

# Exercise 15:

## Foundations of Mathematical, WS24

Zichao Wei

This is **exercise** 15 for Foundations of Mathematical, WS24. Generated on 2025-03-03 with 10 problems per section.

2025-06-09

# 1. Problems

## 1.1. Vector Arithmetic

### 1.1.1. Addition

Find the sum of the following vectors  $\mathbf{u}$  and  $\mathbf{v}$

1.  $\mathbf{u} = \begin{bmatrix} -5 \\ -3 \\ -5 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 6 \\ -10 \\ 6 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
2.  $\mathbf{u} = \begin{bmatrix} -7 \\ 8 \\ -7 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
3.  $\mathbf{u} = \begin{bmatrix} -9 \\ -6 \\ 7 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -10 \\ -4 \\ -7 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
4.  $\mathbf{u} = \begin{bmatrix} 6 \\ -7 \\ 10 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
5.  $\mathbf{u} = \begin{bmatrix} 5 \\ 4 \\ -3 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
6.  $\mathbf{u} = \begin{bmatrix} -9 \\ -6 \\ -3 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -4 \\ -3 \\ 8 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
7.  $\mathbf{u} = \begin{bmatrix} -3 \\ 9 \\ -5 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 7 \\ 8 \\ -2 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
8.  $\mathbf{u} = \begin{bmatrix} 7 \\ 8 \\ -8 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 3 \\ -5 \\ -9 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
9.  $\mathbf{u} = \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 0 \\ -9 \\ 9 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .
10.  $\mathbf{u} = \begin{bmatrix} -3 \\ -8 \\ -6 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 6 \\ 7 \\ -3 \end{bmatrix}$   $\mathbf{u} + \mathbf{v}$ .

### 1.1.2. Subtraction

Find the difference of the following vectors  $\mathbf{u}$  and  $\mathbf{v}$

1.  $\mathbf{u} = \begin{bmatrix} 9 \\ -5 \\ -6 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 6 \\ -3 \\ -2 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
2.  $\mathbf{u} = \begin{bmatrix} -10 \\ 5 \\ 6 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 6 \\ -2 \\ -6 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
3.  $\mathbf{u} = \begin{bmatrix} 3 \\ 9 \\ -1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 4 \\ -9 \\ 3 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
4.  $\mathbf{u} = \begin{bmatrix} 8 \\ -5 \\ -4 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -1 \\ 8 \\ 6 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
5.  $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -6 \\ -6 \\ 3 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .

6.  $\mathbf{u} = \begin{bmatrix} 2 \\ -6 \\ 7 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
7.  $\mathbf{u} = \begin{bmatrix} -7 \\ -2 \\ 5 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
8.  $\mathbf{u} = \begin{bmatrix} 5 \\ 6 \\ -6 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \\ -9 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
9.  $\mathbf{u} = \begin{bmatrix} -8 \\ -4 \\ 2 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -6 \\ 0 \\ 7 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .
10.  $\mathbf{u} = \begin{bmatrix} 5 \\ -4 \\ 8 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$   $\mathbf{u} - \mathbf{v}$ .

### 1.1.3. Scalar Multiplication

Find the scalar product of the following vector  $\mathbf{u}$  and scalar  $k$

1.  $\mathbf{u} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$   $4\mathbf{v}$ .
2.  $\mathbf{u} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$   $10\mathbf{v}$ .
3.  $\mathbf{u} = \begin{bmatrix} -5 \\ -10 \\ -7 \end{bmatrix}$   $8\mathbf{v}$ .
4.  $\mathbf{u} = \begin{bmatrix} -5 \\ 0 \\ -10 \end{bmatrix}$   $2\mathbf{v}$ .
5.  $\mathbf{u} = \begin{bmatrix} 0 \\ -6 \\ -3 \end{bmatrix}$   $-10\mathbf{v}$ .
6.  $\mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$   $10\mathbf{v}$ .
7.  $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$   $1\mathbf{v}$ .
8.  $\mathbf{u} = \begin{bmatrix} 10 \\ -6 \\ 5 \end{bmatrix}$   $4\mathbf{v}$ .
9.  $\mathbf{u} = \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}$   $-5\mathbf{v}$ .
10.  $\mathbf{u} = \begin{bmatrix} 5 \\ -10 \\ 9 \end{bmatrix}$   $-4\mathbf{v}$ .

## 1.2. Matrix Arithmetic

### 1.2.1. Addition

Find the sum of the following matrices  $A$  and  $B$

$$1. \quad A = \begin{bmatrix} -4 & 6 & -3 \\ 1 & -4 & 9 \\ 6 & 5 & -6 \end{bmatrix} \quad (1)$$

and

$$B = \begin{bmatrix} -8 & -9 & -10 \\ 6 & -6 & -8 \\ 2 & -3 & 8 \end{bmatrix} \quad (2)$$

$$2. \quad A = \begin{bmatrix} -2 & 1 & -2 \\ -6 & -6 & 9 \\ -8 & -3 & -7 \end{bmatrix} \quad (3)$$

and

$$B = \begin{bmatrix} 4 & -6 & -2 \\ 4 & -5 & 1 \\ 2 & 3 & 6 \end{bmatrix} \quad (4)$$

$$3. \quad A = \begin{bmatrix} -2 & -6 & 1 \\ 6 & 1 & -9 \\ -1 & -9 & -8 \end{bmatrix} \quad (5)$$

