

Exercise 12:

Foundations of Mathematical, WS24

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This is **exercise** 12 for Foundations of Mathematical, WS24. Generated on 2025-02-10 with 10 problems per section.

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1. Problems

1.1. Vector Arithmetic

1.1.1. Addition

Find the sum of the following vectors \mathbf{u} and \mathbf{v}

1. $\mathbf{u} = \begin{bmatrix} 8 \\ -10 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -10 \\ -10 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} 1 \\ 8 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 10 \\ -9 \\ 10 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \\ 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \\ -8 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 0 \\ -8 \\ 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ -9 \\ 7 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} -9 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -1 \\ 4 \\ -6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 7 \\ 1 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} 7 \\ -10 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -10 \\ 4 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} -7 \\ -10 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -7 \\ -7 \\ -3 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} 7 \\ 3 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ -6 \\ 0 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

1.1.2. Subtraction

Find the difference of the following vectors \mathbf{u} and \mathbf{v}

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 9 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -9 \\ -3 \\ 9 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} -7 \\ -4 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 10 \\ 4 \\ 3 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 5 \\ -4 \\ -8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ 6 \\ -8 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 10 \\ -6 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

6. $\mathbf{u} = \begin{bmatrix} -10 \\ -10 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -5 \\ -6 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 10 \\ 1 \\ -7 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -10 \\ 2 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} -9 \\ -2 \\ 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 0 \\ 7 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

1.1.3. Scalar Multiplication

Find the scalar product of the following vector \mathbf{u} and scalar k

1. $\mathbf{u} = \begin{bmatrix} -1 \\ -5 \\ 0 \end{bmatrix}$ $1\mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} 0 \\ -9 \\ 4 \end{bmatrix}$ $10\mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$ $-8\mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 9 \\ -10 \\ -8 \end{bmatrix}$ $-3\mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} -4 \\ 7 \\ -8 \end{bmatrix}$ $3\mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} 2 \\ -8 \\ -6 \end{bmatrix}$ $-6\mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} 9 \\ -3 \\ -9 \end{bmatrix}$ $0\mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} 5 \\ -5 \\ -10 \end{bmatrix}$ $-4\mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} -9 \\ -9 \\ -1 \end{bmatrix}$ $-8\mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} -5 \\ -10 \\ 10 \end{bmatrix}$ $-8\mathbf{v}$.

1.2. Matrix Arithmetic

1.2.1. Addition

Find the sum of the following matrices A and B

1.
$$A = \begin{bmatrix} 8 & -8 & 1 \\ 0 & -1 & -10 \\ 9 & -4 & -3 \end{bmatrix} \quad (1)$$

and

$$B = \begin{bmatrix} -4 & -2 & -1 \\ -6 & 2 & -6 \\ 8 & 1 & 0 \end{bmatrix} \quad (2)$$

2.
$$A = \begin{bmatrix} -6 & -10 & -3 \\ -5 & 6 & 2 \\ 6 & 8 & 0 \end{bmatrix} \quad (3)$$

and

$$B = \begin{bmatrix} 1 & 1 & -6 \\ -6 & 9 & -7 \\ 6 & 5 & 5 \end{bmatrix} \quad (4)$$

3.
$$A = \begin{bmatrix} -8 & 6 & 9 \\ 4 & 0 & 5 \\ 1 & 8 & 3 \end{bmatrix} \quad (5)$$

and

$$B = \begin{bmatrix} -10 & -6 & -8 \\ 0 & 5 & -5 \\ -4 & 0 & -2 \end{bmatrix} \quad (6)$$

4.
$$A = \begin{bmatrix} 8 & -7 & 1 \\ -2 & -10 & 0 \\ 2 & -2 & -6 \end{bmatrix} \quad (7)$$

and

$$B = \begin{bmatrix} 9 & -4 & 2 \\ 5 & -4 & 6 \\ 0 & 0 & 5 \end{bmatrix} \quad (8)$$

5.
$$A = \begin{bmatrix} -2 & 4 & -10 \\ -3 & 2 & 6 \\ -1 & -9 & 0 \end{bmatrix} \quad (9)$$

and

$$B = \begin{bmatrix} -8 & 1 & -5 \\ -2 & 5 & -10 \\ -1 & 7 & -3 \end{bmatrix} \quad (10)$$

6.
$$A = \begin{bmatrix} -9 & -3 & 1 \\ 0 & 8 & -3 \\ 8 & 5 & -5 \end{bmatrix} \quad (11)$$

and

$$B = \begin{bmatrix} 2 & -3 & -3 \\ 0 & 6 & -4 \\ 5 & -7 & -5 \end{bmatrix} \quad (12)$$

7.

$$A = \begin{bmatrix} 0 & -5 & -4 \\ 3 & -1 & 2 \\ 7 & 3 & -4 \end{bmatrix} \quad (13)$$

and

$$B = \begin{bmatrix} -1 & 3 & -9 \\ 4 & -2 & 5 \\ 1 & -5 & -7 \end{bmatrix} \quad (14)$$

8.

