Exercise 28:

Foundations of Mathematical, WS24

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This is **exercise** 28 for Foundations of Mathematical, WS24. Generated on 2025-06-02 with 10 problems per section.

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1. Problems

1.1. Vector Arithmetic

1.1.1. Addition

Find the sum of the following vectors $\mathbf{u}$ and $\mathbf{v}$

1. $\mathbf{u} = \begin{bmatrix} 8 \\ 5 \\ 10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 4 \\ -10 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} -3 \\ 1 \\ -9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ -1 \\ 0 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 4 \\ -1 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} -7 \\ -5 \\ -9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ 10 \\ -2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} -6 \\ -6 \\ -8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ -3 \\ 2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} 8 \\ -6 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -5 \\ 4 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} -4 \\ 8 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ -9 \\ 2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

1.1.2. Subtraction

Find the difference of the following vectors $\mathbf{u}$ and $\mathbf{v}$

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ -10 \\ 10 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} -7 \\ -5 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 1 \\ 6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} -1 \\ -4 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -3 \\ -4 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} -6 \\ -7 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 9 \\ -5 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ -7 \\ 5 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

6. $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} 4 \\ 8 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ -6 \\ -2 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -1 \\ -4 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ -8 \\ 7 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} 2 \\ 6 \\ -8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \\ 10 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

1.1.3. Scalar Multiplication

Find the scalar product of the following vector $\mathbf{u}$ and scalar k

1. $\mathbf{u} = \begin{bmatrix} 4 \\ -4 \\ -3 \end{bmatrix}$ $7\mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} 6 \\ 1 \\ 9 \end{bmatrix}$ $1\mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} -1 \\ 8 \\ 0 \end{bmatrix}$ $2\mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ -2 \end{bmatrix}$ $-3\mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 0 \\ -5 \\ -10 \end{bmatrix}$ $3\mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} -2 \\ -10 \\ 3 \end{bmatrix}$ $2\mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -7 \\ -4 \\ 10 \end{bmatrix}$ $10\mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -4 \\ 0 \\ -7 \end{bmatrix}$ $-8\mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} -1 \\ -2 \\ -5 \end{bmatrix}$ $3\mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 8 \end{bmatrix}$ $-10\mathbf{v}$.

1.2. Matrix Arithmetic

1.2.1. Addition

Find the sum of the following matrices A and B

1.

$$A = \begin{bmatrix} 1 & -10 & 3 \\ 6 & 2 & 5 \\ 6 & -5 & 4 \end{bmatrix} \tag{1}$$

and

$$B = \begin{bmatrix} 8 & 8 & 8 \\ 6 & -7 & -2 \\ 2 & 4 & 3 \end{bmatrix} \tag{2}$$

2.

$$A = \begin{bmatrix} 2 & 0 & -8 \\ -2 & -7 & -3 \\ 2 & 8 & -9 \end{bmatrix} \tag{3}$$

and

$$B = \begin{bmatrix} -8 & 8 & -6 \\ 9 & -5 & -1 \\ 2 & 8 & 7 \end{bmatrix} \tag{4}$$

3.

$$A = \begin{bmatrix} 9 & 1 & -10 \\ 4 & -2 & -2 \\ 8 & 7 & -4 \end{bmatrix} \tag{5}$$

and

$$B = \begin{bmatrix} 5 & 1 & 4 \\ -8 & -6 & -7 \\ 6 & -1 & -6 \end{bmatrix} \tag{6}$$

4.

$$A = \begin{bmatrix} 8 & -10 & 7 \\ 3 & -5 & 9 \\ 8 & 8 & -3 \end{bmatrix} \tag{7}$$

and

$$B = \begin{bmatrix} -2 & -7 & -3 \\ 7 & -2 & -4 \\ -2 & 3 & -9 \end{bmatrix} \tag{8}$$

5.

$$A = \begin{bmatrix} -8 & 2 & 8 \\ -6 & 0 & 4 \\ 9 & -10 & -8 \end{bmatrix} \tag{9}$$

and

$$B = \begin{bmatrix} 7 & -8 & 3 \\ -10 & 3 & -1 \\ 7 & -6 & -8 \end{bmatrix} \tag{10}$$

6.

$$A = \begin{bmatrix} 5 & -5 & -4 \\ 2 & 9 & -3 \\ 1 & 0 & -2 \end{bmatrix} \tag{11}$$

and

$$B = \begin{bmatrix} -6 & 3 & 2 \\ 9 & 7 & -1 \\ 7 & -6 & 5 \end{bmatrix} \tag{12}$$

7.

