

MIT 2.005-6 Thermofluids Engineering 1-2

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1 Introduction

Definition 1

The systems of interest to the thermo-fluid engineer fall into the following 3 general categories

1. **closed systems** in which the quantity of matter contained within the system *boundary* is fixed for all time
2. **open systems** in which matter contained within the system boundary is allowed to flow across the system boundary
3. **cyclic systems** in which the matter contained within the system boundary is subjected to a series of *processes* that produce no net change in the state of the matter but do result in interaction between the systems the *environment*

Definition 2

uncoupled systems are systems in which thermal and mechanical aspects of system behavior are separate in the absence of dissipation.

Example 3

These include

- (a) Pure conservative mechanical systems: in which work transfer is the only possible energy interaction
- (b) Pure Thermal systems: in which heat transfer is the only possible energy interaction
- (c) Pure Dissipative systems: in which both heat and work transfer can occur but the energy flow by work transfer is always into the system and the energy flow by heat transfer is always out of the system with no possibility of storing energy within the system boundary

Definition 4

On the other hand **coupled systems** are systems in which the thermal and mechanical aspects of system behaviour can each be influenced by both work transfer and heat transfer.

Example 5

These include

- (a) ideal gas
- (b) pure substances

In both cases there is a single form of stored energy that can be changed by both work transfer and heat transfer

Definition 6

Pay attention to the following terminology:

- Everything outside the system boundary is referred to as the **environment**
- The term **state** is used to signify the condition of a system at specific instant
- A **property** is one of those observable macroscopic quantities which are definable at a particular instant without reference to the system's history
- A minimum number of these properties necessary to specify the state of the system constitutes the set of **independent** properties with all others regarded as **dependent** on these

a complete description of a system's state include the values of all its properties

Example 7

The common thermal-fluids quantities that are *properties* include:

pressure, mass, density, temperature, internal energy, enthalpy, entropy, position, velocity, charge, voltage and force

On the other hand the common quantities in thermal-fluids engineering that are *non-properties* are:

heat transfer and work transfer

Definition 8

A system is identified and described by means of mathematical expressions known as **constitutive relations**. These relations express the dependent properties in terms of the independent properties

Definition 9

A **process** occurs when a system undergoes a change of state with or without interactions with its environment. During the change of state, the system passes through a succession of states which forms the **path** of the process

There exists a special class of processes known as **cycles** in which the initial and final states are identical

2 Energy and the first law of thermodynamics

2.1 the first law of thermodynamics for an isolated system

The first law is a generalization of the observed facts about the energy interactions between a system and its environment over a basis of a very large number of systems. The application of the first law requires

1. A characterization of the energy interactions between the system and the environment
2. a means of describing the changes of states. specifically the changes in the stored energy of the systems in terms of the physically observable properties of the system

Definition 10

A system is said to be **isolated** when there are no interactions between the system and its environment even though there are changes occurring within the system boundary.

Definition 11 (1st Law Thermodynamics for an isolated system)

The **1st law of thermodynamics** states that there is no change in the energy stored within the system boundary of an isolated system. Mathematically,

$$(E_2 - E_1)_{\text{isolated}} = 0$$

where E_2 and E_1 is the energy stored within the system boundary in the 1st and 2nd state respectively

The 1st law of thermodynamics is sometimes known as the **principle of conservation of energy**. Energy is an **extensive** property which means that it is additive over all subsystems that makeup the system

2.2 The property energy and energy storage modes

Definition 12

If in the absence of any processes like solid or fluid friction, the thermal and mechanical characteristic of a system remain separate and apart, that is the mechanical characteristics can only be changed by work transfers while the thermal characteristics can only be changed by heat transfers. Such a system is said to be **thermodynamically uncoupled**

Conversely if in the absence of dissipate processes the thermal characteristics can be changed by positive work transfers the system is said to be **thermodynamically coupled**

The significance of the coupling between thermal and mechanical aspects of a system behaviour is to enable us to establish two broad classes of energy storage modes.

1. uncoupled energy storage mode
2. coupled energy storage mode

For uncoupled, the *stored energy* characteristic of that mode can be quantitated in terms of a *single* measurable property via a single constitutive relation or type of interaction(*either* a work transfer *or* heat transfer between the system and environment) - that is all interactions must be of the same type either work or heat transfer.

Conversely it is possible for coupled energy storage modes to have both types of energy interactions simultaneously.

Remark 13. *This means that for each uncoupled mode the value of energy of that mode can be decreased by means of one specific interaction which is either thermal or mechanical depending upon the energy storage mode. The coupled energy storage mode however cannot be described in terms of such simple constitutive relations.*

Example 14

One mode that is unique to thermal-fluid systems is the coupled energy mode known as **internal energy**

2.3 the first law of thermodynamic for a system interacting with another system

Fact 15

Because there are only two possable ways that we can transfer energy across the system boundary namely heat transfer Q and work transfer W we may express energy interactions across the boundary with the stored energy within the boundary by

$$E_2 - E_1 = Q_{1-2} - W_{1-2}$$

where the subscripts 1 and 2 denote the initial and final states for the process respectively.

Note that a positive heat transfer *increases* the stored energy while a positive work transfer *decreases* the stored energy thus the negative sign

Proposition 16

If a system is not isolated is therefore free to interact with another system, the **first law of thermodynamics** states that the net change of energy stored within the system boundary is equal to the net energy transfer across the system boundary

Proof. To see how this was derived from 11 consider the case of a system A interacting with another system B . The two systems taken together form a composite system $A + B$ that is an isolated system so we have

$$(E_2 - E_1)_{A+B} = 0$$

and by virtue of the fact that energy is an extensive property(recall above) we have

$$(E_2 - E_1)_{A+B} = (E_1 - E_2)_A + (E_1 - E_2)_B$$

Then expressing energy interactions across the boundary with stored energy within the boundary using 2.3 we have

$$(E_2 - E_1)_A = (Q^{AB})_{1-2} - (W^{AB})_{1-2}$$

$$(E_2 - E_1)_B = (Q^{BA})_{1-2} - (W^{BA})_{1-2}$$

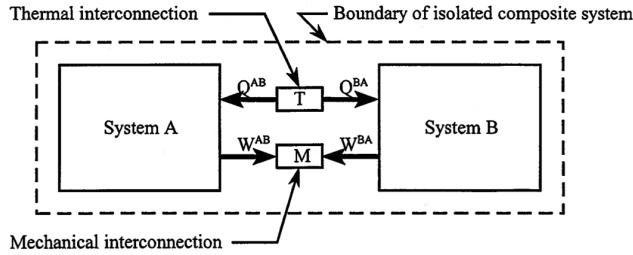


Figure 1: Graphical Representation of Two Interacting Systems

Then combining the 2 equations and assuming the 2 *separated* thermal and mechanical interconnections as shown we have(that is heat transfer and work transfer are independent)

$$(Q^{AB})_{1-2} = -(Q^{BA})_{1-2}$$

$$(W^{AB})_{1-2} = -(W^{BA})_{1-2}$$

2.4 the first law of thermodynamics for a system executing a cycle

We now consider an alternative statement of the first law of thermodynamics, one that will prove that **energy** is indeed a **property** of a system(recall above meaning does not depend on history, only current conditions). We know that in a cycle by definition the system experiences no change in state. If energy is indeed a property then three must be no change in energy stored as well. In that case, it must be independent of process(the method in which change of state is carried out) as well.Consider

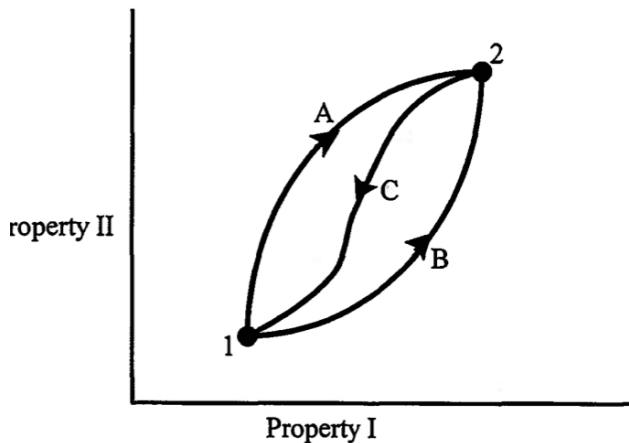


Figure 2: States of a system with two independent

Using 1st law of thermodynamics applied to each cycle, we can show that

$$(E_2 - E_1)_{\text{process } A} = (E_2 - E_1)_{\text{process } B}$$

Hence it is justifiable to assume that stored energy is a property of a system at least in these contexts (recall the 1st and 2nd laws are all empirical that's why we are only using discrete cases to prove it, so this is in no way rigorous full generalization).

2.5 work transfer

Fact 17

In order for a work transfer interaction to occur a system must satisfy

1. the environment must exert a concentrated boundary force on the system
2. there must be a displacement of the system boundary in response to this applied force

Definition 18

The infinitesimal work transfer δW from a system due to concentrated tension boundary force \vec{F} exerted on the system by its environment is given by the negative scalar product of the force vector \vec{F} and the infinitesimal outward displacement vector $d\vec{r}$ of the boundary mass points at the instantaneous point of application of the boundary force. Thus

$$-\delta W = \vec{F} \cdot d\vec{r}$$

Where the negative sign is simply a convention

2.6 Pure Conservative Mechanical Systems: Simple models for uncoupled mechanical energy storage modes

First consider the kinetic energy storage mode: pure translational mass model

Example 19

Compute the work transfer required to increase the translational velocity of a 10kg mass by 1m/s when at an initial velocity of 1m/s. Repeat the calculation for an initial velocity of 10m/s

Solution. The system and processes are modeled as a pure translational mass. Application of the first law gives

$$-W_{1-2} = E_2 - E_1 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

For initial 1m/s

$$-W_{1-2} = \frac{1}{2}(10\text{kg})(2\text{m/s})^2 - \frac{1}{2}(10\text{kg})(1\text{m/sec})^2$$

For initial 10m/s

$$-W_{1-2} = \frac{1}{2}(10\text{kg})(11\text{m/s})^2 - \frac{1}{2}(10\text{kg})(10\text{m/sec})^2$$

Example 20

Compute the work transfer and change in energy when the extension of a simple elastic spring is changed from -0.16m to +0,16m. The spring constant k is 2000N/m.

Solution.

$$E_2 - E_1 = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

$$E_2 - E_1 = \frac{1}{2}(2000N/m)(+0.16m)^2 - \frac{1}{2}(2000N/m)(-0.16m)^2$$

Then by first law

$$-W_{1-2} = E_2 - E_1 = 0$$

The rest of the problems in this chapter is just basic mechanics questions using principle of conservation of energy relating to work transfer by

$$-W_{1-2} = (E_1 - E_1)_{\text{elastic}} + (E_1 - E_1)_{\text{gravitational}} + (E_1 - E_1)_{\text{kinetic}}$$

2.7 first law of thermodynamics for an uncoupled mechanical system

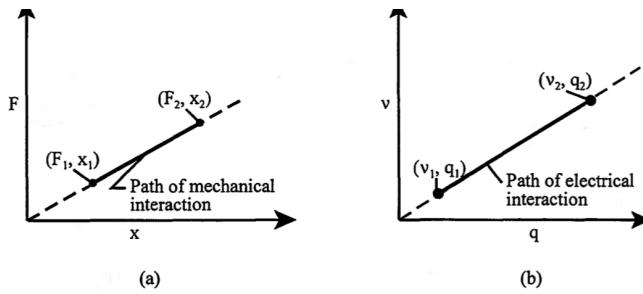


Figure 2.7 Paths of Mechanical and Electrical Interactions in an Uncoupled Electro-mechanical System

As you can see the mechanical and electrical interactions are clearly separate and independent of each other

2.8 first law of thermodynamics for a coupled electro-mechanical system

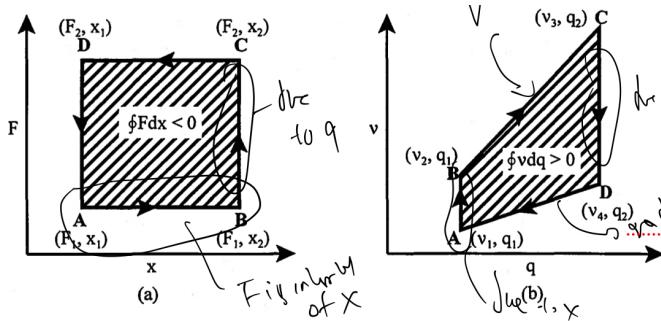


Figure 2.9 Cycle of an Electro-mechanically Coupled System

Consider

$$V = \frac{qX}{\epsilon_0 A}$$

$$F = \frac{Vq}{2X} = \frac{q^2 \epsilon_0}{A}$$

To see why this defers from the uncoupled case consider that v does not just depend on q now but also x . Likewise F also depends on q . Therefore changes in x result in vertical shifts in v and likewise for q for F . These vertical shifts are not present in the uncoupled case.

1. From A to B: With F_1 and q_1 fixed the plate spacing increases from x_1 to x_2
2. From B to C: With x_2 fixed as charge increases from q_1 to q_2 by means of an electrical interaction
3. From C to D: With F_1 and q_2 fixed the plate spacing decreases from x_2 to x_1
4. From D to A: With x_1 fixed as charge increases from q_2 to q_1

3 Equilibrium and the second law of thermodynamics

It is well known that if left to themselves, if left to themselves, i.e isolated from their environment all mechanical systems in motion will eventually run down to a state of static equilibrium. As was with the case with mechanical systems the process of thermal run down to equilibrium always proceeds in the same direction, namely towards uniform temperature. Again the process cannot be reversed without breaking the conditions of isolation so in this sense we say such a process is **irreversible**

Because of the universal nature of this phenomenon, we can attempt to generalize in a manner that allow it to be applied to all systems regardless of their nature much like how the first law of thermodynamics has generalized the universal concept of conservation of energy. In our case the 2nd law of thermodynamics will generalize the universal property of run down to equilibrium.

Recall for the 1st law of thermodynamics we introduced the quantity stored energy within a boundary to do so. For the second law of thermodynamics we introduce a quantity known as **entropy** which measures the "extent of run down to equilibrium". In fact we will show later that any irreversible process generates entropy.

Like energy, it is an extensive property that can be stored within the system and like energy it can be transferred across the system boundary.

3.1 entropy as a property and the second law of thermodynamics

Definition 21

Like the 1st law of thermodynamics expresses the conservation of stored energy in an isolated system the 2nd law of thermodynamic expresses the conservation of entropy of an isolated system but with an important exception - that this only applies to **reversible processes**. Mathematically this means

$$(S_1 - S_2)_{\text{isolated}} \geq 0$$

where $(S_1 - S_2)_{\text{isolated}} = S_{\text{gen}}$ and $S_{\text{gen}} > 0$ for irreversible process and $S_{\text{gen}} = 0$ for reversible process.

That is for changes in entropy we need to consider not just transfer of entropy(which is net zero in an isolated system for reversible processes) but entropy generated if the process is irreversible.

Definition 22

Entropy is transferred into or out of a system by means of a heat transfer only

This distinguishes it from energy which can be transferred by means of both heat and work transfer

Definition 23

We define entropy transfer associated with the heat transfer Q^{AB} for system A as

$$(S_{\text{transfer}})^{AB} = \int_{\text{state } 1_A}^{\text{state } 2_A} \frac{\delta Q^{AB}}{T_A}$$

where T_A is the temperature at that point on the boundary of system A where Q^{AB} crosses. The states 1_A and 2_A are the initial and final states respectively of system A for the heat transfer interaction

similarly for system B we have

$$(S_{\text{transfer}})^{BA} = \int_{\text{state } 1_B}^{\text{state } 2_B} \frac{\delta Q^{BA}}{T_B}$$

Since the point on the boundary where heat transfer and hence entropy transfer occurs is common to the two systems A and B it must be that $T_A = T_B$. Moreover because the composite consisting of systems A and B is isolated recall from above that means $Q^{AB} = -Q^{BA}$ thus it follows that entropy transfers between two interacting systems are equal in magnitude and opposite in sign

$$(S_{\text{transfer}})^{AB} = -(S_{\text{transfer}})^{BA}$$

If the process that resulted in the entropy transfer between the two systems was reversible, there was no entropy generated during the interaction therefore any change in entropy is due to entropy transfer so it follows from 21 that

$$(S_2 - S_1)_{A \text{ reversible}} = -(S_2 - S_1)_{B \text{ reversible}}$$

explicitly this means *for each* system we have

$$(S_1 - S_2)_{\text{reversible}} = (S_{\text{transfer}})_{\text{reversible}}$$

In which case we have

$$(S_1 - S_2)_{\text{reversible}} = \left[\int_{\text{state } 1}^{\text{state } 2} \frac{\delta Q}{T} \right]$$

3.2 The second law of thermodynamics for a system executing a cycle

Just like how we have done for energy we aim to show that entropy change $S_2 - S_1$ is indeed a change in a property of the system. As you would have guessed we employ the use of reversible cycles(because obviously entropy is process dependent if irreversible). Again using 7.5 it can proven that

$$(S_2 - S_1)_{\text{reversible A}} = (S_2 - S_1)_{\text{reversible B}}$$

which shows independence of process as desired *provided the process is reversible*. It then follows that entropy will always have the same value in any given state *regardless* of the nature of the process, reversible or irreversible by which that state was attained.

