

Exercise 10:

Foundations of Mathematical, WS24

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This is **exercise** 10 for Foundations of Mathematical, WS24. Generated on 2025-01-27 with 10 problems per section.

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1. Problems

1.1. Vector Arithmetic

1.1.1. Addition

Find the sum of the following vectors \mathbf{u} and \mathbf{v}

1. $\mathbf{u} = \begin{bmatrix} -6 \\ 3 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ -6 \\ -5 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} -6 \\ -8 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 4 \\ 8 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} 1 \\ -8 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ -4 \\ -10 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 2 \\ -9 \\ 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 10 \\ 8 \\ -6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} -5 \\ -6 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -9 \\ -5 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ -6 \\ -5 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -9 \\ -4 \\ -8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 5 \\ -9 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} -4 \\ 1 \\ -6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 7 \\ -4 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

1.1.2. Subtraction

Find the difference of the following vectors \mathbf{u} and \mathbf{v}

1. $\mathbf{u} = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ -9 \\ -9 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} -7 \\ 9 \\ -10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} -7 \\ -7 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 9 \\ -7 \\ 0 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} -7 \\ 4 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 10 \\ -10 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} -8 \\ -10 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ -5 \\ 0 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

6. $\mathbf{u} = \begin{bmatrix} -5 \\ -1 \\ -9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 2 \\ -1 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -6 \\ 9 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -8 \\ 10 \\ -4 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -2 \\ 9 \\ -6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 7 \\ 6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ -6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} 8 \\ 9 \\ -8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -7 \\ 8 \\ -3 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

1.1.3. Scalar Multiplication

Find the scalar product of the following vector \mathbf{u} and scalar k

1. $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$ $5\mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} 5 \\ 9 \\ -8 \end{bmatrix}$ $2\mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} -7 \\ 2 \\ -6 \end{bmatrix}$ $4\mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ $7\mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 10 \\ 1 \\ 2 \end{bmatrix}$ $6\mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} 7 \\ 0 \\ 6 \end{bmatrix}$ $1\mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -1 \\ -4 \\ 10 \end{bmatrix}$ $0\mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -9 \\ -10 \\ 7 \end{bmatrix}$ $-3\mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} -3 \\ -1 \\ -7 \end{bmatrix}$ $10\mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix}$ $3\mathbf{v}$.

1.2. Matrix Arithmetic

1.2.1. Addition

Find the sum of the following matrices A and B

1.
$$A = \begin{bmatrix} -5 & 1 & -2 \\ 3 & 0 & -3 \\ 5 & 8 & 1 \end{bmatrix} \quad (1)$$

and

$$B = \begin{bmatrix} -4 & -6 & 2 \\ 0 & -4 & 9 \\ 2 & 2 & -7 \end{bmatrix} \quad (2)$$

2.
$$A = \begin{bmatrix} -8 & -5 & -1 \\ -5 & 2 & 5 \\ -3 & -8 & -9 \end{bmatrix} \quad (3)$$

and

$$B = \begin{bmatrix} -8 & 4 & 1 \\ 7 & 7 & -1 \\ 4 & -7 & -8 \end{bmatrix} \quad (4)$$

3.
$$A = \begin{bmatrix} -10 & -10 & -1 \\ 3 & -2 & -6 \\ -8 & 9 & -4 \end{bmatrix} \quad (5)$$

and

$$B = \begin{bmatrix} -6 & 2 & 9 \\ 6 & -9 & -7 \\ -10 & -4 & 3 \end{bmatrix} \quad (6)$$

4.
$$A = \begin{bmatrix} 7 & -5 & -2 \\ -7 & 9 & -1 \\ 4 & 1 & 1 \end{bmatrix} \quad (7)$$

and

$$B = \begin{bmatrix} 0 & 9 & 0 \\ -1 & 1 & 2 \\ -9 & -4 & -8 \end{bmatrix} \quad (8)$$

5.
$$A = \begin{bmatrix} 0 & -4 & 3 \\ -10 & -10 & 9 \\ 8 & -5 & 0 \end{bmatrix} \quad (9)$$

and

$$B = \begin{bmatrix} 1 & -5 & -7 \\ 5 & -6 & 1 \\ -8 & -6 & 8 \end{bmatrix} \quad (10)$$

6.
$$A = \begin{bmatrix} 2 & 1 & 2 \\ -4 & -1 & 6 \\ -3 & -7 & -2 \end{bmatrix} \quad (11)$$

and

$$B = \begin{bmatrix} -6 & 3 & 8 \\ -5 & -8 & 0 \\ 1 & -2 & -2 \end{bmatrix} \quad (12)$$

7.

$$A = \begin{bmatrix} -4 & 2 & 2 \\ 3 & 9 & 9 \\ -9 & 1 & -2 \end{bmatrix} \quad (13)$$

and

$$B = \begin{bmatrix} 8 & 2 & 7 \\ 7 & 5 & 3 \\ -6 & -8 & -9 \end{bmatrix} \quad (14)$$

8.