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 4 & 7 & -2 \\ 7 & 5 & 5 \end{bmatrix} \tag{13}$$

and

$$B = \begin{bmatrix} -9 & 3 & 5 \\ 9 & -1 & 7 \\ -4 & -10 & 4 \end{bmatrix} \tag{14}$$

8.

$$A = \begin{bmatrix} -7 & -3 & 2 \\ -8 & 7 & -4 \\ 4 & -3 & -2 \end{bmatrix} \tag{15}$$

and

$$B = \begin{bmatrix} 0 & 9 & -10 \\ 2 & -5 & -5 \\ 5 & -2 & -5 \end{bmatrix} \tag{16}$$

9.

$$A = \begin{bmatrix} -1 & -9 & 4 \\ -8 & -3 & 6 \\ -7 & -1 & 0 \end{bmatrix} \tag{17}$$

and

$$B = \begin{bmatrix} -4 & -10 & 3 \\ -3 & -8 & 8 \\ -10 & -4 & -2 \end{bmatrix} \tag{18}$$

10.

$$A = \begin{bmatrix} -6 & 0 & -3 \\ 0 & -9 & -3 \\ -4 & 0 & -6 \end{bmatrix} \tag{19}$$

and

$$B = \begin{bmatrix} 5 & 5 & 7 \\ 5 & 2 & 3 \\ -10 & 5 & -7 \end{bmatrix} \tag{20}$$

1.2.2. Subtraction

Find the difference of the following matrices A and B

1.

$$A = \begin{bmatrix} -2 & -5 & 4 \\ 0 & -9 & 4 \\ 3 & 9 & 2 \end{bmatrix} \tag{21}$$

and

$$B = \begin{bmatrix} -2 & 8 & -7 \\ -3 & -8 & -4 \\ 0 & -1 & 9 \end{bmatrix} \tag{22}$$

2.

$$A = \begin{bmatrix} -10 & 4 & -1 \\ 8 & 0 & -9 \\ -6 & 6 & -8 \end{bmatrix} \tag{23}$$

and

$$B = \begin{bmatrix} 9 & -10 & 5 \\ -9 & -1 & 1 \\ -2 & -2 & 8 \end{bmatrix} \tag{24}$$

3.

$$A = \begin{bmatrix} 5 & -10 & -3 \\ 2 & -2 & -4 \\ 3 & -3 & -10 \end{bmatrix} \tag{25}$$

and

$$B = \begin{bmatrix} -1 & -3 & -3 \\ 3 & -6 & 7 \\ 1 & 7 & 8 \end{bmatrix} \tag{26}$$

4.

$$A = \begin{bmatrix} -6 & -9 & 2 \\ -4 & -10 & -2 \\ 0 & 5 & 2 \end{bmatrix} \tag{27}$$

and

$$B = \begin{bmatrix} 4 & -9 & -1 \\ -4 & 8 & -10 \\ -7 & 4 & -4 \end{bmatrix} \tag{28}$$

5.

$$A = \begin{bmatrix} 7 & 6 & -1 \\ 4 & -1 & 0 \\ -10 & -9 & 5 \end{bmatrix} \tag{29}$$

and

$$B = \begin{bmatrix} 8 & 0 & 2 \\ -6 & 3 & 7 \\ -7 & 2 & -4 \end{bmatrix} \tag{30}$$

6.

$$A = \begin{bmatrix} -10 & -7 & 0 \\ -2 & 6 & 6 \\ -4 & -6 & -7 \end{bmatrix} \tag{31}$$

and

$$B = \begin{bmatrix} -2 & -8 & -8 \\ -5 & 7 & 9 \\ 3 & -1 & 4 \end{bmatrix} \tag{32}$$

7.

$$A = \begin{bmatrix} 0 & 4 & 7 \\ -4 & -5 & 6 \\ -6 & -10 & 4 \end{bmatrix} \tag{33}$$

and

$$B = \begin{bmatrix} 0 & -3 & 6 \\ -2 & -6 & 7 \\ -4 & -4 & -4 \end{bmatrix} \tag{34}$$

8.

$$A = \begin{bmatrix} -4 & 8 & -7 \\ 6 & 4 & 0 \\ -1 & -2 & 2 \end{bmatrix} \tag{35}$$

and

$$B = \begin{bmatrix} -3 & -5 & -5 \\ -1 & 6 & -2 \\ 5 & -8 & -10 \end{bmatrix} \tag{36}$$

9.

$$A = \begin{bmatrix} -6 & -7 & -1 \\ 3 & -5 & -6 \\ 3 & -4 & 0 \end{bmatrix} \tag{37}$$

and

$$B = \begin{bmatrix} -5 & 1 & 8 \\ 2 & 9 & 4 \\ -5 & -6 & -3 \end{bmatrix} \tag{38}$$

10.

$$A = \begin{bmatrix} -7 & 5 & -8 \\ -5 & -4 & 2 \\ 4 & -8 & -9 \end{bmatrix} \tag{39}$$

and

$$B = \begin{bmatrix} 6 & 2 & -2 \\ 8 & -9 & 1 \\ 1 & -9 & -7 \end{bmatrix} \tag{40}$$

1.2.3. Multiplication

Find the product of the following matrices A and B

1.

$$A = \begin{bmatrix} -2 & -6 & -5 \\ 2 & -4 & 3 \\ 6 & -6 & -3 \end{bmatrix} \tag{41}$$

and

$$B = \begin{bmatrix} -9 & 3 & 5 \\ 2 & 4 & 3 \\ 4 & 9 & -10 \end{bmatrix} \tag{42}$$