$$A = \begin{bmatrix} -6 & -8 & -6 \\ 7 & 0 & -2 \\ -5 & 8 & -7 \end{bmatrix} \quad (15)$$

and

$$B = \begin{bmatrix} 3 & 6 & -9 \\ 7 & 6 & -7 \\ 9 & 5 & 4 \end{bmatrix} \quad (16)$$

9.

$$A = \begin{bmatrix} -5 & 3 & -2 \\ 2 & -6 & -1 \\ 9 & -2 & -2 \end{bmatrix} \quad (17)$$

and

$$B = \begin{bmatrix} 7 & 8 & 5 \\ -10 & -5 & 5 \\ -10 & 0 & 7 \end{bmatrix} \quad (18)$$

10.

$$A = \begin{bmatrix} 6 & 1 & 4 \\ -9 & 6 & -2 \\ 8 & -4 & 1 \end{bmatrix} \quad (19)$$

and

$$B = \begin{bmatrix} 2 & 2 & 8 \\ 4 & -1 & 9 \\ -6 & 7 & -8 \end{bmatrix} \quad (20)$$

1.2.2. Subtraction

Find the difference of the following matrices A and B

1.
$$A = \begin{bmatrix} -6 & -5 & 4 \\ 9 & 3 & 4 \\ -2 & 5 & -8 \end{bmatrix} \quad (21)$$

and

$$B = \begin{bmatrix} 3 & -8 & 0 \\ 3 & -8 & 4 \\ 2 & -5 & 1 \end{bmatrix} \quad (22)$$

2.
$$A = \begin{bmatrix} -7 & 3 & -8 \\ -3 & 5 & -1 \\ 0 & -1 & -6 \end{bmatrix} \quad (23)$$

and

$$B = \begin{bmatrix} -5 & 2 & -9 \\ -6 & -7 & -1 \\ 6 & -10 & 8 \end{bmatrix} \quad (24)$$

3.
$$A = \begin{bmatrix} 9 & 0 & -6 \\ 8 & 2 & -1 \\ 8 & -4 & -8 \end{bmatrix} \quad (25)$$

and

$$B = \begin{bmatrix} -4 & 9 & -2 \\ -1 & -3 & 3 \\ -10 & -2 & -1 \end{bmatrix} \quad (26)$$

4.
$$A = \begin{bmatrix} 5 & -6 & -1 \\ -5 & 0 & 2 \\ -7 & 2 & -6 \end{bmatrix} \quad (27)$$

and

$$B = \begin{bmatrix} -5 & -4 & -2 \\ 3 & 7 & 9 \\ 4 & 5 & -10 \end{bmatrix} \quad (28)$$

5.
$$A = \begin{bmatrix} -8 & -6 & -9 \\ -5 & -7 & 0 \\ -8 & -10 & 8 \end{bmatrix} \quad (29)$$

and

$$B = \begin{bmatrix} 6 & -2 & 8 \\ -6 & 5 & -10 \\ -7 & 6 & 9 \end{bmatrix} \quad (30)$$

6.
$$A = \begin{bmatrix} 9 & 5 & 2 \\ 1 & 9 & -9 \\ 2 & -3 & -7 \end{bmatrix} \quad (31)$$

and

$$B = \begin{bmatrix} 5 & 3 & 0 \\ -4 & -1 & 7 \\ -7 & 5 & 6 \end{bmatrix} \quad (32)$$

7.

$$A = \begin{bmatrix} -9 & 6 & 9 \\ 9 & -5 & 1 \\ -8 & -6 & 9 \end{bmatrix} \quad (33)$$

and

$$B = \begin{bmatrix} -10 & -1 & -3 \\ -3 & -4 & -1 \\ 6 & -10 & -1 \end{bmatrix} \quad (34)$$

8.

$$A = \begin{bmatrix} 2 & -8 & 4 \\ 0 & -1 & 6 \\ -8 & 0 & -9 \end{bmatrix} \quad (35)$$

and

$$B = \begin{bmatrix} 9 & -9 & -5 \\ 9 & 8 & 2 \\ 8 & 6 & -7 \end{bmatrix} \quad (36)$$

9.

$$A = \begin{bmatrix} 3 & -1 & -8 \\ 3 & -5 & -1 \\ -6 & 5 & 0 \end{bmatrix} \quad (37)$$

and

$$B = \begin{bmatrix} -5 & -7 & -9 \\ 5 & -2 & 3 \\ 9 & 0 & -1 \end{bmatrix} \quad (38)$$

10.

$$A = \begin{bmatrix} 1 & -4 & 8 \\ 6 & -10 & -3 \\ -6 & 3 & -7 \end{bmatrix} \quad (39)$$

and

$$B = \begin{bmatrix} -2 & 2 & -8 \\ -5 & -9 & -7 \\ -1 & 0 & 2 \end{bmatrix} \quad (40)$$

