

# Exercise 22:

## Foundations of Mathematical, WS24

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This is **exercise** 22 for Foundations of Mathematical, WS24. Generated on 2025-04-21 with 10 problems per section.

2025-04-28

# 1. Problems

## 1.1. Vector Arithmetic

### 1.1.1. Addition

Find the sum of the following vectors  $\mathbf{u}$  and  $\mathbf{v}$

1.  $\mathbf{u} = \begin{bmatrix} 1 \\ -8 \\ 4 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
2.  $\mathbf{u} = \begin{bmatrix} 0 \\ -8 \\ -2 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -4 \\ -7 \\ 2 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
3.  $\mathbf{u} = \begin{bmatrix} -5 \\ 7 \\ 3 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -10 \\ 8 \\ -5 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
4.  $\mathbf{u} = \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
5.  $\mathbf{u} = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 6 \\ 6 \\ 4 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
6.  $\mathbf{u} = \begin{bmatrix} -5 \\ -1 \\ -5 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -3 \\ -5 \\ -3 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
7.  $\mathbf{u} = \begin{bmatrix} 3 \\ -4 \\ 8 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 5 \\ 0 \\ -10 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
8.  $\mathbf{u} = \begin{bmatrix} -10 \\ 0 \\ -1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
9.  $\mathbf{u} = \begin{bmatrix} 4 \\ 8 \\ -3 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -7 \\ -4 \\ 4 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
10.  $\mathbf{u} = \begin{bmatrix} 9 \\ 8 \\ 5 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -7 \\ -9 \\ -10 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .

### 1.1.2. Subtraction

Find the difference of the following vectors  $\mathbf{u}$  and  $\mathbf{v}$

1.  $\mathbf{u} = \begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 7 \\ -1 \\ -2 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
2.  $\mathbf{u} = \begin{bmatrix} -4 \\ 10 \\ -3 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
3.  $\mathbf{u} = \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 8 \\ 3 \\ 9 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
4.  $\mathbf{u} = \begin{bmatrix} -4 \\ 5 \\ 7 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -9 \\ 8 \\ -3 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
5.  $\mathbf{u} = \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -8 \\ -7 \\ 5 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .

6.  $\mathbf{u} = \begin{bmatrix} -2 \\ 5 \\ -3 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
7.  $\mathbf{u} = \begin{bmatrix} 10 \\ -4 \\ 6 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
8.  $\mathbf{u} = \begin{bmatrix} -8 \\ 1 \\ -6 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -4 \\ 8 \\ -2 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
9.  $\mathbf{u} = \begin{bmatrix} 9 \\ -1 \\ -1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -8 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
10.  $\mathbf{u} = \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .

### 1.1.3. Scalar Multiplication

Find the scalar product of the following vector  $\mathbf{u}$  and scalar  $k$

1.  $\mathbf{u} = \begin{bmatrix} -6 \\ -7 \\ -3 \end{bmatrix}$   $-1\mathbf{v}$ .
2.  $\mathbf{u} = \begin{bmatrix} -10 \\ 8 \\ -10 \end{bmatrix}$   $-6\mathbf{v}$ .
3.  $\mathbf{u} = \begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix}$   $6\mathbf{v}$ .
4.  $\mathbf{u} = \begin{bmatrix} 0 \\ 9 \\ 4 \end{bmatrix}$   $2\mathbf{v}$ .
5.  $\mathbf{u} = \begin{bmatrix} 7 \\ -5 \\ -10 \end{bmatrix}$   $9\mathbf{v}$ .
6.  $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$   $8\mathbf{v}$ .
7.  $\mathbf{u} = \begin{bmatrix} 1 \\ -5 \\ -1 \end{bmatrix}$   $10\mathbf{v}$ .
8.  $\mathbf{u} = \begin{bmatrix} -9 \\ -3 \\ -4 \end{bmatrix}$   $1\mathbf{v}$ .
9.  $\mathbf{u} = \begin{bmatrix} 9 \\ -8 \\ 4 \end{bmatrix}$   $0\mathbf{v}$ .
10.  $\mathbf{u} = \begin{bmatrix} -5 \\ -8 \\ -3 \end{bmatrix}$   $0\mathbf{v}$ .

## 1.2. Matrix Arithmetic

### 1.2.1. Addition

Find the sum of the following matrices  $A$  and  $B$

$$1. \quad A = \begin{bmatrix} 6 & 5 & 5 \\ -10 & 9 & 4 \\ -7 & -10 & -5 \end{bmatrix} \quad (1)$$

and

$$B = \begin{bmatrix} 2 & 8 & -3 \\ -7 & -1 & -7 \\ -7 & 4 & 5 \end{bmatrix} \quad (2)$$

$$2. \quad A = \begin{bmatrix} -6 & -8 & 0 \\ 4 & -6 & 1 \\ 6 & -3 & -5 \end{bmatrix} \quad (3)$$

and

$$B = \begin{bmatrix} -7 & 2 & -7 \\ 6 & 9 & -9 \\ -4 & -3 & -4 \end{bmatrix} \quad (4)$$

$$3. \quad A = \begin{bmatrix} 0 & 1 & -2 \\ -8 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} \quad (5)$$