Remark 24. You might be confused why "regardless" was used since we only assumed reversible process. You are focusing on the wrong issue. The key thing is that we need to show any cyclic process must be independent of process that achieves it. An irreversible process generates entropy so a cyclic process is not even achievable.

Finally let us relate this to irreversible processes. As a recap that we have considered 2 possible causes of entropy change so far, recall 21 it could be due to entropy generated by irreversibility and due to entropy transfer as recently discussed. With this we can express

$$(S_2 - S_1)_{\text{irreversible process}} = S_{\text{transfer}} + S_{\text{gen}}$$

which means

$$(S_2 - S_1)_{\text{irreversible process}} = \left[\int_{\text{state 1}}^{\text{state 2}} \frac{\delta Q}{T} \right]_{\text{irreversible process}} + S_{\text{gen}}$$

Therefore for a given set of end states 1 and 2 meaning a set of same fixed S_1 and S_2 since entropy is a property of the system which depends not on history/processes but at current conditions of the system, we have

$$(S_2 - S_1)_{\text{irreversible}} - S_{\text{gen}} = \left[\int_{\text{state 1}}^{\text{state 2}} \frac{\delta Q}{T} \right]_{\text{irreversible}} < \left[\int_{\text{state 1}}^{\text{state 2}} \frac{\delta Q}{T} \right]_{\text{reversible}} = (S_1 - S_2)_{\text{reversible}} \quad (1)$$

where

$$(S_1 - S_2)_{\text{irreversible}} = (S_1 - S_2)_{\text{reversible}}$$

the subscript denotes the processes involved to get to these states, it does not affect the value $(S_1 - S_2)$ which depends on the state.

Fact 25

Finally we may now write the second law in the general form

$$(S_2 - S_1) = \int_{\text{state 1}}^{\text{state 2}} \frac{\delta Q}{T} + S_{\text{gen}}$$

where $S_{\text{gen}} > 0$ for irreversible process and $S_{\text{gen}} = 0$ for reversible process.

3.3 temperature and heat transfer

Definition 26 (Thermal Energy Storage Mode: Pure Thermal System Model)

The central features of this model are:

Physical characteristics

$$m = \text{mass}$$

$$C = \text{heat capacity}$$

$$c = C/m = \text{specific heat}$$

Properties

$$T = \text{temperature}$$

$$U = \text{stored thermal energy}$$

$$S = \text{stored entropy}$$

Interaction

$$Q_{1-2} = \text{heat transfer}$$

Energy constitutive relation

$$U_2 - U_1 = C(T_2 - T_1) = mc(T_2 - T_1)$$

Entropy constitutive relation

$$S_2 - S_1 = mc \ln \left(\frac{T_2}{T_1} \right)$$

where we see that

$$C = \frac{dU}{dt}$$

and

$$c = \frac{C}{m} = \frac{1}{m} \frac{dU}{dT} = \frac{du}{dT}$$

where u is the stored energy per unit mass

Remark 27. Note that we have introduced the symbol U to denote the stored energy forms that can be influenced by heat transfer - meaning that it is justifiable to use it as you will soon see in the coupled case when U can be changed by both heat and work transfer.

Because this is a pure thermal system we may write

$$\delta Q_{\text{reversible}} = dU = mc dT$$

The reason for choosing reversible is so we may relate to enthalpy like so

$$(S_2 - S_1)_{\text{reversible}} = \left[\int_{T_1}^{T_2} \frac{\delta Q}{T} \right]_{\text{reversible}} = \left[\int_{T_1}^{T_2} \frac{mc dT}{T} \right]_{\text{reversible}} = mc \ln \left(\frac{T_2}{T_1} \right)$$

Hence the change in entropy between states 1 and 2 becomes

$$S_2 - S_1 = mc \ln \left(\frac{T_2}{T_1} \right)$$

regardless of the path that connects states 1 and 2. Again recall 24 this is not contradictory even though we clearly

avoided the use of irreversible processes for our derivation. This is because all we are saying is that any 2 states may be related like so(i.e when we just choose a reversible path between any 2 states), the focus is not the how it reaches those states.

Before we proceed any further we disgress to discuss a short section on a special case of pure thermal systems

Definition 28 (Heat Reservoir)

A special thermal system known as a **heat reservoir** is one whose mass is so large that heat capacity approaches infinity.

This means

- the system experience heat transfer interactions of any magnitude with negligible changes in temperature.
- no entropy generation during the heat transfer interaction

Remark 29. To see why the 2nd point is true consider that for a process to be reversible and hence generated no entropy it is necessary for the system to be in continuous equilibrium. In the context of entropy change **within** the heat reservoir(HR) that is, $(S_2 - S_1)_{HR}$, that must be true because temperature stays the same in the heat reservoir throughout the heat transfer interaction.(see 30)

However even though the temperature does not change, its state does change in accordance with the amount of energy and entropy transfer associated with the interaction

Let us now discuss heat and entropy transfer in the pure thermal system model. As we know heat transfer involves both the transfer of energy and entropy so it makes sense to talk about this.

Proposition 30

For a pure thermal system, heat transfer down a finite temperature gradient generates entropy and is therefore an irreversible process

Remark 31. We will explore this more in greater detail in future. For now just consider the following simple contexts

Proof. Consider

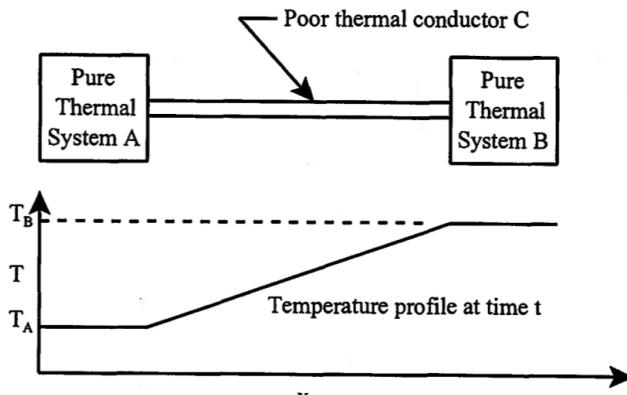


Figure 3: Heat Transfer Interaction between two pure thermal systems via a poor thermal conductor

The first law then gives

$$(U_2 - U_1)_{\text{isolated}} = (U_2 - U_1)_A + (U_2 - U_1)_B = 0$$

basically this means the net change of stored energy within the system boundary is zero for an isolated system. Therfore using the energy constitutive relations for the two systems A and B we have

$$T_2 = \frac{m_{AC_A}(T_A)_1 + m_{BC_B}(T_B)_1}{m_{AC_A} + m_{BC_B}}$$

For entropy recall that

$$(S_2 - S_1)_{\text{isolated}} = (S_2 - S_1)_A + (S_2 - S_1)_B = S_{\text{gen}}$$

where $S_{\text{gen}} > 0$ if irreversible and equal zero if reversible. Now usig our entropy constitutive relations defined earlier(recall it is independent of process) we have

$$m_{AC_A} \ln \left(\frac{T_2}{(T_A)_1} \right) + m_{BC_B} \ln \left(\frac{T_2}{(T_B)_1} \right) = S_{\text{gen}}$$

It can be shown that the LHS of the above becomes

$$m_{AC_A} \ln \left(\frac{T_2}{(T_A)_1} \right) + m_{BC_B} \ln \left(\frac{T_2}{(T_B)_1} \right) \rightarrow C^2 \frac{[(T_B)_1 - (T_A)_1]}{(T_A)_1}$$

and goes to zero as

$$(T_B)_1 - (T_A)_1 \rightarrow 0$$

(to see this recall definition of derivative and notice that $\lim_{\delta x \rightarrow 0} \frac{\delta \ln x}{\delta x} = \frac{d}{dx} \ln(x)$, you can see the resemblance...im lazy show it in full, afterall this is an engineering course). Now this shows that $S_{\text{gen}} > 0$ as desired too. \square

First consider we may write

$$dS = \frac{\delta Q}{T} \quad \Rightarrow \quad \frac{dS}{dt} = \frac{\left(\frac{dQ}{dt}\right)}{T}$$

So for the system above, we may write

$$\begin{aligned} (\dot{S}_{\text{transfer}})_{\text{end B}} &= \frac{-\dot{Q}^{BC}}{T_B} > 0 \\ (\dot{S}_{\text{transfer}})_{\text{end A}} &= \frac{-\dot{Q}^{BC}}{T_A} < 0 \end{aligned}$$

where $\dot{Q}^{BC} > 0$ since energy flows from system B to A due to the temperature difference between the 2 ystems. Notice that $T_B > T_A$ for all states except the final equilibrium state and the rate of entropy transfer out of the thermal conductor at end A is greater in magntitude than the rate of entropy transfer into the conductor at end B thus entropy was generated. We will explore this more in the chapter on heat transfer.

Example 32

A pure thermal system with heat capacity of 500J/K is at an initial temperature of 400k. It is brought into thermal communication with a heat reservoir whose temperature is 300k and the pure thermal system and the heat reservoir are allowed to experience a heat transfer interaction. Find the final equilibrium temperature of the heat reservoir and the pure thermal system. Calculate the heat transfer experienced by the pure thermal system and the entropy generated in the process.

Hint: A heat reservoir can experience heat transfer of any magnitude without changing temperature. So the final equilirbium temperature must be that of the heat reservoir whih is $T_2 = 300K$

Solution. Form an isolated system consisting of the pure thermal system(PTS) and the heat reservoir(HR) together.

For this system there are no interactions with the environment. Thus by first law

$$(E_1 - E_2)_{\text{isolated}} = (U_1 - U_2)_{\text{PTS}} + (U_1 - U_2)_{\text{HR}} = 0$$

with the aid of the energy constitutive relation for the pure thermal model we have

$$(U_2 - U_1)_{\text{HR}} = -(U_2 - U_1)_{\text{PTS}} = -C(T_2 - T_1)_{\text{PTS}} = -(500 \text{J/K})(300 \text{K} - 400 \text{K}) = 5 \times 10^4 \text{J}$$

Then from the first law for the pure thermal system

$$(Q_{1-2})_{\text{PTS}} = (U_1 - U_2)_{\text{PTS}} = -5 \times 10^4 \text{J}$$

Hence it is clear that heat transfer ws from the PTS to the HR. Because there is no entropy generated during the heat transfer interaction in th heat reservoir(refer to your notes)

$$(S_2 - S_1)_{\text{HR}} = \int_{\text{state 1}}^{\text{state 2}} \left(\frac{\delta Q}{T} \right)_{\text{HR}} = \frac{(Q_{1-2})_{\text{HR}}}{T_{\text{HR}}} = \frac{(U_1 - U_2)_{\text{HR}}}{T_{\text{HR}}} = 166.67 \text{J/K}$$

As for the PTS usig the entropy constitutive relation, we have

$$(S_2 - S_1)_{\text{PTS}} = C \ln \left(\frac{T_2}{T_1} \right) = -143.84 \text{J/K}$$

Then the second law for the isolated system then becomes

$$(S_2 - S_1)_{\text{isolated}} = (S_2 - S_1)_{\text{HR}} + (S_2 - S_1)_{\text{PTS}} = 166.67 \text{J/K} - 143.84 \text{J/K} = 22.83 \text{J/K} = S_{\text{gen}}$$

3.4 uncoupled thermodynamic systems

uncoupled thermodynamic systems without mechanical dissipation

Example 33

Consider an uncoupled system model without dissipation. A lead sphere with a mass of 0.1kg free falls a distance of 30m in a vacuum tank while experiencing heat transfer with the cooler tank walls. The initial heat of lead is 133J/KgC . The acceleration of gravity is 9.80m/s^2 . Determine the final velocity of the lead shot and the values of the heat transfer and work transfers involved in the process.

Solution. First we choose the system to be the lead shot in the gravitational field. As such the external work transfer from the cooler tank walls is zero. From constitutive rlations of pure system elemetents we have

$$\begin{aligned} (E_2 - E_1)_{\text{kinetic}} &= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \\ (E_2 - E_1)_{\text{gravitational}} &= mgz_2 - mgz_1 \\ (E_2 - E_1)_{\text{thermal}} &= mc(T_1 - T_2) \end{aligned}$$

By first law we have

$$-W_{1-2} = (E_2 - E_1)_{\text{kinetic}} + (E_2 - E_1)_{\text{kinetic}}$$

and

$$Q_{1-2} = (U_2 - U_1)_{\text{thermal}}$$

Since W_{1-2} (work transfer) is zero it follows that

$$(E_2 - E_1)_{\text{kinetic}} = -(E_2 - E_1)_{\text{gravitational}}$$

now sub in values in calculate the rest...

uncoupled thermodynamic system with mechanical dissipation

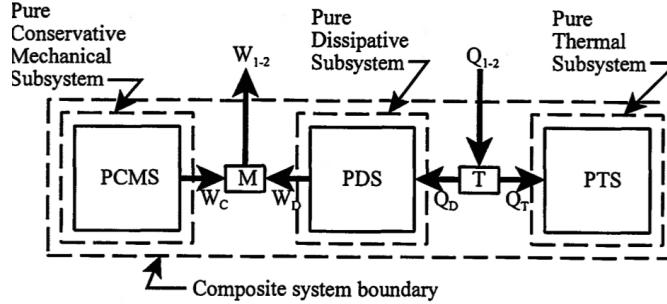


Figure 3.8 Model of Uncoupled Thermodynamic System with Dissipation

Figure 4: notice that $Q_D, W_D < 0$ see the direction of the arrows and recall the convention of the negative sign for work transfer

Consider that the first law expressions for the subsystems are

$$-W_c = (E_2 - E_1)_c$$

$$Q_D - W_D = 0$$

$$Q_T = (U_2 - U_1)_T$$

Notice that the dissipation provides an internal interactions between the mechanical and thermal aspects of the system so the first law does not separate into two independent equations. However this system is still *not coupled* because thermal stored energy of the system can be increased by means of negative work transfer interaction through the mechanism of dissipation but it cannot be decreased by positive work transfer. Hence thermal processes cannot influence the mechanical aspects of system behavior. Consider the interconnection equations

$$-W_c - W_D + W_{1-2} = 0$$

$$-Q_T - Q_D + Q_{1-2} = 0$$

The overall first law of the whole system is

$$Q_{1-2} - W_{1-2} = (E_2 - E_1)_c + (U_2 - U_1)_T$$

and

$$-(W_{1-2} - W_D) = (E_2 - E_1)_c$$

To show how mechanical dissipation generates entropy consider the case where no entropy is transferred into the pure thermal subsystem by heat transfer, that is $Q_{1-2} = 0$ then from the above we have for the first law for the whole system

$$-W_{1-2} = (E_2 - E_1)_c + (U_1 - U_2)_T$$

which means using the energy constitutive relation for pure thermal systems

$$-W_D = (U_2 - U_1)_T = C(T_2 - T_1)_T$$

But since $W_D < 0$ we know this implies $T_2 - T_1 > 0$ (temperature increase) and from the entropy constitutive relation for a pure thermal system $(S_2 - S_1)_T = C \ln \left(\frac{T_2}{T_1} \right) > 0$ (since temperature increase) we have from the second law of thermodynamics

$$(S_2 - S_1)_T = \underbrace{S_{\text{transfer}}}_{=0} + S_{\text{gen}}$$

hence

$$S_{\text{gen}} = C \ln \left(\frac{T_2}{T_1} \right) > 0$$

Example 34

A spring is connected within an oil filled piston-cylinder apparatus as shown below. The spring is slowly extended by pulling on the string. The string is cut and the spring returns to the unextended length. Determine the final temperature and the entropy generated during the run down to equilibrium.