1.2.3. Multiplication

Find the product of the following matrices A and B

$$1. \quad A = \begin{bmatrix} -4 & -9 & 4 \\ -2 & 5 & -9 \\ -8 & -10 & -6 \end{bmatrix} \quad (41)$$

and

$$B = \begin{bmatrix} 6 & -6 & 2 \\ -2 & -1 & 8 \\ 7 & -6 & -3 \end{bmatrix} \quad (42)$$

$$2. \quad A = \begin{bmatrix} -8 & 9 & -3 \\ -1 & -9 & 6 \\ -7 & -1 & 6 \end{bmatrix} \quad (43)$$

and

$$B = \begin{bmatrix} -2 & -3 & -7 \\ -7 & -1 & 1 \\ -3 & 0 & -4 \end{bmatrix} \quad (44)$$

$$3. \quad A = \begin{bmatrix} 0 & -4 & 8 \\ -4 & -4 & -8 \\ -5 & 2 & 5 \end{bmatrix} \quad (45)$$

and

$$B = \begin{bmatrix} 1 & 7 & -3 \\ 9 & 4 & 3 \\ -6 & -1 & 3 \end{bmatrix} \quad (46)$$

$$4. \quad A = \begin{bmatrix} -10 & 7 & 8 \\ -6 & -6 & 4 \\ -6 & 4 & 1 \end{bmatrix} \quad (47)$$

and

$$B = \begin{bmatrix} -4 & -2 & 2 \\ 3 & -1 & 7 \\ 0 & -10 & -6 \end{bmatrix} \quad (48)$$

$$5. \quad A = \begin{bmatrix} -1 & -3 & -5 \\ -3 & -1 & 5 \\ -7 & 6 & -10 \end{bmatrix} \quad (49)$$

and

$$B = \begin{bmatrix} -6 & -9 & 7 \\ 5 & 5 & 2 \\ 5 & -6 & -5 \end{bmatrix} \quad (50)$$

$$6. \quad A = \begin{bmatrix} 6 & -7 & -9 \\ -1 & 3 & 5 \\ -4 & -3 & 7 \end{bmatrix} \quad (51)$$

and

$$B = \begin{bmatrix} -4 & 3 & 1 \\ 7 & -4 & -6 \\ -1 & 1 & -1 \end{bmatrix} \quad (52)$$

7.

$$A = \begin{bmatrix} -7 & -7 & -8 \\ 0 & 4 & -10 \\ 2 & -10 & -9 \end{bmatrix} \quad (53)$$

and

$$B = \begin{bmatrix} -7 & 6 & 3 \\ -4 & -8 & -1 \\ -10 & 5 & 6 \end{bmatrix} \quad (54)$$

8.

$$A = \begin{bmatrix} 4 & 6 & -1 \\ -3 & 2 & -1 \\ -7 & 2 & -2 \end{bmatrix} \quad (55)$$

and

$$B = \begin{bmatrix} -4 & 6 & 2 \\ 9 & -10 & 4 \\ -10 & 1 & -7 \end{bmatrix} \quad (56)$$

9.

$$A = \begin{bmatrix} 4 & 6 & 2 \\ -7 & 4 & 7 \\ 3 & -10 & -5 \end{bmatrix} \quad (57)$$

and

$$B = \begin{bmatrix} -6 & 9 & 8 \\ 3 & 7 & 9 \\ 0 & -1 & 2 \end{bmatrix} \quad (58)$$

10.

$$A = \begin{bmatrix} -9 & -10 & 6 \\ -9 & -1 & 5 \\ 8 & 6 & 7 \end{bmatrix} \quad (59)$$

and

$$B = \begin{bmatrix} -6 & 4 & 5 \\ -10 & -1 & 8 \\ 1 & -8 & -1 \end{bmatrix} \quad (60)$$

1.3. Matrix Properties

1.3.1. Properties

For each matrix A , find:

a) $\text{rank}(A)$

- b) nullity(A)
- c) $\det(A)$
- d) A^{-1} (if exists)
- e) basis of $\ker(A)$

1.
$$A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad (61)$$

2.
$$A = \begin{bmatrix} 1 & 2 & -4 \\ 0 & 3 & 5 \\ 0 & 4 & 7 \end{bmatrix} \quad (62)$$

3.
$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -2 & 1 \end{bmatrix} \quad (63)$$

4.
$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix} \quad (64)$$

5.
$$A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 1 \\ -2 & 3 & 2 \end{bmatrix} \quad (65)$$

6.
$$A = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 2 \\ -7 & 2 & 5 \end{bmatrix} \quad (66)$$

7.
$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 0 & -1 & 2 \end{bmatrix} \quad (67)$$

8.
$$A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix} \quad (68)$$

9.
$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ -2 & 2 & 3 \end{bmatrix} \quad (69)$$

10.
$$A = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 1 & -1 \\ 0 & 2 & 5 \end{bmatrix} \quad (70)$$