and

$$B = \begin{bmatrix} -2 & -7 & -6 \\ 1 & -7 & -8 \\ -3 & -7 & 5 \end{bmatrix} \quad (6)$$

$$4. \quad A = \begin{bmatrix} -3 & -6 & 2 \\ -8 & 9 & 9 \\ 4 & -1 & -8 \end{bmatrix} \quad (7)$$

and

$$B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 7 & 5 \\ -4 & -2 & -1 \end{bmatrix} \quad (8)$$

$$5. \quad A = \begin{bmatrix} 0 & -6 & -5 \\ 5 & -3 & 8 \\ 7 & 6 & -10 \end{bmatrix} \quad (9)$$

and

$$B = \begin{bmatrix} 6 & 4 & -3 \\ -2 & -1 & -4 \\ -7 & 8 & 8 \end{bmatrix} \quad (10)$$

$$6. \quad A = \begin{bmatrix} -4 & 7 & -9 \\ 5 & 2 & 5 \\ 6 & -9 & -3 \end{bmatrix} \quad (11)$$

and

$$B = \begin{bmatrix} -5 & 1 & 8 \\ -9 & -9 & -3 \\ 9 & -9 & 8 \end{bmatrix} \quad (12)$$

7.

$$A = \begin{bmatrix} -9 & -6 & 2 \\ -7 & 0 & -9 \\ -4 & 2 & 3 \end{bmatrix} \quad (13)$$

and

$$B = \begin{bmatrix} 5 & 4 & 3 \\ 1 & -10 & -5 \\ -4 & 2 & 3 \end{bmatrix} \quad (14)$$

8.

$$A = \begin{bmatrix} 3 & -10 & -9 \\ -10 & -1 & -4 \\ 5 & -6 & -7 \end{bmatrix} \quad (15)$$

and

$$B = \begin{bmatrix} -1 & 5 & -7 \\ 4 & 6 & -10 \\ 8 & -7 & 2 \end{bmatrix} \quad (16)$$

9.

$$A = \begin{bmatrix} 3 & -6 & -2 \\ 3 & 0 & -2 \\ -4 & -3 & -6 \end{bmatrix} \quad (17)$$

and

$$B = \begin{bmatrix} 3 & -4 & -6 \\ 3 & 2 & -6 \\ 0 & 8 & -3 \end{bmatrix} \quad (18)$$

10.

$$A = \begin{bmatrix} -4 & 3 & 1 \\ -5 & 3 & 7 \\ -4 & 2 & -8 \end{bmatrix} \quad (19)$$

and

$$B = \begin{bmatrix} -7 & 4 & -3 \\ 4 & 9 & 3 \\ 7 & 1 & -2 \end{bmatrix} \quad (20)$$

### 1.2.2. Subtraction

Find the difference of the following matrices  $A$  and  $B$

$$1. \quad A = \begin{bmatrix} -4 & 7 & 7 \\ -5 & -9 & -3 \\ 4 & 8 & 2 \end{bmatrix} \quad (21)$$

and

$$B = \begin{bmatrix} 0 & 9 & -9 \\ -8 & 0 & 8 \\ 2 & -10 & -6 \end{bmatrix} \quad (22)$$

$$2. \quad A = \begin{bmatrix} -2 & 0 & -1 \\ 7 & 7 & -3 \\ 9 & -10 & 4 \end{bmatrix} \quad (23)$$

and

$$B = \begin{bmatrix} -3 & 6 & 8 \\ 3 & 8 & -7 \\ -7 & -3 & -4 \end{bmatrix} \quad (24)$$

$$3. \quad A = \begin{bmatrix} 0 & -9 & -1 \\ 0 & 2 & -6 \\ -2 & -6 & 2 \end{bmatrix} \quad (25)$$

and

$$B = \begin{bmatrix} -2 & 3 & 4 \\ 9 & -10 & -5 \\ -5 & -8 & 2 \end{bmatrix} \quad (26)$$

$$4. \quad A = \begin{bmatrix} -8 & -3 & -5 \\ 2 & 1 & -8 \\ 4 & -1 & -6 \end{bmatrix} \quad (27)$$

and

$$B = \begin{bmatrix} 7 & -1 & 9 \\ -4 & -1 & 2 \\ 0 & -4 & 5 \end{bmatrix} \quad (28)$$

$$5. \quad A = \begin{bmatrix} -4 & -3 & 2 \\ 7 & 9 & -10 \\ 8 & 6 & 9 \end{bmatrix} \quad (29)$$

and

$$B = \begin{bmatrix} 5 & 6 & 2 \\ -7 & -1 & -8 \\ 4 & -9 & 2 \end{bmatrix} \quad (30)$$

$$6. \quad A = \begin{bmatrix} -9 & 3 & 5 \\ -6 & 5 & -1 \\ -3 & -4 & -7 \end{bmatrix} \quad (31)$$

and

$$B = \begin{bmatrix} 5 & -1 & -3 \\ -5 & 8 & 7 \\ 9 & 4 & 3 \end{bmatrix} \quad (32)$$

7.

$$A = \begin{bmatrix} -4 & -7 & -3 \\ -2 & 9 & 9 \\ 8 & 0 & 6 \end{bmatrix} \quad (33)$$

and

$$B = \begin{bmatrix} 3 & -4 & -3 \\ 0 & 3 & 4 \\ 4 & -7 & 4 \end{bmatrix} \quad (34)$$