2.

$$A = \begin{bmatrix} -8 & 8 & -10 \\ 8 & 8 & 4 \\ 0 & -5 & 1 \end{bmatrix} \tag{43}$$

and

$$B = \begin{bmatrix} 0 & -3 & -6 \\ 6 & -2 & 6 \\ 7 & -4 & 3 \end{bmatrix} \tag{44}$$

3.

$$A = \begin{bmatrix} 6 & 0 & -8 \\ 9 & 3 & 7 \\ -2 & -10 & 5 \end{bmatrix} \tag{45}$$

and

$$B = \begin{bmatrix} -4 & -8 & 8 \\ -1 & -1 & -9 \\ 2 & -6 & -2 \end{bmatrix} \tag{46}$$

4.

$$A = \begin{bmatrix} 9 & -9 & 0 \\ -7 & 2 & 8 \\ -3 & 4 & -4 \end{bmatrix} \tag{47}$$

and

$$B = \begin{bmatrix} 8 & 9 & 8 \\ -2 & 7 & 9 \\ 1 & -6 & -6 \end{bmatrix} \tag{48}$$

5.

$$A = \begin{bmatrix} 0 & -5 & 2 \\ -4 & -3 & -5 \\ -4 & 9 & 9 \end{bmatrix} \tag{49}$$

and

$$B = \begin{bmatrix} 1 & 2 & -9 \\ 8 & 3 & -2 \\ 0 & 0 & -10 \end{bmatrix} \tag{50}$$

6.

$$A = \begin{bmatrix} 2 & -4 & 2 \\ 6 & -9 & -5 \\ -1 & -4 & -3 \end{bmatrix} \tag{51}$$

and

$$B = \begin{bmatrix} -4 & 4 & -3 \\ 5 & -6 & 6 \\ 5 & -4 & -10 \end{bmatrix} \tag{52}$$

7.

$$A = \begin{bmatrix} -3 & -6 & -7 \\ -1 & -4 & -10 \\ 6 & 4 & -5 \end{bmatrix} \tag{53}$$

and

$$B = \begin{bmatrix} 7 & -4 & -4 \\ 9 & -7 & -10 \\ -4 & 8 & 3 \end{bmatrix} \tag{54}$$

8.

$$A = \begin{bmatrix} 2 & 7 & -7 \\ -7 & -8 & 8 \\ 8 & -7 & -6 \end{bmatrix} \tag{55}$$

and

$$B = \begin{bmatrix} -10 & -10 & 6 \\ 2 & -2 & -4 \\ -1 & -7 & -5 \end{bmatrix} \tag{56}$$

9.

$$A = \begin{bmatrix} -10 & 1 & -7 \\ -4 & 0 & 4 \\ -8 & 6 & -6 \end{bmatrix} \tag{57}$$

and

$$B = \begin{bmatrix} 2 & -7 & 8 \\ 1 & -3 & -4 \\ 1 & 8 & -4 \end{bmatrix} \tag{58}$$

10.

$$A = \begin{bmatrix} -1 & 4 & -1 \\ 6 & -7 & 4 \\ 8 & 1 & 0 \end{bmatrix} \tag{59}$$

and

$$B = \begin{bmatrix} 1 & 9 & -1 \\ 8 & 8 & -5 \\ -1 & -10 & 0 \end{bmatrix} \tag{60}$$

1.3. Matrix Properties

1.3.1. Properties

For each matrix A, find:

a) $\operatorname{rank}(A)$

b) nullity(A)

c) det(A)

d) A^{-1} (if exists)

e) basis of ker(A)

1.
$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ -2 & 4 & -4 \end{bmatrix} \tag{61}$$

2.
$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{62}$$

3.
$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{63}$$

4.
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix} \tag{64}$$

5.
$$A = \begin{bmatrix} -3 & 2 & -9 \\ -2 & 1 & -7 \\ -2 & 1 & -6 \end{bmatrix} \tag{65}$$

6.
$$A = \begin{bmatrix} 1 & -3 & 1 \\ -1 & 4 & 0 \\ -2 & 8 & 0 \end{bmatrix} \tag{66}$$

7.
$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -5 \\ 2 & 0 & 3 \end{bmatrix} \tag{67}$$

8.
$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ -2 & -1 & -7 \end{bmatrix} \tag{68}$$

9.
$$A = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix} \tag{69}$$

10.
$$A = \begin{bmatrix} 1 & 7 & 11 \\ 0 & 5 & 8 \\ 0 & -2 & -3 \end{bmatrix} \tag{70}$$

1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix A

1.
$$A = \begin{bmatrix} 3 & 0 & -1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \tag{71}$$

2.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \tag{72}$$

3.
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \tag{73}$$

4.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix} \tag{74}$$

5.
$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} \tag{75}$$

6.
$$A = \begin{bmatrix} -3 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{76}$$

7.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \tag{77}$$

8.
$$A = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \tag{78}$$

9.
$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \tag{79}$$

10.
$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -4 \end{bmatrix} \tag{80}$$

