(0)=(x'(0),y'(0),z'(0))$.That is we are denoting say $x(t)$ to be the restriction of x to the curve or in other words $x\circ a$. Then we have on $S\circ a$,

$$x(0)^2+y(0)^2=1 \xrightarrow{\frac{d}{dt}} 2x(0)x'(0)+2y(0)y'(0)=0$$

Which implies $(x',y',z')\perp(x,y,0)$ at that point. We now choose an orientation for N at p by letting

$$N_p=(N\circ a)_0=(-x(0),-y(0),0)$$

meaning that is points inwards to the centre of the xy plane. Then we have

$$dN_p(x'(0),y'(0),z'(0))=(-x'(0),-y'(0),0)$$

Which is map of the tangent on the curve a to the tangent of the normal of the surface restricted to curve a

$$dN(x'(t),y'(t),z'(t)=(-x'(t),-y'(t),0)$$

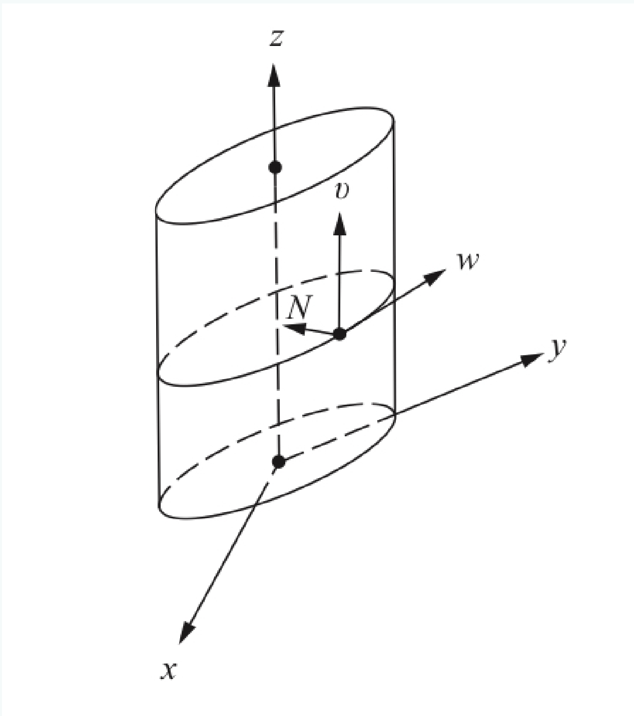

Figure 9: Gauss Map Cylinder

if v is a vector tangent to the clinder and parallel to the z axis then

$$dN(v)=0=0v$$

likewise if w is vector tangent to the clyinder and parallel to the xy plane then

$$dN(w)=-w$$

Example 90

Another way to see the formula is that $dN_p(a'(0)) = DN_p \cdot a'(0)$. It follows that

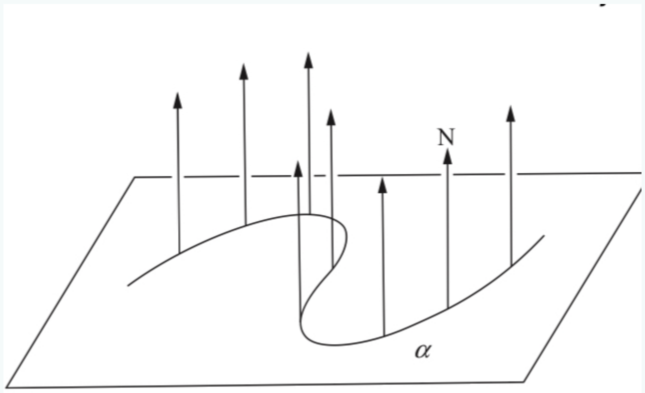

Figure 10: Plane with $dN_p = 0$

Lemma 91

As defined $dN_p : T_pS \to T_pS$ is **self-adjoint** in the sense that

$$\langle dN_p(v), w\rangle = \langle v, dN_p(w)\rangle$$

where $\{x_u, x_v\}$ is a basis of T_pS. Please note that we are assuming a standard euclidean space(i.e is equipped with dot product). That is to say we are only spanning the space with an orthonormal basis.(recall `Artin Algebra 1`)

Proof. Let $a(0) = p, a(t) = x(u(t), v(t)), a'(0) = u'(0)x_u + v'(0)x_v$. Write

$$dN_p(a'(0)) = dN_p(x_u u'(0) + x_v v'(0)) = \left[\frac{d}{dt}(N(u(t), v(t)))\right]_{t=0} = N_u u'(0) + N_v v'(0)$$

recall 71 we are essentially calculating $Df_p \cdot a'(0)$ It can be seen that $dN_p(x_u) = N_u$ and $dN_p(x_v) = N_v$ therefore to prove that dN_p is self-adjoint it suffices to show that

$$\langle N_u, x_v\rangle = \langle x_u, N_v\rangle$$

This is because recall `Artin Algebra 1`, in the standard euclidean space, that is the space equipped with the standard dot product(which is symmetric and positive definite) To prove self adjoint, by definition that means to prove the matrix of transformation dN_p is symmetric supposing x_u, x_v is an orthogonal basis. Observe that $\langle Ae_j, e_i\rangle = a_{ij}$ while $\langle e_j, Ae_i\rangle = a_{ji}$ since we are using an orthonormal basis. Noting that symmetry means $a_{ij} = a_{ji}$ and that we have the analagous situation here $\langle dN_p(x_v), x_u\rangle$ and$\langle dN_p(x_v), x_u\rangle$. Hence it suffices to prove that condition above. Now we know that

$$\langle N, x_u\rangle = \langle N, x_v\rangle = 0$$

since N is by definition perpendicular to the whole tangent space and is clearly spanned by N_v and N_u. Then differiating both sides of this with respect to v and u respectively by chain rule

$$\langle N_v, x_u\rangle + \langle N, x_{uv}\rangle = 0$$
$$\langle N_u, x_v\rangle + \langle N, x_{vu}\rangle = 0$$

but we know that mixed partial derivatives of smooth functions are the same so $\langle N, x_{uv}\rangle = \langle N, x_{vu}\rangle$. That implies

$$\langle x_u, N_v\rangle = \langle N_v, x_u\rangle = \langle N_u, x_v\rangle$$

where the first equality follows because again we are equipped with the dot product in the euclideaen space which is symmetric.