$$A = \begin{bmatrix} -3 & 0 & -3 \\ -1 & -9 & -8 \\ -6 & 7 & 2 \end{bmatrix} \quad (15)$$

and

$$B = \begin{bmatrix} 6 & -4 & 3 \\ -3 & -9 & 6 \\ 7 & 7 & -5 \end{bmatrix} \quad (16)$$

9.

$$A = \begin{bmatrix} -5 & -1 & -4 \\ 2 & -8 & 5 \\ 4 & -2 & 3 \end{bmatrix} \quad (17)$$

and

$$B = \begin{bmatrix} -9 & -6 & 6 \\ 3 & -1 & -2 \\ -10 & -4 & -8 \end{bmatrix} \quad (18)$$

10.

$$A = \begin{bmatrix} -10 & -3 & -2 \\ 7 & -8 & -10 \\ -5 & 0 & 5 \end{bmatrix} \quad (19)$$

and

$$B = \begin{bmatrix} -5 & -3 & -4 \\ -5 & 0 & 1 \\ 6 & -2 & -6 \end{bmatrix} \quad (20)$$

1.2.2. Subtraction

Find the difference of the following matrices A and B

1.
$$A = \begin{bmatrix} 6 & -5 & 1 \\ 3 & 4 & 3 \\ 9 & -1 & -2 \end{bmatrix} \quad (21)$$

and

$$B = \begin{bmatrix} -10 & 5 & -8 \\ 7 & -1 & 7 \\ 0 & 5 & -4 \end{bmatrix} \quad (22)$$

2.
$$A = \begin{bmatrix} 0 & -8 & 6 \\ -10 & 5 & 4 \\ 8 & 2 & -3 \end{bmatrix} \quad (23)$$

and

$$B = \begin{bmatrix} 4 & -10 & 6 \\ -9 & 3 & -3 \\ 2 & 4 & -5 \end{bmatrix} \quad (24)$$

3.
$$A = \begin{bmatrix} -4 & -8 & -1 \\ 8 & 6 & 6 \\ 6 & -4 & 2 \end{bmatrix} \quad (25)$$

and

$$B = \begin{bmatrix} 3 & 0 & -8 \\ 4 & -4 & -1 \\ -6 & -3 & -6 \end{bmatrix} \quad (26)$$

4.
$$A = \begin{bmatrix} -7 & -3 & -1 \\ 9 & 3 & 0 \\ 9 & 5 & -4 \end{bmatrix} \quad (27)$$

and

$$B = \begin{bmatrix} -10 & -3 & -5 \\ -5 & -7 & -7 \\ -2 & -1 & -10 \end{bmatrix} \quad (28)$$

5.
$$A = \begin{bmatrix} 9 & 7 & -3 \\ -8 & -9 & 3 \\ -1 & -4 & 4 \end{bmatrix} \quad (29)$$

and

$$B = \begin{bmatrix} 7 & 9 & 1 \\ 6 & 1 & 8 \\ -6 & 1 & 4 \end{bmatrix} \quad (30)$$

6.
$$A = \begin{bmatrix} -6 & -8 & -3 \\ -3 & 1 & -6 \\ -5 & -10 & -9 \end{bmatrix} \quad (31)$$

and

$$B = \begin{bmatrix} -4 & 7 & 5 \\ 2 & 4 & -2 \\ -5 & 8 & 4 \end{bmatrix} \quad (32)$$

7.

$$A = \begin{bmatrix} 7 & -5 & 2 \\ 4 & -4 & -6 \\ -5 & 7 & -8 \end{bmatrix} \quad (33)$$

and

$$B = \begin{bmatrix} -5 & 1 & 8 \\ -4 & 9 & -5 \\ 0 & 1 & 1 \end{bmatrix} \quad (34)$$

8.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ -7 & 8 & -5 \\ -10 & 0 & 8 \end{bmatrix} \quad (35)$$

and

$$B = \begin{bmatrix} -9 & 1 & 1 \\ -3 & 3 & 9 \\ 9 & -6 & 5 \end{bmatrix} \quad (36)$$

9.

$$A = \begin{bmatrix} 5 & -6 & 5 \\ -3 & -6 & -7 \\ -10 & -1 & 6 \end{bmatrix} \quad (37)$$

and

$$B = \begin{bmatrix} -6 & -9 & -1 \\ -10 & 9 & -9 \\ 4 & -6 & -10 \end{bmatrix} \quad (38)$$

10.

$$A = \begin{bmatrix} 9 & 1 & 6 \\ -8 & -2 & -8 \\ 2 & -3 & 4 \end{bmatrix} \quad (39)$$

and

$$B = \begin{bmatrix} 0 & 1 & 8 \\ 6 & -8 & -4 \\ 1 & 5 & 4 \end{bmatrix} \quad (40)$$

1.2.3. Multiplication

Find the product of the following matrices A and B

1.
$$A = \begin{bmatrix} 7 & 3 & 0 \\ 7 & 6 & -5 \\ 9 & 0 & -1 \end{bmatrix} \quad (41)$$

and

$$B = \begin{bmatrix} 1 & -8 & -8 \\ -2 & -2 & -2 \\ 6 & -7 & 7 \end{bmatrix} \quad (42)$$

