

# Exercise 7:

## Foundations of Mathematical, WS24

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This is **exercise** 7 for Foundations of Mathematical, WS24. Generated on 2025-01-06 with 10 problems per section.

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# 1. Problems

## 1.1. Vector Arithmetic

### 1.1.1. Addition

Find the sum of the following vectors  $\mathbf{u}$  and  $\mathbf{v}$

1.  $\mathbf{u} = \begin{bmatrix} -9 \\ -10 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 7 \\ 1 \\ -8 \end{bmatrix}$
2.  $\mathbf{u} = \begin{bmatrix} 9 \\ 6 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -9 \\ 10 \\ -5 \end{bmatrix}$
3.  $\mathbf{u} = \begin{bmatrix} 7 \\ 4 \\ 10 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}$
4.  $\mathbf{u} = \begin{bmatrix} 5 \\ 8 \\ -8 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix}$
5.  $\mathbf{u} = \begin{bmatrix} 6 \\ 6 \\ -8 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix}$
6.  $\mathbf{u} = \begin{bmatrix} 2 \\ -2 \\ 9 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$
7.  $\mathbf{u} = \begin{bmatrix} -1 \\ -4 \\ 9 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 7 \\ 10 \\ 1 \end{bmatrix}$
8.  $\mathbf{u} = \begin{bmatrix} -10 \\ 10 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -7 \\ -5 \\ -1 \end{bmatrix}$
9.  $\mathbf{u} = \begin{bmatrix} -5 \\ 4 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ 8 \\ 5 \end{bmatrix}$
10.  $\mathbf{u} = \begin{bmatrix} 7 \\ -1 \\ 9 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 5 \\ -9 \\ -10 \end{bmatrix}$

### 1.1.2. Subtraction

Find the difference of the following vectors  $\mathbf{u}$  and  $\mathbf{v}$

1.  $\mathbf{u} = \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 4 \\ -4 \end{bmatrix}$
2.  $\mathbf{u} = \begin{bmatrix} -7 \\ -10 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix}$
3.  $\mathbf{u} = \begin{bmatrix} -6 \\ 1 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -8 \\ -10 \\ 6 \end{bmatrix}$
4.  $\mathbf{u} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 10 \\ -4 \\ -5 \end{bmatrix}$
5.  $\mathbf{u} = \begin{bmatrix} 6 \\ -9 \\ -8 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -5 \\ -3 \\ -10 \end{bmatrix}$

6.  $\mathbf{u} = \begin{bmatrix} 9 \\ 7 \\ 10 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$
7.  $\mathbf{u} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 8 \\ -4 \end{bmatrix}$
8.  $\mathbf{u} = \begin{bmatrix} -1 \\ 5 \\ -6 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 5 \\ -2 \\ 7 \end{bmatrix}$
9.  $\mathbf{u} = \begin{bmatrix} -9 \\ -9 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -7 \\ 6 \\ -10 \end{bmatrix}$
10.  $\mathbf{u} = \begin{bmatrix} -1 \\ -7 \\ -6 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -9 \\ -8 \\ -10 \end{bmatrix}$

### 1.1.3. Scalar Multiplication

Find the scalar product of the following vector  $\mathbf{u}$  and scalar  $k$

1.  $\mathbf{u} = \begin{bmatrix} -7 \\ 4 \\ -6 \end{bmatrix} \quad 3\mathbf{v}.$
2.  $\mathbf{u} = \begin{bmatrix} -3 \\ -9 \\ -9 \end{bmatrix} \quad -2\mathbf{v}.$
3.  $\mathbf{u} = \begin{bmatrix} 8 \\ 7 \\ -8 \end{bmatrix} \quad -7\mathbf{v}.$
4.  $\mathbf{u} = \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix} \quad -2\mathbf{v}.$
5.  $\mathbf{u} = \begin{bmatrix} -10 \\ -1 \\ 10 \end{bmatrix} \quad -6\mathbf{v}.$
6.  $\mathbf{u} = \begin{bmatrix} 7 \\ -1 \\ -5 \end{bmatrix} \quad -6\mathbf{v}.$
7.  $\mathbf{u} = \begin{bmatrix} -4 \\ -5 \\ -2 \end{bmatrix} \quad -8\mathbf{v}.$
8.  $\mathbf{u} = \begin{bmatrix} -1 \\ -6 \\ -8 \end{bmatrix} \quad -7\mathbf{v}.$
9.  $\mathbf{u} = \begin{bmatrix} -10 \\ -4 \\ -3 \end{bmatrix} \quad -5\mathbf{v}.$
10.  $\mathbf{u} = \begin{bmatrix} -4 \\ -2 \\ -9 \end{bmatrix} \quad -4\mathbf{v}.$

