

Exercise 28:

Foundations of Mathematical, WS24

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This is **exercise** 28 for Foundations of Mathematical, WS24. Generated on 2025-06-02 with 10 problems per section.

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1. Problems

1.1. Vector Arithmetic

1.1.1. Addition

Find the sum of the following vectors \mathbf{u} and \mathbf{v}

1. $\mathbf{u} = \begin{bmatrix} 8 \\ 5 \\ 10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 4 \\ -10 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} -3 \\ 1 \\ -9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ -1 \\ 0 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 4 \\ -1 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} -7 \\ -5 \\ -9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ 10 \\ -2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} -6 \\ -6 \\ -8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ -3 \\ 2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} 8 \\ -6 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -5 \\ 4 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} -4 \\ 8 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ -9 \\ 2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

1.1.2. Subtraction

Find the difference of the following vectors \mathbf{u} and \mathbf{v}

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ -10 \\ 10 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} -7 \\ -5 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 1 \\ 6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} -1 \\ -4 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -3 \\ -4 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} -6 \\ -7 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 9 \\ -5 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ -7 \\ 5 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

6. $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} 4 \\ 8 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ -6 \\ -2 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -1 \\ -4 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ -8 \\ 7 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} 2 \\ 6 \\ -8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \\ 10 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

1.1.3. Scalar Multiplication

Find the scalar product of the following vector \mathbf{u} and scalar k

1. $\mathbf{u} = \begin{bmatrix} 4 \\ -4 \\ -3 \end{bmatrix}$ $7\mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} 6 \\ 1 \\ 9 \end{bmatrix}$ $1\mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} -1 \\ 8 \\ 0 \end{bmatrix}$ $2\mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ -2 \end{bmatrix}$ $-3\mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 0 \\ -5 \\ -10 \end{bmatrix}$ $3\mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} -2 \\ -10 \\ 3 \end{bmatrix}$ $2\mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -7 \\ -4 \\ 10 \end{bmatrix}$ $10\mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -4 \\ 0 \\ -7 \end{bmatrix}$ $-8\mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} -1 \\ -2 \\ -5 \end{bmatrix}$ $3\mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 8 \end{bmatrix}$ $-10\mathbf{v}$.

1.2. Matrix Arithmetic

1.2.1. Addition

Find the sum of the following matrices A and B

1.
$$A = \begin{bmatrix} 1 & -10 & 3 \\ 6 & 2 & 5 \\ 6 & -5 & 4 \end{bmatrix} \quad (1)$$

and

$$B = \begin{bmatrix} 8 & 8 & 8 \\ 6 & -7 & -2 \\ 2 & 4 & 3 \end{bmatrix} \quad (2)$$

2.
$$A = \begin{bmatrix} 2 & 0 & -8 \\ -2 & -7 & -3 \\ 2 & 8 & -9 \end{bmatrix} \quad (3)$$

and

$$B = \begin{bmatrix} -8 & 8 & -6 \\ 9 & -5 & -1 \\ 2 & 8 & 7 \end{bmatrix} \quad (4)$$

3.
$$A = \begin{bmatrix} 9 & 1 & -10 \\ 4 & -2 & -2 \\ 8 & 7 & -4 \end{bmatrix} \quad (5)$$

and

$$B = \begin{bmatrix} 5 & 1 & 4 \\ -8 & -6 & -7 \\ 6 & -1 & -6 \end{bmatrix} \quad (6)$$

4.
$$A = \begin{bmatrix} 8 & -10 & 7 \\ 3 & -5 & 9 \\ 8 & 8 & -3 \end{bmatrix} \quad (7)$$

and

$$B = \begin{bmatrix} -2 & -7 & -3 \\ 7 & -2 & -4 \\ -2 & 3 & -9 \end{bmatrix} \quad (8)$$

5.
$$A = \begin{bmatrix} -8 & 2 & 8 \\ -6 & 0 & 4 \\ 9 & -10 & -8 \end{bmatrix} \quad (9)$$

and

$$B = \begin{bmatrix} 7 & -8 & 3 \\ -10 & 3 & -1 \\ 7 & -6 & -8 \end{bmatrix} \quad (10)$$

6.
$$A = \begin{bmatrix} 5 & -5 & -4 \\ 2 & 9 & -3 \\ 1 & 0 & -2 \end{bmatrix} \quad (11)$$

and

$$B = \begin{bmatrix} -6 & 3 & 2 \\ 9 & 7 & -1 \\ 7 & -6 & 5 \end{bmatrix} \quad (12)$$

7.