2.
$$A = \begin{bmatrix} -6 & 8 & -9 \\ -6 & 4 & -8 \\ -8 & 5 & -9 \end{bmatrix} \quad (43)$$

and

$$B = \begin{bmatrix} 7 & -3 & 2 \\ -4 & -8 & 1 \\ -9 & -2 & 7 \end{bmatrix} \quad (44)$$

3.
$$A = \begin{bmatrix} 9 & -1 & 5 \\ 0 & 2 & 8 \\ -7 & 5 & 6 \end{bmatrix} \quad (45)$$

and

$$B = \begin{bmatrix} 2 & 5 & -9 \\ -9 & 5 & -1 \\ 7 & 7 & 3 \end{bmatrix} \quad (46)$$

4.
$$A = \begin{bmatrix} -9 & -7 & -1 \\ 1 & -10 & -1 \\ 8 & 2 & 8 \end{bmatrix} \quad (47)$$

and

$$B = \begin{bmatrix} 4 & -3 & 0 \\ 9 & -6 & -5 \\ 7 & 9 & -7 \end{bmatrix} \quad (48)$$

5.
$$A = \begin{bmatrix} 0 & 3 & 9 \\ 0 & -2 & 4 \\ -3 & -5 & -8 \end{bmatrix} \quad (49)$$

and

$$B = \begin{bmatrix} 7 & 6 & 8 \\ 1 & 1 & 9 \\ -9 & 8 & 8 \end{bmatrix} \quad (50)$$

6.
$$A = \begin{bmatrix} 0 & -1 & -9 \\ 0 & -4 & 9 \\ -10 & -6 & -2 \end{bmatrix} \quad (51)$$

and

$$B = \begin{bmatrix} 2 & -7 & 5 \\ 6 & 7 & 1 \\ 9 & -8 & 4 \end{bmatrix} \quad (52)$$

7.

$$A = \begin{bmatrix} 7 & 9 & -5 \\ -5 & -5 & -6 \\ -9 & -5 & -3 \end{bmatrix} \quad (53)$$

and

$$B = \begin{bmatrix} -5 & -9 & 0 \\ -2 & -1 & 7 \\ -4 & 6 & -4 \end{bmatrix} \quad (54)$$

8.

$$A = \begin{bmatrix} 6 & 6 & -3 \\ -2 & -1 & -3 \\ -3 & 6 & -1 \end{bmatrix} \quad (55)$$

and

$$B = \begin{bmatrix} 7 & -8 & 9 \\ -4 & 2 & -5 \\ -6 & -3 & 3 \end{bmatrix} \quad (56)$$

9.

$$A = \begin{bmatrix} 3 & 9 & 4 \\ -9 & -8 & 4 \\ 0 & 7 & 8 \end{bmatrix} \quad (57)$$

and

$$B = \begin{bmatrix} -10 & 8 & -10 \\ 8 & -4 & -1 \\ -3 & -4 & -5 \end{bmatrix} \quad (58)$$

10.

$$A = \begin{bmatrix} 2 & -2 & -2 \\ -6 & -3 & -6 \\ -2 & -2 & -9 \end{bmatrix} \quad (59)$$

and

$$B = \begin{bmatrix} 3 & 2 & 7 \\ -4 & 5 & -5 \\ 3 & 9 & 0 \end{bmatrix} \quad (60)$$

1.3. Matrix Properties

1.3.1. Properties

For each matrix A , find:

a) $\text{rank}(A)$

- b) $\text{nullity}(A)$
- c) $\det(A)$
- d) A^{-1} (if exists)
- e) basis of $\ker(A)$

1.
$$A = \begin{bmatrix} 3 & 1 & -4 \\ -4 & -1 & 5 \\ 3 & 1 & -4 \end{bmatrix} \quad (61)$$

2.
$$A = \begin{bmatrix} -1 & 3 & 2 \\ -1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \quad (62)$$

3.
$$A = \begin{bmatrix} 3 & 3 & 3 \\ -1 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \quad (63)$$

4.
$$A = \begin{bmatrix} 3 & -7 & 7 \\ 0 & 1 & -1 \\ 2 & -6 & 6 \end{bmatrix} \quad (64)$$

5.
$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad (65)$$

6.
$$A = \begin{bmatrix} 3 & -2 & -7 \\ 2 & -1 & -5 \\ -2 & 2 & 4 \end{bmatrix} \quad (66)$$

7.
$$A = \begin{bmatrix} 1 & -1 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad (67)$$

8.
$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -1 & 5 \\ 0 & 2 & 2 \end{bmatrix} \quad (68)$$

9.
$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (69)$$

10.
$$A = \begin{bmatrix} 2 & 1 & -10 \\ 1 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \quad (70)$$