## 1.2. Matrix Arithmetic

### 1.2.1. Addition

Find the sum of the following matrices  $A$  and  $B$

1.  $A = \begin{bmatrix} -3 & -4 & 0 \\ 5 & 6 & -5 \\ 6 & 0 & -4 \end{bmatrix}$  and  $B = \begin{bmatrix} -6 & -2 & 8 \\ 1 & 5 & -10 \\ 9 & -10 & 8 \end{bmatrix}$

2.  $A = \begin{bmatrix} -5 & -3 & 3 \\ -9 & 8 & -10 \\ 1 & 0 & -10 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & -1 & 7 \\ 7 & 2 & -7 \\ -8 & 1 & -8 \end{bmatrix}$
3.  $A = \begin{bmatrix} 2 & -6 & -6 \\ -6 & -6 & -4 \\ -8 & 6 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & -9 & 9 \\ -3 & 1 & 2 \\ 8 & -8 & 1 \end{bmatrix}$
4.  $A = \begin{bmatrix} 6 & 5 & -4 \\ 8 & -8 & -2 \\ -9 & 0 & -4 \end{bmatrix}$  and  $B = \begin{bmatrix} 8 & -2 & 4 \\ 3 & -8 & -9 \\ -10 & 5 & -6 \end{bmatrix}$
5.  $A = \begin{bmatrix} -2 & -2 & 5 \\ 8 & -2 & -7 \\ -10 & -8 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} -4 & -1 & -2 \\ 7 & 5 & -9 \\ -5 & -7 & 3 \end{bmatrix}$
6.  $A = \begin{bmatrix} -10 & 8 & 3 \\ -6 & -9 & -6 \\ -3 & -5 & -6 \end{bmatrix}$  and  $B = \begin{bmatrix} 9 & 0 & 6 \\ -9 & 2 & 8 \\ 0 & -2 & -3 \end{bmatrix}$
7.  $A = \begin{bmatrix} 7 & -10 & -3 \\ 5 & 9 & 8 \\ 6 & -1 & -7 \end{bmatrix}$  and  $B = \begin{bmatrix} -4 & -7 & 6 \\ -7 & -9 & -1 \\ -4 & 7 & -1 \end{bmatrix}$
8.  $A = \begin{bmatrix} -2 & 9 & -1 \\ 1 & 4 & -9 \\ -5 & -10 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} -10 & 5 & 5 \\ 5 & 9 & 4 \\ 7 & 2 & -4 \end{bmatrix}$
9.  $A = \begin{bmatrix} 3 & -6 & -9 \\ -4 & -2 & -3 \\ -7 & -1 & -5 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & -10 & 3 \\ -2 & 3 & -5 \\ -10 & 2 & 6 \end{bmatrix}$
10.  $A = \begin{bmatrix} -6 & -9 & 9 \\ 8 & 3 & 6 \\ 9 & -10 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} -10 & -4 & 0 \\ 6 & -7 & 3 \\ -8 & 4 & -6 \end{bmatrix}$