and

$$B = \begin{bmatrix} 4 & -1 & 8 \\ 1 & -6 & -10 \\ -4 & 2 & -8 \end{bmatrix} \quad (6)$$

$$4. \quad A = \begin{bmatrix} -4 & -2 & 6 \\ 8 & 8 & -4 \\ -2 & -3 & -1 \end{bmatrix} \quad (7)$$

and

$$B = \begin{bmatrix} -10 & 9 & -1 \\ -1 & 9 & 2 \\ 3 & 6 & -9 \end{bmatrix} \quad (8)$$

$$5. \quad A = \begin{bmatrix} 0 & 7 & 5 \\ -6 & -8 & 3 \\ -7 & 3 & 0 \end{bmatrix} \quad (9)$$

and

$$B = \begin{bmatrix} -8 & 6 & 0 \\ -9 & -10 & 1 \\ -10 & -1 & -2 \end{bmatrix} \quad (10)$$

$$6. \quad A = \begin{bmatrix} -9 & 5 & -6 \\ -5 & 4 & -9 \\ -6 & -10 & -3 \end{bmatrix} \quad (11)$$

and

$$B = \begin{bmatrix} 5 & -7 & -10 \\ -8 & -4 & -10 \\ -2 & -3 & 3 \end{bmatrix} \quad (12)$$

7.

$$A = \begin{bmatrix} -4 & -5 & 6 \\ 6 & 3 & -3 \\ -6 & -6 & -9 \end{bmatrix} \quad (13)$$

and

$$B = \begin{bmatrix} 1 & 4 & -2 \\ -3 & 8 & -4 \\ 4 & 7 & 1 \end{bmatrix} \quad (14)$$

8.

$$A = \begin{bmatrix} -10 & 2 & -7 \\ -10 & 7 & -10 \\ 7 & 6 & -7 \end{bmatrix} \quad (15)$$

and

$$B = \begin{bmatrix} -2 & 1 & -4 \\ 8 & -8 & 9 \\ 2 & 2 & 9 \end{bmatrix} \quad (16)$$

9.

$$A = \begin{bmatrix} -5 & -9 & -9 \\ -4 & 4 & -6 \\ 5 & -10 & -7 \end{bmatrix} \quad (17)$$

and

$$B = \begin{bmatrix} -2 & -1 & -7 \\ 1 & -2 & -4 \\ 7 & 6 & 7 \end{bmatrix} \quad (18)$$

10.

$$A = \begin{bmatrix} 5 & -8 & 8 \\ 5 & 6 & -9 \\ -3 & 2 & -10 \end{bmatrix} \quad (19)$$

and

$$B = \begin{bmatrix} -9 & 3 & -8 \\ 0 & 5 & 4 \\ 0 & -3 & 2 \end{bmatrix} \quad (20)$$

### 1.2.2. Subtraction

Find the difference of the following matrices  $A$  and  $B$

$$1. \quad A = \begin{bmatrix} -5 & 2 & -8 \\ -1 & -7 & -10 \\ -2 & -4 & -9 \end{bmatrix} \quad (21)$$

and

$$B = \begin{bmatrix} -8 & -7 & 7 \\ -1 & -4 & -1 \\ 3 & 1 & -4 \end{bmatrix} \quad (22)$$

$$2. \quad A = \begin{bmatrix} -7 & 3 & -4 \\ 6 & 8 & 8 \\ -3 & -1 & -5 \end{bmatrix} \quad (23)$$

and

$$B = \begin{bmatrix} -5 & 7 & -8 \\ 1 & -1 & -4 \\ -8 & 6 & -7 \end{bmatrix} \quad (24)$$

$$3. \quad A = \begin{bmatrix} 5 & -9 & -7 \\ 4 & -4 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad (25)$$

and

$$B = \begin{bmatrix} -8 & -2 & 7 \\ -8 & -4 & 0 \\ -1 & -4 & -10 \end{bmatrix} \quad (26)$$

$$4. \quad A = \begin{bmatrix} 3 & 8 & -8 \\ -7 & 2 & 1 \\ 2 & 0 & -9 \end{bmatrix} \quad (27)$$

and

$$B = \begin{bmatrix} -10 & 2 & 3 \\ 4 & -9 & 4 \\ -4 & 7 & -9 \end{bmatrix} \quad (28)$$

$$5. \quad A = \begin{bmatrix} 8 & 7 & -8 \\ 6 & 9 & 2 \\ -9 & 9 & -4 \end{bmatrix} \quad (29)$$

and

$$B = \begin{bmatrix} 0 & -7 & 4 \\ 4 & -8 & -2 \\ 6 & 8 & 1 \end{bmatrix} \quad (30)$$

$$6. \quad A = \begin{bmatrix} 6 & -10 & -10 \\ -5 & 0 & 6 \\ -1 & -8 & 2 \end{bmatrix} \quad (31)$$

and

$$B = \begin{bmatrix} 4 & 2 & -9 \\ -5 & 1 & -10 \\ 4 & -10 & 1 \end{bmatrix} \quad (32)$$

7.

$$A = \begin{bmatrix} -9 & 1 & 7 \\ -6 & -3 & -4 \\ 9 & 2 & -1 \end{bmatrix} \quad (33)$$

and

$$B = \begin{bmatrix} -6 & 3 & 3 \\ 6 & -1 & -3 \\ -1 & 9 & -3 \end{bmatrix} \quad (34)$$

