A Check of the Analytical Model of the Adiabatic Expansion Cluster Wind

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ABSTRACT

The power-law approximation is solid, which means there is barely deceleration toward the expanding wind.

1. Introduction

The check the spectral calculation code of my version, we set up a benchmark model as presented in Ji et al (2006, §3.4):

> *As an illustration, we consider a toy model for an adiabatically expanding stellar cluster wind with a constant mass input rate* $\dot{M} \sim 3 \times 10^{-5}\, M_\odot\ yr^{-1}$ *and a constant velocity* $v = 1000\ km\ s^{-1}$*. We assume that the wind is injected at an initial radius of* $r_0 = 0.3\ pc$ *and is heated up to a CIE state with an equilibrium temperature* $T_0 = 5 \times 10^6\ K$*. As the wind expands adiabatically,* ***its temperature drops as*** $T = T_0(r/r_0)^{-4/3}$.

However, The exponential expression looks too simple to me. The fluid follows a static solution that should be more sophisticated. Besides that, I still need to check the exact form describing the adiabatic expansion motion, as well as its approximation in this case, instead of guessing the expression for pressure, velocity and other physical parameters blindly.

2. A Solution to the Adiabatic Expansion

Starting from the basic equations of the fluid mechanics describing a steady flow:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho u r^2\right) = 0, \tag{1}$$

$$\rho u \frac{\mathrm{d}u}{\mathrm{d}r} + \frac{\mathrm{d}P}{\mathrm{d}r} = 0, \tag{2}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\rho u r^2 \left(\frac{1}{2}u^2 + \frac{5}{2}\frac{P}{\rho}\right)\right] = 0. \tag{3}$$

The right hands of those equations are all set to zero because we assume no further mass and energy supply/loss during the adiabatic expansion.

We could easily find that, from Eqn.(1)

$$\rho u r^2 \equiv c_1, \tag{4}$$

in which the constant c_1 could be derived based on the boundary condition at $r = r_0$ (but unimportant), and from Eqn.(3),

$$\frac{1}{2}u^2 + \frac{5}{2}\frac{P}{\rho} \equiv c_2 \Rightarrow \tag{5}$$

$$P = \frac{c_1}{5}\frac{1}{ur^2}\left(2c_2 - u^2\right), \tag{6}$$

and the constant c_2 follows

$$\begin{aligned} 2c_2 &= u_0^2 + 3c_{s0}^2 \\ \text{or} &= u_0^2 + 5kT/\mu m_{\rm H}, \end{aligned}$$

in which $c_{s0} = \sqrt{\gamma P_0/\rho_0} = 221.66$ km s^{-1} is the sound speed at r_0.

Substituting Eqn.(6) into Eqn.(2), we have

$$\begin{aligned} & \rho u r^2 \cdot \tfrac{1}{r^2}\tfrac{{\rm d}u}{{\rm d}r} + \tfrac{c_1}{5}\tfrac{\rm d}{{\rm d}r}\left[\tfrac{1}{ur^2}\left(2c_2 - u^2\right)\right] = 0 \\ \Leftrightarrow\quad & \tfrac{1}{r^2}\tfrac{{\rm d}u}{{\rm d}r} + \tfrac{1}{5}\tfrac{\rm d}{{\rm d}r}\left[\tfrac{2c_2}{r^2 u} - \tfrac{u}{r^2}\right] = 0 \\ \Leftrightarrow\quad & \tfrac{5}{r^2}\tfrac{{\rm d}u}{{\rm d}r} + \left[-\tfrac{4c_2}{r^3 u} - \tfrac{2c_2}{r^2u^2}\tfrac{{\rm d}u}{{\rm d}r} + \tfrac{2u}{r^3} - \tfrac{1}{r^2}\tfrac{{\rm d}u}{{\rm d}r}\right] = 0 \\ \Leftrightarrow\quad & \tfrac{2}{r^2}\left[2 - \tfrac{c_2}{u^2}\right]\tfrac{{\rm d}u}{{\rm d}r} = \tfrac{2}{r^3}\left[\tfrac{2c_2}{u} - u\right] \\ \Rightarrow\quad & \tfrac{2u^2 - c_2}{u^2(2c_2 - u^2)}{\rm d}u^2 = \tfrac{2}{r}{\rm d}r \\ \Leftrightarrow\quad & \left[\tfrac{3}{2c_2 - u^2} - \tfrac{1}{u^2}\right]{\rm d}u = \tfrac{4}{r}{\rm d}r \\ \Rightarrow\quad & -3\ln(2c_2 - u^2) - \ln u^2 + C = 4\ln r. \end{aligned}$$

Again, constant C is derived based on the boundary condition at r_0:

$$C = 3\ln(2c_2 - u_0^2) + \ln u_0^2 + 4\ln r_0, \tag{7}$$

which means the flow's motion follows

$$\left[\frac{2c_2 - u^2}{3c_{s0}^2}\right]^3\left[\frac{u^2}{2c_2 - 3c_{s0}^2}\right] = \left(\frac{r}{r_0}\right)^{-4}, \tag{8}$$

or

$$\left[1 + \frac{u_0^2 - u^2}{3c_{s0}^2}\right]^3\left[\frac{u^2}{u_0^2}\right] = \left(\frac{r}{r_0}\right)^{-4} \tag{9}$$

Replacing $u^2 = 2c_2 - 5kT/\mu m_{\rm H}$ and $c_{s0}^2 = 5kT/3\mu m_{\rm H}$, we have

$$\left(\frac{T}{T_0}\right)^3 \left(\frac{T_c - T}{T_c - T_0}\right) = \left(\frac{r}{r_0}\right)^{-4}, \tag{10}$$

in which $T_0 = 5 \times 10^6$ K at r_0, and $T_c = T_0 + \mu m_{\rm H} u_0^2/5k = 3.9 \times 10^7$ K.