1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix A

1.
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (71)$$

$$2. \quad A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \quad (72)$$

$$3. \quad A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 2 & -4 & 5 \end{bmatrix} \quad (73)$$

$$4. \quad A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 2 & -1 \end{bmatrix} \quad (74)$$

$$5. \quad A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad (75)$$

$$6. \quad A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \quad (76)$$

$$7. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -6 & -3 & 0 \end{bmatrix} \quad (77)$$

$$8. \quad A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (78)$$

$$9. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (79)$$

$$10. \quad A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (80)$$

1.4. Calculus

1.4.1. Limit

Calculate the following limits

1. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} -3x^2 + 2x - 5 \quad (81)$$

2. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x+1)}{x} \quad (82)$$

3. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} -x^3 + 2x^2 + 4x + 2 \quad (83)$$

4. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (84)$$

5. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (85)$$

6. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (86)$$

7. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (87)$$

8. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 0} \frac{\log(x + 1)}{x} \quad (88)$$

9. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (89)$$

10. Calculate the limit of the following expression:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad (90)$$

1.4.2. Derivative

Calculate the derivatives of the following expressions

1. Calculate the derivative of the following expression:

$$\frac{x}{x^2 + 1} \quad (91)$$

2. Calculate the derivative of the following expression:

$$x^2 e^x \quad (92)$$

3. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (93)$$

4. Calculate the derivative of the following expression:

$$\frac{x^2}{x^2 + 1} \quad (94)$$

5. Calculate the derivative of the following expression:

$$\log(x + 1) + \log(x^2 + 1) \quad (95)$$

6. Calculate the derivative of the following expression:

$$x^2 e^x \quad (96)$$

7. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \quad (97)$$

8. Calculate the derivative of the following expression:

$$\log(x^2 + 2) \quad (98)$$

9. Calculate the derivative of the following expression:

$$\frac{x^3}{x^2 + 1} \quad (99)$$

10. Calculate the derivative of the following expression:

$$\frac{x}{x^2 + 1} \quad (100)$$

1.4.3. Integral

Calculate the indefinite and definite integrals of the following expressions

1. Evaluate the improper integral:

$$\int_1^{\infty} \frac{1}{x^2} dx \quad (101)$$

2. Evaluate the improper integral:

$$\int_1^{\infty} e^{-x} dx \quad (102)$$

3. the indefinite integral and evaluate from 2 to 4:

$$\int \sqrt{4 - x^2} dx \quad (103)$$

4. Evaluate the improper integral:

$$\int_1^{\infty} \frac{1}{x^2} dx \quad (104)$$

5. the indefinite integral and evaluate from 1 to 5:

$$\int e^{-x^2} dx \quad (105)$$

6. the indefinite integral and evaluate from 1 to 3:

$$\int -3x^2 - 2x + 1 dx \quad (106)$$

7. Evaluate the improper integral:

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx \quad (107)$$

8. the indefinite integral and evaluate from 2 to 2:

$$\int \frac{1}{\sqrt{1-x^2}} dx \quad (108)$$

9. the indefinite integral and evaluate from 1 to 2:

$$\int \frac{3x+2}{x^2-4} dx \quad (109)$$

10. the indefinite integral and evaluate from 3 to 5:

$$\int 5x^3 - 4x^2 dx \quad (110)$$

1.4.4. Partial Derivative

Calculate the partial derivatives of the following expressions

1. the third order partial derivative of:

$$f(x, y) = x^4 y^3 + 3x^2 y^4 \quad (111)$$

$$\frac{\partial^3 f}{\partial y^3}$$

2. the partial derivatives of the function:

$$f(x, y) = x^3 y^2 - 3x^2 y + 2xy^3 \quad (112)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

3. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \quad (113)$$

where $f = f(u, v)$

4. Given the implicit function:

$$x^2 y + xy^2 - xy = 0 \quad (114)$$

$$\frac{\partial y}{\partial x}$$

5. the mixed partial derivative of:

$$f(x, y) = x^3y^2 + xy^4 \quad (115)$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

6. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \quad (116)$$

where $f = f(u, v)$

7. the partial derivatives of the function:

$$f(x, y) = -\log(xy) + \log(x^3 + y^3) \quad (117)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

8. the partial derivatives of the function:

$$f(x, y) = -\log(xy) + \log(x^3 + y^3) \quad (118)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

9. the third order partial derivative of:

$$f(x, y) = x^4y^3 + 3x^2y^4 \quad (119)$$

$$\frac{\partial^3 f}{\partial y^3}$$

10. the third order partial derivative of:

$$f(x, y) = x^4y^3 + 3x^2y^4 \quad (120)$$

$$\frac{\partial^3 f}{\partial y^3}$$

2. Solutions

2.1. Vector Arithmetic

2.1.1. Addition

$$\begin{bmatrix} 10 \\ -20 \\ -4 \end{bmatrix} + \begin{bmatrix} 11 \\ -1 \\ 9 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 5 \\ -17 \\ 16 \end{bmatrix} + \begin{bmatrix} 10 \\ 6 \\ -12 \end{bmatrix} = \begin{bmatrix} -13 \\ 2 \\ 9 \end{bmatrix}$$
$$\begin{bmatrix} -5 \\ 11 \\ -5 \end{bmatrix} + \begin{bmatrix} 5 \\ -20 \\ 8 \end{bmatrix} = \begin{bmatrix} -14 \\ -17 \\ -10 \end{bmatrix} \quad \begin{bmatrix} 12 \\ -3 \\ -1 \end{bmatrix}$$

