Real Analysis Workbook

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Selected theorems in Rudin's Principles of Real Analysis [1]

Contents

1 Real and Complex Number Systems

Theorem 1 (Archimedean and Density Property of Real Numbers)
if $x, y \in \mathbb{R}$

(a) then there exists a positive integer n such that

$$nx > y$$

(b) then there exists $p \in \mathbb{Q}$ such that

$$x < p < y$$

Proof. use well ordering principle...

Proposition 2

Prove

$$\sup_{y \in Y} \left(\inf_{x \in X} f(x, y) \right) \leq \inf_{x \in X} \left(\sup_{y \in Y} f(x, y) \right)$$

Proof. By definition of suprenum we have

$$\sup_{y \in Y} \left(\inf_{x \in X} f(x, y) \right) - \epsilon < \inf_{x \in X} f(x, y^*)$$

But we also know that for some $y^* \in Y$

$$\inf_{x \in X} f(x, y^*) \leq \inf_{x \in X} \left(\sup_{y \in Y} f(x, y) \right),$$

because $f(x, y^*) \leq \sup_{y \in Y} f(x, y)$ for every x.

Combining the above 2 relations gives

$$\sup_{y \in Y} \left(\inf_{x \in X} f(x, y) \right) - \epsilon < \inf_{x \in X} \left(\sup_{y \in Y} f(x, y) \right).$$

and because this is true for any $\varepsilon > 0$ we have

$$\sup_{y \in Y} \left(\inf_{x \in X} f(x, y) \right) \leq \inf_{x \in X} \left(\sup_{y \in Y} f(x, y) \right).$$

2 Basic Topology

Definition 3 (Metric Space)

A set X that is equipped with a **metric** $d : X \times X \to [0, \infty)$

1. (Identification) $d(x, y) = 0$ if and only if $x = y$
2. (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
3. (Triangle Inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Remark 4. *A subset of a metric space X is a metric space. It is clear that if the above properties hold for $x, y, z \in X$, they will hold in the subset of X*

Example 5

$\mathbb{R}$, R^k are metric spaces.

Definition 6 (Open Relative)

A **neighbourhood/open ball** of point p in metric space X in refers to the set of points in X within radial distance r from point p.

Set E open subset of metric space X is another way of saying E is **open relative** to X. That is for *all* $p \in E$ should there exist any $q \in X$ that can satisfy the following for a given r

$$d(p,q) < r$$

it can only possibly be from E. That is to say all points in E are **interior points** of E in metric space X as required by the definition of **open sets**

Proposition 7

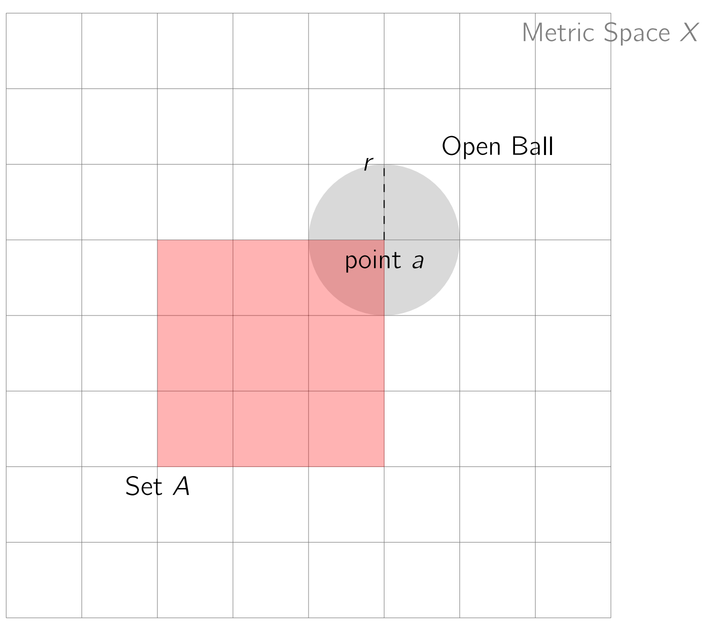
every neighbourhood is an open set

Proof. Consider $d(p,q) < r$, a neighbourhood of p with radius r. For any possible point p in this this neighbourhood there must exist some h for which $d(p,q) < h \leq r$ is satisfied. This certainly is possible by Archimedean property of real numbers. Then by definition every possible point in the neighbourhood is contained in a neighbourhood that is a subset of the parent set, that is the original neighbourhood.

Proposition 8

Every metric space X is an open and closed subset of itself.

Proof. Consider a any set A which is fundamentally just a collection of random points. Then clearly every point in A an interior point in metric space A since no matter which you point of A you define your open ball at, or even how big r is, we can only consider the overlap between the ball and set A since we working in metric space A only. We cannot include/consider any point not in metric space A .Thus every point in A is an interior point of A in metric space A



Recall that set A is closed in metric space X if X/A is open. So equivalently in metric space A we have set A is closed in metric space A if A/A is open. The complement of A in metric space A is the empty set. Thus the set of elements in A but not in the empty set is just A which we have previously proven to be an open subset in metric space A. Hence every metric space A is an open and closed subset of itself.

Remark 9. *However this is not the case in metric space X. Assume now that A is closed in metric space X. Boundary point $a \in A$ is not an interior point of A in the metric space X. There is no $r > 0$ such that the open ball does not contain points in A exclusively in the metric space X. Since in the metric space X, open balls defined are no longer constrained to just consider points of A but instead in X in general.*

Corollary 10

Every metric space X must contain an open or closed subset

Proof. Recall that all neighbourhoods are open . As seen in the above construction, for any point anywhere in X we can define a neighbourhood of it that is a subset of X in metric space X. Moreover X itself can also be possible open subset of metric space X. For closed it is either X itself or the complement of the open balls we defined earlier.

Theorem 11

Let $\{E_a\}$ be a finite or infinite collection of sets E_a

$$(\bigcup_a E_a)^c = \bigcap_a (E_a)^c \tag{1}$$

$$\bigcup_a (E_a)^c = (\bigcap_a E_a)^c \tag{2}$$

Proof. let A and B be the left and right sets in 1. For $x \in A$ it must be a member of all E^c if it is not a member of all E^a. So clearly $A \subseteq B$. Vice versa for $x \in B$ so $B \subseteq A$. Thus $A = B$. The logic is analogous for 2. Simply let $E_b = (E_a)^c$ then take the complement of both sides which will get you an equation in the same form as 1

Lemma 12

$E \subset \mathcal{O}$ if and only $\mathcal{O}^c \subset E^c$

Proof. Notice the following propositions are contrapostives of each other and thus equivalent.

1. Proposition $E \subset \mathcal{O}$ means: for all x in E then x in $\mathcal{O}$
2. Proposition $\mathcal{O}^c \subset E^c$ means: for all x in $\mathcal{O}^c$ then x in E^c

Theorem 13

A set E is open if and only E^c is closed

Proof. Suppose E^c is closed then points not in E^c cannot be a limit point of E^c. Thus there exist a neighbourhood of these points such that they are not in E^c. So they are interior points of E. Next suppose E is open. Since all points on E are interior points there exists a neighbourhood of these points contained within E and thus not containing points in E^c. Then clearly none of the limit points of E^c can be in E, they can only be on E^c itself.

Theorem 14

Let E be a non-empty set of real numbers bounded above. Let y=sup E. Then $y \in \overline{E}$. Hence y in E is it is closed.

Proof. By definition of lowest upper bound, there exists $y - h < x < y$ for every $h > 0$ otherwise $y - h$ will be an upper bound. Thus this makes y a limit point of E.

2.1 Compact Sets

Fact 15

Every subset of a metric space has an open cover, the metric space X itself is one such trivial example see 8. This is why we say *every* open cover must exist a subset subcover to be compact because you can just take X itself and claim it is a finite cover.