8.

$$A = \begin{bmatrix} -10 & 1 & 3 \\ -6 & 0 & -4 \\ -6 & 0 & -6 \end{bmatrix} \quad (35)$$

and

$$B = \begin{bmatrix} 4 & -1 & -4 \\ 3 & 5 & 2 \\ 3 & -5 & -9 \end{bmatrix} \quad (36)$$

9.

$$A = \begin{bmatrix} 3 & 9 & -1 \\ -10 & 6 & 3 \\ 0 & -1 & 5 \end{bmatrix} \quad (37)$$

and

$$B = \begin{bmatrix} 5 & -2 & 6 \\ -8 & 0 & -5 \\ -1 & 8 & 7 \end{bmatrix} \quad (38)$$

10.

$$A = \begin{bmatrix} 7 & 2 & -10 \\ 8 & -6 & 8 \\ -9 & 9 & 9 \end{bmatrix} \quad (39)$$

and

$$B = \begin{bmatrix} 4 & -1 & -6 \\ 8 & 5 & 4 \\ -3 & 2 & -10 \end{bmatrix} \quad (40)$$

### 1.2.3. Multiplication

Find the product of the following matrices  $A$  and  $B$

$$1. \quad A = \begin{bmatrix} -10 & 7 & -5 \\ -1 & -5 & -10 \\ 1 & -6 & -7 \end{bmatrix} \quad (41)$$

and

$$B = \begin{bmatrix} -6 & 5 & -3 \\ -7 & -9 & 8 \\ -9 & -1 & 8 \end{bmatrix} \quad (42)$$

$$2. \quad A = \begin{bmatrix} 4 & -3 & 8 \\ 9 & 8 & -8 \\ 8 & 0 & -9 \end{bmatrix} \quad (43)$$

and

$$B = \begin{bmatrix} 1 & 7 & 8 \\ 5 & -8 & 3 \\ 4 & 6 & -6 \end{bmatrix} \quad (44)$$

$$3. \quad A = \begin{bmatrix} 0 & 5 & -4 \\ 3 & -9 & 7 \\ -3 & 6 & 8 \end{bmatrix} \quad (45)$$

and

$$B = \begin{bmatrix} 1 & 8 & 5 \\ -8 & -2 & -1 \\ -6 & -9 & -9 \end{bmatrix} \quad (46)$$

$$4. \quad A = \begin{bmatrix} -3 & -6 & -4 \\ -8 & -10 & -2 \\ 5 & -5 & 2 \end{bmatrix} \quad (47)$$

and

$$B = \begin{bmatrix} 8 & -2 & -6 \\ -10 & -7 & 7 \\ 8 & 1 & -10 \end{bmatrix} \quad (48)$$

$$5. \quad A = \begin{bmatrix} -6 & 9 & -7 \\ 7 & -7 & -8 \\ -10 & 5 & -7 \end{bmatrix} \quad (49)$$

and

$$B = \begin{bmatrix} -5 & 8 & -8 \\ -7 & -4 & -5 \\ 0 & 3 & -5 \end{bmatrix} \quad (50)$$

$$6. \quad A = \begin{bmatrix} 6 & -3 & 3 \\ -8 & 6 & 3 \\ 3 & 3 & 6 \end{bmatrix} \quad (51)$$



and

$$B = \begin{bmatrix} 0 & -10 & -8 \\ -4 & -3 & 1 \\ 1 & -2 & 0 \end{bmatrix} \quad (52)$$

7.

$$A = \begin{bmatrix} -6 & 5 & 7 \\ -8 & -8 & -9 \\ 5 & 2 & -4 \end{bmatrix} \quad (53)$$

and

$$B = \begin{bmatrix} 1 & 1 & -3 \\ -8 & 5 & -9 \\ -10 & 3 & -5 \end{bmatrix} \quad (54)$$

8.

$$A = \begin{bmatrix} -5 & -6 & -4 \\ -1 & 2 & -7 \\ -3 & -6 & 4 \end{bmatrix} \quad (55)$$

and

$$B = \begin{bmatrix} -4 & 0 & 6 \\ 9 & -4 & 8 \\ 7 & 3 & -2 \end{bmatrix} \quad (56)$$

9.

$$A = \begin{bmatrix} 2 & -8 & 2 \\ -6 & -5 & -10 \\ -6 & -10 & 8 \end{bmatrix} \quad (57)$$

and

$$B = \begin{bmatrix} 7 & -4 & -9 \\ -1 & -8 & -5 \\ -3 & -7 & -9 \end{bmatrix} \quad (58)$$

10.

$$A = \begin{bmatrix} 7 & 5 & -10 \\ 4 & 4 & -6 \\ -8 & 0 & -10 \end{bmatrix} \quad (59)$$

and

$$B = \begin{bmatrix} 8 & -7 & 8 \\ 8 & 7 & 5 \\ -1 & 3 & 2 \end{bmatrix} \quad (60)$$

## 1.3. Matrix Properties

### 1.3.1. Properties

For each matrix  $A$ , find:

a)  $\text{rank}(A)$

- b) nullity( $A$ )
- c)  $\det(A)$
- d)  $A^{-1}$  (if exists)
- e) basis of  $\ker(A)$

