ENGR520 - Spring 2025 **Due: 11:59pm Tuesday April 29^{th}**
Homework 2

Exercise 2–1: We will apply dynamic mode decomposition (DMD) to analyze the two-dimensional fluid flow past a circular cylinder. This flow exhibits periodic vortex shedding, and it is a canonical flow that is used to understand more sophisticated flows, such as over aircraft wings.

Using the provided template, load the cylinder flow vorticity data. There are 150 vorticity field snapshots, each of dimension 199×449. Thus, reshaping each flow field into a tall skinny vector and arranging into columns of a matrix, we obtain a data matrix with dimensions 89351×150. A single flow field snapshot is shown in Fig. 1.

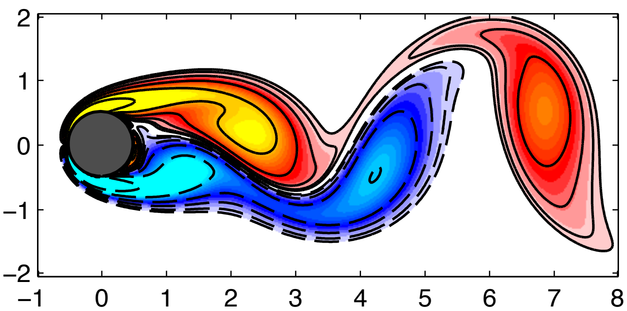

Figure 1: Single snapshot of the vorticity field for the fluid flow past a two-dimensional cylinder. This flow is periodic and conserves energy.

- Compute DMD on this cylinder flow dataset and plot the DMD eigenvalues in the complex plane. Use $r = 21$ modes, which corresponds to truncating and keeping only the first 21 modes in the singular value decomposition. This means you should have 21 DMD eigenvalues. DMD eigenvalues come in complex conjugate pairs and should be on the unit circle in the complex plane; what does it mean physically for a discrete-time linear system to have eigenvalues on the unit circle?
- Now add white noise with standard deviation of 1%, 10%, and 20% of the 2-norm of the data matrix divided by the square root of the number of points in the matrix, and repeat the exercise above. Plot a snapshot of the flow at each noise level, and plot the new DMD eigenvalues on the same plot as the DMD eigenvalues for the clean data. In what way(s) are the DMD eigenvalues misleading of the physics of the cylinder flow? In other words, if you had no knowledge of the DMD eigenvalues of the clean data, what conclusions would you draw from the noisy DMD eigenvalues?
- Now use the clean data, but take a subset of the flow field snapshots so that you only sample an incomplete period of vortex shedding (i.e., only sample so that you have 75% of a vortex shedding cycle). Repeat the exercise above and plot the DMD eigenvalues.

You can use the `PyDMD` package for any of this assignment.

In a future homework, we will use *physics-informed DMD* (PI-DMD) to incorporate physics into the DMD regression to improve the results for noisy and partial data. What are some potential aspects of the physics that we might try to include?

Exercise 2–2: We will now explore PINNs on two example vector fields that have the incompressible property (i.e., $\nabla \cdot \mathbf{u} = 0$). We will first consider a linear flow field, given by

$$\mathbf{u}(x, y, t) = \begin{bmatrix} u(x, y, t) \\ v(x, y, t) \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

This is the vector field obtained by a simple quadratic potential $\phi(x, y) = \frac{1}{2}(x^2 + y^2)$, where the vector field is obtained as

$$\mathbf{u}(x, y, t) = \begin{bmatrix} u(x, y, t) \\ v(x, y, t) \end{bmatrix} = \begin{bmatrix} -\partial\phi/\partial y \\ \partial\phi/\partial x \end{bmatrix}.$$

The vector field is shown in Fig. 2. The domain is $[-1, 1] \times [-1, 1]$.