8.

$$A = \begin{bmatrix} 8 & 8 & -7 \\ 4 & -5 & 9 \\ -9 & 0 & -2 \end{bmatrix} \quad (35)$$

and

$$B = \begin{bmatrix} -2 & 9 & -3 \\ -10 & -3 & 2 \\ -6 & -3 & -1 \end{bmatrix} \quad (36)$$

9.

$$A = \begin{bmatrix} -6 & 6 & -1 \\ 6 & 8 & 9 \\ -4 & -4 & -4 \end{bmatrix} \quad (37)$$

and

$$B = \begin{bmatrix} -4 & -1 & 8 \\ 8 & -8 & -7 \\ 3 & -10 & -8 \end{bmatrix} \quad (38)$$

10.

$$A = \begin{bmatrix} 2 & -2 & -2 \\ -1 & -4 & 7 \\ 1 & -1 & 0 \end{bmatrix} \quad (39)$$

and

$$B = \begin{bmatrix} 3 & 2 & -5 \\ -5 & -4 & -10 \\ -2 & 0 & -5 \end{bmatrix} \quad (40)$$

### 1.2.3. Multiplication

Find the product of the following matrices  $A$  and  $B$

1. 
$$A = \begin{bmatrix} -1 & 0 & -1 \\ -7 & 4 & -7 \\ 1 & 0 & 4 \end{bmatrix} \quad (41)$$

and

$$B = \begin{bmatrix} -6 & 4 & 0 \\ -3 & -1 & 3 \\ -8 & 3 & 4 \end{bmatrix} \quad (42)$$

2. 
$$A = \begin{bmatrix} 0 & -6 & 0 \\ 1 & 2 & -1 \\ -6 & -1 & -6 \end{bmatrix} \quad (43)$$

and

$$B = \begin{bmatrix} 9 & -3 & 6 \\ -10 & -2 & 6 \\ 2 & -3 & -6 \end{bmatrix} \quad (44)$$

3. 
$$A = \begin{bmatrix} 7 & 3 & -6 \\ 9 & -3 & -5 \\ -8 & -10 & -5 \end{bmatrix} \quad (45)$$

and

$$B = \begin{bmatrix} -2 & 7 & 8 \\ 0 & 2 & 1 \\ -6 & 4 & -6 \end{bmatrix} \quad (46)$$

4. 
$$A = \begin{bmatrix} -1 & -9 & 0 \\ -8 & -3 & -10 \\ 9 & -4 & 4 \end{bmatrix} \quad (47)$$

and

$$B = \begin{bmatrix} 6 & 6 & -4 \\ 3 & -9 & 3 \\ -10 & -2 & -6 \end{bmatrix} \quad (48)$$

5. 
$$A = \begin{bmatrix} -9 & -7 & 3 \\ -4 & -4 & -4 \\ -8 & 3 & -1 \end{bmatrix} \quad (49)$$

and

$$B = \begin{bmatrix} 6 & -4 & -5 \\ 7 & 6 & 5 \\ -2 & 7 & -7 \end{bmatrix} \quad (50)$$

6. 
$$A = \begin{bmatrix} 6 & -5 & 2 \\ -8 & -2 & 7 \\ 7 & -6 & 7 \end{bmatrix} \quad (51)$$



and

$$B = \begin{bmatrix} 7 & 3 & 5 \\ 8 & -2 & -7 \\ -1 & -8 & 4 \end{bmatrix} \quad (52)$$

7.

$$A = \begin{bmatrix} -8 & 4 & -3 \\ 6 & 0 & -10 \\ 7 & -6 & -10 \end{bmatrix} \quad (53)$$

and

$$B = \begin{bmatrix} -2 & 4 & -5 \\ -10 & 5 & 2 \\ 8 & 6 & -2 \end{bmatrix} \quad (54)$$

8.

$$A = \begin{bmatrix} -6 & -9 & -1 \\ -7 & 1 & -1 \\ 9 & 6 & 8 \end{bmatrix} \quad (55)$$

and

$$B = \begin{bmatrix} -2 & 9 & 2 \\ -9 & -1 & 5 \\ -8 & 2 & 5 \end{bmatrix} \quad (56)$$

9.

$$A = \begin{bmatrix} -2 & 1 & 6 \\ -3 & -3 & 3 \\ 6 & -3 & 1 \end{bmatrix} \quad (57)$$

and

$$B = \begin{bmatrix} -3 & 9 & 0 \\ 9 & 0 & -4 \\ 8 & 2 & -9 \end{bmatrix} \quad (58)$$

10.

$$A = \begin{bmatrix} 5 & 1 & 3 \\ -8 & -9 & -3 \\ -1 & -6 & 7 \end{bmatrix} \quad (59)$$

and

$$B = \begin{bmatrix} 3 & -3 & -3 \\ 7 & 2 & 4 \\ -5 & -9 & -2 \end{bmatrix} \quad (60)$$

## 1.3. Matrix Properties

### 1.3.1. Properties

For each matrix  $A$ , find:

a)  $\text{rank}(A)$

- b)  $\text{nullity}(A)$
- c)  $\det(A)$
- d)  $A^{-1}$  (if exists)
- e) basis of  $\ker(A)$

1. 
$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & -3 \end{bmatrix} \quad (61)$$