Definition 16

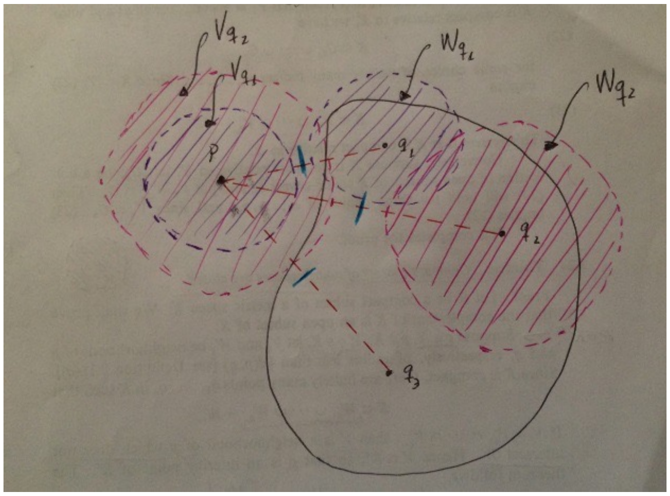
A subset K of a metric space X is **compact** if every open cover of K contains a *finite* subcover. That is if $\{G_a\}$ is open cover of K then there must be a finite number of indices $\{a_1, \ldots a_n\}$ such that

$$K \subset G_{a_1} \ldots G_{a_n}$$

Theorem 17

Compact subsets of metric spaces are closed

Proof. We prove this by proving that its complement is open. Let K be a compact subset of metric space X. Then there exists a finite covering of open subsets such that $K \subseteq W_{q1} \cup W_{q2} \cup W_{q3} \ldots W_{qn}$. We now prove that every point in its complement has a neighbourhood contained within the complement, making them all interior points. Let p be such a point in the complement. Let V_{qi} be the neighbourhood of p such that both V_{qi} and W_{qi} have a radius of less than $\frac{1}{2}d(p, q_i)$, meaning a size such that both V_{qi} and W_{qi} don't intersect. Then p will be contained in a neighbourhood of $V_{q1} \cap V_{q2} \cap V_{q3} \ldots V_{qn}$ in the complement.



Now because we can construct such a neighborhood for any $p \in K^c$ we conclude that K^c must be open.

Theorem 18

closed subspaces of compact sets are compact

Proof. Suppose $F \subset K \subset X$ where X is a metric space and K is a compact set. Let V_a be an open cover of F. Suppose F is closed then its complement is open. Thus V_a adjoined to F^c is an open cover of K. Since K is compact it means V_a must also be finite as by definition, every open cover must have a finite cover to be considered compact.

Theorem 19

If $\{K_\alpha\}$ is a collection of compact subsets and every finite intersection of a finite sub-collection of $\{K_\alpha\}$ is nonempty then $\bigcap K_\alpha$ is nonempty

Proof. Proof by contradiction: Suppose (event P) every finite intersection of a finite sub-collection of $\{K_\alpha\}$ is nonempty and $\{K_\alpha\}$ is a collection of compact subsets (event $\overline{Q}$) and $\bigcap K_\alpha$ is empty. Then there exists K_1 such that it has no elements that belongs to the intersection of all other sets.

$$K_1 \not\subset (K_{a_1} \cap \dots K_{a_n})$$

This implies, $K_1 \subset (K_{a_1} \cap \dots K_{a_n})^c$ which is equivalent to

$$K_1 \subset (K_{a_1}^c \cup \dots K_{a_n}^c)$$

Note that since compact each K_a is closed so its complement is open. Clearly $(K_{a_1} \cap \dots K_{a_n})^c$ must be a finite open subcover because K_1 is compact. Hence the corresponding $(K_{a_1} \cap \dots K_{a_n})$ is finite. So we can construct a intersection of a finite sub collection $K_1 \cap K_{a_1} \cap \dots K_{a_n}$ that is empty which is a contradiction.

Remark 20. *Note we used compactness to be assured of a finite subcover in our construction. However an astute reader may question if it is possible to contruct in another way a similar finite sub collection without using compact sets. There is no obvious evidence about this in . Turns out* compactness is essential *as we shall see.(not required to prove the above theorem though)*

Lemma 21

If $\bigcap K_\alpha$ is nonempty when every finite intersection of a finite sub-collection of $\{K_\alpha\}$ is nonempty, then $\{K_\alpha\}$ is a collection of compact subsets

Proof. Prove by contradiction: suppose (event P): $\bigcap K_\alpha$ is nonempty when every finite intersection of a finite sub-collection of $\{K_\alpha\}$ is nonempty and (event $\overline{Q}$): $\{K_\alpha\}$ is a collection of non-compact sets.Consider by cases of non-compact sets:

1. an open subset $K_n = \{(0, 1/n) | n \in \mathbb{N}^+\}$

   Clearly every finite intersection is non-empty. Simply pick the largest n in the intersection and $\frac{1}{n+1}$ is definitely inside. However the intersection over all is empty since the existence of x in $0 < x < \frac{1}{n}$ as $n \to \infty$ violates **Archimedian Property**. Since this implies there exist $\exists x > 0, \forall n \in \mathbb{Z}^+$ such that $nx < 1$ which is a clear contradiction. Alternatively, the limit of $\lim_{n \to \infty}(\frac{1}{n}) = 0$ also by Archimedian property.

2. an unbounded subset $K_n = \{[n, \infty) | n \in \mathbb{N}^+\}$

   Again the every finite intersection is non-empty. Simply pick the biggest n. However the intersection over all is empty since if it did, there exists a real number m such that $m > n, \forall n$ which again violates Archimedian property.

Theorem 22
if E is an infinite subset of compact set K, then E has a limit point in K

Proof. Let us prove this by the contra-positive. Suppose E has no limit points in K. Then for every point in K there exists a neighbourhood that does not contain points in E where these neighbourhoods form an open cover of K. Hence at best, these neighbourhoods only contain 1 point if they were in E. Given that E is a infinite, then clearly no finite collection of such neighbourhoods can cover E and consequently K too since E is a subset of K.

Theorem 23
If I_n is a sequence of intervals in R^1 such that $I_n \subset I_{n+1}$ then $\bigcap_1^\infty I_n$ is not empty

Proof. Let a_n and b_n be the lower and upper bounds of each interval I_n respectively. Then clearly $a_n \leq b_n$ fpr each interval. Thus $\sup a_n \leq \inf b_n$ given that $I_n \subset I_{n+1}$. This corresponds to the smallest interval that is contained in all intervals. Since E is non empty and bounded above by b_n, clearly there exists $x \in \{\sup a_n \leq x \leq \inf b_n\}$ where $x = \sup a_n$.

Theorem 24
every K-cell which are defined to be all points $\mathbf{x} = (x_1, \ldots, x_K)$ such that $a_j \leq x_j \leq b_j (1 \leq j \leq k)$ i.e bounded intervals of $\mathbf{x} \in \mathbb{R}^k$ is compact.

Proof. Suppose to get a contradiction that there exist an open cover $\{G_a\}$ of I with no finite sub-cover of I. This means if we for some sub-interval say $I \supset I_1 \supset I_2 \supset I_3 \ldots I_n$ that is obtained by recusively doing splitting each k cell into 2 by

$$[a_j, c_j] \quad \text{and} \quad [c_j, b_j]$$

where $c_j = (a_j + b_j)/2$ so the sizes of each such interval follow $2^{-n}\delta$ where

$$\delta = \left\{ \sum^k (b_j - a_j)^2 \right\}^{1/2} \qquad \text{L2 norm size of the original k cell essentially}$$

there must be at exist at least one I_n that has no finite sub-cover. However by 23, I_n cannot be empty. Let x^* be a point in all intervals. Define some open neighbourhood G_a of size r such that it is centered at x^*. So given x^*

$$|x^* - y| < r, \quad \forall y \in G_a$$

Let the size of $I_n < \epsilon$. Thus if $\epsilon < r$. Then there exists an open cover of radius r that covers I_n. By Archimedian property of real numbers for every $\epsilon > 0$, there must exist an n that satisfies this, that is

$$2^{-n}\delta < r$$

contradicting our hypothesis. Thus every k-cell can be covered by a finite sub cover and is thus compact.