### 1.2.2. Subtraction

Find the difference of the following matrices  $A$  and  $B$

1.  $A = \begin{bmatrix} 8 & -5 & 9 \\ 9 & 3 & 0 \\ -1 & 1 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} 7 & 9 & -9 \\ -2 & 5 & 7 \\ 7 & 3 & -10 \end{bmatrix}$
2.  $A = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 1 & -6 \\ 9 & -9 & -9 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & -7 & 4 \\ 9 & -7 & 2 \\ 4 & 4 & 7 \end{bmatrix}$
3.  $A = \begin{bmatrix} -3 & -4 & -7 \\ -8 & -4 & 2 \\ -7 & -8 & -7 \end{bmatrix}$  and  $B = \begin{bmatrix} -8 & 6 & -8 \\ 7 & -5 & -1 \\ 8 & 9 & 5 \end{bmatrix}$
4.  $A = \begin{bmatrix} 3 & 6 & 7 \\ 8 & -6 & -4 \\ -8 & -7 & -10 \end{bmatrix}$  and  $B = \begin{bmatrix} 6 & 6 & 4 \\ 7 & -2 & -6 \\ 8 & 3 & -4 \end{bmatrix}$
5.  $A = \begin{bmatrix} -2 & 8 & -6 \\ 5 & 1 & 1 \\ 4 & -3 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} -4 & 7 & 9 \\ 2 & -7 & -8 \\ -6 & 0 & -8 \end{bmatrix}$
6.  $A = \begin{bmatrix} 4 & 8 & -4 \\ 5 & 5 & 7 \\ 5 & 8 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} -9 & -3 & 6 \\ -8 & -7 & -7 \\ 8 & 8 & -6 \end{bmatrix}$
7.  $A = \begin{bmatrix} -3 & 3 & 8 \\ 1 & 7 & -3 \\ 1 & -10 & -8 \end{bmatrix}$  and  $B = \begin{bmatrix} 9 & 6 & -1 \\ 5 & 8 & -1 \\ 8 & 2 & -7 \end{bmatrix}$
8.  $A = \begin{bmatrix} -7 & -10 & 7 \\ 3 & 0 & 4 \\ -9 & 2 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} -4 & 5 & 1 \\ -1 & 5 & 0 \\ 5 & -3 & 3 \end{bmatrix}$

$$9. A = \begin{bmatrix} 8 & 5 & 3 \\ -1 & 3 & 3 \\ 2 & -3 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} -7 & -7 & -8 \\ 1 & 2 & -10 \\ 0 & -2 & -6 \end{bmatrix}$$

$$10. A = \begin{bmatrix} 8 & -4 & -8 \\ -2 & -4 & 1 \\ 9 & -10 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -10 & 5 & -7 \\ -8 & 5 & 2 \\ 4 & 8 & 8 \end{bmatrix}$$

### 1.2.3. Multiplication

Find the product of the following matrices  $A$  and  $B$

$$1. A = \begin{bmatrix} -3 & -10 & 5 \\ -2 & 2 & 9 \\ -1 & 5 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -4 & -10 & -4 \\ 8 & -7 & 1 \\ -5 & 9 & 7 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 0 & -2 & -3 \\ -4 & 7 & 0 \\ 0 & 5 & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -4 & -5 & 5 \\ -6 & 4 & 3 \\ -9 & -5 & -3 \end{bmatrix}$$

$$3. A = \begin{bmatrix} 1 & -5 & -5 \\ -9 & -6 & 2 \\ 1 & 4 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 2 & -6 \\ -3 & -1 & 0 \\ 0 & -3 & 9 \end{bmatrix}$$

$$4. A = \begin{bmatrix} 7 & 9 & -9 \\ -6 & -2 & -2 \\ 5 & 3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -10 & 2 & 3 \\ -8 & -3 & 0 \\ -1 & 4 & 6 \end{bmatrix}$$

$$5. A = \begin{bmatrix} 0 & -8 & -9 \\ 7 & 4 & -4 \\ -6 & -10 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -9 & -9 \\ -10 & 4 & -5 \\ 2 & -4 & -4 \end{bmatrix}$$

$$6. A = \begin{bmatrix} 8 & 1 & -3 \\ 1 & 5 & -5 \\ -2 & 5 & -10 \end{bmatrix} \text{ and } B = \begin{bmatrix} -10 & 4 & 9 \\ 6 & -3 & -7 \\ 5 & 7 & 7 \end{bmatrix}$$

$$7. A = \begin{bmatrix} 5 & 4 & -6 \\ 0 & -8 & 6 \\ -8 & 8 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -6 & 2 & -4 \\ -3 & -3 & -5 \\ -8 & -3 & -3 \end{bmatrix}$$

$$8. A = \begin{bmatrix} 6 & 5 & -7 \\ 4 & -5 & -8 \\ 1 & 3 & -10 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 2 & -9 \\ 9 & -2 & -10 \\ -9 & 7 & -8 \end{bmatrix}$$

$$9. A = \begin{bmatrix} 0 & -9 & -10 \\ -4 & 3 & 1 \\ -3 & 8 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -9 & 2 & -3 \\ -4 & 9 & 6 \\ 4 & -1 & -4 \end{bmatrix}$$

$$10. A = \begin{bmatrix} -5 & 4 & 7 \\ 9 & -8 & 8 \\ -1 & -6 & -8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 & 3 & -10 \\ -6 & -7 & -3 \\ 2 & -7 & -4 \end{bmatrix}$$

