Fourier Analysis Workbook

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June 2024

selected theorems from Stein and Shakarchi [1] [2]

Contents

1 Readings

1. Read Chapter 1 of Stein and Shakarchi, Fourier Analysis **done**
2. Read Chapter 2 of Stein and Shakarchi, Fourier Analysis. **done**
3. Continue reading and reviewing Chapter 2 of Stein and Shakarchi, Fourier Analysis. **done**
4. Read and review Chapter 3 of Stein and Shakarchi, Fourier Analysis.
5. Read and review Chapter 1.1 and 1.2 of Real Analysis by Stein and Shakarchi up through the section on Exterior Measure. Also read the section on construction of a non-measurable set at the end of Section 1.3. Fourier Analysis. **done**
6. Read and review Chapter 1 of Real Analysis by Stein and Shakarchi **done**
7. Read and review Chapter 2, Section 1 of Real Analysis by Stein and Shakarchi.**done**
8. Read and review Chapter 3, Section 1 and 2 of Real Analysis by Stein and Shakarchi. **done**
9. Read and review Chapter 5 of Stein and Shakarchi, Fourier Analysis.
10. Read and review Chapter 6 of Stein and Shakarchi, Fourier Analysis.

2 Basic Properties of Fourier Series

2.1 Preliminaries

Definition 1
F is a function on a circle if it can be written as a function $f(\theta)$

$$f(\theta) = F(e^{i\theta})$$

where clearly f is periodic

$$f(\theta) = f(\theta + 2\pi)$$

Remark 2. *so if f has period of 2π if the interval of θ is restricted to a lenght of 2π such as $[0, 2\pi], [-\pi, \pi]$ it still captures the initial function F on the circle.*

Definition 3

If f is an integrable function on an interval $[a, b]$ with length(that is, $b - a = L$), then we define:

The nth **fourier coefficient** is given by

$$\hat{f}(n) = \frac{1}{L}\int_a^b f(x)e^{-2\pi inx/L}dx$$

While the **fourier series** is given by where it serves as an *approximate* of f

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx/L}$$

The nth **partial sum of the fourier series** is given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx/L}$$

Definition 4

The **trigonometric series** is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L}$$

while the **trigonometric polynomial** is when the trigonometric series involves only finitely many non-zero terms

Remark 5. *the fourier series belongs to the extended family of trigonometric series*

2.2 Uniqueness of Fourier series

2.3 Convolutions

Definition 6 (Convolution)

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy$$

Proposition 7

Convultions satisfy

(i) $f * (g + h) = (f * g) + (f * h)$

(ii) $(cf) * g = c(f * g) = f * (cg)$ for any $c \in \mathbb{C}$

(iii) $f * g = g * f$

Proof. (i) and (ii) is clear by the linearity of integrals. (iii) can be seen by change of variable which is allowed by riemann integral recall `rudin 6.19`

2.4 Good Kernels

Definition 8 (Good Kernels)

A family of kernels $\{K_n(x)\}$ on the circle is said to be a family of **good kernels** if it satisfies

(a) For all $n \geq 1$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1$$

(b) There exists $M > 0$ such that for all $n \geq 1$

$$\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M$$

(c) for every $\delta > 0$

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|\, dx \to 0$$

as $n \to \infty$

Definition 9

We define **convolution** $f * g$ on $[-\pi, \pi]$ by

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy$$

Theorem 10

Suppose $\{K_n(x)\}$ is a family of good kernels and f is integrable on the circle. Then

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

as $N \to \infty$ whenever f is continuous at x. If f is continuous everywhere, then the above limit is uniform

Proof.

$$\begin{aligned}
|(f*K_n)(x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)f(x-y)dy - f(x)\right| &(1)\\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)f(x-y)dy - \int_{-\pi}^{\pi} K_n(y)f(x)dy\right| &(2)\\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]dy\right| &(3)\\
&\leq \frac{1}{2\pi}\int_{|y|\leq\delta} |K_n(y)||f(x-y)-f(x)|dy + \frac{1}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)||f(x-y)-f(x)|dy &(4)\\
&\leq \frac{\varepsilon}{2\pi}\int_{|y|\leq\delta} |K_n(y)|dy + \frac{2B}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|dy &(5)\\
&\leq \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} |K_n(y)|dy + \frac{2B}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|dy &(6)\\
&\frac{M\varepsilon}{2\pi} \qquad \text{as } n \to \infty &(7)
\end{aligned}$$

Since f is integrable on a circle and continuous at every x, so $f \in L^2(C[-\pi, \pi])$. Recall for such functions the lebesgue and riemann integral agree. Firstly, (2) follows by property (1) of good kernels. The ability to split into different ranges and apply triangle inequality in (2),(3),(4) follows by properties of lebesgue integrals.(5) follows by continuity that we can pick for everyε such there exists a δ such that $f(x-y) - f(x) \leq \varepsilon$ if $|y| \leq \delta$. B represents the bound on each function which exists due to continuous functions on a closed bounded interval. Finally the ability to expand (6) to the whole interval follows from the fact $\int_F f \leq \int_E f$ if $E \subset F$ and f is a non-negative lebesgue integrable function. Thus the absolute valued $|K_n|$ definitely satisfies this. (7) follows by property (3) of good kernels so the second term disappears. $\square$

2.5 Cesaro and Abel summability

Definition 11

Let a series of complex numbers be given by

$$c_0 + c_1 + c_2 \ldots = \sum_{k=1}^{\infty} c_k$$

and define its partial sum by

$$s_n = \sum_{k=1}^{n}$$

$$\sigma_N = \frac{s_0 + s_1 + \ldots + s_{N-1}}{N}$$

This is known as the **Nth Cesaro sum** of the series $\sum c_n$ or the **Nth Cesaro mean** of the sequence $\{s_k\}$ If

$$\sigma_N \to \sigma, N \to \infty$$

we say $\sum c_n$ is **cesaro summable** to σ

3 Convergence of fourier series

Fact 12

In $\mathbb{R}^d, \mathbb{C}^d, \ell^2(\mathbb{Z})$, these vector space with their inner products and norms satisfy

1. Inner product is strictly **positive-definite**
2. The vector space is **complete**

Theorem 13 (Riemann-Lebesgue)

Let $f_k = \langle f, e^{2\pi i k x}\rangle = \int f(x) e^{-2\pi i k x} dx$ where S^1 is the circle, that is $f(x)$ is a periodic function.
Let $f \in L^1(S^1)$. Then $f_k \to 0$ as $k \to \infty$

Proof.

$$f_k = \int f(x + \frac{1}{2k}) e^{-2\pi i k (x + \frac{1}{2k})} dx = -\int f(x + \frac{1}{2k}) e^{-2\pi i k x} dx$$

where we used $e^{\pi i} = -1$. Then we have

$$f_k = \frac{1}{2}\int (f(x) - f(x + \frac{1}{2k}))e^{-2\pi ikx}dx$$

by triangle inequality we have

$$|f_k| \le \frac{1}{2}\int \left|(f(x) - f(x + \frac{1}{2k}))\right| dx$$

since $1 = |e^{-2\pi ikx}|$ then clearly $f_k \to 0$ as $k \to \infty$. We move the limit in like this since $f \in L^1(S^1)$ which implies $|f| < \infty, \forall x \in S^1$ and that S^1 being a closed set of real numbers is measurable.Therefore by dominated convergence theorem we may bring the limits into the integral. $\int \lim_{n\to\infty} \left|f(x) - f(x + \frac{1}{2k})\right|$

Corollary 14

Let $f \in L^1(S^1)$. As $m \to \infty$ we have

$$\int f(x)\sin(2\pi mx)dx \to 0$$

and

$$\int f(x)\cos(2\pi mx)dx \to 0$$

Proof. Consider that each of these can expressed as sums of $\int f(x)e^{-2\pi ikx}dx$ then by linearity of limits everything goes to zero.

4 fourier transform

Definition 15

A function on $\mathbb{R}$ is said to have **moderate decrease** if f is continuous and there exists a constant $A > 0$ so that

$$|f(x)| \le \frac{A}{1 + x^2}$$

for all $x \in \mathbb{R}$

5 Measure Theory

5.1 Preliminaries

Definition 16

distance between 2 sets is given by

$$d(x, y) = \inf |x - y|$$

where inf is taken over all $x \in E$ and $y \in F$

Definition 17

A closed rectangle is where every interval it consists of is closed and vice versa for open rectangle

- (closed): $R = [a_1, b_1] \times \dots [a_d, b_d]$
- (open): $R = (a_1, b_1) \times \dots (a_d, b_d)$

the volume of a rectangle regardless closed or open in $\mathbb{R}^d$ where $d = 1, 2, 3$ is defined as

$$|R| = (b_1 - a_1) \dots (b_d - a_d)$$

A cube is where every $(b_j - a_j), j \in [1, d]$ is the same, equal to some $l > 0$.

$$|R| = l^d$$

Remark 18. *Essentially rectangles are multi-intervals in $\mathbb{R}^1$ or $\mathbb{R}^2$ or $\mathbb{R}^3$. When d=1, "volume" is length. when d=2, "volume" is area. Rectangles are almost disjoint if their interiors are disjoint. Almost because the boundaries are not.*

Theorem 19

if $R = \bigcup_{k=1}^{N} R_k$ where R is an almost disjoint union of finitely many rectangles then

$$|R| = \sum_{k=1}^{N} |R_k|$$

Proof. basically involves the area of rectangle is equal to the sum of all disjoint rectangles that make it up. A very common sense thing but needs a more rigorous proof of visual contruction as seen in *Shakarchi*

Corollary 20

if $R \subseteq \bigcup_{k=1}^{N} R_k$ where R is union of finitely many rectangles(need not be disjoint)

$$|R| \leq \sum_{k=1}^{N} |R_k|$$

Proof. essentially a slight modification of the above argument to allow for overlaps of rectangles. So you may double count these areas of overlap.

Theorem 21

Every open subset $\mathcal{O} \in \mathbb{R}^d$ can be written uniquely as a countable union of almost disjoint closed cubes

Proof. See `stein` for a rigourous visual proof. However in essence we essentially can approximate an open cube with countably many almost disjoint closed cubes that are infinitely small. Clearly we don't include the boundary for the approximate hence we estimating for open cube. Countably because all of them can be written on a grid.

