# Exercise 9:

# Foundations of Mathematical, WS24

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This is **exercise** 9 for Foundations of Mathematical, WS24. Generated on 2025-01-20 with 10 problems per section.

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# 1. Problems

## 1.1. Vector Arithmetic

### 1.1.1. Addition

Find the sum of the following vectors $\mathbf{u}$ and $\mathbf{v}$

1. $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ -10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

2. $\mathbf{u} = \begin{bmatrix} -5 \\ -1 \\ -9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

3. $\mathbf{u} = \begin{bmatrix} -10 \\ -4 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

4. $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -7 \\ -1 \\ 0 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

5. $\mathbf{u} = \begin{bmatrix} -9 \\ -1 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -8 \\ 0 \\ -3 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

6. $\mathbf{u} = \begin{bmatrix} 5 \\ 9 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ -6 \\ 7 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

7. $\mathbf{u} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

8. $\mathbf{u} = \begin{bmatrix} 0 \\ -3 \\ 10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ -4 \\ 6 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

9. $\mathbf{u} = \begin{bmatrix} 9 \\ -4 \\ 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -10 \\ 2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

10. $\mathbf{u} = \begin{bmatrix} -10 \\ 5 \\ 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 10 \\ 6 \\ -2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

### 1.1.2. Subtraction

Find the difference of the following vectors $\mathbf{u}$ and $\mathbf{v}$

1. $\mathbf{u} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ -8 \\ 3 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

2. $\mathbf{u} = \begin{bmatrix} -5 \\ 5 \\ -10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

3. $\mathbf{u} = \begin{bmatrix} -6 \\ -1 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 7 \\ -3 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

4. $\mathbf{u} = \begin{bmatrix} -8 \\ -2 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 10 \\ 8 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

5. $\mathbf{u} = \begin{bmatrix} 10 \\ 4 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -8 \\ -6 \\ 0 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

6. $\mathbf{u} = \begin{bmatrix} 0 \\ -10 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -9 \\ -5 \\ -7 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

7. $\mathbf{u} = \begin{bmatrix} 0 \\ 7 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -7 \\ 9 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

8. $\mathbf{u} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

9. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 10 \\ 9 \\ -2 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

10. $\mathbf{u} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ 5 \\ -7 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

### 1.1.3. Scalar Multiplication

Find the scalar product of the following vector $\mathbf{u}$ and scalar $k$

1. $\mathbf{u} = \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix}$ $-5\mathbf{v}$.

2. $\mathbf{u} = \begin{bmatrix} 9 \\ 0 \\ 10 \end{bmatrix}$ $-3\mathbf{v}$.

3. $\mathbf{u} = \begin{bmatrix} 2 \\ -10 \\ -8 \end{bmatrix}$ $-2\mathbf{v}$.

4. $\mathbf{u} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$ $-6\mathbf{v}$.

5. $\mathbf{u} = \begin{bmatrix} 1 \\ 9 \\ 5 \end{bmatrix}$ $0\mathbf{v}$.

6. $\mathbf{u} = \begin{bmatrix} 3 \\ -5 \\ -7 \end{bmatrix}$ $-8\mathbf{v}$.

7. $\mathbf{u} = \begin{bmatrix} 4 \\ 10 \\ 8 \end{bmatrix}$ $10\mathbf{v}$.

8. $\mathbf{u} = \begin{bmatrix} 10 \\ -10 \\ 0 \end{bmatrix}$ $7\mathbf{v}$.

9. $\mathbf{u} = \begin{bmatrix} -8 \\ 5 \\ -1 \end{bmatrix}$ $3\mathbf{v}$.

10. $\mathbf{u} = \begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix}$ $4\mathbf{v}$.

## 1.2. Matrix Arithmetic

### 1.2.1. Addition

Find the sum of the following matrices $A$ and $B$

1.

$$A = \begin{bmatrix} -8 & -10 & -7 \\ 9 & -3 & -6 \\ -1 & -9 & 3 \end{bmatrix} \tag{1}$$

and

$$B = \begin{bmatrix} 4 & 3 & 3 \\ 1 & 3 & -4 \\ -5 & 6 & 6 \end{bmatrix} \tag{2}$$

2.

$$A = \begin{bmatrix} -9 & -5 & -6 \\ 9 & 8 & -8 \\ -8 & -8 & 6 \end{bmatrix} \tag{3}$$

and

$$B = \begin{bmatrix} -7 & 0 & -8 \\ 2 & 0 & 1 \\ 9 & 1 & -4 \end{bmatrix} \tag{4}$$

3.

$$A = \begin{bmatrix} 7 & 9 & 3 \\ -6 & -9 & 4 \\ -2 & -4 & 7 \end{bmatrix} \tag{5}$$

and

$$B = \begin{bmatrix} 6 & -8 & 3 \\ -9 & -1 & -9 \\ -6 & -4 & 0 \end{bmatrix} \tag{6}$$

4.