2.1.2. Subtraction

$$\begin{bmatrix} 8 \\ 12 \\ -13 \end{bmatrix} - \begin{bmatrix} -17 \\ -8 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -10 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 10 \\ -10 \\ 0 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} -16 \\ -10 \\ -1 \end{bmatrix} - \begin{bmatrix} -15 \\ -7 \\ 13 \end{bmatrix} = \begin{bmatrix} -19 \\ -7 \\ 9 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 3 \\ -11 \end{bmatrix} - \begin{bmatrix} -5 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \\ -13 \end{bmatrix}$$

2.1.3. Scalar Multiplication

$$1: \begin{bmatrix} -1 \\ -5 \\ 0 \end{bmatrix} \quad 2: \begin{bmatrix} 0 \\ -90 \\ 40 \end{bmatrix} \quad 3: \begin{bmatrix} 8 \\ 24 \\ -32 \end{bmatrix} \quad 4: \begin{bmatrix} -27 \\ 30 \\ 24 \end{bmatrix} \quad 5: \begin{bmatrix} -12 \\ 21 \\ -24 \end{bmatrix}$$
$$6: \begin{bmatrix} -12 \\ 48 \\ 36 \end{bmatrix} \quad 7: \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad 8: \begin{bmatrix} -20 \\ 20 \\ 40 \end{bmatrix} \quad 9: \begin{bmatrix} 72 \\ 72 \\ 8 \end{bmatrix} \quad 10: \begin{bmatrix} 40 \\ 80 \\ -80 \end{bmatrix}$$

2.2. Matrix Arithmetic

2.2.1. Addition

1:

$$\begin{bmatrix} 4 & -10 & 0 \\ -6 & 1 & -16 \\ 17 & -3 & -3 \end{bmatrix} \quad (121)$$

1:

$$\begin{bmatrix} -5 & -9 & -9 \\ -11 & 15 & -5 \\ 12 & 13 & 5 \end{bmatrix} \quad (122)$$

1:

$$\begin{bmatrix} -18 & 0 & 1 \\ 4 & 5 & 0 \\ -3 & 8 & 1 \end{bmatrix} \quad (123)$$

1:

$$\begin{bmatrix} 17 & -11 & 3 \\ 3 & -14 & 6 \\ 2 & -2 & -1 \end{bmatrix} \quad (124)$$

1:

$$\begin{bmatrix} -10 & 5 & -15 \\ -5 & 7 & -4 \\ -2 & -2 & -3 \end{bmatrix} \quad (125)$$

1:

$$\begin{bmatrix} -7 & -6 & -2 \\ 0 & 14 & -7 \\ 13 & -2 & -10 \end{bmatrix} \quad (126)$$

1:

$$\begin{bmatrix} -1 & -2 & -13 \\ 7 & -3 & 7 \\ 8 & -2 & -11 \end{bmatrix} \quad (127)$$

1:

$$\begin{bmatrix} -3 & -2 & -15 \\ 14 & 6 & -9 \\ 4 & 13 & -3 \end{bmatrix} \quad (128)$$

1:

$$\begin{bmatrix} 2 & 11 & 3 \\ -8 & -11 & 4 \\ -1 & -2 & 5 \end{bmatrix} \quad (129)$$

1:

$$\begin{bmatrix} 8 & 3 & 12 \\ -5 & 5 & 7 \\ 2 & 3 & -7 \end{bmatrix} \quad (130)$$

2.2.2. Subtraction

1:

$$\begin{bmatrix} -9 & 3 & 4 \\ 6 & 11 & 0 \\ -4 & 10 & -9 \end{bmatrix} \quad (131)$$

1:

$$\begin{bmatrix} -2 & 1 & 1 \\ 3 & 12 & 0 \\ -6 & 9 & -14 \end{bmatrix} \quad (132)$$

1:

$$\begin{bmatrix} 13 & -9 & -4 \\ 9 & 5 & -4 \\ 18 & -2 & -7 \end{bmatrix} \quad (133)$$

1:

$$\begin{bmatrix} 10 & -2 & 1 \\ -8 & -7 & -7 \\ -11 & -3 & 4 \end{bmatrix} \quad (134)$$

1:

$$\begin{bmatrix} -14 & -4 & -17 \\ 1 & -12 & 10 \\ -1 & -16 & -1 \end{bmatrix} \quad (135)$$

1:

$$\begin{bmatrix} 4 & 2 & 2 \\ 5 & 10 & -16 \\ 9 & -8 & -13 \end{bmatrix} \quad (136)$$

1:

$$\begin{bmatrix} 1 & 7 & 12 \\ 12 & -1 & 2 \\ -14 & 4 & 10 \end{bmatrix} \quad (137)$$

1:

$$\begin{bmatrix} -7 & 1 & 9 \\ -9 & -9 & 4 \\ -16 & -6 & -2 \end{bmatrix} \quad (138)$$