Definition 92
Let II_p be the quadratic form defined in $T_p(S)$ by

$$II_p(v) = -\langle dN_p(v), v\rangle = -\langle v, dN_p(v)\rangle$$

This is called the **second fundamental form** of S at p.

You might question how come this is well defined. We will prove later that the second fundamental form is actually **self-adjoint** meaning recall `artin algebra 1`

$$\langle Tv, w\rangle = \langle v, Tw\rangle$$

for all v, w in a euclidean/hermitian space.

Definition 93
Let C be a regular curve in S and $p \in S$ be a point on it. Let k be the curvature of C at p and set $\cos\theta = \langle n, N\rangle$ where n is the normal to C and N is the normal vector to S at p. The number $k_n = k\cos\theta$ is called the **normal curvature** of $C \subset S$ at p.

Essentially k_n is the *length* of projection of the vector kn over the normal to the surface at p with a sign given by the orientation N of S at p.

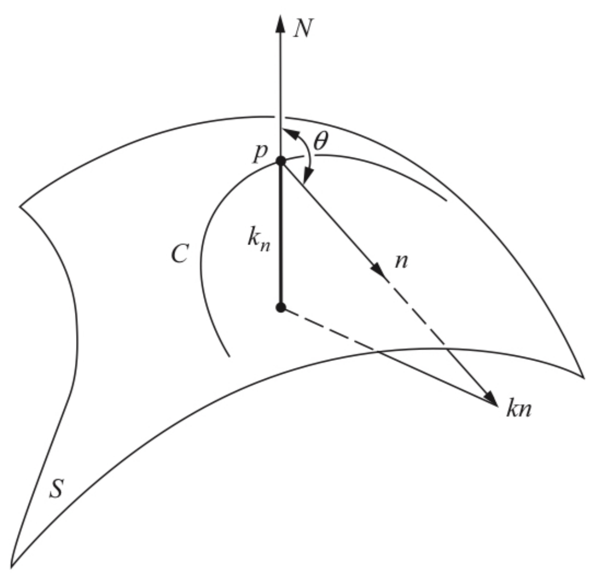

Figure 11: Normal Curvature

Theorem 94 (Meusnier)
The normal curvature of a curve C at p depends only on the **tangent line** at the curve.

Proof. Again we denote $N(s)$ to be the restriction of normal vector N to the curve $a(s)$. Then we have $\langle N(s), a'(s)\rangle = 0$ hence

$$\langle N(s), a''(s)\rangle = -\langle N'(s), a'(s)\rangle$$

upon application of product rule when differeiating dot products. Therebefore

$$\begin{aligned} II_p(a'(0)) &= -\langle dN_p(a'(0)), a'(0)\rangle \\ &= -\langle N'(0), a'(0)\rangle \\ &= \langle N(0), a''(0)\rangle \\ &= \langle N, kn\rangle(p) = k_n(p) \end{aligned}$$

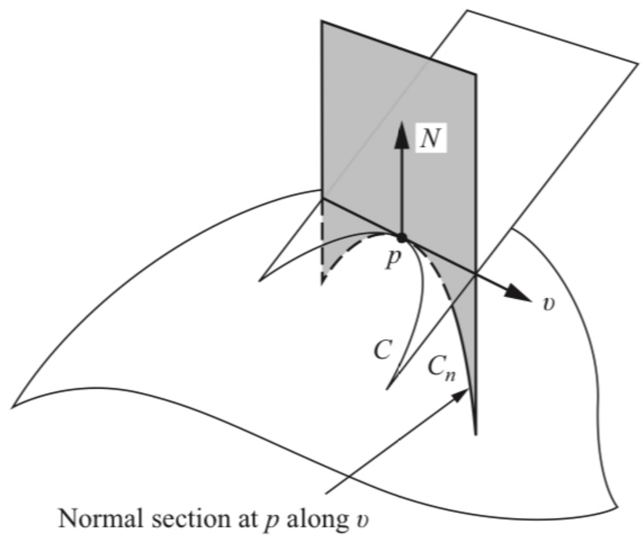

Figure 12: Meusnier theorem: C and C_n have the same normal curvature at p along v

Hence the theorem follows because in $\langle N, kn\rangle$, for the same point N is clearly the same and kn only depends on the tangent of each curve.

Let us return to linear self adjoint map dN_p. We recall that this implies there exists for each $p \in S$ an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that $dN_p(e_1) = -k_1 e_1$ and $dN_p(e_2) = -k_2 e_2$. Moreover k_1 and k_2 where $k_1 \geq k_2$ are the maximum and minimum of the second fundamental form II_p restricted to the unit circle of $T_p(S)$. That is they are the extreme values of the normal cruvature at p.

Definition 95
The maximum normal curvature k_1 and the minimum k_2 are called the **principle curvatures** at p. The corresponding directions given by e_1 and e_2 respectively are called the **principle directions** at p.

Definition 96

If a regular connected curve C on S is such that for all $p \in C$ the tangent line of C is a principal direction at p then C is said to be the **line of curvature** of S

Proposition 97 (Condition to be line of curvature)

A necessary and sufficient condition for a connected regular curve C on S to be a *line of curvature* of S is that

$$N'(t) = \lambda(t)a'(t)$$

for any parameterization $a(t)$ of C where $N(t) = N \circ a(t)$ and $\lambda(t)$ is a differentiable function of t. In this case $-\lambda(t)$ is the (principal) curvature along $a'(t)$

Proof. Observe that if $a'(t)$ is contained in a principal direction then $a'(t)$ is an eigenvector of dN. Recall $dN_p(e_1) = \lambda_1$ where e_1 is the basis for one of the principal directions.

$$dN(a'(t)) = N'(t) = \lambda(t)a'(t)$$

Let us consider an application of the second fundamental form

Example 98

The knowledge of the principal curvatures at p allow us to compute easily the normal curvature along a given direction of $T_p(S)$. Consider $v \in T_p(S)$ where $|v| = 1$(i.e restricted to a unit circle). Then we can express every v as

$$v = e_1 \cos\theta + e_2 \sin\theta$$

since e_1, e_2 constitute an orthonormal basis of $T_p(S)$. Then to find k_n along v we use the 2nd fundamental formula and notice

$$\begin{aligned}
k_n = II_p(v) &= -\langle dN_p(v), v\rangle \\
&= -\langle dN_p(e_1 \cos\theta + e_2 \sin\theta), e_1 \cos\theta + e_2 \sin\theta\rangle \\
&= \langle e_1 k_1 \cos\theta + e_2 k_2 \sin\theta, e_1 \cos\theta + e_2 \sin\theta\rangle \\
&= k_1 \cos^2\theta + k_2 \sin^2\theta
\end{aligned}$$

This last expression is known as **Euler's formula**

Definition 99

Let $p \in S$ and let $dN_p : T_p(S) \to T_p(S)$ be the differential of the Gauss map. The determinant of dN_p is the **gaussian curvature** K of S at p. The negative of half of the trace of dN_p is called the **mean curvature** H of S at p. So in terms of *principal curvatures* we can write

$$K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2}$$

Notice that being a diagonal matrix(as symmetrical and orthonormal basis of eigenvectors), the determinant of the self adjoint operator dN is just the product $k_1 k_2 = H$.