$$A = \begin{bmatrix} -10 & -10 & 3 \\ 0 & -7 & 1 \\ 8 & 3 & 4 \end{bmatrix} \tag{7}$$

and

$$B = \begin{bmatrix} 9 & 8 & -1 \\ 6 & -6 & -6 \\ -1 & -8 & -7 \end{bmatrix} \tag{8}$$

5.

$$A = \begin{bmatrix} 7 & -2 & -10 \\ 6 & 3 & 2 \\ -5 & 1 & -10 \end{bmatrix} \tag{9}$$

and

$$B = \begin{bmatrix} -1 & -2 & 1 \\ 6 & -1 & 8 \\ 3 & -1 & -6 \end{bmatrix} \tag{10}$$

6.

$$A = \begin{bmatrix} 3 & -2 & -5 \\ -6 & 6 & -8 \\ -2 & -4 & 3 \end{bmatrix} \tag{11}$$

and

$$B = \begin{bmatrix} 8 & -10 & 0 \\ 8 & -2 & -10 \\ -9 & 1 & -4 \end{bmatrix} \tag{12}$$

7.

$$A = \begin{bmatrix} 2 & 4 & -8 \\ 3 & 2 & -3 \\ 8 & 7 & -8 \end{bmatrix} \tag{13}$$

and

$$B = \begin{bmatrix} -7 & -6 & 5 \\ -4 & -3 & 7 \\ 1 & -8 & 9 \end{bmatrix} \tag{14}$$

8.

$$A = \begin{bmatrix} 4 & -8 & 8 \\ -10 & 1 & 1 \\ 2 & -1 & 8 \end{bmatrix} \tag{15}$$

and

$$B = \begin{bmatrix} 8 & -9 & 4 \\ -6 & -3 & -9 \\ 2 & -6 & 0 \end{bmatrix} \tag{16}$$

9.

$$A = \begin{bmatrix} 3 & -4 & -5 \\ -3 & -3 & 7 \\ 9 & 2 & 7 \end{bmatrix} \tag{17}$$

and

$$B = \begin{bmatrix} -7 & 0 & -6 \\ 0 & 3 & -2 \\ -4 & -3 & -8 \end{bmatrix} \tag{18}$$

10.

$$A = \begin{bmatrix} -8 & -4 & 4 \\ -4 & 2 & -7 \\ -5 & 5 & -3 \end{bmatrix} \tag{19}$$

and

$$B = \begin{bmatrix} -3 & 4 & -4 \\ 7 & -5 & -9 \\ 1 & 7 & -5 \end{bmatrix} \tag{20}$$

### 1.2.2. Subtraction

Find the difference of the following matrices $A$ and $B$

1.
$$A = \begin{bmatrix} -1 & 5 & 0 \\ 9 & -4 & -9 \\ 2 & 1 & -8 \end{bmatrix} \tag{21}$$

and

$$B = \begin{bmatrix} 0 & 0 & -1 \\ 6 & 7 & -4 \\ 2 & 9 & 9 \end{bmatrix} \tag{22}$$

2.
$$A = \begin{bmatrix} -4 & -7 & 5 \\ -2 & 9 & 5 \\ -2 & 0 & 5 \end{bmatrix} \tag{23}$$

and

$$B = \begin{bmatrix} 2 & -6 & -8 \\ -3 & 8 & 7 \\ -9 & -8 & 6 \end{bmatrix} \tag{24}$$

3.
$$A = \begin{bmatrix} -8 & 8 & -5 \\ 8 & -10 & -5 \\ 9 & 3 & -8 \end{bmatrix} \tag{25}$$

and

$$B = \begin{bmatrix} -4 & 4 & 8 \\ -5 & -10 & 4 \\ 9 & 8 & 8 \end{bmatrix} \tag{26}$$

4.
$$A = \begin{bmatrix} 9 & 5 & 8 \\ -1 & 2 & -9 \\ 6 & -10 & 3 \end{bmatrix} \tag{27}$$

and

$$B = \begin{bmatrix} 2 & 4 & -6 \\ -7 & -6 & -6 \\ 5 & -8 & -6 \end{bmatrix} \tag{28}$$

5.
$$A = \begin{bmatrix} 6 & 0 & 8 \\ 5 & 6 & -6 \\ -6 & 4 & 9 \end{bmatrix} \tag{29}$$

and

$$B = \begin{bmatrix} -5 & 3 & -1 \\ -5 & 5 & 8 \\ -6 & 9 & 0 \end{bmatrix} \tag{30}$$

6.
$$A = \begin{bmatrix} -1 & -1 & -6 \\ 0 & 8 & -2 \\ 1 & -9 & -1 \end{bmatrix} \tag{31}$$

and

$$B = \begin{bmatrix} -2 & -3 & -4 \\ -7 & -9 & -8 \\ -4 & -4 & 3 \end{bmatrix} \tag{32}$$

7.

$$A = \begin{bmatrix} 4 & -7 & 1 \\ 7 & 5 & 3 \\ -3 & 8 & -1 \end{bmatrix} \tag{33}$$

and

$$B = \begin{bmatrix} 8 & 0 & 9 \\ 3 & 6 & 1 \\ -1 & -9 & 0 \end{bmatrix} \tag{34}$$

