Geometry of Manifolds I

Ian Poon

September 2024

Selected theorems from `MIT 18.965 Geometry of Manifolds 1` and `Do Carmo Riemann Geometry`. This is my first graduate math module :)

Contents

1 Review of manifolds

1.1 tangent spaces

Recall in `MIT 18.905 Differential Geometry` you have delealt with regular surfaces in $\mathbb{R}^3$. We would now like to extend our study of surfaces to $\mathbb{R}^n$

Definition 1

A **differentiable manifold** of dimension n is a set M a family of injective mappings $x_a : U_a \subset \mathbb{R}^n \to M$ of open sets U_a of $\mathbb{R}^n$ into M such that

1. $\bigcup_a x_a(U_a) = M$
2. for any pair a, b with $x_a(U_a) \cap x_b(U_b) = W \neq \emptyset$, the sets $x_a^{-1}(W)$ and $x_b^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $x_b^{-1} \circ x_a$ are differentiable
3. the family $\{(U_a, x_a)\}$ is maximal relative to the conditions (1) and (2)

Definition 2

A coordinate system is defined by

$$(x_a(U_a), x_a^{-1})$$

where essentially x^{-1} is called a **coordinate chart** while $x_a(U_a)$ is called the **coordinate neighborhood** at p.

Definition 3

A family $\{(x_a(U_a), x_a^{-1})\}$ of charts is called **maximal** if you cannot add any more (U_a, x_a) to this family such that conditions (1) and (2) remain satisfied.

In other words, any chart that is compatible with the charts already in the family must already be included in the family.

Definition 4

A **global coordinate system** means there exists a single coordinate chart that covers the whole manifold M

Example 5

The euclidean space is one such manifold that admits a global coordinate system. Just consider the identity map

$$I : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n)$$

Clearly this map is a diffeomorphism. Note that this precisely defines the **cartesian coordinate system**(recall your tensor analysis notes)

However not all manifolds have a global coordinate system and thus the study of this math subject in the first place. Consider

Example 6

Recall the S^2 sphere is a manifold. However it cannot be covered by a single coordinate patch. Suppose there is an open set in $\mathbb{R}^2$ diffeomorphic to the whole sphere, note sphere is compact, in particular closed and bounded in $\mathbb{R}^3$ By continuity of inverse of the parametrization, the sphere should be open. Hence we know there is an proper subset in $\mathbb{R}^3$ clopen, which contradicts to connectedness of $\mathbb{R}^3$ which says the only clopen sets are $\mathbb{R}^3$ itself or the empty set(recall rudin)

If you are interested, in future(or perhaps when you learn topology) you might come acrross the **whitney embedding theorem** which gives a lower bound on the dimension of open sets that can do this.

Definition 7

The pair (U_a, x_a) is called the **paramterization** of M at p while $x_a(U_a)$ is called the **coordinate neighborhood** of p(as mentioned earlier). A family $\{(U_a, x_a)\}$ satisfying (1) and (2) is called a **differentiable structure** on M.

Fact 8

To make clear:

- A **coordinate chart** maps a subset of a manifold to an open subset of $\mathbb{R}^n$
- In contrast, the **paramterization** maps an open subset $U_a \subset \mathbb{R}^n$ to a neighbourhood on M known as the **coordinate patch**

Definition 9

Let M_1^n and M_2^m be differentiable manifolds. A mapping $\phi : M_1 \to M_2$ is **differentiable** at $p \in M_1$ if given a paramterization $y : V \subset \mathbb{R}^m \to M_2$ at $\phi(p)$ there exists a paramterization $x : U \subset \mathbb{R}^n \to M_1$ at p such that $\phi(x(U)) \subset y(V)$ and the mapping

$$y^{-1} \circ \phi \circ x : U \subset \mathbb{R}^n \to \mathbb{R}^m$$

is differentiable at p^{-1}

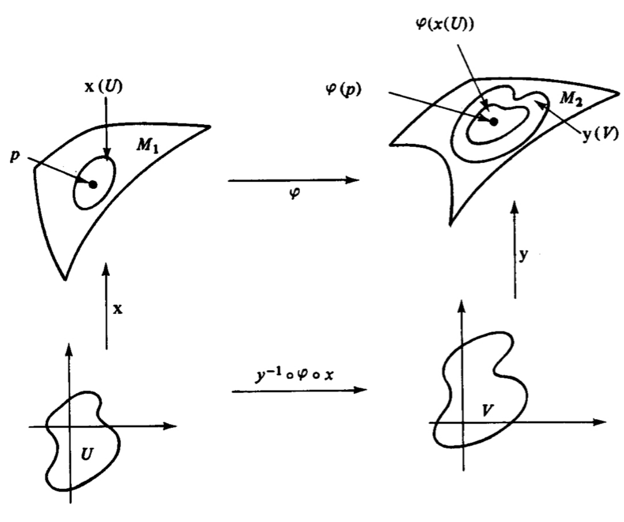

Figure 1: Differentiable mapping between manifolds

Refer back to `differential geometry do carmo` for visualization if necessary. Very similar situation in there. Essentially this is composition of differentiable functions. By multivariable chain rule, it must be differentiable.

Definition 10

Let M be a differentiable manifold. Let a differentiable curve on M be defined by $a : (-\varepsilon, \varepsilon) \to M$ where $a(0) = p \in M$ and $\mathcal{D}$ be the set of functions on M that are differentiable at p. The **tangent vector** to the curve a at $t = 0$ is a function $a'(0) : \mathcal{D} \to \mathbb{R}$ given by

$$a'(0)f = \frac{d(f \circ a)}{dt}|_{t=0}, \quad f \in \mathcal{D}$$

Tje set of all tangent vectors to M at p is indicated by T_pM

Upon comparison with that of $\mathbb{R}^3$regular surfaces in `MIT 18.950 Differential Geometry` you will notice that this time we did not directly define $a'(0)$ to be the tensor vector but instead made a' be a correspondence of f. The reason is that we want to express directly in terms of the local coordinates of the curve. That is we want either $(a(t))'$ itself or $(f \circ a)'(t)$

Fact 11

If we choose a parameterization $x : U \to M^n$ at $p = x(0) = a(0)$ we can express the function f and the curve a in ths paramterization by

$$f \circ x(q) = f(x_1, \dots, x_n), \quad q = (x_1, \dots, x_n) \in U$$

and we know the curve on U is given by

$$\beta = x^{-1} \circ a(t) = (x_1(t), \dots, x_n(t))$$

Therefore since

$$f \circ a = (f \circ x) \circ (x^{-1} \circ a)(t)$$

we have

$$a'(0)f = \frac{d}{dt}(f \circ a)|_{t=0} = \frac{d}{dt} f(x_1(t), \dots, x_n(t))|_{t=0}$$

this expression can be written as(removing f by comparison)

$$a'(0) = \sum_{i=1}^{n} x_i'(0)\left(\frac{\partial}{\partial x_i}\right) = \left(\sum_i x_i'(0)\left(\frac{\partial}{\partial x_i}\right)_0\right) = \underbrace{\left(\frac{\partial}{\partial x_i}\right)}_{1\times n \text{ matrix}} \underbrace{(x_i')(0)}_{n\times 1 \text{ matrix}} = dx_0(\beta'(0))$$

We also see that

$$\left\{\left(\frac{\partial}{\partial x_1}\right)_0, \dots, \left(\frac{\partial}{\partial x_n}\right)_0\right\}$$

is a basis of T_pM

Again refer back to `differential geometry do carmo` for visualization if necessary. Very similar situation in there just without the f.

Proposition 12

Let M_1^n and M_2^m be differentiable manifolds and let $\phi : M_1 \to M_2$ be a differentiable mapping. For every $p \in M_1$ and for every $v \in T_pM_1$ choose a differentiable curve $a : (-\varepsilon, \varepsilon) \to M_1$ with $a(0) = p, a'(0) = v$. Take $\beta = \phi \circ a$. The mapping $d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$ given by $d\phi_p(v) = \beta'(0)$ is a linear mapping that does not depend on the choice of a.

Proof. Again refer back to `differential geometry do carmo` for visualization if necessary. Very similar situation in there. But for completeness sake we can easily see:

First observe that $\alpha(t)$ and $\beta(t)$ are just curves in M_1 and M_2 respectively. Let $x : U \to M_1$ and $y : V \to M_2$ be parameterizations at p and $\phi(p)$ respectively. Then we have $y^{-1} \circ \phi \circ x : U \to V$ which we can write as

$$y^{-1} \circ \phi \circ x(q) = (y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n))$$

where

$$q = (x_1, \dots, x_n) \in U, \quad (y_1, \dots, y_m) \in V$$

On the other hand for $x^{-1} \circ a : (-\varepsilon, \varepsilon) \to U$ we can write

$$x^{-1} \circ a(t) = \underbrace{(x_1(t), \dots, x_n(t))}_{\text{curve in } U}$$