3.5 thermal dissipation

Previously we have seen how the process of mechanical dissipation in the uncoupled thermodynamic system leads to generation of entropy. We now explore what is meant by "thermal dissipation"

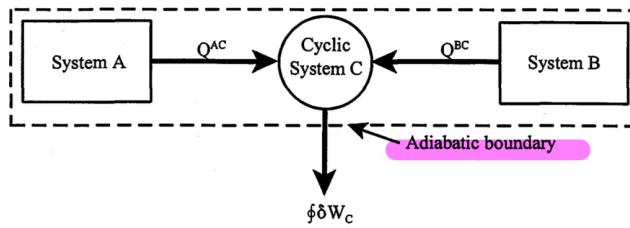


Figure 3.9 Thermal Equilibration between Two Systems Via a Cyclic System

Figure 5: note that C operates in reversible cycles and notice $\delta W_C > 0$ notice again the convention and the arrow direction

Consider that by the second law of thermodynamics since the overall system goes through reversible processes it must be that

$$(S_2 - S_1)_{A+B+C} = (S_2 - S_1)_A + (S_2 - S_1)_B + \oint dS_C = 0$$

and since entropy is a property

$$\oint dS_C = 0$$

according to the first law for the adiabatic ($Q_{1-2} = 0$) system A+B+C we have

$$-(W_{1-2})_{A+B+C} = (U_2 - U_1)_{A+B+C} = (U_2 - U_1)_A + (U_2 - U_1)_B + \oint dU_C$$

where

$$\oint dU_C = 0$$

since stored energy is also a property so it follows that

$$-(W_{1-2})_{A+B+C} = (U_2 - U_1)_A + (U_2 - U_1)_B = \oint \delta W_C$$

but we also know

$$\oint \delta Q_C - \oint \delta W_C = \oint dU_C = 0$$

which then implies

$$\oint \delta W_C = \oint \delta Q_C > 0$$

Remark 35. This is in direct contrast to 30 where run down to thermal equilibrium happened irreversibly and hence had no heat transfer(instead it took the form of entropy generated)

This is what we meant by **thermal dissipation** which refers to work transfer that could have been realized had the equilibrium process been carried out reversibly like so.

4 simple models for thermal-fluid systems

Definition 36

Pressure P at a point is defined as

$$P = - \lim_{\Delta A \rightarrow \Delta A_c} \frac{\Delta F_n}{\Delta A} = - \frac{d\vec{F}_n}{d\vec{A}} = - \frac{d\vec{F}_n}{\vec{n} dA}$$

where ΔA is a small area enclosing the point while ΔA_c is the smallest area over which we can consider the fluid as a continuum(i.e as matter distributed in space in a continuous manner rather than as discrete molecules) and ΔF_n is the component of the boundary force on ΔA normal to ΔA and is taken to be positive when the fluid is in tension and \vec{n} is the outward normal to the area dA . Then

$$d\vec{F}_n = -P\vec{n}dA$$

Example 37

A tornado was reported in a town. The atmospheric pressure at the earth's surface was recorded as 850mb(millibars)

- (a) Compute the value of this pressure in pascals
- (b) what would have been the height of a mercury filled barometer at this pressure at a location where $g = 9.80m/s^2$, $p_{Hg} = 13,560kg/m^3$
- (c) A building with a flat roof $10m \times 10m$ is filled with air at a pressure of $P_{\text{inside}} = 1.01325 \times 10^5 N/m^2$ as the tornado passes over. There is insufficient time to ventilate the building as the tornado approaches. Compute the net force exerted on the roof as the tornado passes overhead

Solution. 1bar = 10^5 Pa, use $h = \frac{P_{\text{atm}}}{pg}$ and for the last part $F_{+z} = P_{\text{inside}}A$ while $F_{-z} = -P_{\text{atm}}A$ then calculate $F_{\text{net}} = F_{-z} + F_z$

Fact 38

Strategy summary thus far

1. Identify the systems
2. Relate Stored energy change to heat and work transfer for each system element
3. Relate inter-system connections
4. apply equilibrium conditions

4.1 fluid models

Definition 39 (Inviscid Incompressible Fluid Model)

The central features of this model are:

Physical characteristics

m = mass

ρ = density = mass/unit volume

v = specific volume = $1/\rho$ = volume/unit mass

c = specific heat

μ = viscosity = 0

k = thermal conductivity

Properties

P = pressure

V = volume

T = temperature

U = stored thermal energy

S = stored entropy

Property constitutive relation

$$p = \text{constant} \quad \text{or} \quad v = \text{constant}$$

Interaction

W_{1-2} = work transfer

Q_{1-2} = heat transfer

Energy constitutive relation

$$(E_2 - E_1)_{\text{kinetic}} = \frac{1}{2}m\theta_2^2 - \frac{1}{2}m\theta_1^2$$

$$(E_2 - E_1)_{\text{gravitational}} = mg(z_2 - z_1)$$

$$(U_2 - U_1)_{\text{thermal}} = mc(T_2 - T_1)$$

Entropy constitutive relation

$$S_1 - S_2 = mc_v \ln\left(\frac{T_2}{T_1}\right) + mR \ln\left(\frac{V_2}{V_1}\right)$$

and knowing $\frac{P_2 V_2}{P_1 V_1} = \frac{T_2}{T_1}$ from above we may substitute this in to get

$$S_1 - S_2 = m(c_v + R) \ln\left(\frac{T_2}{T_1}\right) - mR \ln\left(\frac{P_2}{P_1}\right)$$

and

$$S_1 - S_2 = mc_v \ln\left(\frac{P_2}{P_1}\right) + m(c_v + R) \ln\left(\frac{V_2}{V_1}\right)$$

for the **viscous incompressible fluid model** it is the exact same as the above except the $\mu \neq 0$ under the physical characteristics.

Definition 40 (Ideal Gas Model)

The central features of this model are:

Physical characteristics

$$m = \text{mass}$$

$$\bar{R} = \text{universal gas constant} = 8.3134 \text{ kJ/kg-mole K}$$

$$c_v = \text{specific heat at constant volume}$$

$$M = \text{molecular weight}$$

$$R = \frac{\bar{R}}{M} = \text{gas constant}$$

Properties

$$P = \text{pressure}$$

$$V = \text{volume}$$

$$T = \text{temperature}$$

$$U = \text{stored thermal energy}$$

$$S = \text{stored entropy}$$

Property constitutive relation

$$PV = mRT$$

Interaction

$$Q_{1-2} = \text{heat transfer}$$

Energy constitutive relation

$$U_2 - U_1 = C(T_2 - T_1) = mc_v(T_2 - T_1)$$

Entropy constitutive relation

$$S_1 - S_2 = mc_v \ln\left(\frac{T_2}{T_1}\right) + mR \ln\left(\frac{V_2}{V_1}\right)$$

and knowing $\frac{P_2 V_2}{P_1 V_1} = \frac{T_2}{T_1}$ from above we may substitute this in to get

$$S_1 - S_2 = m(c_v + R) \ln\left(\frac{T_2}{T_1}\right) - mR \ln\left(\frac{P_2}{P_1}\right)$$

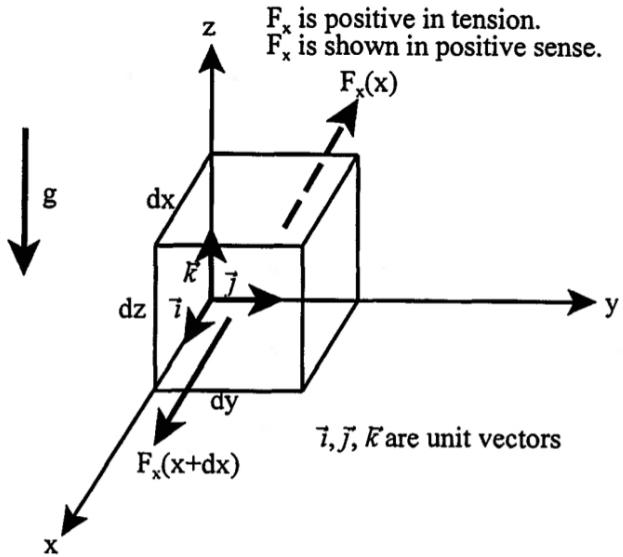
and

$$S_1 - S_2 = mc_v \ln\left(\frac{P_2}{P_1}\right) + m(c_v + R) \ln\left(\frac{V_2}{V_1}\right)$$

4.2 Pressure distribution in a static fluid

First let pressure at any location in space be defined by

$$P = P(x, y, z)$$



Consider forces in the x-direction

$$F_x(x) = -PdA = -Pdydz$$

in which case we have

$$dF_x/dx = -\frac{\partial P}{\partial x} dydz$$

so

$$dF_x = -\frac{\partial P}{\partial x} dxdydz$$

Similarly for the y and z directions we have

$$dF_y = -\frac{\partial P}{\partial y} dxdydz$$

$$dF_z = -\frac{\partial P}{\partial z} dxdydz$$

Then the net surface force vector $d\vec{F}_{\text{surface}}$ is

$$d\vec{F}_{\text{surface}} = \left(-\vec{i}\frac{\partial P}{\partial x} - \vec{j}\frac{\partial P}{\partial y} - \vec{k}\frac{\partial P}{\partial z} \right) dxdydz$$

so

$$\nabla P = -\frac{d\vec{F}_{\text{surface}}}{dxdydz}$$

which is a gradient of the scalar field P . Similarly the body force due to the gravitational field is given by

$$d\vec{F}_{\text{body}} = pg\vec{k}dxdydz$$

And so

$$d\vec{F}_{\text{net}} = d\vec{F}_{\text{surface}} + d\vec{F}_{\text{body}} = \vec{a}dm = p\vec{a}dxdydz$$

by newton's law. Then the equation governing the pressure distribution of the fluid becomes

$$\boxed{-\nabla P + p\vec{g} = p\vec{a}}$$

We now consider an example scenario

Example 41

In the static case there are no net forces so

$$-\nabla P + p\vec{g} = 0$$

and in the case when gravity is only aligned in the z direction that is

$$g_x = 0, g_y = 0, g_z = -g$$

then we have altogether

$$\frac{\partial P}{\partial x} = 0, \frac{\partial P}{\partial y} = 0, \frac{\partial P}{\partial z} = -pg$$

Fact 42

For the above scenario we see that

1. there is no shear stress in a fluid in static equilibrium
2. the normal stress(pressure) in the fluid varies only with distance measured in direction parallel to the gravity vector(typically the vertical direction)
3. pressure is independent of the shape of the container
4. pressure increases with depth of fluid

4.3 hydrostatic forces on a plane submerged surface

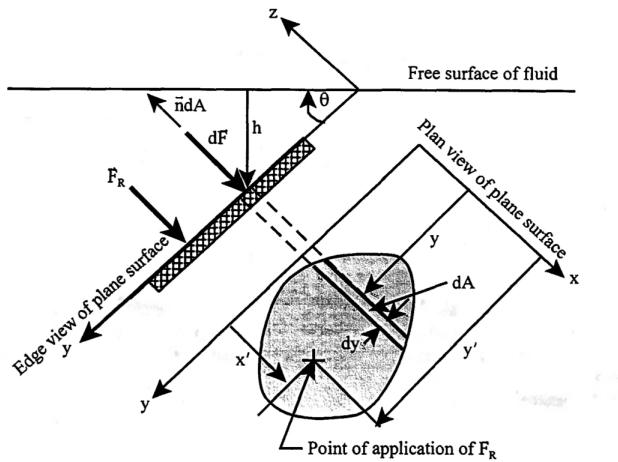


Figure 4.7 Plane Submerged Surface

Now consider

$$\begin{aligned}
 \frac{dP}{dh} &= pg \\
 P &= P_0 + pgh \\
 F_R &= \int_A P dA \\
 &= P_0 A + pg \int_A h dA \\
 &= P_0 A + \sin \theta \int_A y dA
 \end{aligned}$$

Since $y_{CG} = \frac{1}{A} \int y dA$ we have

$$F_R = P_0 A + (pg \sin \theta) y_{CG} A$$

but $y_{CG} \sin \theta = h_{CG}$ so we have

$$F_R = P_0 A + pgh_{CG} A = (P_0 + pgh_{CG}) A = P_{CG} A$$

Magnitude of the F_R : This shows that under conditions of hydrostatic equilibrium, the force acting on the plane submerged surface is equal to the product of the hydrostatic pressure at the centroid(COM) of the surface and the area of the surface.

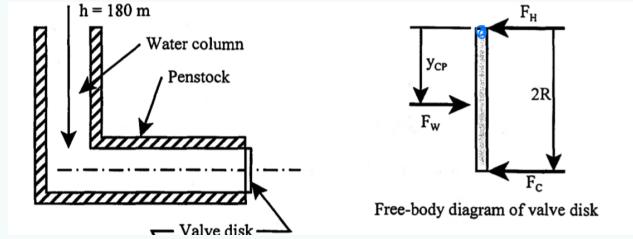
Location of F_R : The force acts through the **center of pressure** whose coordinates are given by

$$\begin{aligned}
 x'F_R &= x' \int_A P dA = \int_A x P dA \\
 y'F_R &= y' \int_A P dA = \int_A y P dA
 \end{aligned}$$

Remark 43. Note the locations of the center of mass and center of pressure are not necessarily the same

Example 44

Consider



Solution. From the free body diagram we have

$$\sum M_{\text{hinge}} = F_c(2R) - F_w y_{CP} = 0$$

then using the above relation for magnitude of resultant force, we have

$$F_c = \frac{F_w y_{CP}}{2R} = \frac{pg(h_0 + R \sin \theta) \pi R^2 y_{CP}}{2R}$$

where we have R in the calculation of F_C because that is precisely the where the COM of the valve disk is(mid way of $2R$).

4.4 hydrostatic force on curved submerged surface

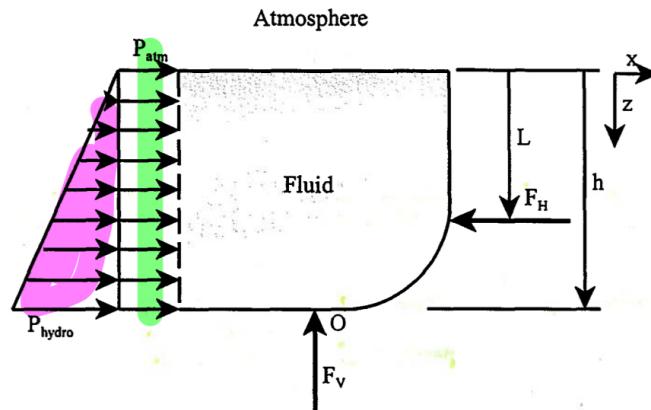


Figure 4.8 Forces on a Body of Fluid with Curved Boundaries

Remark 45. Note that there is a atmospheric pressure equal the green highlighted portion on the right hand side too acting in the opposite direction so the pressure due to atmospheric on both sides cancel

Taking the equations of equilibrium we have

$$\sum F_x = \int_0^h P_{atm} dx + \int_0^h P_{hydro} dz - F_H - P_{atm}h = 0$$

So we have

$$F_H = P_{atm}h + \int_0^h \rho g z dz - P_{atm}h = \rho g \frac{h^2}{2}$$

As for moments we have

$$\sum M_0 = P_{atm}h \left(\frac{h}{2}\right) + \int_0^h P_{hydro}(h-z) dz - F_H(h-L) - P_{atm}h \left(\frac{h}{2}\right) = 0$$

then substituting the expressions from above yields

$$\int_0^h \rho g z h dz - \int_0^h \rho g z^2 dz = F_H(h-L) = \rho g \frac{h^2}{2}(h-L)$$

on solving we obtain

$$L = \frac{2}{3}h$$

4.5 forces on submerges objects of uniform density: buoyancy and stability

Consider

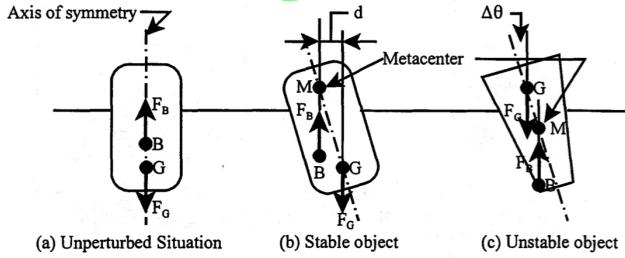


Figure 4.10 Floating Object in Unperturbed and Perturbed States

Definition 46

the **metacenter** is the point at which the line of action of the buoyancy force intersects the vertical axis of the center of gravity

See that in the above diagram in (c), it is unstable because the moment $F_G d$ acts to increase further the rotation of the vessel in the direction of overturning. In contrast in (b) it decreases instead while for (a) there is no net moment. B, G stand for center of buoyancy and center of gravity respectively.