Corollary 25
Every bounded set in R^k is compact.

Theorem 26

If a set in R^k has one of the following properties, it has the other 2

(a) E is closed and bounded

(b) E is compact

(c) Every infinite subset of E has a limit point in E

Proof. It is clear that (a) implies (b) since every k-cell is compact. Then (b) implies (c) by 22. Now it remains to show that (c) implies (a) so all 3 imply each other bidirectionally(circular proof among the 3). So now suppose (c) holds.

- (c) implies E is bounded

  Prove by contradiction: If unbounded there exists a sub-sequence X_n in E that does not converge and hence has no limit point as we will learn later

- (c) implies E is closed

  Prove by contradiction: If not closed, there exists a limit point of E not in E which we define as x_0. Then there exists a sub-sequence $\{X_n\}$ in E defined by $|x_n - x_0| < \epsilon$ that converges to x_0 as we shall learn later. We will learn later that limits of sub sequences are unique so this sub-sequence will have no limit points in E

2.2 Perfect Sets

Theorem 27

Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable

Proof. see rudin...

Definition 28

Two subsets A and B of a metric space X is **separated** if both $A \cap \overline{B}$ and $\overline{B} \cap A$ is empty. Where $\overline{A}$ refers to the closure of A and likewise for B. A set $E \subset X$ is said to be **connected** if E is not a union of two non-empty separated sets.

Theorem 29

Separated sets are disjoint but disjoint sets may not be separated

Proof. By definition the former holds since

$$A \cap B \subset A \cap \overline{B} = \emptyset$$

and

$$A \cap B \subset \overline{B} \cap A = \emptyset$$

For the latter consider the intervals $[0, 1]$ and $(1, 2)$. While they are disjoint, they are not separated as 1 is a limit point of $(1, 2)$.

3 Numerical Sequences and Series

Theorem 30

Let $\{p_n\}$ be a sequence in metric space X

(a) limits of sequences are unique

(b) E is a subset of metric space X and if p is a limit point of E, then there is a sequence in E such that

$$p = \lim_{n\to\infty} p_n$$

Proof. To prove (a), let $d(p_n, q) < \epsilon/2$ and $d(p_n, q') < \epsilon/2$ for $n > N$ and $n > M$ respectively. Let $n \geq \max(N, M)$ then $d(q, q') \leq d(p_n, q') + d(p_n, q') \leq \epsilon$. The only solution is $d(q, q') = 0$ to satisfy $\forall\epsilon > 0$ thus $q = q'$. To prove (b) simply define $d(p_n, q) < \frac{1}{n}$ which will always exist we suppose q is a limit point of E. There certainly exist $n \geq N$ such that $\frac{1}{n} \leq \epsilon$

Theorem 31

$\{p_n\}$ converges to p if and only every subsequence converges to p.

Proof. Let $\{p_{n_k}\}$ be a subsequence of $\{p_n\}$. Let $m \geq N$ such that $\{p_n\}$ converges. By definition of subsequence $n_m \geq m$ so $\{p_{n_k}\}$ certainly converges too

Theorem 32 (Equivalent definitions of compact using sequences)

Let $\{p_n\}$ be a sequence in compact metric space X

(a) a subsequence of $\{p_n\}$ converges to a point of X

(b) Every bounded sequence in R^k contains a convergent sequence

Proof. Let $E \subset X$ be the range of $\{p_n\}$. If E is infinite, there exists a limit point of this subset in X by 22 Thus by 30 there is a subsequence of E that converges to it.Namely we can do

$$d(p_{n_i}, q) < \frac{1}{n_i}$$

which definitely is possible as q is a limit point.

To prove (b),we know that the range of this bounded sequence in R^k is compact by 24. Thus (a) follows

Theorem 33

Let $\{p_n\}$ be a sequence in metric space X All sub-sequential limits of sequence $\{p_n\}$ form a closed subset of X

Theorem 34 (Squeeze theorem)

Suppose

$$a_n \leq x_n \leq b_n$$

and $\lim a_n = \lim b_n = x$ Then $\lim x_n = x$

Proof. Consider $n \geq \max(M_1, M_2)$ corresponding to $\{a_n\}$ and $\{b_n\}$ such that $|a_n - x| < \varepsilon$ and $|b_n - x| < \varepsilon$

$$x - \varepsilon < a_n \leq x_n \leq b_n < x + \varepsilon$$

which implies $|x_n - x| < \varepsilon$ too

Definition 35 (Cauchy)
A sequence $\{P_n\}$ is a *metric space* X is Cauchy if for every $\varepsilon > 0$ there exists integer N where

$$d(p_n, p_m) < \varepsilon$$

if $n, m \geq N$

Definition 36 (Diameter)
Let E be a nonempty subset of metric space X. Then the **diameter** is defined by

$$\text{diam}(E) = \sup d(p, q)$$

for $p, q \in E$

Proposition 37
$\{P_n\}$ is a cauchy sequence if and only

$$\lim_{N \to \infty} E_N = 0$$

where E_N consists of elements of $\{P_n\}$ where $n \geq N$

Proof. $\lim_{N \to \infty} E_N = 0$ implies for every $\varepsilon > 0$ there exist K

$$\text{diam}(E_N) < \varepsilon$$

for all $N \geq K$. Then by definition of diam,

$$d(p_n, p_m) < \varepsilon$$

for all $n, m \geq N \geq K$. To prove from the other direction we have by definition of cauchy sequence, for every ε there exists N where

$$d(p_n, p_m) < \varepsilon$$

for all $n, m \geq N$ Thus $\sup_{n,m>N} d(p_n, p_m) < \varepsilon$ so

$$\text{diam}(E_N) < \varepsilon$$

. Now note that

$$E_{N+1} \subset E_N \Rightarrow \text{diam}(E_{N+1}) \leq \text{diam}(E_N) < \varepsilon$$

Thus by induction we have $\text{diam}(E_k) < \varepsilon, \forall k \geq N$

Lemma 38

Consider the following subsets in metric space X

(a) if $\overline{E}$ is the closure of E then

$$diam(\overline{E}) = diam(E)$$

(b) if $\{K_n\}$ is a sequence of compact sets that decreases to K and $\lim_{n\to\infty} diam(K_n) = 0$ then K contains only 1 point

Proof. (a) can be proven from the definitions given above. (b) can be proven by considering 19 that K is non-empty. But if K contains more than one point, we have $\text{diam}(K_n) \geq \text{diam}(K) > 0$ for all n. Then we have

$$\lim_{n\to\infty} \text{diam}(K_n) \geq \liminf \text{diam}(K_n) > 0$$

which is a contradiction

Theorem 39

Cauchy Sequences satisfy the following:

(a) In any metric space X , every convergent sequence is Cauchy

(b) If X is a compact metric space and if $\{P_n\}$ is a Cauchy Sequence in X then $\{P_n\}$ converges to some point in X

(c) In $\mathbb{R}^k$ every Cauchy sequence converges to a point in $\mathbb{R}^k$

Proof. for (a) apply triangle inequality and use the definition of limits of sequences. For (b) consider like before E_N which consists of elements of $\{P_n\}$ where $n \geq N$. Now take its closure $\overline{E}_N$ since closed subsets of compacts sets are compact. We know that

$$\overline{E}_{N+1} \subset \overline{E}_N$$

since

$$E_{n+1} \subset E_n \subset \overline{E}_n \quad \text{and} \quad E'_{n+1} \subset E'_n \subset \overline{E}_n \quad \Rightarrow \quad E_{n+1} \cup E'_{n+1} = \overline{E}_{n+1} \subset \overline{E}_n$$

as well as

$$\lim_{n\to\infty} \text{diam}(\overline{E}_n) = 0$$

since $\text{diam}(\overline{E}_n) = \text{diam}(E_n)$ for all n

Now let $\overline{E} = \bigcap_{N=1}^{\infty} \overline{E}_N$. From the above theorem we know $\overline{E}$ contains exactly 1 element which we denote as p that is found in $\overline{E}_N$ and thus E_N for all N. Hence by definition of cauchy sequence, for every $\varepsilon > 0$ there exists integer N where

$$d(p_n, p_m) < \varepsilon$$

if $n, m \geq N$ Then for no matter what N we get for each ε, we can replace p_m with p such that

$$d(p_n, p) < \varepsilon$$

if $n \geq N$ which implies convergence of $\{P_n\}$ since p is unique. For (c) the proof is the exact same just that we are not restricted to compact spaces. So taking closure is not enough. We need to prove that cauchy sequences in R^k are bounded as well to ensure compactness and reuse (b) which turns out they are. See `rudin` for proof.