Therefore for $y^{-1} \circ \beta : (-\varepsilon, \varepsilon) \to V$ we can write

$$y^{-1} \circ \beta(t) = \underbrace{(y_1(x_1(t), \dots, x_n(t)), \dots, y_m(x_1(t), \dots, x_n(t))}_{\text{curve in } V})$$

so we have

$$\beta'(0)f = \frac{d}{dt}(f \circ \beta)|_{t=0} = \frac{d}{dt}(f \circ y) \circ (y^{-1} \circ \beta)(t)|_{t=0} = f(y_1(x_1(t), \dots, x_n(t)), \dots, y_m(x_1(t), \dots, x_n(t)))|_{t=0}$$

so like the previous example on comparison and removing f we have

$$\beta'(0) = (\sum_{i=1}^{n} \frac{\partial y_1}{\partial x_i} x_i'(0), \dots, \sum_{i=1}^{n} \frac{\partial y_m}{\partial x_i} x_i'(0))$$

writing the above in matrix form we have

$$\beta'(0) = d\phi_{a(0)}(a'(0)) = d\phi_p(v) = \left(\frac{\partial y_i}{\partial x_j} \right) (x_j'(0))$$

where $\left(\frac{\partial y_i}{\partial x_j} \right)$ is an $m \times n$ matrix while $(x_j'(0))$ is a column matrix with n elements(consider that it corresponds to the columns of the $m \times n$ matrix).So in matrix multiplication which is row times column it must be that $(x_j'(0))$ is an n-element column matrix)

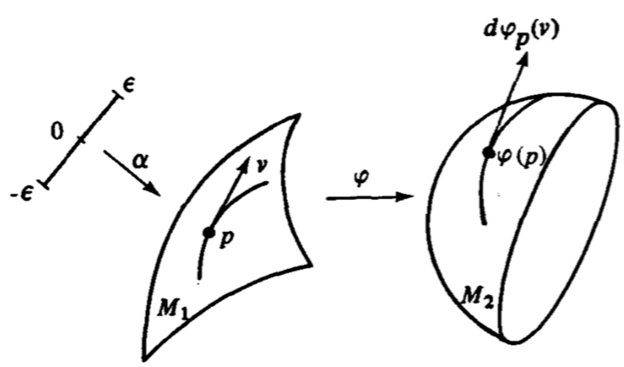

Figure 2: Map between tangent spaces of manifolds

Remark 13. *From the above 2 secenarios you can clearly see how the defitition in* `munkres` *given by* $df_q(v) = Df(q) \cdot v$

where the $\cdot$ here refers to the matrix product came about once again!

Theorem 14

Let $\phi : M_1^n \to M_2^n$ be a differentiable mapping and let $p \in M_1$ be such that $d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$ is an ismorphism(bijective linear map)

Proof. Again done before in `Differential Geometry Do Carmo`, essentially inverse function theorem. $\square$

The rest of the other stuff like the $\mathbb{R}^n$ version of regular surfaces, images of regular values are manifolds are all done before in `MIT 18.101 Analysis on manifolds`. However we would cover one more important example of manifolds.

Example 15 (Tangent bundle)

Let M^n be a differentiable manifold and let

$$TM = \{(p, v); p \in M, v \in T_pM\}$$

We will show that TM can be given differentiable structure of $2n$(recall 1) in which case we will call TM a **tangent bundle**. First let $\{U_a, x_a\}$ be the maximal differentiable structure on M. Denote $(x_1^a, \ldots, x_n^a)$ the coordinates of U_a and $\left\{\frac{\partial}{\partial x_1^a}, \ldots, \frac{\partial}{\partial x_n^a}\right\}$ the associated bases to the tangent spaces of each $x_a(U_a)$. For every a define

$$y_a : U_a \times \mathbb{R}^n \to TM$$

by

$$y_a(x_1^a, \ldots, x_n^a, u_1, \ldots, u_n) = (x_a(x_1^a, \ldots, x_n^a), \sum_{i=1}^n u_i \frac{\partial}{\partial x_i^a})$$

where $(u_1, \ldots, u_n) \in \mathbb{R}^n$. Then since $\bigcup_a x_a(U_a) = M$and $(dx_a)_q(\mathbb{R}^n) = T_{x_a(q)}M, \quad q \in U_a$ we have that

$$\bigcup_a y_a(U_a \times \mathbb{R}^n) = TM$$

so we have satisfied one of the required conditions. For the other condition let

$$(p, v) \in y_a(U_a \times \mathbb{R}^n) \cap y_\beta(U_\beta \times \mathbb{R}^n)$$

then

$$(p, v) = (x_a(q_a), dx_a(v_a)) = (x_\beta(q_\beta), dx_\beta(v_\beta))$$

where $q_a \in U_a, q_\beta \in U_\beta, v_a, v_\beta \in \mathbb{R}^n$ Therefore

$$y_\beta^{-1} \circ y_a(q_a, v_a) = y_\beta^{-1}(x_a(q_a), dx_a(v_a)) = ((x_\beta^{-1} \circ x_a)(q_a), d(x_\beta^{-1} \circ x_a)(v_a))$$

Since $x_\beta^{-1} \circ x_a$ is differentiable so must $d(x_\beta^{-1} \circ x_a)$ and hence so must $y_\beta^{-1} \circ y_a$ as desired.

Remark 16. *You could see that*

$$y_a : \overbrace{U_a}^{\text{point}} \times \underbrace{\mathbb{R}^n}_{\text{coefficients the tangent vector at that point will take}} \to TM$$

1.2 vector fields and brackets

Definition 17

A **vector field** X on a differentiable manifold M is correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. In other words, it is the mapping $X : M \to TM$.Considering a paramterization $x : U \subset \mathbb{R}^n \to M$ we then express the vector field like so

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i} \in T_pM$$

where each $a_i : U \to \mathbb{R}$ is a function on U and $\left\{ \frac{\partial}{\partial x_i} \right\}$ is the basis associated to x. The field is **differentiable** if this mapping is differentiable.

Recall that TM refers to the **tangent bundle** 15 and $\mathcal{D} = \{\text{set of differentiabe functions on M}\})$. Now we let $\mathcal{F}$ be the set of functions in general on M. Also occasionally it is convenient to think of the vector field as a mapping $X : \mathcal{D} \to \mathcal{F}$ is defined like so

$$(Xf)(p) = \sum_i a_i(p) \frac{\partial f}{\partial x_i}(p)$$

This is exactly what we did when we defined the **tangent vector** above.

Lemma 18

Let X and Y be differentiable vector fields on a differentiable manifold M. Thenn there exists a unique vector field Z such that for all $f \in \mathcal{D}$, we have

$$Zf = (XY - YX)f$$

Consider Let

$$X = \sum_i a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_j b_j \frac{\partial}{\partial x_j}$$

Then we have by the chain rule and definition above

$$XYf = X(\sum_j b_j \frac{\partial f}{\partial x_j}) = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i x_j}$$

$$YXf = Y(\sum_i a_j i \frac{\partial f}{\partial x_i}) = \sum_{i,j} b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} + \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i x_j}$$

Then we get an expression for Z like so

$$Zf = XYf - YXf = \sum_{i,j} (a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}) \frac{\partial f}{\partial x_j}$$

It is clear to see Z is unique regardless of f for a given X, Y.

Definition 19

The vector field Z given above 18 is called the **bracket** which we denote as

$$[X, Y] = XY - YX = Z$$

Since X, Y are differentiable it is obvious that Z is as well

Proposition 20

If X, Y, Z are differentiable vector fields on M and a, b are real numbers and f, g are differentiable functons then

(a) $[X, Y] = -[Y, X]$(anticommutativity)

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$(linearity)

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$(Jacobi Identity)

(d) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$

Proof. (a) and (b) are obvious from the definition. For (c) just expand everything out and you should get the desired result. The first few steps should look like

$$[[X, Y], Z] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX$$

Now do it for the other terms. Finally for (d)

$$[fX, gY] = fX(gY) - gY(fX) = fgXY + fX(g)Y - gfYX - gY(f)X = fg[X, Y] + fX(g)X - gY(f)X$$

the second equality follow by chain rule(notice the partial derivative in the definition of vector field)

Theorem 21

Let X be a differentiable vector field on a differentiable manifold M ad let $p \in M$. Then there exists a neighbourhood $U \subset M$ of p an interval $(-\delta, \delta)$, $\delta > 0$ and a differentiable mapping $\phi : (-\delta, \delta) \times U \to M$ such that the curve $t \to \phi(t, q)$, $t \in (-\delta, \delta)$, $q \in U$ is the *unique* curve which satisfies $\frac{\partial \phi}{\partial t} = X(\phi(t, q))$ and $\phi(0, q) = q$

Remark 22. *It is common to use the notation $\phi_t(q) = \phi(t, q)$ and call $\phi_t : U \to M$ the* **local flow** *of X.*

Proof. damn i actually know this! Refer back to the appendix section on flows and ODEs in your `18.905` differential geometry of curves and surfaces notes.

Proposition 23

Let X, Y be differentiable vector fields on a differentiable manifold M , let $p \in M$ and let ϕ_t be the local flow

2 Riemannian Metric

Definition 24

A **Riemannian metric**(also known as **Riemannian structure**) on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $\langle , \rangle_p$(that is, a symmetric, bilinear, positive definite form) on the tangent space T_pM where if $x : U \subset \mathbb{R}^n \to M$ is a system of coordinates around p with $x(x_1, \ldots, x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = dx_q(0, 1 \ldots, 1, \ldots, 0)$ then $\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \rangle_q = g_{ij}(x_1, \ldots, x_n)$ is a differentiable function on U

It is clear this definition is independent of choice of coordinate system. The function g_{ij}(which equals g_{ji}) is called the **local representation** of the Riemann metric in the coordinate system.