5.2 The Exterior Measure

Definition 22

the **exterior measure** of E is defined by

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

where the infinum is taken over all countable covering of E by *closed cubes* Q_j

$$E \subseteq \bigcup_{j=1}^{\infty} Q_j$$

Theorem 23

If $A \subset \mathbb{R}$ is countable then $m^*(A) = 0$

Proof. By definition of countable we have to consider:

Case (i) When A is finite:

Follows the same logic below. We can just define a sum of intervals to any of an arbituary lenght to get any desired ε where $m^*(A) \leq \varepsilon$

Case (ii) When A is countably infinite:

Let $A = \{a_n : n \in \mathbb{N}\}$. For each n define an interval

$$I_n = (a_n - \frac{\varepsilon}{2^{2n+1}}, a_n + \frac{\varepsilon}{2^{2n+1}})$$

Then $A \subset \bigcup_{n=1}^{\infty} I_n$ hence we have

$$m^*(A) \leq \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

Remark 24. *However this is not a bidirectional proposition. For example* **cantor sets** *are uncountable and have measure zero*

Example 25

The measure of $\mathbb{Q}$ is zero.

Theorem 26

the exterior measure of a point and an empty set is zero

Proof. from the definition of exterior measure we see that its values lies in the range $0 \leq m_*(E) \leq \infty$. Since a point is basically a cube with zero volume which covers itself, that is definitely the infinum of all possible coverings. Similary, the exterior measure of an empty set is also zero. Not covered by anything is the same thing as covered by zero volume cube

Theorem 27

the exterior measure of a *closed cube* is equal to its volume

Proof. Suppose Q is a closed cube. A covering can be closed or open. Q can cover itself hence $m_*(Q) \leq |Q|$ by definition of infinum. Since Q is a closed cube(bounded by intervals), it is *compact*. For any covering of closed cubes Q_j of Q, we can find a covering of open cubes S_j of Q where $|Q_j| \leq |S_j| \leq |Q_j| + \epsilon$ for every $\epsilon > 0$. To see why, consider a closed interval [a,b]. The open interval (a,b) cannot contain [a,b], only the open interval $(a - \delta/2, b + \delta/2)$ can for some $\delta > 0$. Since the length of cube given by $f(l) = l$ is a continuous function and the product of continuous functions is continuous(recall `rudin 4.4`),the volume of the cube given by $g(l) = f(l)^d = l^d$ is also continuous. Hence

$$|l_i - l_f| \leq \delta \rightarrow |l_i^d - l_f^d| \leq \epsilon$$

By 20, we have

$$|Q| \leq \sum_{j=1}^{N} |S_j| \leq \sum_{j=1}^{\infty} |S_j| \leq (1 + \epsilon) \sum_{j=1}^{\infty} |Q_j|$$

Since ϵ is arbiturary the only solution to all such ϵ is where $|Q| \leq \sum_{j=1}^{\infty} |Q_j|$. Since $|Q|$ is now clearly a lower bound of all coverings of closed cubes of Q, we have by definition of infinum, $|Q| \leq m_*(Q)$. Hence proving $|Q| = m_*(Q)$

Corollary 28

the exterior measure of a *open cube* is equal to its volume

Proof. Suppose Q is an open cube. From 17, we see that $|Q| = |\overline{Q}|$. So it follows $m_*(Q) \leq |\overline{Q}| = |Q|$. Suppose Q_0 is a closed cube contained within Q where $|Q| \leq |Q_0| + \epsilon$. Then from 27, $|Q| - \epsilon \leq |Q_0| \leq m_*(Q_0) \leq m_*(Q)$. Similary, since ϵ is arbitrary we conclude $|Q| \leq m_*(Q)$, proving $|Q| = m_*(Q)$

Corollary 29

the exterior measure of a rectangle equals its volume

Proof. we can approximate any open rectangle with a countable union of disjoint closed cubes. As for closed rectangle simply approximate it to some very "close" open rectangle like we have done previously

5.2.1 Properties of Exterior Measure

Theorem 30 (Monotonicity)

If $E_1 \subseteq E_2$ then $m_*(E_1) \leq m_*(E_2)$

Proof. Any covering of E_2 is also a covering of E_1. Thus the possible coverings of E_1 is a superset of that of E_2 thus taking the infinum of these coverings we obtain $m_*(E_1) \leq m_*(E_2)$ (QED)

Theorem 31 (Countable sub-additivity)

if $E = \bigcup_{j=1}^{\infty} E_j$ then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$

Remark 32. *We assume that $m_*(E) < \infty$ or the inquality won't make sense. How can infinity be less than anything?*

Proof. Assume each E_j has a covering $\bigcup_{k=1}^{\infty} Q_{k,j}$ of closed cubes

$$
\begin{aligned}
m_*(E) \leq \sum_{j,k} |Q_{k,j}| &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |Q_{k,j}| \\
&\leq \sum_{j=1}^{\infty}(m_*(E_j) + \frac{\epsilon}{2^j}) \\
&= \sum_{j=1}^{\infty} m_*(E_j) + \epsilon
\end{aligned}
$$

Theorem 33

$m_*(E) = \inf m_*(\mathcal{O})$ where the infinum is taken over all open sets $\mathcal{O}$ containing E

Proof. From *monotoncity* it is clear

$$m_*(E) \leq \inf m_*(\mathcal{O})$$

Let E have a covering $\bigcup_{k=1}^{\infty} Q_j$ of closed cubes. Once again by definition of exterior measure we have

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \frac{\epsilon}{2}$$

then let Q_j^0 be an open cube containing Q_j such that $|Q_j| \leq |Q_j^0| \leq |Q_j| + \frac{\epsilon}{2^{j+1}}$.
Then $\mathcal{O} = \bigcap_{j=1}^{\infty} Q_j^0$ is open and

$$
\begin{aligned}
m_*(\mathcal{O}) \leq \sum_{j=1}^{\infty} &= \sum_{j=1}^{\infty} |Q_j^0| \\
&\leq \sum_{j=1}^{\infty}(|Q_j| + \frac{\epsilon}{2^{j+1}}) \\
&\leq \sum_{j=1}^{\infty} |Q_j| + \frac{\epsilon}{2} \\
&\leq m_*(E) + \epsilon
\end{aligned}
$$

Theorem 34

If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$ then

$$m_*(E) = m_*(E_1) + m_*(E_2)$$

Proof. Due to *sub-additivity*, we have

$$m_*(E) \leq m_*(E_1) + m_*(E_2)$$

Let $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes. Since there exists $\delta > 0$ where $d(E_1, E_2) > \delta > 0$, if we choose diameter of Q_j to be less than δ we can have

$$E_1 \subset \bigcup_{j \in J_1}^{\infty} Q_j$$

and

$$E_2 \subset \bigcup_{j \in J_2}^{\infty} Q_j$$

where $J_1 \cap J_2 = 0$ are disjoint partitions of integers. Hence we have

$$m_*(E_1) + m_*(E_2) \leq \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| \leq \sum_{j} |Q_j| \leq m_*(E) + \epsilon$$

Theorem 35

If set E is the countable union of *almost disjoint cubes* where $E = \bigcup_{j=1}^{\infty} Q_j$ then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j| = \sum_{j=1}^{\infty} m_*(Q_j)$$

Let $\tilde{Q}_j$ be a cube strictly contained in Q_j such that $|Q_j| \leq |\tilde{Q}_j| + \frac{\epsilon}{2^j}$ which is possible again by the continuity of volume cubes as a function of length of sides as seen previously. So now we $\{\tilde{Q}_j\}$ are *disjoint cubes* not just almost disjoint cubes which allows for the intersection of boundaries. Since if disjoint, the distance between sets is surely non-zero, by repeated application of the above theorem we get

$$m_*(\bigcup_{j=1}^{N} \tilde{Q}_j) = \sum_{j=1}^{N} m_*(\tilde{Q}_j) = \sum_{j=1}^{N} |\tilde{Q}_j| \geq \sum_{j=1}^{N} (|Q_j| - \frac{\epsilon}{2^j}) = \sum_{j=1}^{N} |Q_j| - \epsilon$$

Then equality follows knowing that this holds for all n including $n \to \infty$ and all $\epsilon > 0$. Then equality follows after *countable sub-additivity* to prove the inequality in the other direction.

Fact 36

However despite all this we still cannot conclude that if $E_1 \cup E_2$ is a disjoint union in $\mathbb{R}^d$ then

$$m_*(E) = m_*(E_1) + m_*(E_2)$$

This is because $E_1 \cap E_2 = \emptyset \to d(E_1, E_2) > 0$, is not necessarily true. We only were able to apply it to disjoint cubes because they were measurable. With this in mind, we now will learn how the concept of **measurability** of a set determines precisely if such a proposition is true for a given set.

Example 37

Consider the disjoint sets $E_1 = \mathbb{N}$ and $E_2 = \{n + \frac{1}{n}, n \in \mathbb{N}\}$. However

$$d = (E_1, E_2) = \inf d(x, y) = \lim_{n \to \infty} \frac{1}{n} = 0$$

for $x \in E_1, y \in E_2$.

5.3 Measurable Sets and the Lebesgue Measure

5.3.1 properties of measure

Definition 38

A subset E of $\mathbb{R}^d$ is **Lebesgue Measurable** or simply **measurable** if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ where $E \subseteq \mathcal{O}$ and

$$m_*(E - \mathcal{O}) \leq \epsilon$$

We then define its **Lebesgue Measure** or simply **measure** to be

$$m_*(E) = m(E)$$

Theorem 39

Every open set in $\mathbb{R}^d$ is measurable

Proof. Let $E = \mathcal{O}$. Then $m_*(E - \mathcal{O}) = m_*(\emptyset) = 0 \leq \epsilon$ which is certainly true

Theorem 40

If $m_*(E) = 0$, then E is measurable

Proof. by 33, we know that $m_*(E) = \inf m_*(\mathcal{O})$. By definition of infinum, we have

$$m_*(E) \leq m_*(\mathcal{O}) \leq m_*(E) + \epsilon$$

So we have $m_*(\mathcal{O} - E) \leq m_*(\mathcal{O}) \leq \epsilon$ by *monotonicity* property

Theorem 41

A countable union of measurable sets is measurable

Proof. Suppose $E = \bigcup_{j=1}^{\infty} E_j$ where each E_j is measurable. Hence we can find for each j

$$m_*(E_j - \mathcal{O}) \leq \frac{\epsilon}{2^j}$$

Since $\mathcal{O} = \bigcup_{j=1}^{\infty} \mathcal{O}_j$ is open, $E \subset \mathcal{O}$ and $(\mathcal{O} - E) \subset \bigcup_{j=1}^{\infty}(\mathcal{O}_j - E_j)$ we have by *monotonicity* and *sub-additivity* that

$$m_*(\mathcal{O} - E) \leq m_*(\bigcup_{j=1}^{\infty}(\mathcal{O}_j - E_j) \leq \sum_{j=1}^{\infty}(\mathcal{O}_j - E_j) \leq \epsilon$$