Remark 40. *You can clearly see the motivation of the proof (b) and its lemmas by considering how its reverse engineered from the definition of cauchy sequences and then having to make sure this p is unique etc.*

Definition 41 (Complete)
A metric space is complete if every Cauchy Sequence converges in it

We immediately see from 3(c) that $\mathbb{R}^k$ is complete so sets like $\mathbb{C}$ which is basically a $\mathbb{R}^2$ is complete. It also clearly shows that whether or not a cauchy sequence and convergent sequence implies each other bidirectionally will depend on whether the metric space is complete. If not only 3(a) is true.

Example 42
The metric space of rational numbers is not complete since there exists a sequence of rational numbers that converges, hence it is cauchy.

$$e = \sum_{n=1}^{\infty} \frac{1}{n!}$$

However it converges to a point outside the set of rational numbers since e is irrational but the not the sequence of partial sums $\{\sum_{n=1}^{N} \frac{1}{n!}\}$.

3.1 basic series operations

Fact 43
It can be proven by induction that

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}$$

Let $b = 1$ to get geometric series

Fact 44
Multiplication of series

$$\sum_n a_n z^n \sum_n b_n z^n = (a_0 + a_1 z^1 \ldots)(a_0 + a_1 z^1 \ldots) = \sum_n c_n z^n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

To see this, since the subscript on each coefficient a_i, b_i is associated with the power of z it multiplies: z^i within each bracket. Then simply consider that c_n is summing all pairs of coefficients that contribute to z^n in the full expansion.

3.2 limit superior and inferior

Theorem 45

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of all subsequential limits of $\{S_n\}$ and $s^* = \limsup s_n$ and $s_* = \liminf s_n$. Then

(a) $s^* \in E$

(b) If $x > s^*$ there exists an integer N such that $n \geq N$ implies $s_n < x$

Proof. For (a), consider the 3 possible values of s^*. If $s^* = +\infty$ then E is not bounded above so there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \to +\infty$. If s^* is real then E is bounded above, at least one subsequential limit exists. If $s^* = -\infty$, then E contains only one element, that is $-\infty$ because s^* is the value of the largest. subsequential limit. In all cases $s^* \in E$. For (b), the proposition means there is only a finite number of $s_n \leq x$. Therefore there exists N such that $n \geq N$ we will have $s_n < x$. To be clear this proposition says nothing about when $n < N$. In that case, s_n could be bigger or smaller than x, we don't care. So lets prove by contradiction that there exists infinite elements of $\{s_n\}$ where $s_n \geq x$. Then we can find some infinite subseqence s_{n_k} whose elements are all $\geq x$. Suppose s_{n_k} is bounded above so we have

$$x \leq \lim s_{n_k} = \limsup s_{n_k} \leq \limsup s_n = s^*$$

but this is a contradiction to our assumption that $x \geq s^*$. Now if s_{n_k} is not bounded above, then $s^* = +\infty$. But ∞ cant be lesser than x so again contradiction.

Corollary 46

We also have $s_* \in E$ and if $x < s_*$ then there exists integer N such that $n \geq N$ implies $s_n < x$

Proof. The proof is similar to above. For (a), if $s_* = -\infty$ E not bounded below, so there exists $s_{n_k} \to -\infty$. If s_* then E bounded below so there is at least one subsequential limit. If $s_* = \infty$ then only one element in E that is ∞ because s_* represents the smallest subsequential limit. For (b) again prove by contradiction and let $\{s_{n_k}\}$ be an infinite sequence whose elements are all $\leq x$. Suppose s_{n_k} is bounded below so we have

$$x \geq \lim s_{n_k} = \liminf s_{n_k} \geq \liminf s_n = s_*$$

which is a contradiction to our assumption that $x < s_*$. If s_{n_k} is not bounded below, we have $x < -\infty$ which is again a contradiction.

3.3 Root and Ratio Tests

Theorem 47

Consider

(a) if $p > 0$ then
$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

(b) if $p > 0$ then
$$\lim_{n\to\infty} \sqrt[n]{p} = 1$$

(c) if $|x| < 1$ then
$$\lim_{n\to\infty} x^n = 0$$

(d)
$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Proof. For (a) Simply directly take

$$\frac{1}{n^p} < \varepsilon$$
$$\frac{1}{\varepsilon} < n^p$$
$$\frac{1}{\varepsilon^{\frac{1}{p}}} < n$$

This obvously exists by **archimedian property** of real numbers. For (b) consider that positive unique square root theorem we have $\sqrt[n]{p} > 1$ if $p > 1$. Hence

$$x_n = \sqrt[n]{p} - 1 > 0 \tag{1}$$
$$1 + nx_n \leq (x_n + 1)^n = p \tag{2}$$
$$0 < x_n \leq \frac{p-1}{n} \tag{3}$$

Where the LHS of (2) is simply the first 2 terms in binomial expansion which consists of all positive terms since both $1, x_n > 0$. Taking limits of all terms in (3), by squeeze theorem we get

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (\sqrt[n]{p} - 1) = 0$$

So (b) follows. For (c) just consider the geometric series and then let $n = m$ like 48 for cauchy criterion for series. For (d) again use binomial theorem like so

$$x_n = \sqrt[n]{n} - 1 > 0 \tag{4}$$
$$\frac{n(n-1)}{2} \leq (x_n + 1)^n = n \tag{5}$$
$$0 < x_n \leq \frac{2}{n-1} \tag{6}$$

in (5) we use the 3rd term in the binomial series now because the 1st two terms aren't conclusive by squeeze theorem since

$$0 < x_n \leq \frac{n-1}{n} \quad \Rightarrow \quad 0 < \lim x_n \leq 1$$

In squeeze theorem we want both sides of the inquality to be the same!

Theorem 48 (Cauchy Criterion for series)
We use the fact that every convergent series is cauchy. $\sum a_n$ converges if and only if every $\varepsilon > 0$ there exists an integer N such that

$$\left|\sum_{k=n}^{m} a_k\right| = \left|\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k\right| \leq \varepsilon$$

for $m \geq n \geq N$

Corollary 49
If $\sum a_n$ converges then

$$\lim_{n\to\infty} a_n = 0$$

Proof. Letting $n = m$ in our cauchy criterion for series we see that

$$|a_n| \leq \varepsilon$$

for $n \geq N$ must be satisfied too.

Remark 50. *Note that this only follows after the assumption of convergent $\to$ cauchy $\to$ then this. This proof is not bidirectional, it cannot be used to prove convergence of series for example*

$$\sum \frac{1}{n}$$

individual terms converge but not the series.

Theorem 51
$\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$

Proof. Cant use any of the root/ratio tests or above results for common series. Gotta use of method with some $\sum 2^k a_{2^k}$ thingy...see `rudin` for more.

Theorem 52 (Root Test)
Given $\sum a_n$ put

$$\alpha = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

Then

(a) if $\alpha < 1$ then $\sum a_n$ converges

(b) if $\alpha > 1$ then $\sum a_n$ diverges

(c) if $\alpha = 1$ the test is inconclusive

Proof. For (a).If $\alpha < 1$ we find β so that $\alpha < \beta < 1$ by density of real numbers and hence by 45integer N such that

$$\sqrt[n]{|a_n|} < \beta$$

for $n \geq N$. So we have

$$|a_n| < \beta^n$$

but because $0 < \beta < 1$, we have by 47

$$\sum |a_n| < \sum \beta^n < \infty$$

For (b) if $\alpha > 1$, there clearly exists a infinite subsequence of $\sqrt[n_k]{|a_{n_k}|} \to \alpha > 1$ Therefore taking removing square roots we see that all such $|a_{n_k}| > 1$ so clearly diverges because we cannot satisfy the required $|a_n| \to 0$ for convergence of $\sum |a_n|$ see 48 For (c) consider

$$\sum \frac{1}{n}, \sum \frac{1}{n^2}$$

Recall 47 for both $\alpha = 1$ but the first diverges and the second converges.