2. 
$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & -2 \\ -2 & -4 & 1 \end{bmatrix} \quad (62)$$

3. 
$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad (63)$$

4. 
$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad (64)$$

5. 
$$A = \begin{bmatrix} 1 & -2 & -2 \\ 2 & -3 & -6 \\ -1 & 2 & 3 \end{bmatrix} \quad (65)$$

6. 
$$A = \begin{bmatrix} 5 & 2 & -9 \\ 2 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \quad (66)$$

7. 
$$A = \begin{bmatrix} 5 & -2 & 3 \\ -2 & 1 & -1 \\ -1 & 0 & 0 \end{bmatrix} \quad (67)$$

8. 
$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad (68)$$

9. 
$$A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & -2 \\ 0 & -1 & -2 \end{bmatrix} \quad (69)$$

10. 
$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad (70)$$

### 1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix  $A$

1. 
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 0 \end{bmatrix} \quad (71)$$

$$2. \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix} \quad (72)$$

$$3. \quad A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad (73)$$

$$4. \quad A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \quad (74)$$

$$5. \quad A = \begin{bmatrix} -7 & -12 & 0 \\ -4 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (75)$$

$$6. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \quad (76)$$

$$7. \quad A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad (77)$$

$$8. \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix} \quad (78)$$

$$9. \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad (79)$$

$$10. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -2 \\ -1 & 1 & -1 \end{bmatrix} \quad (80)$$

## 1.4. Calculus

### 1.4.1. Limit

Calculate the following limits

1. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (81)$$

2. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (82)$$

3. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (83)$$

4. Calculate the limit of the following expression:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \quad (84)$$

5. Calculate the limit of the following expression:

$$\lim_{x \rightarrow -3} 2x^2 - x + 3 \quad (85)$$

6. Calculate the limit of the following expression:

$$\lim_{x \rightarrow -1} 5x^2 - 3x - 4 \quad (86)$$

7. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x + 1)}{x} \quad (87)$$

8. Calculate the limit of the following expression:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \quad (88)$$

9. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (89)$$

10. Calculate the limit of the following expression:

$$\lim_{x \rightarrow -1} -x^3 - 4x^2 - 2x - 1 \quad (90)$$

### 1.4.2. Derivative

Calculate the derivatives of the following expressions

1. Calculate the derivative of the following expression:

$$\frac{x^3}{x^2 + 1} \quad (91)$$

2. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (92)$$

3. Calculate the derivative of the following expression:

$$x^4 \quad (93)$$

4. Calculate the derivative of the following expression:

$$\log(x^2 - 1) \quad (94)$$

5. Calculate the derivative of the following expression:

$$\frac{x^3}{x^2 + 1} \quad (95)$$

6. Calculate the derivative of the following expression:

$$\frac{x^3}{x^2 + 1} \quad (96)$$

7. Calculate the derivative of the following expression:

$$e^{x^2-2} \quad (97)$$

8. Calculate the derivative of the following expression:

$$\log(x) \quad (98)$$

9. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (99)$$

10. Calculate the derivative of the following expression:

$$\frac{x^2}{x^2 + 1} \quad (100)$$

### 1.4.3. Integral

Calculate the indefinite and definite integrals of the following expressions

1. Evaluate the improper integral:

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx \quad (101)$$

2. Evaluate the improper integral:

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx \quad (102)$$

3. the indefinite integral and evaluate from 2 to 4:

$$\int x^3 \log(x) dx \quad (103)$$

4. the indefinite integral and evaluate from 2 to 5:

$$\int x^3 \log(x) dx \quad (104)$$

5. the indefinite integral and evaluate from 2 to 2:

$$\int x \sqrt{x^2 + 1} dx \quad (105)$$

6. the indefinite integral and evaluate from 2 to 2:

$$\int -x^2 - 2dx \quad (106)$$

7. the indefinite integral and evaluate from 2 to 4:

$$\int \sqrt{4 - x^2} dx \quad (107)$$

8. the indefinite integral and evaluate from 2 to 5:

$$\int e^{-x^2} dx \quad (108)$$

9. the indefinite integral and evaluate from 4 to 4:

$$\int -3x^2 - 2dx \quad (109)$$

10. the indefinite integral and evaluate from 3 to 5:

$$\int 4x^3 - 4x^2 + 5x - 4dx \quad (110)$$

#### 1.4.4. Partial Derivative

Calculate the partial derivatives of the following expressions

1. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \quad (111)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

2. the second order partial derivative of:

$$f(x, y) = x^4y^3 + 3x^2y^4 \quad (112)$$

$$\frac{\partial^2 f}{\partial x^2}$$

3. the mixed partial derivative of:

$$f(x, y) = x^3y^2 + xy^4 \quad (113)$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

4. the mixed partial derivative of:

$$f(x, y) = x^3y^2 + xy^4 \quad (114)$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

5. the partial derivatives of the function:

$$f(x, y) = -\log(xy) + \log(x^3 + y^3) \quad (115)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

6. Given the implicit function:

$$x^2y + xy^2 - xy = 0 \quad (116)$$

$$\frac{\partial y}{\partial x}$$

7. the partial derivatives of the function:

$$f(x, y) = -\log(xy) + \log(x^3 + y^3) \quad (117)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

8. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \quad (118)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

9. the third order partial derivative of:

$$f(x, y) = x^4y^3 + 3x^2y^4 \quad (119)$$

$$\frac{\partial^3 f}{\partial y^3}$$

10. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \quad (120)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