Lemma 42

If F is closed, K is compact and these sets are disjoint then $d(F, K) > 0$

Proof. To prove the contrary suppose $d(F, K) = 0$, distance between sets is non-negative. This implies there exists a sequence $a_n \in F$ and $b_n \in K$ such $d(a_n, b_n) \to 0$ as $n \to \infty$. Since for all $a \in F, b \in K, \epsilon > 0$ there must exist

$$0 = \inf d(a, b) \leq d(a_n, b_n) \leq \inf d(a, b) + \epsilon$$

where we can contruct

$$d(a_n, b_n) \leq \frac{1}{n}$$

which converges to zero as $n \to \infty$ Since K is compact there exists a subsequence b_{n_k} of b_n that converges to some point $b \in K$. Hence by triangle inequality

$$d(a_{n_k}, b) \leq d(a_{n_k}, b_{n_k}) + d(b_{n_k}, b) \to 0$$

as $k \to \infty$ Since F is closed, all convergent sequences converge to a point in F. Recall `rudin 2.41` Hence, $b \in F$ which contradicts $F \cup K = \emptyset$

Theorem 43

closed sets in $\mathbb{R}^d$ are measurable

Proof. Consider a closed set F. We first consider compact sets.By 33 we can select an open set $\mathcal{O}$ where $F \subset \mathcal{O}$ and $m_*(\mathcal{O}) \leq m_*(F) + \epsilon$. Then $(\mathcal{O} - F)$ is open given that it is equal $\mathcal{O} \cap F^c$, a finite intersection of open sets. Then by 21 there exists almost disjoint closed cubes Q_j such that

$$\mathcal{O} - F = \bigcup_{j=1}^{\infty} Q_j$$

Then by 35 we have

$$m_*(\mathcal{O} - F) = \sum_{j=1}^{\infty} m_*(Q_j)$$

We now aim to show the above expression $\leq \epsilon$ by relating it to $m_*(\mathcal{O}) - m_*(F) \leq \epsilon$. First, we know that $(\bigcup_{j=1}^{\infty} Q_j) \cup F = \mathcal{O}$. To be able to use 34 to represent the measure of a union of sets as the sum of the measure of its component sets, we make use of 42 by letting $K = \bigcup_{j=1}^{N} Q_j$ which is compact because it is a finite union of closed sets(which gaurantees it is closed unlike countable union recall `rudin 2.24`) and it is bounded as it is made of cubes. Then we have

$$(\bigcup_{j=1}^{N} Q_j) \cup F \subset \mathcal{O}$$

By *monotonicity* we have

$$\begin{aligned}
m_*(\mathcal{O}) &\geq m_*(K \cup F) \\
&= m_*(K) + m_*(F) \\
&= \sum_{j=1}^{N} m_*(Q_j) + m_*(F)
\end{aligned}$$

Which holds for any n including $n \to \infty$. So putting everything together

$$m_*(\mathcal{O} - F) \leq \sum_{j=1}^{N} m_*(Q_j) \leq m_*(\mathcal{O}) - m_*(F) \leq \epsilon$$

Lemma 44

$E \subset \mathcal{O}$ if and only $\mathcal{O}^c \subset E^c$

Proof. Notice the following propositions are contrapostives of each other and thus equivalent.

1. Proposition $E \subset \mathcal{O}$ means: for all x in E then x in $\mathcal{O}$

2. Proposition $\mathcal{O}^c \subset E^c$ means: for all x in $\mathcal{O}^c$ then x in E^c

Theorem 45

The complement of a measurable set is measurable

Proof. Consider measurable set E and open set $\mathcal{O}$ where $E \subset \mathcal{O}$ and thus $m_*(\mathcal{O} - E) \leq \epsilon$ by definition. Then $\mathcal{O}^c \subset E^c$ if we consider the contrapositive. Given that $E^c = \mathcal{O}^c \cup (E^c - \mathcal{O}^c)$. Clearly to show E^c is measurable we need to show $E^c - \mathcal{O}^c$ is measurable so we have a union of measurable sets which is measurable. Using $\mathcal{O} - E = \mathcal{O} \cap E^c = E^c - \mathcal{O}^c$ thus $m_*(E^c - \mathcal{O}^c) \leq \epsilon$ for all $\epsilon > 0$. Clearly the only solution is $m_*(E^c - \mathcal{O}^c) = 0$ which implies $E^c - \mathcal{O}^c$ is measurable by 40.

Theorem 46

A countable intersection of measurable sets is measurable

Proof. recall `rudin 2.24` that

$$(\bigcup_a (E_a)^c)^c = \bigcap_a (E_a)$$

Then the conclusion follows knowing that the countable union of measurable sets is measurable and so is its complement.

Lemma 47

Subsets of bounded sets are bounded

Proof. Consider subsets A, B in a metric space X. By definition if A is bounded, then

$$d(a, x) \leq M, \forall a \in A, \exists x \in X, \exists M \in \mathbb{R}$$

Since $B \subset A$,

$$d(b, x) \leq M, \forall b \in B, \exists x \in X, \exists M \in \mathbb{R}$$

is certainly true

Remark 48. *By the same logic, the trivial fact that subsets of disjoint sets are also disjoint follows*

Corollary 49 (Countable Additivity)

If $E_1, E_2 \dots$ are *disjoint* measurable sets and $E = \bigcup_{j=1}^{\infty} E_j$ then

$$m(E) = \sum_{j=1}^{\infty} m(E_j)$$

Proof. The clear approach is to somehow make use of 42, and repeatedly apply it to obtain the result. Thus we try a refinement of E_j that is bounded. By 46, we know E^c is measurable too. Thus we can find for each $E_j^c \subset \mathcal{O}_j$:

$$m_*(\mathcal{O}_j - E_j^c) = m_*(\mathcal{O}_j \cap E_j) = m_*(E_j - \mathcal{O}_j^c) \leq \frac{\epsilon}{2^j}$$

where $\mathcal{O}_j$ is an open set and $\mathcal{O}_j^c$ is a closed set which we denote as F_j. We also know that $F_j \subset E_j$. Since F_j is a subset of bounded and disjoint set E_j by 47 F_j both closed, bounded and disjoint. Hence F_j is compact and disjoint.

Firstly we apply 42 repeatedly like so

$$\sum_{j=1}^{N} m(F_j) = m(\bigcup_{j=1}^{N} F_j)$$

and as usual finite union to ensure it is closed and hence compact for this relation to be valid. Secondly by *countable sub-addivity* and knowing that $E_j = (E_j - F_j) \cup F_j$ we have

$$m(E_j) - m(F_j) \leq m(E_j - F_j) \leq \frac{\epsilon}{2^j}$$

Using the fact that $\bigcup_{j=1}^{N} F_j \subset \bigcup_{j=1}^{N} E_j \subset E$, we are finally able to split out $m(E_j)$ and relate its sum to $m(E)$ via the use of compact sets F_j

$$m(E) \geq \sum_{j=1}^{N} m(F_j) \geq \sum_{j=1}^{N} m(E_j) - \epsilon$$

Since ϵ is arbituary and applies to any N including up to infinity we have

$$m(E) \geq \sum_{j=1}^{\infty} m(E_j)$$

The inequality in the other direction is proven by directly applying countable sub-additivity to $E = \bigcup_{j=1}^{\infty} E_j$

It now remains to show we can always find a bounded refinement for any E that is a countable disjoint union of measurable sets. Cubes are bounded and measurable if you recall. Using any sequence of cubes $Q_j \nearrow \mathbb{R}^d$(see 51) and letting $S_k = Q_k - Q_{k-1}$ can define $E_{j,k} = E_j \cap S_k$. Subsets of disjoint sets are disjoint, likewise for bounded and countable intersections of measurable sets are measurable. Thus we know every $E_{j,k}$ is disjoint, measurable and bounded. We also know intersections of countable sets are countable. So we can redefine E as the countable union

$$E = \bigcup_{j,k} E_{j,k}$$

where $E_j = \bigcup_{k=1}^{\infty} E_{j,k}$ Putting everything together we get

$$m(E) = \sum_{j,k} m(E_{j,k}) = \sum_{j} \sum_{k} m(E_{j,k}) = \sum_{j} m(E_j)$$

Definition 50

For the following we first denote that

- E_k increases to E means $E_k \subset E_{k+1}$ and $E = \bigcup_{k=1}^{\infty} E_k$ which we write as

$$E_k \nearrow E$$

- E_k decreases to E means $E_{k+1} \subset E_k$ and $E = \bigcap_{k=1}^{\infty} E_k$ which we write as

$$E_k \searrow E$$

Corollary 51 (continuity of measure)
Suppose $E_1, E_2 \ldots$ are measurable subsets of $\mathbb{R}^d$.

(i) If $E_k \nearrow E$ or

(ii) If $E_k \searrow E$ and $m(E_k) < \infty$ for some k

Then:

$$m(E) = \lim_{n\to\infty} m(E_n)$$

Proof. For (i) Construct disjoint measurable subsets $G_k = E_k - E_{k-1}$. Then we have by *countable additivity*

$$m(E) = \lim_{n\to\infty} m(\bigcup_{k=1}^{N} G_k) = \lim_{n\to\infty} \sum_{k=1}^{N} m(G_k)$$

For (ii) Construct disjoint sets $G_k = E_k - E_{k-1}$ for each k. Because every subsequent E_k is a subset of E_n

$$\bigcap_{k=n}^{\infty} E_k = \bigcap E_k = E$$

which is the set common between all subsequent subsets and E_n and by *countable additivity*

$$m(\bigcup_{k=n}^{\infty} G_k) = \sum_{k=n}^{\infty} m(G_k)$$

which is the set difference of all subsequent subsets with E_n Putting everything together we have

$$m(E_n) = m(E) + m(\sum_{k=n}^{\infty} G_k) \tag{8}$$

Since there exists some n where $m(E_n) \leq \infty$ as assumed in the proposition this implies:

1. by monotoncity we have $m(E) < m(E_n) \leq \infty$
2. $m(\sum_{k=n}^{\infty} G_k)$ converges for that n to some finite value. Hence it is clear from

$$\sum_{k=n}^{\infty} m(G_k) = \sum_{k=1}^{\infty} m(G_k) - \sum_{k=1}^{n} m(G_k)$$

that $\sum_{k=1}^{\infty} G_k$ converges since all terms on both RHS and LHS here mut be finite. Thus there exists an N in which for all $n > N$,

$$\sum_{k=n}^{\infty} m(G_k) = \sum_{k=1}^{\infty} m(G_k) - \sum_{k=1}^{n} m(G_k) \leq \epsilon$$