1. 
$$A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} \quad (61)$$

2. 
$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \quad (62)$$

3. 
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \quad (63)$$

4. 
$$A = \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 2 \\ -2 & 6 & 4 \end{bmatrix} \quad (64)$$

5. 
$$A = \begin{bmatrix} 1 & -4 & 4 \\ 1 & -3 & 2 \\ -1 & 2 & 1 \end{bmatrix} \quad (65)$$

6. 
$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (66)$$

7. 
$$A = \begin{bmatrix} 1 & -4 & 6 \\ 0 & -1 & 2 \\ 1 & -4 & 5 \end{bmatrix} \quad (67)$$

8. 
$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & -1 \\ 1 & -2 & 0 \end{bmatrix} \quad (68)$$

9. 
$$A = \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (69)$$

10. 
$$A = \begin{bmatrix} 1 & 2 & -8 \\ 1 & -1 & -3 \\ -1 & 2 & 1 \end{bmatrix} \quad (70)$$

### 1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix  $A$

1. 
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix} \quad (71)$$

$$2. \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -2 & 0 \end{bmatrix} \quad (72)$$

$$3. \quad A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -4 \\ 1 & -2 & 2 \end{bmatrix} \quad (73)$$

$$4. \quad A = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \quad (74)$$

$$5. \quad A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (75)$$

$$6. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad (76)$$

$$7. \quad A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (77)$$

$$8. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \quad (78)$$

$$9. \quad A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad (79)$$

$$10. \quad A = \begin{bmatrix} 2 & 2 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix} \quad (80)$$

## 1.4. Calculus

### 1.4.1. Limit

Calculate the following limits

1. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (81)$$

2. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} -x - 2 \quad (82)$$

3. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (83)$$

4. Calculate the limit of the following expression:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \quad (84)$$

5. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} 3x^2 - 4x \quad (85)$$

6. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (86)$$

7. Calculate the limit of the following expression:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \quad (87)$$

8. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x + 1)}{x} \quad (88)$$

9. Calculate the limit of the following expression:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \quad (89)$$

10. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x + 1)}{x} \quad (90)$$

### 1.4.2. Derivative

Calculate the derivatives of the following expressions

1. Calculate the derivative of the following expression:

$$x^2 \log(x) \quad (91)$$

2. Calculate the derivative of the following expression:

$$x^4 \quad (92)$$

3. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (93)$$

4. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (94)$$

5. Calculate the derivative of the following expression:

$$x^4 \quad (95)$$

6. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (96)$$

7. Calculate the derivative of the following expression:

$$\frac{x^3}{x^2 + 1} \quad (97)$$

8. Calculate the derivative of the following expression:

$$x^2 \log(x) \quad (98)$$

9. Calculate the derivative of the following expression:

$$\log(x + 1) + \log(x^2 + 1) \quad (99)$$

10. Calculate the derivative of the following expression:

$$x^3 \log(x) \quad (100)$$

### 1.4.3. Integral

Calculate the indefinite and definite integrals of the following expressions

1. Evaluate the improper integral:

$$\int_1^{\infty} \frac{1}{x^2} dx \quad (101)$$

2. the indefinite integral and evaluate from 4 to 4:

$$\int -x^4 + 3x^3 + 4x^2 - 4x dx \quad (102)$$

3. Evaluate the improper integral:

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx \quad (103)$$

4. the indefinite integral and evaluate from 4 to 5:

$$\int \frac{\sin(x)}{x} dx \quad (104)$$

5. the indefinite integral and evaluate from 2 to 4:

$$\int \frac{x}{x^2 - 5x + 6} dx \quad (105)$$

6. Evaluate the improper integral:

$$\int_1^{oo} e^{-x} dx \quad (106)$$

7. the indefinite integral and evaluate from 2 to 5:

$$\int e^x \sin(x) dx \quad (107)$$

8. the indefinite integral and evaluate from 4 to 4:

$$\int 2x^4 - 3x^3 - x^2 - 3x + 3 dx \quad (108)$$

9. the indefinite integral and evaluate from 2 to 5:

$$\int \frac{1}{(x-2)(x+1)} dx \quad (109)$$

10. Evaluate the improper integral:

$$\int_1^{oo} e^{-x} dx \quad (110)$$

#### 1.4.4. Partial Derivative

Calculate the partial derivatives of the following expressions

1. the mixed partial derivative of:

$$f(x, y) = x^3 y^2 + xy^4 \quad (111)$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

2. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \quad (112)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

3. the mixed partial derivative of:

$$f(x, y) = x^3 y^2 + xy^4 \quad (113)$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

4. Given the implicit function:

$$x^2 y + xy^2 - xy = 0 \quad (114)$$

$$\frac{\partial y}{\partial x}$$

5. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \quad (115)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

6. the mixed partial derivative of:

$$f(x, y) = x^3y^2 + xy^4 \quad (116)$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

7. the second order partial derivative of:

$$f(x, y) = x^4y^3 + 3x^2y^4 \quad (117)$$

$$\frac{\partial^2 f}{\partial x^2}$$

8. the partial derivatives of the function:

$$f(x, y) = x^3y^2 - 3x^2y + 2xy^3 \quad (118)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