Definition 25
A differentiable manifold given a Riemannian metric will be called a **Riemannian manifold**

Definition 26
Recall, let M and N be Riemannian manifolds. A diffeomorphism $f : M \to N$ is called an **isometry** if

$$\langle u, v\rangle_p = \langle df_p(u), df_p(v)\rangle_{f(p)}$$

for all $p \in M$ and $u, v \in T_pM$

We will provide some examples of *Riemannian manifolds*

Example 27 (Euclidean Space)
The differentiable manifold $M = \mathbb{R}^n$ with $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, \ldots, 1, \ldots, 0)$ has the metric given by $\langle e_i, e_j\rangle$. $\mathbb{R}^n$ is called the **Euclidean space** of dimension n

Example 28 (Isometric Immersion)
Let $f : M^n \to N^{n+k}$ be an immersion, that is, f is differentiable and $df_p : T_pM \to T_{f(p)}N$ is injective for all p in M. If N has a *Riemannian structure* then f **induces** a riemannian structure on M if we define

$$\langle u, v\rangle_p = \langle df_p(u), df_p(v)\rangle_{f(p)}$$

Since injective, if $df_p(u) = df_p(v)$ then $u = v$. Then $\langle df_p(u), df_p(v)\rangle_{f(p)} > 0$ since by definition a riemannian metric is postive definite and so is $\langle u, v\rangle$ since it has the same value.

3 connections

We shall indicate $\mathfrak{X}(M)$ as the set of a vector fields of class C^∞ on M and $\mathcal{D}(M)$ to be the ring of real valued functions of class C^∞ defined on M.

3.1 Affine Connections

Definition 29 (Affine connection)

An **affine connection** ∇ on a differentiable manifold M is a mapping

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

which is denoted by $(X, Y) \xrightarrow{\nabla} \nabla_X Y$ which satisfes the following properties

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

(ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$

(iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$

in which $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in \mathcal{D}(M)$

Proposition 30

Let M be a differentiable manifold with an affine connection ∇. There exists a unique correspondence which associates to a vector field V along the differenentiable curve $c : I \to M$ another vector field $\frac{DV}{dt}$ along c called the **covariant derivative** of V along c such that

(a) $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$

(b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$ where W is a vector field along c and f is a differentiable function on I

(c) If V is induced by a vector field $Y \in \mathfrak{X}(M)$ meaning $V(t) = Y(c(t))$ then $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$

Proof. We first contruct a possible correspondance satisfying (a)-(c). Let $x : U \subset \mathbb{R}^n \to M$ be a system of coordinates such that $c(I) \cap x(U) \neq \emptyset$ and $(x_1(t), x_2(t), \ldots, x_n(t))$ be the local expression of curve $c(t), t \in I$. Let $X_i = \frac{\partial}{\partial x_i}$. Then we can express the field V locally as

$$V = \sum_j v^j X_j, j = 1, \ldots, n$$

where $v^j = v^j(t)$ and $X_j = X_j(c(t))$. By (a) we may move the sum out and by (b) we can apply "product rule" on covariant derivatives

$$\frac{DV}{dt} = \sum_j \frac{dv^j}{dt} X_j + \sum_j v^j \frac{DX_j}{dt}$$

Now by (c) and property (i) in 29 where we get

$$\frac{DX_j}{dt} = \nabla_{dc/dt} X_j = \nabla_{(\sum \frac{dx_i}{dt} X_i)} X_j = \sum_i \frac{dx_i}{dt} \nabla_{X_i} X_j, \quad i, j = 1, \ldots, n$$

Therefore upon combining the 2 equations we have here we get

$$\frac{DV}{dt} = \sum_j \frac{dv^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} v^j \nabla_{X_i} X_j$$

This is unique because assume there is another correspondance $\frac{\tilde{D}}{dt}$ satisfying (a)-(c) in this exact same region and same system of coordinates(i.e x is still the paramterization function). However whatever form it takes, it must also satisfy

$$\frac{\tilde{D}V}{dt} = \sum_j \frac{dv^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} v^j \nabla_{X_i} X_j$$

because our correspondance above was derived by direct definition. Hence for the same V, it must be that

$$\frac{DV}{dt} = \frac{\tilde{D}V}{dt}$$

as desired. Next we aim to show existence. First we need to show that this is true for all of M and not just a some local coordinate neighbourhood. Recall so far we have only defined it on some portion of $c(I)$(i.e local). Suppose now we have $y(W) \cap x(U) \cap c(t) \neq \emptyset$. Then for all local coordinates of $c(t)$ in this area of intersection, regardless of x or y, $\frac{DV}{dt}$ will be expressed as above due to uniqueness of correspondance. But notice

$$X_j(c(t)) = X_j((x_1(t), x_2(t), \dots, x_n(t))) = X_j((y_1(t), x_2(t), \dots, y_m(t)))$$

. Obviously same for $v(t)$. So this definition is clearly independent of choice of paramterization so we can extend this definition to all over M. Finally we need to show that our correspondence satisfies the propositions(so far we only have given the propositions are true then our correspondance must look like so). Essentially it is not bijective yet. Consider the below. Suggestion: prove property (c) first, it simplifies the proof of the rest by a lot.

Proposition 31

Show explicitly that our correspondence satisfies property (c) above. That is show that $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$ given $V(t) = Y(c(t))$

Proof. Since $V(t) = Y(c(t))$, this means that the vector field V along the curve c is just the vector field Y evaluated at the points of the curve.

Conceptually the action of covariant derivative along the curve is that if $V(t) = Y(c(t))$, then $\frac{DV}{dt}$ is defined by how the vector field Y changes along the curve $c(t)$. In other words, $\frac{DV}{dt}$ measures how Y changes in the direction of the curve's tangent vector $\frac{dc}{dt}$.

Let the tangent vector field $\frac{dc}{dt}$ be written in local coordinates as $\frac{dc}{dt} = \sum_i \frac{dx_i}{dt} \frac{\partial}{\partial x_i}$.

By linearity of the connection in the first argument (Property i), we have:

$$\nabla_{\frac{dc}{dt}} Y = \nabla_{\sum_i \frac{dx_i}{dt} \frac{\partial}{\partial x_i}} Y = \sum_i \frac{dx_i}{dt} \nabla_{\frac{\partial}{\partial x_i}} Y.$$

Using the vector field expression $Y = \sum_k Y^k X_k$, we calculate $\nabla_{\frac{\partial}{\partial x_i}} Y$:

$$\nabla_{\frac{\partial}{\partial x_i}} Y = \nabla_{\frac{\partial}{\partial x_i}} \left(\sum_k Y^k X_k \right).$$

By applying linearity (Property ii):

$$\nabla_{\frac{\partial}{\partial x_i}} Y = \sum_k \nabla_{\frac{\partial}{\partial x_i}} (Y^k X_k).$$

Now, using the Leibniz rule (Property iii):

$$\nabla_{\frac{\partial}{\partial x_i}} (Y^k X_k) = Y^k \nabla_{\frac{\partial}{\partial x_i}} X_k + \frac{\partial Y^k}{\partial x_i} X_k.$$

Substituting back, we have:

$$\nabla_{\frac{\partial}{\partial x_i}} Y = \sum_k \left(\frac{\partial Y^k}{\partial x_i} X_k + Y^k \nabla_{\frac{\partial}{\partial x_i}} X_k \right).$$

Substituting this result into the earlier expression for $\nabla_{\frac{dc}{dt}} Y$:

$$\nabla_{\frac{dc}{dt}} Y = \sum_i \frac{dx_i}{dt} \left(\sum_k \left(\frac{\partial Y^k}{\partial x_i} X_k + Y^k \nabla_{\frac{\partial}{\partial x_i}} X_k \right) \right).$$

Simplify this as:

$$\nabla_{\frac{dc}{dt}} Y = \sum_k \left(\sum_i \frac{dx_i}{dt} \frac{\partial Y^k}{\partial x_i} \right) X_k + \sum_k \left(\sum_i \frac{dx_i}{dt} Y^k \nabla_{\frac{\partial}{\partial x_i}} X_k \right).$$

From the formula for $\frac{DV}{dt}$:

$$\frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx_i}{dt} \Gamma_{ij}^k \right) X_k.$$

since

$\frac{dv^k}{dt} = \sum_i \frac{\partial Y^k}{\partial x_i} \frac{dx_i}{dt}$ and the term $\sum_{i,j} v^j \frac{dx_i}{dt} \Gamma_{ij}^k$ corresponds exactly to the term $\sum_i \frac{dx_i}{dt} Y^j \nabla_{\frac{\partial}{\partial x_i}} X_k$ due to how Christoffel symbols Γ_{ij}^k encode the connection coefficients $\nabla_{\frac{\partial}{\partial x_i}} X_k$.

Thus, we've shown that:

$$\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y,$$

□

Proposition 32

Show explicitly that our correspondence satisfies (a) and (b)

Proof. First we prove (a) Let $V(t)$ and $W(t)$ be vector fields along the curve $c(t)$.

By the definition of the covariant derivative along the curve:

$$\frac{D}{dt}(V + W) = \frac{D}{dt}(Y(c(t)) + Z(c(t))),$$

where $V(t) = Y(c(t))$ and $W(t) = Z(c(t))$ for vector fields Y and Z on the manifold.

By property (ii) of the affine connection:

$$\nabla_{\frac{dc}{dt}}(Y + Z) = \nabla_{\frac{dc}{dt}} Y + \nabla_{\frac{dc}{dt}} Z.$$

Applying this directly:

$$\frac{D}{dt}(V + W) = \nabla_{\frac{dc}{dt}}(Y + Z) = \nabla_{\frac{dc}{dt}} Y + \nabla_{\frac{dc}{dt}} Z.$$

Since $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$ and $\frac{DW}{dt} = \nabla_{\frac{dc}{dt}} Z$, we obtain:

$$\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}.$$

This proves property (a).