Theorem 53 (Ratio test)

The series $\sum a_n$

(a) converges if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$

(b) dinverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$ where n_0 is some fixed integer

Proof. If (a) holds as usual by 45 we can find $\beta < 1$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta$$

for $n \geq N$. Then expanding our inequalities we have

$$\begin{aligned} |a_{N+1}| &< \beta \, |a_N| \\ |a_{N+2}| &< \beta \, |a_{N+1}| \\ \beta \, |a_{N+2}| &< \beta^2 \, |a_N| \end{aligned}$$

so by induction we have

$$|a_{N+p}| < \beta^p \, |a_N|$$

where $p = N - n, \forall n \geq N$ therefore we have

$$\sum |a_{N+p}| < \sum \beta^p \, |a_N| < \infty$$

so the conclusion follows. For (b) this implies

$$\ldots |a_{n_0+2}| \geq |a_{n_0+1}| \geq |a_{n_o}|$$

so clearly divergent by 48.

Fact 54

Note that the root test is more powerful than the ratio test because it can be shown with similar arguments from above that

$$\limsup \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n}$$

Hence if root test shows convergence so will the ratio test and same for divergence.

Theorem 55 (Radius of convergence)
The **radius of convergence** R defined by

$$\frac{1}{R} = \lim_{n\to\infty} \sup |c_n|^{\frac{1}{n}} \in [0, \infty]$$

Where

$$\sum_{n\geq 0} c_n z^n$$

is convergent whenever $|z| < R$ and divergent whenever $|z| > R$

Proof. Simply put $a_n = c_n z^n$ then apply **root test**. Let

$$\alpha = \limsup \sqrt[n]{|a_n|} = |z| \limsup \sqrt[n]{|c_n|}$$

The condition for convergence and divergence are $\alpha < 1$ and $\alpha > 1$ respectively so we may find the range of values $|z|$ can take by

$$|z| < R(\text{convergence}) \text{ and } |z| > R(\text{divergence})$$

where

$$R = \frac{1}{\limsup \sqrt[n]{|c_n|}}$$

Remark 56. *You could also get radius of convergence using ratio test of its coefficients too which is analogously defined by*

$$\frac{1}{R} = \limsup \frac{c_{n+1}}{c_n}$$

but we prefer to use more powerful tests hehe

4 Continuity

Definition 57
Suppose X and Y are metric spaces, $E \subset X, p \in E$ and f maps E into Y. Then f is said to be continuous at p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$

Theorem 58
A mapping f of a metric space X into metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y

Proof. Suppose f is continuous on X and V is an open set. We aim to show $f^{-1}(V)$ is open.Suppose $p \in X$ and $f(p) \in V$ which implies $p \in f^{-1}(V)$ by definition of inverse image. Since V is open there exist a $\epsilon > 0$ such that

$d_Y(y, f(p)) < \epsilon$ where $y \in V$. Since f is continuous, there exists δ for every such ϵ such that $d_X(x, p) < \delta$ and that $y = f(x)$. Since $y = f(x) \in V$ we can again conclude that $x \in f^{-1}(V)$. Then we know that for every p we can find a neighbourhood of p in $f^{-1}(V)$

Conversely suppose $f^{-1}(V)$ and V is open. We aim to show f is continuous. Define open set V as set of all y satisfying $d_Y(y, f(p)) < \epsilon$ for some fixed $p \in X$ and $\epsilon > 0$. This is open as all neighbourhoods are open if you recall. Since $f^{-1}(V)$ is open there exist $d_X(x, p) < \delta$ for every such p and ϵ too where $x \in f^{-1}(V)$. Hence $f(x) = y$ as $f(x) \in V$ given that inverse mappings are 1 to 1.

Corollary 59

A mapping f of a metric space X into metric space Y is continuous on X if and only if $f^{-1}(V)$ is closed in X for every closed set V in Y

Proof.

$$f^{-1}(A^C) = (f^{-1}(A))^c$$

This is always true as for inverse to exists there must be a 1 to 1 mapping. See your notes for more. Having established this we can let A be an open set so A^c is a closed set and the result follows

Theorem 60

Suppose f is a continuous mapping of *compact metric space* X into metric space Y. Then $f(X)$ is compact

Corollary 61

If f is a continuous mapping on R^k then $f(X)$ is closed and bounded. Thus f is bounded.

Proof. From the above, we know $f(X)$ is compact so it is closed and bounded, then $M = \sup f(X)$ and $m = \inf f(X)$ must exist on $f(X)$.

Definition 62

Let f be mapping of metric space X into Y. We say that f is **uniformly continuous** on X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \varepsilon$$

for all p and q in X for which $d_X(p, q) < \delta$

Here is a graphical visualization of the difference between uniform continuous and continuous.

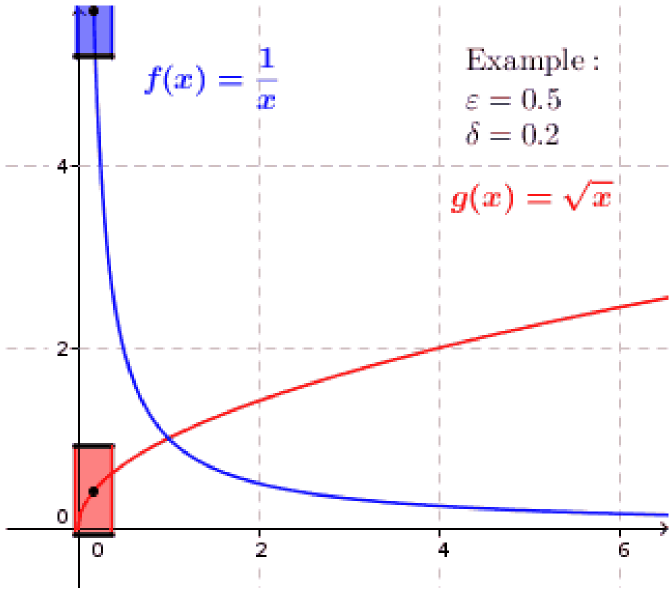


Figure 1: animated gif sourced from wikipedia

notice that

- blue function $f(x)$ is only continuous
- but the red function $\sqrt{x}$ is uniformly continuous

In this case we are considering $\varepsilon = 0.5$ so the half height of the boxes are $2 \times \varepsilon = 1$ is the height of the box while $\delta = 0.2$ is the half width of the box so $\delta \times 2 = 0.4$ is the width of the box. Now notice uniform continuous on X means that for every $\varepsilon > 0$ we may find a δ for every $p \in X$ such that

$$d_Y(f(p), f(q) < \varepsilon$$

if $d_X(p, q) < \delta$ and that this *same* δ works for all p. Now you see clearly for the red function, it never touches the top or bottom of the box as the box travels along the path specified by the graph via its centroid. On the other hand see that for the blue function this however is not the case meaning that for points $d(p, q) < \delta$(i.e those that lie within the width of the box) do not satisfy their corresponding a output $d(f(p), f(q))$(vertical) difference corresponding the height of the box. Therefore this is not δ that works for all x and ε(also visually you can see that you cant find such a δ given the near vertical region of the blue graph)

Remark 63. *From here you should have a better intuition on why the word "uniform" is used. You could say that x and y are mapped in not just a continuous manner but also in a more "consistent" manner*

Theorem 64

Let f be a continuous mapping of a compact metric space X into metric space Y then f is uniformly continuous on X