1.4. Calculus

1.4.1. Limit

Calculate the following limits

1. Calculate the limit of the following expression:

$$\lim_{x \to 0} -4x^3 + 4x^2 + 3x - 1 \tag{81}$$

2. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x+1)}{x} \tag{82}$$

3. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x+1)}{x} \tag{83}$$

4. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x+1)}{x} \tag{84}$$

5. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x+1)}{x} \tag{85}$$

6. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{86}$$

7. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{87}$$

8. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x+1)}{x} \tag{88}$$

9. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{89}$$

10. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x+1)}{x} \tag{90}$$

1.4.2. Derivative

Calculate the derivatives of the following expressions

1. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \tag{91}$$

2. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \tag{92}$$

3. Calculate the derivative of the following expression:

$$\log(x+1) + \log(x^2+1) \tag{93}$$

4. Calculate the derivative of the following expression:

$$\frac{x^3}{x^2+1} \tag{94}$$

5. Calculate the derivative of the following expression:

$$xe^x \tag{95}$$

6. Calculate the derivative of the following expression:

$$\log(x^2-1) \tag{96}$$

7. Calculate the derivative of the following expression:

$$\log(x+1)+\log(x^2+1) \tag{97}$$

8. Calculate the derivative of the following expression:

$$\frac{x^2}{x^2+1} \tag{98}$$

9. Calculate the derivative of the following expression:

$$\log(x+1)+\log(x^2+1) \tag{99}$$

10. Calculate the derivative of the following expression:

$$\frac{x^3}{x^2+1} \tag{100}$$

1.4.3. Integral

Calculate the indefinite and definite integrals of the following expressions

1. the indefinite integral and evaluate from 1 to 3:

$$\int \frac{\sin(x)}{x} dx \tag{101}$$

2. Evaluate the improper integral:

$$\int_1^{oo} \frac{1}{\sqrt{x}} dx \tag{102}$$

3. the indefinite integral and evaluate from 1 to 4:

$$\int e^x \sin(x) dx \tag{103}$$

4. the indefinite integral and evaluate from 2 to 5:

$$\int x\sqrt{x^2+1} dx \tag{104}$$

5. the indefinite integral and evaluate from 2 to 3:

$$\int x^3 \log(x) dx \tag{105}$$

6. the indefinite integral and evaluate from 3 to 4:

$$\int \frac{1}{(x-2)(x+1)} dx \tag{106}$$

7. the indefinite integral and evaluate from 1 to 1:

$$\int -5x^2 - 5x + 2 dx \tag{107}$$

8. the indefinite integral and evaluate from 2 to 4:

$$\int e^x \sin(x) dx \tag{108}$$

9. the indefinite integral and evaluate from 4 to 5:

$$\int x^3 \log(x) dx \tag{109}$$

10. the indefinite integral and evaluate from 1 to 3:

$$\int x\sqrt{x^2+1} dx \tag{110}$$

1.4.4. Partial Derivative

Calculate the partial derivatives of the following expressions

1. the mixed partial derivative of:

$$f(x, y) = x^3 y^2 + x y^4 \tag{111}$$

   $\frac{\partial^2 f}{\partial x \partial y}$

2. Given the implicit function:

$$x^2 y + x y^2 - x y = 0 \tag{112}$$

   $\frac{\partial y}{\partial x}$

3. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \tag{113}$$

   where $f = f(u, v)$

4. the partial derivatives of the function:

$$f(x, y) = x^3 y^2 - 3x^2 y + 2x y^3 \tag{114}$$

   $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

5. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \tag{115}$$

   where $f = f(u, v)$

6. the second order partial derivative of:

$$f(x, y) = x^4 y^3 + 3x^2 y^4 \tag{116}$$

$\frac{\partial^2 f}{\partial x^2}$

7. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \tag{117}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

8. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \tag{118}$$

   where $f = f(u, v)$

9. Given the implicit function:

$$x^2 y + xy^2 - xy = 0 \tag{119}$$

$\frac{\partial y}{\partial x}$

10. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \tag{120}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

2. Solutions

2.1. Vector Arithmetic

2.1.1. Addition

$\begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -8 \\ 0 \\ -9 \end{bmatrix}$ $\begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix}$ $\begin{bmatrix} 8 \\ 7 \\ 2 \end{bmatrix}$ $\begin{bmatrix} -12 \\ 5 \\ -11 \end{bmatrix}$

$\begin{bmatrix} -12 \\ -9 \\ -6 \end{bmatrix}$ $\begin{bmatrix} 15 \\ -3 \\ -5 \end{bmatrix}$ $\begin{bmatrix} -5 \\ 12 \\ 4 \end{bmatrix}$ $\begin{bmatrix} -7 \\ -1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$