Example 47

Consider

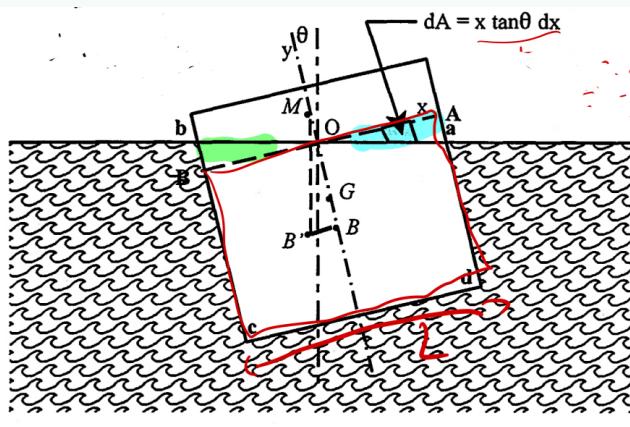
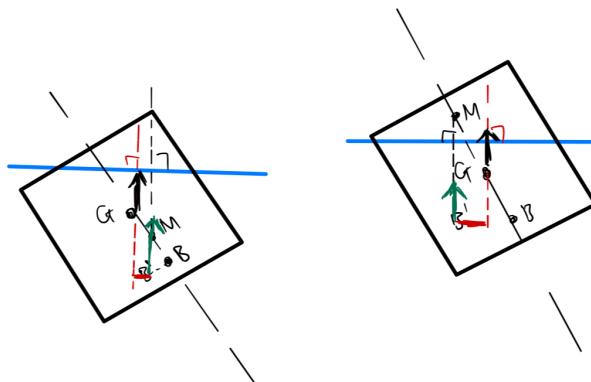


Figure 4E.8b

Solution. See that distance $MB < GB$ is a stability condition. See this as taking moments through the COM where the axis goes into the page. Then the net moments are due to the green arrows(buoyancy force)



4.6 pressure distribution in a fluid in rigid body motion

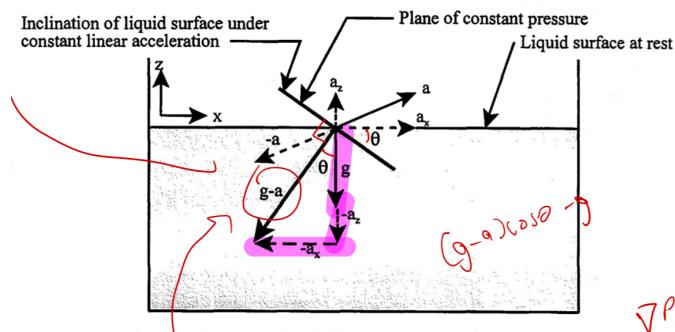


Figure 4.11 Tank of Liquid Under Constant Linear Acceleration

Example 48

Consider

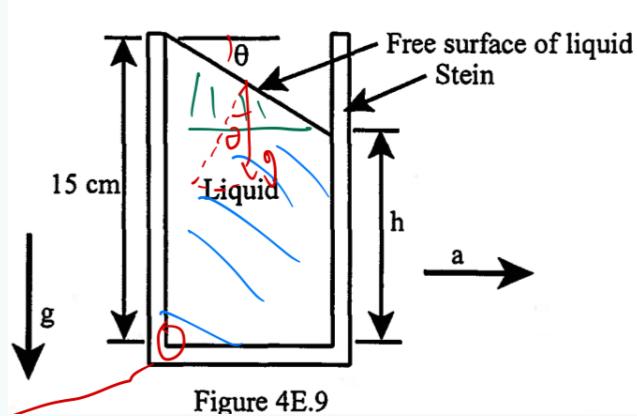


Figure 4E.9

Example 49

Consider

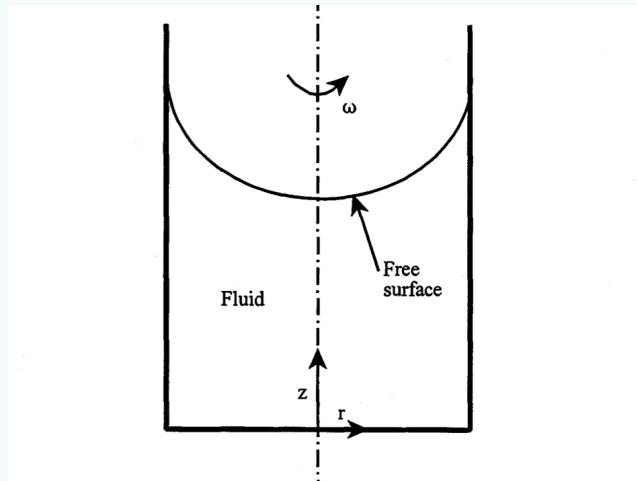


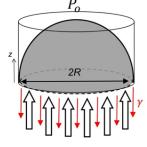
Figure 4.12 Cylindrical Container of Fluid in Rotation About Its Centerline

4.7 surface tension

Young-Laplace Relationship

IMPERIAL 
White §1.7, p28

- Consider a spherical droplet of radius R and internal pressure P_i .
- Goal:** relate P_i to R, γ and P_o .
- Perform force balance on hemisphere sectioned through equator, neglecting body forces.



$$\sum F_z = 0 = P_i \pi R^2 - \gamma 2 \pi R - P_o \pi R^2$$

Spherical surface: $P_i = P_o + \frac{2\gamma}{R}$

Key results:

- P_i increases with γ
- P_i decreases with R

As any 2-D surface can have two radii of curvature, R_1 and R_2 ,

In general: $P_i - P_o = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \gamma(\kappa_1 + \kappa_2)$ where $\kappa_i = \frac{1}{R_i}$ is the "curvature"

Applies equally well to bubbles and droplets.

Question 50. why is F_z contribution by P_0 just the projected area much like for submerged slanted and vertical walls from earlier?

Proof. Using spherical coordinates if you recall:

- $r = R$: The radius is constant for the hemisphere.
- $\theta \in [0, \pi/2]$: The polar angle ranges from the north pole to the equator.
- $\phi \in [0, 2\pi]$: The azimuthal angle spans around the equator.

we will prove that the vertical component of the force exerted by the external pressure P_0 on the hemispherical surface is $F_z = P_0 \pi R^2$, we proceed as follows:

1. **Setup the Geometry:** Consider a hemispherical surface of radius R , with uniform external pressure P_0 acting normal to the surface. The infinitesimal vertical force contributed by an element of the surface is:

$$dF_z = -P_0 \cos \theta dA,$$

where:

- $dA = R^2 \sin \theta d\theta d\phi$ is the surface area element in spherical coordinates,
- $\cos \theta$ accounts for the vertical (z-axis) component of the force.
- negative sign because it is downwards

2. **Expression for Total Vertical Force:** Integrate dF_z over the entire hemispherical surface:

$$F_z = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} -P_0 \cos \theta R^2 \sin \theta d\theta d\phi.$$

3. **Simplify the Integrals:** Separate the integration into the ϕ - and θ -components:

$$F_z = -P_0 R^2 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi/2} \sin \theta \cos \theta d\theta.$$

- The ϕ -integral evaluates to:

$$\int_{\phi=0}^{2\pi} d\phi = 2\pi.$$

- For the θ -integral, use the trigonometric identity $\sin \theta \cos \theta = \frac{1}{2} \sin(2\theta)$:

$$\int_{\theta=0}^{\pi/2} \sin \theta \cos \theta d\theta = \frac{1}{2} \int_{\theta=0}^{\pi/2} \sin(2\theta) d\theta.$$

The integral of $\sin(2\theta)$ is:

$$\int \sin(2\theta) d\theta = -\frac{1}{2} \cos(2\theta).$$

Evaluate this from $\theta = 0$ to $\theta = \pi/2$:

$$\int_{\theta=0}^{\pi/2} \sin(2\theta) d\theta = -\frac{1}{2} [\cos(\pi) - \cos(0)] = -\frac{1}{2} [-1 - 1] = 1.$$

Thus:

$$\int_{\theta=0}^{\pi/2} \sin \theta \cos \theta d\theta = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

4. **Combine Results:** Substitute the results of the ϕ - and θ -integrals into the expression for F_z :

$$F_z = -P_0 R^2 (2\pi) \left(\frac{1}{2}\right).$$

Simplify:

$$F_z = -P_0 \pi R^2.$$

5. **Interpretation:** The negative sign indicates the force is in the downward (negative z -direction). Therefore, the magnitude of the vertical force is:

$$F_z = P_0 \pi R^2.$$

This matches the expression for the external pressure force acting over the projected circular area of the hemisphere. \square

5 work transfer interactions in thermal-fluid systems

5.1 moving boundaries in a channel of liquid

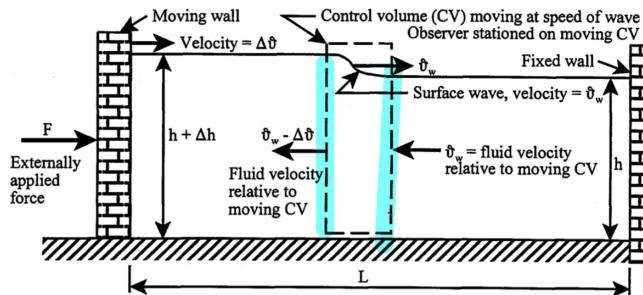


Figure 5.2 Open Channel of Liquid Bounded by Moving Wall on One End

Remark 51. this control volume looks the same to the observer at all times as it moves across

From the point of view of an observer on the control volume moving at the speed of the wave θ_w the surface wave is moving relative towards he/she at θ_w while the back is moving away from you at $\theta_w - \Delta\theta_w$ where the $-\Delta\theta_w$ the wall moving towards you makes you feel like you are moving towards the wall which is opposite in direction to the θ in which you are traveling at . We describe the behaviour entering the leaving in terms of two fundamental principals

conservation of mass

$$p(bh)\theta_w - p[b(h + \Delta h)](\theta_w - \Delta\theta) = 0$$

which describes the masses of the unit planes of width b (the blue highlighted parts) in terms of height and density. Expanding we have

$$\theta_w h - \theta_w h - \theta_w \Delta h + h \Delta\theta + \Delta\theta \Delta h = 0$$

ignoring second order small terms like $\Delta\theta \Delta h$ we obtain

$$\frac{\Delta\theta}{\theta_w} = \frac{\Delta h}{h}$$

conservation of momentum

From **newton's second law of motion** we have

$$\sum \vec{F} = \frac{d}{dt}(m\vec{\theta})$$

but because the flow is steady this implies $\dot{\theta} = 0$ so our product rule differentiation gets

$$F_b - F_f = \dot{m}\theta_{\text{out}} - \dot{m}\theta_{\text{in}}$$

where F_f and F_b is the force on the face of the control volume facing towards and away from the moving wall. Note that we have also used $\dot{m} = p(bh)\theta_w = p[b(h + \Delta h)](\theta_w - \Delta\theta)$ as found from above. Now notice

$$-F_f = (P_{\text{avg}}A)_f = \left(pg \frac{h}{2} \right) (bh) = \left(pgb \frac{h^2}{2} \right)$$

and

$$-F_b - (P_{\text{avg}}A)_b = \left(pg \frac{h + \Delta h}{2} \right) [b(h + \Delta h)] = \left(pgb \frac{(h + \Delta h)^2}{2} \right)$$

where P_{avg} comes from $(\int_0^h pgh)/h$ (or just visualize the straight line graph of p against h) Finally putting these together we have into our equation for conservation for momentum we have

$$\left(pgb \frac{h^2}{2} \right) - \left[pgb \frac{(h + \Delta h)^2}{2} \right] = p(bh)\theta_w[(\theta_w - \Delta\theta) - \theta_w]$$

and upon expanding and again ignoring 2nd order small terms we have

$$\theta_w^2 = gh$$

then using $\frac{\Delta\theta}{\theta_w} = \frac{\Delta h}{h}$ from above we obtain

$$\Delta h = \Delta\theta \sqrt{\frac{h}{g}}$$

This means the greater the piston velocity the larger the amplitude of the disturbance. Notice that since constant velocity for balance of forces we have

$$-F = F_b = P_{\text{avg}}A = \left(\frac{pg(h + \Delta h)}{2} \right) [b(h + \Delta h)] = pgb \left(\frac{h^2 + 2h\Delta h + (\Delta h)^2}{2} \right)$$

and again neglecting second order small terms we have

$$-F = pgb \frac{h^2}{2} \left(1 + 2 \frac{\Delta h}{h} \right) = pgb \frac{h^2}{2} \left(1 + 2 \frac{\Delta\theta}{\theta_w} \right)$$

so we have

$$-F = pgb \frac{h^2}{2} + pgbh^2 \frac{\Delta\theta}{\theta_w}$$

where the first term on the RHS represents the **quasi-static force** acting on the moving wall, i.e. the force that the fluid would exert on the wall if the wall were stationary and the second term on the RHS represents the relative dynamic force necessary to accelerate the fluid contained between the moving wall and the wave front.

Definition 52

The **quasi static process** is a model for a dynamic process in which the state of the system is changing at a rate which is slow compared to the rate at which the system approaches the equilibrium by means of energy transfer and entropy transfer processes internal to the system boundary

Example 53

reversible processes are a special subset of the class of quasi-static processes since by definition

1. a reversible process generates no entropy
2. a reversible process is a sequence of equilibrium states
3. a reversible process can proceed in the reverse direction just as readily as in the forward direction
4. a reversible process takes an infinitely long time to be carried out

5.2 reversible processes

Definition 54

a **mechanical matching element** is one that maintains mechanical equilibrium with precisely the correct force in the system regardless of position

Example 55

Consider

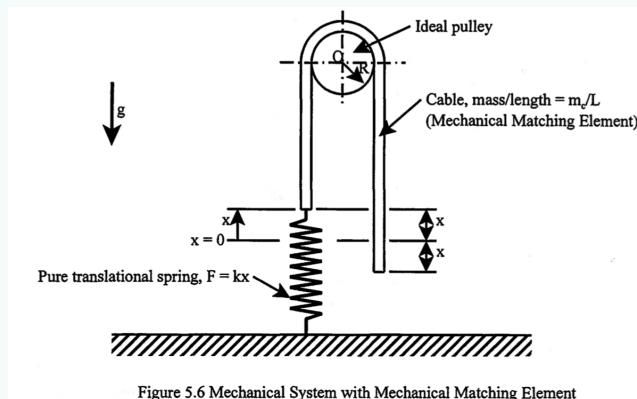


Figure 5.6 Mechanical System with Mechanical Matching Element

Balance forces to get

$$\left(\frac{m_c}{L}\right)gx = 2kx$$

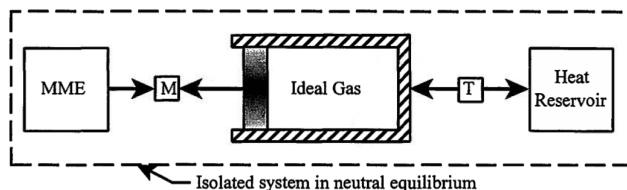
then a spring that satisfies

$$k = \left(\frac{m_c}{L}\right)2g$$

will always be in equilibrium regardless of the value of x .

5.3 neutral equilibrium

Let us now extend the neutral equilibrium to beyond just the mechanical case. We now specify that conditions for neutral equilibrium for both the mechanical and heat transfer interactions now.