## 1.3. Matrix Properties

### 1.3.1. Properties

For each matrix  $A$ , find:

- $\text{rank}(A)$
- $\text{nullity}(A)$
- $\det(A)$
- $A^{-1}$  (if exists)
- basis of  $\ker(A)$

$$1. A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2.  $A = \begin{bmatrix} 1 & 0 & -1 \\ -5 & 1 & 5 \\ -2 & 0 & 3 \end{bmatrix}$
3.  $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$
4.  $A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 2 & -1 & 8 \end{bmatrix}$
5.  $A = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 1 & 2 \\ 1 & 4 & 5 \end{bmatrix}$
6.  $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 3 \\ 2 & -2 & -1 \end{bmatrix}$
7.  $A = \begin{bmatrix} 5 & 5 & -12 \\ 0 & 1 & 2 \\ -2 & -2 & 5 \end{bmatrix}$
8.  $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$
9.  $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}$
10.  $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 5 & 10 \\ 0 & 2 & 4 \end{bmatrix}$

### 1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix  $A$

1.  $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$
2.  $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & 0 & 0 \end{bmatrix}$
3.  $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$
4.  $A = \begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & 6 \\ 2 & 0 & 3 \end{bmatrix}$
5.  $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
6.  $A = \begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$
7.  $A = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$
8.  $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix}$

$$9. \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$10. \quad A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 2. Solutions

### 2.1. Vector Arithmetic

#### 2.1.1. Addition

$$\begin{bmatrix} -2 \\ -9 \\ -5 \end{bmatrix} + \begin{bmatrix} 0 \\ 16 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 18 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 15 \\ -5 \end{bmatrix} + \begin{bmatrix} 15 \\ 13 \\ -7 \end{bmatrix} = \begin{bmatrix} 18 \\ 28 \\ -12 \end{bmatrix}$$
$$\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -17 \\ 5 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -8 \\ 12 \\ 9 \end{bmatrix} + \begin{bmatrix} 12 \\ -10 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix}$$

#### 2.1.2. Subtraction

$$\begin{bmatrix} 8 \\ 1 \\ 9 \end{bmatrix} - \begin{bmatrix} -4 \\ -7 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ -11 \end{bmatrix} \quad \begin{bmatrix} -6 \\ 11 \\ 11 \end{bmatrix} - \begin{bmatrix} 11 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} -17 \\ 17 \\ 9 \end{bmatrix}$$
$$\begin{bmatrix} 11 \\ 10 \\ 11 \end{bmatrix} - \begin{bmatrix} 6 \\ -8 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ 7 \\ -13 \end{bmatrix} \quad \begin{bmatrix} -2 \\ -15 \\ 13 \end{bmatrix} - \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -10 \\ -16 \\ 9 \end{bmatrix}$$

#### 2.1.3. Scalar Multiplication

$$1: \begin{bmatrix} -21 \\ 12 \\ -18 \end{bmatrix} \quad 2: \begin{bmatrix} 6 \\ 18 \\ 18 \end{bmatrix} \quad 3: \begin{bmatrix} -56 \\ -49 \\ 56 \end{bmatrix} \quad 4: \begin{bmatrix} -2 \\ 8 \\ -20 \end{bmatrix} \quad 5: \begin{bmatrix} 60 \\ 6 \\ -60 \end{bmatrix}$$
$$6: \begin{bmatrix} -42 \\ 6 \\ 30 \end{bmatrix} \quad 7: \begin{bmatrix} 32 \\ 40 \\ 16 \end{bmatrix} \quad 8: \begin{bmatrix} 7 \\ 42 \\ 56 \end{bmatrix} \quad 9: \begin{bmatrix} 50 \\ 20 \\ 15 \end{bmatrix} \quad 10: \begin{bmatrix} 16 \\ 8 \\ 36 \end{bmatrix}$$