## 2. Solutions

### 2.1. Vector Arithmetic

#### 2.1.1. Addition

$$\begin{bmatrix} 1 \\ -13 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 11 \\ -9 \end{bmatrix} = \begin{bmatrix} -19 \\ -10 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 14 \\ -7 \\ 10 \end{bmatrix} + \begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 23 \\ -7 \\ 11 \end{bmatrix}$$
$$\begin{bmatrix} -13 \\ -9 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ 17 \\ -7 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 10 \\ 3 \\ -17 \end{bmatrix} + \begin{bmatrix} -4 \\ -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -9 \end{bmatrix}$$

#### 2.1.2. Subtraction

$$\begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix} - \begin{bmatrix} -16 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 19 \\ -9 \\ -16 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 18 \\ -4 \end{bmatrix} - \begin{bmatrix} 9 \\ -13 \\ -10 \end{bmatrix} = \begin{bmatrix} -10 \\ 31 \\ 6 \end{bmatrix}$$
$$\begin{bmatrix} 3 \\ -10 \\ 8 \end{bmatrix} - \begin{bmatrix} -9 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ -7 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -4 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$$

#### 2.1.3. Scalar Multiplication

$$1: \begin{bmatrix} -24 \\ 16 \\ 8 \end{bmatrix} \quad 2: \begin{bmatrix} -30 \\ 30 \\ 10 \end{bmatrix} \quad 3: \begin{bmatrix} -40 \\ -80 \\ -56 \end{bmatrix} \quad 4: \begin{bmatrix} -10 \\ 0 \\ -20 \end{bmatrix} \quad 5: \begin{bmatrix} 0 \\ 60 \\ 30 \end{bmatrix}$$
$$6: \begin{bmatrix} 20 \\ -40 \\ 30 \end{bmatrix} \quad 7: \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \quad 8: \begin{bmatrix} 40 \\ -24 \\ 20 \end{bmatrix} \quad 9: \begin{bmatrix} -25 \\ -40 \\ -10 \end{bmatrix} \quad 10: \begin{bmatrix} -20 \\ 40 \\ -36 \end{bmatrix}$$

### 2.2. Matrix Arithmetic

#### 2.2.1. Addition

1:

$$\begin{bmatrix} -12 & -3 & -13 \\ 7 & -10 & 1 \\ 8 & 2 & 2 \end{bmatrix} \quad (121)$$

1:

$$\begin{bmatrix} 2 & -5 & -4 \\ -2 & -11 & 10 \\ -6 & 0 & -1 \end{bmatrix} \quad (122)$$

1:

$$\begin{bmatrix} 2 & -7 & 9 \\ 7 & -5 & -19 \\ -5 & -7 & -16 \end{bmatrix} \quad (123)$$

1:

$$\begin{bmatrix} -14 & 7 & 5 \\ 7 & 17 & -2 \\ 1 & 3 & -10 \end{bmatrix} \quad (124)$$



1:

$$\begin{bmatrix} -8 & 13 & 5 \\ -15 & -18 & 4 \\ -17 & 2 & -2 \end{bmatrix} \quad (125)$$

1:

$$\begin{bmatrix} -4 & -2 & -16 \\ -13 & 0 & -19 \\ -8 & -13 & 0 \end{bmatrix} \quad (126)$$

1:

$$\begin{bmatrix} -3 & -1 & 4 \\ 3 & 11 & -7 \\ -2 & 1 & -8 \end{bmatrix} \quad (127)$$

1:

$$\begin{bmatrix} -12 & 3 & -11 \\ -2 & -1 & -1 \\ 9 & 8 & 2 \end{bmatrix} \quad (128)$$

1:

$$\begin{bmatrix} -7 & -10 & -16 \\ -3 & 2 & -10 \\ 12 & -4 & 0 \end{bmatrix} \quad (129)$$

1:

$$\begin{bmatrix} -4 & -5 & 0 \\ 5 & 11 & -5 \\ -3 & -1 & -8 \end{bmatrix} \quad (130)$$

### 2.2.2. Subtraction

1:

$$\begin{bmatrix} 3 & 9 & -15 \\ 0 & -3 & -9 \\ -5 & -5 & -5 \end{bmatrix} \quad (131)$$

1:

$$\begin{bmatrix} -2 & -4 & 4 \\ 5 & 9 & 12 \\ 5 & -7 & 2 \end{bmatrix} \quad (132)$$

1:

$$\begin{bmatrix} 13 & -7 & -14 \\ 12 & 0 & 1 \\ 0 & 5 & 11 \end{bmatrix} \quad (133)$$

1:

$$\begin{bmatrix} 13 & 6 & -11 \\ -11 & 11 & -3 \\ 6 & -7 & 0 \end{bmatrix} \quad (134)$$

1:

$$\begin{bmatrix} 8 & 14 & -12 \\ 2 & 17 & 4 \\ -15 & 1 & -5 \end{bmatrix} \quad (135)$$

1:

$$\begin{bmatrix} 2 & -12 & -1 \\ 0 & -1 & 16 \\ -5 & 2 & 1 \end{bmatrix} \quad (136)$$

1:

$$\begin{bmatrix} -3 & -2 & 4 \\ -12 & -2 & -1 \\ 10 & -7 & 2 \end{bmatrix} \quad (137)$$