Taking limits on all sides of (1):

$$\lim_{n\to\infty} m(E_n) = \lim_{n\to\infty} m(E) + \lim_{n\to\infty} \sum_{k=n}^{\infty} m(G_k)$$
$$= m(E) + 0$$

□

Remark 52. *We note that in 51 finiteness for some $m(E_k)$ is essential.*

Example 53
Consider $E_n = [n, \infty)$
$$m(E) = m(\bigcup E_n) = m(\emptyset) = 0 \neq \lim_{n\to\infty} m(E_n) = \infty$$
since $m(E_n) = \infty$ for every n

Definition 54
We denote that $E \triangle F$ is the symmetric difference between sets E and F
$$E \triangle F = (E - F) \cup (F - E)$$

Theorem 55
Suppose E is a measurable subset of $\mathbb{R}^d$. Then for every $\epsilon > 0$:

(i) There exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \epsilon$

(ii) There exists a closed set F with $F \subset E$ and $m(E - F) \leq \epsilon$

(iii) If $m(E)$ is finite, there exists compact set K with $K \subset E$ and $m(E - K) \leq \epsilon$

(iv) If $m(E)$ is finite, there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that $m(E \triangle F) \leq \epsilon$

Proof. For (i) it follows from the definition of measurable sets. For (ii) consider
$$m_*(\mathcal{O} - E^c) = m_*(\mathcal{O} \cap E) = m_*(E - \mathcal{O}^c) \leq \epsilon$$
where $\mathcal{O}$ is an open set containing measurable set E.For (iii) consider a closed subset F of measurable set E where $m_*(E - F) \leq \epsilon$ and closed balls B_n with radius n. Let $K_n = F \cap B_n$ which is compact since finite union closed sets closed and subsets bounded sets are bounded. Firstly $(F \cap B_n) \subset (F \cap B_{n+1})$ so $(E - K_n) \supset (E - K_{n+1})$. Hence $\{E - K_n\} \searrow E - F$ .Secondly $m(E - K_n) < m(E) < \infty$ for all n by *monotoncity*. Thus by 51 we have:
$$\lim_{n\to\infty} m(E - K_n) = m(E - F)$$
Hence there exists an N where
$$m(E - K_n) \leq \epsilon$$
for all $n \geq N$

Theorem 56 (Inner regularity of Lebesgue Measure)
Let $E \subset \mathbb{R}^d$ be Lebesgue measurable. Show that
$$m(E) = \sup\{m(K) : K \subset E, K \text{compact}\}.$$

Proof. We have already done the case when E is bounded(i.e (iii) of 55). When E is unbounded, there are two cases:

1. $m(E) = +\infty$
2. $m(E) < +\infty$

Now we handle with the first case. Consider the closed ball in $\mathbb{R}^d$ $A_m = \{x \in \mathbb{R}^d : |x| \leq m\}$. Then

$$E = \cup_{m=1}^{\infty} E_m$$

with $E_m = E \cap A_m$, and $E_1 \subset E_2 \subset \cdots \subset \mathbb{R}^d$. By monotone convergence theorem for measurable sets(51),

$$\lim_{n\to\infty} m(E_n) = m(E) = +\infty.$$

Note that since E_m is bounded, by the case you've done, for every m, we have a compact set $K_m \subset E_m \subset E$ such that $m(K_m) + 1 \geq m(E_m) \to \infty$. This is because a closed set in $\mathbb{R}^d$ is measurable(43). So now we have a finite intersection of measurable sets which is measurable(46).Hence we must indeed find such a compact set for every E_m due to 55 (ii). Subset of a bounded set is bounded. Closed and bounded sets in $\mathbb{R}^d$ are compact if you recall basic real analysis.Hence we have

$$\sup\{m(K) : K \subset E, K\text{compact}\} = +\infty = m(E).$$

Now assume that $m(E) < +\infty$. For any $\varepsilon > 0$, we can choose N such that

$$m(E) \leq m(E_N) + \varepsilon/2.$$

We also have a compact set $K \subset E_N \subset E$ with

$$m(E_N) \leq m(K) + \varepsilon/2$$

since E_N is bounded. It follows that

$$m(E) \leq m(K) + \varepsilon.$$

which is clearly just the definition of the suprenum We are done.

5.3.2 sigma algebras and Borel sets

Definition 57 (σ-algebras)
a ***σ*-algebra** of sets is a collection of subsets in $\mathbb{R}^d$ that is closed under countable unions, intersections and complements

Definition 58 (Borel Sets)
Borel *σ*-algebra in $\mathbb{R}^d$ denoted by $\mathcal{B}_{\mathbb{R}^d}$ is the smallest σ-algebra that contains all *open* sets

By smallest we mean $\mathcal{B}_{\mathbb{R}^d} \subset S$ for all $S = \{\sigma\text{-algebra that contains all open sets}\}$. This also means the Borel set is the intersection of all such S.

Example 59
Consider the following **Borel Sets**.

(i) open and closed sets

(ii) countable intersection of open sets(G_δ sets)

(iii) countable union of closed sets(F_σ sets)

5.3.3 Construction of Non Measurable Set

Proposition 60 (Axiom of Choice)
Suppose E is a set and $\{E_\alpha\}$ is a collection of non-empty subsets of E.

5.4 Measurable Functions

5.4.1 Definitions

Definition 61
A **real valued** function f on $\mathbb{R}^d$ takes the form

$$-\infty \leq f(x) \leq \infty$$

where $f(x)$ belongs to the extended real numbers.
A **finite valued** function f takes the form

$$-\infty < f(x) < \infty$$

for all x on $\mathbb{R}^d$ or equivalently for $a, b \in \mathbb{R}$

$$a < f(x) < b \quad \text{or} \quad a \leq f(x) < b \quad \text{or} \quad a < f(x) \leq b \quad \text{or} \quad a \leq f(x) \leq b$$

where as shown, whether the inequality is strict or not is immaterial.
A **complex valued** function f takes the form

$$\mathrm{Re}(f) + i\,\mathrm{Im}(f)$$

where $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are real valued functions

Definition 62
A function f is *defined on a measurable subset* E of $\mathbb{R}^d$ is **measurable** if for all $a \in \mathbb{R}$ the set

$$f^{-1}([-\infty, a)) = \{x \in E : f(x) < a\}$$

is measurable. We denote such a set as $\{f < a\}$

Note that this also implies that the following are sets are measurable

1. $f^{-1}([-\infty, a])$

$$\{f \leq a\} = \bigcap_{k}^{\infty}\{f < a + \frac{1}{k}\}$$

   From the original definition, we know $\{f < a + \frac{1}{k}\}$ is measurable since a can take any value and countable intersection measurable sets measurable

2. $f^{-1}([a, \infty])$

$$\{f \geq a\} = \{f < a\}^c$$

   Taking the complements of above measurable sets and that complements of measurable sets are measurable

3. $f^{-1}((a,\infty])$

$$\{f > a\} = \{f \leq a\}^c$$

   Same logic as above

4. $f^{-1}((a,b))$

$$\{a < f < b\} = \{f < b\} \cap \{f > a\}$$

   For all $a, b \in \mathbb{R}$. Finite intersection measurable sets. Recall countable can be countably infinite or finite.

5. $f^{-1}([a,b))$

$$\{a \leq f < b\} = \{f < b\} \cap \{f \geq a\}$$

   Same logic as above

6. $f^{-1}(\infty)$

$$f^{-1}(\infty) = \bigcap_{n=1}^{\infty} f^{-1}((n,\infty])$$

   Countable intersections measurable sets measurable

7. $f^{-1}(-\infty)$

$$f^{-1}(-\infty) = \bigcap_{n=1}^{\infty} f^{-1}([-\infty,n))$$

   Same logic as above

8. $f^{-1}([-\infty,a))$

$$\{f < a\} = \bigcup_{k}^{\infty} \{f \leq a - \frac{1}{k}\}$$

   Recall countable union of measurable sets measurable. With this last item, we now have a circular proof among these propositions. Any one of these sets being measurable will imply all of the rest are too.

Theorem 63

Let $E \subset \mathbb{R}$ be measurable and suppose $f, g : E \to \mathbb{R}$ are two measurable functions and $c \in \mathbb{R}$. Then

(i) cf

(ii) $f + g$

(iii) fg

are all measurable functions

Remark 64. *For (iii) we are considering the product of functions not the composition. We make the distinction in notation by* $f(x)g(x) \Leftrightarrow fg$ *and* $f(g(x)) \Leftrightarrow f \circ g$

Proof. For(i) consider $\{cf(x) > a\} = \{f(x) > \frac{a}{c}\}$ and recall a is arbitrary since f is measurable. For (ii) consider $\{f + g > a\} = \{f > \ a - g\} = \{f > r > a - g\}$ where $r \in \mathbb{Q}$. This exists by the **density property of real numbers**. Thus

$$\{f + g > a\} = \{f > r\} \cup \{g > a - r\}$$

and since intersection of measurable sets are measurable $f + g$ is measurable. For (iii) consider $\{f^2 > a\}$. Clearly

$\{f > \sqrt{a}\}$ or $\{f < -\sqrt{a}\}$. So taking the union of the preimages, we have

$$\{f^2 > a\} = \{f > \sqrt{a}\} \cup \{f < -\sqrt{a}\}$$

which is again measurable. Now consider

$$fg = \frac{1}{2}((f+g)^2 - (f-g)^2$$

which is measurable considering the above properties.

Theorem 65

The following functions are measurable

$$\sup_n f_n(x) \quad \inf_n f_n(x) \quad \limsup_{n\to\infty} f_n(x) \quad \liminf_{n\to\infty} f_n(x) \quad \lim_{n\to\infty} f_n(x)$$

Proof. $\{\sup_n f_n > a\} = \bigcup_n \{f_n > a\}$ as for every $x \in \{\sup_n f_n > a\}$ the greatest $f_k(x)$ over all possible k is bigger than a. This means *there exists* a k for which $f_k(x) > a$ for every x in the subset. Thus we take the union $\{\inf_n f_n > a\} = \bigcap_n \{f_n > a\}$ as for every $x \in \{\inf_n f_n > a\}$ the smallest $f_k(x)$ over all possible k is bigger than a. This means $f_k(x) > a$ *for all* k for every x in the subset.Thus we take the intersection. Recall that

$$\limsup_{n\to\infty} f_n(x) = \inf_k \{\sup_{n\geq k} f_n\}$$

Then we have

$$\{\inf_k \{\sup_{n\geq k} f_n\} > a\} = \bigcap_k \{\sup_{n\geq k} f_n > a\} = \bigcap_k \bigcup_{n\geq k} \{f_n > a\}$$

Which is cleary measurable. Same logic applies to $\liminf_{n\to\infty} f_n(x)$ and definitely to $\lim_{n\to\infty} f_n(x)$ which equals $\liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$ if it exists.