### 2.2. Matrix Arithmetic

#### 2.2.1. Addition

$$1: \begin{bmatrix} -9 & -6 & 8 \\ 6 & 11 & -15 \\ 15 & -10 & 4 \end{bmatrix} \quad 2: \begin{bmatrix} -3 & -4 & 10 \\ -2 & 10 & -17 \\ -7 & 1 & -18 \end{bmatrix} \quad 3: \begin{bmatrix} 1 & -15 & 3 \\ -9 & -5 & -2 \\ 0 & -2 & 0 \end{bmatrix} \quad 4: \begin{bmatrix} 14 & 3 & 0 \\ 11 & -16 & -11 \\ -19 & 5 & -10 \end{bmatrix} \quad 5: \begin{bmatrix} -6 & -3 & 3 \\ 15 & 3 & -16 \\ -15 & -15 & 6 \end{bmatrix}$$
$$6: \begin{bmatrix} -1 & 8 & 9 \\ -15 & -7 & 2 \\ -3 & -7 & -9 \end{bmatrix} \quad 7: \begin{bmatrix} 3 & -17 & 3 \\ -2 & 0 & 7 \\ 2 & 6 & -8 \end{bmatrix} \quad 8: \begin{bmatrix} -12 & 14 & 4 \\ 6 & 13 & -5 \\ 2 & -8 & 2 \end{bmatrix} \quad 9: \begin{bmatrix} 3 & -16 & -6 \\ -6 & 1 & -8 \\ -17 & 1 & 1 \end{bmatrix} \quad 10: \begin{bmatrix} -16 & -13 & 9 \\ 14 & -4 & 9 \\ 1 & -6 & -4 \end{bmatrix}$$

#### 2.2.2. Subtraction

$$1: \begin{bmatrix} 1 & -14 & 18 \\ 11 & -2 & -7 \\ -8 & -2 & 16 \end{bmatrix} \quad 2: \begin{bmatrix} 3 & 8 & -6 \\ -10 & 8 & -8 \\ 5 & -13 & -16 \end{bmatrix} \quad 3: \begin{bmatrix} 5 & -10 & 1 \\ -15 & 1 & 3 \\ -15 & -17 & -12 \end{bmatrix} \quad 4: \begin{bmatrix} -3 & 0 & 3 \\ 1 & -4 & 2 \\ -16 & -10 & -6 \end{bmatrix} \quad 5: \begin{bmatrix} 2 & 1 & -15 \\ 3 & 8 & 9 \\ 10 & -3 & 7 \end{bmatrix}$$
$$6: \begin{bmatrix} 13 & 11 & -10 \\ 13 & 12 & 14 \\ -3 & 0 & 9 \end{bmatrix} \quad 7: \begin{bmatrix} -12 & -3 & 9 \\ -4 & -1 & -2 \\ -7 & -12 & -1 \end{bmatrix} \quad 8: \begin{bmatrix} -3 & -15 & 6 \\ 4 & -5 & 4 \\ -14 & 5 & 0 \end{bmatrix} \quad 9: \begin{bmatrix} 15 & 12 & 11 \\ -2 & 1 & 13 \\ 2 & -1 & 11 \end{bmatrix} \quad 10: \begin{bmatrix} 18 & -9 & -1 \\ 6 & -9 & -1 \\ 5 & -18 & -6 \end{bmatrix}$$

#### 2.2.3. Multiplication

$$1: \begin{bmatrix} -93 & 145 & 37 \\ -21 & 87 & 73 \\ 29 & 2 & 30 \end{bmatrix} \quad 2: \begin{bmatrix} 39 & 7 & 3 \\ -26 & 48 & 1 \\ 24 & 50 & 33 \end{bmatrix} \quad 3: \begin{bmatrix} 18 & 22 & -51 \\ -9 & -18 & 72 \\ -9 & 4 & -24 \end{bmatrix} \quad 4: \begin{bmatrix} -133 & -49 & -33 \\ 78 & -14 & -30 \\ -75 & 5 & 21 \end{bmatrix} \quad 5: \begin{bmatrix} 62 & 4 & 76 \\ -20 & -31 & -67 \\ 86 & -6 & 84 \end{bmatrix}$$
$$6: \begin{bmatrix} -89 & 8 & 44 \\ -5 & -46 & -61 \\ 0 & -93 & -123 \end{bmatrix} \quad 7: \begin{bmatrix} 6 & 16 & -22 \\ -24 & 6 & 22 \\ 40 & -34 & -2 \end{bmatrix} \quad 8: \begin{bmatrix} 138 & -47 & -48 \\ 47 & -38 & 78 \\ 122 & -74 & 41 \end{bmatrix} \quad 9: \begin{bmatrix} -4 & -71 & -14 \\ 28 & 18 & 26 \\ -5 & 66 & 57 \end{bmatrix} \quad 10: \begin{bmatrix} -55 & -92 & 10 \\ 145 & 27 & -98 \\ 11 & 95 & 60 \end{bmatrix}$$