2.1.2. Subtraction

$\begin{bmatrix} 4 \\ 14 \\ -12 \end{bmatrix}$ $\begin{bmatrix} -15 \\ -6 \\ -3 \end{bmatrix}$ $\begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix}$ $\begin{bmatrix} -8 \\ -12 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ -9 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$ $\begin{bmatrix} -4 \\ 14 \\ -5 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 4 \\ -10 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 1 \\ -18 \end{bmatrix}$ $\begin{bmatrix} 6 \\ 3 \\ -6 \end{bmatrix}$

2.1.3. Scalar Multiplication

1: $\begin{bmatrix} 28 \\ -28 \\ -21 \end{bmatrix}$ 2: $\begin{bmatrix} 6 \\ 1 \\ 9 \end{bmatrix}$ 3: $\begin{bmatrix} -2 \\ 16 \\ 0 \end{bmatrix}$ 4: $\begin{bmatrix} -21 \\ -6 \\ 6 \end{bmatrix}$ 5: $\begin{bmatrix} 0 \\ -15 \\ -30 \end{bmatrix}$

6: $\begin{bmatrix} -4 \\ -20 \\ 6 \end{bmatrix}$ 7: $\begin{bmatrix} -70 \\ -40 \\ 100 \end{bmatrix}$ 8: $\begin{bmatrix} 32 \\ 0 \\ 56 \end{bmatrix}$ 9: $\begin{bmatrix} -3 \\ -6 \\ -15 \end{bmatrix}$ 10: $\begin{bmatrix} 10 \\ -10 \\ -80 \end{bmatrix}$

2.2. Matrix Arithmetic

2.2.1. Addition

1:

$$\begin{bmatrix} 9 & -2 & 11 \\ 12 & -5 & 3 \\ 8 & -1 & 7 \end{bmatrix} \tag{121}$$

1:

$$\begin{bmatrix} -6 & 8 & -14 \\ 7 & -12 & -4 \\ 4 & 16 & -2 \end{bmatrix} \tag{122}$$

1:

$$\begin{bmatrix} 14 & 2 & -6 \\ -4 & -8 & -9 \\ 14 & 6 & -10 \end{bmatrix} \tag{123}$$

1:

$$\begin{bmatrix} 6 & -17 & 4 \\ 10 & -7 & 5 \\ 6 & 11 & -12 \end{bmatrix} \tag{124}$$

1:

$$\begin{bmatrix} -1 & -6 & 11 \\ -16 & 3 & 3 \\ 16 & -16 & -16 \end{bmatrix} \tag{125}$$

1:

$$\begin{bmatrix} -1 & -2 & -2 \\ 11 & 16 & -4 \\ 8 & -6 & 3 \end{bmatrix} \tag{126}$$

1:

$$\begin{bmatrix} -9 & 1 & 4 \\ 13 & 6 & 5 \\ 3 & -5 & 9 \end{bmatrix} \tag{127}$$

1:

$$\begin{bmatrix} -7 & 6 & -8 \\ -6 & 2 & -9 \\ 9 & -5 & -7 \end{bmatrix} \tag{128}$$

1:

$$\begin{bmatrix} -5 & -19 & 7 \\ -11 & -11 & 14 \\ -17 & -5 & -2 \end{bmatrix} \tag{129}$$

1:

$$\begin{bmatrix} -1 & 5 & 4 \\ 5 & -7 & 0 \\ -14 & 5 & -13 \end{bmatrix} \tag{130}$$

2.2.2. Subtraction

1:

$$\begin{bmatrix} 0 & -13 & 11 \\ 3 & -1 & 8 \\ 3 & 10 & -7 \end{bmatrix} \tag{131}$$

1:

$$\begin{bmatrix} -19 & 14 & -6 \\ 17 & 1 & -10 \\ -4 & 8 & -16 \end{bmatrix} \tag{132}$$

1:

$$\begin{bmatrix} 6 & -7 & 0 \\ -1 & 4 & -11 \\ 2 & -10 & -18 \end{bmatrix} \tag{133}$$

1:

$$\begin{bmatrix} -10 & 0 & 3 \\ 0 & -18 & 8 \\ 7 & 1 & 6 \end{bmatrix} \tag{134}$$

1:

$$\begin{bmatrix} -1 & 6 & -3 \\ 10 & -4 & -7 \\ -3 & -11 & 9 \end{bmatrix} \tag{135}$$

1:

$$\begin{bmatrix} -8 & 1 & 8 \\ 3 & -1 & -3 \\ -7 & -5 & -11 \end{bmatrix} \tag{136}$$

1:

$$\begin{bmatrix} 0 & 7 & 1 \\ -2 & 1 & -1 \\ -2 & -6 & 8 \end{bmatrix} \tag{137}$$

1:

$$\begin{bmatrix} -1 & 13 & -2 \\ 7 & -2 & 2 \\ -6 & 6 & 12 \end{bmatrix} \tag{138}$$

1:

$$\begin{bmatrix} -1 & -8 & -9 \\ 1 & -14 & -10 \\ 8 & 2 & 3 \end{bmatrix} \tag{139}$$

1:

$$\begin{bmatrix} -13 & 3 & -6 \\ -13 & 5 & 1 \\ 3 & 1 & -2 \end{bmatrix} \tag{140}$$