1:

$$\begin{bmatrix} 10 & -1 & -4 \\ 14 & -2 & 7 \\ -3 & 3 & -1 \end{bmatrix} \quad (138)$$

1:

$$\begin{bmatrix} -2 & 7 & -9 \\ -2 & 16 & 16 \\ -7 & 6 & 4 \end{bmatrix} \quad (139)$$

1:

$$\begin{bmatrix} -1 & -4 & 3 \\ 4 & 0 & 17 \\ 3 & -1 & 5 \end{bmatrix} \quad (140)$$

### 2.2.3. Multiplication

1:

$$\begin{bmatrix} 14 & -7 & -4 \\ 86 & -53 & -16 \\ -38 & 16 & 16 \end{bmatrix} \quad (141)$$

1:

$$\begin{bmatrix} 60 & 12 & -36 \\ -13 & -4 & 24 \\ -56 & 38 & -6 \end{bmatrix} \quad (142)$$

1:

$$\begin{bmatrix} 22 & 31 & 95 \\ 12 & 37 & 99 \\ 46 & -96 & -44 \end{bmatrix} \quad (143)$$

1:

$$\begin{bmatrix} -33 & 75 & -23 \\ 43 & -1 & 83 \\ 2 & 82 & -72 \end{bmatrix} \quad (144)$$

1:

$$\begin{bmatrix} -109 & 15 & -11 \\ -44 & -36 & 28 \\ -25 & 43 & 62 \end{bmatrix} \quad (145)$$

1:

$$\begin{bmatrix} 0 & 12 & 73 \\ -79 & -76 & 2 \\ -6 & -23 & 105 \end{bmatrix} \quad (146)$$

1:

$$\begin{bmatrix} -48 & -30 & 54 \\ -92 & -36 & -10 \\ -34 & -62 & -27 \end{bmatrix} \quad (147)$$

1:

$$\begin{bmatrix} 101 & -47 & -62 \\ 13 & -66 & -14 \\ -136 & 91 & 88 \end{bmatrix} \quad (148)$$

1:

$$\begin{bmatrix} 63 & -6 & -58 \\ 6 & -21 & -15 \\ -37 & 56 & 3 \end{bmatrix} \quad (149)$$

1:

$$\begin{bmatrix} 7 & -40 & -17 \\ -72 & 33 & -6 \\ -80 & -72 & -35 \end{bmatrix} \quad (150)$$

## 2.3. Matrix Properties

### 2.3.1. Properties

#### Solution

#### Row Operations:

$$\text{Step 1: } r_1 := r_1 - (-2)r_2 \quad \begin{bmatrix} 1 & 0 & 4 & | & 1 & 2 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & -2 & -3 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_3 := r_3 - (-2)r_2 \quad \begin{bmatrix} 1 & 0 & 4 & | & 1 & 2 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 2 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (4)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & 1 & -6 & -4 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 2 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_2 := r_2 - (2)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & 1 & -6 & -4 \\ 0 & 1 & 0 & | & 0 & -3 & -2 \\ 0 & 0 & 1 & | & 0 & 2 & 1 \end{bmatrix}$$

#### Results:

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

#### Solution

#### Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-2)r_1 \quad \begin{bmatrix} 1 & 2 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & -2 & | & 2 & 1 & 0 \\ -2 & -4 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_3 := r_3 - (-2)r_1 \quad \begin{bmatrix} 1 & 2 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & -2 & | & 2 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (2)r_2 \quad \begin{bmatrix} 1 & 0 & 4 & | & -3 & -2 & 0 \\ 0 & 1 & -2 & | & 2 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_1 := r_1 - (4)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & -11 & -2 & -4 \\ 0 & 1 & -2 & | & 2 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 5: } r_2 := r_2 - (-2)r_3 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & -2 & -4 \\ 0 & 1 & 0 & 6 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right]$$

**Results:**

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

**Solution**

**Row Operations:**

$$\text{Step 1: } r_1 := r_1 - (3)r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -3 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_3 := r_3 - (-1)r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -3 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

**Results:**

a)  $\text{rank}(A) = 2$

b)  $\text{nullity}(A) = 1$

c)  $\det(A) = 0$

d)  $A^{-1} = \text{does not exist}$

e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

**Solution**

**Row Operations:**

$$\text{Step 1: } r_2 := r_2 - (-2)r_1 \quad \left[ \begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_1 := r_1 - (2)r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & -3 & -2 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_1 := r_1 - (4)r_3 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -2 & -4 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 4: } r_2 := r_2 - (-2)r_3 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -2 & -4 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

**Results:**

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

**Solution****Row Operations:**

Step 1:  $r_2 := r_2 - (2)r_1$   $\left[ \begin{array}{ccc|ccc} 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array} \right]$

Step 2:  $r_3 := r_3 - (-1)r_1$   $\left[ \begin{array}{ccc|ccc} 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right]$

Step 3:  $r_1 := r_1 - (-2)r_2$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & -6 & -3 & 2 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right]$

Step 4:  $r_1 := r_1 - (-6)r_3$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & 6 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right]$

Step 5:  $r_2 := r_2 - (-2)r_3$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & 6 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right]$

**Results:**

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 3 & 0 & 2 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

**Solution****Row Operations:**

Step 1:  $r_1 := 1/5r_1$   $\left[ \begin{array}{ccc|ccc} 1 & 2/5 & -9/5 & 1/5 & 0 & 0 \\ 2 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$