8.

$$A = \begin{bmatrix} -5 & -10 & 9 \\ -9 & 9 & 2 \\ 9 & 9 & 2 \end{bmatrix} \tag{35}$$

and

$$B = \begin{bmatrix} 4 & -2 & -7 \\ -8 & 2 & -6 \\ -5 & -8 & -6 \end{bmatrix} \tag{36}$$

9.

$$A = \begin{bmatrix} 2 & 2 & -8 \\ -2 & -2 & -2 \\ -4 & -9 & -1 \end{bmatrix} \tag{37}$$

and

$$B = \begin{bmatrix} 6 & -9 & 4 \\ 8 & -3 & 3 \\ -8 & 4 & -2 \end{bmatrix} \tag{38}$$

10.

$$A = \begin{bmatrix} 6 & 8 & 8 \\ 3 & 8 & -9 \\ -3 & -6 & -7 \end{bmatrix} \tag{39}$$

and

$$B = \begin{bmatrix} -9 & 4 & 4 \\ -9 & 0 & 4 \\ 0 & 9 & 1 \end{bmatrix} \tag{40}$$

### 1.2.3. Multiplication

Find the product of the following matrices $A$ and $B$

1.

$$A = \begin{bmatrix} -10 & 1 & -10 \\ -2 & -8 & 9 \\ 6 & 9 & 9 \end{bmatrix} \tag{41}$$

and

$$B = \begin{bmatrix} -8 & 9 & 2 \\ 4 & 1 & -5 \\ 4 & -7 & -5 \end{bmatrix} \tag{42}$$

2.

$$A = \begin{bmatrix} -4 & -3 & 9 \\ 1 & 3 & 1 \\ -8 & -8 & -7 \end{bmatrix} \tag{43}$$

and

$$B = \begin{bmatrix} -2 & 0 & 6 \\ -10 & 8 & -3 \\ 1 & -9 & 1 \end{bmatrix} \tag{44}$$

3.

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 8 & -7 & 3 \\ -10 & 9 & 7 \end{bmatrix} \tag{45}$$

and

$$B = \begin{bmatrix} 1 & 8 & 6 \\ 0 & -4 & 2 \\ 5 & -4 & 6 \end{bmatrix} \tag{46}$$

4.

$$A = \begin{bmatrix} 3 & 9 & -7 \\ 4 & -4 & 3 \\ -5 & -8 & -8 \end{bmatrix} \tag{47}$$

and

$$B = \begin{bmatrix} -7 & -10 & -5 \\ -4 & -5 & 0 \\ 6 & -10 & 7 \end{bmatrix} \tag{48}$$

5.

$$A = \begin{bmatrix} -2 & 8 & 0 \\ 9 & -2 & -1 \\ -10 & -8 & -1 \end{bmatrix} \tag{49}$$

and

$$B = \begin{bmatrix} -3 & 5 & 3 \\ -6 & -2 & -6 \\ -10 & -1 & 3 \end{bmatrix} \tag{50}$$

6.

$$A = \begin{bmatrix} -6 & -3 & -5 \\ 4 & 0 & 0 \\ -4 & -6 & -5 \end{bmatrix} \tag{51}$$

and

$$B = \begin{bmatrix} -9 & -1 & 8 \\ -1 & -4 & 5 \\ 4 & 6 & -2 \end{bmatrix} \quad (52)$$

7.

$$A = \begin{bmatrix} -2 & 9 & 2 \\ 6 & 3 & -9 \\ 9 & -6 & 2 \end{bmatrix} \quad (53)$$

and

$$B = \begin{bmatrix} -5 & 4 & -3 \\ -6 & -4 & 1 \\ 9 & -5 & 0 \end{bmatrix} \quad (54)$$

8.

$$A = \begin{bmatrix} 4 & -4 & -1 \\ 0 & 2 & -9 \\ 6 & -8 & 0 \end{bmatrix} \quad (55)$$

and

$$B = \begin{bmatrix} 5 & -10 & 3 \\ 7 & 8 & -6 \\ -7 & 7 & -3 \end{bmatrix} \quad (56)$$

9.

$$A = \begin{bmatrix} 0 & -3 & 5 \\ 8 & 0 & 7 \\ 4 & -10 & 9 \end{bmatrix} \quad (57)$$

and

$$B = \begin{bmatrix} 5 & -4 & -10 \\ 5 & 8 & -9 \\ -7 & -2 & 5 \end{bmatrix} \quad (58)$$

10.

$$A = \begin{bmatrix} -4 & 0 & -1 \\ 5 & -4 & 1 \\ 1 & -7 & -1 \end{bmatrix} \quad (59)$$

and

$$B = \begin{bmatrix} -7 & -1 & 5 \\ -10 & 2 & -8 \\ -10 & -5 & -6 \end{bmatrix} \quad (60)$$