Proof. Let $\varepsilon > 0$ be given. Since f is continuous we can associate to each point $p \in X$ a positive number $\phi(p)$ such that

$$q \in X, d_X(p, q) < \phi(p) \quad \text{implies} \quad d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$$

5 Differentiation

Theorem 65

Let f be defined on $[a, b]$ if f is differentiable at a point $x \in [a, b]$ then f is continuous at x

Proof. first consider

$$\lim_{t\to x}(f(t) - f(x)) = \lim_{t\to x}\left(\frac{f(t) - f(x)}{t - x} \cdot (t - x)\right)$$

which implies upon rearrangment and the definition of $f'(x)$

$$\lim_{t\to x} f(x) = f'(x)(0) + f(x) = f(x)$$

as desired

Theorem 66

Suppose f, g are defined on $[a, b]$ and are differentiable at a piont $x \in [a, b]$ then $f + g, fg, f/g$ are differentiable at x and

(a) $(f + g)'(x) = f'(x) + g'(x)$

(b) $(fg)' = f'(x)g(x) + f(x)g'(x)$

(c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$

Theorem 67

Let f be defined on $[a, b]$ if f has a local maximum at a point $x \in (a, b)$ and if $f'(x)$ exists then $f'(x) = 0$. The analagous statement for minima is also true due to this

Proof. Choose δ such that

$$f(q) \leq f(p)$$

for all $q \in X$ with $d(p, q) < \delta$, that is $f(p)$ is a local maximum. To be well defined this local area must satisfy

$$a < x - \delta < x < x + \delta < b$$

too. Consider the two cases. If $x - \delta < t < x$ then by definition of local maximum

$$\frac{f(t) - f(x)}{t - x} \geq 0$$

and so when $t \to x$ we have $f'(x) \geq 0$. Similarly when $x < t < x + \delta$ then

$$\frac{f(t) - f(x)}{t - x} \leq 0$$

which shows that $f'(x) \leq 0$ hence $f'(x) = 0$

Theorem 68 (Generalized mean value theorem)
If f and g are continuous rel functions on $[a, b]$ which are differentiable in (a, b) then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

Proof. Put

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t) \quad (a \leq t \leq b)$$

then h is clearly continuous as it is the sum and product of continuous functions.Same reasoning for differentiability too Then since h is continuous on $[a, b]$ which is a compact set we know that it must achieve it's maximum in $x \in (a, b)$. Then from the previous theorem we know that $h'(x) = 0$.Therefore knowing that

$$h'(t) = [f(b) - f(a)]g'(t) - [g(b) - g(a)]f'(t)$$

when $t = x$ upon rearrangment the theorem clearly follows.

Corollary 69 (Mean value theorem)
If f is a real continuous function on $[a, b]$ which is differentiable in (a, b) then there is a pont $x \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(x)$$

Proof. Just take $g(x) = x$ in the previous theorem

Theorem 70 (Chain Rule Differentiation)
Suppose f is continuous on [a,b], $f'(x)$ exists at some point $x \in [a, b]$, g is defined on interval I which contains the range of f and g is differentiable at the point $f(x)$. If $h(t) = f(g(t))$ $(a \leq t \leq b)$ then $h'(x) = g'(f(x))f'(x)$

Proposition 71
Let $f : \mathbb{R} \to \mathbb{R}$ be a derivable function. f' is uniformly continuous in $\mathbb{R}$ Prove that $[n(f(x+1/n) - f(x))]$ converges uniformly to $f'(x)$

Proof. The difference quotient is equal to $f'(\xi)$ for some $\xi \in (x, x+1/n)$ by the mean value theorem. Now because of the uniform continuity of the derivative you have that for each $\epsilon > 0 \exists \delta > 0$ that $|f'(x) - f'(y)| \leq \epsilon \forall x, y : |x - y| \leq \delta$. Chose $1/n \leq \delta$ and you have it.

6 Integration

For this section we denote $\mathcal{R}$ to be the set of Riennman Integrable functions.

Theorem 72
If P^* is a refinement of P then

$$L(P, f) \leq L(P^*, f) \tag{7}$$

$$U(P^* f) \leq U(P, f) \tag{8}$$

Proof. Simply consider that a refinement will only affect the intervals when an existing interval is split into further subintervals when taking the total sum of sup(s) and inf(s) over each interval. Recall that Let $A \subset B$ then we have:

$$\inf(B) \leq \inf(A) \tag{9}$$
$$\sup(B) \geq \sup(B) \tag{10}$$

So we clearly when split into further subintervals the the sum of sup(s) will be lesser while the sum of inf(s) will be greater. $\square$

Theorem 73

$f \in \mathcal{R}$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U(P, f, a) - L(P, f, a) < \varepsilon$$

Proof. Pretty obvious from definition that $f \in \mathcal{R}$ we must have

$$\overline{\int} f = \underline{\int} f = \int f$$

Definition 74

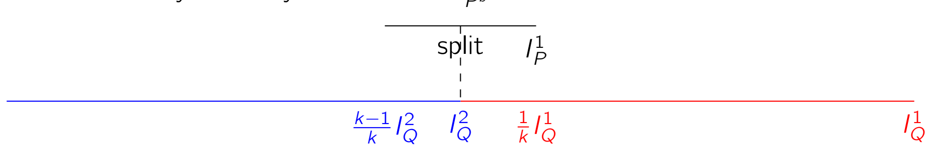
Let P be a partition of I. Let **maxsize**(P) refer to the maximum value of $\ell(I_P^i)$ over all possible i where superscript(i) is the index of the subinterval in the partiton and the subscript(P) is the partition it belongs to. When the superscript or subscript is ommitted it simply refers to any arbituary i or P respectively. Similarly define for **minsize**(P).The function $\boldsymbol{\ell}$ calculates the size of the interval

Lemma 75

Let $|f| \leq M$ and P, Q be partions on interval I. Assume maxsize$(\mathcal{P}) \leq \frac{1}{k}$minsize$(\mathcal{Q})$ then

$$U(P, f) \leq U(Q, f) + \frac{2M}{k}\ell(I)$$
$$L(P, f) \geq L(Q, f) - \frac{2M}{k}\ell(I)$$

Proof. Let P_1 be a common refinement between partions P and Q. When an interval of P is contained in an interval of Q, it is already an interval in P_1. However when interval of P is not contained in an interval of Q, it is split into 2 intervals on P_1 when calculting the sum of maximums/minimums over each interval. Let P^b be the collection of such intervals of P_1. Now consider any arbituary such interval $I_{P^b}^1$.



Clearly the overlap of I_P^1 on each possible I_Q is less than $\frac{1}{k}$ of each I_Q since the max size of $I_{P^b}^1$ cannot exceed $\frac{1}{k}$ of any I_Q. So now since $\sum_j \ell(I_Q^j) = \ell(I)$ then $\sum_i \ell(I_{P^b}^i) \leq \frac{\ell(I)}{k}$. Thus we have

$$|U(P,f) - U(P_1,f)| = U(P,f) - U(P_1,f) \leq \sum_i 2M\ell(I_{P^b}^i) \leq 2M\frac{\ell(I)}{k} \tag{11}$$

$$|L(P,f) - L(P_1,f)| = L(P_1,f) - L(P,F) \leq \sum_i 2M\ell(I_{P^b}^i) \leq 2M\frac{\ell(I)}{k} \tag{12}$$

Again to reiterate, we only need consider, intervals in P^b since the sum of maximums/minimums over intervals in $\{P^b\}^c$ are taken in the same way over the same interval in P so their difference is just zero.By the above 2 results respectively we obtain

$$U(P,f) \leq U(P_1,f) + \frac{2M}{k}\ell I \tag{13}$$

$$L(P,f) \geq L(P_1,f) - \frac{2M}{k}\ell I \tag{14}$$

and we obtain our desired conclusion knowing that $U(P_1,f) \leq U(Q,f)$ and $L(P_1,f) \geq L(Q,f)$.