1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix A

$$1. \quad A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (71)$$

$$2. \quad A = \begin{bmatrix} 1 & 0 & 2 \\ -6 & -3 & -14 \\ 4 & 2 & 9 \end{bmatrix} \quad (72)$$

$$3. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (73)$$

$$4. \quad A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix} \quad (74)$$

$$5. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (75)$$

$$6. \quad A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (76)$$

$$7. \quad A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad (77)$$

$$8. \quad A = \begin{bmatrix} 5 & -8 & 0 \\ 2 & -3 & 0 \\ 2 & -3 & 0 \end{bmatrix} \quad (78)$$

$$9. \quad A = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (79)$$

$$10. \quad A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \quad (80)$$

1.4. Calculus

1.4.1. Limit

Calculate the following limits

1. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (81)$$

2. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (82)$$

3. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x + 1)}{x} \quad (83)$$

4. Calculate the limit of the following expression:

$$\lim_{x \rightarrow -3} -2 \quad (84)$$

5. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x + 1)}{x} \quad (85)$$

6. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x + 1)}{x} \quad (86)$$

7. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x + 1)}{x} \quad (87)$$

8. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 3} 3x^3 + 4x^2 - 4x - 1 \quad (88)$$

9. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (89)$$

10. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} 3x^2 + 4x + 4 \quad (90)$$

1.4.2. Derivative

Calculate the derivatives of the following expressions

1. Calculate the derivative of the following expression:

$$\log(x + 1) + \log(x^2 + 1) \quad (91)$$

2. Calculate the derivative of the following expression:

$$\log(x) \quad (92)$$

3. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (93)$$

4. Calculate the derivative of the following expression:

$$\log(x+1) + \log(x^2+1) \quad (94)$$

5. Calculate the derivative of the following expression:

$$x^2 e^x \quad (95)$$

6. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (96)$$

7. Calculate the derivative of the following expression:

$$e^x \quad (97)$$

8. Calculate the derivative of the following expression:

$$x^3 e^x \quad (98)$$

9. Calculate the derivative of the following expression:

$$\frac{x^2}{x^2+1} \quad (99)$$

10. Calculate the derivative of the following expression:

$$\log(x+1) + \log(x^2+1) \quad (100)$$

1.4.3. Integral

Calculate the indefinite and definite integrals of the following expressions

1. the indefinite integral and evaluate from 2 to 3:

$$\int \sqrt{4-x^2} dx \quad (101)$$

2. the indefinite integral and evaluate from 2 to 5:

$$\int e^{-x^2} dx \quad (102)$$

3. the indefinite integral and evaluate from 2 to 2:

$$\int e^{-x^2} dx \quad (103)$$

4. the indefinite integral and evaluate from 1 to 3:

$$\int e^x \sin(x) dx \quad (104)$$

5. the indefinite integral and evaluate from 2 to 3:

$$\int \frac{x}{x^2-5x+6} dx \quad (105)$$

6. the indefinite integral and evaluate from 4 to 4:

$$\int \frac{1}{x \log(x)} dx \quad (106)$$

7. the indefinite integral and evaluate from 1 to 3:

$$\int x^4 + 5x^3 + 5x^2 + x + 3 dx \quad (107)$$

8. the indefinite integral and evaluate from 2 to 4:

$$\int x^3 \log(x) dx \quad (108)$$

9. the indefinite integral and evaluate from 3 to 4:

$$\int \frac{1}{x^2 + 1} dx \quad (109)$$

10. the indefinite integral and evaluate from 4 to 5:

$$\int \sqrt{4 - x^2} dx \quad (110)$$

1.4.4. Partial Derivative

Calculate the partial derivatives of the following expressions

1. the partial derivatives of the function:

$$f(x, y) = x^3 y^2 - 3x^2 y + 2xy^3 \quad (111)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

2. the second order partial derivative of:

$$f(x, y) = x^4 y^3 + 3x^2 y^4 \quad (112)$$

$$\frac{\partial^2 f}{\partial x^2}$$

3. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2 + y^2} \quad (113)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

4. the mixed partial derivative of:

$$f(x, y) = x^3 y^2 + xy^4 \quad (114)$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

5. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2 + y^2} \quad (115)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

6. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \quad (116)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

7. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \quad (117)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

8. Given the implicit function:

$$x^2y + xy^2 - xy = 0 \quad (118)$$

$$\frac{\partial y}{\partial x}$$

9. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \quad (119)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

10. Given the implicit function:

$$x^2y + xy^2 - xy = 0 \quad (120)$$

$$\frac{\partial y}{\partial x}$$

2. Solutions

2.1. Vector Arithmetic

2.1.1. Addition

$$\begin{bmatrix} -9 \\ -3 \\ -12 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \\ 15 \end{bmatrix} = \begin{bmatrix} 7 \\ -12 \\ -15 \end{bmatrix} + \begin{bmatrix} 8 \\ -15 \\ 12 \end{bmatrix} = \begin{bmatrix} 11 \\ 16 \\ -11 \end{bmatrix}$$
$$\begin{bmatrix} -3 \\ 0 \\ 11 \end{bmatrix} + \begin{bmatrix} -1 \\ -11 \\ -7 \end{bmatrix} = \begin{bmatrix} -9 \\ -6 \\ -5 \end{bmatrix} + \begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ -10 \end{bmatrix}$$