Step 2:  $r_2 := r_2 - (2)r_1$   $\left[ \begin{array}{ccc|ccc} 1 & 2/5 & -9/5 & 1/5 & 0 & 0 \\ 0 & 1/5 & -2/5 & -2/5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$

$$\text{Step 3: } r_2 := 5r_2 \begin{bmatrix} 1 & 2/5 & -9/5 & | & 1/5 & 0 & 0 \\ 0 & 1 & -2 & | & -2 & 5 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_1 := r_1 - (2/5)r_2 \begin{bmatrix} 1 & 0 & -1 & | & 1 & -2 & 0 \\ 0 & 1 & -2 & | & -2 & 5 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

**Results:**

a)  $\text{rank}(A) = 2$

b)  $\text{nullity}(A) = 1$

c)  $\det(A) = 0$

d)  $A^{-1}$  = does not exist

e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

**Solution**

**Row Operations:**

$$\text{Step 1: } r_1 := 1/5r_1 \begin{bmatrix} 1 & -2/5 & 3/5 & | & 1/5 & 0 & 0 \\ -2 & 1 & -1 & | & 0 & 1 & 0 \\ -1 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_2 := r_2 - (-2)r_1 \begin{bmatrix} 1 & -2/5 & 3/5 & | & 1/5 & 0 & 0 \\ 0 & 1/5 & 1/5 & | & 2/5 & 1 & 0 \\ -1 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_3 := r_3 - (-1)r_1 \begin{bmatrix} 1 & -2/5 & 3/5 & | & 1/5 & 0 & 0 \\ 0 & 1/5 & 1/5 & | & 2/5 & 1 & 0 \\ 0 & -2/5 & 3/5 & | & 1/5 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_2 := 5r_2 \begin{bmatrix} 1 & -2/5 & 3/5 & | & 1/5 & 0 & 0 \\ 0 & 1 & 1 & | & 2 & 5 & 0 \\ 0 & -2/5 & 3/5 & | & 1/5 & 0 & 1 \end{bmatrix}$$

$$\text{Step 5: } r_1 := r_1 - (-2/5)r_2 \begin{bmatrix} 1 & 0 & 1 & | & 1 & 2 & 0 \\ 0 & 1 & 1 & | & 2 & 5 & 0 \\ 0 & -2/5 & 3/5 & | & 1/5 & 0 & 1 \end{bmatrix}$$

$$\text{Step 6: } r_3 := r_3 - (-2/5)r_2 \begin{bmatrix} 1 & 0 & 1 & | & 1 & 2 & 0 \\ 0 & 1 & 1 & | & 2 & 5 & 0 \\ 0 & 0 & 1 & | & 1 & 2 & 1 \end{bmatrix}$$

$$\text{Step 7: } r_1 := r_1 - r_3 \begin{bmatrix} 1 & 0 & 0 & | & 0 & 0 & -1 \\ 0 & 1 & 1 & | & 2 & 5 & 0 \\ 0 & 0 & 1 & | & 1 & 2 & 1 \end{bmatrix}$$

$$\text{Step 8: } r_2 := r_2 - r_3 \begin{bmatrix} 1 & 0 & 0 & | & 0 & 0 & -1 \\ 0 & 1 & 0 & | & 1 & 3 & -1 \\ 0 & 0 & 1 & | & 1 & 2 & 1 \end{bmatrix}$$

**Results:**

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

### Solution

#### Row Operations:

Step 1:  $r_1 := r_1 - (2)r_2$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$

#### Results:

a)  $\text{rank}(A) = 2$

b)  $\text{nullity}(A) = 1$

c)  $\det(A) = 0$

d)  $A^{-1}$  = does not exist

e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

### Solution

#### Row Operations:

Step 1:  $r_2 := r_2 - (-2)r_1$   $\left[ \begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 & 0 & 1 \end{array} \right]$

Step 2:  $r_1 := r_1 - (3)r_2$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & -4 & -5 & -3 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 & 0 & 1 \end{array} \right]$

Step 3:  $r_3 := r_3 - (-1)r_2$   $\left[ \begin{array}{ccc|ccc} 1 & 0 & -4 & -5 & -3 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{array} \right]$

#### Results:

a)  $\text{rank}(A) = 2$

b)  $\text{nullity}(A) = 1$

c)  $\det(A) = 0$

d)  $A^{-1}$  = does not exist

e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

### Solution

#### Row Operations:



$$\text{Step 1: } r_1 := 1/3r_1 \quad \left[ \begin{array}{ccc|ccc} 1 & -1/3 & 2/3 & 1/3 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_2 := r_2 - (4)r_1 \quad \left[ \begin{array}{ccc|ccc} 1 & -1/3 & 2/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & -2/3 & -4/3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_2 := 3r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & -1/3 & 2/3 & 1/3 & 0 & 0 \\ 0 & 1 & -2 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 4: } r_1 := r_1 - (-1/3)r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

### Results:

a)  $\text{rank}(A) = 2$

b)  $\text{nullity}(A) = 1$

c)  $\det(A) = 0$

d)  $A^{-1}$  does not exist

e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$

### 2.3.2. RREF

#### Solution

#### Elementary Row Operations:

(1)  $r_3 := r_3 - (2)r_1$

$$\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{array} \right]$$

(2)  $r_3 := r_3 - (-2)r_2$

$$\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

#### Result:

$$\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

#### Solution

#### Elementary Row Operations:

(1)  $r_3 := r_3 + (-1)r_1$

$$\left[ \begin{array}{ccc} 1 & 1 & 0 \\ 2 & 3 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