Let $V(t) = Y(c(t))$ be a vector field along the curve, and let f be a differentiable function on the interval I. By definition, we need to compute:

$$\frac{D}{dt}(fV) = \frac{D}{dt}(fY(c(t))).$$

Applying property (iii) of the affine connection, $\nabla_X(fY) = f\nabla_X Y + X(f)Y$, with $X = \frac{dc}{dt}$, we get:

$$\nabla_{\frac{dc}{dt}}(fY) = f\nabla_{\frac{dc}{dt}}Y + \left(\frac{dc}{dt}\right)(f)Y.$$

- The first term, $f\nabla_{\frac{dc}{dt}}Y$, corresponds to $f\frac{DV}{dt}$ since $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}}Y$. - The second term, $\left(\frac{dc}{dt}\right)(f)Y$, corresponds to the derivative of f along the curve.

$\left(\frac{dc}{dt}\right)(f)$ is the directional derivative of f along the curve, which can be written as:

$$\left(\frac{dc}{dt}\right)(f) = \frac{df}{dt}.$$

Therefore, we have:

$$\nabla_{\frac{dc}{dt}}(fY) = f\nabla_{\frac{dc}{dt}}Y + \frac{df}{dt}Y.$$

Combining these results:

$$\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}.$$

proves (b) as desired □

Definition 33

Let M be a differentiable manifold with an affine connection ∇. A vector field V along a curve $c : I \to M$ is called **parallel** when $\frac{DV}{dt} = 0$ for all $t \in I$

Definition 34

We make a new point about Affine connections, that by definition $\nabla_{X_i}X_j = \sum_k \Gamma_{ij}^k X_k$ where Γ_{ij}^k are the coefficients of the affine connection(recall which is map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$) called **christoffel symbols**). And clearly they are differentiable functions too since by definition $\mathfrak{X}(M)$ is the set of C^∞ vector fields.

Proposition 35

Let M be a differentiable manifold with an affine connection ∇. Let $c : I \to M$ be a differentiable curve in M and let V_0 be a vector tangent to M at $c(t_0)$, $t_0 \in I$.($V_0 \in T_{c(t_0)}$) Then there exists a unique parallel vector field V along c such that $V(t_0) = V_0$.

Proof. Let $V_0 = \sum_j v_0^j X_j$. Using the christoffel symbols we have and solving directly $\frac{Dv}{dt} = 0$

$$\frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx_i}{dt}\Gamma_{ij}^k\right) X_k = 0$$

Where we essentially have $n \times n$ system of first order ODEs with solution $v^k(t), k = 1, \dots, n$ where each must satisfy the IVP $v^k(t_0) = v_0^k$. By theory of ODEs(refer to `MIT 18.905` notes appendix) we know there exists a unique solution $\square$

$V(t)$ is known as the **parallel transport** of $V(t_0)$ along c. But $V(t)$ may or not be in $T_{c(t)}$. We will explore more of this in reimannian connections below

3.2 Reinmann Connections

Definition 36

Let M be a differentiable manifold with an affine connection ∇ and a *reinmannian metric* $\langle , \rangle$(recall above). A connection is said to be **compatible** with metric $\langle , \rangle$ when for any smooth curve c and any pair of parallel vector feilds P and P' alongn c we have $\langle P, P' \rangle$ =constant

Proposition 37

Let M be a riemannian manifold. A connection ∇ on M is compatible with a metric if and only if for any vector fields V and W along the diffrentiable curve $c : I \to M$ we have

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle, \quad t \in I$$

Proof. For the forward direction it is obvious if you consider

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle = \langle 0, W\rangle + \langle V, 0\rangle = 0$$

Let $\{V_0^1, \dots, V_0^n\}$ be an orthonormal basis of $T_{c(t_0)}$. Since each V_0^i is a vector of $T_{c(t_0)}$, for each of them, there exists a unique parallel vector field P_i along c such that $V_0^i = P_i(t_0)$. But because they are compatable with the metric, every $P_i(t)$ at point $c(t)$ in $T_{c(t)}$ remain orthogonal to each other so they are in fact still an orthonormal basis of $T_{c(t)}$.(since constant dot product means angles and lenghts preserved). Recall we are assuming vector fields refer to tangent fields(those that map points to tangent space tangent space and we assume they are differentiable). Therefore we may express any 2 arbituary vector fields along c by

$$V = \sum_i v^i P_i \quad W = \sum_i w^i P_i, i = 1, \dots, n$$

where v^i and w^i are differentiable functions on I. It follows that

$$\frac{DV}{dt} = \sum_i \frac{dv^i}{dt} P_i \quad \text{and} \quad \frac{DW}{dt} = \sum_i \frac{dw^i}{dt} P_i$$

because by definition of parallel vector fields $\frac{DP_i}{dt} = 0$ so we omit the first term in the correspondance relation of covariant derivative(recall above). Therefore

$$\begin{aligned}\langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle &= \sum_i \left(\frac{dv^i}{dt} w^i + \frac{dw^i}{dt} v^i\right) \\ &= \frac{d}{dt}\left(\sum_i v^i w^i\right) = \frac{d}{dt}\langle V, W\rangle\end{aligned}$$

as desired

Corollary 38

A connection ∇ on a Riemannian manifold M is compatbile with the metric if and only if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Proof. Suppose...to be continued

Definition 39

An affine connection ∇ on a smooth manifold M is said to be **symmetric** when

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all $X, Y \in \mathfrak{X}(M)$

This in other words says that

$$\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j]$$

which is equal zero since recall the definition of brackets above

$$[X_i, X_j] = X_i X_j - X_j X_i$$

but since $X_i = \frac{\partial}{\partial x_i}$ and that mixed partial derivatives commute the above is just the zero vector field. Hence because by recall above by definition $\Gamma_{ij}^k = \nabla_{X_i} X_j$. So we are implying $\Gamma_{ij}^k = \Gamma_{ji}^k$

Theorem 40 (Levi Civita)

Given a Riemannian manifold M there exists a unique affine connection ∇ on M satisfying the conditions

(a) ∇ is symmetric

(b) ∇ is compatible with the Riemannian metric

Fact 41

We may express key results from above in chirstoffel symbols by

$$\sum_{\ell} \Gamma_{ij}^{\ell} g_{\ell k} = \frac{1}{2}\left\{\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij}\right\}$$

where the matrix given by the *riemannian metric*, $g_{ij} = \langle X_i, X_j\rangle$. Since X_i are basis vectors of the tangent space, this matrix (g_{km}) is onvertible and we can express its inverse as (g^{km}). Then Also as recap we have earlier also found $\frac{DV}{dt}$ in terms of chirstoffel symbols too

$$\frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx_i}{dt} \Gamma_{ij}^k\right) X_k$$

4 Geosedics: Convex neighborhoods

4.1 The geodesic flow

Definition 42

let M be a Riemannian manifold together with its Riemannian connection. A parameterized curve $\gamma : I \to M$ is a **geodesic** at $t_0 \in I$ if $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ at the point t_0. γ is a geodesic if this is true for all $t \in I$. If $[a, b] \subset I$ and $\gamma : I \to M$ is a geodesic, then the restriction γ to $[a, b]$ is called a **geodesic segment** joining $\gamma(a)$ to $\gamma(b)$.

Hence if $\gamma : I \to M$ is a geodesic then

$$\frac{d}{dt}\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = 2\langle\frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = 0$$

where the first equality follows from 37 and the symmetry of inner product 24 while the second equality follows because of compability of rienmannian metric with the metric in our case. That is since $\frac{d\gamma}{dt}$ is clearly a parallel vector field so $\frac{d}{dt}\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle$ must be zero recall above

Assume from now on that $\left|\frac{d\gamma}{dt}\right| = c \neq 0$. Then recall from `MIT 18.950` the arc length s of γ is given by

$$s(t) = \int_{t_0}^{t}\left|\frac{d\gamma}{dt}\right| dt = c(t - t_0)$$

We say that the geodesic γ is **normalized** if $c = 1$.

Let us explore the local equations satisfied by the geodesic γ in a system of coordinates (U, x) about $\gamma(t_0)$. In U a curve γ given by

$$\gamma(t) = (x_1(t), \dots, x_n(t))$$

by definition will be a geodesic if and only if

$$0 = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \sum_k \left(\frac{d^2x_k}{dt^2} + \sum_{i,j}\Gamma_{ij}^k\frac{dx_i}{dt}\frac{dx_j}{dt}\right)\frac{\partial}{\partial x^k}$$

the last equality will make sense in a bit when we use tangent bundles. Proceeding from here we have a 2nd order system

$$\frac{d^2x_k}{dt^2} + \sum_{i,j}\Gamma_{ij}^k\frac{dx_i}{dt}\frac{dx_j}{dt} = 0, \quad k = 1, \dots, n$$

Like how we do for linear differential equations we try to set a system of 1 order ODEs so we can put it in a matrix to solve and that by ODE theory(see appendix in `MIT 18.950`) we know exists a unique solution to. Indeed we can do by considering the tangent bundle TM(recall 15). Any differentiable curve $\gamma \to \gamma(t)$ in M determines a curve $t \to (\gamma(t), \frac{d\gamma}{dt}(t))$ in TM specifically

$$t \to (x_1(t), \dots, x_n(t), \underbrace{\frac{dx_1(t)}{dt}, \dots, \frac{dx_n(t)}{dt}}_{\text{clearly the coordinates of } \gamma'(t)})$$

which is exactly what we are looking far. Hence if γ is geodesic on TU the curve satisfies the system

$$\begin{cases} \frac{dx_k}{dt} = y_k \\ \frac{dy_k}{dt} = -\sum_{i,j}\Gamma_{ij}^k y_i y_j \end{cases}$$

where $y_i(t) = \frac{dx_i(t)}{dt}$. Hence as expected by ODE theory we can claim the following theorem

Theorem 43

let M be a Riemannian manifold together with its Riemannian connection. If X is a C^∞ vector field on the open set V in the manifold M and $p \in V$ then there exist an open set $V_0 \subset V, p \in V_0$ a number $\delta > 0$ and a C^∞ mapping $\phi : (-\delta, \delta) \times V_0 \to V$ such that the curve $t \to \phi(t, q), t \in (-\delta, \delta)$ is the unique trajectory of X which at the instant $t = 0$ passes through the point q for every $q \in V_0$

The mapping $\phi_t : V_0 \to V$ given by $\phi_t(q) = \phi(t, q)$ is called the **flow** of X on V.