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 4 & 7 & -2 \\ 7 & 5 & 5 \end{bmatrix} \quad (13)$$

and

$$B = \begin{bmatrix} -9 & 3 & 5 \\ 9 & -1 & 7 \\ -4 & -10 & 4 \end{bmatrix} \quad (14)$$

8.

$$A = \begin{bmatrix} -7 & -3 & 2 \\ -8 & 7 & -4 \\ 4 & -3 & -2 \end{bmatrix} \quad (15)$$

and

$$B = \begin{bmatrix} 0 & 9 & -10 \\ 2 & -5 & -5 \\ 5 & -2 & -5 \end{bmatrix} \quad (16)$$

9.

$$A = \begin{bmatrix} -1 & -9 & 4 \\ -8 & -3 & 6 \\ -7 & -1 & 0 \end{bmatrix} \quad (17)$$

and

$$B = \begin{bmatrix} -4 & -10 & 3 \\ -3 & -8 & 8 \\ -10 & -4 & -2 \end{bmatrix} \quad (18)$$

10.

$$A = \begin{bmatrix} -6 & 0 & -3 \\ 0 & -9 & -3 \\ -4 & 0 & -6 \end{bmatrix} \quad (19)$$

and

$$B = \begin{bmatrix} 5 & 5 & 7 \\ 5 & 2 & 3 \\ -10 & 5 & -7 \end{bmatrix} \quad (20)$$

1.2.2. Subtraction

Find the difference of the following matrices A and B

$$1. \quad A = \begin{bmatrix} -2 & -5 & 4 \\ 0 & -9 & 4 \\ 3 & 9 & 2 \end{bmatrix} \quad (21)$$

and

$$B = \begin{bmatrix} -2 & 8 & -7 \\ -3 & -8 & -4 \\ 0 & -1 & 9 \end{bmatrix} \quad (22)$$

$$2. \quad A = \begin{bmatrix} -10 & 4 & -1 \\ 8 & 0 & -9 \\ -6 & 6 & -8 \end{bmatrix} \quad (23)$$

and

$$B = \begin{bmatrix} 9 & -10 & 5 \\ -9 & -1 & 1 \\ -2 & -2 & 8 \end{bmatrix} \quad (24)$$

$$3. \quad A = \begin{bmatrix} 5 & -10 & -3 \\ 2 & -2 & -4 \\ 3 & -3 & -10 \end{bmatrix} \quad (25)$$

and

$$B = \begin{bmatrix} -1 & -3 & -3 \\ 3 & -6 & 7 \\ 1 & 7 & 8 \end{bmatrix} \quad (26)$$

$$4. \quad A = \begin{bmatrix} -6 & -9 & 2 \\ -4 & -10 & -2 \\ 0 & 5 & 2 \end{bmatrix} \quad (27)$$

and

$$B = \begin{bmatrix} 4 & -9 & -1 \\ -4 & 8 & -10 \\ -7 & 4 & -4 \end{bmatrix} \quad (28)$$

$$5. \quad A = \begin{bmatrix} 7 & 6 & -1 \\ 4 & -1 & 0 \\ -10 & -9 & 5 \end{bmatrix} \quad (29)$$

and

$$B = \begin{bmatrix} 8 & 0 & 2 \\ -6 & 3 & 7 \\ -7 & 2 & -4 \end{bmatrix} \quad (30)$$

$$6. \quad A = \begin{bmatrix} -10 & -7 & 0 \\ -2 & 6 & 6 \\ -4 & -6 & -7 \end{bmatrix} \quad (31)$$

and

$$B = \begin{bmatrix} -2 & -8 & -8 \\ -5 & 7 & 9 \\ 3 & -1 & 4 \end{bmatrix} \quad (32)$$

7.

$$A = \begin{bmatrix} 0 & 4 & 7 \\ -4 & -5 & 6 \\ -6 & -10 & 4 \end{bmatrix} \quad (33)$$

and

$$B = \begin{bmatrix} 0 & -3 & 6 \\ -2 & -6 & 7 \\ -4 & -4 & -4 \end{bmatrix} \quad (34)$$

8.

$$A = \begin{bmatrix} -4 & 8 & -7 \\ 6 & 4 & 0 \\ -1 & -2 & 2 \end{bmatrix} \quad (35)$$

and

$$B = \begin{bmatrix} -3 & -5 & -5 \\ -1 & 6 & -2 \\ 5 & -8 & -10 \end{bmatrix} \quad (36)$$

9.

$$A = \begin{bmatrix} -6 & -7 & -1 \\ 3 & -5 & -6 \\ 3 & -4 & 0 \end{bmatrix} \quad (37)$$

and

$$B = \begin{bmatrix} -5 & 1 & 8 \\ 2 & 9 & 4 \\ -5 & -6 & -3 \end{bmatrix} \quad (38)$$

10.

$$A = \begin{bmatrix} -7 & 5 & -8 \\ -5 & -4 & 2 \\ 4 & -8 & -9 \end{bmatrix} \quad (39)$$

and

$$B = \begin{bmatrix} 6 & 2 & -2 \\ 8 & -9 & 1 \\ 1 & -9 & -7 \end{bmatrix} \quad (40)$$