Example 5.6

Consider

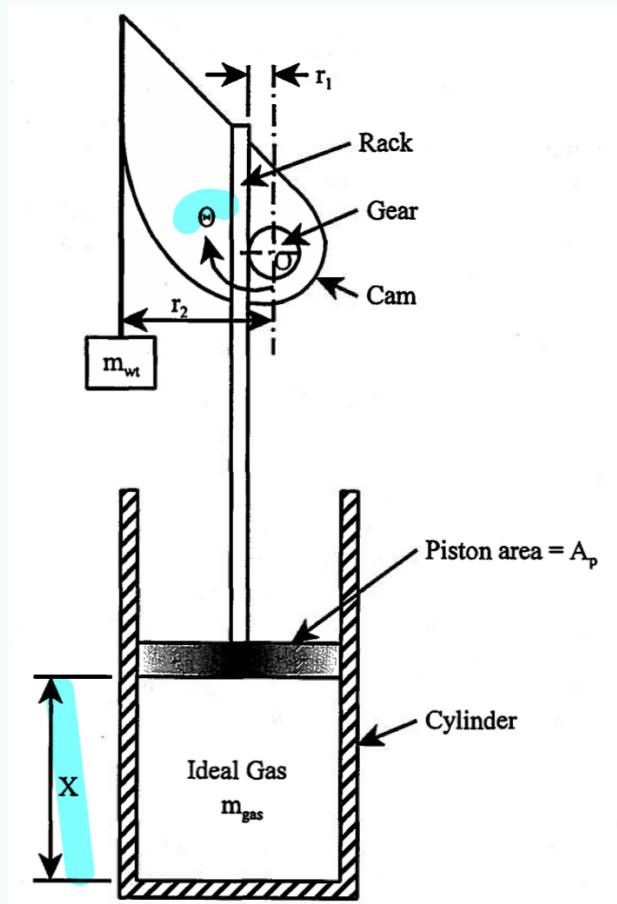


Figure 5.9 Ideal Gas with Mechanical Matching Element

From condition for rotational equilibrium it follows that

$$r_1 F_{\text{piston}} = m_{wt} g r_2$$

and for the rack and gear combination

$$r_1 \Theta = X$$

Thus

$$F_{\text{piston}} = \frac{m_{wt} g r_2 \Theta}{X}$$

For an ideal gas the property constitutive relation gives

$$P_{\text{gas}} = \frac{m_{gas} R T}{V_{\text{gas}}} = \frac{m_{gas} R T}{X A_p}$$

In which case combining the 2 noticing that F_{piston} is the normal reaction force of the pressure in the cylinder on the rack we have

$$F_{\text{piston}} = \frac{m_{gas} R T}{X}$$

and so combining with earlier conditions

$$r_2\Theta = \frac{m_{\text{gas}}RT}{m_{\text{wt}}g}$$

where X is cancelled out. I.e the piston that satisfies this design requirement makes it a **mechanical matching element**. Before we layout how we ensure thermal equilibrium too first consider what happens when the mass m_{wt} is increased by a small amount δm_{wt} .

1. the force exerted by the cam on the piston will be greater than the force of the gas on the piston face
2. so the piston moves down, decreasing the volume. But that means there is negative work transfer for the gas which then increases its internal energy
3. being an ideal gas, this means its temperature increases too
4. the energy and temperature increase only stops when its rate is equal to rate of heat transfer from the gas to the reservoir
5. in that case temperature and energy remains the same as the piston continues to move down

We may express such a temperature increase like so

$$r_2\Theta = \frac{m_{\text{gas}}R(T+\delta T)}{(m_{\text{wt}} + \delta m_{\text{wt}})g}$$

recall this is the relation that must be satisfied for the mechanical matching element. Then we have

$$\delta T = \frac{r_2\Theta(m_{\text{wt}} + \delta m_{\text{wt}})g}{m_{\text{gas}}R} - T$$

where T is the temperature of the heat reservoir. It is clear now that if δm_{wt} is small so is δT . Therefore under such conditions the process becomes **quasi-static** as the gas stays constant at T which is the temperature of the heat reservoir and thus is in thermal equilibrium with it.

Remark 57. *Reminder that*

- *thermal equilibrium = same temperature*
- *mechanical equilibrium = balance of forces and moments*

In which case now the work and heat transfer for the ideal gas is given by

$$\begin{aligned} W_{1-2} &= \int_{V_1}^{V_2} PdV \\ W_{1-2} &= \int_{V_1}^{V_2} \frac{mRT}{V} dV = mRT \ln \left(\frac{V_2}{V_1} \right) \\ Q_{1-2} &= W_{1-2} + (U_2 - U_1) = W_{1-2} = mRT \ln \left(\frac{V_2}{V_1} \right) \end{aligned}$$

where the last relation follows by first law and by *energy constitutive relation* for ideal gases(recall earlier) where we have $U_2 - U_1 = C(T_2 - T_1)$ which in this case is clearly 0.

Proposition 58

The above describes an *reversible* isothermal process for an ideal gas

Proof. With the system in equilibrium at state 2 we decrease the mas m_{wt} by an infinitesimal amount dm_{wt} (which then implies $\delta T \approx 0$ so thermal equilibrium again). Then the system will execute a quasi static expansion since we still

have both thermal and mechanical equilibrium as still using same piston. Then we may use the same equations above to get

$$W_{2-1} = mRT \ln\left(\frac{V_1}{V_2}\right)$$

so clearly $\oint \delta W = 0$ and $\oint \delta Q = 0$. Now because temperature is constant(thermal equilibrium) net heat transfer for the cycle must be zero

$$\oint \frac{\delta Q}{T} = \frac{1}{T} \oint \delta Q = \oint dS + \oint \delta S_{\text{gen}} = 0$$

However recall from earlier that entropy is property and since our system just executed a cycle it must be that

$$\oint dS = 0$$

all these prove that

$$\oint \delta S_{\text{gen}} = 0$$

which proves the proposition

Example 59

Consider

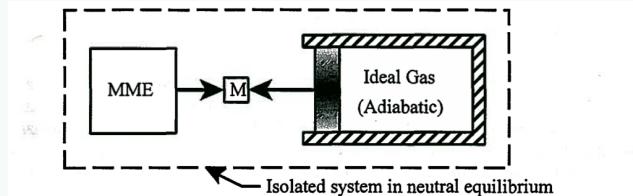


Figure 5.14 Ideal Gas and Mechanical Matching Element in Neutral Equilibrium

Example 60

Consider

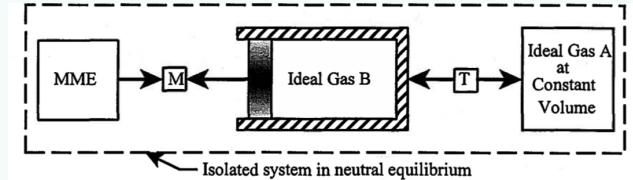


Figure 5.15 Ideal Gas at Constant Volume, Ideal Gas in a Piston-Cylinder, and a Mechanical Matching Element in Neutral Equilibrium

Example 61

Consider

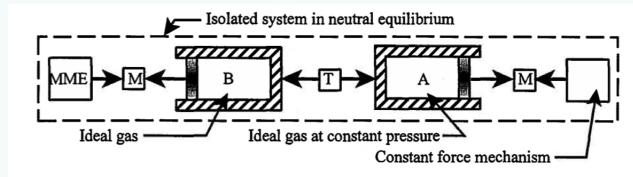


Figure 5.16 Ideal Gas at Constant Pressure in Neutral Equilibrium

5.4 limitations imposed by the second law of thermodynamics

Recall from 1 that

$$(\delta Q)_{\text{reversible}} = T dS$$

and

$$(\delta Q)_{\text{irreversible}} < T dS$$

And so from first law we have

$$(W_{1-2})_{\text{rev}} = \int_1^2 T dS - (U_2 - U_1)$$

and

$$(W_{1-2})_{\text{irrev}} < \int_1^2 T dS - (U_2 - U_1)$$

Now consider 2 special cases

Example 62

Isothermal process(can be reversible or irreversible)

$$(W_{1-2})_{\text{isothermal}} \leq \int_1^2 T dS - (U_2 - U_1)$$

which we can write as

$$(W_{1-2})_{\text{isothermal}} \leq -[(U - TS)_2 - (U - TS)_1] = -(F_1 - F_2)$$

we call $U - TS$ the **helmholtz free energy** denoted by F which corresponds to the amount of energy which is "free" be be extracted as a work transfer

Example 63

Isothermal process(can be reversible or irreversible) and constant pressure

$$(W_{1-2})_{\text{isothermal}} \leq \int_1^2 T dS - (U_2 - U_1)$$

which we can write as

$$(W_{1-2})_{\text{isothermal}} \leq -[(U - TS)_2 - (U - TS)_1] = -(F_1 - F_2)$$

we call $U - TS$ the **helmholtz free energy** denoted by F which corresponds to the amount of energy which is "free" be be extracted as a work transfer

6 heat transfer interactions in thermal fluid systems

6.1 modes of heat transfer

conduction

In gases energy is transferred from one location to another by virtue of the collisions between the gas molecules.

Theorem 64

The **fourier conduction law** states that the heat transfer rate i.e the energy flowing per unit time \dot{Q} is directly proportional to the temperature gradient and the cross sectional area of the gas through which the energy is flowing

$$\dot{Q} = -kA \left(\frac{dT}{dx} \right)$$

where $\frac{dT}{dx}$ is the temperature gradient and k is the constant of proportionality known as the **thermal conductivity**, a physical characteristic of the gas.

In the case of liquids, the mechanisms of conduction is quite similar to that in gases so its conduction rate is also given by the fourier conduction law except that the molecules are much closer together and the forces of attraction are much stronger. Hence the thermal conductivity of liquids are significantly larger than the values for gases.

For solids the heat transfer rate is once again given by the fourier conduction law except that the values of thermal conductivity for solids are typically much higher than the values for liquids

convection

Theorem 65

In the case of convection(bulk movement heat transfer) the heat transfer rate can be calculated by **newton's law of cooling**

$$\dot{Q} = h_c A_s (T_s - T_\infty)$$

where the **convection heat transfer coefficient** h_c is a quantity that depends upon the nature of the flow field and the various temperatures of the participating media

radiation

The term radian implies energy transfer by virtue of some sort of wave phenomenon

Theorem 66

Stefen and Boltzmann showed that

$$\dot{Q}_{12} = \varepsilon_1 A_1 (\sigma T_1^2 - \sigma T_2^2)$$

where σ is the **Stefan-Boltzmann radiation constant** and its value is $\sigma = 5.67 \times 10^{-8} W/m^2 K^4$

Note that with some rearrangement letting $T_1 \approx T_2$ we have

$$\begin{aligned}\dot{Q}_{12} &= \varepsilon_1 A_1 \sigma (T_1^2 + T_2^2)(T_1 + T_2)(T_1 - T_2) \\ \dot{Q} &= \varepsilon_1 A_1 \sigma (4T_m^3)(T_1 - T_2) \\ \dot{Q}_{12} &\approx A_1 h_r (T_1 - T_2)\end{aligned}$$

where $h_r = 4\varepsilon_1 \sigma T_m^3$ is called the **radiation heat transfer coefficient** and $T_m = \frac{T_1+T_2}{2}$

6.2 thermal resistance

Upon inspection of the fourier conduction law, newton law of cooling and the above radiation equation we see that they bear a strong resemblance to **Ohm's Law**

$$I = \frac{(\phi_1 - \phi)2}{\mathfrak{R}}$$

Thus by analogy we can write for heat transfer rate in the steady state

$$\dot{Q} = \frac{T_1 - T_2}{R_{th}}$$

where R_{th} stands for thermal resistance. In which case for a steady state condution heat trasfer normal to plane slab of thickness L we have

$$R_{cond} = \frac{L}{kA}$$

and for convection heat transfer we have

$$R_{conv} = \frac{1}{h_c A}$$

and for radiation heat transfer we have

$$R_{rad} = \frac{L}{kh_r}$$

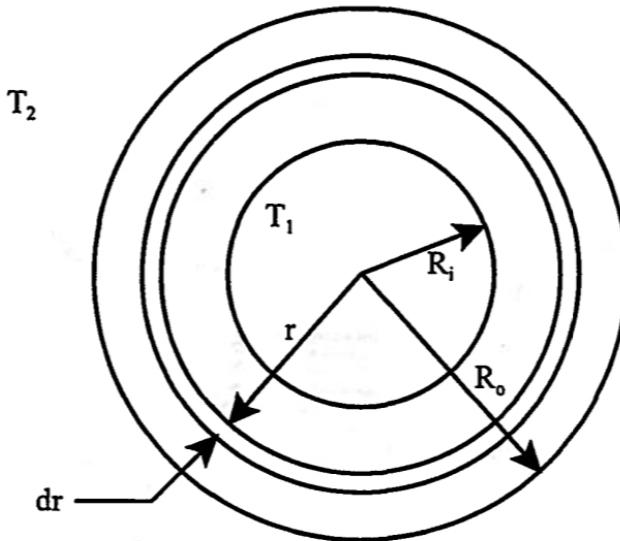


Figure 6.3 Radial Heat Transfer in a Cylinder

Assuming a steady rate of energy flow that is $\dot{Q} = \text{constant}$, T being independent of T and from fourier law of conduction we have

$$\begin{aligned}\dot{Q} &= -2\pi r L k \left(\frac{dT}{dr} \right) \\ \frac{\dot{Q}}{2\pi k L} &= -r \left(\frac{dT}{dr} \right) = C_1 \\ \left(\frac{dT}{dr} \right) &= -\frac{C_1}{r} \\ T &= C_1 \ln r + C_2\end{aligned}$$

with boundary conditions $T(R_i) = T_1$ and $T(R_o) = T_2$ we then have

$$T_1 = -C_1 \ln R_i + C_2$$

$$T_2 = -C_1 \ln R_o + C_2$$

this is just a system of equation with unknowns C_1, C_2 so solving for C_1 we get

$$C_1 = \frac{T_1 - T_2}{\ln \frac{R_o}{R_i}}$$

substituting this back into our expression for \dot{Q} earlier we have

$$\dot{Q} = \frac{2\pi k L (T_1 - T_2)}{\ln \frac{R_o}{R_i}}$$

then by virtue of our analogy with ohm's law above observe that

$$R_{th} = \frac{\ln \frac{R_o}{R_i}}{2\pi k L}$$

Example 67

Consider pipe carrying hot fluid with convection on the inside and radiation on the outside

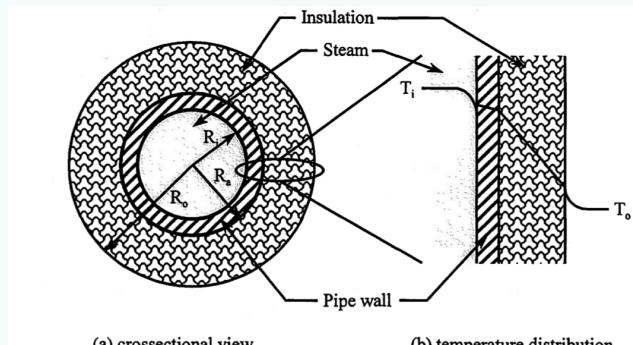


Figure 6.4 Insulated Steam Pipe

Solution. See that

$$R_1 = \frac{1}{h_{c,i} 2\pi R_i L}$$

R_1 is just follows the usual $\frac{1}{h_c A}$ for convection recall above. The rest is just application of radial heat transfer in a cylinder which we just explored prior to this

$$R_2 = \frac{\ln \frac{R_a}{R_i}}{2\pi k_a L}$$

$$R_3 = \frac{\ln \frac{R_o}{R_a}}{2\pi k_b L}$$

$$R_4 = \frac{1}{(h_{c,o}) 2\pi R_o L}$$

$$\dot{Q} = \overline{U} A (T_1 - T_2) = \frac{(T_1 - T_2)}{\sum R_i}$$

where \overline{U} is the overall heat transfer coefficient

6.3 conduction heat transfer and entropy generation in te uncoupled thermal fluid system

At some instant of time t let $T(x, y, z)$ be the temprerature distribution of a *pure thermal system* element. Let $\dot{q}(x)$ eb the rate of energy transfer per unit cross sectional area by conduction heat transfer at the location x (also known as **heat flux**) and let $j(x)$ be the concomitant rate of entropy transfer per unit cross sectional area(also known as **entropy flux**) for the x direction where

$$j(x) = \frac{\dot{Q}(x)/A}{T(x)} = \frac{\dot{q}(x)}{T(x)}$$

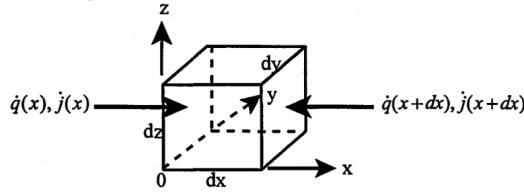


Figure 6.5 x -component of Heat Flux and Entropy Flux in Elemental Volume

Let \dot{q}_{gen} be the rate at which energy is "dissipated" irreversibly per unit mass by all energy storaage modes other than thermal and u be the thermal energy per unit mass. Then the first law for the volume element gives

$$\begin{aligned} [\dot{q}(x) - \dot{q}(x + dx)]dydz + [\dot{q}(y) - \dot{q}(y + dy)]dxdz + [\dot{q}(z) - \dot{q}(z + dz)]dxdy \\ = p\frac{\partial u}{\partial t}dxdydz + p\dot{q}_{\text{gen}}dxdydz \end{aligned}$$

Now this is because clearly any net energy transfer either ends up as thermal stores energy or dissipated. A similar concept will be used later in *Reynolds Transport Theorem* discussed in the next chapter. Now from basic multivariable calculus we have

$$\begin{aligned} \dot{q}(x + dx) &= \dot{q}(x) + \frac{\partial \dot{q}(x)}{\partial x} dx \\ \dot{q}(y + dy) &= \dot{q}(y) + \frac{\partial \dot{q}(y)}{\partial y} dy \\ \dot{q}(z + dz) &= \dot{q}(z) + \frac{\partial \dot{q}(z)}{\partial z} dz \end{aligned}$$

But we also know that fourier conduction law gives

$$\begin{aligned} \dot{q}(x) &= -k\frac{\partial T}{\partial x} \\ \dot{q}(y) &= -k\frac{\partial T}{\partial y} \\ \dot{q}(z) &= -k\frac{\partial T}{\partial z} \end{aligned}$$

and the energy constitutive relation for the pure thermal system gives $u - u_0 = c(T - T_0)$ then substituting these relations into our earlier relation gives

$$\frac{\partial T}{\partial t} = \frac{1}{pc} \left[\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) \right] + \frac{\dot{q}_{\text{gen}}}{c}$$

if the thermal conductivity is **isotropic**(that is same value regardless of direction) we have

$$\frac{\partial^2 T}{\partial t} = \frac{k}{\rho c} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{\dot{q}_{\text{gen}}}{c}$$

where the **thermal diffusivity** is denoted by $\alpha = \frac{k}{\rho c}$..to be continued

7 open thermal fluid systems

So far we have focused our attention on **closed systems** that constitute a single piece of matter known as the **control mass**, analyzing its properties via its time history as the process occurs. This is what is known as the **langarian approach**. In contrast we now look at another class of thermal fluid systems known as **open systems** that do not contain a fixed quantity of matter of known identity. Instead of looking at the time history of a specific object we look at a collection of objects as they move through a specific location in space. This is what is known as the **eulerian approach**

Example 68

When we sit on the bank of a river and watch the water flow we might look at specific features in the flow that are seemingly stable in front of us such as the small wave that remains in front of a rock as the flow strikes it. However instead in the Eulerian view we might be more interested in the array of velocities in the water as it flows around the rock.