Proposition 66

a *complex valued* function f is measurable if both $Re(f)$ and $Im(f)$ are measurable.

Proof. Recall the definition of measurable function in $\mathbb{R}^d$. Complex numbers are basically 2 element tuples so they can be treated as $\mathbb{R}^2$. Consider

$$\{f < (a, b)\} = \{\operatorname{Im} f < b\} \cap \{\operatorname{Re} f < a\}$$

Where

- $\{\operatorname{Im} f < b\}$ is simply all 2 elements tuples whose second element is less than b
- $\{\operatorname{Re} f < a\}$ is simply all 2 elements tuples whose first element is less than a

Clearly if $\operatorname{Re} f$ and $\operatorname{Im} f$ measurable then f is measurable □

Definition 67 (characteristic function)

Let E, F be measurable sets. We define **characteristic function** $f : E \to \mathbb{R}$ by:

$$\chi_F(x) = \begin{cases} 1 & x \in F \\ 0 & x \notin F \end{cases}$$

Proposition 68

Characteristic functions are measurable

Proof. Consider

$$f^{-1}((a,\infty]) = \{f > a\} = \begin{cases} \emptyset & a \geq 1 \\ E \cap F & 0 \leq a < 1 \\ E & a < 0 \end{cases}$$

$\chi_F(x)$ is certainly non-negative for whole domain by definition so the preimage is just the whole domain E.Note that a here is not the value of the function but the value in which the value of the function is bigger than. $\chi_F(x)$ is bigger than 0 if preimage is the domain that is also in E. However it is certainly impossible for $\chi_F(x)$ to be greater than 1 by definition therefore the preimage here is the empt set. Now because all 3 preimages, $\emptyset, E \cap F, E$ are measurable which correspond to all values of a, the characteristic function is measurable

Definition 69 (simple function)

A measurable function $\phi : E \to \mathbb{C}$ is **simple** if range(ϕ) has a finite number of elements.

Therefore we can represent range(ϕ)=$\{a_1, a_2, \dots a_N\}$ for some $N \in \mathbb{N}$ and each a_i is distinct by definition of set. Then we can define the sets

$$A_i = \phi^{-1}(\{a_i\})$$

which certainly are all each measurable.We just have to find an interval (a, b) that contains just that particular a_i only $\{a < a_i < b\}$ which is possible for all a_i by density property of real numbers applied separately to Re(f) and Im(f). We note further that $A_i \cap A_j = \emptyset$ if $i \neq j$ since by definition of function, there can only be 1 mapping per element in domain to be well defined(although the corresponding mapped item in the codomain need not be unique). Given that each $x \in A_i$ or $x \in A_j$ maps to the distinct values a_i and a_j respectively when $i \neq j$, the intersection must be empty or there exist an element in the domain that has 2 mappings and thus the function is not well-defined anymore.Alternatively just a do a vertical line test on at each distinct value of $f(x)$. Can only intersect graph once to be a well defined function. Hence we can thus we can represent f as

$$f = \sum_{k=1}^{N} a_k \chi_{A_k}$$

where each A_k is a measurable set of finite measure and a_k are constants.For each x, $f(x)$ will be equal a single a_i. Clearly *simple functions are also measurable* as they are a combination of indicator functions which are measurable

Definition 70 (step function)

We define **step function** as

$$f = \sum_{k=1}^{N} a_k \chi_{R_k}$$

where each R_k is a rectangle and a_k are constants.

They need not be disjoint however as you will learn later that simple functions are independent of representation.

Definition 71

We denote that two functions f and g defined on set E are equal **almost everywhere** as

$$f(x) = g(x) \quad \text{a.e. } x \in E$$

if the set $\{x \in E : f(x) \neq g(x)\}$ has measure zero

Theorem 72

if $f, g : E \to [-\infty, \infty]$ satisfy $f = g$ a.e on E, and f is measurable then g is measurable

Proof. Let $N = \{x \in E : f(x) \neq g(x)\}$ and by assumption $m(N) = 0$. Then

$$N_a = \{x \in N : g(x) > a\} \subset N$$

by subadditivity $m(N_a) \leq m(N) = 0$ so $m(N_a) = 0$ too as measure is non-negative. Then since

$$\{g > a\} = (\{f > a\} \cap N^c) \cup N_a$$

clearly g is measurable (QED) □

5.4.2 Approximation by simple or step functions

Theorem 73

Suppose $f : E \to [0, \infty]$ is a nonnegative measurable function on $\mathbb{R}$ then there exists a sequence of simple functions $\{\phi_n\}$ such that

1. is non-negative and increasing:
$$\phi_k < \phi_{k+1}$$
2. converges point-wise to f for all $x \in E$:
$$\lim_{n\to\infty} \phi_n(x) = f(x), \forall x \in E$$
3. For all $B > 0$, $\phi_n \to f$ converges uniformly on the set $\{x \in E : f(x) \leq B\}$, basically where f is bounded.

Proof. This means

$$|\phi_n(x) - f(x)| < \epsilon, \forall x \in E$$

for $n > N$. Thus for every x, the $\phi_n(x)$ we define must get closer and closer to the actual $f(x)$ as $n \to \infty$. Let's say this difference 2^{-n} which clearly gets abrituary close to zero when $n \to \infty$. Hence for every n, we could split the range of $f(x)$ into windows of size 2^{-n} such that every corresponding codomain of x to each window we have

$$E_n^k = \{x \in E : k2^{-n} < f(x) \leq (k+1)2^{-n}\}$$

where $k \in \mathbb{N}, 0 \leq k \leq 2^{2n} - 1$ enumerates through this windows. Then we define our simple function to be

$$\phi_n = \sum_{k=0}^{2^{2n}-1} ((k2^{-n})\chi_{E_n^k}) + 2^n \chi_{F_n}$$

Clearly this is a floor function for each corresponding to each window thus $\phi_n(x)$ is within 2^{-n} of $f(x)$ for every x as desired.Moreover as each window is disjoint, E_n^k is disjoint as required for a well defined function. To see why simply do a vertical line test in at the pair of boundary points of any range window. Obviously every E_k^n is a measurable set as f is measurable. Here $F_n = f^{-1}((2^n, \infty])$ which lets the floor function accomodate x not in domains corresponding to the windows. Why this specific range of k? The lowerbound of k is zero because $f(x) > 0$ as a nonnegative function. The reason why we cant just k enumerate to infinity and just forgo the need for F_n is because our functions are limited to simple functions, so it is necessary that the sum is *finite* so the range of our ϕ_n specifically is of the form $\{0, (1)(2^{-n}), (2)(2^{-n}) \ldots (2^{2n} - 1)(2^{-n})\}$. Because for every $x \in F_n$, $\phi_n(x)$ is not necessarily within 2^{-n} of $f(x)$, we need to make sure this $F_n \subset E$ gets super small as $n \to \infty$ to satisfy point wise convergence on the whole of E. That is why we chose the upper bound on k to be $2^{2n} - 1$ such that subbing it into $(k+1)2^{-n}$ to get the lowerbound on set corresponding to the domain of F_n, we get 2^n as shown in our definition of F_n. Clearly $(2^n, \infty] \to \emptyset$ as $n \to \infty$. This is pointwise, not uniform convergence as how large an n is required depends on the possible x values which then depends on how large the range this function is.

Coincidentally, the $\phi_n(x)$ we defined is also non-negative and increasing. To show this we need to have for any fixed x, $\phi_k < \phi_{k+1}$. Consider 3 subsets of E where x can possibly belong to.

(Case 1) $x \in E_n^k$. Then, to relate it to $n+1$ we can consider

$$k2^{-n} < f(x) \leq (k+1)2^{-n} \Rightarrow 2k2^{-(n+1)} < f(x) \leq (2k+2)2^{-(n+1)}$$

Hence $f(x)$ is in either when we consider interval lengths of 2^{-n}

$$2k2^{-(n+1)} < f(x) \leq (2k+1)2^{-(n+1)}$$

or

$$(2k+1)2^{-(n+1)} < f(x) \leq (2k+2)2^{-(n+1)}$$

Then this same x in ϕ_n vs ϕ_{n+1} is say the latter interval is:

$$\phi_n(x) = k2^{-n} = (2k)2^{n+1} < (2k+1)2^{-(n+1)} = \phi_{n+1}(x)$$

Verify in the same way for the former interval

(Case 2) $x \in F_n$ Then, Use the same methods as above

Thus proving property (2).

As for property (3), just pick an N where $\{x \in E : f(x) \leq B\}$ is contained in $\{x \in E : f(x) \leq 2^N\}$. This clearly exists by archimedian property of real numbers. This is *uniform convergence* because we have a single n that works for *all* x in this set. □

We now attempt to extend this to negative measurable functions as well

Corollary 74
Let $E \subset \mathbb{R}$ be measurable and $f : E \to \mathbb{C}$ be measurable. Then there exists a sequence of simple functions $\{\phi_n\}$ such that

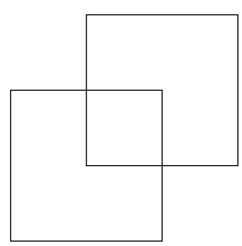
1. is non-negative and increasing:
$$|\phi_k| < |\phi_{k+1}|$$

2. converges point-wise to f for all $x \in E$:
$$\lim_{n\to\infty} \phi_n(x) = f(x), \forall x \in E$$

3. For all $B > 0$, $\phi_n \to f$ converges uniformly on the set $\{x \in E : |f(x)| \leq B\}$, basically where f is bounded.