Definition 100

A point of a surface S is called

1. Elliptic if $\det(dN_p) > 0$
2. Hyperbolic if $\det(dN_p) < 0$
3. Parabolic if $\det(dN_p) = 0$ with $dN_p \neq 0$
4. Planar if $dN_p = 0$

Notice that this classification does not depend on the oreentation since even if orientation changes that is $(-k_2)(-k_1) = H$ the determinant is unchanged.

Definition 101

If at $p \in S, k_1 = k_2$ then p is called an **umbilical point** of S. In particular the planar points $(k_1 = k_2 = 0)$ are umbilical points.

As seen earlier we know $dN_p = 0$ for a plane and sphere. That is dN_p is the zero matrix. Clearly for unit circle neighbourhood in T_pS, we have $\lambda_1 = \lambda_2 = 0$.

Proposition 102

If all points of a connected surface S are umbilical points then S is either contained in a sphere or in a plane

Proof. Let $p \in S$ and $x(u, v)$ be a parameterization of S at p such that the coordinate neighbourhood V is connected. To be continued... $\square$

5.3 The gauss map in local coordinates

For the next section for the parameterizations $x : U \subset \mathbb{R}^2 \to S$, assume the orientation

$$N = \frac{x_u \wedge x_v}{|x_u \wedge x_v|}$$

let $x(u, v)$ be the parameterization at a point $p \in S$ on S and let a curve on S be $a(t) = x(u(t), v(t))$ with $a(0) = p$. So

$$a' = x_u u' + x_v v'$$

and

$$dN(a') = N'(u(t), v(t)) = N_u u' + N_v v'$$

where

$$N_u = a_{11} x_u + a_{21} x_v$$

and

$$N_v = a_{12} x_u + a_{22} x_v$$

since N_u an N_v belong to $T_p(S)$ so

$$dN(a') = (a_{11} u' + a_{12} v') x_u + (a_{21} u' + a_{22} v') x_v$$

and hence we may write

$$dN\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & b_{22} \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}$$

Remark 103. *also recall $dN_p : T_pS \to T_pS$ is self adjoint, if x_u, x_v is an orthonormal basis of T_pS then the matrix A(the matrix of the map dN_p is symmetric*

also recall that maps from tangent spaces to another are linear so this matrix form should be expected. You can immediately see the parallel with 71 where A is in fact the jacobian matrix of the paramterization N_p.

We are now ready to express the terms of the **second fundamental form**. See that with this basis $\{x_u, x_v\}$ we have

$$\begin{aligned} II_p(a') &= -\langle dN(a'), a'\rangle = -\langle N_u u + N_v v', x_u u' + x_v v'\rangle \\ &= e(u')^2 + 2fu'v' + g(v')^2 \end{aligned}$$

where since $\langle N, x_u\rangle = \langle N, x_v\rangle = 0$ we have by product rule

$$e = -\langle N_u, x_u\rangle = \langle N, x_{uu}\rangle$$

$$f = -\langle N_v, x_u\rangle = \langle N, x_{uv}\rangle = \langle N, x_{vu}\rangle = -\langle N_u, x_v\rangle$$

$$g = -\langle N_v, x_v\rangle = \langle N, x_{vv}\rangle$$

so we have relating to 1st fundemental form where recall we have

$$E = \langle x_u, x_u\rangle$$

$$F = \langle x_u, x_v\rangle$$

$$G = \langle x_v, x_v\rangle$$

we then have

$$-f = \langle N_u, x_v\rangle = a_{11}F + a_{21}G$$

$$-f = \langle N_v, x_u\rangle = a_{12}E + a_{22}F$$

$$-e = \langle N_u, x_u\rangle = a_{11}E + a_{21}F$$

$$-g = \langle N_u v, x_v\rangle = a_{12}F + a_{22}G$$

and so in matrix form we have

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & b_{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

hence

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{21} & b_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

and substituting

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

so it can checked that we have

$$a_{11} = \frac{fF - eG}{EG - F^2}$$
$$a_{12} = \frac{gF - fG}{EG - F^2}$$
$$a_{21} = \frac{eF - fE}{EG - F^2}$$
$$a_{22} = \frac{fF - gE}{EG - F^2}$$

these are known as the **equations of weingarten**

Now let us use our new found calculated terms of the 2nd fundamental form and relate this to variables that we have defined(such as mean,normal and gaussian curvature) and see how we can calculate them given $\mathbb{R}^2$ parameterizations, much like how the 1st fundamental form helped us calculate measures like arc length and area.

First let us relate this to the **mean curvature**(H) and **gaussian curvature**(K) recall 99

We can already calculate immediately by defintion

$$K = \det A = \frac{eg - f^2}{EG - F^2}$$

while for mean curvature recall that $-k_1, -k_2$ are the eigenvalues of dN.