## 1.3. Matrix Properties

### 1.3.1. Properties

For each matrix $A$, find:

a) rank($A$)

b) nullity($A$)

c) det($A$)

d) $A^{-1}$ (if exists)

e) basis of ker($A$)

1.
$$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{61}$$

2.
$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & -3 \\ 1 & 2 & 1 \end{bmatrix} \tag{62}$$

3.
$$A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & -3 & 5 \\ -2 & -4 & 5 \end{bmatrix} \tag{63}$$

4.
$$A = \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -1 \\ 3 & 2 & -8 \end{bmatrix} \tag{64}$$

5.
$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -4 & -4 \end{bmatrix} \tag{65}$$

6.
$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \tag{66}$$

7.
$$A = \begin{bmatrix} 3 & 1 & 7 \\ -1 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix} \tag{67}$$

8.
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \tag{68}$$

9.
$$A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \tag{69}$$

10.
$$A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix} \tag{70}$$

### 1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix $A$

1.
$$A = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ -2 & 2 & -7 \end{bmatrix} \tag{71}$$

2.
$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{72}$$

3.
$$A = \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \tag{73}$$

4.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \tag{74}$$

5.
$$A = \begin{bmatrix} -3 & 4 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{75}$$

6.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \tag{76}$$

7.
$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -4 & 5 \end{bmatrix} \tag{77}$$

8.
$$A = \begin{bmatrix} 1 & 0 & -2 \\ -4 & 1 & 6 \\ 2 & 0 & -3 \end{bmatrix} \tag{78}$$

9.
$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{79}$$

10.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \tag{80}$$

## 1.4. Calculus

### 1.4.1. Limit

Calculate the following limits

1. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x+1)}{x} \tag{81}$$

2. Calculate the limit of the following expression:

$$\lim_{x \to -3} x^2 + 4x + 1 \tag{82}$$

3. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{83}$$

4. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x+1)}{x} \tag{84}$$

5. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x+1)}{x} \tag{85}$$

6. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x+1)}{x} \tag{86}$$

7. Calculate the limit of the following expression:

$$\lim_{x \to 1} 2x - 3 \tag{87}$$

8. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{88}$$

9. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{89}$$

10. Calculate the limit of the following expression:

$$\lim_{x \to 1} -5x^3 - x^2 + 2x - 2 \tag{90}$$

### 1.4.2. Derivative

Calculate the derivatives of the following expressions

1. Calculate the derivative of the following expression:

$$\log(x+1) + \log(x^2+1) \tag{91}$$

2. Calculate the derivative of the following expression:

$$x^4 \tag{92}$$

3. Calculate the derivative of the following expression:

$$\log(x+1) + \log(x^2+1) \tag{93}$$

4. Calculate the derivative of the following expression:

$$\log(x^2+3) \tag{94}$$

5. Calculate the derivative of the following expression:

$$e^x \tag{95}$$

6. Calculate the derivative of the following expression:

$$\log(x+1) + \log(x^2+1) \tag{96}$$

7. Calculate the derivative of the following expression:

$$\log(x) \tag{97}$$

8. Calculate the derivative of the following expression:

$$\log(x+1) + \log(x^2+1) \tag{98}$$

9. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \tag{99}$$

10. Calculate the derivative of the following expression:

$$x^2 e^x \tag{100}$$

### 1.4.3. Integral

Calculate the indefinite and definite integrals of the following expressions

1. the indefinite integral and evaluate from 3 to 5:

$$\int \frac{1}{\sqrt{1-x^2}} dx \tag{101}$$

2. the indefinite integral and evaluate from 3 to 5:

$$\int e^{-x^2} dx \tag{102}$$

3. the indefinite integral and evaluate from 1 to 2:

$$\int x^4 + 2x^3 - 2x^2 + 2x \, dx \tag{103}$$

4. the indefinite integral and evaluate from 2 to 2:

$$\int e^{-x^2} dx \tag{104}$$

5. the indefinite integral and evaluate from 2 to 5:

$$\int -x^3 - 2x^2 + 3x - 3 \, dx \tag{105}$$

6. the indefinite integral and evaluate from 2 to 3:

$$\int x^2 e^x \, dx \tag{106}$$

7. the indefinite integral and evaluate from 1 to 2:

$$\int \frac{1}{x^2 + 1} dx \tag{107}$$

8. the indefinite integral and evaluate from 2 to 4:

$$\int \frac{e^x}{x} dx \tag{108}$$

9. the indefinite integral and evaluate from 3 to 4:

$$\int 2x^3 + 2x^2 - 3x - 4 dx \tag{109}$$

10. the indefinite integral and evaluate from 1 to 1:

$$\int 5x^2 + 2x - 3 dx \tag{110}$$

### 1.4.4. Partial Derivative

Calculate the partial derivatives of the following expressions

1. the third order partial derivative of:

$$f(x, y) = x^4 y^3 + 3x^2 y^4 \tag{111}$$

$\frac{\partial^3 f}{\partial y^3}$

2. the partial derivatives of the function:

$$f(x, y) = x^3 y^2 - 3x^2 y + 2xy^3 \tag{112}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