9. the partial derivatives of the function:

$$f(x, y) = x^3y^2 - 3x^2y + 2xy^3 \quad (119)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

10. the partial derivatives of the function:

$$f(x, y) = x^3y^2 - 3x^2y + 2xy^3 \quad (120)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

## 2. Solutions

### 2.1. Vector Arithmetic

#### 2.1.1. Addition

$$\begin{bmatrix} 8 \\ -5 \\ 10 \end{bmatrix} + \begin{bmatrix} -4 \\ -15 \\ 0 \end{bmatrix} = \begin{bmatrix} -15 \\ 15 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 13 \\ -6 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \\ -5 \end{bmatrix}$$
$$\begin{bmatrix} -8 \\ -6 \\ -8 \end{bmatrix} + \begin{bmatrix} 8 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}$$

#### 2.1.2. Subtraction

$$\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 11 \\ -5 \end{bmatrix} = \begin{bmatrix} -3 \\ -11 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 5 \\ -3 \\ 10 \end{bmatrix} - \begin{bmatrix} 5 \\ 14 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -17 \\ 13 \end{bmatrix}$$
$$\begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} - \begin{bmatrix} 17 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -24 \\ 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -4 \\ -7 \\ -4 \end{bmatrix} - \begin{bmatrix} 7 \\ -4 \\ 7 \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -11 \end{bmatrix}$$

#### 2.1.3. Scalar Multiplication

$$1: \begin{bmatrix} 6 \\ 7 \\ 3 \end{bmatrix} \quad 2: \begin{bmatrix} 60 \\ -48 \\ 60 \end{bmatrix} \quad 3: \begin{bmatrix} -30 \\ 12 \\ 24 \end{bmatrix} \quad 4: \begin{bmatrix} 0 \\ 18 \\ 8 \end{bmatrix} \quad 5: \begin{bmatrix} 63 \\ -45 \\ -90 \end{bmatrix}$$
$$6: \begin{bmatrix} 16 \\ 8 \\ -40 \end{bmatrix} \quad 7: \begin{bmatrix} 10 \\ -50 \\ -10 \end{bmatrix} \quad 8: \begin{bmatrix} -9 \\ -3 \\ -4 \end{bmatrix} \quad 9: \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad 10: \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

## 2.2. Matrix Arithmetic

### 2.2.1. Addition

1:

$$\begin{bmatrix} 8 & 13 & 2 \\ -17 & 8 & -3 \\ -14 & -6 & 0 \end{bmatrix} \quad (121)$$

1:

$$\begin{bmatrix} -13 & -6 & -7 \\ 10 & 3 & -8 \\ 2 & -6 & -9 \end{bmatrix} \quad (122)$$

1:

$$\begin{bmatrix} -2 & -6 & -8 \\ -7 & -6 & -7 \\ 5 & -3 & 8 \end{bmatrix} \quad (123)$$

1:

$$\begin{bmatrix} -4 & -5 & 4 \\ -8 & 16 & 14 \\ 0 & -3 & -9 \end{bmatrix} \quad (124)$$



1:

$$\begin{bmatrix} 6 & -2 & -8 \\ 3 & -4 & 4 \\ 0 & 14 & -2 \end{bmatrix} \quad (125)$$

1:

$$\begin{bmatrix} -9 & 8 & -1 \\ -4 & -7 & 2 \\ 15 & -18 & 5 \end{bmatrix} \quad (126)$$

1:

$$\begin{bmatrix} -4 & -2 & 5 \\ -6 & -10 & -14 \\ -8 & 4 & 6 \end{bmatrix} \quad (127)$$

1:

$$\begin{bmatrix} 2 & -5 & -16 \\ -6 & 5 & -14 \\ 13 & -13 & -5 \end{bmatrix} \quad (128)$$

1:

$$\begin{bmatrix} 6 & -10 & -8 \\ 6 & 2 & -8 \\ -4 & 5 & -9 \end{bmatrix} \quad (129)$$

1:

$$\begin{bmatrix} -11 & 7 & -2 \\ -1 & 12 & 10 \\ 3 & 3 & -10 \end{bmatrix} \quad (130)$$

### 2.2.2. Subtraction

1:

$$\begin{bmatrix} -4 & -2 & 16 \\ 3 & -9 & -11 \\ 2 & 18 & 8 \end{bmatrix} \quad (131)$$

1:

$$\begin{bmatrix} 1 & -6 & -9 \\ 4 & -1 & 4 \\ 16 & -7 & 8 \end{bmatrix} \quad (132)$$

1:

$$\begin{bmatrix} 2 & -12 & -5 \\ -9 & 12 & -1 \\ 3 & 2 & 0 \end{bmatrix} \quad (133)$$

1:

$$\begin{bmatrix} -15 & -2 & -14 \\ 6 & 2 & -10 \\ 4 & 3 & -11 \end{bmatrix} \quad (134)$$

1:

$$\begin{bmatrix} -9 & -9 & 0 \\ 14 & 10 & -2 \\ 4 & 15 & 7 \end{bmatrix} \quad (135)$$

1:

$$\begin{bmatrix} -14 & 4 & 8 \\ -1 & -3 & -8 \\ -12 & -8 & -10 \end{bmatrix} \quad (136)$$