1.2.3. Multiplication

Find the product of the following matrices A and B

$$1. \quad A = \begin{bmatrix} -2 & -6 & -5 \\ 2 & -4 & 3 \\ 6 & -6 & -3 \end{bmatrix} \quad (41)$$

and

$$B = \begin{bmatrix} -9 & 3 & 5 \\ 2 & 4 & 3 \\ 4 & 9 & -10 \end{bmatrix} \quad (42)$$

$$2. \quad A = \begin{bmatrix} -8 & 8 & -10 \\ 8 & 8 & 4 \\ 0 & -5 & 1 \end{bmatrix} \quad (43)$$

and

$$B = \begin{bmatrix} 0 & -3 & -6 \\ 6 & -2 & 6 \\ 7 & -4 & 3 \end{bmatrix} \quad (44)$$

$$3. \quad A = \begin{bmatrix} 6 & 0 & -8 \\ 9 & 3 & 7 \\ -2 & -10 & 5 \end{bmatrix} \quad (45)$$

and

$$B = \begin{bmatrix} -4 & -8 & 8 \\ -1 & -1 & -9 \\ 2 & -6 & -2 \end{bmatrix} \quad (46)$$

$$4. \quad A = \begin{bmatrix} 9 & -9 & 0 \\ -7 & 2 & 8 \\ -3 & 4 & -4 \end{bmatrix} \quad (47)$$

and

$$B = \begin{bmatrix} 8 & 9 & 8 \\ -2 & 7 & 9 \\ 1 & -6 & -6 \end{bmatrix} \quad (48)$$

$$5. \quad A = \begin{bmatrix} 0 & -5 & 2 \\ -4 & -3 & -5 \\ -4 & 9 & 9 \end{bmatrix} \quad (49)$$

and

$$B = \begin{bmatrix} 1 & 2 & -9 \\ 8 & 3 & -2 \\ 0 & 0 & -10 \end{bmatrix} \quad (50)$$

$$6. \quad A = \begin{bmatrix} 2 & -4 & 2 \\ 6 & -9 & -5 \\ -1 & -4 & -3 \end{bmatrix} \quad (51)$$

and

$$B = \begin{bmatrix} -4 & 4 & -3 \\ 5 & -6 & 6 \\ 5 & -4 & -10 \end{bmatrix} \quad (52)$$

7.

$$A = \begin{bmatrix} -3 & -6 & -7 \\ -1 & -4 & -10 \\ 6 & 4 & -5 \end{bmatrix} \quad (53)$$

and

$$B = \begin{bmatrix} 7 & -4 & -4 \\ 9 & -7 & -10 \\ -4 & 8 & 3 \end{bmatrix} \quad (54)$$

8.

$$A = \begin{bmatrix} 2 & 7 & -7 \\ -7 & -8 & 8 \\ 8 & -7 & -6 \end{bmatrix} \quad (55)$$

and

$$B = \begin{bmatrix} -10 & -10 & 6 \\ 2 & -2 & -4 \\ -1 & -7 & -5 \end{bmatrix} \quad (56)$$

9.

$$A = \begin{bmatrix} -10 & 1 & -7 \\ -4 & 0 & 4 \\ -8 & 6 & -6 \end{bmatrix} \quad (57)$$

and

$$B = \begin{bmatrix} 2 & -7 & 8 \\ 1 & -3 & -4 \\ 1 & 8 & -4 \end{bmatrix} \quad (58)$$

10.

$$A = \begin{bmatrix} -1 & 4 & -1 \\ 6 & -7 & 4 \\ 8 & 1 & 0 \end{bmatrix} \quad (59)$$

and

$$B = \begin{bmatrix} 1 & 9 & -1 \\ 8 & 8 & -5 \\ -1 & -10 & 0 \end{bmatrix} \quad (60)$$

1.3. Matrix Properties

1.3.1. Properties

For each matrix A , find:

a) $\text{rank}(A)$

- b) $\text{nullity}(A)$
- c) $\det(A)$
- d) A^{-1} (if exists)
- e) basis of $\ker(A)$

1.
$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ -2 & 4 & -4 \end{bmatrix} \quad (61)$$

2.
$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (62)$$

3.
$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (63)$$

4.
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix} \quad (64)$$

5.
$$A = \begin{bmatrix} -3 & 2 & -9 \\ -2 & 1 & -7 \\ -2 & 1 & -6 \end{bmatrix} \quad (65)$$

6.
$$A = \begin{bmatrix} 1 & -3 & 1 \\ -1 & 4 & 0 \\ -2 & 8 & 0 \end{bmatrix} \quad (66)$$

7.
$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -5 \\ 2 & 0 & 3 \end{bmatrix} \quad (67)$$

8.
$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ -2 & -1 & -7 \end{bmatrix} \quad (68)$$