3. the second order partial derivative of:

$$f(x, y) = x^4 y^3 + 3x^2 y^4 \tag{113}$$

$\frac{\partial^2 f}{\partial x^2}$

4. the third order partial derivative of:

$$f(x, y) = x^4 y^3 + 3x^2 y^4 \tag{114}$$

$\frac{\partial^3 f}{\partial y^3}$

5. the third order partial derivative of:

$$f(x, y) = x^4 y^3 + 3x^2 y^4 \tag{115}$$

$\frac{\partial^3 f}{\partial y^3}$

6. Given the implicit function:

$$x^2 y + xy^2 - xy = 0 \tag{116}$$

$$\frac{\partial y}{\partial x}$$

7. the mixed partial derivative of:

$$f(x, y) = x^3 y^2 + xy^4 \tag{117}$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

8. the mixed partial derivative of:

$$f(x, y) = x^3 y^2 + xy^4 \tag{118}$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

9. the partial derivatives of the function:

$$f(x, y) = x^3 y^2 - 3x^2 y + 2xy^3 \tag{119}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

10. the partial derivatives of the function:

$$f(x, y) = x^3 y^2 - 3x^2 y + 2xy^3 \tag{120}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

# 2. Solutions

## 2.1. Vector Arithmetic

### 2.1.1. Addition

$$\begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} \quad \begin{bmatrix} -11 \\ -3 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 6 \\ 12 \end{bmatrix} \quad \begin{bmatrix} -5 \\ -6 \\ -4 \end{bmatrix} \quad \begin{bmatrix} -17 \\ -1 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \\ 15 \end{bmatrix} \quad \begin{bmatrix} -7 \\ -1 \\ -2 \end{bmatrix} \quad \begin{bmatrix} -6 \\ -7 \\ 16 \end{bmatrix} \quad \begin{bmatrix} 10 \\ -14 \\ 11 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 11 \\ 7 \end{bmatrix}$$

### 2.1.2. Subtraction

$$\begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -14 \\ 1 \\ -2 \end{bmatrix} \quad \begin{bmatrix} -2 \\ -8 \\ 9 \end{bmatrix} \quad \begin{bmatrix} -4 \\ -12 \\ -11 \end{bmatrix} \quad \begin{bmatrix} 18 \\ 10 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 9 \\ -5 \\ 4 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 14 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 11 \\ -7 \\ -5 \end{bmatrix} \quad \begin{bmatrix} -7 \\ -11 \\ 8 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -6 \\ 6 \end{bmatrix}$$

### 2.1.3. Scalar Multiplication

1: $\begin{bmatrix} 5 \\ 10 \\ 20 \end{bmatrix}$ 2: $\begin{bmatrix} -27 \\ 0 \\ -30 \end{bmatrix}$ 3: $\begin{bmatrix} -4 \\ 20 \\ 16 \end{bmatrix}$ 4: $\begin{bmatrix} 6 \\ 12 \\ -24 \end{bmatrix}$ 5: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

6: $\begin{bmatrix} -24 \\ 40 \\ 56 \end{bmatrix}$ 7: $\begin{bmatrix} 40 \\ 100 \\ 80 \end{bmatrix}$ 8: $\begin{bmatrix} 70 \\ -70 \\ 0 \end{bmatrix}$ 9: $\begin{bmatrix} -24 \\ 15 \\ -3 \end{bmatrix}$ 10: $\begin{bmatrix} 20 \\ 24 \\ 40 \end{bmatrix}$

## 2.2. Matrix Arithmetic

### 2.2.1. Addition

1:

$$\begin{bmatrix} -4 & -7 & -4 \\ 10 & 0 & -10 \\ -6 & -3 & 9 \end{bmatrix} \tag{121}$$

1:

$$\begin{bmatrix} -16 & -5 & -14 \\ 11 & 8 & -7 \\ 1 & -7 & 2 \end{bmatrix} \tag{122}$$

1:

$$\begin{bmatrix} 13 & 1 & 6 \\ -15 & -10 & -5 \\ -8 & -8 & 7 \end{bmatrix} \tag{123}$$

1:

$$\begin{bmatrix} -1 & -2 & 2 \\ 6 & -13 & -5 \\ 7 & -5 & -3 \end{bmatrix} \tag{124}$$

1:

$$\begin{bmatrix} 6 & -4 & -9 \\ 12 & 2 & 10 \\ -2 & 0 & -16 \end{bmatrix} \tag{125}$$

1:

$$\begin{bmatrix} 11 & -12 & -5 \\ 2 & 4 & -18 \\ -11 & -3 & -1 \end{bmatrix} \tag{126}$$

1:

$$\begin{bmatrix} -5 & -2 & -3 \\ -1 & -1 & 4 \\ 9 & -1 & 1 \end{bmatrix} \tag{127}$$

1:

$$\begin{bmatrix} 12 & -17 & 12 \\ -16 & -2 & -8 \\ 4 & -7 & 8 \end{bmatrix} \tag{128}$$