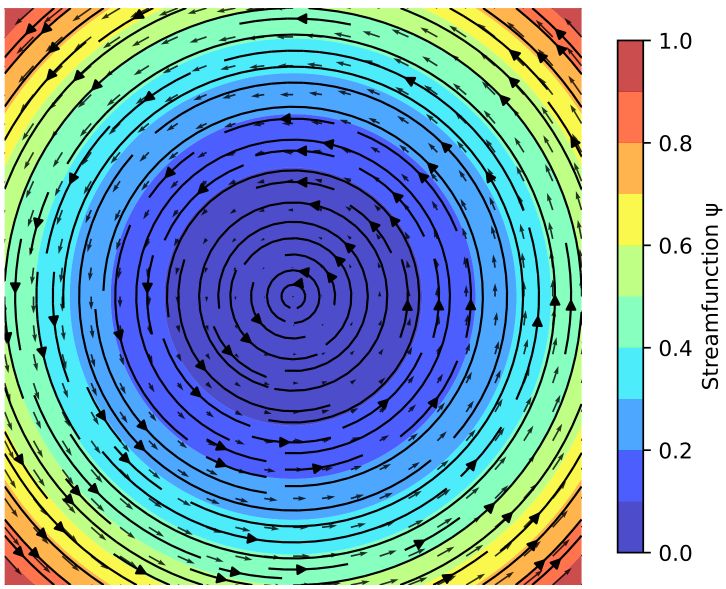

Figure 2: Vector field for linear vortex.

The second vector field is the Taylor-Green vortex. Although this flow is often defined on a domain of $[0, 2\pi] \times [0, 2\pi]$, we will scale the x and y axes to make the domain $[0, 1] \times [0, 1]$. The vector field is given by

$$\mathbf{u}(x, y, t) = \begin{bmatrix} u(x, y, t) \\ v(x, y, t) \end{bmatrix} = \begin{bmatrix} \sin(2\pi x)\cos(2\pi y) \\ -\cos(2\pi x)\sin(2\pi y) \end{bmatrix}.$$

This is the flow field obtained from the stream function

$$\phi(x, y) = \frac{1}{2\pi}\sin(2\pi x)\sin(2\pi y).$$

The vector field is shown in Fig. 3.

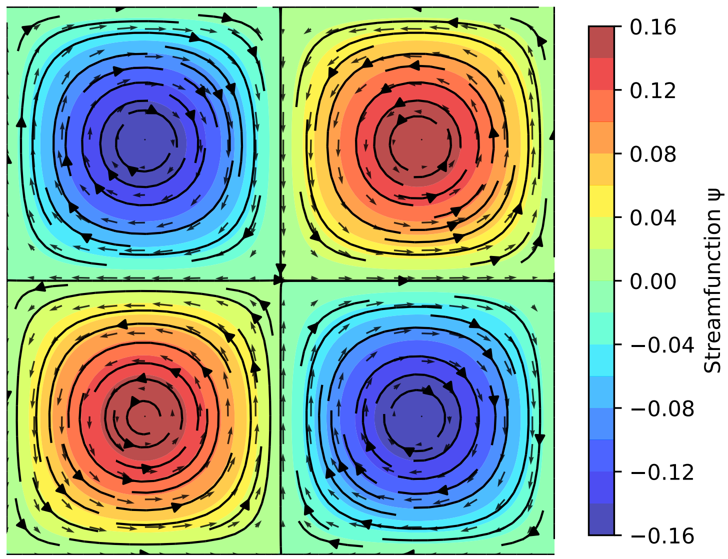

Figure 3: Vector field for Taylor-Green vortex.

Design a simple neural network with 3 hidden layers and 32 units per layer. The network has 2 inputs, x and y, and 2 outputs, u and v. The PDE loss is the divergence term. One option for taking derivatives is `torch.autograd.grad`.

Do each of the exercises below for the linear flow field and the Taylor-Green vortex field. For each, plot the true flow, the reconstructions, and the errors, including a colorbar for each image.

- Write down your loss function, including any learning rates or parameters, and in a few words, explain the purpose of each term.
- For both flow fields, use a coarse uniform sampling, with 10 points in the x direction and 10 points in the y direction (so 100 points total). Compare the results of linear interpolation and your PINN to fill in a higher resolution grid of 100×100.
- Now use the same coarse grid, but only sample the left half of the domain. How do the linear interpolation and PINN predictions work in the extrapolation region (i.e., the right half of the domain).
- Because the flow is symmetric about the middle of the domain, it is possible to add another custom loss term to enforce mirror symmetry. Write down this loss term, add this to the PINN, and see how the extrapolation behaves now. Discuss your results.

Hints: Before you start this problem, take a moment to think about how to structure and modularize your code for efficiency, re-using most of your code across all exercises. If coding a neural net or defining the loss function from scratch is intimidating, ask for help!