### 2.3. Matrix Properties



### 2.3.1. Properties

#### Solution

##### Row Operations:

$$\text{Step 1: } r_1 := r_1 - (-1)r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

##### Results:

- a)  $\text{rank}(A) = 2$
- b)  $\text{nullity}(A) = 1$
- c)  $\det(A) = 0$
- d)  $A^{-1}$  = does not exist
- e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

#### Solution

##### Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-5)r_1 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ -2 & 0 & 3 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_3 := r_3 - (-2)r_1 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_1 := r_1 - (-1)r_3 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right]$$

##### Results:

- a)  $\text{rank}(A) = 3$
- b)  $\text{nullity}(A) = 0$
- c)  $\det(A) = 0$
- d)  $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$
- e)  $\ker(A) = \{\mathbf{0}\}$

#### Solution

##### Row Operations:

$$\text{Step 1: } r_1 := r_1 - r_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

##### Results:

- a)  $\text{rank}(A) = 2$
- b)  $\text{nullity}(A) = 1$
- c)  $\det(A) = 0$
- d)  $A^{-1}$  = does not exist
- e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

### Solution

#### Row Operations:

$$\text{Step 1: } r_3 := r_3 - (2)r_1 \quad \begin{bmatrix} 1 & -1 & 5 & | & 1 & 0 & 0 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 1 & -2 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := r_1 - (-1)r_2 \quad \begin{bmatrix} 1 & 0 & 3 & | & 1 & 1 & 0 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 1 & -2 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_3 := r_3 - r_2 \quad \begin{bmatrix} 1 & 0 & 3 & | & 1 & 1 & 0 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & -2 & -1 & 1 \end{bmatrix}$$

#### Results:

- a)  $\text{rank}(A) = 2$
- b)  $\text{nullity}(A) = 1$
- c)  $\det(A) = 0$
- d)  $A^{-1}$  = does not exist
- e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

### Solution

#### Row Operations:

$$\text{Step 1: } r_3 := r_3 - r_1 \quad \begin{bmatrix} 1 & 5 & 6 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & -1 & -1 & | & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := r_1 - (5)r_2 \quad \begin{bmatrix} 1 & 0 & -4 & | & 1 & -5 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & -1 & -1 & | & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_3 := r_3 - (-1)r_2 \quad \begin{bmatrix} 1 & 0 & -4 & | & 1 & -5 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 1 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_1 := r_1 - (-4)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & -3 & -1 & 4 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 1 & 1 \end{bmatrix}$$

$$\text{Step 5: } r_2 := r_2 - (2)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & -3 & -1 & 4 \\ 0 & 1 & 0 & | & 2 & -1 & -2 \\ 0 & 0 & 1 & | & -1 & 1 & 1 \end{bmatrix}$$

**Results:**

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

**Solution****Row Operations:**

Step 1:  $r_2 := r_2 - (-2)r_1 \quad \begin{bmatrix} 1 & 0 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 2 & 1 & 0 \\ 2 & -2 & -1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2:  $r_3 := r_3 - (2)r_1 \quad \begin{bmatrix} 1 & 0 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 2 & 1 & 0 \\ 0 & -2 & 3 & | & -2 & 0 & 1 \end{bmatrix}$

Step 3:  $r_3 := r_3 - (-2)r_2 \quad \begin{bmatrix} 1 & 0 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 2 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 2 & 1 \end{bmatrix}$

Step 4:  $r_1 := r_1 - (-2)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & 5 & 4 & 2 \\ 0 & 1 & -1 & | & 2 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 2 & 1 \end{bmatrix}$

Step 5:  $r_2 := r_2 - (-1)r_3 \quad \begin{bmatrix} 1 & 0 & 0 & | & 5 & 4 & 2 \\ 0 & 1 & 0 & | & 4 & 3 & 1 \\ 0 & 0 & 1 & | & 2 & 2 & 1 \end{bmatrix}$