Theorem 75
Suppose $f : E \to \mathbb{R}^d$ is measurable. Then there exists a sequence of step functions $\{\psi_k\}$ that converges pointwise to $f(x)$ for almost every x

Proof. By 73 we know that f can be approximated by simple functions. Hence f will approximately take the form of some $f = \sum_{i=1}^{N} a_i \chi_{E_i}$ where E are disjoint measurable sets and a_i is some distinct value in the finite sized range of f. Now consider each individual $f = \chi_E$ defined on E corresponding to any arbitrary a. We attempt find a set of rectangles $\{R_j\}$ approximately equal to E. In doing so we aim to define a new approximate form $f(x) = \sum_{j=1}^{M} \chi_{R_j}(x)$ which is basically like the previous approximate of f but now defined on $\bigcup_{j=1}^{M} R_j$ and as a step function instead of on E and as a simple function.Then we aim to show that there exists a sequence of step functions which are defined on these rectangles which converges pointwise almost anywhere to $f(x)$ for $x \in E$ because the set difference between the rectangles and E is negligible. First to construct such rectangles, by 55(iv) we can find closed cubes Q_j for every ϵ
$$m(E \triangle \bigcup_{j=1}^{N} Q_j) < \epsilon$$
. Consider a grid by extending the sides of each cube into one another such that we are able to get almost disjoint rectangles R_j. Then we consider rectangles slightly smaller rectangles $\tilde{R}_j$ that are disjoint. Such that the measure is approximately the same.



So we have
$$m(E \triangle \bigcup_{j=1}^{N} \tilde{R}_j) < \epsilon$$
. Because we can find such closed cubes for every ϵ define
$$m(E \triangle \bigcup_{j=1}^{N_k} \tilde{R}_j^k) < \frac{1}{2^k}$$

Because $f(x) = \sum_{j=1}^{M} \chi_{R_j}(x)$ now being a step and measurable function by reapplication of 73 we can define a sequence of simple functions which in this case are step functions because they converge to a step function like so $\psi_k(x) = \sum_{j=1}^{N_k} \chi_{\tilde{R}_j^k}(x)$ which is clearly defined on $\bigcup_{j=1}^{N_k} \tilde{R}_j^k$ as we constructed above. Now we aim to show the set $F = \{\lim_{k\to\infty} \psi_k \neq f(x) = \chi_E\}$ has measure zero . The reason for using the approximate form $f(x) = \chi_E$ is that we know this function converges point-wise on E by 73. To proceed we need to define

$$E_k = \{x : f(x) \neq \psi_k(x)\}$$
$$F_K = \bigcup_{j>K}^{\infty} E_j$$

and thus(recall how limsup is defined)

$$F = \bigcap_{K=1}^{\infty} F_K$$

We know that $E_k = E \triangle \bigcup_{j=1}^{N_k} \tilde{R}_j^k$. If you are in the symmetrical difference of the 2 sets, then clearly one of the functions will output 1 while the other will be zero.You can prove from the other direction too. Hence, by subadditity we have $m(E_k) \leq 2^{-k}$ and $m(F_K) \leq \sum_{j>K} E_j \leq 2^{-K}$. Hence $m(F) = 0$(QED) □

5.4.3 Littlewood's three principles

Fact 76 (Littlewood's Three Principles)

Before **measure theory**, its precursor follows 3 assertions made by `Littlewood`

1. Every set is *nearly* a finite union of intervals
2. Every function is *nearly* continuous
3. Every convergent sequence is *nearly* uniformly convergent

Where we now know that the sets and functions referred to above are *measurable*

`Littlewood`'s principle 3 can be attested by:

Theorem 77 (Egorov)

Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$ and assume $f_k \to f$ almost everywhere on E. Given $\epsilon > 0$ we can find a set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) \leq \epsilon$ and $f_k \to f$ uniformly on A_ϵ

Proof. Firstly, like 55, $m(E) < \infty$ is essential. For example $f_n(x) = \frac{x}{n}$ converges to zero but not uniformly on unbounded set. Now Let us attempt to construct A_ϵ directly. We first define

$$E_k^n = \bigcap_{j>k} \{x \in E : |f_j(x) - f(x)| < \frac{1}{2^n}\}$$

This is essentially finds the set of x for which *all* $f_j(x), j > k$ is within $\frac{1}{2^n}$ of $f(x)$. Because $f_k \to f$ uniformly on our A_e we consider

$$|\lim_{j\to\infty} f_j - f| < \epsilon$$

means for every ϵ, in this case $\frac{1}{2^n}$, there must exist such a k corresponding to each n which we denote as k_n. Then we know $A_\epsilon = \bigcap^n E^n_{k_n}$ because every x in this A_ϵ must satisfy the above for *for all* n as mentioned,

Because $f_k \to f$ almost everywhere we must have by definition,

$$m(\{\lim_{j\to\infty} f_j \neq f\}) = 0$$

Since f_j converges uniformly to f on A_ϵ this means

$$m(\{A_\epsilon \triangle E\}) = m(\{E - A_\epsilon\}) = 0$$

Because clearly $A_\epsilon \subset E$ and $x \in E - A_\epsilon$ is where f_k does not converge to f. Hence

$$m(E - A_\epsilon) < \epsilon$$

Remark 78. *Note that if only need pointwise not uniform convergence $A_\epsilon = \bigcup^n E^n_{k_n}$. Because for every $\frac{1}{2^n}$ we defined our set such that we can find k_n to converge within this when $x \in E^n_{k_n}$ which is a subset of our A_ϵ here. Hence every n chosen is dependent on x too for convergence.*

□

`Littlewood`'s principle 2 can be attested by:

Theorem 79 (Lusin)
Suppose f is measurable and finite valued on E with E of finite measure. Then for every $\epsilon > 0$ there exists a closed set F_ϵ with

$$F_\epsilon \subset E \quad \text{and} \quad m(E - F_\epsilon) \leq \epsilon$$

and such that $f|_{F_\epsilon}$ is continuous

6 Integration Theory

Fact 80
We aim to make a definition of integration more general than Riemann integration. For example, Riemann Integration cannot deal with every subset of $\mathbb{R}$.

Example 81
Consider a function $f = \chi_{\mathbb{Q}}$ defined by on an arbituary interval of real numbers:

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Hence $\chi(x)$. Recall a function is **riemann integrable** only if there exists a partion P for every ε where

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

However if you consider $\chi_Q(x)$ over some arbituary interval of $\mathbb{R}$, for any partition we use, $U(P, f, \alpha) - L(P, f, \alpha)$ is *always* equal 1 since between any real numbers there will always be rational number by the **density property of archimedian numbers**. So the sup and inf of $f(x)$ over any paritiaion is 1 and 0 respectively. Thus this function is *not riemann integrable*.

Another way to see this is to recall fron `rudin`

Fact 82

Let $-\infty < a < b < \infty$ and $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is not Riemann integrable if and only if it is *discontinuous almost everywhere*

In other words, the set of continuities points has measure zero. Notice that this function is discontinuous at every point. Also recall

Fact 83

f is not continuous at a if there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists x with $0 < |x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$

Choose arbitrary $a \in \mathbb{R}$ in the specified interval of real numbers as defined by the domain of our function. Let $f(a)$ be either 1 or 0 depending on whether a is rational or not. Now for any δ there will exist an x within δ of a that is of different nature to a by the density property of real numbers once again. Then there exists ε satisfying this within $[0, 1]$. Hence the set of continuous points for this function is empty. Recall that the empty set has measure zero.

6.1 The Lebesgue Integral

Definition 84

the support of f is defined by the set

$$\operatorname{supp} f(x) = \{x : f(x) \neq 0\}$$

Proposition 85

set $\operatorname{supp}(f)$ is measurable

Proof. Consider

$$f^{-1}(\mathbb{R} - 0) = f^{-1}[-\infty, 0) \cup f^{-1}(0, \infty]$$

Definition 86

Define the class of non-negative measurable functions defined on measurable set $E \subset \mathbb{R}$ as

$$L^+(E) = \{f : E \to [0, \infty]\}$$

Definition 87 (Lebesgue Integral for non-negative simple functions)
Suppose $\phi \in L^+(E)$ is a simple function such that $\phi = \sum_{j=1}^n a_j\chi_{A_i}$ where $\cup_{j=1}^m A_i = E$ is a disjoint union of measurable sets. Then the **Lebesgue integral** of ϕ is

$$\int_E \phi = \sum_{j=1}^n a_j m(A_j) \in [0, \infty]$$

6.1.1 simple functions

Theorem 88 (linearity of Lebesgue integrals of simple functions)
Suppose ϕ, ψ are two simple functions . Then for any $c \geq 0$ we have the following indentities:

1. $\int_E c\phi = c\int_E \phi$
2. $\int_E(\phi + \psi) = \int_E \phi + \int_E \psi$
3. $\int_E \phi \leq \int_E \psi$ if $\phi \leq \psi$
4. if $F \subset E$ is measurable , then $\int_F \phi = \int_E \chi_F\phi \leq \int_E \phi$
5. $\int_{E\cup F} \phi = \int_E \phi + \int_F \phi$ if $E \cup F$ is a disjoint union of measurable sets

Proof. let $\phi = \sum_{j=1}^n a_j\chi_{A_i}$ and $\psi = \sum_{k=1}^m b_k\chi_{B_k}$. For (1), $c\phi = \sum_{j=1}^n (ca_j)\chi_{A_i}$ as every output in the range is now multiplied by c. Then by definition:

$$\int_E c\phi = \sum_{j=1}^n ca_j m(A_j) = c\sum_{j=1}^n a_j m(A_j) = c\int_E \phi$$

. For (2), notice that

$$E = \bigcup_{j=1}^n A_j = \bigcup_{k=1}^m B_k \Rightarrow A_j = \bigcup_{k=1}^m (A_j \cap B_k), B_k = \bigcup_{j=1}^n (A_j \cap B_k)$$

. Then note that

$$\int_E (\phi + \psi) = \sum_{j=1}^n \sum_{k=1}^m (a_j + b_k)\chi_{A_j \cap B_k}$$

Apply *countable additivity* of measuers to the LHS for prove equality. For (3) $\phi \leq \psi$ means $a_j \leq b_k$. Then by comparison we see

$$\int_E \phi = \sum_{j,k} a_j m(A_j \cap B_k) \leq \sum_{j,k} b_j m(A_j \cap B_k) = \int_E \psi$$

. For (4)

$$\begin{aligned}\int_E \phi\chi_F &= \int_E \left(\sum_{j=1}^{n} a_j\chi_{A_i}\right)\chi_F \\ &= \int_E \left(\sum_{j=1}^{n} a_j\chi_{A_i}\chi_F\right) \\ &= \int_E \left(\sum_{j=1}^{n} a_j\chi_{A_i\cap F}\right) \\ &= \sum_{j=1}^{n} a_j m(A_i\cap F) \\ &= \int_F \phi\end{aligned}$$

By *monotonicity* of measures we have $m(A_i \cap F) \leq m(A_i)$ hence

$$\int_F \phi = \sum_{j=1}^{n} a_j m(A_i\cap F) \leq \sum_{j=1}^{n} a_j m(A_i) = \int_E \phi$$

For (5) consider that

$$\chi_{E\cup F}\phi = \chi_E\phi + \chi_E\phi$$

□

6.1.2 non negative measurable functions

Earlier we defined the lebesgue intergral for non-negative simple functions. Now we define it $f \in L^+(E)$ in general. Because we know that every non-negative measurable function can be represented a pointwise convergence sequence of increasing simple functions, we can intuitively the class of Lebesgue integrals like so. Because increasing sequence, sup is the lim here. But we are clearly assuming $\lim_{n\to\infty}\int_E \phi_n = \int_E f$ when $\phi_n \to f$. As we shall see in **monotone convergence theorem**, this actually works!