Theorem 76

For any bounded function f on $[a,b]$, we have

$$\lim_{||P||\to 0} U(f,P) = \sup_P U(f,P) = \overline{\int} f$$

$$\lim_{||P||\to 0} L(f,P) = \inf_P L(f,P) = \underline{\int} f$$

Proof. Suppose $\{P_v\}$ is a sequence of partitions, each splitting I into v intervals such that

$$\text{maxsize}(P_v) \to 0$$

We aim to prove that then as $v \to \infty$

$$U(P_v,f) \to \overline{\int} f \tag{15}$$

$$L(P_v,f) \to \underline{\int} f \tag{16}$$

Consider an abituary partition Q. By definition it satisfies

$$\overline{\int} f \leq U(Q,f) \leq \overline{\int} f + \varepsilon$$

$$\underline{\int} f \geq L(Q,f) \geq \underline{\int} f - \varepsilon$$

Now pick N such that for $v \geq N$ we have

$$\text{maxsize}(P_v) \leq \varepsilon \text{ minsize}Q$$

Then by the previous lemma we have

$$U(P_v,f) \leq U(Q,f) + \varepsilon 2M\ell(I)$$

$$L(P_v,f) \geq L(Q,f) - \varepsilon 2M\ell(I)$$

Then combining with the above yields

$$\overline{\int} f = \inf_P U(f,P) \le U(P_v, f) \le U(Q,f) + \varepsilon(2M\ell(I)) \le \overline{\int} f + \varepsilon(1 + 2M\ell(I))$$
$$\underline{\int} f = \sup_P L(f,P) \ge L(P_v, f) \ge L(Q,f) - \varepsilon(2M\ell(I)) \ge \underline{\int} f - \varepsilon(1 + 2M\ell(I))$$

which gets as desired

$$0 \le U(P_v, f) - \overline{\int} f \le \varepsilon \tag{17}$$
$$0 \ge L(P_v, f) - \underline{\int} f \ge -\varepsilon \tag{18}$$

Corollary 77

It follows that if f is reinmann integrable,

$$\int f = \lim_{v\to\infty} \sum_{k=1}^{v} f(\xi_{vk})\ell(I_{P_v}^k)$$

for arbitrary $\xi_{vk} \in I_{P_v}^k$ where $f(\xi_{vk})$ is the maximum or minimum in that interval depending whether we are calculating for $\lim_{||P||\to 0} U(f,P)$ or $\lim_{||P||\to 0} L(f,P)$ respectively. This exists by the **mean value theorem**

Theorem 78

Let $-\infty < a < b < \infty$ and $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if it is *continuous almost everywhere*

Proof. refer to your notes `(Munkres)Introduction to Manifolds & Analysis` under the integration section.

Corollary 79

Suppose f is bounded on $[a,b]$, has finitely many points of discontinuity on $[a,b]$ and α is continuous at point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$

Theorem 80

Let $f \in \mathcal{R}$ on $[a,b]$. For $a \le x \le b$ put

$$F(x) = \int_a^x f(t)dt$$

Then F is continuous on $[a,b]$. Furthermore if f is continuous at a point x_0 of $[a,b]$ then F is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$

Theorem 81 (Fundamental Theorem of Calculus)
If $f \in \mathcal{R}$ on $[a, b]$ and if there exists a differentiable function F on $[a, b]$ such that $F' = f$(recall in `complex analysis` this is the analogue of the *primitive*) then

$$\int_a^b f(x)dx = F(b) - F(a)$$

7 Sequences and series of functions

7.1 uniform convergence

Fact 82 (Uniform vs Non uniform)
Consider the differences. See for every as a "for loop"

Uniform continuity: $\lvert f(x) - f(y)\rvert < \varepsilon; \lvert x - y\rvert < \delta; x, y \in E$	
Continuous at p	For every ε there exists for $p \in E$, a δ
Continuous on E	For every ε there exists for every $x \in E$, a δ
Uniformly Continuous	For every ε there exists a δ for all x

Uniform convergence: $\lvert f_n - f\rvert < \varepsilon; n \geq N; x \in E$	
Convergent at p	For every ε there exists for $p \in E$ an N
Pointwise Convergent	For every ε there exists for evert $x \in E$ an N
Uniformly Convergent	For every ε there exists an N for all x

Definition 83 (Bounded Function)
A **bounded function** in R^k is one where

$$\sup_{x \in X} |f(x)| \leq M$$

for some $M \in \mathbb{R}$

Theorem 84
Suppose

$$\lim_{n \to \infty} f_n(x) = f(x), \forall x \in E$$

Put

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|$$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$

Proof. By definition uniform convergence there exists for every ε there exists a N where $|f(x) - f_n(x)| < \varepsilon$ for $n \geq N$ for all(not just for each) $x \in [a, b]$. That is to say for one ε we have one δ that works for all x. Clearly sup satisfies this.

Theorem 85

Suppose $f_n \to f$ uniformly on set E in a metric space. Let x be a limit point of E, and suppose

$$\lim_{t \to x} f_n(t) = A_n \tag{1}$$

where $(n = 1, 2, 3 \ldots)$, then $\{A_n\}$ converges and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n \tag{2}$$

Proof. By **cauchy criterion for uniform convergence** there exists

$$|f_n(t) - f_m(t)| \leq \epsilon$$

for $n, m \geq N$. Let $t \to x$ then by (1)

$$|A_n - A_m| \leq \epsilon$$

implying A_n is convergent by Cauchy criterion thus

$$|A_n - A| \leq \epsilon$$

Where $\lim_{n \to \infty} A_n = A$.Then we arrive at our desired expression by triangle inequality under the same conditions $t \to x$ and $n \to \infty$

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \leq \epsilon$$

this implies (2) as $|f(t) - A|$ is independent of n.

Corollary 86

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

Where the order of limits is clearly shown to be immaterial

Proof. From (1) in the previous theorem

$$\lim_{t \to x} f_n(t) = A_n \tag{3}$$

$$\lim_{n \to \infty} \lim_{t \to x} f_n(t) = \lim_{n \to \infty} A_n \tag{4}$$

Substituting (2) in the previous theorem into (4)

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t) \tag{5}$$

since $f_n \to f$ then $\lim_{n \to \infty} f_n(t) = f(t)$ for all $t \in E$ so substituting into (5) we have

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t) \tag{6}$$

Corollary 87

If $\{f_n\}$ is a sequence of continuous functions and if $f_n \to f$ uniformly on E, then f is continuous on E

Proof. By contiuity of each $f_n(x)$ we have $A_n = \lim_{t\to x} f_n(t) = f_n(x)$

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n = \lim_{n\to\infty} f_n(x) = f(x)$$

Theorem 88

Suppose $f_n \in \mathfrak{R}(a)$ on $[a, b]$ for $n = 1, 2, 3 \ldots$ and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathfrak{R}(a)$ on $[a, b]$ and

$$\int_a^b f\,da = \lim_{n\to\infty} \int_a^b f_n da$$

where $\mathfrak{R}(a)$ indicates the set of Reinmann Integrable functions

Proof. To show that we can swap the intergrals under assumption of uniform convergence let

$$\varepsilon_n = \sup |f_n(x) - f(x)|$$

the suprenum is taken over $a \le x \le b$. By definition of sup that implies we have

$$|f_n - f| \le \varepsilon_n \quad \Rightarrow -\varepsilon_n \le f_n - f \le \varepsilon_n$$

therefore we have

$$f_n - \varepsilon_n \le f \le f_n + \varepsilon_n$$

because $f_n \in \mathfrak{R}(a)$ and my monotonicity we have

$$\int_a^b (f_n - \varepsilon_n)da \le \underline{\int} f\,da \le \overline{\int} f\,da \le \int_a^b (f_n + \varepsilon_n)da$$

therefore we have

$$0 \le \overline{\int} f\,da - \underline{\int} f\,da \le 2\varepsilon_n[a(b) - a(a)]$$

which follows when you subtract the first half of the integrals from second half. Because subtracting smaller number with greater number and subtracting greater number with smaller number preserves the inequality we obtain this. The zero on the left hand side can be seen from the middle inequality directly.We know that $\varepsilon_n \to 0$ as $n \to 0$ since $f_n \to f$ uniformly on $[a, b]$. □

Example 89

Here we illustrate why uniform convergence is essential. Consider the sequence of functions $f_n(x)$ on $[0, 1]$:

$$f_n(x) = \begin{cases} n & \text{if } 0 \leq x \leq \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < x \leq 1. \end{cases}$$

Pointwise Limit: For any $x \in (0, 1]$, $f(x) = \lim_{n\to\infty} f_n(x) = 0$. Thus,

$$f(x) = 0 \quad \text{for all } x \in [0, 1].$$

Integral of the Limit Function:

$$\int_0^1 f(x)\, dx = \int_0^1 0\, dx = 0.$$

Limit of the Integrals:

$$\int_0^1 f_n(x)\, dx = 1, \quad \text{so} \quad \lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 1.$$

Since $0 \neq 1$, uniform convergence is required to swap limits and integrals.