1:

$$\begin{bmatrix} -4 & -4 & -11 \\ -3 & 0 & 5 \\ 5 & -1 & -1 \end{bmatrix} \tag{129}$$

1:

$$\begin{bmatrix} -11 & 0 & 0 \\ 3 & -3 & -16 \\ -4 & 12 & -8 \end{bmatrix} \tag{130}$$

### 2.2.2. Subtraction

1:

$$\begin{bmatrix} -1 & 5 & 1 \\ 3 & -11 & -5 \\ 0 & -8 & -17 \end{bmatrix} \tag{131}$$

1:

$$\begin{bmatrix} -6 & -1 & 13 \\ 1 & 1 & -2 \\ 7 & 8 & -1 \end{bmatrix} \tag{132}$$

1:

$$\begin{bmatrix} -4 & 4 & -13 \\ 13 & 0 & -9 \\ 0 & -5 & -16 \end{bmatrix} \tag{133}$$

1:

$$\begin{bmatrix} 7 & 1 & 14 \\ 6 & 8 & -3 \\ 1 & -2 & 9 \end{bmatrix} \tag{134}$$

1:

$$\begin{bmatrix} 11 & -3 & 9 \\ 10 & 1 & -14 \\ 0 & -5 & 9 \end{bmatrix} \tag{135}$$

1:

$$\begin{bmatrix} 1 & 2 & -2 \\ 7 & 17 & 6 \\ 5 & -5 & -4 \end{bmatrix} \tag{136}$$

1:

$$\begin{bmatrix} -4 & -7 & -8 \\ 4 & -1 & 2 \\ -2 & 17 & -1 \end{bmatrix} \tag{137}$$

1:

$$\begin{bmatrix} -9 & -8 & 16 \\ -1 & 7 & 8 \\ 14 & 17 & 8 \end{bmatrix} \tag{138}$$

1:

$$\begin{bmatrix} -4 & 11 & -12 \\ -10 & 1 & -5 \\ 4 & -13 & 1 \end{bmatrix} \tag{139}$$

1:

$$\begin{bmatrix} 15 & 4 & 4 \\ 12 & 8 & -13 \\ -3 & -15 & -8 \end{bmatrix} \tag{140}$$

### 2.2.3. Multiplication

1:

$$\begin{bmatrix} 44 & -19 & 25 \\ 20 & -89 & -9 \\ 24 & 0 & -78 \end{bmatrix} \tag{141}$$

1:

$$\begin{bmatrix} 47 & -105 & -6 \\ -31 & 15 & -2 \\ 89 & -1 & -31 \end{bmatrix} \tag{142}$$

1:

$$\begin{bmatrix} 45 & -4 & 84 \\ 23 & 80 & 52 \\ 25 & -144 & 0 \end{bmatrix} \tag{143}$$

1:

$$\begin{bmatrix} -99 & -5 & -64 \\ 6 & -50 & 1 \\ 19 & 170 & -31 \end{bmatrix} \tag{144}$$

1:

$$\begin{bmatrix} -42 & -26 & -54 \\ -5 & 50 & 36 \\ 88 & -33 & 15 \end{bmatrix} \tag{145}$$

1:

$$\begin{bmatrix} 37 & -12 & -53 \\ -36 & -4 & 32 \\ 22 & -2 & -52 \end{bmatrix} \tag{146}$$

1:

$$\begin{bmatrix} -26 & -54 & 15 \\ -129 & 57 & -15 \\ 9 & 50 & -33 \end{bmatrix} \tag{147}$$

1:

$$\begin{bmatrix} -1 & -79 & 39 \\ 77 & -47 & 15 \\ -26 & -124 & 66 \end{bmatrix} \tag{148}$$

1:

$$\begin{bmatrix} -50 & -34 & 52 \\ -9 & -46 & -45 \\ -93 & -114 & 95 \end{bmatrix} \tag{149}$$

1:

$$\begin{bmatrix} 38 & 9 & -14 \\ -5 & -18 & 51 \\ 73 & -10 & 67 \end{bmatrix} \tag{150}$$

## 2.3. Matrix Properties

### 2.3.1. Properties

**Solution**

**Row Operations:**

Step 1: $r_1 := r_1 - (-4)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 4 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_2 := r_2 - (-1)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 4 \\ 0 & 1 & 0 & | & 0 & 1 & 1 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

**Results:**

a) $\operatorname{rank}(A) = 3$

b) $\operatorname{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

**Solution**

**Row Operations:**

Step 1: $r_2 := r_2 - (-2)r_1$ $\begin{bmatrix} 1 & 2 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & 2 & 1 & 0 \\ 1 & 2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_3 := r_3 - r_1$ $\begin{bmatrix} 1 & 2 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & 2 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$

Step 3: $r_1 := r_1 - (2)r_2$ $\begin{bmatrix} 1 & 0 & 6 & | & -3 & -2 & 0 \\ 0 & 1 & -3 & | & 2 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$

Step 4: $r_1 := r_1 - (6)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 3 & -2 & -6 \\ 0 & 1 & -3 & | & 2 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$