Remark 104. *Recall that $dN_p : T_pS \to T_pS$ is self adjoint. Therefore there exists an orthonormal basis in which $A = A^*$. And since orthonormal basis, clearly A will be diagonal too in that case and the trace will consist of eigenvalues. Now we can proceed to find the eigenvalues, despite not knowing what this diagonalizing orthonormal basis of eigenvectors are. Because recall similar matrices $P^{-1}AP$ where $\mathbf{B}' = \mathbf{B}P$, that is P is the base change matrix, the eigenvalues are the* same*!*

In which case solving for the eigenvalues k,

$$\det \begin{pmatrix} a_{11} + k & a_{21} \\ a_{21} & b_{22} + k \end{pmatrix} = 0$$

where we get the characteristic equation

$$k^2 + k(a_{11} + a_{22}) + a_{11}a_{22} - a_{21}a_{12} = 0$$

Then noting that the sum of roots from this polynomial which is just the trace of the matrix recall `artin algebra 1` is just $a_{11} + a_{22}$ .So

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}$$

Notice that $a_{11}a_{22} - a_{21}a_{12} = \det A$ which happens to be the produt of roots(k_1k_2) so it makes sense in the definition of gaussian curvature that $K = k_1k_2 = \det A$ from the *equations of weingarten*. Again relating this back to principle curvatures, we know these eigenvalues k_1 and k_2 are the maximum and minimum normal curvatures/values of the second fundamental form restricted to the unit circle of V respectively(see **euler formula** from earlier).

Problem 105

Calculate the gaussian curvature of the points of the torus which is parameterized by

$$x(u, v)((a + r\cos u)\cos v, (a + r\cos u)\sin v), r\sin u), 0 < u < 2\pi, 0 < v < 2\pi$$

Solution. A rough approach is as follows

1. find the expressions for $x_u, x_v, x_{uu}, x_{uv}, x_{vv}$ via partial differentiation
2. which allows you to find E, F, G of the first fundamental form
3. now find $N = \frac{x_u \wedge x_v}{|x_u \wedge x_v|}$
4. which combined with our knowledge of E, F, G allows you to find e, f, g of the second fundamental form
5. finally recall that
$$K = \det A = \frac{eg - f^2}{EG - F^2}$$
upoon calculation we get our desired result

You may verify that the answer is
$$K = \frac{\cos u}{r(a + r\cos u)}$$
refer back to `do Carmo` page 159 if necessary. ☐

Finally our calculations for 2nd fundamental form could be related back to 97 too for which although defined we haven't actually covered an explicit way to calculate the variables involved.

Theorem 106 (Condition for principal directions)
Fix a point $p \in S$. Then $u'x_u + v'x_v \in T_p(S)$ is an eigenvector of dN_p if and only if
$$\det \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{pmatrix} = 0$$
That is to say if every $p \in C$ where C is a curve parameterized by $a(t) = x(u(t), v(t))$ satisfies this, then it is a line of curvature.

Proof. First recall 97 that a necessary condition for being a line of curvature is that
$$dN(a'(t)) = \lambda(t)a'(t)$$
We then know $u'(t), v'(t)$ satisfies the equations
$$\frac{fF - eG}{EG - F^2}u' + \frac{gF - fG}{EG - F^2}v' = \lambda u'$$
$$\frac{eF - fE}{EG - F^2}u' + \frac{fF - gE}{EG - F^2}v' = \lambda v'$$
by eliminating λ in the above system we obtain a differential equation of the lines of curvature
$$(fE - ef)(u')^2 + (gE - eG)u'v' + (gF - fG)(v')^2 = 0$$
which may be written in the form given in the theorem as desired

6 intrinsic geometry of surfaces

6.1 isometries conformal maps

Definition 107

A diffeomoprhism $\phi : S \to \overline{S}$ is an **isometry** if for all $p \in S$ and all pairs $w_1, w_2 \in T_p(S)$ we have

$$\langle w_1, w_2 \rangle_p = \langle d\varphi_p(w_1), d\varphi_p(w_2) \rangle_{\varphi(p)}$$

The surfaces S and $\overline{S}$ are then said to be isometric

This means the first fundamental form $I_p : T_pS \to \mathbb{R}$ is preserved as we see that

$$I_p(w) = \langle w, w \rangle_p = \langle d\varphi_p(w), d\varphi_p(w) \rangle_{\varphi(p)} = I_{\varphi(p)}(d\varphi_p(w))$$

Definition 108

A map $\varphi : V \to \overline{S}$ of a neighbourhood V of $p \in S$ is a **local isometry** at p if there exists a neighbourhood $\overline{V}$ of $\varphi(p) \in \overline{S}$ such that $\varphi : V \to \overline{V}$ is an isometry. We say S is locally isometric to S' if one can find a local isometry into S' at every point $p \in S$

Definition 109

Properties preserved uder local isometry are said to be **intrinsic**. For example the first fundamental form is intrinsic.

Proposition 110 (Criterion for local isomertry)

Assume the existence of parameterizations $x : U \to S$ an $\overline{x} : U \to \overline{S}$ such that $E = \overline{E}, F = \overline{F}, G = \overline{G}$ in U. Then the map $\varphi = \overline{x} \circ x^{-1} : x(U) \to \overline{S}$ is a local isometry

Proof. Consider

$$w = x_u u' + x_v v'$$

and

$$d\phi_p(w) = \overline{x}_u u' + \overline{u}_v v'$$

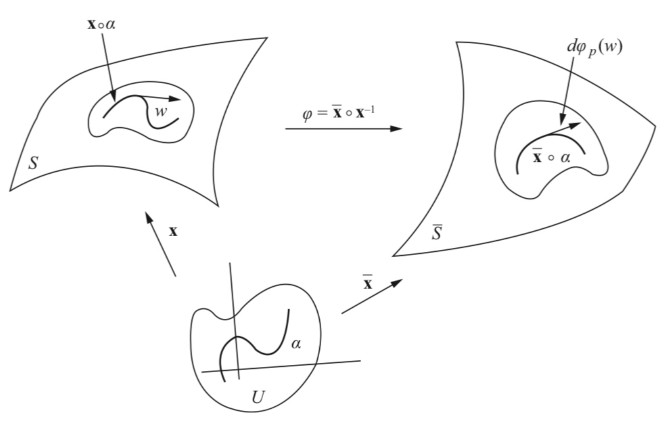

Figure 13: Isometry

so

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2$$

$$I_{\phi(p)}(d\phi_p(w)) = \overline{E}(u')^2 + 2\overline{F}u'v' + \overline{G}(v')^2$$

which are both equal as desired since $E = \overline{E}, F = \overline{F}, G = \overline{G}$ in U. □

Definition 111

A diffeomorphism $\phi : S \to \overline{S}$ is called a conformal map if for all $p \in S$ and all $v_1, v_2 \in T_pS$ we have

$$\langle d\phi_p(v), d\phi_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle_p$$

where λ^2 is a nowhere zero differentiable function on S. Then the surfaces S and $\overline{S}$ are said to be **conformal**

Definition 112

A map $\phi : V \to \overline{S}$ of a neighborhood V of $p \in S$ into $\overline{S}$ is a **local conformal** map at p if there exists a neighbourhood $\overline{V}$ of $\phi(p)$ such that $\phi : V \to \overline{V}$ is a conformal map. if this is true for all $p \in S$ then S is said to be **locally conformal** to $\overline{S}$.