Definition 89 (Lebesgue Integral of non-negative measurable functions)
Suppose $f \in L^+(E)$. Then the lebesgue intergral of f is

$$\int_E f = \sup\{\int_E \phi : \phi \in L^+(E) \text{ simple}, \phi \leq f\}$$

Proposition 90
if $E \subset \mathbb{R}$ is measurable set with $m(E) = 0$ then for all $f \in L^+(E)$, we have $\int_E f = 0$

Proof. Consider an arbituary $\phi \leq f$ defined on E. We know that

$$\int_E \phi = \sum_{j=1}^{n} a_j m(A_j) \in [0,\infty]$$

and since $\bigcup_{j=1}^{m} A_j = E$ it follows from *monotonicity* that $m(A_j) \leq m(E) = 0$ for all j. So all $\int_E \phi$ defined on E where $m(E) = 0$ is zero. Hence the supremum over all possible simple functions will also be zero.

Theorem 91 (Monotone convergence theorem)
If $\{f_n\}$ is a increasing sequence of functions in $L^+(E)$ where $f_n \to f$ pointwise on E. Then,

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. Since $f_k \le f_{k+1}$ then $\int_E f_k \le \int_E f_{k+1}$. Since $f_k \to f$ pointwise on E, f must be be the supernum of the sequence $\{f_k\}$ for each $x \in E$ where $f_n \le f$ for all n. Hence, $\int_E f_k \le \int_E f$ for all n too, thus $\int_E f$ is an upperbound on the sequence of $\{\int_E f_k\}$. Now taking the limit on the sequence $\{\int_E f_k\}$ which is the supernum once again since it is monotonic increasing sequence, by definition of supernum we have

$$\lim_{n\to\infty} \int_E f_n \le \int_E f$$

We now proceed to prove the inequality in the other direction, namely $\lim_{n\to\infty} \int_E f_n \ge \int_E f \ge \int_E \phi$ for any arbituary $\phi \le f$ like how $L^+(E)$ is defined. First we know that there must exist some n for every ε such that $\phi - \varepsilon \le f - \varepsilon < f_n \le f$ for each $x \in E$ by definition suprenum and since $\phi \le f$. In other words for a given fixed ε we must have that every x must lie in some E_n defined by

$$E_n = \{x \in E : f_n(x) \ge (1-\varepsilon)\phi(x)\}$$

where $\varepsilon \in (0, 1)$ and that

$$E = \bigcup_{k=1}^{\infty} E_n$$

Directly relating this to integrals we have

$$\int_E f_n \ge \int_{E_n} f_n \ge \int_{E_n} (1-\varepsilon)\phi = (1-\varepsilon)\sum_{j=1}^{m} a_j m(A_j \cap E_n)$$

Now taking limits on everything

$$\lim_{n\to\infty} \int_E f_n \ge \lim_{n\to\infty} \int_{E_n} f_n \ge \lim_{n\to\infty} (1-\varepsilon)\sum_{j=1}^{m} a_j m(A_j \cap E_n) = (1-\varepsilon)\sum_{j=1}^{m} a_j m(A_j) = (1-\varepsilon)\int_E \phi$$

since $E_1 \subset E_2 \subset \ldots$ so we have $m(A_j \cap E_n) \to (A_j)$ by the *continuity of Lebesgue measure*. Now because ε is abiturary we have proven the inquality in the other direction as desired.

Corollary 92
Let $f \in L^+(E)$ and let $\{\phi_n\}$ be a sequence of simple functions such that $0 \le \phi_1 \le \phi_2 \ldots$ and $\phi_n \to f$ pointwise then

$$\int_E f = \lim_{n\to\infty} \int_E \phi_n$$

Proof. simply replace $\{f_n\}$ above with $\{\phi_n\}$ since it is also a member of $L^+(E)$. Now you can clearly see why our definition in 89 makes sense!!

Corollary 93
if $f, g \in L^+(E)$ then $\int_E (f+g) = \int_E f + \int_E g$

Proof. Consider 2 sequences of increasing simple functions in $L^+(E)$, $\{\phi_n\}$ and $\{\psi_n\}$ such that they converge pointwise to f and g respectively. This exists by 73. Thus we have

$$\int_E (f+g) = \lim_{n\to\infty} \int_E (\phi_n + \psi_n) = \lim_{n\to\infty} \int_E \phi_n + \lim_{n\to\infty} \int_E \psi_n$$

by application of *linearity of limits* and *linearity of Lebesgue integral of simple functions*

Corollary 94

$$\int_E \sum_n f_n = \sum_n \int_E f_n$$

Proof. by induction on 93

Proposition 95

if $f, g \in L^+(E)$ and $c \geq 0$ we have the following indentities

1. $\int_E f \leq \int_E g$ if $f \leq g$
2. $\int_E cf = c \int_E f$
3. $\int_F f \leq \int_E f$ if $F \subset E$
4. $\int_{E\cup F} f = \int_E f + \int_F f$ if $E \cup F$ is a disjoint union of measurable sets

Proof. For (1) consider:

$$\int_E f = \sup_{\phi<f} \int_E \phi \leq \sup_{\phi<g} \int_E \phi = \int_E g$$

since $\forall x \in E, \phi < f < g$ hence

$$\{\phi | \phi < f, x \in E\} \subset \{\phi | \phi < g, x \in E\}$$

For (2) use *monotone convergence theorem* . For (3) consider

$$\int_F f = \sup_{\phi<f} \int_F \phi \leq \sup_{\phi<f} \int_E \phi = \int_E f$$

since $F \subset E$. For(4) consider the sequence of simple functions $\{\phi_n\}$ where $\phi_n \to f$ pointwise on $E \cup F$ and *linearity of limits* and *linearity of simple functions*

$$\int_{E\cup F} f = \lim_{n\to\infty} \int_{E\cup F} \phi_n = \lim_{n\to\infty} \left(\int_E \phi_n + \int_F \phi_n \right) = \lim_{n\to\infty} \int_E \phi_n + \lim_{n\to\infty} \int_F \phi_n = \int_E f + \int_F f$$

since $\phi_n \to f$ pointwise on both E and F as they are subsets of $E \cup F$

Lemma 96

If $f \leq g$ almost anywhere on E, then $\int_E f \leq \int_E g$

Proof. Define the set

$$F = \{f \leq g\} \quad F^c = \{f > g\}$$

by assumption, $m(F^c) = 0$. Then recall 90 and consider

$$\int_E f = \int_F f + \int_{F^c} f = \int_F f \leq \int_F g = \int_F g + \int_{F^c} g = \int_E g$$

Theorem 97

If $f \in L^+(E)$ then $\int_E f = 0$ if and only if $f = 0$ almost everywhere on E

Proof. The zero function is a simple function since the range as only 1 element, that is zero. Thus $\int_E 0 = 0m(E) = 0$. Now, we mimic 96 but this time we define the set

$$F = \{f = 0\} \quad F^c = \{f \neq 0\}$$

by assumption, $m(F^c) = 0$. Then recall 90 and consider

$$\int_E f = \int_F f + \int_{F^c} f = \int_F f = \int_F g = \int_F 0 + \int_{F^c} 0 = \int_E 0 = 0$$

To prove the direction direction, that is to show that $\int_E f = 0 \to f = 0$ almost everywhere on E consider that

$$F^c = \{f \neq 0\} = \{f > 0\}$$

since $f \in L^+(E)$. Then we have

$$0m(F^c) = \int_{F^c} 0 \leq \int_{F^c} f \leq \int_E f = 0$$

But we cannot conclude anything about $m(F^c)$ unless we get rid of the zero preceding it. So we recall that we can represent F^c alternatively like so

$$\{f > 0\} = \bigcup_n^\infty \{f \geq \frac{1}{n}\}$$

where we define $F_n^c = \{f \geq \frac{1}{n}\}$ so we now have

$$\frac{1}{n} m(F_n^c) = \int_{F_n^c} \frac{1}{n} \leq \int_{F_n^c} f \leq \int_E f = 0$$

Clearly every $m(F_n^c) = 0$. Given that $F_1^c \subset F_2^c \subset \dots$ by the continuity of lebesgue measure we have

$$m(F^c) = \lim_{n\to\infty} m(F_n^c) = 0$$

□

Theorem 98

If $\{f_n\}$ is an increasing sequence in $L^+(E)$ for almost all $x \in E$ and $\lim_{n\to\infty} f_n(x) = f(x)$ then

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

Proof. Let F be the of $x \in E$ such that these 2 assumptions hold. Hence we have for almost all $x \in E$

$$f - \chi_F f = 0$$
$$f_n - \chi_F f_n = 0$$

so by 97 and *monotone convergence theorem* we have

$$\int_E f = \int_E f\chi_F = \int_F f = \lim_{n\to\infty} \int_F f_n = \lim_{n\to\infty} \int_E f_n$$

Remark 99. *With this we are able to relax the conditions for monotone convergence. But there exists an even more powerful convergence theorem as we shall see later*

Theorem 100 (Fatou's Lemma)
Let $\{f_n\}$ be a sequence in $L^+(E)$ Then

$$\int_E \liminf_{n\to\infty} f_n(x) \leq \liminf_{n\to\infty} \int_E f_n(x)$$

Proof. The objective is to bring out both the lim and the inf from the integral. Now recall that

$$\liminf f_n(x) = \sup_{n\geq 1}[\inf_{k\geq n} f_k(x)]$$

and recall that $\{\inf_{k\geq n} f_k(x)\}_n$ is an increasing sequence. We also know that the sup of an increasing sequence is basically the lim of the sequence so we have

$$\lim_{n\to\infty} \inf_{k\geq n} f_k(x) = \sup_{n\geq 1}[\inf_{k\geq n} f_k(x)]$$

Now applying *monotone convergence theorem* we have

$$\int_E \liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_E \inf_{k\geq n} f_k(x)$$

Now we try to bring out inf. Comparing

$$\int_E \inf_{k\geq n} f \leq \int_E f_j$$

is true for all $j \geq n$ thus we have

$$\int_E \inf_{k\geq n} f \leq \inf_{j\geq n} \int_E f_j$$

Now take $\lim_{n\to\infty}$ on both sides and apply *monotone convergence theorem* again. Then the conclusion follows. □

6.1.3 Lebesgue Integrable

We now aim to extend the definition of lebesgue integrals defined over non-negative measurable functions to measurable functions in general

Definition 101
Let $E \subset \mathbb{R}$ be measurable. A measurable function $f : E \to \mathbb{R}$ is **Lebesgue integrable** over E if $\int_E |f| < \infty$ Note that this is the L^1 space which you will learn more about in L^p space theory

Remark 102. *Notice how the class of lebesgue intergrals has changed. At first we had defined over integrals of simple functions where they were finite sums of measures of disjoint measurable sets. Then they were defined over the integrals of functions that can be expressed as suprenum of a sequence of integrals of simple functions. Now the the integral of our function is defined by being bounded using the absolute value.*

Definition 103

The **Lebesgue integral** of an integrable function $f : E \to \mathbb{R}$ is

$$\int_E f = \int_E f^+ - \int_E f^-$$

where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are the positive and negative parts of the function respectively.