2.1.2. Subtraction

$$\begin{bmatrix} 2 \\ 11 \\ 14 \end{bmatrix} - \begin{bmatrix} -10 \\ 13 \\ -10 \end{bmatrix} = \begin{bmatrix} -16 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -7 \\ -6 \\ 18 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \\ 8 \end{bmatrix}$$
$$\begin{bmatrix} -13 \\ -3 \\ -8 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -12 \end{bmatrix} + \begin{bmatrix} -8 \\ -4 \\ 11 \end{bmatrix} = \begin{bmatrix} 15 \\ 1 \\ -5 \end{bmatrix}$$

2.1.3. Scalar Multiplication

$$1: \begin{bmatrix} 10 \\ 10 \\ -15 \end{bmatrix} \quad 2: \begin{bmatrix} 10 \\ 18 \\ -16 \end{bmatrix} \quad 3: \begin{bmatrix} -28 \\ 8 \\ -24 \end{bmatrix} \quad 4: \begin{bmatrix} -21 \\ -7 \\ 7 \end{bmatrix} \quad 5: \begin{bmatrix} 60 \\ 6 \\ 12 \end{bmatrix}$$
$$6: \begin{bmatrix} 7 \\ 0 \\ 6 \end{bmatrix} \quad 7: \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad 8: \begin{bmatrix} 27 \\ 30 \\ -21 \end{bmatrix} \quad 9: \begin{bmatrix} -30 \\ -10 \\ -70 \end{bmatrix} \quad 10: \begin{bmatrix} 6 \\ -18 \\ -12 \end{bmatrix}$$

2.2. Matrix Arithmetic

2.2.1. Addition

1:

$$\begin{bmatrix} -9 & -5 & 0 \\ 3 & -4 & 6 \\ 7 & 10 & -6 \end{bmatrix} \quad (121)$$

1:

$$\begin{bmatrix} -16 & -1 & 0 \\ 2 & 9 & 4 \\ 1 & -15 & -17 \end{bmatrix} \quad (122)$$

1:

$$\begin{bmatrix} -16 & -8 & 8 \\ 9 & -11 & -13 \\ -18 & 5 & -1 \end{bmatrix} \quad (123)$$

1:

$$\begin{bmatrix} 7 & 4 & -2 \\ -8 & 10 & 1 \\ -5 & -3 & -7 \end{bmatrix} \quad (124)$$

1:

$$\begin{bmatrix} 1 & -9 & -4 \\ -5 & -16 & 10 \\ 0 & -11 & 8 \end{bmatrix} \quad (125)$$

1:

$$\begin{bmatrix} -4 & 4 & 10 \\ -9 & -9 & 6 \\ -2 & -9 & -4 \end{bmatrix} \quad (126)$$

1:

$$\begin{bmatrix} 4 & 4 & 9 \\ 10 & 14 & 12 \\ -15 & -7 & -11 \end{bmatrix} \quad (127)$$

1:

$$\begin{bmatrix} 3 & -4 & 0 \\ -4 & -18 & -2 \\ 1 & 14 & -3 \end{bmatrix} \quad (128)$$

1:

$$\begin{bmatrix} -14 & -7 & 2 \\ 5 & -9 & 3 \\ -6 & -6 & -5 \end{bmatrix} \quad (129)$$

1:

$$\begin{bmatrix} -15 & -6 & -6 \\ 2 & -8 & -9 \\ 1 & -2 & -1 \end{bmatrix} \quad (130)$$

2.2.2. Subtraction

1:

$$\begin{bmatrix} 16 & -10 & 9 \\ -4 & 5 & -4 \\ 9 & -6 & 2 \end{bmatrix} \quad (131)$$

1:

$$\begin{bmatrix} -4 & 2 & 0 \\ -1 & 2 & 7 \\ 6 & -2 & 2 \end{bmatrix} \quad (132)$$

1:

$$\begin{bmatrix} -7 & -8 & 7 \\ 4 & 10 & 7 \\ 12 & -1 & 8 \end{bmatrix} \quad (133)$$

1:

$$\begin{bmatrix} 3 & 0 & 4 \\ 14 & 10 & 7 \\ 11 & 6 & 6 \end{bmatrix} \quad (134)$$

1:

$$\begin{bmatrix} 2 & -2 & -4 \\ -14 & -10 & -5 \\ 5 & -5 & 0 \end{bmatrix} \quad (135)$$

1:

$$\begin{bmatrix} -2 & -15 & -8 \\ -5 & -3 & -4 \\ 0 & -18 & -13 \end{bmatrix} \quad (136)$$

1:

$$\begin{bmatrix} 12 & -6 & -6 \\ 8 & -13 & -1 \\ -5 & 6 & -9 \end{bmatrix} \quad (137)$$

1:

$$\begin{bmatrix} 10 & 4 & -1 \\ -4 & 5 & -14 \\ -19 & 6 & 3 \end{bmatrix} \quad (138)$$

1:

$$\begin{bmatrix} 11 & 3 & 6 \\ 7 & -15 & 2 \\ -14 & 5 & 16 \end{bmatrix} \quad (139)$$