9.
$$A = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix} \quad (69)$$

10.
$$A = \begin{bmatrix} 1 & 7 & 11 \\ 0 & 5 & 8 \\ 0 & -2 & -3 \end{bmatrix} \quad (70)$$

1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix A

$$1. \quad A = \begin{bmatrix} 3 & 0 & -1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad (71)$$

$$2. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad (72)$$

$$3. \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \quad (73)$$

$$4. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix} \quad (74)$$

$$5. \quad A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} \quad (75)$$

$$6. \quad A = \begin{bmatrix} -3 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (76)$$

$$7. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \quad (77)$$

$$8. \quad A = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad (78)$$

$$9. \quad A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad (79)$$

$$10. \quad A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -4 \end{bmatrix} \quad (80)$$

1.4. Calculus

1.4.1. Limit

Calculate the following limits

1. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} -4x^3 + 4x^2 + 3x - 1 \quad (81)$$

2. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (82)$$

3. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (83)$$

4. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (84)$$

5. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (85)$$

6. Calculate the limit of the following expression:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \quad (86)$$

7. Calculate the limit of the following expression:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \quad (87)$$

8. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (88)$$

9. Calculate the limit of the following expression:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \quad (89)$$

10. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (90)$$

1.4.2. Derivative

Calculate the derivatives of the following expressions

1. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (91)$$

2. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (92)$$

3. Calculate the derivative of the following expression:

$$\log(x+1) + \log(x^2+1) \quad (93)$$

4. Calculate the derivative of the following expression:

$$\frac{x^3}{x^2 + 1} \quad (94)$$

5. Calculate the derivative of the following expression:

$$xe^x \quad (95)$$

6. Calculate the derivative of the following expression:

$$\log(x^2 - 1) \quad (96)$$

7. Calculate the derivative of the following expression:

$$\log(x + 1) + \log(x^2 + 1) \quad (97)$$

8. Calculate the derivative of the following expression:

$$\frac{x^2}{x^2 + 1} \quad (98)$$

9. Calculate the derivative of the following expression:

$$\log(x + 1) + \log(x^2 + 1) \quad (99)$$

10. Calculate the derivative of the following expression:

$$\frac{x^3}{x^2 + 1} \quad (100)$$

1.4.3. Integral

Calculate the indefinite and definite integrals of the following expressions

1. the indefinite integral and evaluate from 1 to 3:

$$\int \frac{\sin(x)}{x} dx \quad (101)$$

2. Evaluate the improper integral:

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx \quad (102)$$

3. the indefinite integral and evaluate from 1 to 4:

$$\int e^x \sin(x) dx \quad (103)$$

4. the indefinite integral and evaluate from 2 to 5:

$$\int x \sqrt{x^2 + 1} dx \quad (104)$$

5. the indefinite integral and evaluate from 2 to 3:

$$\int x^3 \log(x) dx \quad (105)$$

6. the indefinite integral and evaluate from 3 to 4:

$$\int \frac{1}{(x-2)(x+1)} dx \quad (106)$$

7. the indefinite integral and evaluate from 1 to 1:

$$\int -5x^2 - 5x + 2 dx \quad (107)$$

8. the indefinite integral and evaluate from 2 to 4:

$$\int e^x \sin(x) dx \quad (108)$$

9. the indefinite integral and evaluate from 4 to 5:

$$\int x^3 \log(x) dx \quad (109)$$

10. the indefinite integral and evaluate from 1 to 3:

$$\int x \sqrt{x^2 + 1} dx \quad (110)$$

1.4.4. Partial Derivative

Calculate the partial derivatives of the following expressions

1. the mixed partial derivative of:

$$f(x, y) = x^3 y^2 + xy^4 \quad (111)$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

2. Given the implicit function:

$$x^2 y + xy^2 - xy = 0 \quad (112)$$

$$\frac{\partial y}{\partial x}$$

3. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \quad (113)$$

where $f = f(u, v)$

4. the partial derivatives of the function:

$$f(x, y) = x^3 y^2 - 3x^2 y + 2xy^3 \quad (114)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

5. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \quad (115)$$

where $f = f(u, v)$

6. the second order partial derivative of:

$$f(x, y) = x^4 y^3 + 3x^2 y^4 \quad (116)$$

$$\frac{\partial^2 f}{\partial x^2}$$

7. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2 + y^2} \quad (117)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

8. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \quad (118)$$

where $f = f(u, v)$

9. Given the implicit function:

$$x^2 y + xy^2 - xy = 0 \quad (119)$$

$$\frac{\partial y}{\partial x}$$

10. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2 + y^2} \quad (120)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