Step 5: $r_2 := r_2 - (-3)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 3 & -2 & -6 \\ 0 & 1 & 0 & | & -1 & 1 & 3 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$

**Results:**

a) $\operatorname{rank}(A) = 3$

b) $\operatorname{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

**Solution**

**Row Operations:**

Step 1: $r_2 := r_2 - (-2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ -2 & -4 & 5 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_3 := r_3 - (-2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$

Step 3: $r_1 := r_1 - (2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$

Step 4: $r_1 := r_1 - (-4)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & 4 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$

Step 5: $r_2 := r_2 - r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & 4 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$

**Results:**

a) $\operatorname{rank}(A) = 3$

b) $\operatorname{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

**Solution**

**Row Operations:**

Step 1: $r_3 := r_3 - (3)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array}\right]$

Step 2: $r_1 := r_1 - (2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array}\right]$

Step 3: $r_3 := r_3 - (-4)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 4 & 1 \end{array}\right]$

**Results:**

a) rank($A$) = 2

b) nullity($A$) = 1

c) det($A$) = 0

d) $A^{-1} =$ does not exist

e) ker($A$) = span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

## Solution

### Row Operations:

Step 1: $r_1 := r_1 - r_2$ $\begin{bmatrix} 1 & 0 & -2 & | & 1 & -1 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & -4 & -4 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_3 := r_3 - (-4)r_2$ $\begin{bmatrix} 1 & 0 & -2 & | & 1 & -1 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 4 & 1 \end{bmatrix}$

### Results:

a) rank($A$) = 2

b) nullity($A$) = 1

c) det($A$) = 0

d) $A^{-1} =$ does not exist

e) ker($A$) = span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

## Solution

### Row Operations:

Step 1: $r_1 := r_1 - (2)r_2$ $\begin{bmatrix} 1 & 0 & 4 & | & 1 & -2 & 0 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_1 := r_1 - (4)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 1 & -2 & -4 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 3: $r_2 := r_2 - (-2)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 1 & -2 & -4 \\ 0 & 1 & 0 & | & 0 & 1 & 2 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

### Results:

a) rank($A$) = 3

b) nullity($A$) = 0

c) det($A$) = 0

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

**Solution**

**Row Operations:**

Step 1: $r_1 := 1/3 r_1$ $\begin{bmatrix} 1 & 1/3 & 7/3 & | & 1/3 & 0 & 0 \\ -1 & 0 & -3 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_2 := r_2 - (-1)r_1$ $\begin{bmatrix} 1 & 1/3 & 7/3 & | & 1/3 & 0 & 0 \\ 0 & 1/3 & -2/3 & | & 1/3 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 3: $r_3 := r_3 - r_1$ $\begin{bmatrix} 1 & 1/3 & 7/3 & | & 1/3 & 0 & 0 \\ 0 & 1/3 & -2/3 & | & 1/3 & 1 & 0 \\ 0 & 2/3 & -4/3 & | & -1/3 & 0 & 1 \end{bmatrix}$

Step 4: $r_2 := 3 r_2$ $\begin{bmatrix} 1 & 1/3 & 7/3 & | & 1/3 & 0 & 0 \\ 0 & 1 & -2 & | & 1 & 3 & 0 \\ 0 & 2/3 & -4/3 & | & -1/3 & 0 & 1 \end{bmatrix}$

Step 5: $r_1 := r_1 - (1/3)r_2$ $\begin{bmatrix} 1 & 0 & 3 & | & 0 & -1 & 0 \\ 0 & 1 & -2 & | & 1 & 3 & 0 \\ 0 & 2/3 & -4/3 & | & -1/3 & 0 & 1 \end{bmatrix}$

Step 6: $r_3 := r_3 - (2/3)r_2$ $\begin{bmatrix} 1 & 0 & 3 & | & 0 & -1 & 0 \\ 0 & 1 & -2 & | & 1 & 3 & 0 \\ 0 & 0 & 0 & | & -1 & -2 & 1 \end{bmatrix}$

**Results:**

a) $\mathrm{rank}(A) = 2$

b) $\mathrm{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} =$ does not exist

e) $\ker(A) = \mathrm{span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

**Solution**

**Row Operations:**

Step 1: $r_2 := r_2 - r_1$ $\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & -1 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$

**Results:**

a) $\mathrm{rank}(A) = 2$

b) $\mathrm{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} =$ does not exist

**e)**
$$\ker(A) = \text{span} \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

**Solution**

**Row Operations:**

Step 1: $r_1 := r_1 - (-2)r_2$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Step 2: $r_2 := r_2 - (-2)r_3$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

**Results:**

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d)
$$A^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

e) $\ker(A) = \{\mathbf{0}\}$

**Solution**

**Row Operations:**

Step 1: $r_2 := r_2 - (-3)r_1$
$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Step 2: $r_3 := r_3 - r_1$
$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

Step 3: $r_1 := r_1 - r_2$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

**Results:**

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} = $ does not exist

**e)**
$$\ker(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

**2.3.2. RREF**

**Solution**

**Elementary Row Operations:**

(1) $r_3 := r_3 - (2)r_1$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) $r_1 := r_1 - (2)r_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