1:

$$\begin{bmatrix} 9 & 0 & -2 \\ -14 & 6 & -4 \\ 1 & -8 & 0 \end{bmatrix} \quad (140)$$

2.2.3. Multiplication

1:

$$\begin{bmatrix} 1 & -62 & -62 \\ -35 & -33 & -103 \\ 3 & -65 & -79 \end{bmatrix} \quad (141)$$

1:

$$\begin{bmatrix} 7 & -28 & -67 \\ 14 & 2 & -64 \\ 5 & 2 & -74 \end{bmatrix} \quad (142)$$

1:

$$\begin{bmatrix} 62 & 75 & -65 \\ 38 & 66 & 22 \\ -17 & 32 & 76 \end{bmatrix} \quad (143)$$

1:

$$\begin{bmatrix} -106 & 60 & 42 \\ -93 & 48 & 57 \\ 106 & 36 & -66 \end{bmatrix} \quad (144)$$

1:

$$\begin{bmatrix} -78 & 75 & 99 \\ -38 & 30 & 14 \\ 46 & -87 & -133 \end{bmatrix} \quad (145)$$

1:

$$\begin{bmatrix} -87 & 65 & -37 \\ 57 & -100 & 32 \\ -74 & 44 & -64 \end{bmatrix} \quad (146)$$

1:

$$\begin{bmatrix} -33 & -102 & 83 \\ 59 & 14 & -11 \\ 67 & 68 & -23 \end{bmatrix} \quad (147)$$

1:

$$\begin{bmatrix} 36 & -27 & 15 \\ 8 & 23 & -22 \\ -39 & 39 & -60 \end{bmatrix} \quad (148)$$

1:

$$\begin{bmatrix} 30 & -28 & -59 \\ 14 & -56 & 78 \\ 32 & -60 & -47 \end{bmatrix} \quad (149)$$

1:

$$\begin{bmatrix} 8 & -24 & 24 \\ -24 & -81 & -27 \\ -25 & -95 & -4 \end{bmatrix} \quad (150)$$

2.3. Matrix Properties

2.3.1. Properties

Solution

Row Operations:

$$\text{Step 1: } r_1 := 1/3r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 1/3 & -4/3 & 1/3 & 0 & 0 \\ -4 & -1 & 5 & 0 & 1 & 0 \\ 3 & 1 & -4 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_2 := r_2 - (-4)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 1/3 & -4/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & -1/3 & 4/3 & 1 & 0 \\ 3 & 1 & -4 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_3 := r_3 - (3)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 1/3 & -4/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & -1/3 & 4/3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right]$$

$$\text{Step 4: } r_2 := 3r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 1/3 & -4/3 & 1/3 & 0 & 0 \\ 0 & 1 & -1 & 4 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right]$$

$$\text{Step 5: } r_1 := r_1 - (1/3)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 4 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := -1r_1 \quad \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_2 := r_2 - (-1)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_3 := r_3 - (-1)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array} \right]$$

$$\text{Step 4: } r_2 := -1r_2 \begin{bmatrix} 1 & -3 & -2 & | & -1 & 0 & 0 \\ 0 & 1 & 1 & | & 1 & -1 & 0 \\ 0 & -2 & -2 & | & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Step 5: } r_1 := r_1 - (-3)r_2 \begin{bmatrix} 1 & 0 & 1 & | & 2 & -3 & 0 \\ 0 & 1 & 1 & | & 1 & -1 & 0 \\ 0 & -2 & -2 & | & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Step 6: } r_3 := r_3 - (-2)r_2 \begin{bmatrix} 1 & 0 & 1 & | & 2 & -3 & 0 \\ 0 & 1 & 1 & | & 1 & -1 & 0 \\ 0 & 0 & 0 & | & 1 & -2 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := 1/3r_1 \begin{bmatrix} 1 & 1 & 1 & | & 1/3 & 0 & 0 \\ -1 & 0 & 0 & | & 0 & 1 & 0 \\ -1 & -1 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_2 := r_2 - (-1)r_1 \begin{bmatrix} 1 & 1 & 1 & | & 1/3 & 0 & 0 \\ 0 & 1 & 1 & | & 1/3 & 1 & 0 \\ -1 & -1 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_3 := r_3 - (-1)r_1 \begin{bmatrix} 1 & 1 & 1 & | & 1/3 & 0 & 0 \\ 0 & 1 & 1 & | & 1/3 & 1 & 0 \\ 0 & 0 & 0 & | & 1/3 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_1 := r_1 - r_2 \begin{bmatrix} 1 & 0 & 0 & | & 0 & -1 & 0 \\ 0 & 1 & 1 & | & 1/3 & 1 & 0 \\ 0 & 0 & 0 & | & 1/3 & 0 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := 1/3r_1 \begin{bmatrix} 1 & -7/3 & 7/3 & | & 1/3 & 0 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 2 & -6 & 6 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_3 := r_3 - (2)r_1 \begin{bmatrix} 1 & -7/3 & 7/3 & | & 1/3 & 0 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & -4/3 & 4/3 & | & -2/3 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (-7/3)r_2 \begin{bmatrix} 1 & 0 & 0 & | & 1/3 & 7/3 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & -4/3 & 4/3 & | & -2/3 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_3 := r_3 - (-4/3)r_2 \begin{bmatrix} 1 & 0 & 0 & | & 1/3 & 7/3 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & -2/3 & 4/3 & 1 \end{bmatrix}$$