The geometric meaning of this definition is that the angles not necessarily the lenghts are preserved by conformal maps. For example

$$\cos\theta \frac{\langle a', \beta'\rangle}{|a'|\,|\beta'|}$$

$$\cos\overline{\theta} = \frac{\langle d\phi(a'), d\phi(\beta')\rangle}{|d\phi(a')|\,|d\phi(\beta')|} = \frac{\lambda^2\langle a', \beta'\rangle}{\lambda^2\,|a'|\,|\beta'|} = \cos\theta$$

Proposition 113 (Criterion for local conformality)

Assume the existence of parameterizations $x : U \to S$ an $\overline{x} : U \to \overline{S}$ such that $E = \lambda^2\overline{E}, F = \lambda^2\overline{F}, G = \lambda^2\overline{G}$ in U where λ^2 is a nowhere zero differentiable function on U. Then the map $\varphi = \overline{x} \circ x^{-1} : x(U) \to \overline{S}$ is locally conformal

Proof. Just work out similar to how we did for proving the criterion for local isometry above.

6.2 the gauss theorem and the equation of compatibility

By expressing derivatives of the vectors x_u, x_v and N in the basis $\{x_u, x_v, N\}$ we obtain

$$x_{uu} = \Gamma^1_{11}x_u + \Gamma^2_{11}x_v + L_1N \tag{1}$$

$$x_{uv} = \Gamma^1_{12}x_u + \Gamma^2_{12}x_v + L_2N \tag{2}$$

$$x_{vu} = \Gamma^1_{21}x_u + \Gamma^2_{21}x_v + \overline{L}_2N \tag{3}$$

$$x_{vv} = \Gamma^1_{22}x_u + \Gamma^2_{22}x_v + L_3N \tag{4}$$

$$N_u = a_{11}x_u + a_{21}x_v \tag{5}$$

$$N_u = a_{12}x_u + a_{22}x_v \tag{6}$$

where Γ^i_{kj} and L_i and $\overline{L}_i$ are some coefficients of our basis vectors to be determined while a_{ij} are the coefficients obtained from the map of dN_p as in the second fundamental form discussed previously.

Let us first find L_i and $\overline{L}_i$. Notice that upon taking the inner product of (1) to (4) with N we obtain immediately

$$\begin{aligned} e &= \langle N, x_{uu}\rangle = L_1\langle N, N\rangle = L_1 \\ f &= \langle N, x_{uv}\rangle = L_2\langle N, N\rangle = L_2 \\ f &= \langle N, x_{vu}\rangle = \overline{L}_2\langle N, N\rangle = \overline{L}_2 \\ g &= \langle N, x_{vv}\rangle = L_3\langle N, N\rangle = L_3 \end{aligned}$$

Since N is a unit vector and e, f, g are coefficients of the second fundamental form.

As for Γ^i_{kj} which we call the **Christoffel symbols** they are obtained by taking the inner product of (1) to (4) with x_u and x_v in which we obtain

$$\begin{cases} \langle x_{uu}, x_u\rangle = \Gamma^1_{11}E + \Gamma^2_{11}F = \frac{1}{2}E_u \\ \langle x_{uu}, x_v\rangle = \Gamma^1_{11}F + \Gamma^2_{11}G = F_u - \frac{1}{2}E_v \end{cases} \tag{1}$$

$$\begin{cases} \langle x_{uv}, x_u\rangle = \Gamma^1_{12}E + \Gamma^2_{12}F = \frac{1}{2}E_v \\ \langle x_{uv}, x_v\rangle = \Gamma^1_{12}F + \Gamma^2_{12}G = \frac{1}{2}G_u \end{cases} \tag{2}$$

$$\begin{cases} \langle x_{vv}, x_u\rangle = \Gamma^1_{22}E + \Gamma^2_{22}F = F_v - \frac{1}{2}G_u \\ \langle x_{vv}, x_v\rangle = \Gamma^1_{22}F + \Gamma^2_{22}G = \frac{1}{2}G_v \end{cases} \tag{3}$$

where we have grouped these equations like so since in each pair, the system of equations has determinant $EG - F^2 \neq 0$(since two unknowns 2 equations each so full rank)

Theorem 114 (Gauss Theorema Egregium)
The gaussian curvature K of a surface is invariant by local isometries

Theorem 115 (Bonnet)
Let E, F, G, e, f, g be differentiable functions $V \to \mathbb{R}$ with $V \subset \mathbb{R}^2$. Assume $E, G > 0$ that Gauss and Codazzi holds and moreover $EG - F^2 > 0$ then

(a) for every $q \in V$ there is a neighborhood U of V such that a diffeomorphism $x : U \to \mathbb{R}^3$ such that $x(U)$ is a surface having the above coeffients

(b) this x is unique up to translation and orthogonal rotation

6.3 Appendix: vector fields(Noob ver)

Definition 116 (Differentiable Vector field)
Recall a (tangent) **vector field** in an open set $U \subset S$ of a regular surface S is a correspondence w that assigns to each $p \in U$ a vector $w(p) \in T_p(S)$. The vector field w is **differentiable** at p if for some pasteurization $x(u, v)$ in p the components a and b for $w = ax_u + bx_v$ in the basis $\{x_u, x_v\}$ are differentiable functions at p.

Example 117

The **trajectory** refers to a differentiable parameterized curve $a(t) = (x(t), y(t)), t \in I$ such that $a'(t) = w(a(t))$. In other words a set connected points that can be the domain of our vector field map.