1:

$$\begin{bmatrix} -7 & -3 & 0 \\ -2 & 6 & 5 \\ 4 & 7 & 2 \end{bmatrix} \quad (137)$$

1:

$$\begin{bmatrix} -14 & 2 & 7 \\ -9 & -5 & -6 \\ -9 & 5 & 3 \end{bmatrix} \quad (138)$$

1:

$$\begin{bmatrix} -2 & 11 & -7 \\ -2 & 6 & 8 \\ 1 & -9 & -2 \end{bmatrix} \quad (139)$$

1:

$$\begin{bmatrix} 3 & 3 & -4 \\ 0 & -11 & 4 \\ -6 & 7 & 19 \end{bmatrix} \quad (140)$$

### 2.2.3. Multiplication

1:

$$\begin{bmatrix} 56 & -108 & 46 \\ 131 & 50 & -117 \\ 99 & 66 & -107 \end{bmatrix} \quad (141)$$

1:

$$\begin{bmatrix} 21 & 100 & -25 \\ 17 & -49 & 144 \\ -28 & 2 & 118 \end{bmatrix} \quad (142)$$

1:

$$\begin{bmatrix} -16 & 26 & 31 \\ 33 & -21 & -39 \\ -99 & -108 & -93 \end{bmatrix} \quad (143)$$

1:

$$\begin{bmatrix} 4 & 44 & 16 \\ 20 & 84 & -2 \\ 106 & 27 & -85 \end{bmatrix} \quad (144)$$

1:

$$\begin{bmatrix} -33 & -105 & 38 \\ 14 & 60 & 19 \\ 15 & -121 & 90 \end{bmatrix} \quad (145)$$

1:

$$\begin{bmatrix} 15 & -57 & -51 \\ -21 & 56 & 70 \\ -6 & -51 & -21 \end{bmatrix} \quad (146)$$

1:

$$\begin{bmatrix} -116 & 40 & -62 \\ 146 & -75 & 141 \\ 29 & 3 & -13 \end{bmatrix} \quad (147)$$

1:

$$\begin{bmatrix} -62 & 12 & -70 \\ -27 & -29 & 24 \\ -14 & 36 & -74 \end{bmatrix} \quad (148)$$

1:

$$\begin{bmatrix} 16 & 42 & 4 \\ -7 & 134 & 169 \\ -56 & 48 & 32 \end{bmatrix} \quad (149)$$

1:

$$\begin{bmatrix} 106 & -44 & 61 \\ 70 & -18 & 40 \\ -54 & 26 & -84 \end{bmatrix} \quad (150)$$

## 2.3. Matrix Properties

### 2.3.1. Properties

#### Solution

#### Row Operations:

$$\text{Step 1: } r_1 := r_1 - r_2 \quad \begin{bmatrix} 1 & 0 & -5 & | & 1 & -1 & 0 \\ 0 & 1 & 3 & | & 0 & 1 & 0 \\ 0 & -1 & -2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_3 := r_3 - (-1)r_2 \quad \begin{bmatrix} 1 & 0 & -5 & | & 1 & -1 & 0 \\ 0 & 1 & 3 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (-5)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & 1 & 4 & 5 \\ 0 & 1 & 3 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_2 := r_2 - (3)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & 1 & 4 & 5 \\ 0 & 1 & 0 & | & 0 & -2 & -3 \\ 0 & 0 & 1 & | & 0 & 1 & 1 \end{bmatrix}$$

#### Results:

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

#### Solution

#### Row Operations:

$$\text{Step 1: } r_2 \leftrightarrow r_3 \quad \begin{bmatrix} 1 & -1 & -1 & | & 1 & 0 & 0 \\ 0 & -1 & 2 & | & 0 & 0 & 1 \\ 0 & 0 & 0 & | & 0 & 1 & 0 \end{bmatrix}$$

$$\text{Step 2: } r_2 := -1r_2 \quad \begin{bmatrix} 1 & -1 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & -2 & | & 0 & 0 & -1 \\ 0 & 0 & 0 & | & 0 & 1 & 0 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (-1)r_2 \quad \begin{bmatrix} 1 & 0 & -3 & | & 1 & 0 & -1 \\ 0 & 1 & -2 & | & 0 & 0 & -1 \\ 0 & 0 & 0 & | & 0 & 1 & 0 \end{bmatrix}$$

#### Results:

a)  $\text{rank}(A) = 2$

b)  $\text{nullity}(A) = 1$

c)  $\det(A) = 0$

d)  $A^{-1}$  does not exist

e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

**Solution**

**Row Operations:**

Step 1:  $r_1 := r_1 - (2)r_2$  
$$\begin{bmatrix} 1 & 0 & 7 & | & 1 & -2 & 0 \\ 0 & 1 & -3 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

Step 2:  $r_1 := r_1 - (7)r_3$  
$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & -2 & -7 \\ 0 & 1 & -3 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

Step 3:  $r_2 := r_2 - (-3)r_3$  
$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & -2 & -7 \\ 0 & 1 & 0 & | & 0 & 1 & 3 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

**Results:**

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

**Solution**

**Row Operations:**

Step 1:  $r_3 := r_3 - (-2)r_1$  
$$\begin{bmatrix} 1 & -4 & -4 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & -2 & -4 & | & 2 & 0 & 1 \end{bmatrix}$$