**Results:**

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

**Solution****Row Operations:**

Step 1:  $r_1 := 1/5r_1 \quad \begin{bmatrix} 1 & 1 & -12/5 & | & 1/5 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ -2 & -2 & 5 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2:  $r_3 := r_3 - (-2)r_1 \quad \begin{bmatrix} 1 & 1 & -12/5 & | & 1/5 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 1/5 & | & 2/5 & 0 & 1 \end{bmatrix}$

$$\text{Step 3: } r_1 := r_1 - r_2 \begin{bmatrix} 1 & 0 & -22/5 & | & 1/5 & -1 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 1/5 & | & 2/5 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_3 := 5r_3 \begin{bmatrix} 1 & 0 & -22/5 & | & 1/5 & -1 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 0 & 5 \end{bmatrix}$$

$$\text{Step 5: } r_1 := r_1 - (-22/5)r_3 \begin{bmatrix} 1 & 0 & 0 & | & 9 & -1 & 22 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & 0 & 5 \end{bmatrix}$$

$$\text{Step 6: } r_2 := r_2 - (2)r_3 \begin{bmatrix} 1 & 0 & 0 & | & 9 & -1 & 22 \\ 0 & 1 & 0 & | & -4 & 1 & -10 \\ 0 & 0 & 1 & | & 2 & 0 & 5 \end{bmatrix}$$

### Results:

a)  $\text{rank}(A) = 3$

b)  $\text{nullity}(A) = 0$

c)  $\det(A) = 0$

d)  $A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{bmatrix}$

e)  $\ker(A) = \{\mathbf{0}\}$

### Solution

#### Row Operations:

$$\text{Step 1: } r_1 := r_1 - (3)r_2 \begin{bmatrix} 1 & 0 & 6 & | & 1 & -3 & 0 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 1 & -2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_3 := r_3 - r_2 \begin{bmatrix} 1 & 0 & 6 & | & 1 & -3 & 0 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & -1 & 1 \end{bmatrix}$$

### Results:

a)  $\text{rank}(A) = 2$

b)  $\text{nullity}(A) = 1$

c)  $\det(A) = 0$

d)  $A^{-1} = \text{does not exist}$

e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$

### Solution

#### Row Operations:

$$\text{Step 1: } r_1 := -1r_1 \begin{bmatrix} 1 & -1 & -2 & | & -1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & -1 & -2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_3 := r_3 - r_1 \begin{bmatrix} 1 & -1 & -2 & | & -1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 1 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (-1)r_2 \begin{bmatrix} 1 & 0 & -1 & | & -1 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 1 & 0 & 1 \end{bmatrix}$$

**Results:**

a)  $\text{rank}(A) = 2$

b)  $\text{nullity}(A) = 1$

c)  $\det(A) = 0$

d)  $A^{-1}$  = does not exist

e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

**Solution**

**Row Operations:**

$$\text{Step 1: } r_2 := 1/5r_2 \begin{bmatrix} 1 & -2 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & 1/5 & 0 \\ 0 & 2 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := r_1 - (-2)r_2 \begin{bmatrix} 1 & 0 & 4 & | & 1 & 2/5 & 0 \\ 0 & 1 & 2 & | & 0 & 1/5 & 0 \\ 0 & 2 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_3 := r_3 - (2)r_2 \begin{bmatrix} 1 & 0 & 4 & | & 1 & 2/5 & 0 \\ 0 & 1 & 2 & | & 0 & 1/5 & 0 \\ 0 & 0 & 0 & | & 0 & -2/5 & 1 \end{bmatrix}$$

**Results:**

a)  $\text{rank}(A) = 2$

b)  $\text{nullity}(A) = 1$

c)  $\det(A) = 0$

d)  $A^{-1}$  = does not exist

e)  $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

### 2.3.2. RREF

**Solution**

**Elementary Row Operations:**

(1)  $r_3 := r_3 + (-1)r_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 - r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_3 := r_3 - r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 - (4)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_3 := r_3 - (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_2 := r_2 + (-1)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 + (-2)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_2 := r_2 + (-2)r_3$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_1 := r_1 + (-1)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_2 := r_2 - r_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_3 := r_3 + (-1)r_1$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 - r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_2 := r_2 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_2 := r_2 + (-1)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_1 := r_1 + (-2)r_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_2 := r_2 + (-1)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$(2) \ r_2 := r_2 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

**Elementary Row Operations:**

$$(1) \ r_3 := r_3 + (-2)r_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$(2) \ r_3 := r_3 - (-2)r_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Result:**

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$