(1) $r_1 := r_1 + (-1)r_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) $r_2 := r_2 - r_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

(1) $r_1 := r_1 - (2)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_2 := r_2 + (-2)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_2 := r_2 + (-1)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

(1) $r_2 := r_2 + (-1)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

(2) $r_3 := r_3 - (2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

(1) $r_1 := r_1 + (-2)r_2$

$$\begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 + (-2)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_2 := r_2 - r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

(1) $r_3 := r_3 + (-1)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_2 := r_2 - r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_2 := r_2 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

(1) $r_3 := r_3 - (2)r_2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$$

(2) $r_3 := r_3 - (2)r_2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

(1) $r_2 := r_2 - (2)r_3$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

(2) $r_3 := r_3 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

(1) $r_1 := r_1 + (-2)r_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 + (-1)r_3$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(3) $r_3 := r_3 + (-1)r_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

(1) $r_2 := r_2 - (2)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

(2) $r_3 := r_3 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## 2.4. Calculus

### 2.4.1. Limit

The limit is:

$$1 \tag{151}$$

The limit is:

$$-2 \tag{152}$$

The limit is:

$$e \tag{153}$$

The limit is:

$$1 \tag{154}$$

The limit is:

$$1 \tag{155}$$

The limit is:

$$1 \tag{156}$$

The limit is:

$$-1 \tag{157}$$

The limit is:

$$e \tag{158}$$

The limit is:

$$e \tag{159}$$

The limit is:

$$-6 \tag{160}$$

### 2.4.2. Derivative

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \tag{161}$$

The derivative is:

$$4x^3 \tag{162}$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \tag{163}$$

The derivative is:

$$\frac{2x}{x^2 + 3} \tag{164}$$

The derivative is:

$$e^x \tag{165}$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \tag{166}$$

The derivative is:

$$\frac{1}{x} \tag{167}$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \tag{168}$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \tag{169}$$

The derivative is:

$$x^2 e^x + 2xe^x \tag{170}$$

### 2.4.3. Integral

The indefinite integral is:

$$\operatorname{asin}(x) \tag{171}$$

Definite integral from 3 to 5:

$$\operatorname{asin}(5) - \operatorname{asin}(3) \tag{172}$$

The indefinite integral is:

$$\frac{\sqrt{\pi}\,\operatorname{erf}(x)}{2} \tag{173}$$

Definite integral from 3 to 5:

$$-\frac{\sqrt{\pi}\,\operatorname{erf}(3)}{2} + \frac{\sqrt{\pi}\,\operatorname{erf}(5)}{2} \tag{174}$$

The indefinite integral is:

$$\frac{x^5}{5} + \frac{x^4}{2} - \frac{2x^3}{3} + x^2 \tag{175}$$

Definite integral from 1 to 2:

$$\frac{361}{30} \tag{176}$$

The indefinite integral is:

$$\frac{\sqrt{\pi}\,\operatorname{erf}(x)}{2} \tag{177}$$

Definite integral from 2 to 2:

$$0 \tag{178}$$

The indefinite integral is:

$$-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{3x^2}{2} - 3x \tag{179}$$

Definite integral from 2 to 5:

$$-\frac{831}{4} \tag{180}$$

The indefinite integral is:

$$\left(x^2 - 2x + 2\right)e^x \tag{181}$$

Definite integral from 2 to 3:

$$-2e^2 + 5e^3 \tag{182}$$

The indefinite integral is:

$$\text{atan}\,(x) \tag{183}$$

Definite integral from 1 to 2:

$$-\frac{\pi}{4} + \text{atan}\,(2) \tag{184}$$

The indefinite integral is:

$$\text{Ei}\,(x) \tag{185}$$

Definite integral from 2 to 4:

$$-\,\text{Ei}\,(2) + \text{Ei}\,(4) \tag{186}$$

The indefinite integral is:

$$\frac{x^4}{2} + \frac{2x^3}{3} - \frac{3x^2}{2} - 4x \tag{187}$$

Definite integral from 3 to 4:

$$\frac{293}{3} \tag{188}$$

The indefinite integral is:

$$\frac{5x^3}{3} + x^2 - 3x \tag{189}$$

Definite integral from 1 to 1:

$$0 \tag{190}$$

### 2.4.4. Partial Derivative

$$\frac{\partial^3 f}{\partial y^3} = 6x^2\left(x^2 + 12y\right) \tag{191}$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \tag{192}$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \tag{193}$$

$$\frac{\partial^2 f}{\partial x^2} = 6y^3(2x^2 + y) \tag{194}$$

$$\frac{\partial^3 f}{\partial y^3} = 6x^2(x^2 + 12y) \tag{195}$$

$$\frac{\partial^3 f}{\partial y^3} = 6x^2(x^2 + 12y) \tag{196}$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \tag{197}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \tag{198}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \tag{199}$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \tag{200}$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \tag{201}$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \tag{202}$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \tag{203}$$