Remark 104. *Both f^+ and f^- are clearly in $L^+(E)$. In paritcular, $|f| = f^+ + f^-$ and $f = f^+ - f^-$. Thus this definition follows by the linearity of non-negative functions that we have proven earlier*

Like all the classes of lebesgue intergrals we defined earlier, our definition here satisfies the usual properties of lebesgue integrals as we shall see.

Proposition 105

Suppose $f, g : E \to \mathbb{R}$ are integrable then

1. for all $c \in \mathbb{R}$, cf is integrable with $\int_E cf = c \int_E f$
2. the sum $f + g$ is integrable with $\int_E (f + g) = \int_E f + \int_E g$
3. if A, B are disjoint measurable sets then $\int_{A \cup B} f = \int_A f + \int_B f$

Proof. for (2) we have

$$\int_E |f + g| \leq \int_E |f| + |g| = \int_E |f| + \int_E |g| \leq \infty$$

So $(f + g)$ is indeed integrable. To prove $\int_E (f + g) = \int_E f + \int_E g$ simply split

$$f + g = (f^+ + g^+) - (f^- + g^-)$$

rearrange and apply *additivity lebesgue integrals of non-negative measurable functions*. The proofs for the rest follow the same strategy of splitting the function into its positive and negative parts then applying properties of lebesgue integrals of non-negative measurable functions

Fact 106 (full form of triangle inequality)

By the triangle inequality for any $a, b \in \mathbb{R}$ the following is always true

$$|a \pm b| \leq |a| + |b|$$

Proof. For the negative case

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$$

Proposition 107

Suppose $f, g : E \to \mathbb{R}$ are measurable functions. Then we have

1. if f is integrable then $|\int_E f| \leq \int_E |f|$
2. if g is integrable and $f = g$ almost everywhere then f is integrable and $\int_E f = \int_E g$
3. if f, g are integrable and $f \leq g$ almost everywhere then $\int_E f \leq \int_E g$

Proof. (1) follows from the *full form of triangle inquality*

$$\left|\int_E f\right| = \left|\int_E (f^+ - f^-)\right| = \left|\int_E f^+ - \int_E f^-\right| \leq \int_E f^+ + \int_E f^- = \int_E (f^+ + f^-) = \int_E |f|$$

□

Fact 108

Finally we move on the more powerful theorem as promised earlier. Recall that till now we had **monotone convergence theorem** that basically needed a monotonic sequence of functions that converge pointwise. But in the following theorem known as the **dominated convergence theorem**, this sequence of functions need not be monotonic.They only need to be convergent point-wise and bounded by some nonnegative integrable function *almost everywhere*

Theorem 109 (Dominated Convergence Theorem)

Let $g : E \to [0, \infty)$ be a nonnegative integrable function and let $\{f_n\}_n$ be a sequence of real-valued mearsurable functions such that (1)$|f_n| \leq g$ almost everywhere for all n and (2) there exists a function $f : E \to \mathbb{R}$ so that $f_n(x) \to f(x)$ pointwise almost everywhere on E. Then

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. Consider

$$\left|\int_E f_n\right| \leq \int_E |f_n| \leq \int_E g < \infty$$

. Since $\left\{\int_E f_n\right\}_n$ is bounded sequence(both above and below due to the absolute value), then there exists a finite lim sup and lim sup if you recall **Bonzalno Weinstrass Theorem** in foundational `real analysis` courses. So the obvious approach now is to prove they are equal for the limit to exist. Finish the rest of the proof using **Fatou's Lemma**... □

Fact 110

$C^n([a, b])$ denotes the continuity of the nth derivative in $[a, b]$. Thus in this case where $n = 1$, it denotes continuity of f over $[a, b]$.

Theorem 111

Let $f \in C([a, b])$ for some real numbers $a < b$. Then $\int_{[a,b]} f = \int_b^a f(x)dx$. That is to say the **Lebesgue** and **Riemann** on the left and right of this equality respectively agree.

Proof. Our approach is to construct a sequence of functions $\{F_n\}_n$ and show that

$$\lim_{n\to\infty} F_n = \int_b^a f(x)dx \tag{1}$$

$$\lim_{n\to\infty} F_n = \int_{[a,b]} f \tag{2}$$

And that because the limit of sequences are unique, $\int_{[a,b]} f = \int_b^a f(x)dx$ as desired. For this to happen in (1), we can infer that F_n must be the upper/lower Reinmann Integral corresponding to some partition n, in which case the sup or inf or lim over all possible n will get the Reinmann Integral. For (2) to happen, we can infer that this same F_n must also be a sequence of some integral of measurable function,s $\left\{\int_{[a,b]} f_n\right\}$ that converge to the Lebesgue Integral under the right conditions. So we begin our construction as follows

For (1), the Reinmann Integral suppose we have our partitions defined by

$$\underline{x}^n = \left\{a = x_0^n, x_1^n \dots x_{m_n}^n = b\right\}$$

Recall from basic `real analysis`, because f is continuous over a closed and bounded interval we know that if we suppose the norm/size of the partitions defined by $||\underline{x}^n|| = \max_{1 \leq j \leq m_n} \left|x_j^n - x_{j-1}^n\right|$ goes to 0 as $n \to \infty$ then by *theory of Rienmann Integration*, we know that the both the lower or upper Rienmann sums converge to the Rienmann Integral. So we define

$$F_n = \sum_{j=1}^{m_n} f(\xi_j^n)(x_j^n - x_{j-1}^n) \tag{3}$$

$$\lim_{n\to\infty} F_n = \int_b^a f(x)dx \tag{4}$$

where $\xi_j^n \in (x_j^n - x_{j-1}^n)$ is the point where the minimum in the interval is achieved which exists by **mean value theorem**.

For (2), the Lebesgue Integral we notice that

$$F_n = \sum_{j=1}^{m_n} f(\xi_j^n)(x_j^n - x_{j-1}^n) = \sum_{j=1}^{m_n} f(\xi_j^n)m([x_j^n - x_{j-1}^n])$$

Immediately we see that can define our measurable functions $\{f_n\}_n$ like so

$$F_n = \int_{[a,b]} f_n = \int_{[a,b]} \sum_{j=1}^{m_n} f(\xi_j^n)\chi_{[x_j^n - x_{j-1}^n]}$$

We now attempt to show that we do indeed have the conditions required for **dominated convergence theorem**. Firstly, we know f_n is bounded because it is a continuous function defined over closed and bounded interval as we can see from $f_n = f(\xi_j^n)\chi_{[x_j^n - x_{j-1}^n]}$. Hence it follows that $\int_{[a,b]} f$ must be bounded as well and thus *Lebesgue integrable* if we define $\lim_{n\to\infty} f_n = f$. Now finally we need to show this limit here is at least *pointwise almost everywhere*. Because f is continuous over $[a, b]$, we know that means there exists for some $\delta > 0$ where $|x - y| < \delta,\ x, y \in [a, b]$ then $|f(x) - f(y)| < \varepsilon$ for all $\varepsilon > 0$ for that x, y. The goal is to somehow relate relate this to $\forall\varepsilon, \exists M, |f_n(x) - f(x)| < \varepsilon, \forall x \geq M$

Firstly we know that partitions gets smaller as $n \to \infty$. Hence we can possibly let x, y to be in some interval $[x_j^n - x_{j-1}^n]$ corresponding to some partition $\underline{x}^n$ where $||\underline{x}_n|| < \delta$ for $n \geq M$

Secondly, we know that $f_n(x) = f(\xi_j^n)$ for some interval in some partition.

So we can try some $x, y = \xi_j^n$ in that interval to get our desired form

$$|f(\xi^n) - f(x)| = |f_n(x) - f(x)| < \varepsilon \tag{5}$$

$$\left|\xi_j^n - x\right| < \delta \tag{6}$$

for $n \geq M$ as desired. We now have to verify our δ for our chosen x, y satisfies all $\varepsilon > 0$ Because by definition of inf we have

$$f(\xi_j^n) \leq f(x) < f(\xi_j^n) + \varepsilon, x \in (x_j^n - x_{j-1}^n)$$

Hence it sure does! Now lets check pointwise convergence criteria. Firstly we need to ensure x is definitely inside an interval and not equal some partition point. If you recall the mean value theorem assumes $\xi_j^n \in (x_j^n - x_{j-1}^n)$ since differentiability at endpoints is not a given because end points are not necessarily continuous(you cannot test for continuity without points arbitrarily close to it). As such our inf relations we made earlier only applies to $(x_j^n - x_{j-1}^n)$ not $[x_j^n - x_{j-1}^n]$ as shown above. So we need to exclude $\bigcup_{n=1}^{\infty} \underline{x}^n$ which is a set of measure zero, because it is a countable union of countable sets which is countable. Thus we have pointwise convergence almost anywhere.

Corollary 112

A function $f : \mathbb{R} \to \mathbb{C}$ is lebesgue integrable if and only it is measurable and there exists a lebesgue integrable function F such that $|f| \leq F$ almost everywhere.

Proof. We know that any measurable function can be estimated by a sequence of simple functions that converge to it. Now if $|f| \leq F$ almost everywhere, then $|\phi_n| \leq F$ almost everywhere We now have condition for *dominated convergence theorem*

References

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