Fact 69

Consider

1. In the langarian point of view the object of interest is a control mass while in the eulerian approach the object of interest is the control volume.
2. mass is allowed to pass through the surface of the control volume via the control surface while the control mass contains a fixed amount of mass

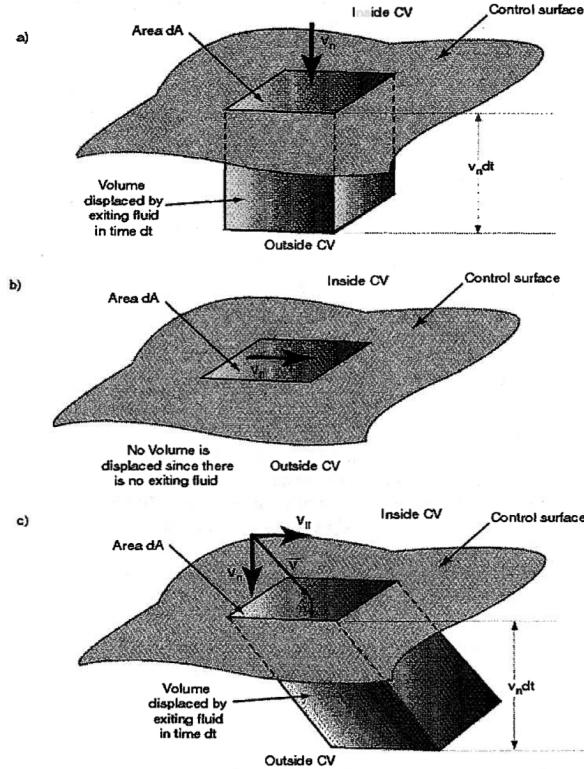


Figure 8.2 Schematic diagram of the flow through a differential area dA

Proposition 70

Consider

$$\frac{dM_{CM}}{dt} = \frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho (\vec{\theta}_r \cdot \vec{n}) dA$$

The first term on the RHS is the time rate of change of the mass in the control volume while the second term on the right is the mass flow rate out of the control volume (while $\vec{\theta}_r$, \vec{n} is flow rate and the unit normal to the control surface a). So if the second term is positive the flow is into and if negative it is flow out.

This makes sense as the net change of mass in the system is composed of the the net change of mass in the control volume and the mass flowing in and out of it. Think about it if flow in and out not equal then it must be accumulating or draining from the control volume. If the flow in and out equal then the change in mass of the control volume must be zero.

Theorem 71 (Conservation of mass)

Basically states that $\frac{dM_{CM}}{dt} = 0$

$$0 = \frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho (\vec{\theta}_r \cdot \vec{n}) dA$$

7.1 Reynolds transport theorem

Recall that so far we have found a way to use the langarian approach to account for mass of fluid in a control volume. Let us now try to generalize this result to what is known as the **Reynolds Transport Theorem** which allows us to find

transformation equations not just for mass but other extensive properties (can sum over components of the system) like energy, entropy, linear momentum and angular momentum too.

Theorem 72 (Reynolds Transport Theorem)

Suppose B are any of the extensive properties mentioned above then

$$\frac{dB_{CM}}{dt} = \frac{d}{dt} \int_{CV} pbdV + \int_{CS} pb(\vec{\theta}_r \cdot \vec{n})dA$$

where b represents the associated mass specific quantity where $b = B/M$

Example 73

For example the previous subsection derived the case for $B = M$ so we had $b = M/M = 1$

7.2 first law of thermodynamics for the control volume

Recall the first law may be expressed in the form

$$Q - W = (E_2 - E_1)$$

so we have on differentiation

$$\dot{Q} - \dot{W} = \frac{dE_{CM}}{dt}$$

suppose that $B = E$ so $b = E/M = e$. Then the **reynold's transport theorem** gives

$$\frac{dE_{CM}}{dt} = \dot{Q} - \dot{W} = \underbrace{\frac{d}{dt} \int_{CV} pedV}_{\frac{dE_{CV}}{dt}} + \int_{CS} pe(\vec{\theta}_r \cdot \vec{n})dA$$

but as discussed earlier there are the energy storage modes all divided by m : kinetic $\theta^2/2$ GPE gz and internal energy(thermal) u so we have

$$e = u + \frac{\theta^2}{2} + gz \quad (1)$$

The flow of energy associated with the mass flow into and out of the control volume leads to additional energy interactions that were not present in the case of the control mass.(consider 69). In particular we can separate this work transfer into 3 components

1. \dot{W}_P due to pressure acting on the mass in motion at the ports of entry and exit
2. \dot{W}_{shaft} due to shear stresses acting on solid elements in motion at the control surface
3. $\dot{W}_{\text{fluid shear}}$ due to shear stresses arising from the action of viscosity in the fluid

So altogether we write an express for the work transfer \dot{W} as

$$\dot{W} = \dot{W}_P + \dot{W}_{\text{shaft}} + \dot{W}_{\text{fluid shear}} \quad (2)$$

we now make sense of each of these terms on the RHS with examples

Example 74

The first term of the RHS \dot{W}_P is known as the **flow work transfer**.

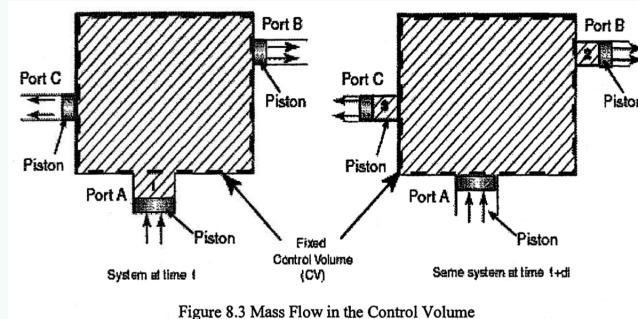


Figure 8.3 Mass Flow in the Control Volume

$$\text{Then } \dot{W}_P = \dot{m}_B P_B v_B + \dot{m}_C P_C v_C - \dot{m}_A P_A v_A = \sum_{\text{out}} (\dot{m} P v) - \sum_{\text{in}} (\dot{m} P v)$$

So in general(for the case of infinite ports) we may write this as

$$\dot{W}_P = \int_{CS} p(Pv)(\vec{\theta}_r \cdot \vec{n}) dA \quad (3)$$

where \vec{n} is the outwardly directed normal.

Fact 75

So putting equation (1),(2),(3) together into Reynold's transport theorem we get a new form for the first law of thermodynamics namely

$$\frac{dE_{CM}}{dt} = \dot{Q} - \dot{W} = \frac{d}{dt} \int_{CV} pe dV + \int_{CS} pe(\vec{\theta}_r \cdot \vec{n}) dA$$

which gets

$$\frac{d}{dt} \int_{CV} \left(u + \frac{\theta^2}{2} + gz \right) dV = \dot{Q} - \dot{W}_{\text{shaft}} - \int_{CS} p \left(\underbrace{u + Pv}_{=h} + \frac{\theta^2}{2} + gz \right) (\vec{\theta}_r \cdot \vec{n}) dA$$

where h is the specific enthalpy and the blue term corresponds to the \dot{W}_P term in \dot{W} . Also notice that we have dropped the $\dot{W}_{\text{fluid shear}}$ term as we have assumed that the control surface that is coincident with the solid surface is stationary. Alternatively when there are finite number of ports we write

$$\frac{dE_{CV}}{dt} = \dot{Q} - \dot{W}_{\text{shaft}} + \underbrace{\sum_{\text{in}} \dot{m} \left(h + \frac{\theta^2}{2} + gz \right)}_{\text{in brackets}} - \underbrace{\sum_{\text{out}} \dot{m} \left(h + \frac{\theta^2}{2} + gz \right)}_{\text{in brackets}}$$

where the terms in the brackets corresponds to the control surface integral above for the infinite ports case.

Example 76

A pump is used to pump water out of a well and into a storage tank as shown in the figure below. The pressure in the storage tank is maintained at a constant absolute pressure of $5 \times 10^5 \text{ Pa}$ using a flexible membrane and a large volume of air as shown. The pump is placed 100m below the surface of the earth. Assume that the temperature of the water entering the pump is the same as the temperature of the water entering the tank. Assume that all heat transfers are negligible.

- How much electrical power is required to pump water into the storage tank at volumetric rate of 7 litres/min if the water surface is 50m below the surface of the earth?
- How much electrical power is required if the water surface is at the surface of the earth itself?

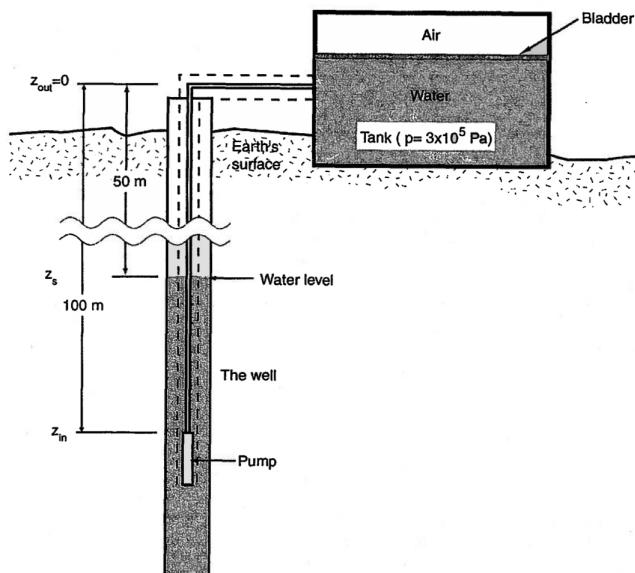


Figure 8E.4 Artesian Well

Solution. We first select the control volume to be the dashed line indicated above. Now knowing that the mass of the control mass is conserved we may apply 71 which gets

$$0 = \frac{d}{dt} \int_{CV} p dV + \int_{CS} p(\vec{\theta}_r \cdot \vec{n}) dA = \frac{d}{dt} \int_{CV} p dV - \dot{m}_{well} + \dot{m}_{tank}$$

where \dot{m}_{well} is the mass flow rate of water into the pump while \dot{m}_{tank} is the mass flow rate into the tank. But observe that

$$\frac{d}{dt} \int_{CV} p_w dV = \frac{d}{dt} [p_w V_w + p_p V_p + p_{air} V_{air} + p_m V_m] = 0$$

where V_w, V_p, V_{air}, V_m are the volumes of water, plastic, air and metal inside the control volume respectively. Now clearly since steady state the volume of water won't change and obvious same goes for the other materials. Therefore our continuity equation simplifies to

$$\dot{m}_{well} = \dot{m}_{tank}$$

which we denote to be \dot{m} . As for the first law we express

$$\frac{dE_{CV}}{dt} = -\dot{W}_{shaft} + \dot{m} \left(h + \frac{\theta^2}{2} + gz \right)_{in} - \dot{m} \left(h + \frac{\theta^2}{2} + gz \right)_{out}$$

where we have dropped the heat transfer term as assumed in the question. But observe that

$$\frac{dE_{CV}}{dt} = \frac{d}{dt} \int_{CV} p \left(u + \frac{v^2}{2} + gz \right) dV = 0$$

since in the steady flow case, the system in the control volume looks exactly the same no matter the instance in time.

So our first law reduces to

$$\dot{W}_{\text{shaft}} = \dot{m} \left(h + \frac{\theta^2}{2} + gz \right)_{\text{in}} - \dot{m} \left(h + \frac{\theta^2}{2} + gz \right)_{\text{out}}$$

now because our fluid can be modelled as an incompressible fluid by 39 we have from the energy constitutive relation that

$$h = u + Pv = c(T - T_0) + Pv$$

where u, v is the specific internal energy and volume respectively. So our first law becomes

$$\dot{W}_{\text{shaft}} = \dot{m}c(T_{\text{in}} - T_{\text{out}}) + \dot{m}v(P_{\text{in}} - P_{\text{out}}) + \dot{m}g(z_{\text{in}} - z_{\text{out}})$$

but the question assumes the temperature of the water entering the tank is the same as that leaving the tank so the first term disappears. Also knowing that the specific volume is just the reciprocal of density if you recall and that

$$P_{\text{in}} = P_{\text{atm}} + p_w g(z_s - z_{\text{in}})$$

where $z_s = -50m$ we finally find \dot{W}_{shaft} via

$$\dot{W}_{\text{shaft}} = \dot{m} \left[\left(\frac{P_{\text{atm}} - P_{\text{out}}}{p_w} \right) + g(z_s - z_{\text{out}}) \right]$$

for (b) to find the case when the water level is at the surface simply sub $z_s = 0$ instead.