2. Solutions

2.1. Vector Arithmetic

2.1.1. Addition

$$\begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ -9 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \\ -9 \end{bmatrix}$$
$$\begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 8 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} -12 \\ -9 \\ -6 \end{bmatrix} + \begin{bmatrix} 15 \\ -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \\ -11 \end{bmatrix}$$
$$\begin{bmatrix} -5 \\ 12 \\ 4 \end{bmatrix} + \begin{bmatrix} -7 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -12 \\ 11 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} + \begin{bmatrix} -7 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ 5 \end{bmatrix}$$

2.1.2. Subtraction

$$\begin{bmatrix} 4 \\ 14 \\ -12 \end{bmatrix} - \begin{bmatrix} -15 \\ -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 19 \\ 20 \\ -9 \end{bmatrix}$$
$$\begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} -8 \\ -12 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} - \begin{bmatrix} -4 \\ 14 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 8 \end{bmatrix}$$
$$\begin{bmatrix} -1 \\ 4 \\ -10 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \\ -18 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 8 \end{bmatrix}$$
$$\begin{bmatrix} 5 \\ 1 \\ -18 \end{bmatrix} - \begin{bmatrix} 6 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -12 \end{bmatrix}$$

2.1.3. Scalar Multiplication

$$1: \begin{bmatrix} 28 \\ -28 \\ -21 \end{bmatrix} \quad 2: \begin{bmatrix} 6 \\ 1 \\ 9 \end{bmatrix} \quad 3: \begin{bmatrix} -2 \\ 16 \\ 0 \end{bmatrix} \quad 4: \begin{bmatrix} -21 \\ -6 \\ 6 \end{bmatrix} \quad 5: \begin{bmatrix} 0 \\ -15 \\ -30 \end{bmatrix}$$
$$6: \begin{bmatrix} -4 \\ -20 \\ 6 \end{bmatrix} \quad 7: \begin{bmatrix} -70 \\ -40 \\ 100 \end{bmatrix} \quad 8: \begin{bmatrix} 32 \\ 0 \\ 56 \end{bmatrix} \quad 9: \begin{bmatrix} -3 \\ -6 \\ -15 \end{bmatrix} \quad 10: \begin{bmatrix} 10 \\ -10 \\ -80 \end{bmatrix}$$

2.2. Matrix Arithmetic

2.2.1. Addition

1:

$$\begin{bmatrix} 9 & -2 & 11 \\ 12 & -5 & 3 \\ 8 & -1 & 7 \end{bmatrix} \quad (121)$$

1:

$$\begin{bmatrix} -6 & 8 & -14 \\ 7 & -12 & -4 \\ 4 & 16 & -2 \end{bmatrix} \quad (122)$$

1:

$$\begin{bmatrix} 14 & 2 & -6 \\ -4 & -8 & -9 \\ 14 & 6 & -10 \end{bmatrix} \quad (123)$$

1:

$$\begin{bmatrix} 6 & -17 & 4 \\ 10 & -7 & 5 \\ 6 & 11 & -12 \end{bmatrix} \quad (124)$$

1:

$$\begin{bmatrix} -1 & -6 & 11 \\ -16 & 3 & 3 \\ 16 & -16 & -16 \end{bmatrix} \quad (125)$$

1:

$$\begin{bmatrix} -1 & -2 & -2 \\ 11 & 16 & -4 \\ 8 & -6 & 3 \end{bmatrix} \quad (126)$$

1:

$$\begin{bmatrix} -9 & 1 & 4 \\ 13 & 6 & 5 \\ 3 & -5 & 9 \end{bmatrix} \quad (127)$$

1:

$$\begin{bmatrix} -7 & 6 & -8 \\ -6 & 2 & -9 \\ 9 & -5 & -7 \end{bmatrix} \quad (128)$$

1:

$$\begin{bmatrix} -5 & -19 & 7 \\ -11 & -11 & 14 \\ -17 & -5 & -2 \end{bmatrix} \quad (129)$$

1:

$$\begin{bmatrix} -1 & 5 & 4 \\ 5 & -7 & 0 \\ -14 & 5 & -13 \end{bmatrix} \quad (130)$$

2.2.2. Subtraction

1:

$$\begin{bmatrix} 0 & -13 & 11 \\ 3 & -1 & 8 \\ 3 & 10 & -7 \end{bmatrix} \quad (131)$$

1:

$$\begin{bmatrix} -19 & 14 & -6 \\ 17 & 1 & -10 \\ -4 & 8 & -16 \end{bmatrix} \quad (132)$$

1:

$$\begin{bmatrix} 6 & -7 & 0 \\ -1 & 4 & -11 \\ 2 & -10 & -18 \end{bmatrix} \quad (133)$$

1:

$$\begin{bmatrix} -10 & 0 & 3 \\ 0 & -18 & 8 \\ 7 & 1 & 6 \end{bmatrix} \quad (134)$$

1:

$$\begin{bmatrix} -1 & 6 & -3 \\ 10 & -4 & -7 \\ -3 & -11 & 9 \end{bmatrix} \quad (135)$$