2.2.3. Multiplication

1:

$$\begin{bmatrix} -14 & -75 & 22 \\ -14 & 17 & -32 \\ -78 & -33 & 42 \end{bmatrix} \tag{141}$$

1:

$$\begin{bmatrix} -22 & 48 & 66 \\ 76 & -56 & 12 \\ -23 & 6 & -27 \end{bmatrix} \tag{142}$$

1:

$$\begin{bmatrix} -40 & 0 & 64 \\ -25 & -117 & 31 \\ 28 & -4 & 64 \end{bmatrix} \tag{143}$$

1:

$$\begin{bmatrix} 90 & 18 & -9 \\ -52 & -97 & -86 \\ -36 & 25 & 36 \end{bmatrix} \tag{144}$$

1:

$$\begin{bmatrix} -40 & -15 & -10 \\ -28 & -17 & 92 \\ 68 & 19 & -72 \end{bmatrix} \tag{145}$$

1:

$$\begin{bmatrix} -18 & 24 & -50 \\ -94 & 98 & -22 \\ -31 & 32 & 9 \end{bmatrix} \tag{146}$$

1:

$$\begin{bmatrix} -47 & -2 & 51 \\ -3 & -48 & 14 \\ 98 & -92 & -79 \end{bmatrix} \tag{147}$$

1:

$$\begin{bmatrix} 1 & 15 & 19 \\ 46 & 30 & -50 \\ -88 & -24 & 106 \end{bmatrix} \tag{148}$$

1:

$$\begin{bmatrix} -26 & 11 & -56 \\ -4 & 60 & -48 \\ -16 & -10 & -64 \end{bmatrix} \tag{149}$$

1:

$$\begin{bmatrix} 32 & 33 & -19 \\ -54 & -42 & 29 \\ 16 & 80 & -13 \end{bmatrix} \tag{150}$$

2.3. Matrix Properties

2.3.1. Properties

Solution

Row Operations:

Step 1: $r_3 := r_3 - (-2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \end{array}\right]$

Step 2: $r_1 := r_1 - (-2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} = \text{does not exist}$

e) $\ker(A) = \text{span}\left\{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}\right\}$

Solution

Row Operations:

Step 1: $r_1 \leftrightarrow r_2$ $\left[\begin{array}{ccc|ccc} -1 & 4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_1 := -1r_1$ $\left[\begin{array}{ccc|ccc} 1 & -4 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Step 3: $r_1 := r_1 - (-4)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} = \text{does not exist}$

e) $\ker(A) = \text{span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$

Solution

Row Operations:

Step 1: $r_1 := r_1 - (2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_1 := r_1 - (-5)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 5 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

Step 3: $r_2 := r_2 - r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 5 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 0

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_3 := r_3 - (3)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$

Step 2: $r_1 := r_1 - r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$

Step 3: $r_1 := r_1 - (-1)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$

Step 4: $r_2 := r_2 - r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 0

d) $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_1 := -1/3r_1$ $\begin{bmatrix} 1 & -2/3 & 3 & | & -1/3 & 0 & 0 \\ -2 & 1 & -7 & | & 0 & 1 & 0 \\ -2 & 1 & -6 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_2 := r_2 - (-2)r_1$ $\begin{bmatrix} 1 & -2/3 & 3 & | & -1/3 & 0 & 0 \\ 0 & -1/3 & -1 & | & -2/3 & 1 & 0 \\ -2 & 1 & -6 & | & 0 & 0 & 1 \end{bmatrix}$

Step 3: $r_3 := r_3 - (-2)r_1$ $\begin{bmatrix} 1 & -2/3 & 3 & | & -1/3 & 0 & 0 \\ 0 & -1/3 & -1 & | & -2/3 & 1 & 0 \\ 0 & -1/3 & 0 & | & -2/3 & 0 & 1 \end{bmatrix}$

Step 4: $r_2 := -3r_2$ $\begin{bmatrix} 1 & -2/3 & 3 & | & -1/3 & 0 & 0 \\ 0 & 1 & 3 & | & 2 & -3 & 0 \\ 0 & -1/3 & 0 & | & -2/3 & 0 & 1 \end{bmatrix}$

Step 5: $r_1 := r_1 - (-2/3)r_2$ $\begin{bmatrix} 1 & 0 & 5 & | & 1 & -2 & 0 \\ 0 & 1 & 3 & | & 2 & -3 & 0 \\ 0 & -1/3 & 0 & | & -2/3 & 0 & 1 \end{bmatrix}$

Step 6: $r_3 := r_3 - (-1/3)r_2$ $\begin{bmatrix} 1 & 0 & 5 & | & 1 & -2 & 0 \\ 0 & 1 & 3 & | & 2 & -3 & 0 \\ 0 & 0 & 1 & | & 0 & -1 & 1 \end{bmatrix}$

Step 7: $r_1 := r_1 - (5)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 1 & 3 & -5 \\ 0 & 1 & 3 & | & 2 & -3 & 0 \\ 0 & 0 & 1 & | & 0 & -1 & 1 \end{bmatrix}$

