ENGR 510 ODE Homework

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Exercise 1

$$\dot{x} = \lambda x, \quad x(0) = 1$$

1.a

General solution by separation of variables:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda x \\
\frac{\mathrm{d}x}{x} &= \lambda \mathrm{d}t \\
\ln(x) &= \lambda t + C \\
x(t) &= e^{\lambda t + C} \\
x(t) &= k e^{\lambda t}
\end{aligned}$$

Applying the initial condition:

$$\begin{aligned}
x(0) &= k e^{\lambda \cdot 0} \\
1 &= k
\end{aligned}$$

Therefore, the particular solution is:

$$\boxed{x(t) = e^{\lambda t}}$$

1.b

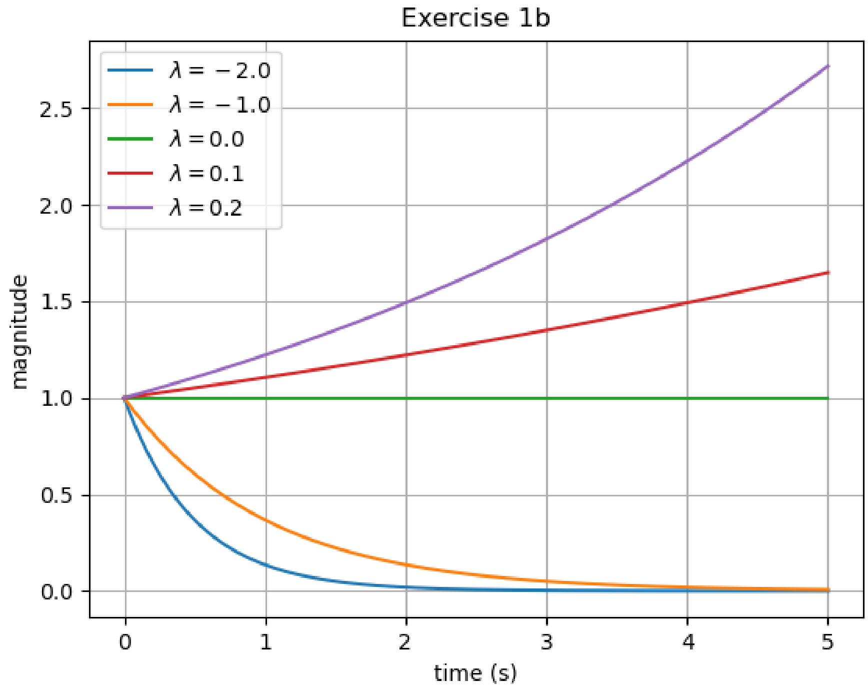

Figure 1: Solution to $\dot{x} = \lambda x$ for various values of λ

1.c

Positive values of λ diverge to infinity, negative values of λ converge to zero, and $\lambda = 0$ remains constant.

Exercise 2

$$\ddot{x} - \lambda x = 0$$

2.a

Guess $x(t) = e^{lt}$:

$$x(t) = e^{lt}$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = le^{lt}$$
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = l^2e^{lt}$$

Generate characteristic polynomial by substitution into the ODE:

$$
\begin{aligned}
l^2 e^{lt} - \lambda e^{lt} &= 0 \\
e^{\lambda t}\left(l^2 - \lambda\right) &= 0 \\
l^2 - \lambda &= 0
\end{aligned}
$$

Solutions to the characteristic polynomial are $l = \pm\sqrt{\lambda}$. The general solution is the linear combination of the particular solutions:

$$x(t) = C_1 e^{\sqrt{\lambda}t} + C_2 e^{-\sqrt{\lambda}t}$$

The first derivative is:

$$\dot{x}(t) = C_1\sqrt{\lambda}e^{\sqrt{\lambda}t} - C_2\sqrt{\lambda}e^{-\sqrt{\lambda}t}$$

Applying initial conditions given by $x(0)$ and $\dot{x}(0)$:

$$
\begin{aligned}
x(0) &= C_1 + C_2 \\
\dot{x}(0) &= C_1\sqrt{\lambda} - C_2\sqrt{\lambda}
\end{aligned}
$$

Solving the system of equations yields $C_1 = \frac{1}{2}\left(x(0) + \frac{\dot{x}(0)}{\sqrt{\lambda}}\right)$ and $C_2 = \frac{1}{2}\left(x(0) - \frac{\dot{x}(0)}{\sqrt{\lambda}}\right)$. Thus, the general solution to the ODE is

$$\boxed{x(t) = \frac{1}{2}\left(x(0) + \frac{\dot{x}(0)}{\sqrt{\lambda}}\right)e^{\sqrt{\lambda}t} + \frac{1}{2}\left(x(0) - \frac{\dot{x}(0)}{\sqrt{\lambda}}\right)e^{-\sqrt{\lambda}t}}$$

2.b

When $\lambda > 0$, all quantities are real and the solution diverges due to the positive exponent in the first term. When $\lambda < 0$, the exponents are purely imaginary so oscillatory behavior without damping ensues.