Results:

- a) $\text{rank}(A) = 2$
- b) $\text{nullity}(A) = 1$
- c) $\det(A) = 0$
- d) A^{-1} = does not exist
- e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - r_1 \begin{bmatrix} 1 & -1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := r_1 - (-1)r_2 \begin{bmatrix} 1 & 0 & -1 & | & 0 & 1 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

Results:

- a) $\text{rank}(A) = 2$
- b) $\text{nullity}(A) = 1$
- c) $\det(A) = 0$
- d) A^{-1} = does not exist
- e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := 1/3r_1 \begin{bmatrix} 1 & -2/3 & -7/3 & | & 1/3 & 0 & 0 \\ 2 & -1 & -5 & | & 0 & 1 & 0 \\ -2 & 2 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_2 := r_2 - (2)r_1 \begin{bmatrix} 1 & -2/3 & -7/3 & | & 1/3 & 0 & 0 \\ 0 & 1/3 & -1/3 & | & -2/3 & 1 & 0 \\ -2 & 2 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_3 := r_3 - (-2)r_1 \begin{bmatrix} 1 & -2/3 & -7/3 & | & 1/3 & 0 & 0 \\ 0 & 1/3 & -1/3 & | & -2/3 & 1 & 0 \\ 0 & 2/3 & -2/3 & | & 2/3 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_2 := 3r_2 \begin{bmatrix} 1 & -2/3 & -7/3 & | & 1/3 & 0 & 0 \\ 0 & 1 & -1 & | & -2 & 3 & 0 \\ 0 & 2/3 & -2/3 & | & 2/3 & 0 & 1 \end{bmatrix}$$

$$\text{Step 5: } r_1 := r_1 - (-2/3)r_2 \begin{bmatrix} 1 & 0 & -3 & | & -1 & 2 & 0 \\ 0 & 1 & -1 & | & -2 & 3 & 0 \\ 0 & 2/3 & -2/3 & | & 2/3 & 0 & 1 \end{bmatrix}$$

$$\text{Step 6: } r_3 := r_3 - (2/3)r_2 \begin{bmatrix} 1 & 0 & -3 & | & -1 & 2 & 0 \\ 0 & 1 & -1 & | & -2 & 3 & 0 \\ 0 & 0 & 0 & | & 2 & -2 & 1 \end{bmatrix}$$

Results:

- a) $\text{rank}(A) = 2$
- b) $\text{nullity}(A) = 1$
- c) $\det(A) = 0$
- d) A^{-1} = does not exist
- e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := r_1 - (-1)r_2 \begin{bmatrix} 1 & 0 & -6 & | & 1 & 1 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := r_1 - (-6)r_3 \begin{bmatrix} 1 & 0 & 0 & | & 1 & 1 & 6 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_2 := r_2 - (-1)r_3 \begin{bmatrix} 1 & 0 & 0 & | & 1 & 1 & 6 \\ 0 & 1 & 0 & | & 0 & 1 & 1 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

Results:

- a) $\text{rank}(A) = 3$
- b) $\text{nullity}(A) = 0$
- c) $\det(A) = 0$
- d) $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-2)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_2 := 1/3r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2/3 & 1/3 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_1 := r_1 - (2)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -1/3 & -2/3 & 0 \\ 0 & 1 & 1 & 2/3 & 1/3 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 4: } r_3 := r_3 - (2)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -1/3 & -2/3 & 0 \\ 0 & 1 & 1 & 2/3 & 1/3 & 0 \\ 0 & 0 & 0 & -4/3 & -2/3 & 1 \end{array} \right]$$

Results:

- a) $\text{rank}(A) = 2$
- b) $\text{nullity}(A) = 1$
- c) $\det(A) = 0$
- d) A^{-1} = does not exist
- e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := -1r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_2 := r_2 - r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_2 := -1r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Results:

- a) $\text{rank}(A) = 3$
- b) $\text{nullity}(A) = 0$
- c) $\det(A) = 0$
- d) $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$
- e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := 1/2r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 1/2 & -5 & 1/2 & 0 & 0 \\ 1 & 1 & -6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_2 := r_2 - r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 1/2 & -5 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1 & -1/2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_2 := 2r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 1/2 & -5 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 4: } r_1 := r_1 - (1/2)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & -1 & 0 \\ 0 & 1 & -2 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 5: } r_1 := r_1 - (-4)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 4 \\ 0 & 1 & -2 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 6: } r_2 := r_2 - (-2)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 4 \\ 0 & 1 & 0 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 3 & -5 & 2 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

2.3.2. RREF

Solution

Elementary Row Operations:

(1) $r_1 := r_1 + (-1)r_2$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

(2) $r_2 := r_2 + (-1)r_1$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

Result:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

Solution

Elementary Row Operations:

$$(1) \ r_2 := r_2 - (2)r_3$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 4 & 2 & 9 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_2$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_2 := r_2 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 + (-1)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-2)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_2 := r_2 - r_3$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-1)r_2$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_2 := r_2 + (-1)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_3 := r_3 + (-1)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

(3) $r_3 := r_3 - r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 + (-2)r_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_2 := r_2 - (2)r_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_1 := r_1 - r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 + (-1)r_3$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-2)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_2 := r_2 - (2)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 + (-1)r_2$$

$$\begin{bmatrix} 5 & -8 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-2)r_2$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_2 := r_2 + (-2)r_1$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_2 := r_2 - r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_2 := r_2 - r_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(2) $r_3 := r_3 - r_1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.4. Calculus

2.4.1. Limit

The limit is:

$$1 \quad (151)$$

The limit is:

$$2 \quad (152)$$

The limit is:

$$1 \quad (153)$$

The limit is:

$$-2 \quad (154)$$

The limit is:

$$1 \quad (155)$$

The limit is:

$$1 \quad (156)$$

The limit is:

$$1 \quad (157)$$

The limit is:

$$104 \quad (158)$$

The limit is:

$$2 \quad (159)$$

The limit is:

$$11 \quad (160)$$

2.4.2. Derivative

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \quad (161)$$

The derivative is:

$$\frac{1}{x} \quad (162)$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \quad (163)$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \quad (164)$$

The derivative is:

$$x^2e^x + 2xe^x \quad (165)$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \quad (166)$$

The derivative is:

$$e^x \quad (167)$$

The derivative is:

$$x^3e^x + 3x^2e^x \quad (168)$$

The derivative is:

$$-\frac{2x^3}{(x^2 + 1)^2} + \frac{2x}{x^2 + 1} \quad (169)$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \quad (170)$$

2.4.3. Integral

The indefinite integral is:

$$\frac{x\sqrt{4-x^2}}{2} + 2 \operatorname{asin}\left(\frac{x}{2}\right) \quad (171)$$

Definite integral from 2 to 3:

$$-\pi + 2 \operatorname{asin}\left(\frac{3}{2}\right) + \frac{3\sqrt{5}i}{2} \quad (172)$$

The indefinite integral is:

$$\frac{\sqrt{\pi} \operatorname{erf}(x)}{2} \quad (173)$$

Definite integral from 2 to 5:

$$-\frac{\sqrt{\pi} \operatorname{erf}(2)}{2} + \frac{\sqrt{\pi} \operatorname{erf}(5)}{2} \quad (174)$$

The indefinite integral is:

$$\frac{\sqrt{\pi} \operatorname{erf}(x)}{2} \quad (175)$$

Definite integral from 2 to 2:

$$0 \quad (176)$$

The indefinite integral is:

$$\frac{e^x \sin(x)}{2} - \frac{e^x \cos(x)}{2} \quad (177)$$

Definite integral from 1 to 3:

$$-\frac{e \sin(1)}{2} + \frac{e \cos(1)}{2} + \frac{e^3 \sin(3)}{2} - \frac{e^3 \cos(3)}{2} \quad (178)$$

The indefinite integral is:

$$3 \log(x-3) - 2 \log(x-2) \quad (179)$$

Definite integral from 2 to 3:

$$-\infty \quad (180)$$

The indefinite integral is:

$$\log(\log(x)) \quad (181)$$

Definite integral from 4 to 4:

$$0 \quad (182)$$

The indefinite integral is:

$$\frac{x^5}{5} + \frac{5x^4}{4} + \frac{5x^3}{3} + \frac{x^2}{2} + 3x \quad (183)$$

Definite integral from 1 to 3:

$$\frac{3026}{15} \quad (184)$$

The indefinite integral is:

$$\frac{x^4 \log(x)}{4} - \frac{x^4}{16} \quad (185)$$

Definite integral from 2 to 4:

$$-15 - 4 \log(2) + 64 \log(4) \quad (186)$$

The indefinite integral is:

$$\operatorname{atan}(x) \quad (187)$$

Definite integral from 3 to 4:

$$- \operatorname{atan}(3) + \operatorname{atan}(4) \quad (188)$$

The indefinite integral is:

$$\frac{x\sqrt{4-x^2}}{2} + 2 \operatorname{asin}\left(\frac{x}{2}\right) \quad (189)$$

Definite integral from 4 to 5:

$$-4\sqrt{3}i + 2 \operatorname{asin}\left(\frac{5}{2}\right) - 2 \operatorname{asin}(2) + \frac{5\sqrt{21}i}{2} \quad (190)$$

2.4.4. Partial Derivative

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \quad (191)$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \quad (192)$$

$$\frac{\partial^2 f}{\partial x^2} = 6y^3(2x^2 + y) \quad (193)$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (194)$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (195)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \quad (196)$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (197)$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (198)$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (199)$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (200)$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (201)$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (202)$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \quad (203)$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (204)$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \quad (205)$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \quad (206)$$