Step 8: $r_2 := r_2 - (3)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 1 & 3 & -5 \\ 0 & 1 & 0 & | & 2 & 0 & -3 \\ 0 & 0 & 1 & | & 0 & -1 & 1 \end{bmatrix}$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 48

d) $A^{-1} = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -4 \\ 0 & -1 & 1 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_2 := r_2 - (-1)r_1$ $\begin{bmatrix} 1 & -3 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ -2 & 8 & 0 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_3 := r_3 - (-2)r_1$ $\begin{bmatrix} 1 & -3 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ 0 & 2 & 2 & | & 2 & 0 & 1 \end{bmatrix}$

Step 3: $r_1 := r_1 - (-3)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 4 & 3 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 & 1 \end{array}\right]$

Step 4: $r_3 := r_3 - (2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 4 & 3 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{array}\right]$

Results:

a) rank(A) = 2

b) nullity(A) = 1

c) det(A) = 320

d) $A^{-1} = \text{does not exist}$

e) ker(A) = span $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

Step 1: $r_2 := r_2 - (-1)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_3 := r_3 - (2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$

Step 3: $r_1 := r_1 - r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -1 \\ 0 & 1 & -4 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$

Step 4: $r_2 := r_2 - (-4)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & -7 & 1 & 4 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 0

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_3 := r_3 - (-2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 3 & 2 & 0 & 1 \end{array}\right]$

Step 2: $r_3 := r_3 - (-1)r_2$ $\begin{bmatrix} 1 & 0 & 5 & | & 1 & 0 & 0 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 1 & 1 \end{bmatrix}$

Step 3: $r_1 := r_1 - (5)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & -9 & -5 & -5 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 1 & 1 \end{bmatrix}$

Step 4: $r_2 := r_2 - (-2)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & -9 & -5 & -5 \\ 0 & 1 & 0 & | & 4 & 3 & 2 \\ 0 & 0 & 1 & | & 2 & 1 & 1 \end{bmatrix}$

Results:

a) $\mathrm{rank}(A) = 3$

b) $\mathrm{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_1 := r_1 - (-5)r_2$ $\begin{bmatrix} 1 & 0 & -1 & | & 1 & 5 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & -3 & 3 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_3 := r_3 - (-3)r_2$ $\begin{bmatrix} 1 & 0 & -1 & | & 1 & 5 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 3 & 1 \end{bmatrix}$

Results:

a) $\mathrm{rank}(A) = 2$

b) $\mathrm{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} = \text{does not exist}$

e) $\ker(A) = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

Step 1: $r_2 := 1/5 r_2$ $\begin{bmatrix} 1 & 7 & 11 & | & 1 & 0 & 0 \\ 0 & 1 & 8/5 & | & 0 & 1/5 & 0 \\ 0 & -2 & -3 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_1 := r_1 - (7)r_2$ $\begin{bmatrix} 1 & 0 & -1/5 & | & 1 & -7/5 & 0 \\ 0 & 1 & 8/5 & | & 0 & 1/5 & 0 \\ 0 & -2 & -3 & | & 0 & 0 & 1 \end{bmatrix}$

Step 3: $r_3 := r_3 - (-2)r_2$ $\begin{bmatrix} 1 & 0 & -1/5 & | & 1 & -7/5 & 0 \\ 0 & 1 & 8/5 & | & 0 & 1/5 & 0 \\ 0 & 0 & 1/5 & | & 0 & 2/5 & 1 \end{bmatrix}$

Step 4: $r_3 := 5r_3$ $\begin{bmatrix} 1 & 0 & -1/5 & | & 1 & -7/5 & 0 \\ 0 & 1 & 8/5 & | & 0 & 1/5 & 0 \\ 0 & 0 & 1 & | & 0 & 2 & 5 \end{bmatrix}$

Step 5: $r_1 := r_1 - (-1/5)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 1 & -1 & 1 \\ 0 & 1 & 8/5 & | & 0 & 1/5 & 0 \\ 0 & 0 & 1 & | & 0 & 2 & 5 \end{bmatrix}$

Step 6: $r_2 := r_2 - (8/5)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 1 & -1 & 1 \\ 0 & 1 & 0 & | & 0 & -3 & -8 \\ 0 & 0 & 1 & | & 0 & 2 & 5 \end{bmatrix}$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 0

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

2.3.2. RREF

Solution

Elementary Row Operations:

(1) $r_1 := r_1 - r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(2) $r_2 := r_2 - (2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(3) $r_3 := r_3 - (2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_3 := r_3 + (-2)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 - r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_3 := r_3 - (2)r_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_2 := r_2 + (-2)r_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_1 := r_1 + (-1)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_2 := r_2 + (-1)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

(2) $r_3 := r_3 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

(3) $r_3 := r_3 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_2 := r_2 - r_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(2) $r_1 := r_1 + (-1)r_3$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(3) $r_3 := r_3 - r_2$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 - (2)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_2 := r_2 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_3 := r_3 - (2)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