1:

$$\begin{bmatrix} 8 & 6 & 1 \\ -2 & -3 & -4 \\ -15 & 5 & 1 \end{bmatrix} \quad (139)$$

1:

$$\begin{bmatrix} 3 & -6 & 16 \\ 11 & -1 & 4 \\ -5 & 3 & -9 \end{bmatrix} \quad (140)$$

2.2.3. Multiplication

1:

$$\begin{bmatrix} 22 & 9 & -92 \\ -85 & 61 & 63 \\ -70 & 94 & -78 \end{bmatrix} \quad (141)$$

1:

$$\begin{bmatrix} -38 & 15 & 77 \\ 47 & 12 & -26 \\ 3 & 22 & 24 \end{bmatrix} \quad (142)$$

1:

$$\begin{bmatrix} -84 & -24 & 12 \\ 8 & -36 & -24 \\ -17 & -32 & 36 \end{bmatrix} \quad (143)$$

1:

$$\begin{bmatrix} 61 & -67 & -19 \\ 6 & -22 & -78 \\ 36 & -2 & 10 \end{bmatrix} \quad (144)$$

1:

$$\begin{bmatrix} -34 & 24 & 12 \\ 38 & -8 & -48 \\ 22 & 153 & 13 \end{bmatrix} \quad (145)$$

1:

$$\begin{bmatrix} -64 & 37 & 57 \\ 20 & -10 & -24 \\ -12 & 7 & 7 \end{bmatrix} \quad (146)$$

1:

$$\begin{bmatrix} 157 & -26 & -62 \\ 84 & -82 & -64 \\ 116 & 47 & -38 \end{bmatrix} \quad (147)$$

1:

$$\begin{bmatrix} 48 & -37 & 39 \\ 40 & -39 & 9 \\ 66 & -64 & 8 \end{bmatrix} \quad (148)$$

1:

$$\begin{bmatrix} -6 & 76 & 90 \\ 54 & -42 & -6 \\ -48 & -38 & -76 \end{bmatrix} \quad (149)$$

1:

$$\begin{bmatrix} 160 & -74 & -131 \\ 69 & -75 & -58 \\ -101 & -30 & 81 \end{bmatrix} \quad (150)$$

2.3. Matrix Properties

2.3.1. Properties

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-1)r_1 \quad \begin{bmatrix} 1 & -2 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := r_1 - (-2)r_2 \quad \begin{bmatrix} 1 & 0 & 0 & | & 3 & 2 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := 1/3r_2 \quad \begin{bmatrix} 1 & 2 & -4 & | & 1 & 0 & 0 \\ 0 & 1 & 5/3 & | & 0 & 1/3 & 0 \\ 0 & 4 & 7 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := r_1 - (2)r_2 \quad \begin{bmatrix} 1 & 0 & -22/3 & | & 1 & -2/3 & 0 \\ 0 & 1 & 5/3 & | & 0 & 1/3 & 0 \\ 0 & 4 & 7 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_3 := r_3 - (4)r_2 \quad \begin{bmatrix} 1 & 0 & -22/3 & | & 1 & -2/3 & 0 \\ 0 & 1 & 5/3 & | & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & | & 0 & -4/3 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_3 := 3r_3 \quad \begin{bmatrix} 1 & 0 & -22/3 & | & 1 & -2/3 & 0 \\ 0 & 1 & 5/3 & | & 0 & 1/3 & 0 \\ 0 & 0 & 1 & | & 0 & -4 & 3 \end{bmatrix}$$

$$\text{Step 5: } r_1 := r_1 - (-22/3)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & 1 & -30 & 22 \\ 0 & 1 & 5/3 & | & 0 & 1/3 & 0 \\ 0 & 0 & 1 & | & 0 & -4 & 3 \end{bmatrix}$$

$$\text{Step 6: } r_2 := r_2 - (5/3)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & 1 & -30 & 22 \\ 0 & 1 & 0 & | & 0 & 7 & -5 \\ 0 & 0 & 1 & | & 0 & -4 & 3 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & -8 & 6 \\ 0 & -1 & 1 \\ 0 & -4 & 3 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_1 \leftrightarrow r_2$ $\begin{bmatrix} -1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \\ 1 & -2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_1 := -1r_1$ $\begin{bmatrix} 1 & -2 & 1 & | & 0 & -1 & 0 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \\ 1 & -2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 3: $r_3 := r_3 - r_1$ $\begin{bmatrix} 1 & -2 & 1 & | & 0 & -1 & 0 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 1 & 1 \end{bmatrix}$

Step 4: $r_1 := r_1 - (-2)r_2$ $\begin{bmatrix} 1 & 0 & -1 & | & 2 & -1 & 0 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 1 & 1 \end{bmatrix}$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

Step 1: $r_2 := r_2 - (-1)r_1$ $\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 1 & 1 & 2 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_3 := r_3 - r_1$ $\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$