1:

$$\begin{bmatrix} -8 & 1 & 8 \\ 3 & -1 & -3 \\ -7 & -5 & -11 \end{bmatrix} \quad (136)$$

1:

$$\begin{bmatrix} 0 & 7 & 1 \\ -2 & 1 & -1 \\ -2 & -6 & 8 \end{bmatrix} \quad (137)$$

1:

$$\begin{bmatrix} -1 & 13 & -2 \\ 7 & -2 & 2 \\ -6 & 6 & 12 \end{bmatrix} \quad (138)$$

1:

$$\begin{bmatrix} -1 & -8 & -9 \\ 1 & -14 & -10 \\ 8 & 2 & 3 \end{bmatrix} \quad (139)$$

1:

$$\begin{bmatrix} -13 & 3 & -6 \\ -13 & 5 & 1 \\ 3 & 1 & -2 \end{bmatrix} \quad (140)$$

2.2.3. Multiplication

1:

$$\begin{bmatrix} -14 & -75 & 22 \\ -14 & 17 & -32 \\ -78 & -33 & 42 \end{bmatrix} \quad (141)$$

1:

$$\begin{bmatrix} -22 & 48 & 66 \\ 76 & -56 & 12 \\ -23 & 6 & -27 \end{bmatrix} \quad (142)$$

1:

$$\begin{bmatrix} -40 & 0 & 64 \\ -25 & -117 & 31 \\ 28 & -4 & 64 \end{bmatrix} \quad (143)$$

1:

$$\begin{bmatrix} 90 & 18 & -9 \\ -52 & -97 & -86 \\ -36 & 25 & 36 \end{bmatrix} \quad (144)$$

1:

$$\begin{bmatrix} -40 & -15 & -10 \\ -28 & -17 & 92 \\ 68 & 19 & -72 \end{bmatrix} \quad (145)$$

1:

$$\begin{bmatrix} -18 & 24 & -50 \\ -94 & 98 & -22 \\ -31 & 32 & 9 \end{bmatrix} \quad (146)$$

1:

$$\begin{bmatrix} -47 & -2 & 51 \\ -3 & -48 & 14 \\ 98 & -92 & -79 \end{bmatrix} \quad (147)$$

1:

$$\begin{bmatrix} 1 & 15 & 19 \\ 46 & 30 & -50 \\ -88 & -24 & 106 \end{bmatrix} \quad (148)$$

1:

$$\begin{bmatrix} -26 & 11 & -56 \\ -4 & 60 & -48 \\ -16 & -10 & -64 \end{bmatrix} \quad (149)$$

1:

$$\begin{bmatrix} 32 & 33 & -19 \\ -54 & -42 & 29 \\ 16 & 80 & -13 \end{bmatrix} \quad (150)$$

2.3. Matrix Properties

2.3.1. Properties

Solution

Row Operations:

$$\text{Step 1: } r_3 := r_3 - (-2)r_1 \quad \begin{bmatrix} 1 & -2 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := r_1 - (-2)r_2 \quad \begin{bmatrix} 1 & 0 & 4 & | & 1 & 2 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 2 & 0 & 1 \end{bmatrix}$$

Results:

- a) $\text{rank}(A) = 2$
- b) $\text{nullity}(A) = 1$
- c) $\det(A) = 0$
- d) A^{-1} = does not exist
- e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 \leftrightarrow r_2 \quad \begin{bmatrix} -1 & 4 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := -1r_1 \quad \begin{bmatrix} 1 & -4 & -1 & | & 0 & -1 & 0 \\ 0 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (-4)r_2 \quad \begin{bmatrix} 1 & 0 & 3 & | & 4 & -1 & 0 \\ 0 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

Results:

- a) $\text{rank}(A) = 2$
- b) $\text{nullity}(A) = 1$
- c) $\det(A) = 0$
- d) A^{-1} = does not exist
- e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := r_1 - (2)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_1 := r_1 - (-5)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 5 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_2 := r_2 - r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 5 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_3 := r_3 - (3)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_1 := r_1 - r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_1 := r_1 - (-1)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array} \right]$$

$$\text{Step 4: } r_2 := r_2 - r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := -1/3 r_1 \quad \left[\begin{array}{ccc|ccc} 1 & -2/3 & 3 & -1/3 & 0 & 0 \\ -2 & 1 & -7 & 0 & 1 & 0 \\ -2 & 1 & -6 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_2 := r_2 - (-2)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & -2/3 & 3 & -1/3 & 0 & 0 \\ 0 & -1/3 & -1 & -2/3 & 1 & 0 \\ -2 & 1 & -6 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_3 := r_3 - (-2)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & -2/3 & 3 & -1/3 & 0 & 0 \\ 0 & -1/3 & -1 & -2/3 & 1 & 0 \\ 0 & -1/3 & 0 & -2/3 & 0 & 1 \end{array} \right]$$