Step 2:  $r_1 := r_1 - (-4)r_2$  
$$\begin{bmatrix} 1 & 0 & 4 & | & 1 & 4 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & -2 & -4 & | & 2 & 0 & 1 \end{bmatrix}$$

Step 3:  $r_3 := r_3 - (-2)r_2$  
$$\begin{bmatrix} 1 & 0 & 4 & | & 1 & 4 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 2 & 2 & 1 \end{bmatrix}$$

**Results:**

a)  $\text{rank}(A) = 2$

b)  $\text{nullity}(A) = 1$

c)  $\det(A) = 0$

d)  $A^{-1}$  does not exist

e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

### Solution

#### Row Operations:

Step 1:  $r_2 := r_2 - r_1$   $\left[ \begin{array}{ccc|ccc} 1 & -4 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array} \right]$

Step 2:  $r_3 := r_3 - (-1)r_1$   $\left[ \begin{array}{ccc|ccc} 1 & -4 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & -2 & 5 & 1 & 0 & 1 \end{array} \right]$

Step 3:  $r_1 := r_1 - (-4)r_2$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 4 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & -2 & 5 & 1 & 0 & 1 \end{array} \right]$

Step 4:  $r_3 := r_3 - (-2)r_2$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 4 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array} \right]$

Step 5:  $r_1 := r_1 - (-4)r_3$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 12 & 4 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array} \right]$

Step 6:  $r_2 := r_2 - (-2)r_3$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 12 & 4 \\ 0 & 1 & 0 & -3 & 5 & 2 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array} \right]$

### Results:

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

### Solution

#### Row Operations:

Step 1:  $r_1 := r_1 - (-1)r_2$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$

Step 2:  $r_1 := r_1 - (4)r_3$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -4 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$

Step 3:  $r_2 := r_2 - r_3$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -4 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$

### Results:

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

### Solution

#### Row Operations:

Step 1:  $r_3 := r_3 - r_1$   $\left[ \begin{array}{ccc|ccc} 1 & -4 & 6 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right]$

Step 2:  $r_2 := -1r_2$   $\left[ \begin{array}{ccc|ccc} 1 & -4 & 6 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right]$

Step 3:  $r_1 := r_1 - (-4)r_2$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -4 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right]$

Step 4:  $r_3 := -1r_3$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -4 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array} \right]$

Step 5:  $r_1 := r_1 - (-2)r_3$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & -2 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array} \right]$

Step 6:  $r_2 := r_2 - (-2)r_3$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & -2 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array} \right]$

#### Results:

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

### Solution

#### Row Operations:

Step 1:  $r_2 := r_2 - (-2)r_1$   $\left[ \begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{array} \right]$

Step 2:  $r_3 := r_3 - r_1$   $\left[ \begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right]$

$$\text{Step 3: } r_1 := r_1 - (-2)r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & -2 & 5 & 2 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right]$$

**Results:**

- a)  $\text{rank}(A) = 2$
- b)  $\text{nullity}(A) = 1$
- c)  $\det(A) = 0$
- d)  $A^{-1}$  = does not exist
- e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

**Solution**

**Row Operations:**

$$\text{Step 1: } r_1 := r_1 - (-2)r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

**Results:**

- a)  $\text{rank}(A) = 2$
- b)  $\text{nullity}(A) = 1$
- c)  $\det(A) = 0$
- d)  $A^{-1}$  = does not exist
- e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$

**Solution**

**Row Operations:**

$$\text{Step 1: } r_2 := r_2 - r_1 \quad \left[ \begin{array}{ccc|ccc} 1 & 2 & -8 & 1 & 0 & 0 \\ 0 & -3 & 5 & -1 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_3 := r_3 - (-1)r_1 \quad \left[ \begin{array}{ccc|ccc} 1 & 2 & -8 & 1 & 0 & 0 \\ 0 & -3 & 5 & -1 & 1 & 0 \\ 0 & 4 & -7 & 1 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_2 := -1/3r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 2 & -8 & 1 & 0 & 0 \\ 0 & 1 & -5/3 & 1/3 & -1/3 & 0 \\ 0 & 4 & -7 & 1 & 0 & 1 \end{array} \right]$$

$$\text{Step 4: } r_1 := r_1 - (2)r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & -14/3 & 1/3 & 2/3 & 0 \\ 0 & 1 & -5/3 & 1/3 & -1/3 & 0 \\ 0 & 4 & -7 & 1 & 0 & 1 \end{array} \right]$$

$$\text{Step 5: } r_3 := r_3 - (4)r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & -14/3 & 1/3 & 2/3 & 0 \\ 0 & 1 & -5/3 & 1/3 & -1/3 & 0 \\ 0 & 0 & -1/3 & -1/3 & 4/3 & 1 \end{array} \right]$$



$$\text{Step 6: } r_3 := -3r_3 \begin{bmatrix} 1 & 0 & -14/3 & | & 1/3 & 2/3 & 0 \\ 0 & 1 & -5/3 & | & 1/3 & -1/3 & 0 \\ 0 & 0 & 1 & | & 1 & -4 & -3 \end{bmatrix}$$

$$\text{Step 7: } r_1 := r_1 - (-14/3)r_3 \begin{bmatrix} 1 & 0 & 0 & | & 5 & -18 & -14 \\ 0 & 1 & -5/3 & | & 1/3 & -1/3 & 0 \\ 0 & 0 & 1 & | & 1 & -4 & -3 \end{bmatrix}$$

$$\text{Step 8: } r_2 := r_2 - (-5/3)r_3 \begin{bmatrix} 1 & 0 & 0 & | & 5 & -18 & -14 \\ 0 & 1 & 0 & | & 2 & -7 & -5 \\ 0 & 0 & 1 & | & 1 & -4 & -3 \end{bmatrix}$$