7.2 Weierstrass Approximation theorem

Theorem 90 (Weierstrass Approximation theorem)

If $f \in C([a, b])$ there exists a sequence of polynomials $\{P_n\}$ such that

$$P_n \to f$$

uniformly on $[a, b]$

Lemma 91

Let

$$Q_n(x) = c_n(1 - x^2)^n$$

and c_n be such that

$$c_n = \left(\int_{-1}^1 (1 - x^2)^n dx\right)^{-1} > 0$$

Then

1. $\forall n \int_{-1}^1 Q_n = 1$
2. $\forall Q_n(x) \geq 0$ on $[-1, 1]$ and
3. $\forall \delta \in (0, 1), Q_n \to 0$ uniformly on $\delta \leq |x| \leq 1$

1 and 2 are clear from definition. For (3) first estimate the magntidude of c like so

$$\begin{aligned}\frac{1}{c_n} &= \int_{-1}^{1}(1-x^2)^n dx \\ &= 2\int_{0}^{1}(1-x^2)^n dx \\ &> 2\int_{0}^{\frac{1}{\sqrt{n}}}(1-x^2)^n dx \\ &\geq 2\int_{0}^{\frac{1}{\sqrt{n}}}(1-nx^2) dx \\ &= 2\left(\frac{1}{\sqrt{n}} - \frac{n}{3}\cdot n^{-\frac{3}{2}}\right) \\ &= \frac{4}{3}\sqrt{n} > \sqrt{n}\end{aligned}$$

where the 3rd equality follows by binomial expansion done earlier in 47. Then let $\delta > 0$. Note that

$$\lim_{n\to\infty}(\sqrt{n}(1-\delta^2)^n)^{\frac{1}{n}} = \lim_{n\to\infty}(n^{\frac{1}{n}})^{\frac{1}{2}}(1-\delta^2) = (1-\delta^2) < 1$$

where $\lim_{n\to\infty}(n^{\frac{1}{n}}) = 1$ is a limit we covered before earlier if you recall. Therefore since we know $(1-\delta^2) < 1$ then

$$\lim_{n\to\infty}\sqrt{n}(1-\delta^2)^n = 0$$

which again by 47 is convergent. Therefore we may choose M such that

$$\sqrt{n}(1-\delta^2)^n < \varepsilon$$

for all $n \geq M$. Then under this conditions $\forall \delta \leq |x| \leq 1$ we have

$$\left|c_n(1-x^2)^n\right| < \sqrt{n}(1-x^2)^n \leq \sqrt{n}(1-\delta^2)^n < \varepsilon$$

□

Lemma 92
The following is a polynomial

$$P_n(x) = \int_0^1 f(t)Q_n(t-x)dt$$

Proof.

Remark 93. *Credits to* `ChatGPT` *I was too lazy to expand out*

Define the Polynomial $Q_n(u)$:

Suppose $Q_n(u)$ is a polynomial of degree n in u. We can write:

$$Q_n(u) = \sum_{k=0}^{n} a_k u^k,$$

where a_k are coefficients that may depend on t, but are constants with respect to x.

Substitute $u = t - x$:

Substitute $u = t - x$ into $Q_n(u)$:

$$Q_n(t - x) = \sum_{k=0}^{n} a_k (t - x)^k.$$

Set Up the Integral:

We need to evaluate:

$$P_n(x) = \int_0^1 f(t) Q_n(t - x)\, dt.$$

Substitute $Q_n(t - x)$:

$$P_n(x) = \int_0^1 f(t) \left(\sum_{k=0}^{n} a_k (t - x)^k \right) dt.$$

Distribute the Integral:

Since the integral and sum can be interchanged:

$$P_n(x) = \sum_{k=0}^{n} a_k \int_0^1 f(t)(t - x)^k\, dt.$$

Calculate the Integral:

Let's calculate $\int_0^1 f(t)(t - x)^k\, dt$. For each fixed k, the integrand $f(t)(t - x)^k$ is a polynomial in x. Expand $(t - x)^k$:

$$(t - x)^k = \sum_{j=0}^{k} \binom{k}{j} t^{k-j} (-x)^j.$$

Thus:

$$\int_0^1 f(t)(t - x)^k\, dt = \int_0^1 f(t) \left(\sum_{j=0}^{k} \binom{k}{j} t^{k-j} (-x)^j \right) dt.$$

By interchanging the sum and the integral:

$$\int_0^1 f(t)(t - x)^k\, dt = \sum_{j=0}^{k} \binom{k}{j} (-x)^j \int_0^1 f(t) t^{k-j}\, dt.$$

Let:

$$I_{k-j} = \int_0^1 f(t) t^{k-j}\, dt.$$

Therefore:

$$\int_0^1 f(t)(t - x)^k\, dt = \sum_{j=0}^{k} \binom{k}{j} (-x)^j I_{k-j}.$$

Combine Results:

Substitute this back into $P_n(x)$:

$$P_n(x) = \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} \binom{k}{j} (-x)^j I_{k-j} \right).$$

Reorder the sums:

$$P_n(x) = \sum_{k=0}^{n} \sum_{j=0}^{k} a_k \binom{k}{j} (-x)^j I_{k-j}.$$

This is a polynomial in x, as it is expressed as a finite sum of terms involving powers of x with coefficients $a_k \binom{k}{j} I_{k-j}$, which are constants with respect to x for each term.

Thus, $P_n(x)$ is indeed a polynomial in x. $\square$

Now back to proving wiersstrass approximation theorem let

$$P_n(x) = \int_0^1 f(t)Q_n(t-x)dt$$

Observe that for $x \in [0,1]$ we have by change of variables

$$\begin{aligned} P_n(x) &= \int_0^1 f(t)Q_n(t-x)dt \\ &= \int_{-x}^{1-x} f(x+t)Q_n(t)dt \\ &= \int_0^1 f(x+t)Q_n(t)dt \end{aligned}$$

much like the convolution. Then consider

$$\begin{aligned} |P_n(x) - f(x)| &= \left| \int_{-1}^1 (f(x-t) - f(t))Q_n(t)dt \right| \\ &\leq \int_{-1}^1 |f(x-t) - f(t)Q_n(t)|dt \\ &\leq \int_{|t|\leq t} |f(x-t) - f(x)|Q_n(t)dt + \int_{\delta \leq |t| \leq 1} |f(x-t) - f(x)|Q_n(t)dt \\ &\leq \int_{|t|\leq t} |f(x-t) - f(x)|Q_n(t)dt + \int_{\delta \leq |t| \leq 1} |f(x-t) - f(t)|Q_n(t)dt \\ &< \varepsilon \end{aligned}$$

We could bound $\int_{|t|\leq t} Q_n(t)$ because we know $Q_n(t) > 0$ on $t \in [0,1], \forall n$ and that $\int_{-1}^1$ from the above lemma. It is pretty obvious to see how to bound the rest. Since f continuous there exists $|f(x) - f(y)| < \varepsilon, |x-y| < \delta$ etc kind of thing.

References

[1] Walter Rudin. *Principles of mathematical analysis*. 3d ed. International series in pure and applied mathematics. New York: McGraw-Hill, 1976. ISBN: 978-0-07-054235-8.