Lemma 44 (Uniqueness)

There exists a unique vector field G on TM whose trajectories are of the form $t \to (\gamma(t), \gamma'(t))$ where γ is geosedic on M

Proof. If γ is geodesic that it clearly statisfies the 2×2 linear system of ODE we defined above. Then by ODE theory we know that a uniques and existence as desired follows.

Definition 45

The vector field G defined above is called the **geodesic flow** on TM.

Lemma 46 (Homogenity of a geodesic)

If the geodesic $\gamma(t, q, v)$ is defined on the intervl $(-\delta, \delta)$ then the geodesic $\gamma(t, q, av), a \in \mathbb{R}, a > 0$ is defined on the interval $(-\frac{\delta}{a}, \frac{\delta}{a})$ and

$$\gamma(t, q, av) = \gamma(at, q, v)$$

Proof. Let $h : (-\frac{\delta}{a}, \frac{\delta}{a}) \to M$ be a curve given by $h(t) = \gamma(at, q, v)$. Then $h(0) = q$ and $\frac{dh}{dt}(0) = av$. In addition since $h'(t) = a\gamma'(at, q, v)$

$$\frac{D}{dt}\left(\frac{dh}{dt}\right) = \nabla_{h'(t)} h'(t) = a^2 \underbrace{\nabla_{\gamma'(at,q,v)}\gamma'(at, q, v)}_{\frac{D}{dt}(\gamma')} = 0$$

The first equality follows by property (c) in 30. See that the vector field is already in the form "takes the local coordinates" of the curve. The last equality follows by definition of a geodesic.Therefore we have proven that h is geodesic passing through q with velocity av at $t = 0$. However Since $h(0) = \gamma(0) = q$, by uniqueness the vector field whose geodesic has trajectories of the form in 44, this implies $h(t) = \gamma(t)$ for all t(refer to appendix of `MIT 18.950` if necessary). Then this implies

$$h(t) = \gamma(at, q, v) = \gamma(t, q, av)$$

where the 1st equality is just the definition of $h(t)$ as in the lemma while the 2nd equality is just the original $\gamma(t)$.

Proposition 47

Given $p \in M$ there exists a neighborhood V of p in M a numer $\varepsilon > 0$ and a C^∞ mapping $\gamma : (-2, 2) \times \mathcal{U} \to M, \mathcal{U} = \{(q, w) \in TM; q \in V, w \in T_qM, |w| < \varepsilon\}$ such that $t \to \gamma(t, q, w), t \in (-2, 2)$ is the unique geodesic of M which at the instant $t = 0$ passes through q with velocity w for every $q \in V$ and for every $w \in T_qM$ with $|w| < \varepsilon$

Proof. The set $\mathcal{U}$ is yet another open set on the manifold M so reapplication of 43 yields yet another family of unique continuous solutions in a subset of the manifold. By homogenity of the geodesic we may scale the interval of the definition and the velocity of the geodesic both ways or opposite directions so we can certainly get a specific range like $(-2, 2)$ here and still have uniqueness and smoothness etc.

Definition 48

Let $p \in M$ and let $\mathcal{U} \subset TM$ be an open set given by the previous proposition. Then the map $\exp : \mathcal{U} \to M$ given by

$$\exp(q, v) = \gamma(1, q, v) = \gamma(|v|, q, \frac{v}{|v|}), \quad (q, v) \in \mathcal{U}$$

is called the **exponential map** on $\mathcal{U}$

The exponential map is differentiable, $\exp_p(v) = \exp(p, v) = \gamma(1)$ and $\gamma(0) = \exp_q(0) = q$ and $\gamma'(0) = v$

Proposition 49

Given $q \in M$ there exists $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0) \subset T_q M \to M$ is a diffeomorphism of $B_\varepsilon(0)$ onto an open subset of M

Proof. Consider the curve $a(t) = tv, v \in T_p(S)$. Where $a(0) = 0$ and $a'(0) = v$

$$\begin{aligned} d(\exp_q)_0(v) = D(\exp_q)_0 \cdot v = \frac{d}{dt}(\exp_q(tv))|_{t=0} = \frac{d}{dt}(\gamma(1, q, tv)|_{t=0} \\ = \frac{d}{dt}(\gamma(t, q, v)|_{t=0} = v \end{aligned}$$

Where tv is just an expression of points in a ball centered at zero $(\exp_p \circ a) = \exp_p(tv)$ and also has tangent vector at $t = 0$ recall 10. The main idea that since it is the identity map at the center of the ball where $t = 0$ by inverse function theorem the proposition follows.

Remark 50. *This makes sense since(as you will see below) γ is essentially a unique geodesic traced out by t.And we know by definition $\gamma'(0) = \frac{d}{dt}(\gamma(t, q, v)|_{t=0} = v$*

4.2 Minimizing properties of geodesics

Definition 51

A **piecewise differentiable curve** is a continuous mapping $c : [a, b] \to M$ of a closed interval $[a, b] \subset \mathbb{R}$ into M satisfying the condition that there exists a partition $a = t_0 < t_1 < \ldots < t_k = b$ of $[a, b]$ such that the restrictions $c|_{t_i, t_{i+1}}, \quad i = 0, \ldots, k-1$ are differentiable. We say that c joins the points $c(a)$ and $c(b)$

More specifically recalling Rudin $c(t), t \in (t_i.t_{i+1})$ is continuous differentiable. While at the endpoints $c(t_i), i = 1, \ldots, n$ the one sided limits exists.

Lemma 52

Once again M is a differentiable manifold with a symmetric connection. Consider **paramterized surface** in M by a smooth map $F : A \subset \mathbb{R}^2 \to M$. Let (u, v) be the local coordinates then

$$\frac{D}{\partial t}\frac{\partial F}{\partial s} = \frac{D}{\partial s}\frac{\partial F}{\partial t}$$

If V is a vector field along $F : A \to M$ then $\frac{DV}{\partial t}(t, s_0)$ is the covariant derivative along the curve $u \to F(t, s_0)$ of the restriction of V to this curve. Similarly $\frac{DV}{\partial s}(t_0, s)$ is the covariant derivative along the curve $u \to F(t_0, s)$ of the restriction of V to this curve. For fixed s_0, t_0 respectively.

Proof. Consider

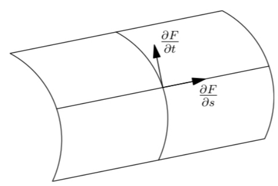

Figure 3: Parameterized Surface

Writing $F = (F_1, \ldots, F_n)$ and using summation convention we have

$$\begin{aligned}
\frac{\partial F}{\partial s} &= \frac{\partial F_i}{\partial s}\frac{\partial}{\partial x_i} \\
\frac{\partial F}{\partial t} &= \frac{\partial F_j}{\partial s}\frac{\partial}{\partial x_j} \\
\frac{D}{\partial t}\frac{\partial F}{\partial s} &= \frac{D}{\partial t}\left(\frac{\partial F_i}{\partial s}\frac{\partial}{\partial x_i}\right) \\
&= \frac{\partial^2 F_i}{\partial s \partial t}\frac{\partial}{\partial x_i} + \frac{\partial F}{\partial s}\frac{D}{\partial t}\frac{\partial}{\partial x_i} \\
&= \frac{\partial^2 F_i}{\partial s \partial t}\frac{\partial}{\partial x_i} + \frac{\partial F}{\partial s}\nabla_{\frac{\partial F_j}{\partial t}\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i} \\
&= \frac{\partial^2 F_i}{\partial s \partial t}\frac{\partial}{\partial x_i} + \frac{\partial F_j}{\partial t}\frac{\partial F_i}{\partial s}\Gamma^k_{ji}\frac{\partial}{\partial x_k}
\end{aligned}$$

where the second last line is application of property (c) of 30. Recall that for riemannan connection $\Gamma^k_{ji} = \Gamma^k_{ij}$. Hence the whole last line is symmetric. It could be product rule expansion either with respect to s or t so the lemma follows.