Remark 77. Essentially to solve this question we simply solved for conservation of energy and mass using the reynold transport theorem form

7.3 the second law of thermodynamics for the control volume

Recall the seond law may be stated as

$$(S_2 - S_1)_{CM} = \sum_i \left(\frac{Q}{T} \right)_i + S_{\text{gen}}$$

where i corresponds to different regions of the boundary where heat transfer occurs. So we have on differentiation

$$\frac{dS_{CM}}{dt} = \sum_i \left(\frac{\dot{Q}}{T} \right)_i + \dot{S}_{\text{gen}}$$

Since entropy is an extensive entropy applying **reynold's transport theorem** again by letting $S = B$ we have

$$\frac{dS_{CM}}{dt} = \underbrace{\frac{d}{dt} \int_{CV} ps dV}_{\frac{dS_{CV}}{dt}} + \int_{CS} ps(\vec{\theta}_r \cdot \vec{n}) dA = \sum_i \left(\frac{\dot{Q}}{T} \right)_i + \dot{S}_{\text{gen}}$$

where s is the "specific entropy". Like how we did for the 1st law we may also rearrange these terms to get

$$\frac{dS_{CV}}{dt} = \sum_i \left(\frac{\dot{Q}}{T} \right)_i + \sum_{\text{in}} (\dot{m}s) - \sum_{\text{out}} (\dot{m}s) + \dot{S}_{\text{gen}}$$

Example 78

In the interest of improving fuel efficiency of large trunk engines it has been suggested that a heat engine could be installed on the exhaust system on the truck engine. The design objective would be to use the energy remaining in the products of combustion to provide electrical power for auxiliary systems. The first analysis is always to check if the available energy in the fluid stream is worth the effort of building a device to extract useful energy. If the mass flow rate of the exhaust is $\dot{m} = 0.04 \text{ kg/s}$ and its pressure and temperature are $P = 10^5 \text{ Pa}$ and $T_{\text{out}} = 540 \text{ C}$ respectively what is the maximum power that can be generated from the exhaust gases? Assume that the exhaust gas can be modelled as an ideal gas with $c_v = 716 \text{ J/KgK}$ and $R = 287 \text{ J/KgK}$

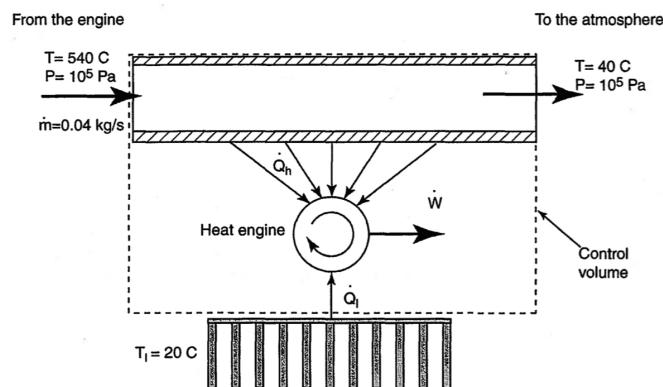


Figure 8E.9 Schematic Diagram of the Exhaust Gas Heat Engine

7.4 bernoulli equation in steady flow

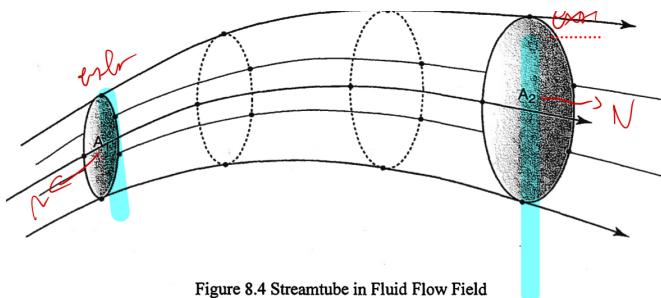


Figure 8.4 Streamtube in Fluid Flow Field

Figure 6: The curves are known as streamlines. These are the paths that fluid particles will follow in steady flow. The control surfaces chosen are A_1 (entrance) and A_2 (exit) and the control volume is the area between A_1, A_2

We first assume that

1. no heat transfer across the control surface so flow is adiabatic
2. stream tube has diameter small enough such that we may assume fluid properties(u, h, s, v)s are uniform over the cross sectional areas A_1, A_2
3. no devices requiring shaft work transfer across the control the control surface

4. no viscous dissipation in this flow: that is flow is reversible

5. fluid is incompressible

Now the continuity(conservation of mass) equation then gives

$$0 = \frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho (\vec{\theta}_r \cdot \vec{n}) dA$$

due to steady flow. Clearly this also implies time integral of the volume is zero too. Therefore we may reduce this to

$$\int_{A_1} \rho (\vec{\theta}_r \cdot \vec{n}) dA + \int_{A_1} \rho (\vec{\theta}_r \cdot \vec{n}) dA = -\dot{m}_1 + \dot{m}_2 = 0$$

so we conclude that $\dot{m}_1 = \dot{m}_2 = \dot{m}$. Now applying the second law to the control volume we get

$$\frac{dS_{CV}}{dt} = \sum_i \left(\frac{\dot{Q}}{T} \right)_i + \sum_{in} (\dot{m}s) - \sum_{out} (\dot{m}s) + \dot{S}_{gen}$$

but by assumption

- since flow is steady $\frac{dS_{CV}}{dt} = 0$
- since flow is adiabatic $\sum_i \left(\frac{\dot{Q}}{T} \right)_i = 0$
- since flow is reversible $\dot{S}_{gen} = 0$

so we obtain

$$\dot{m}(s_{in} - s_{out}) = \dot{m}(s_1 - s_2) = \dot{m}c \ln \frac{T_1}{T_2} = 0$$

using the entropy constitutive relation for pure thermal systems(which is applicable because our fluid is assumed incompressible), Therefore we conclude that $T_1 = T_2$. Finally using the first law

$$\frac{dE_{CV}}{dt} = \dot{Q} - \dot{W}_{shaft} + \underbrace{\sum_{in} \dot{m} \left(h + \frac{\theta^2}{2} + gz \right) - \sum_{out} \dot{m} \left(h + \frac{\theta^2}{2} + gz \right)}_{\text{constant } T}$$

where by assumption clearly $\dot{Q} = \dot{W}_{shaft} = 0$ our equation reduces to

$$\begin{aligned} \left(h + \frac{\theta^2}{2} + gz \right)_{A_1} &= \left(h + \frac{\theta^2}{2} + gz \right)_{A_2} \\ \left(u + Pv + \frac{\theta^2}{2} + gz \right)_{A_1} &= \left(u + Pv + \frac{\theta^2}{2} + gz \right)_{A_2} \\ \left(cT + \frac{P}{\rho} + \frac{\theta^2}{2} + gz \right)_{A_1} &= \left(cT + \frac{P}{\rho} + \frac{\theta^2}{2} + gz \right)_{A_2} \\ \left(\frac{P}{\rho} + \frac{\theta^2}{2} + gz \right)_{A_1} &= \left(\frac{P}{\rho} + \frac{\theta^2}{2} + gz \right)_{A_2} \end{aligned}$$

again recalling that the specific volume is the reciprocal of the density and the final equation follows since T is the same on both sides. This is essentially the steady flow **Bernoulli equation** named after Daniel Bernoulli.

Fact 79

So summarize and recap the assumptions we used here are

1. incompressible flow
2. steady flow
3. negligible fluid viscosity
4. no shaft work transfer
5. no heat transfer

7.5 reversible flow: the Bernoulli equation in unsteady flow

Consider

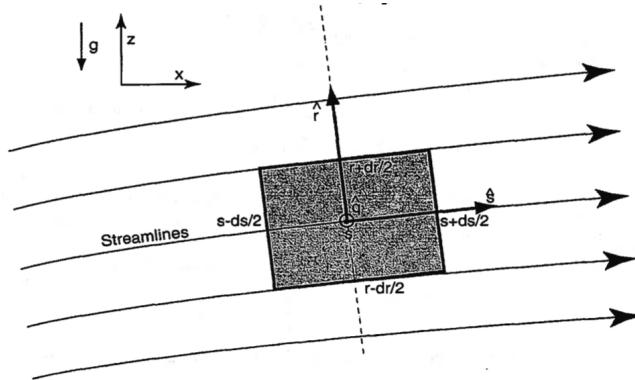


Figure 8.7 A Differential Fluid Particle

By newton's second law of motion the force acting on a particle F is equal to the change of the momentum of the particle that is

$$\vec{F} = \frac{d\vec{\theta}}{dt} = m\vec{a}$$

now since the viscosity of the fluid is negligible the s component of newton's second law for the fluid element is then

$$F_s = P \underbrace{\left(s - \frac{ds}{2} \right)}_{\text{function argument}} dp dr - P \left(s - \frac{ds}{2} \right) dq dr + \underbrace{pd s d r d q}_{\text{mass}} (\vec{g} \cdot \vec{s}) \quad (1)$$

Remark 80. Here we mean that $P(\dots)$ is the pressure function and \dots are arguments.

where the first and second term are the pressure forces on the left and right faces of the fluid element respectively while the third term is due to the gravitational forces. Now consider that the velocity term is function of both position s on the streamline as well as t time. We denote it as $\theta(s, t)$. Then

$$\frac{d\theta}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\theta[s(t + \Delta t), t + \Delta t] - \theta[s(t), t]}{\Delta t} \quad (2)$$

now expanding this as a double taylor series we have

$$\theta[s(t + \Delta t), t + \Delta t] = \theta(s + \frac{ds}{dt}\Delta t, t + \Delta t) = \theta(s, t) + \frac{\partial\theta}{\partial s} \frac{ds}{dt} \Delta t + \frac{\partial\theta}{\partial t} \Delta t \quad (3)$$

now substituting equation (2) into (3) we find that the acceleration term may found found as

$$a = \frac{d\theta}{dt} = \frac{\partial\theta}{\partial s} \frac{ds}{dt} + \frac{\partial\theta}{\partial t} = \theta \frac{\partial\theta}{\partial s} = \frac{\partial\theta}{\partial t}$$

now finally combining this we (1) we have that

$$\underbrace{P \left(s - \frac{ds}{2} \right) dq dr - P \left(s + \frac{ds}{2} \right) dq dr}_{\approx -\frac{dP}{dt} ds} + pdsdrdq(\vec{g} \cdot \vec{s}) = pdsdrdq \left[\theta \frac{\partial\theta}{\partial s} + \frac{\partial\theta}{\partial t} \right]$$

(again recall that $P(\dots)$ is a function) dividing this equation by the mass of fluid element $pdsdrdq$ and taking the limit as ds approaches zero we get

$$0 = \frac{\partial\theta}{\partial t} + \frac{1}{p} \frac{dP}{ds} + \theta \frac{\partial\theta}{\partial s} - (\vec{g} \cdot \vec{s})$$

so intergrating along the streamline we have

$$0 = \int_1^2 \frac{\partial\theta}{\partial t} ds + \frac{1}{p} \int_1^2 \frac{dP}{ds} ds + \int_1^2 \theta \frac{\partial\theta}{\partial s} ds - \int_1^2 (\vec{g} \cdot \vec{s}) ds$$

notice that

$$-\int_1^2 (\vec{g} \cdot \vec{s}) ds = g(z_2 - z_1)$$

therefore putting together we have that

$$0 = \int_1^2 \frac{\partial\theta}{\partial t} ds + \frac{P_2 - P_1}{p} + \frac{\theta_2^2 - \theta_1^2}{2} + g(z_1 - z_2)$$

this is the **Bernoulli equation for unsteady flow**

Remark 81. Notice that if the flow is steady then the first term $\int_1^2 \frac{\partial\theta}{\partial t} ds = 0$ and the eqution reduces to

$$\frac{P_1}{p} + \frac{\theta_1^2}{2} + gz_1 = \frac{P_2}{p} + \frac{\theta_2^2}{2} + gz_2$$

which is the steady flow bernoulli equation we discussed earlier

Example 82

A constant diameter U tube as shown below contains a low visocity, imcompressible liquid. Initially one side of the tube is at higher pressure so that there is a height difference h between the two free surfaces of the liquid. The valve is opened to the atmosphere and the pressure on that side of the U tube immediately falls to atmospheric pressure

- (a) assuming no dissipation, what is the subsequent motion of the fluid?
- (b) what is the period of the oscillation?

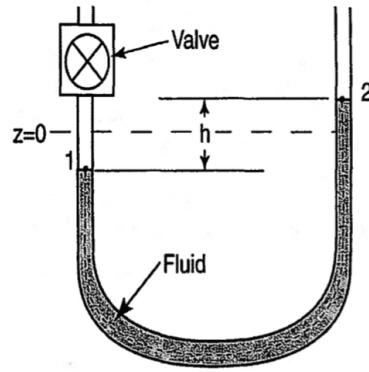


Figure 8E.15 Fluid-filled U-tube

Now consider a fixed point in time. Applying the time dependent bernoulli equation we have that

$$0 = \int_1^2 \frac{\partial \theta}{\partial t} ds + \frac{P_2 - P_1}{\rho} + \frac{\theta_2^2 - \theta_1^2}{2} + g(z_2 - z_1) = L \frac{\partial \theta}{\partial t} + g(z_2 - z_1)$$

this is because in the absence of viscous damping the acceleration is constant throughout s . By the continuity the velocities at point 1 and 2 are the same too. Therefore denoting the distance between point 1 and 2 by L we have the equation above. Note that the pressures at point 1 and 2 are equal by assumption as stated in the question. Now because we $z = 0$ as shown in the diagram , z_1, z_2 are equal in magnitude and opposite in direction. So putting all these together our bernoulli equation simplifies to

$$\frac{d^2 z_2}{t^2} + \frac{2g}{L} z_2 = 0$$

and this is equation in SHM form whose solution is

$$z_2(t) = \frac{h}{2} \cos \left(\sqrt{\frac{2g}{L}} t \right)$$

after which (b) can be easily solved

$$T = \frac{2\pi}{w} = 2\pi \sqrt{\frac{L}{2g}}$$

7.6 Effect of streamline curvature on the pressure in the flow field

Consider

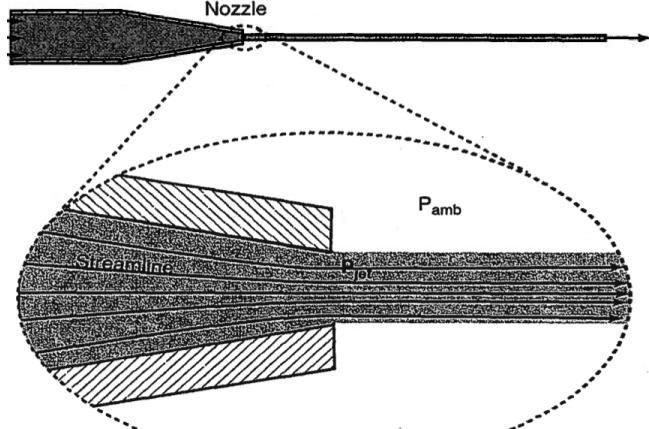


Figure 8.8 Streamlines in a Jet Discharging From a Nozzle

Like did earlier we similarly want to calculate forces on the fluid element but this time in the r direction. So we do

$$F_r = P \left(r - \frac{dr}{2} \right) dqds - P \left(r + \frac{dr}{2} \right) dqds - pd s dr dq (\vec{g} \cdot \vec{r})$$

clearly this is equal to

$$-pd s dr dq \frac{\theta^2}{R} = P \left(r - \frac{dr}{2} \right) dqds - P \left(r + \frac{dr}{2} \right) dqds - pd s dr dq (\vec{g} \cdot \vec{r})$$

where R is the radius of the local curvature of the streamline. Then similar to before we divide through by $p d s dr dq$ and take the limit as dr goes to zero to obtain

$$\frac{dP}{dr} = p \frac{\theta^2}{R} - p(\vec{g} \cdot \vec{r})$$

now in the case of a jet steam with a narrow opening we assume that the effect of gravity is negligible and that near the nozzle shear forces are not yet established substantially (we will explore this fully later) then the second term in the above vanishes and since straight line local curvature is infinity that is we take $R \rightarrow \infty$ therefore $\frac{dP}{dr} = 0$. That is to say near the nozzle tip in the radial direction the pressure is uniform so we must have that

$$P_{\text{jet}} = P_{\text{amb}}$$

7.7 equation of linear momentum for a control volume

Momentum $M_{CM}\vec{\theta}$ is clearly an extensive property (can be added over subsystems) therefore we may apply RTT to get

$$F = \frac{d}{dt}(\vec{P}) = \frac{d}{dt} \int_{CV} p \vec{\theta} dV + \int_{CS} p \vec{\theta} (\vec{\theta}_r \cdot \vec{n}) dA \quad (1)$$

where $\vec{P} = M_{CM}\vec{\theta}$ by newton's second law.

Remark 83. This time we have vector equation. This is unlike our previous cases, where used RTT for energy, entropy, mass where we got a scalar equation

Therefore say for the x direction we have

$$F_x = \frac{d}{dt}(\vec{P}) = \frac{d}{dt} \int_{CV} p\vec{\theta}_x dV + \int_{CS} p\vec{\theta}_x (\vec{\theta}_r \cdot \vec{n}) dA$$

note that the net force acting the control volume as reflected in equation (1) can be split into the following components

$$F = \frac{d}{dt}(\vec{P}) = \frac{d}{dt} \int_{CV} p\vec{\theta} dV + \int_{CS} p\vec{\theta} (\vec{\theta}_r \cdot \vec{n}) dA = \underbrace{\int_{CS} (-P\vec{n}) dA}_1 + \underbrace{\int_{CS} (\vec{\tau}) dA}_2 + \underbrace{\int_{CV} p\vec{g} dV}_3$$

where these terms correspond to

1. sum of forces due to pressure applied on CS of the CV
2. sum of shear forces acting on the CS of the CV
3. gravitational force that acts on all mass in the CV at any instant of time

Example 84

A fire house is held by a firefighter in the configuration below. The diameter at the end of the nozzle is $D_n = 2\text{cm}$. If the mass flow rate through the house is 500kg/min what is magnitude of the force the firefighter must place on the nozzle to keep it stationary?