2.c

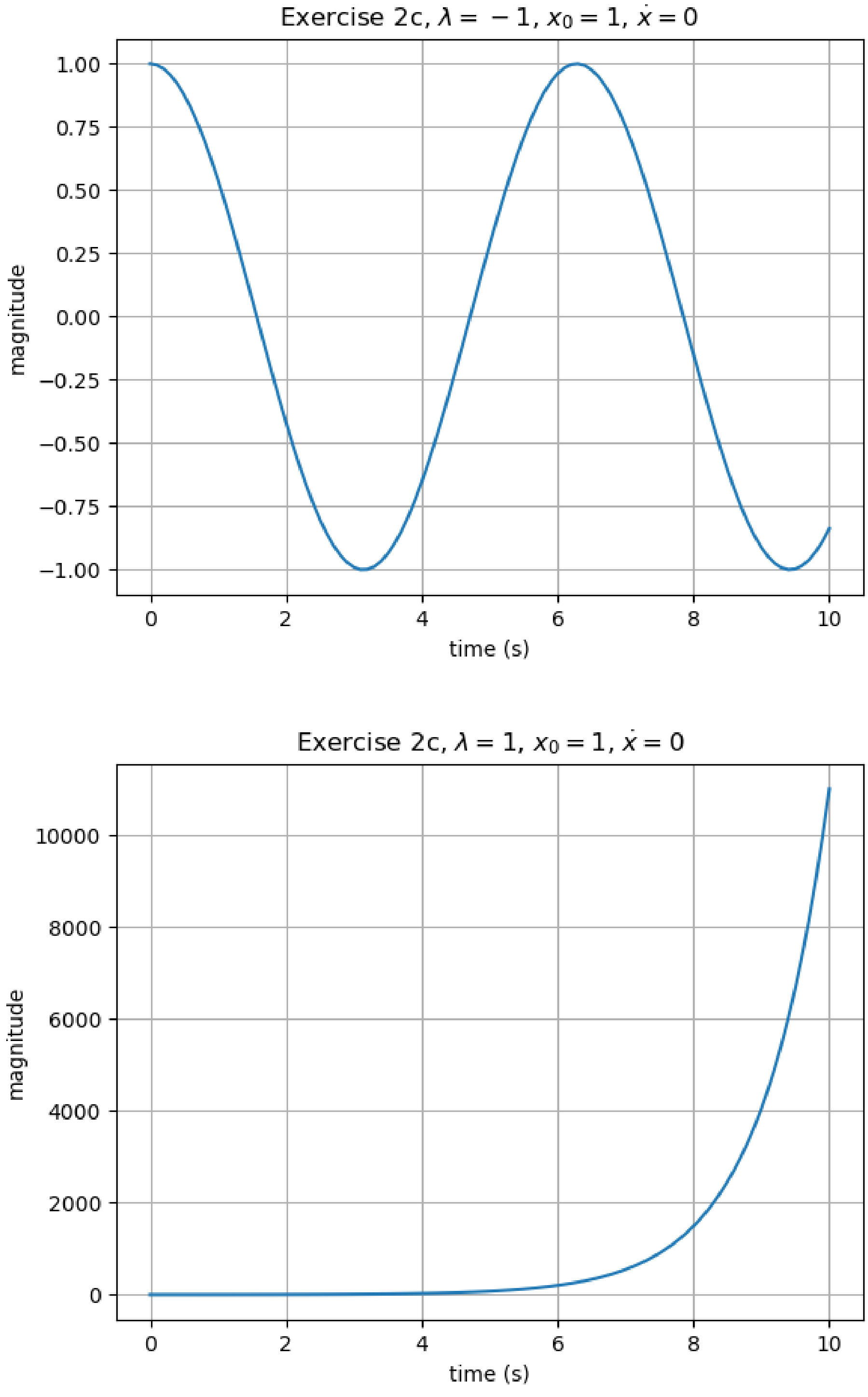

Figure 2: Solution to $\ddot{x} = \lambda x$ for various values of λ

Exercise 3

3.a

The eigenvalues $(-1, -3)$ are negative real numbers, so the exponential result converges without oscillation.

3.b

The eigenvalues $(0 + 2i, 0 - 2i)$ are purely imaginary which results a system that is purely oscillatory with no damping.

3.c

The eigenvalues (-2, 4) have at least one with a positive real component, so the exponential solution diverges.