Lemma 53 (Gauss)

Once again M is a differentiable manifold with a symmetric connection. Let $p \in M$ and let $v \in T_pM$ such that $\exp_p v$ is defined. Let $w \in T_pM \approx T_v(T_pM)$. Then

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle = \langle v, w\rangle$$

Proof. Let $w = w_T + w_N$ where w_T is parallel to v and w_N is normal to v. Since $d\exp_p$ is linear we have

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(v)\rangle = \langle (d\exp_p)_v(v), (d\exp_p)_v(w_N)\rangle + \langle (d\exp_p)_v(v), (d\exp_p)_v(w_T)\rangle$$

where

$$\langle v, w_T \rangle + \langle v, w_N \rangle = \langle v, w \rangle$$

. Consider the case for w_T. Since the exponential map is defined, its corresponding geosdesic γ satisfies $\exp_p(v) = \gamma(1, p, v)$, $d\exp_p(v) = \gamma'(1)$ where $\exp_p(0) = \gamma(0) = p$ and $v = \gamma'(0)$.In which we case we can get as desired

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w_T) \rangle = \langle v, w_T \rangle$$

since from 37 that vector fields along the curve preserve inner products. so we have letting $w_T = av$ since parallel

$$a\langle \gamma'(0), \gamma'(0) \rangle = a\langle \gamma'(1), \gamma'(1) \rangle$$

since $\frac{d}{dt}\langle \gamma(t), \gamma(t) \rangle = 0$.

It now remains show w_N term is null because the lemma implies

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w_N) \rangle = \underbrace{\langle v, w_N \rangle}_{\text{orthogonal}} = 0$$

First we define the paramterized curve.

$$a : [-\varepsilon, \varepsilon] \times [0, 1] \to T_pM, \quad (s, t) \mapsto tv + tsw_N$$

this choice of curve is so that we are able to express its various partial derivatives like so

$$a(0, 1) = v, \quad \frac{\partial a}{\partial t}(s, t) = v + sw_N, \quad \frac{\partial a}{\partial s}(0, t) = tw_N$$

Then we may write

$$f : [-\varepsilon, \varepsilon] \times [0, 1] \to M, \quad (s, t) \mapsto \exp_p(tv + tsw_N)$$

Notice that for fixed s_0 ranging $f(t)$ over t traces out a single geosedic because each $exp_p(tv + ts_0w_N)$ is a point on some geosedic defined by $\gamma(1, p, tv + ts_0w_N) = \gamma(t, p, v + s_0w_N)$. So by uniqueness since all these corresponding geodesics start at the same point *and* start with the same velocity(recall we have a 2×2 ODE system), they must be the same geodesic and that if f is smooth on t we should be able to trace out the geodesic.

Now calculate

$$d(\exp_p(v))_v = d(\exp_p)_{a(0,1)}\left(\frac{\partial a}{\partial t}(0, 1)\right) = \frac{\partial}{\partial t}\left(\exp_p \circ a(s, t)\right)|_{t=1,s=0} = \frac{\partial f}{\partial t}(0, 1)$$

and

$$d(\exp_p(w_N))_v = d(\exp_p)_{a(0,1)}\left(\frac{\partial a}{\partial s}(0, 1)\right) = \frac{\partial}{\partial s}\left(\exp_p \circ a(s, t)\right)|_{t=1,s=0} = \frac{\partial f}{\partial s}(0, 1)$$

Hence

$$\langle d(\exp_v(v))_v, d(\exp_p(w_N))_v \rangle = \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \rangle(0, 1)$$

We can show that the RHS is actuall zero. Consider

$$\frac{\partial}{\partial t}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle = \langle \frac{D}{\partial t}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle + \langle \frac{\partial f}{\partial s}, \frac{D}{\partial t}\frac{\partial f}{\partial t} \rangle$$

by 37. Then by the symmetry of the connection from 52 we have

$$\langle \frac{D}{\partial t}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle = \langle \frac{D}{\partial s}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle = \frac{1}{2}\frac{\partial}{\partial s}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle = 0$$

However the maps $t \mapsto f(s, t)$ are goedesics recall above and the fact that each partial derivative treats s like a constant) hence $\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle$ is a constant so proposition follows.

Definition 54

If $\exp_p$ is a diffeomorphism of a neighbourhood of V of the origin of T_pM then $\exp V = U$ is called a **normal neighborhood** of p If $B_\varepsilon(0)$ is such that $\overline{B_\varepsilon}(0) \subset V$ we call $\exp_p B_\varepsilon(0) = B_\varepsilon(p)$ the **normal ball**(or the **geodesic ball**)

Proposition 55 (Geodesics Locally minimize arc length)

Once again M is a differentiable manifold with a symmetric connection. Let $p \in M, U$ be a normal neighborhood of p and $B \subset U$ be a normal ball of center p. Let $\gamma : [0,1] \to B$ be geodesic segment with $\gamma(0) = p$. If $c : [0,1] \to M$ is any piecewise differentiable curve joining $\gamma(0)$ to $\gamma(1)$ then $\ell(y) \leq \ell(c)$ and if equality holds then $\gamma([0,1]) = c([0,1])$

Proof. Suppose initially that $c([0,1]) \subset B$. Since $\exp_p$ is a diffeomorphism on U, the curve $c(t)$ for $t \neq 0$ can be writttten uniquely as $\exp_p(r(t) \cdot v(t)) = f(r(t), t)$ where $t \to v(t)$ is a curve in T_pM with $|v(t)| = 1$(since measuring angles) and $r : (0,1] \to \mathbb{R}$ is a positive piecewise differentiable function(we can suppose that if $t_1 \in (0,1]$ then $c(t_1) \neq p$; otherwise ignore the interval $[0, t_1)$).This is valid because a composition of smooth functions is still smooth. It follows that except for a finite number of points,

$$\frac{dc}{dt} = \frac{\partial f}{\partial r} r'(t) + \frac{\partial f}{\partial t}$$

Since

$$c'(t) = \frac{d}{ds}|_{s=0} f(r(t+s), t+s)$$

then by chain rule(applied to each slot separately like so)

$$c'(t) = df(r(t), t) \cdot (r'(t), 1)$$

where $(r'(t), 1)$ is the tangent vector to the curve $s \mapsto (r(s), s)$ at $s = t$. Hence by definiiont of partial derivatives in differential geometry(by default with respect to t recall apply multivariable chainrule separtely to each slot to get the below)

$$c'(t) = \frac{\partial f}{\partial r}(r(t), t) \cdot (r'(t), 0) + \frac{\partial f}{\partial t}(r(t), t) \cdot (0, 1)$$

which is just $c'(t) = \frac{\partial f}{\partial r} \cdot r'(t) + \frac{\partial f}{\partial t}$ Now noticing that

$$\frac{\partial f}{\partial r}(r, t) = d \exp_p(r \times v(t)) \cdot v(t)$$

$$\frac{\partial f}{\partial t}(r, t) = d \exp_p(r \times v(t)) \cdot r \times v'(t)$$

Also see that the function $f(., t) : r \mapsto \exp_p(r \times v(t))$ is a geodesic(due to the exp map and the velocity scaled linearly. recall above too) so all its tangent vectors have thet same lenght. At $r = 0$ the tangent vector is $v(t)$ which is 1 by definition of polar decomposition. At $r > 0$ the tangent vector to the curve is $\frac{\partial f}{\partial r}$ so $||\frac{\partial f}{\partial r}|| = 1$. This is also implies that $v(t) \perp v'(t)$ and notice that $r \times v'(t)$ is just a scalar product(r is 1D). Hence $\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle = 0$ since they are orthogonal. Hence from our expression of $c'(t)$ we can get

$$(1) : \left|\frac{dc}{dt}\right|^2 = |r'(t)|^2 + \left|\frac{\partial f}{\partial t}\right|^2 \geq |r'(t)|^2$$

by triangle inequality and so

$$(2) : \int_{\varepsilon}^{1} \left|\frac{dc}{dt}\right| \geq \int_{\varepsilon}^{1} |r'(t)|\, dt \geq \int_{\varepsilon}^{1} r'(t)dt = r(1) - r(\varepsilon)$$

Taking $\varepsilon \to 0$ we obtain $\ell(c) \geq \ell(\gamma)$ because $r(1) = \ell(\gamma)$.It is clear that if the inequality (1) or the second inequality (2) is strict then $\ell(c) > \ell(\gamma)$. If $\ell(c) = \ell(\gamma)$ then $\left|\frac{\partial f}{\partial t}\right| = 0$ that is $v(t)$ =const and $|r'(t)| = r'(t) > 0$. It follows that c is a monotonic reparameterization of γ hence $c([0,1]) = \gamma([0,1])$. If $c([0,1])$ is not contained in B consider the irst point $t_1 \in (0,1)$ for which $c(t_1)$ belongs to the boundary of B. If p is the radius of the geodesic ball B we have

$$\ell(c) \geq \ell_{[0,t_1]}(c) \geq q > \ell(\gamma)$$

□

Theorem 56

Once again M is a differentiable manifold with a symmetric connection. For any $p \in M$ there exists a neighbourhood W of p and a number $\delta > 0$ such that for every $q \in W$, $\exp_q$ is a diffeomorphism on $B_\delta(0) \subset T_qM$ and $\exp_q(B_\delta(0)) \supset W$ that is W is a normal neighborhood of each of its points

Proof. Consider $F(p,v) = (G_1(p,v), G_2(p,v))$ with $G_1(p,v) = p$ and $G_2(p,v) = \exp_p(v)$ Then we have the jacobian matrix $dF_{p,0}$

$$d_pG_1(p,0) = I \text{ and } d_vG_1(p,0) = 0$$

$$d_p \exp_p(0) = I \text{ and } d_v \exp_p(0) = I$$

Hence we have

$$dF_{(p,0)} = \begin{bmatrix} d_pG_1 & d_vG_1 \\ d_p \exp_q & d_v \exp_q \end{bmatrix} = \begin{bmatrix} \frac{\partial G_1}{\partial p} & \frac{\partial G_1}{\partial v} \\ \frac{\partial G_2}{\partial p} & \frac{\partial G_2}{\partial v} \end{bmatrix} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix},$$

Clearly this matrix is invertible(diagonal all identity) so by inverse function theorem the theorem follows.