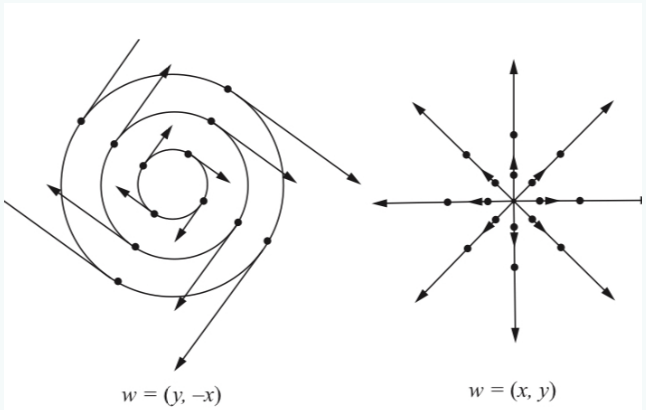

Figure 14: Vector fields

Notice that the magnitude of each vector in both of these vector fields are $\sqrt{y^2 + x^2} = r$(euclidean distance).Each dot corresponds to the point inputted into the vector field map and each arrow represents the output of that map. For instance the trajectory passing through a point (x_0, y_0) is the straight line $a(t) = (x_0 e^t, y_0 e^t), t \in \mathbb{R}$ while it is the circle $\beta(t) = (r \sin t, r \cos t), t \in \mathbb{R}, r^2 = x_0^2 + y_0^2$ respectively in the vector fields $w = (x, y)$ and $w = (y, -x)$ as shown.Too see why observe that

$$a = (x(t), y(t)) = (x_0 e^t, y_0 e^t) \quad \rightarrow \quad a'(t) = (x'(t), y'(t)) = (x_0 e^t, y_0 e^t)$$

$$\beta = (x(t), y(t)) = (r \cos t, r \sin t) \quad \rightarrow \quad \beta'(t) = (x'(t), y'(t)) = (r \cos t, -r \sin t)$$

In other words in the language of **ordinary differential equations**, the vector field w defines a system of differential equations

$$\frac{dx}{dt} = a(x, y)$$
$$\frac{dy}{dt} = b(x, y)$$

and the trajectory of w denoted by $(x(t), y(t))$ is a solution to this set of equations.

Definition 118

A **field of directions** r is an open set $U \subset \mathbb{R}^2$ which assigns to each $p \in U$ a line $r(p) \in \mathbb{R}^2$ passing through p. r is said to be differeneitable at $p \in U$ if there exists a nonzero differeentiable vector field w defined in a neighbourhood $V \subset U$ of p such that for each $q \in V, w(q) \neq 0$ is a basis of $r(q)$.

Definition 119

A regular connected curve $C \subset U$ is an integral curve of a field of directions r defined in $U \subset \mathbb{R}^2$ if $r(q)$ is the tangent line to C at q for all $q \in C$

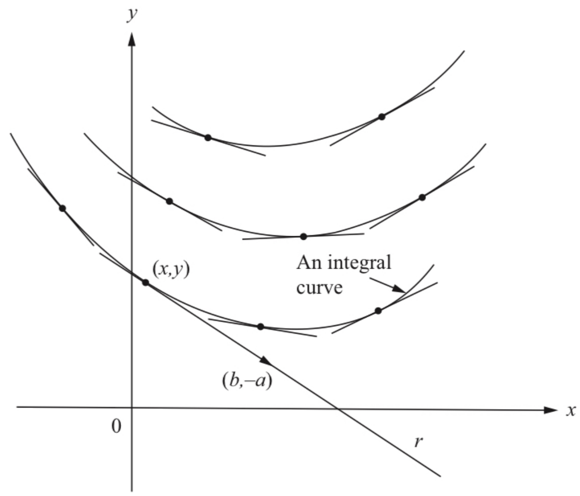

Figure 15: Integral Curve

6.4 Parallel transport Geodesics

We will now assume vector field refers to tangent vector fields for the this section. Earlier we have discussed the derivative of vectors in a plane. We we will now extend this discussion for the "derivative" of vectors in a surface.

Definition 120 (Covariant Derivative)

Let w be a differentiable *vector field* and let $y \in T_pS$ be a tangent vector realized by $y = a'(0)$ for $a : (\varepsilon, \varepsilon) \to S$ parameterized. Let $w(t)$ be the restriction the vector field w to the curve. The **covariant derivative** is the projection of $(w \circ a)'(0) = \frac{dw}{dt}$ onto the tangent plane and denoted by $(D_y w)(p)$. We also denoted it by $(\frac{Dw}{dt})(0)$ if a is given.

Remark 121. *In contrast, recall we defined our tangent vector to be*

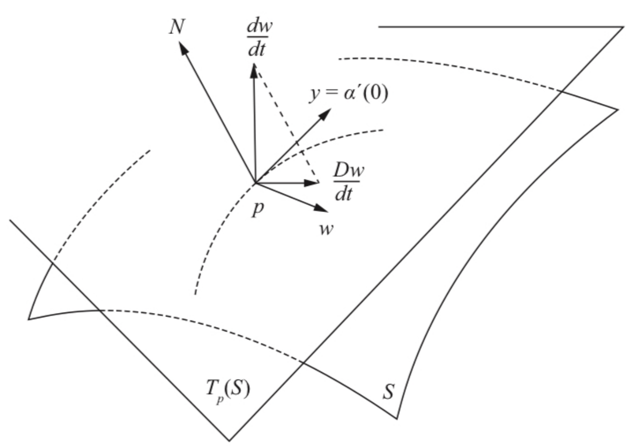

Figure 16: Covariant Derivative

Let $x(u(t), v(t)) = a(t)$ be the expression of the curve a and let

$$w(t) = a(u(t), v(t))x_u + b(u(t), v(t))x_v = a(t)x_u + b(t)x_v$$

be the restriction o the vector field to a. Then

$$\frac{dw}{dt} = a(x_{uu}u' + x_{uv}v') + b(x_{vu}u' + x_{vv}v') + a'x_u + b'x_v$$

Since $\frac{Dw}{dt}$ is the component $\frac{dw}{dt}$ on the tangent plane we use the expressions derived for x_u, x_v and N in the basis $\{x_u, x_v, N\}$ earlier to get after dropping of the normal components

$$\frac{Dw}{dt} = (a' + \Gamma^1_{11}au' + \Gamma^1_{12}av' + \Gamma^1_{12}bu' + \Gamma^1_{22}bv')x_u + (b' + \Gamma^2_{11}au' + \Gamma^2_{12}av' + \Gamma^2_{12}bu' + \Gamma^2_{22}bv')x_v \tag{4}$$

this clearly shows $\frac{Dw}{dt}$ only depends on the vector (u', v') not on curve a itself.