3. Diagram and Approximation

In Eqn.(9), if the expression of flow velocity[1], u, is adopted as in 1,000 km s^{-1}–1,070 km s^{-1}, the first and second brackets will be in the range of 1.0–1.145 and 1.0–4.77×10^{-6}, respectively. Therefore, we could make a approximation as

$$u = \left[u_0^2 + 3c_{s0}^2 - 3c_{s0}^2 \left(r/r_0\right)^{-4/3}\right]^{0.5}. \tag{11}$$

Similar analysis to the solution of flow temperature returns

$$T = T_0 \left(\frac{r}{r_0}\right)^{-4/3}. \tag{12}$$

Actually, they are quite good approximation of less than 5% accuracy, as seen in Fig. 1.

[1]The solution of $u \leq u_0$ is discarded due to weird variation as a function of radius: thanks for Niu Shu pointing that out! However, in the $u \geq u_0$ case, u should not exceed $\sqrt{u_0^2 + 3c_{s0}^2} = 1071.17$ km s^{-1}.

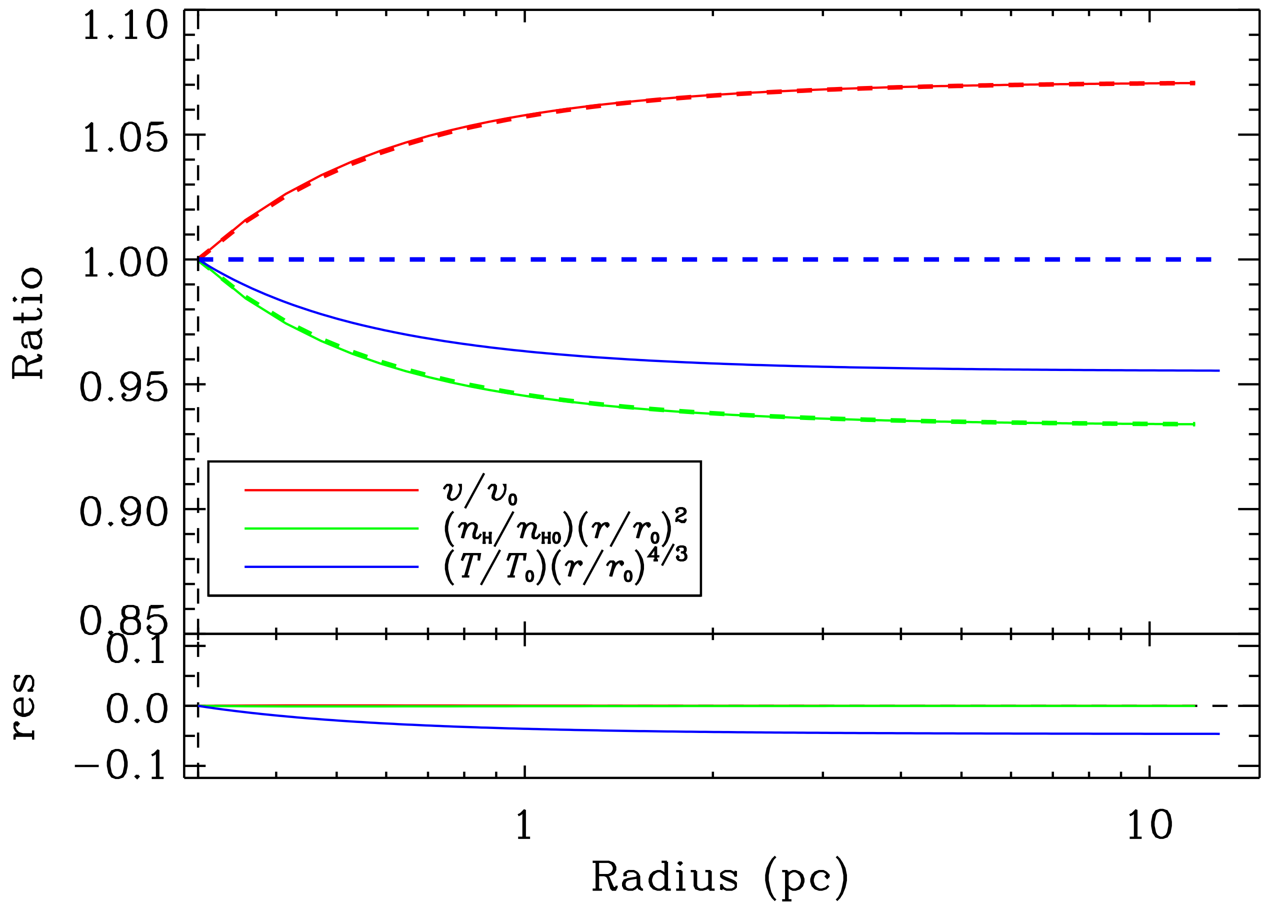

Fig. 1.— Radial profile of the flow velocity (red), density (green), and temperature (blue) of the adiabatic expansion. The solid lines follows the analytical solution in §2, and the dashed lines present the exponential approximations in §3. Their differences are plotted in the lower panel. The black dashed line marks the beginning radius r_0 of 0.3 pc.