$$\text{Step 4: } r_2 := -3r_2 \quad \left[\begin{array}{ccc|ccc} 1 & -2/3 & 3 & -1/3 & 0 & 0 \\ 0 & 1 & 3 & 2 & -3 & 0 \\ 0 & -1/3 & 0 & -2/3 & 0 & 1 \end{array} \right]$$

$$\text{Step 5: } r_1 := r_1 - (-2/3)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & -2 & 0 \\ 0 & 1 & 3 & 2 & -3 & 0 \\ 0 & -1/3 & 0 & -2/3 & 0 & 1 \end{array} \right]$$

$$\text{Step 6: } r_3 := r_3 - (-1/3)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & -2 & 0 \\ 0 & 1 & 3 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

$$\text{Step 7: } r_1 := r_1 - (5)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -5 \\ 0 & 1 & 3 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

$$\text{Step 8: } r_2 := r_2 - (3)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -5 \\ 0 & 1 & 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 48$

d) $A^{-1} = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -4 \\ 0 & -1 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-1)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ -2 & 8 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_3 := r_3 - (-2)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_1 := r_1 - (-3)r_2 \begin{bmatrix} 1 & 0 & 4 & | & 4 & 3 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ 0 & 2 & 2 & | & 2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_3 := r_3 - (2)r_2 \begin{bmatrix} 1 & 0 & 4 & | & 4 & 3 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & -2 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 320$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-1)r_1 \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & -4 & | & 1 & 1 & 0 \\ 2 & 0 & 3 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_3 := r_3 - (2)r_1 \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & -4 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - r_3 \begin{bmatrix} 1 & 0 & 0 & | & 3 & 0 & -1 \\ 0 & 1 & -4 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_2 := r_2 - (-4)r_3 \begin{bmatrix} 1 & 0 & 0 & | & 3 & 0 & -1 \\ 0 & 1 & 0 & | & -7 & 1 & 4 \\ 0 & 0 & 1 & | & -2 & 0 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_3 := r_3 - (-2)r_1 \begin{bmatrix} 1 & 0 & 5 & | & 1 & 0 & 0 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & -1 & 3 & | & 2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_3 := r_3 - (-1)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array} \right]$$

$$\text{Step 3: } r_1 := r_1 - (5)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & -5 & -5 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array} \right]$$

$$\text{Step 4: } r_2 := r_2 - (-2)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & -5 & -5 \\ 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := r_1 - (-5)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 5 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -3 & 3 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_3 := r_3 - (-3)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 5 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := 1/5r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 7 & 11 & 1 & 0 & 0 \\ 0 & 1 & 8/5 & 0 & 1/5 & 0 \\ 0 & -2 & -3 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_1 := r_1 - (7)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1/5 & 1 & -7/5 & 0 \\ 0 & 1 & 8/5 & 0 & 1/5 & 0 \\ 0 & -2 & -3 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_3 := r_3 - (-2)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1/5 & 1 & -7/5 & 0 \\ 0 & 1 & 8/5 & 0 & 1/5 & 0 \\ 0 & 0 & 1/5 & 0 & 2/5 & 1 \end{array} \right]$$

$$\text{Step 4: } r_3 := 5r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1/5 & 1 & -7/5 & 0 \\ 0 & 1 & 8/5 & 0 & 1/5 & 0 \\ 0 & 0 & 1 & 0 & 2 & 5 \end{array} \right]$$

$$\text{Step 5: } r_1 := r_1 - (-1/5)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 8/5 & 0 & 1/5 & 0 \\ 0 & 0 & 1 & 0 & 2 & 5 \end{array} \right]$$

$$\text{Step 6: } r_2 := r_2 - (8/5)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -3 & -8 \\ 0 & 0 & 1 & 0 & 2 & 5 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

2.3.2. RREF

Solution

Elementary Row Operations:

(1) $r_1 := r_1 - r_3$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{array} \right]$$

(2) $r_2 := r_2 - (-2)r_1$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{array} \right]$$

(3) $r_3 := r_3 - (-2)r_1$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

Result:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

Solution

Elementary Row Operations:

(1) $r_3 := r_3 + (-2)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_1 := r_1 - r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 + (-2)r_3$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_1 := r_1 + (-1)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_2 := r_2 + (-1)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_3 := r_3 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_2 := r_2 - r_3$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-1)r_3$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$(3) \ r_3 := r_3 - r_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$(2) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-2)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_1 := r_1 + (-2)r_3$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 - r_3$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.4. Calculus