Fact 122

It can be seen that covariant derivatives is a generalization of the usual derivative of vectors in a plane. Using the above equation for $\frac{Dw}{dt}$ and assuming now that S is a plane and noting that 78 that there exists a paramterization of any plane in $\mathbb{R}^3$ where the first fundamental form gives $E = G = 1$ and $F = 0$, on solving the system of equations in 6.2 we have

$$\begin{cases} \langle x_{uu}, x_u \rangle = \Gamma^2_{11} = 0 \\ \langle x_{uu}, x_v \rangle = \Gamma^1_{11} = 0 \end{cases} \tag{5}$$

$$\begin{cases} \langle x_{uv}, x_u \rangle = \Gamma^2_{12} = 0 \\ \langle x_{uv}, x_v \rangle = \Gamma^1_{12} = 0 \end{cases} \tag{6}$$

$$\begin{cases} \langle x_{vv}, x_u \rangle = \Gamma^2_{22} = 0 \\ \langle x_{vv}, x_v \rangle = \Gamma^1_{22} = 0 \end{cases} \tag{7}$$

since by product rule(taking partial derivative with respect to $\mu = u$ or $\mu = v$)

$$\langle x_u, x_u \rangle = 0 \quad \Rightarrow \quad 2\langle x_{u\mu}, x_u \rangle = 0$$

similary

$$\langle x_v, x_v \rangle = 0 \quad \Rightarrow \quad 2\langle x_{v\mu}, x_v \rangle = 0$$

Finally besides the Christoffel symbols, in the plane, the second derivatives of the surface parametrization $x(u, v)$ point entirely within the plane itself(see below for more explanation), and the normal vector N is constant. As a result:

$$\langle N, x_{uu} \rangle = \langle N, x_{uv} \rangle = \langle N, x_{vv} \rangle = 0,$$

Hence in the case where S is the plane

$$\frac{Dw}{dt} = \frac{dw}{dt}$$

Definition 123

Let a be a parameterized curve and w a vector field defined on it. We say a is **parallel** if $\frac{Dw}{dt} = 0$ everywhere on a.

Example 124

Let $a(t)$ be a curve on S which can be seen as a trajectory of a point moving on S. Then $a'(t)$ is its speed and $a''(s)$ is its acceleration. Then intuitively $\frac{Da'}{dt}$ is the tangentional component of the acceleration $a''(t)$.

Fact 125

An alternative way to think of 122 is to oberseve figure 16 and you will see this makes sense geometrically. Explicity this follows from:

If the surface S is a plane, then:

- The tangent plane at any point on the surface is simply the plane itself.
- The normal vector $\mathbf{N}$ to the surface is constant and perpendicular to the plane at all points.

Now, consider a curve $\mathbf{a}(s)$ that lies on the plane, parameterized by arc length s:

- The tangent vector $\mathbf{t}(s)$ to the curve is the derivative of the curve, $\mathbf{t}(s) = \mathbf{a}'(s)$, and this vector lies in the plane because the curve itself is on the plane.
- The normal vector $\mathbf{n}(s)$ to the curve is perpendicular to the tangent vector $\mathbf{t}(s)$ and also lies in the plane because both $\mathbf{a}(s)$ and $\mathbf{t}(s)$ are in the plane.

Since the curve lies entirely on the surface (which is a plane):

- Both the tangent vector $\mathbf{t}(s)$ and the normal vector $\mathbf{n}(s)$ of the curve lie in the plane.
- The plane itself is the tangent plane to the surface at every point.

Thus, both $\mathbf{t}(s)$ and $\mathbf{n}(s)$ lie in the tangent plane at each point on the surface because the tangent plane is the surface itself. Therefore suppose like in the previous example $w(t) = a'(t)$(i.e the tangent vector field). Then $w'(t) = a''(t)$. We know by definition the covariant derivative is the projection of $w'(t)$ onto the tangent plane but $a''(t)$ already lies entirely in the tangent plane for a in the case where S is the plane! So the covariant derivative and the usual derivative of vectors on the plane agree!

Proposition 126

Let w and v be parallel vector fields along curve a. Then $\langle w(t), v(t)\rangle$ is fixed.

Proof. To say that vector field w is parallel along a means that $\frac{dw}{dt}$ is normal to the plane tangent to the surface at $a(t)$(so there is no component that can be projected to the tangent plane hence $\frac{Dw}{dt} = 0$). That is to say

$$\langle v(t), w'(t)\rangle = 0$$

recall this is must be true true as both v, w are vector fields which map points to vectors in $T_p(S)$. Similary for the parallel vector field v we have

$$\langle v'(t), w(t)\rangle = 0$$

therefore

$$\langle v(t), w(t)\rangle = \langle v'(t), w(t)\rangle + \langle v(t), w'(t)\rangle = 0$$

so $\langle w(t), v(t)\rangle$ is constant

Proposition 127

Let $a : I \to S$ be a parameterized curve in S and let $w_0 \in T_{a(t_0)}(S)$, $t_0 \in I$. Then there exists a unique parallel vector field $w(t)$

In other words for every curve a on a surface S, there is *one* unique parallel vector field along it.

With the above proposition we are now ready to talk about parallel transport of a vector along a parameterized curve.

Definition 128

Let $a : I \to S$ be a parameterized curve and $w_0 \in T_{a(t_0)}(S) \in I$. Let w be the parallel vector field along a with $w(t_0) = w_0$. The vector $w(t_1)$, $t_1 \in I$ is called the **parallel transport** of w_0 along a at the point t_1.

Proposition 129

Consider the following properties of parallel transport

1. First two points $p, q \in S$ and a paramterized curve $a : I \to S$ with $a(0) = p$ and $a(1) = q$. Then the map $P_a : T_p(S) \to T_q(S)$ that assigns each $v \in T_p(S)$ is parallel transport along a at q is a *linear isometry*
2. If two surfaces S and $\overline{S}$ are tangent along a paramterized curve a and w_0 is a vector of $T_{a(t_0)}(S) = T_{a(t_0)}(\overline{S})$ then $w(t)$ is the parallel transport of w_0 relative to S if and only if $w(t)$ is the parallel transport of w_0 relative to $\overline{S}$

Proof. Let $w(t), v(t)$ be parallel vector fields along a. For (1) to be considered a linear isometry we must have

$$\langle w(0), v(0)\rangle = \langle P_a(w(0)), P_a(v(0)\rangle = \langle w(1), v(1)\rangle$$

where the last equality follows since

$$w(0) \in T_{a(0)}S \quad \text{and} \quad v(0) \in T_{a(0)}S$$

and upon a map by P_a, by definition we have the corresponding parallel transports along a

$$w(1) \in T_{a(1)}S \quad \text{and} \quad v(1) \in T_{a(1)}S$$

However by 126 we know $\langle w(t), v(t)\rangle$ is constant.