### Results:

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 1 \\ 1 & -4 & -3 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

### 2.3.2. RREF

#### Solution

#### Elementary Row Operations:

(1)  $r_2 := r_2 + (-2)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix}$$

(2)  $r_3 := r_3 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

#### Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

#### Solution

#### Elementary Row Operations:

(1)  $r_3 := r_3 - r_1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(2)  $r_1 := r_1 - (2)r_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_2 := r_2 - r_3$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_2 := r_2 - (2)r_1$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-1)r_1$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$(3) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_3 := r_3 + (-2)r_2$$

$$\begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_2 := r_2 - r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution****Elementary Row Operations:**

$$(1) \ r_2 := r_2 + (-1)r_1$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 + (-2)r_1$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution****Elementary Row Operations:**

$$(1) \ r_3 := r_3 - r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 - r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution****Elementary Row Operations:**

$$(1) \ r_1 := r_1 - (2)r_3$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_1 := r_1 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution****Elementary Row Operations:**

$$(1) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 + (-2)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution****Elementary Row Operations:**

$$(1) \ r_1 := r_1 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_1 := r_1 - (2)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution****Elementary Row Operations:**

$$(1) \ r_2 := r_2 + (-1)r_3$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

$$(2) \ r_1 := r_1 - r_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

$$(3) \ r_3 := r_3 - r_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 2.4. Calculus

### 2.4.1. Limit

The limit is:

$$1 \quad (151)$$

The limit is:

$$-2 \quad (152)$$

The limit is:

$$2 \quad (153)$$

The limit is:

$$e \quad (154)$$

The limit is:

$$0 \quad (155)$$

The limit is:

$$2 \quad (156)$$

The limit is:

$$e \quad (157)$$

The limit is:

$$1 \quad (158)$$

The limit is:

$$e \quad (159)$$

The limit is:

$$1 \quad (160)$$

### 2.4.2. Derivative

The derivative is:

$$2x \log(x) + x \quad (161)$$

The derivative is:

$$4x^3 \quad (162)$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \quad (163)$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \quad (164)$$

The derivative is:

$$4x^3 \quad (165)$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \quad (166)$$

The derivative is:

$$-\frac{2x^4}{(x^2 + 1)^2} + \frac{3x^2}{x^2 + 1} \quad (167)$$

The derivative is:

$$2x \log(x) + x \quad (168)$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \quad (169)$$

The derivative is:

$$3x^2 \log(x) + x^2 \quad (170)$$

### 2.4.3. Integral

The improper integral converges to:

$$1 \quad (171)$$

The indefinite integral is:

$$-\frac{x^5}{5} + \frac{3x^4}{4} + \frac{4x^3}{3} - 2x^2 \quad (172)$$

Definite integral from 4 to 4:

$$0 \quad (173)$$

The improper integral converges to:

$$\infty \quad (174)$$

The indefinite integral is:

$$\text{Si}(x) \quad (175)$$

Definite integral from 4 to 5:

$$-\text{Si}(4) + \text{Si}(5) \quad (176)$$

The indefinite integral is:

$$3 \log(x-3) - 2 \log(x-2) \quad (177)$$

Definite integral from 2 to 4:

$$\text{NaN} \quad (178)$$

The improper integral converges to:

$$e^{-1} \quad (179)$$

The indefinite integral is:

$$\frac{e^x \sin(x)}{2} - \frac{e^x \cos(x)}{2} \quad (180)$$

Definite integral from 2 to 5:

$$\frac{e^5 \sin(5)}{2} - \frac{e^5 \cos(5)}{2} - \frac{e^2 \sin(2)}{2} + \frac{e^2 \cos(2)}{2} \quad (181)$$

The indefinite integral is:

$$\frac{2x^5}{5} - \frac{3x^4}{4} - \frac{x^3}{3} - \frac{3x^2}{2} + 3x \quad (182)$$

Definite integral from 4 to 4:

$$0 \quad (183)$$

The indefinite integral is:

$$\frac{\log(x-2)}{3} - \frac{\log(x+1)}{3} \quad (184)$$

Definite integral from 2 to 5:

$$\infty \quad (185)$$

The improper integral converges to:

$$e^{-1} \quad (186)$$

#### 2.4.4. Partial Derivative

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \quad (187)$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (188)$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (189)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \quad (190)$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \quad (191)$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (192)$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (193)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \quad (194)$$

$$\frac{\partial^2 f}{\partial x^2} = 6y^3(2x^2 + y) \quad (195)$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \quad (196)$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \quad (197)$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \quad (198)$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \quad (199)$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \quad (200)$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \quad (201)$$