(2) $r_3 := r_3 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 - (2)r_2$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(2) $r_3 := r_3 - (2)r_2$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_3 := r_3 - (2)r_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) $r_1 := r_1 + (-2)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) $r_1 := r_1 + (-2)r_3$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 - r_3$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

(2) $r_3 := r_3 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.4. Calculus

2.4.1. Limit

The limit is:

$$-1 \tag{151}$$

The limit is:

$$1 \tag{152}$$

The limit is:

$$1 \tag{153}$$

The limit is:

$$1 \tag{154}$$

The limit is:

$$1 \tag{155}$$

The limit is:

$$e \tag{156}$$

The limit is:

$$e \tag{157}$$

The limit is:

$$1 \tag{158}$$

The limit is:

$$e \tag{159}$$

The limit is:

$$1 \tag{160}$$

2.4.2. Derivative

The derivative is:

$$2xe^{x^2} + 2e^{2x} \tag{161}$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \tag{162}$$

The derivative is:

$$\frac{2x}{x^2+1} + \frac{1}{x+1} \tag{163}$$

The derivative is:

$$-\frac{2x^4}{\left(x^2+1\right)^2} + \frac{3x^2}{x^2+1} \tag{164}$$

The derivative is:

$$xe^x + e^x \tag{165}$$

The derivative is:

$$\frac{2x}{x^2-1} \tag{166}$$

The derivative is:

$$\frac{2x}{x^2+1} + \frac{1}{x+1} \tag{167}$$

The derivative is:

$$-\frac{2x^3}{\left(x^2+1\right)^2} + \frac{2x}{x^2+1} \tag{168}$$

The derivative is:

$$\frac{2x}{x^2+1} + \frac{1}{x+1} \tag{169}$$

The derivative is:

$$-\frac{2x^4}{\left(x^2+1\right)^2} + \frac{3x^2}{x^2+1} \tag{170}$$

2.4.3. Integral

The indefinite integral is:

$$\text{Si}\left(x\right) \tag{171}$$

Definite integral from 1 to 3:

$$-\text{Si}\left(1\right) + \text{Si}\left(3\right) \tag{172}$$

The improper integral converges to:

$$\infty \tag{173}$$

The indefinite integral is:

$$\frac{e^{x} \sin(x)}{2} - \frac{e^{x} \cos(x)}{2} \tag{174}$$

Definite integral from 1 to 4:

$$\frac{e^{4} \sin(4)}{2} - \frac{e \sin(1)}{2} + \frac{e \cos(1)}{2} - \frac{e^{4} \cos(4)}{2} \tag{175}$$

The indefinite integral is:

$$\frac{x^{2} \sqrt{x^{2} + 1}}{3} + \frac{\sqrt{x^{2} + 1}}{3} \tag{176}$$

Definite integral from 2 to 5:

$$-\frac{5 \sqrt{5}}{3} + \frac{26 \sqrt{26}}{3} \tag{177}$$

The indefinite integral is:

$$\frac{x^{4} \log(x)}{4} - \frac{x^{4}}{16} \tag{178}$$

Definite integral from 2 to 3:

$$-\frac{65}{16} - 4 \log(2) + \frac{81 \log(3)}{4} \tag{179}$$

The indefinite integral is:

$$\frac{\log(x - 2)}{3} - \frac{\log(x + 1)}{3} \tag{180}$$

Definite integral from 3 to 4:

$$-\frac{\log(5)}{3} + \frac{\log(2)}{3} + \frac{\log(4)}{3} \tag{181}$$

The indefinite integral is:

$$-\frac{5 x^{3}}{3} - \frac{5 x^{2}}{2} + 2 x \tag{182}$$

Definite integral from 1 to 1:

$$0 \tag{183}$$

The indefinite integral is:

$$\frac{e^x \sin(x)}{2} - \frac{e^x \cos(x)}{2} \tag{184}$$

Definite integral from 2 to 4:

$$\frac{e^4 \sin(4)}{2} - \frac{e^2 \sin(2)}{2} + \frac{e^2 \cos(2)}{2} - \frac{e^4 \cos(4)}{2} \tag{185}$$

The indefinite integral is:

$$\frac{x^4 \log(x)}{4} - \frac{x^4}{16} \tag{186}$$

Definite integral from 4 to 5:

$$-64 \log(4) - \frac{369}{16} + \frac{625 \log(5)}{4} \tag{187}$$

The indefinite integral is:

$$\frac{x^2 \sqrt{x^2 + 1}}{3} + \frac{\sqrt{x^2 + 1}}{3} \tag{188}$$

Definite integral from 1 to 3:

$$-\frac{2\sqrt{2}}{3} + \frac{10\sqrt{10}}{3} \tag{189}$$

2.4.4. Partial Derivative

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \tag{190}$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \tag{191}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \tag{192}$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \tag{193}$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \tag{194}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \tag{195}$$

$$\frac{\partial^2 f}{\partial x^2} = 6y^3(2x^2 + y) \tag{196}$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{197}$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{198}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \tag{199}$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \tag{200}$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{201}$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{202}$$