For (2), consider that $\frac{Dw}{dt}$ of w is the same for both surfaces since the same tangent vector w_0 and tangent plane is involved.

Definition 130

A nonconstant parameterized curve $\gamma : I \to S$ is said to be a **geosedic** at $t \in I$ if the field of its tagent vectors $\gamma'(t)$ is parallel along γ at t.

By 126 we immediately see that $|y'(t)| = \langle y'(t), y'(t)\rangle$ =constant

Definition 131

A regular connected curve C in S is said to be a geodesic if for every $p \in C$ the paramterization $a(s)$ of a coordinate neighbourhood of p by the arc length s is a parameterized geodesic; that is $a'(s)$ is a prallel vector field along $a(s)$

Definition 132

Let w be a differentiable field of unit vectors along a paramterized curve $a : I \to S$ on an oriented surface S. Since $w(t), t \in I$ is unit vector field, $(\frac{dw}{dt})(t)$ is normal to $w(t)$ and therefore

$$\frac{Dw}{dt} = \lambda(N \wedge w(t))$$

The real number $\lambda = \lambda(t)$ denoted by $[\frac{Dw}{dt}]$ is called the **algebraic value of the covariant derivative** of w at t.

Observe that the sign of $[\frac{Dw}{dt}]$ depends on the orientation of $S(\pm N$ clearly affects the polarity of the calculated value) and that $[\frac{Dw}{dt}] = \langle \frac{dw}{dt}, N \wedge w \rangle$(quite obvious see that it's in the form $a = \lambda b$ where $a \parallel b$)

Definition 133

Let C be an oriented regular curve contained in an oriented surface S and let $a(s)$ be a paramterization of C in a neighbourhood of $p \in S$ by the arc length s. The algebraic value of the covariant derivative $[Da'(s)/ds] = k_g$ of $a'(s)$ at p is called the **geodesic curvature** of C at p

We can immediately see that

$$k^2 = k_g^2 + k_n^2$$

given that k_g and k_n(normal curvature recall above) are the tangential and normal components of the curvature

6.5 The exponential map. geodesic polar coordinates

Lemma 134

If the geodesic $\gamma(t, v)$ is defined for $t \in (-\varepsilon, \varepsilon)$ then the geodesic $\sigma(t, \lambda v), \lambda \in \mathbb{R}, \lambda > 0)$ is defined for $t \in (-\varepsilon/\lambda, \varepsilon/\lambda)$ and $\gamma(t, \lambda v) = \gamma(\lambda t, v)$

Proof. Let $a : (-\varepsilon/\lambda, \varepsilon/\lambda) \to S$ be a paramterized curve defined by $a(t) = \gamma(\lambda t)$. Then $a(0) = \gamma(0), a'(0) = \lambda\gamma'(0)$ amd by the linearity of D 4 we have

$$D_{a'(t)} a'(t) = \lambda^2 D_{\gamma'(t)} \gamma'(t) = 0$$

It follows that a is a geodesic with initial conditions $\gamma(0), \lambda\gamma'(0)$ and by uniqueness it must be equal to the original defintion $a(t) = \gamma(\lambda t)$ so

$$a(t) = \gamma(t, \lambda v) = \gamma(\lambda t, v)$$

Remark 135. *This means intuitively since the "speed" of the geosedic is constant(the component of $\frac{dw}{dt}$ in the tangential plane is constant) we are able to traverse its trace is a prescribed time of our choice. For example we want to traverse the same entire trace defined by $a(-\varepsilon, \varepsilon)$ in a shorter time interval of $(-\varepsilon/\lambda, \varepsilon\lambda)$ by doing $a(\lambda t)$*

Proposition 136

Given $p \in S$ there exists an $\varepsilon > 0$ such that $\exp_p$ is defined and differenitable in the interior B_ε of a disk of raidus ε of $T_p(S)$ with center in the origin

Proof. It is clear for every direction of T_pS it is possible. From the previous lemma you could take a small enough v so that the interval of the definition $\gamma(t, v)$ contains 1 so that $\gamma(1, v) = \exp_p(v)$ is defined. To show that this reducetion can be made uniformly in all directions we will come back after the next subsection...to be continued

Proposition 137
$\exp_p : B_\varepsilon \subset T_pS \to S$ is a diffeomorphism in a neighborhood $U \subset B$ of the origin 0 of T_pS

Proof. Consider the curve $a(t) = tv, v \in T_pS$. So $a(0) = 0$ and $a'(0) = v$. Hence the curve $(\exp_p \circ a)(t) = \exp_p(tv)$ has at $t = 0$

$$\frac{d}{dt}(\exp_p(tv))|_{t=0} = \frac{d}{dt}(\gamma(t, v))|_{t=0} = v$$

□

In a system of normal coordinates centred in p the geodesics that pass through p are the images by $\exp_p$ of the lines $u = at, v = bt$ which pass through the origin of T_pS. Observe also that at p the coefficients of the first fundamental form in such a system are given by $E(p) = G(p) = 1, F(p) = 0$

7 Global differential geometry

7.1 Complete Surfaces: Hopf-Rinow Theorem

Definition 138
A regular(connected) surface S is said to be **extendable** if there exists a regular (connected) surface $\overline{S}$ such that $S \subset \overline{S}$ as a proper subset. If there exists no such $\overline{S}$, S is said to be nonextendable.

Definition 139
If $v \in T_p(S), v \neq 0$ is such that $\gamma(|v|, v/|v|) = \gamma(t, v)$ is defined we set

$$\exp_p(v) = \gamma(1, v) \quad \text{and } \exp_p(0) = p$$

Definition 140
A regular surface S is said to be **complete** when for every point $p \in S$ an paramterized geodesic $\gamma : [0, \varepsilon) \to S$ of S starting from $p = \gamma(0)$ may be extended into a paramterized geodesic $\tilde{y} : R \to S$ defined on the entire line R

7.2 First & Second Variation Arc Length: Bonnet theorem