Step 3: $r_1 := r_1 - r_2$ $\begin{bmatrix} 1 & 0 & -2 & | & 0 & -1 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$

Step 4: $r_1 := r_1 - (-2)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & -2 & -1 & 2 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$

$$\text{Step 5: } r_2 := r_2 - (3)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 1 & 0 & 4 & 1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-1)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ -2 & 3 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_3 := r_3 - (-2)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 3 & -2 & 2 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_3 := r_3 - (3)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -3 & 1 \end{array} \right]$$

$$\text{Step 4: } r_1 := r_1 - (-2)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -6 & 2 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -3 & 1 \end{array} \right]$$

$$\text{Step 5: } r_2 := r_2 - (-1)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -6 & 2 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 & -3 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-3)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ -7 & 2 & 5 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_3 := r_3 - (-7)r_1 \begin{bmatrix} 1 & 0 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 3 & 1 & 0 \\ 0 & 2 & -2 & | & 7 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_3 := r_3 - (2)r_2 \begin{bmatrix} 1 & 0 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 3 & 1 & 0 \\ 0 & 0 & 0 & | & 1 & -2 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-2)r_1 \begin{bmatrix} 1 & -2 & 3 & | & 1 & 0 & 0 \\ 0 & -1 & 2 & | & 2 & 1 & 0 \\ 0 & -1 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_2 := -1r_2 \begin{bmatrix} 1 & -2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & -2 & | & -2 & -1 & 0 \\ 0 & -1 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (-2)r_2 \begin{bmatrix} 1 & 0 & -1 & | & -3 & -2 & 0 \\ 0 & 1 & -2 & | & -2 & -1 & 0 \\ 0 & -1 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_3 := r_3 - (-1)r_2 \begin{bmatrix} 1 & 0 & -1 & | & -3 & -2 & 0 \\ 0 & 1 & -2 & | & -2 & -1 & 0 \\ 0 & 0 & 0 & | & -2 & -1 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_3 := r_3 - r_1 \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := r_1 - (-2)r_2 \quad \begin{bmatrix} 1 & 0 & -3 & | & 1 & 2 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (-3)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & -2 & 2 & 3 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_2 := r_2 - (-1)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & -2 & 2 & 3 \\ 0 & 1 & 0 & | & -1 & 1 & 1 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-1)r_1 \quad \begin{bmatrix} 1 & -1 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 1 & 1 & 0 \\ -2 & 2 & 3 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_3 := r_3 - (-2)r_1 \quad \begin{bmatrix} 1 & -1 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (-1)r_2 \quad \begin{bmatrix} 1 & 0 & -2 & | & 2 & 1 & 0 \\ 0 & 1 & -1 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_1 := r_1 - (-2)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & 6 & 1 & 2 \\ 0 & 1 & -1 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 5: } r_2 := r_2 - (-1)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & 6 & 1 & 2 \\ 0 & 1 & 0 & | & 3 & 1 & 1 \\ 0 & 0 & 1 & | & 2 & 0 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-2)r_1 \begin{bmatrix} 1 & -1 & -1 & | & 1 & 0 & 0 \\ 0 & -1 & -3 & | & 2 & 1 & 0 \\ 0 & 2 & 5 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_2 := -1r_2 \begin{bmatrix} 1 & -1 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & -2 & -1 & 0 \\ 0 & 2 & 5 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (-1)r_2 \begin{bmatrix} 1 & 0 & 2 & | & -1 & -1 & 0 \\ 0 & 1 & 3 & | & -2 & -1 & 0 \\ 0 & 2 & 5 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_3 := r_3 - (2)r_2 \begin{bmatrix} 1 & 0 & 2 & | & -1 & -1 & 0 \\ 0 & 1 & 3 & | & -2 & -1 & 0 \\ 0 & 0 & -1 & | & 4 & 2 & 1 \end{bmatrix}$$

$$\text{Step 5: } r_3 := -1r_3 \begin{bmatrix} 1 & 0 & 2 & | & -1 & -1 & 0 \\ 0 & 1 & 3 & | & -2 & -1 & 0 \\ 0 & 0 & 1 & | & -4 & -2 & -1 \end{bmatrix}$$

$$\text{Step 6: } r_1 := r_1 - (2)r_3 \begin{bmatrix} 1 & 0 & 0 & | & 7 & 3 & 2 \\ 0 & 1 & 3 & | & -2 & -1 & 0 \\ 0 & 0 & 1 & | & -4 & -2 & -1 \end{bmatrix}$$

$$\text{Step 7: } r_2 := r_2 - (3)r_3 \begin{bmatrix} 1 & 0 & 0 & | & 7 & 3 & 2 \\ 0 & 1 & 0 & | & 10 & 5 & 3 \\ 0 & 0 & 1 & | & -4 & -2 & -1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} -7 & -4 & -2 \\ 2 & 1 & 1 \\ -4 & -2 & -1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