Remark 57. *Notational matters, subscript on function vs on differential*

$$d\phi_p(v) = \textit{brings tangent vector } v \textit{ to } T_{\phi(p)} \textit{ by default it is with respect to } t$$

$$d_pf(v) = \textit{brings tangent vector } v \textit{ to as specified by } F \textit{ to some } T_q \textit{ is a differential with respect to } p$$

Corollary 58

Once again M is a differentiable manifold with a symmetric connection. If a piecewise differentiable curve $\gamma : [a,b] \to M$ with paramter proportional to arc lenght has lenght less or equal to the lenght of any other piecewise differentiable curve joining $\gamma(a)$ to $\gamma(b)$ then γ is geodesic. In particular γ is regular.

Proof. By the last proposition on local minimizing geodesics 55 then it must have the same lenght as radial geodesic joining these two points. Again by the last proposition, then it must equal that radial geodesic.

5 Curvature

5.1 Tensors on Riemannian Manifolds

Definition 59

A **tensor** of order r on a Riemannian manifold is a multilinear mapping:

$$T : \underbrace{\mathfrak{X}(M) \times \ldots \mathfrak{X}(M)}_{\text{r factors}} \to \mathcal{D}(M)$$

This means that given $Y_1, \ldots, Y_r \in \mathfrak{X}(M), T(Y_1, \ldots, Y_r)$ is a differentiable function on M and that T is linear in each argument. That is

$$T(Y_1, \ldots, fX + gY, \ldots, Y_r) = fT(Y_1, \ldots, X, \ldots, Y_r) + gT(Y_1, \ldots, Y, \ldots, Y_r)$$

for all $X, Y \in \mathfrak{X}(M), f, g \in \mathcal{D}(M)$

6 Complete manifolds: Hopf Rinow and hadamard theorems

7 Variations of Energy

Definition 60

Let $c : [0, a] \to M$ be a piecewise differentiable curve in manifold M. A **variation** of c is a continuous mapping $f : (-\varepsilon, \varepsilon) \times [0, a] \to M$ such that

(a) $f(0, t) = c(t), t \in [0, a]$

(b) there exists a subdvision of $[0, a]$ by points $0 = t_0 < t_1 < \ldots t_{k+1} = a$ such that the restriction of f to each $(-\varepsilon, \varepsilon) \times [t_i, t_{i+1}], i = 0, \ldots, k$ is differentiable(in other words f piecewise differentiable on $[0, a]$)

Definition 61

A variation is said to be **proper** if

$$f(s, 0) = c(0) \quad \text{and} \quad f(s, a) = c(a)$$

for all $s \in (-\varepsilon, \varepsilon)$. If f is differentiable the variation is said to be **differentiable**

Definition 62

For each $s \in (-\varepsilon, \varepsilon)$ the paramterized curve $f_s : [0, a] \to M$ given by $f_s(t) = f(s, t)$ is called a **curve in the variation**. In this way a veriation determines a family $f_s(t)$ of neighbouring curves f $f_0(t) = c(t)$ and a variation is proper if and only if the curves of this family have the same initial point $c(0)$ and the same endpoint $c(a)$.

Definition 63

We call the paramterized differentiable curve given by $f_t(s) = f(s,t)$ with t fixed a **traversal curve of the variation**. The velocity of a traversal curve at $s = 0$ defined by $V(t) = \frac{\partial f}{\partial s}(0,t)$ is a (piece-wise differentiable) vector field along $c(t)$ and is called the **variational field** of f.

Proposition 64

Given a piecewise differentiable field $V(t)$ along a piecewise differentiable curve $c : [0,a] \to M$ there exists a variation $f : (-\varepsilon, \varepsilon) \times [0,a] \to M$ of c such that $V(t)$ is the variational field of f; in addition if $V(0) = V(a) = 0$ it is possible to choose f as a proper variation

Proof. Since $c[0,a]$ is compact it is possibe to find a $\delta > 0$ such that $\exp_{c(t)}, t \in [0,a]$ is well-defined for all $v \in T_{c(t)}M$ with $|v| < \delta$. Now consider each $c(t)$. By 56 we know that by inverse function theorem there exists a totally normal neighbourhood W_t(recall basically a neighbourhood where $\exp_{c(t)}$ is a diffeomorphism) of each $c(t)$ and the number $\delta_t > 0$ governing the size of this neighbourhood. Because $c([0,a])$ compact, since the union of all $W(t)$ of each $c(t)$ is an open cover of it, we know there exists a finite number of W_t that forms a finite subcover of $c([0,a])$. Hence taking $\delta = \min(\delta_1, \ldots, \delta_n)$ where $\delta_i > 0$ we see that any $|v| < \delta$ in $T_{c(t)}M$ will certainly lie in a totally normal neighbourhood where $\exp_{c(t)}$ is defined.

Now consider $N = \max_{t\in[0,a]} |V(t)|, \varepsilon < \frac{\delta}{N}$ which is possible continuous differentiable function on a compact set is bounded. Also define

$$f(s,t) = \exp_{c(t)}(sV(t)), s \in (-\varepsilon, \varepsilon), t \in [0,a]$$

By choice of ε we know this function is well defined because it ensures $|sV(t)| < \delta$. Notice that

$$\exp_{c(t)} sV(t) = \gamma(1, c(t), sV(t)) = \gamma(s, c(t), V(t))$$

Now as γ is a geodesic which we know is a solution to a system of ODE solving $\frac{DV}{dt} = 0$ so you know by theory of ODEs it is continuous differentiable with respect to initial data i.e $\gamma(0,t) = \exp_{c(t)}(0) = c(t)$ is $C^k, k > 0$. Therefore this implies piecewise differentiable 51. Also see that

$$\frac{\partial f}{\partial s}(0,1) = \frac{d}{ds}(\exp_{c(t)} sV(t))|_{s=0} = (d\exp_{c(t)})_0 V(t) = V(t)$$

as desired. Finally see that if $V(0) = V(a) = 0$

$$\frac{\partial f}{\partial s}(0,0) = V(0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial s}(0,a) = V(a) = 0$$

recall that by definition $f(0,t) = c(t)$. Since $\frac{\partial f}{\partial s}(0,a) = \frac{\partial c(a)}{\partial s} = \frac{\partial f}{\partial s}(0,0) = \frac{\partial c(0)}{\partial s} = 0$ it follows that

$$f(s,0) = c(0) \quad \text{and} \quad f(s,a) = c(a)$$

for all s so f is proper ∎

Example 65

Consider the following variational field on a riemann manifold

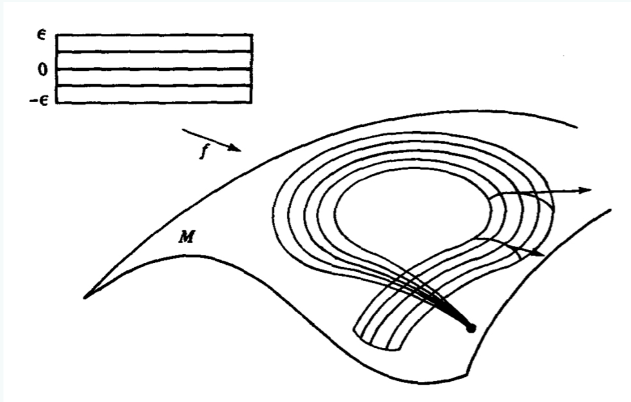

Figure 4: Variational field

See that at the end points, the change in variational fied with respect to $s \in (-\varepsilon, \varepsilon)$ is zero so the curves in the variation $f_s(t)$ for each s converge at the end points.

To compare the arc lenght of c with the arc lenght of neighboring curves in a variation $f : (-\varepsilon, \varepsilon) \times [0, a] \to M$ of c we define a function

$$L(s) = \int_0^a \left| \frac{\partial f}{\partial t} \right| dt, \quad s \in (-\varepsilon, \varepsilon)$$

That is $L(s)$ is the lenght of the curve $f_s(t)$. It will be convenient to work with what we call the **energy function** $E(s)$ given by

$$E(s) = \int_0^a \left| \frac{\partial f}{\partial t}(s, t) \right|^2 dt, \quad s \in (-\varepsilon, \varepsilon)$$

Fact 66

Let $c : [0, a] \to M$ be a curve and let

$$L(c) = \int_0^a \left| \frac{dc}{dt} \right| dt \quad \text{and} \quad E(c) = \int_0^a \left| \frac{dc}{dt} \right|^2 dt$$

Putting $f = 1$ and $g = \left| \frac{dc}{dt} \right|$ in the *schawarz inquality*:

$$\left(\int_0^a fg dt \right)^2 \leq \int_0^a f^2 dt \cdot \int_0^a g^2 dt$$

we then obtain

$$L(c)^2 \leq aE(c)$$

where clearly we get equality if and only if g is a constant.Since in this case we get an integral of 2 constants. The second inequality follows because γ is the minimal geodesic which by definition has the minimal arc lenghth. The last inquality also follows from the previous fact.