(2)  $r_2 := r_2 + (-2)r_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-1)r_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_2 := r_2 + (-1)r_3$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$(2) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_1 := r_1 + (-2)r_2$$

$$\begin{bmatrix} 1 & 2 & 0 \\ -4 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 - (2)r_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_2 := r_2 - (2)r_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_2 := r_2 - r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_1 := r_1 + (-1)r_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 + (-1)r_1$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-1)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_2 := r_2 - r_3$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-1)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_3 := r_3 + (-1)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_2 := r_2 - (-2)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_2 := r_2 - r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 2.4. Calculus

### 2.4.1. Limit

The limit is:

$$1 \quad (151)$$

The limit is:

$$1 \quad (152)$$

The limit is:

$$2 \quad (153)$$

The limit is:

$$e \quad (154)$$

The limit is:

$$24 \quad (155)$$

The limit is:

$$4 \quad (156)$$

The limit is:

$$1 \quad (157)$$

The limit is:

$$e \quad (158)$$

The limit is:

$$2 \quad (159)$$

The limit is:

$$-2 \quad (160)$$

### 2.4.2. Derivative

The derivative is:

$$-\frac{2x^4}{(x^2 + 1)^2} + \frac{3x^2}{x^2 + 1} \quad (161)$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \quad (162)$$

The derivative is:

$$4x^3 \quad (163)$$

The derivative is:

$$\frac{2x}{x^2 - 1} \quad (164)$$

The derivative is:

$$-\frac{2x^4}{(x^2 + 1)^2} + \frac{3x^2}{x^2 + 1} \quad (165)$$

The derivative is:

$$-\frac{2x^4}{(x^2 + 1)^2} + \frac{3x^2}{x^2 + 1} \quad (166)$$

The derivative is:

$$2xe^{x^2-2} \quad (167)$$

The derivative is:

$$\frac{1}{x} \quad (168)$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \quad (169)$$

The derivative is:

$$-\frac{2x^3}{(x^2 + 1)^2} + \frac{2x}{x^2 + 1} \quad (170)$$

### 2.4.3. Integral

The improper integral converges to:

$$\infty \quad (171)$$

The improper integral converges to:

$$\infty \quad (172)$$

The indefinite integral is:

$$\frac{x^4 \log(x)}{4} - \frac{x^4}{16} \quad (173)$$

Definite integral from 2 to 4:

$$-15 - 4\log(2) + 64\log(4) \quad (174)$$

The indefinite integral is:

$$\frac{x^4 \log(x)}{4} - \frac{x^4}{16} \quad (175)$$

Definite integral from 2 to 5:

$$-\frac{609}{16} - 4 \log(2) + \frac{625 \log(5)}{4} \quad (176)$$

The indefinite integral is:

$$\frac{x^2 \sqrt{x^2 + 1}}{3} + \frac{\sqrt{x^2 + 1}}{3} \quad (177)$$

Definite integral from 2 to 2:

$$0 \quad (178)$$

The indefinite integral is:

$$-\frac{x^3}{3} - 2x \quad (179)$$

Definite integral from 2 to 2:

$$0 \quad (180)$$

The indefinite integral is:

$$\frac{x \sqrt{4 - x^2}}{2} + 2 \operatorname{asin} \left( \frac{x}{2} \right) \quad (181)$$

Definite integral from 2 to 4:

$$-\pi + 2 \operatorname{asin} (2) + 4\sqrt{3}i \quad (182)$$

The indefinite integral is:

$$\frac{\sqrt{\pi} \operatorname{erf} (x)}{2} \quad (183)$$

Definite integral from 2 to 5:

$$-\frac{\sqrt{\pi} \operatorname{erf} (2)}{2} + \frac{\sqrt{\pi} \operatorname{erf} (5)}{2} \quad (184)$$

The indefinite integral is:

$$-x^3 - 2x \quad (185)$$

Definite integral from 4 to 4:

$$0 \quad (186)$$

The indefinite integral is:

$$x^4 - \frac{4x^3}{3} + \frac{5x^2}{2} - 4x \quad (187)$$

Definite integral from 3 to 5:

$$\frac{1336}{3} \quad (188)$$

#### 2.4.4. Partial Derivative

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (189)$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (190)$$

$$\frac{\partial^2 f}{\partial x^2} = 6y^3(2x^2 + y) \quad (191)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \quad (192)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \quad (193)$$

$$\frac{\partial f}{\partial x} = \frac{3x^2}{x^3 + y^3} - \frac{1}{x} \quad (194)$$

$$\frac{\partial f}{\partial y} = \frac{3y^2}{x^3 + y^3} - \frac{1}{y} \quad (195)$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \quad (196)$$

$$\frac{\partial f}{\partial x} = \frac{3x^2}{x^3 + y^3} - \frac{1}{x} \quad (197)$$

$$\frac{\partial f}{\partial y} = \frac{3y^2}{x^3 + y^3} - \frac{1}{y} \quad (198)$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (199)$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (200)$$

$$\frac{\partial^3 f}{\partial y^3} = 6x^2(x^2 + 12y) \quad (201)$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (202)$$



$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (203)$$