2.3.2. RREF**Solution****Elementary Row Operations:**

(1) $r_3 := r_3 + (-2)r_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

(2) $r_3 := r_3 - r_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 + (-2)r_2$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_2$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_1 := r_1 - r_2$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 + (-2)r_1$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$$

$$(2) \ r_3 := r_3 - (-2)r_2$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_1 := r_1 - r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_2 := r_2 - r_1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_2 := r_2 - r_3$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 + (-1)r_3$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-1)r_2$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$(3) \ r_3 := r_3 + (-2)r_2$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 - r_1$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-1)r_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_1 := r_1 - r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

$$(2) \ r_3 := r_3 - r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_2 := r_2 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 - r_3$$

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_2 := r_2 - r_3$$

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_2 := r_2 - r_1$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 + (-1)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 + (-1)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution**Elementary Row Operations:**

$$(1) \ r_1 := r_1 - (2)r_3$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-2)r_3$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.4. Calculus**2.4.1. Limit**

The limit is:

$$-5 \qquad (151)$$

The limit is:

$$1 \qquad (152)$$

The limit is:

$$7 \qquad (153)$$

The limit is:

$$2 \qquad (154)$$

The limit is:

$$2 \quad (155)$$

The limit is:

$$2 \quad (156)$$

The limit is:

$$2 \quad (157)$$

The limit is:

$$1 \quad (158)$$

The limit is:

$$2 \quad (159)$$

The limit is:

$$2 \quad (160)$$

2.4.2. Derivative

The derivative is:

$$-\frac{2x^2}{(x^2 + 1)^2} + \frac{1}{x^2 + 1} \quad (161)$$

The derivative is:

$$x^2 e^x + 2x e^x \quad (162)$$

The derivative is:

$$2x e^{x^2} + 2e^{2x} \quad (163)$$

The derivative is:

$$-\frac{2x^3}{(x^2 + 1)^2} + \frac{2x}{x^2 + 1} \quad (164)$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \quad (165)$$

The derivative is:

$$x^2 e^x + 2x e^x \quad (166)$$

The derivative is:

$$2x e^{x^2} + 2e^{2x} \quad (167)$$

The derivative is:

$$\frac{2x}{x^2 + 2} \quad (168)$$

The derivative is:

$$-\frac{2x^4}{(x^2 + 1)^2} + \frac{3x^2}{x^2 + 1} \quad (169)$$

The derivative is:

$$-\frac{2x^2}{(x^2 + 1)^2} + \frac{1}{x^2 + 1} \quad (170)$$

2.4.3. Integral

The improper integral converges to:

$$1 \quad (171)$$

The improper integral converges to:

$$e^{-1} \quad (172)$$

The indefinite integral is:

$$\frac{x\sqrt{4-x^2}}{2} + 2 \operatorname{asin} \left(\frac{x}{2} \right) \quad (173)$$

Definite integral from 2 to 4:

$$-\pi + 2 \operatorname{asin} (2) + 4\sqrt{3}i \quad (174)$$

The improper integral converges to:

$$1 \quad (175)$$

The indefinite integral is:

$$\frac{\sqrt{\pi} \operatorname{erf} (x)}{2} \quad (176)$$

Definite integral from 1 to 5:

$$-\frac{\sqrt{\pi} \operatorname{erf} (1)}{2} + \frac{\sqrt{\pi} \operatorname{erf} (5)}{2} \quad (177)$$

The indefinite integral is:

$$-x^3 - x^2 + x \quad (178)$$

Definite integral from 1 to 3:

$$-32 \quad (179)$$

The improper integral converges to:

$$\infty \quad (180)$$

The indefinite integral is:

$$\arcsin(x) \quad (181)$$

Definite integral from 2 to 2:

$$0 \quad (182)$$

The indefinite integral is:

$$2 \log(x-2) + \log(x+2) \quad (183)$$

Definite integral from 1 to 2:

$$-\infty - 2i\pi \quad (184)$$

The indefinite integral is:

$$\frac{5x^4}{4} - \frac{4x^3}{3} \quad (185)$$

Definite integral from 3 to 5:

$$\frac{1648}{3} \quad (186)$$

2.4.4. Partial Derivative

$$\frac{\partial^3 f}{\partial y^3} = 6x^2(x^2 + 12y) \quad (187)$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \quad (188)$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \quad (189)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \quad (190)$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \quad (191)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y(3x^2 + 2y^2) \quad (192)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \quad (193)$$

$$\frac{\partial f}{\partial x} = \frac{3x^2}{x^3 + y^3} - \frac{1}{x} \quad (194)$$

$$\frac{\partial f}{\partial y} = \frac{3y^2}{x^3 + y^3} - \frac{1}{y} \quad (195)$$

$$\frac{\partial f}{\partial x} = \frac{3x^2}{x^3 + y^3} - \frac{1}{x} \quad (196)$$

$$\frac{\partial f}{\partial y} = \frac{3y^2}{x^3 + y^3} - \frac{1}{y} \quad (197)$$

$$\frac{\partial^3 f}{\partial y^3} = 6x^2(x^2 + 12y) \quad (198)$$

$$\frac{\partial^3 f}{\partial y^3} = 6x^2(x^2 + 12y) \quad (199)$$