Solution. Consider

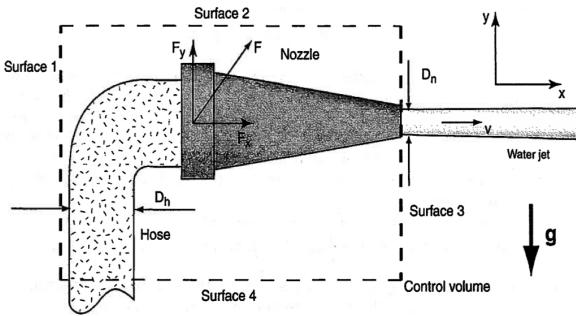


Figure 8E.17 Nozzle on Fire Hose

The pressure integral can be expanded over the four surfaces 1, 2, 3 and 4 as shown in Figure 8E.17.

so we have

$$F_x = \frac{d}{dt} \int_{CV} p\vec{\theta}_x dV + \int_{CS} p\vec{\theta}_x (\vec{\theta}_r \cdot \vec{n}) dA = \left[\underbrace{\int_{CS} (-P\vec{n}) dA}_1 + \underbrace{\int_{CS} (\vec{\tau}) dA}_2 + \underbrace{\int_{CV} p\vec{g} dV}_3 \right]_x$$

consider the 1st term. clearly the contribution due to surface 2 and 4 is zero because the outward normal is perpendicular to the x direction. Now for surface 3 where the jet nozzle is at there is no pressure gradient because the pressure of the water in and outside the jet must be equal(recall 7.6) so the pressure over surface 3 and 1 are uniform and both equal to P_{atm} . But considering their outward normals these effects clearly cancel out as $A_1 = A_3$. Next by assumption of uniform there are no shear stresses so the 2nd term disappear. Finally the direction of gravity is in the y direction so the 3rd term disappears as well. Furthermore again by assumption of steady flow the volume in the control volume is

time invariant so the 1st blue term disappears. So finally on solving what is remaining we get

$$F_{x,\text{firefighter}} = \rho \theta_{\text{jet}}^2 = \frac{\pi D_n^2}{4} = 221N$$

Example 85

A jet of water strikes a plate as shown in below. The stream separates into an upward moving stream along the plate and a downward moving stream along the plate. The streams on the plate are confined by walls so that all movement of the streams in position 2 and 3 are in the y direction. The velocity and area of the incoming steam in θ_1 and A_1

- (a) what is the velocity at position 2 and 3?
- (b) what is the pressure at position 4?
- (c) what are the areas at position 2 and 3?
- (d) what is the force in the x direction on the fixed support due to the fluid flow?
- (e) what is the force in the horizontal direction on the fixed support due to the fluid flow?

Example 8E.19: A jet of water strikes a plate as shown in Figure 8E.19. The stream

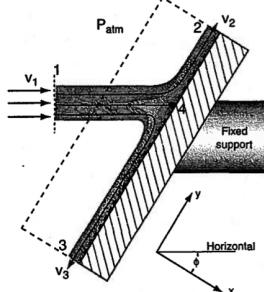


Figure 8E.19 Flow on a Grooved Plate

7.8 equation of linear momentum for a control volume with rectilinear acceleration

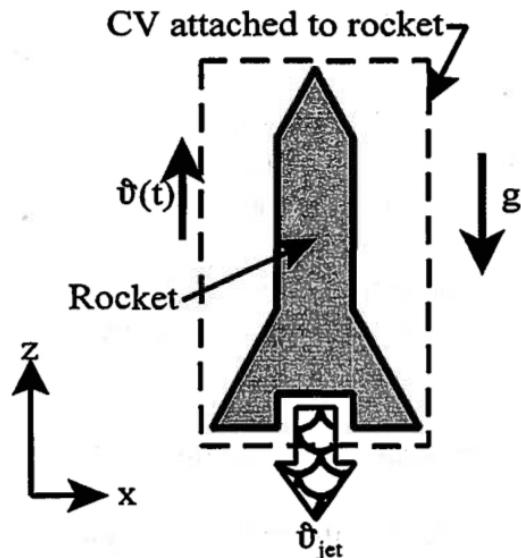


Figure 8E.23 Accelerating Rocket

7.9 equation of angular momentum for a control volume

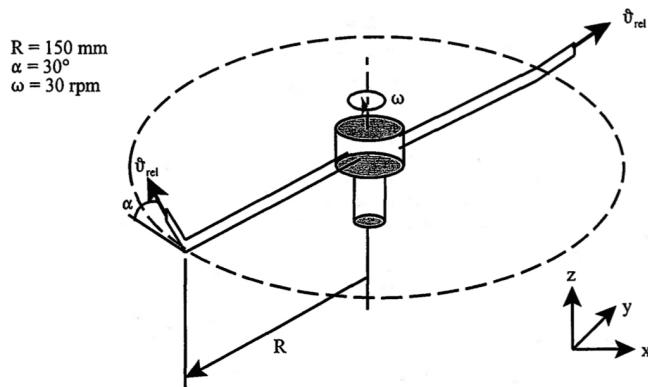


Figure 8E.25a Lawn Sprinkler

8 viscosity and the transfer of momentum in a fluid

8.1 viscosity

Consider a viscous incompressible fluid

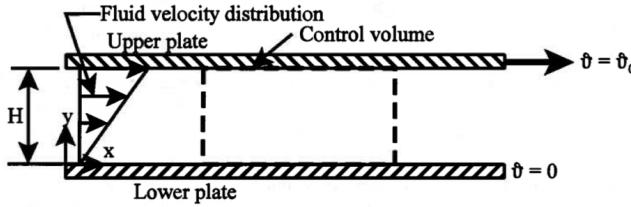


Figure 9.1 Fluid Layer Between Two Flat Plates

assuming no slip condition that is we think of the fluid as many thin layers parallel to the plates such that the fluid in contact with the plates adheres to the plates. Now due to viscosity, for any given layer, the layer above will drag in the forward direction. As a result the motion is given by

$$\frac{d\theta_y}{dy} = \frac{\theta_0}{H} \quad (1)$$

Proposition 86

For the viscous most commonly found in thermal fluid systems the shear stress is found to be linearly proportional to the velocity gradient

$$\tau_{yx} = \mu \frac{d\theta_x}{dy}$$

in the one dimensional situation previously. This relationship is known as the **newtonian fluid model**

Now consider that we take the control volume like so

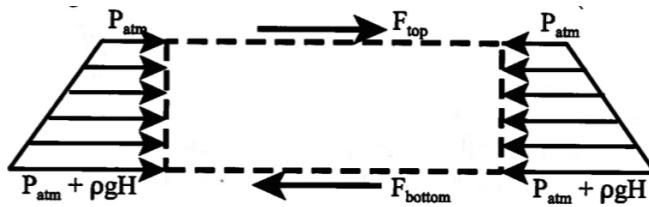


Figure 9.1b Forces on the Control Volume of Figure 9.1a

Figure 7: dimension L in the x direction: H in the y direction and b in the z direction

By assumption of steady state we have

$$\dot{m}_{in} = \dot{m}_{out} = \dot{m} = \int_0^H \rho \theta_x(y) b dy$$

where $\theta_x(y)$ is the fluid velocity as a function of distance y measured normal to the plate. By (1) on integration we get

$$\theta_x = \frac{\theta_0}{H} y$$

and so we have

$$\dot{m} - \int_0^H \rho \frac{\theta_0}{H} b y dy = \rho b H \frac{\theta_0}{2}$$

now expressing the sum of forces we have that

$$\sum F_x = \left(P_{atm} + \rho g \frac{H}{2} \right) H b - \left(P_{atm} + \rho g \frac{H}{2} \right) H b + F_{top} - F_{bottom}$$

but we know from 86 that the shear stress acting on the top and bottom surfaces of the fluid control volume ar the same since the velocity gradient is constant throughout the fluid layer. That is

$$F_{\text{top}} - F_{\text{bottom}} = \tau_{yx} bL - \tau_{yx} bL = 0$$

that is there is no net force acting on the fluid control volume.

Remark 87. *This is to be expected after all in steady flow, we expect that there should be no net momentum flow*

Now lets analyze the thermodynamics of this. For the first flow for this steady flow situation with no pressure drop in the x direction(so $\dot{W}_p = 0$ and also it is obvious in the x direction we have $\dot{W}_{\text{shaft}} = 0$ we have

$$\dot{Q} - \dot{W}_{\text{shear}} = \dot{m}(u_{\text{out}} - u_{\text{in}}) = \dot{m}c(T_{\text{out}} - T_{\text{in}})$$

where u is the specific internal energy. The RHS is given by the energy constitutive relation for incompressible fluids.

Now let us explore the work transfer for this. From the definition of work transfer(that is we restrict out attention to only the boundary mass points direct interaction with the surroundings) and that the bottom layer is stationary(no slip condition and that the lower plate is stationary), we only need to consider the top layer of boundary mass points. So we have that

$$\dot{W}_{\text{shear}} = \frac{\delta W_{\text{shear}}}{dt} - (F_x)_{\text{shear}} = -F_{\text{top}} \frac{dx}{dt} = -\tau_{yx} A \theta_0 = -\tau_{yx} bL \theta_0$$

now substituting equations (1) and (2) into this we have

$$\dot{W}_{\text{shear}} = -\mu \frac{\theta_0^2}{H} bL$$

and since our flow is adiabatic we may drop the \dot{Q} term above to get

$$T_{\text{out}} = T_{\text{in}} + \frac{2\mu\theta_0 L}{pcH^2}$$

now using the entropy constitutive relation for the incompressible fluid model we have

$$\dot{S}_{\text{gen}} = \dot{m}c(s_{\text{out}} - s_{\text{in}}) = p\theta_0 bc \frac{H}{2} \ln \left(\frac{T_{\text{out}}}{T_{\text{in}}} \right) = p\theta_0 bc \frac{H}{2} \ln \left(1 + \frac{2\mu\theta_0 L}{pcT_{\text{in}}H^2} \right)$$

In particular we have shown that

Proposition 88

The greater the viscoity of the fluid the greater the rate of entropy generation

8.2 the total derivative

Definition 89

We define the new differential operator

$$\frac{D}{Dt} \equiv \left(\frac{d}{dt} \right)_{\text{CM}}$$

Example 90

This is what we mean by a **total derivative**. In this case suppose we take the total derivative with respect to temperature function $T(x, y, z, t)$ where x, y, z are the spacial coordinates and t is time. So by chain rule we have

$$\begin{aligned}\left(\frac{dT}{dt}\right)_{cm} &= \frac{\partial T}{\partial t} + \left(\frac{dx}{dt}\right)_{cm} \frac{\partial T}{\partial x} + \left(\frac{dy}{dt}\right)_{cm} \frac{\partial Y}{\partial t} + \left(\frac{dz}{dt}\right)_{cm} \frac{\partial T}{\partial z} \\ &= \frac{\partial T}{\partial t} + \theta_x \frac{\partial T}{\partial x} + \theta_y \frac{\partial Y}{\partial y} + \theta_z \frac{\partial T}{\partial z}\end{aligned}$$

This equation here is known as the **material/substantial derivative** and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \theta_x \frac{\partial}{\partial x} \theta_x \frac{\partial}{\partial x} \theta_x \frac{\partial}{\partial x} + \theta_y \frac{\partial}{\partial y} + \theta_z \frac{\partial}{\partial z}$$

is the total derivative. In essence it describes the total time rate of change of a property

Now it is clear that we may write the above more compactly as

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{\theta} \cdot \nabla}$$

where $\cdot \nabla$ represents the divergence not a scalar product. Just like how we use RTT to express conservation of mass, momentum etc lets see how we can use the total derivative to do something similar.

8.3 conservation of mass

Recall previously we showed that RTT applied to a control mass ges

$$\frac{d}{dt} \int_{CV} p dV + \int_{CS} p (\vec{\theta} \cdot \vec{n}) dA = 0$$

but now we consider a rigid control volume *fixed* in space. So then this implies limits of the volume integral are clearly independent of time so we may differentiate under the integral by lebniz integral rule where we obtain

$$\int_{CV} \frac{\partial p}{\partial t} dV + \int_{CS} p (\vec{\theta} \cdot \vec{n}) dA = 0 \quad (1)$$

We now consider the 2 dimensional control volume

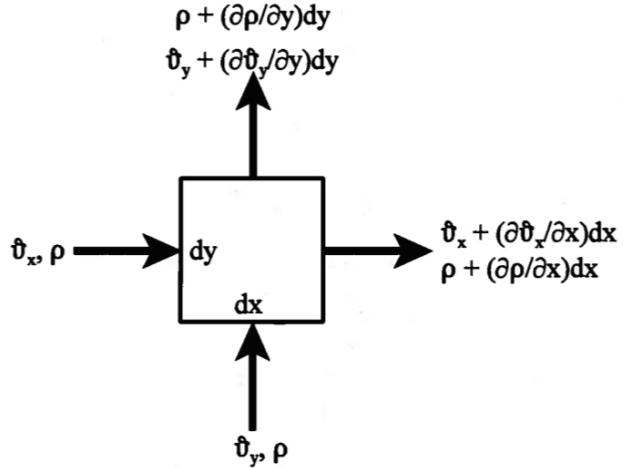


Figure 9.4 Differential Fluid Element Fixed in Space

Figure 8: notice the linear approximations used for dimensions

Then the first term of the above can be written as

$$\int_{CV} \frac{\partial p}{\partial t} dV \approx \frac{\partial p}{\partial t} dx dy \quad (2)$$

and the second term becomes

$$\int_{CS} p(\vec{\theta} \cdot \vec{n}) dA = \sum_i (p_i A_i \theta_i)_{out} - \sum_i (p_i A_i \theta_i)_{in} \quad (3)$$

from the diagram we also see that

$$\int_i (p_i A_i \theta_i)_{out} = \left[\left(p + \frac{\partial p}{\partial x} dx \right) \left(\theta_x + \frac{\partial \theta_x}{\partial x} dx \right) \right] dy + \left[\left(p + \frac{\partial p}{\partial y} dy \right) \left(\theta_y + \frac{\partial \theta_y}{\partial y} dy \right) \right] dx$$

and that

$$\sum_i (p_i A_i \theta_i)_{in} = p \theta_x dy + p \theta_y dx$$

then expanding the above terms (and ignoring higher order terms like $dxdy$) and subbing them into (3) we get

$$\int_{CS} p(\vec{\theta} \cdot \vec{n}) dA = \frac{\partial p \theta_x}{\partial x} dx dy + \frac{\partial p \theta_y}{\partial y} dy$$

we now divide this by $dxdy$ getting

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x} (p \theta_x) + \frac{\partial}{\partial y} (p \theta_y) = 0 \quad (4)$$

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So we can clearly write (4) compactly as

$$\boxed{\frac{\partial p}{\partial t} + \nabla \cdot (p\vec{\theta})}$$

again $\nabla \cdot$ refers to the divergence not scalar product. Alternatively we may expand (4) via product rule differentiation to get

$$\frac{\partial p}{\partial t} + \theta_x \frac{\partial p}{\partial x} + \theta_y \frac{\partial p}{\partial y} + p \frac{\partial \theta_x}{\partial x} + p \frac{\partial \theta_y}{\partial y} = 0$$

but hey the terms of blue is literally the definition of the total derivative(recall above) and so we may write

$$\boxed{\frac{\partial p}{\partial t} + p \nabla \cdot \vec{\theta} = 0}$$

8.4 equation of linear momentum for an inviscid fluid(Euler equation)

Now moving on from masses we now express for the forces instead using the total differential as promised. Consider the differential element below

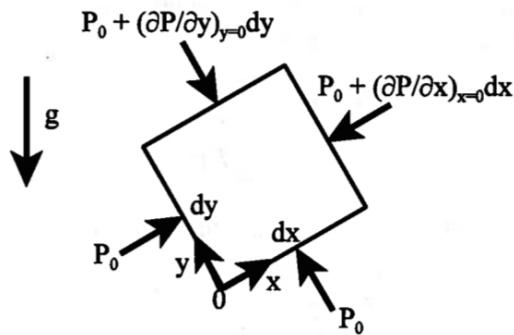


Figure 9.5 Forces Acting on a Differential Fluid Element

Figure 9: again notice the linear approximations used for dimensions

now summing forces we have that