Lemma 67
Let $p, q \in M$ and let $\gamma : [0, a] \to M$ be a minimizing geodesic joining to p to q. Then for all curves $c : [0, a] \to M$ joining p to q

$$E(\gamma) \leq E(c)$$

with equality holding if and only if c is a minimizing geodesic

Proof. Consider

$$aE(\gamma) = (L(\gamma))^2 \leq (L(c))^2 \leq aE(c)$$

where first equality follows from

$$\frac{d}{dt}|\gamma'| = \frac{d}{dt}\langle\gamma', \gamma'\rangle = \langle\frac{D\gamma}{dt}, \gamma'\rangle + \langle\gamma', \frac{D\gamma}{dt}\rangle = 0$$

So according to the previous fact, $g = |\gamma'|$ =constant. Hence if $E(\gamma) = E(c)$ then $L(c) = L(\gamma)$ which implies c is the minimizing geodesic by 58

Proposition 68 (First Variation of the energy of a curve)
Let $c : [0, a] \to M$ be a piecewise differentiable curve and let $f : (-\varepsilon, \varepsilon) \times [0, a] \to M$ be a variation of c. If $E : (-\varepsilon, \varepsilon) \to \mathbb{R}$ ise a the energy of f then

$$\frac{1}{2}E'(0) = \int_0^a \langle V(t), \frac{D}{dt}\frac{dc}{dt}\rangle dt - \sum_{i=1}^{k}\langle v(t_i), \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-)\rangle - \langle V(0), \frac{dc}{dt}(0)\rangle + \langle V(a), \frac{dc}{dt}(a)\rangle$$

where $V(t)$ is the variational field of f and

$$\frac{dc}{dt}(t_i^+) = \lim_{t \to t_i^+}\frac{dc}{dt} \quad \frac{dc}{dt}(t_i^-) = \lim_{t \to t_i^-}\frac{dc}{dt}$$

Proof. By definition

$$E(s) = \int_0^a \langle\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle dt = \sum_{i=0}^{k}$$

8 Appendix: Tensor Analysis

Material in this section is referenced from `Manifolds, Tensor Analysis and applications 2nd Edition` by Abraham,Marsden,Ratiu

Consider a banach space E and its dual E^*(recall the set of all $\mathcal{L}(E)$ maps that is $\ell : V \to \mathbb{R}$). Then for any $v \in E$ and $a \in E^*$ we have

$$v = \sum_{i=1}^{n}\langle e^i, v\rangle e_i \quad \text{and} \sum_{i=1}^{n} \quad a = \langle a, e_i\rangle e^i$$

then employing **summation convention** we drop the summation symbol and let an index on the upper and lower levels imply it instead.

$$v = \langle e^i, v\rangle e_i \quad \text{and} \quad a = \langle a, e_i\rangle e^i$$

Definition 69

For a vector space E we put

$$T^r_s(E) = L^{r+s}(\underbrace{E^*, \ldots, E^*}_{\text{r copies}}, \underbrace{E, \ldots, E}_{\text{s copies}}; \mathbb{R})$$

Elements of $T^r_s(E)$ are called **tensors** on E, **contravariant** of order r and **covariant** of order s. Or simply, **type** (r, s).

Give $t_1 \in T^{r_1}_{s_1}$ and $t_2 \in T^{r_2}_{s_2}$ we have the **tensor product** $t_1 \otimes t_2 \in T^{r_1+r_2}_{s_1+s_1}$

$$\begin{aligned}(t_1, \times, t_2)&(\beta^1, \ldots, \beta^{r_1}, \gamma^1, \ldots, \gamma^{r_2}, f_1, \ldots, f_{s_1}, g_1, \ldots, g_{s_2}) \\ &= t_1(\beta^1, \ldots, \beta^{r_1}, f_1, \ldots, f_{s_1})t_2(\gamma^1, \ldots, \gamma^{r_2}, g_1, \ldots, g_{s_2})\end{aligned}$$

Where $\beta^j, \gamma^j \in E^*$ and $f_j, g_j \in E$

Example 70

Note these special cases

$$T^1_0(E) = E, \quad T^0_1(E) = E^*, \quad T^0_2(E) = L(E; E^*), \quad T^1_1(E) = L(E; E), \quad T^0_0(E; F) = F$$

Proposition 71

Let E be an n-dimensional vector space. If $\{e_1, \ldots, e_n\}$ is a basis of E and $\{e^1, \ldots, e^n\}$ is a basis of E^* then

$$\{e_{i_1} \otimes \ldots \otimes e_{e_{i_r}} \otimes e^{j_1} \otimes \ldots \otimes e^{j_s} | i_1, \ldots, i_r, j_1, \ldots, j_s = 1, \ldots, n\}$$

is a basis of $T^r_s(E)$ and thus $\dim(T^r_s(E)) = n^{r+s}$

Definition 72 (Interior product)

The **interior product** of a vector $v \in E$ with a tensor $t \in T^r_s(E; F)$ is the $(r, s-1)$ type F-valued tensor defined by

$$(\mathbf{i}_v t)(\beta^1, \ldots, \beta^r, v_1 \ldots, v_{s-1}) = t(\beta^1, \ldots, \beta^r, v, v_1, \ldots, v_{s-1})$$

The **interior product** of a vector $\beta \in E^*$ with a tensor $t \in T^r_s(E; F)$ is the $(r-1, s)$ type F-valued tensor defined by

$$(\mathbf{i}^\beta t)(\beta^1, \ldots, \beta^r, v_1 \ldots, v_{s-1}) = t(\beta, \beta^1, \ldots, \beta^{r-1}, v_1, \ldots, v_{s-1})$$

Clearly $\mathbf{i}_v : T^r_s(E; F) \to T^r_{s-1}(E; F)$ and $\mathbf{i}^\beta : T^r_s(E; F) \to T^{r-1}_s(E; F)$ are linear continuous maps as are $v \mapsto \mathbf{i}_v$ and $\beta \mapsto \mathbf{i}^\beta$.

To see this consider that $\mathbf{i}_v$ inserted a fixed v into one of the slots, effective reducing the available s slots originally by 1

$$t(\beta^1, \ldots, \beta^r, v, v_1, \ldots, v_{s-1}) \in T^r_{s-1}(E; F) \quad \text{vs} \quad t(\beta^1, \ldots, \beta^r, v_s, v_1, \ldots, v_{s-1}) \in T^r_s(E; F)$$

Likewise $\mathbf{i}^\beta$ inserted a fixed β into one of the slots, effective reducing the available r slots originally by 1

$$t(\beta, \beta^1, \ldots, \beta^{r-1}, v_1, \ldots, v_{s-1}) \in T^{r-1}_s(E; F) \quad \text{vs} \quad t(\beta_r, \beta^1, \ldots, \beta^{r-1}, v_1, \ldots, v_{s-1}) \in T^r_s(E; F)$$

Hence we have

$$\mathbf{i}_{e_k}(e_{i_1}\otimes\ldots\otimes e_{i_r}\otimes e^{j_1}\otimes\ldots\otimes e^{j_s}) = \delta_k^{j_1} e_{i_1}\otimes\ldots\otimes e_{i_r}\otimes e^{j_2}\otimes\ldots\otimes e^{j_s}$$

and

$$\mathbf{i}^{e_k}(e_{i_1}\otimes\ldots\otimes e_{i_r}\otimes e^{j_1}\otimes\ldots\otimes e^{j_s}) = \delta_{i_1}^{k} e_{i_2}\otimes\ldots\otimes e_{i_r}\otimes e^{j_1}\otimes\ldots\otimes e^{j_s}$$

Definition 73

Let $\dim(E) = n$. The contraction of the kth contravariant with the ℓth covariant index or short the (k,ℓ) **contraction** is the family of linear maps $C_\ell^k : T_s^r(E;F) \to T_{s-1}^{r-1}(E)$ defined for any pair of natural numers $r, s \geq 1$ by

$$C_\ell^k(T^{i_1,\ldots,i_s}_{j_1,\ldots j_r} e_{i_1}\otimes\ldots\otimes e_{i_s}\otimes e^{j_1}\otimes\ldots\otimes e^{j_r})$$
$$= T^{i_1,\ldots,i_{k-1},p,i_{k+1},\ldots,i_s}_{j_1,\ldots,j_{\ell-1},p,j_{\ell+1},\ldots,j_r} e_{i_1}\otimes,\ldots,\widehat{e}_{i_k},\ldots\otimes e_{i_s}\otimes e^{j_1}\otimes\ldots,\widehat{e}^{j_\ell},\ldots,\otimes e^{j_r}$$

Definition 74

Suppose E is a finite dimensional real inner product space with basis $\{e_1,\ldots,e_n\}$ and the corresponding dual basis $\{e^1,\ldots,e^n\}$ in E^*. Using the inner product with matrix denoted by $[g_{ij}]$ so $g_{ij} = \langle\langle e_i, e_j\rangle\rangle$ we get the isomorphism

$$^\flat : E \to E^* \text{given by } x \mapsto \langle\langle x, .\rangle\rangle \quad \text{and its inverse} \quad ^\sharp : E^* \to E$$

The matrix of $\flat$ is $[g_{ij}]$ that is

$$(x^\flat)_i = g_{ij}x^j$$

he matrix of $\flat$ is $[g_{ij}]$ that is

$$(a^\sharp)^i = g^{ij}a_j$$

where x^j and a_j are components of e and a respectively. We call $^\flat$ the **index lowering operator** and $^\sharp$ the **index raising operator**

This makes sense, afterall if you recall that there is a unique correspondence between dual bases and its corresponding vector space. Explcitly you could see each "dot product field" $\langle x_i, .\rangle$(see that if it dots other bases other than itself it will be zero) as a unique correspondence between a base and dual base.

Example 75

$$g^{jk}g_{ik} = \underbrace{g^{ik}g_{ki}}_{\text{only true if symmetric}} = \delta_i^j$$

Example 76

Order matters

$$t_i^j = g^{jk}t_{ik}$$

compared with

$$t_i^j = g^{ij}t_{ki}$$

only equivalent if symmetric