2.4.1. Limit

The limit is:

$$-1 \tag{151}$$

The limit is:

$$1 \tag{152}$$

The limit is:

$$1 \tag{153}$$

The limit is:

$$1 \tag{154}$$

The limit is:

$$1 \tag{155}$$

The limit is:

$$e \tag{156}$$

The limit is:

$$e \tag{157}$$

The limit is:

$$1 \tag{158}$$

The limit is:

$$e \tag{159}$$

The limit is:

$$1 \tag{160}$$

2.4.2. Derivative

The derivative is:

$$2xe^{x^2} + 2e^{2x} \quad (161)$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \quad (162)$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \quad (163)$$

The derivative is:

$$-\frac{2x^4}{(x^2 + 1)^2} + \frac{3x^2}{x^2 + 1} \quad (164)$$

The derivative is:

$$xe^x + e^x \quad (165)$$

The derivative is:

$$\frac{2x}{x^2 - 1} \quad (166)$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \quad (167)$$

The derivative is:

$$-\frac{2x^3}{(x^2 + 1)^2} + \frac{2x}{x^2 + 1} \quad (168)$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \quad (169)$$

The derivative is:

$$-\frac{2x^4}{(x^2 + 1)^2} + \frac{3x^2}{x^2 + 1} \quad (170)$$

2.4.3. Integral

The indefinite integral is:

$$\text{Si}(x) \quad (171)$$

Definite integral from 1 to 3:

$$-\text{Si}(1) + \text{Si}(3) \quad (172)$$

The improper integral converges to:

$$\infty \quad (173)$$

The indefinite integral is:

$$\frac{e^x \sin(x)}{2} - \frac{e^x \cos(x)}{2} \quad (174)$$

Definite integral from 1 to 4:

$$\frac{e^4 \sin(4)}{2} - \frac{e \sin(1)}{2} + \frac{e \cos(1)}{2} - \frac{e^4 \cos(4)}{2} \quad (175)$$

The indefinite integral is:

$$\frac{x^2 \sqrt{x^2 + 1}}{3} + \frac{\sqrt{x^2 + 1}}{3} \quad (176)$$

Definite integral from 2 to 5:

$$-\frac{5\sqrt{5}}{3} + \frac{26\sqrt{26}}{3} \quad (177)$$

The indefinite integral is:

$$\frac{x^4 \log(x)}{4} - \frac{x^4}{16} \quad (178)$$

Definite integral from 2 to 3:

$$-\frac{65}{16} - 4 \log(2) + \frac{81 \log(3)}{4} \quad (179)$$

The indefinite integral is:

$$\frac{\log(x-2)}{3} - \frac{\log(x+1)}{3} \quad (180)$$

Definite integral from 3 to 4:

$$-\frac{\log(5)}{3} + \frac{\log(2)}{3} + \frac{\log(4)}{3} \quad (181)$$

The indefinite integral is:

$$-\frac{5x^3}{3} - \frac{5x^2}{2} + 2x \quad (182)$$

Definite integral from 1 to 1:

$$0 \quad (183)$$

The indefinite integral is:

$$\frac{e^x \sin(x)}{2} - \frac{e^x \cos(x)}{2} \quad (184)$$

Definite integral from 2 to 4:

$$\frac{e^4 \sin(4)}{2} - \frac{e^2 \sin(2)}{2} + \frac{e^2 \cos(2)}{2} - \frac{e^4 \cos(4)}{2} \quad (185)$$

The indefinite integral is:

$$\frac{x^4 \log(x)}{4} - \frac{x^4}{16} \quad (186)$$

Definite integral from 4 to 5:

$$-64 \log(4) - \frac{369}{16} + \frac{625 \log(5)}{4} \quad (187)$$

The indefinite integral is:

$$\frac{x^2 \sqrt{x^2 + 1}}{3} + \frac{\sqrt{x^2 + 1}}{3} \quad (188)$$

Definite integral from 1 to 3:

$$-\frac{2\sqrt{2}}{3} + \frac{10\sqrt{10}}{3} \quad (189)$$

2.4.4. Partial Derivative

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \quad (190)$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \quad (191)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \quad (192)$$

$$\frac{\partial f}{\partial x} = 3x^2 y^2 - 6xy + 2y^3 \quad (193)$$

$$\frac{\partial f}{\partial y} = 2x^3 y - 3x^2 + 6xy^2 \quad (194)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \quad (195)$$

$$\frac{\partial^2 f}{\partial x^2} = 6y^3(2x^2 + y) \quad (196)$$

$$\frac{\partial f}{\partial x} = 2x(x + y)e^{x^2+y^2} + e^{x^2+y^2} \quad (197)$$

$$\frac{\partial f}{\partial y} = 2y(x + y)e^{x^2+y^2} + e^{x^2+y^2} \quad (198)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \quad (199)$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \quad (200)$$

$$\frac{\partial f}{\partial x} = 2x(x + y)e^{x^2+y^2} + e^{x^2+y^2} \quad (201)$$

$$\frac{\partial f}{\partial y} = 2y(x + y)e^{x^2+y^2} + e^{x^2+y^2} \quad (202)$$