

Exercise 2:

Foundations of Mathematical, WS24

Zichao Wei

This is **exercise** 2 for Foundations of Mathematical, WS24. Generated on 2024-12-02 with 10 problems per section.

2025-03-24

1. Problems

1.1. Vector Arithmetic

1.1.1. Addition

Find the sum of the following vectors \mathbf{u} and \mathbf{v}

1. $\mathbf{u} = \begin{bmatrix} -5 \\ 7 \\ -10 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 6 \\ -10 \\ 1 \end{bmatrix}$
2. $\mathbf{u} = \begin{bmatrix} 9 \\ 2 \\ -8 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}$
3. $\mathbf{u} = \begin{bmatrix} -8 \\ 9 \\ -6 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ -2 \\ 10 \end{bmatrix}$
4. $\mathbf{u} = \begin{bmatrix} 10 \\ 5 \\ -8 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$
5. $\mathbf{u} = \begin{bmatrix} -2 \\ 10 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$
6. $\mathbf{u} = \begin{bmatrix} -6 \\ -8 \\ -6 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 8 \\ -6 \end{bmatrix}$
7. $\mathbf{u} = \begin{bmatrix} 1 \\ 10 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -8 \\ 2 \\ -1 \end{bmatrix}$
8. $\mathbf{u} = \begin{bmatrix} 5 \\ 9 \\ -3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 6 \\ -5 \\ -2 \end{bmatrix}$
9. $\mathbf{u} = \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 7 \\ -7 \\ -8 \end{bmatrix}$
10. $\mathbf{u} = \begin{bmatrix} 6 \\ 2 \\ -10 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$

1.1.2. Subtraction

Find the difference of the following vectors \mathbf{u} and \mathbf{v}

1. $\mathbf{u} = \begin{bmatrix} 8 \\ -5 \\ 6 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix}$
2. $\mathbf{u} = \begin{bmatrix} -10 \\ -2 \\ -7 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ 10 \\ 4 \end{bmatrix}$
3. $\mathbf{u} = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix}$
4. $\mathbf{u} = \begin{bmatrix} -9 \\ 9 \\ 9 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 8 \\ 3 \\ 8 \end{bmatrix}$
5. $\mathbf{u} = \begin{bmatrix} 6 \\ 4 \\ -4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 0 \\ -9 \end{bmatrix}$

6. $\mathbf{u} = \begin{bmatrix} -7 \\ 5 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 7 \\ -2 \\ -7 \end{bmatrix}$
7. $\mathbf{u} = \begin{bmatrix} -5 \\ 5 \\ -7 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$
8. $\mathbf{u} = \begin{bmatrix} 5 \\ 8 \\ -6 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 9 \\ 4 \\ 3 \end{bmatrix}$
9. $\mathbf{u} = \begin{bmatrix} 5 \\ 8 \\ 9 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 8 \\ -2 \\ -9 \end{bmatrix}$
10. $\mathbf{u} = \begin{bmatrix} 8 \\ 4 \\ -8 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -7 \\ 0 \\ 10 \end{bmatrix}$

1.1.3. Scalar Multiplication

Find the scalar product of the following vector \mathbf{u} and scalar k

1. $\mathbf{u} = \begin{bmatrix} 8 \\ -3 \\ 3 \end{bmatrix} - 2\mathbf{v}.$
2. $\mathbf{u} = \begin{bmatrix} -8 \\ 0 \\ -10 \end{bmatrix} 5\mathbf{v}.$
3. $\mathbf{u} = \begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} 3\mathbf{v}.$
4. $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} 5\mathbf{v}.$
5. $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \\ 5 \end{bmatrix} 0\mathbf{v}.$
6. $\mathbf{u} = \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix} 2\mathbf{v}.$
7. $\mathbf{u} = \begin{bmatrix} -6 \\ 7 \\ 10 \end{bmatrix} 10\mathbf{v}.$
8. $\mathbf{u} = \begin{bmatrix} -4 \\ 10 \\ 1 \end{bmatrix} 1\mathbf{v}.$
9. $\mathbf{u} = \begin{bmatrix} 1 \\ -9 \\ 1 \end{bmatrix} -10\mathbf{v}.$
10. $\mathbf{u} = \begin{bmatrix} -10 \\ -7 \\ 2 \end{bmatrix} 7\mathbf{v}.$

1.2. Matrix Arithmetic

1.2.1. Addition

Find the sum of the following matrices A and B

1. $A = \begin{bmatrix} -6 & -1 & 6 \\ 3 & 9 & 8 \\ 6 & -4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & -9 \\ -7 & -3 & 2 \\ -7 & 1 & -2 \end{bmatrix}$

2. $A = \begin{bmatrix} -6 & 4 & 4 \\ 9 & -6 & -1 \\ -10 & -6 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 7 & -8 \\ 6 & -1 & 6 \\ -9 & 6 & 5 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & -3 & 8 \\ -4 & -6 & 9 \\ 1 & -9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ -8 & -1 & 5 \\ 6 & -5 & -6 \end{bmatrix}$
4. $A = \begin{bmatrix} 0 & -2 & 9 \\ -3 & 4 & 9 \\ -8 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 & -4 \\ -8 & 8 & -10 \\ -5 & -3 & 1 \end{bmatrix}$
5. $A = \begin{bmatrix} 9 & -10 & -5 \\ 3 & -9 & 2 \\ -3 & 6 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -8 & 2 & -2 \\ 5 & -9 & 2 \\ -9 & -1 & 1 \end{bmatrix}$
6. $A = \begin{bmatrix} -10 & 3 & 1 \\ 5 & 6 & 8 \\ 4 & -5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -8 & -2 \\ -10 & 9 & -8 \\ 5 & 0 & 7 \end{bmatrix}$
7. $A = \begin{bmatrix} -3 & -10 & -1 \\ -2 & 1 & -1 \\ -1 & 5 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 & 0 \\ 6 & -10 & 9 \\ -1 & -5 & 8 \end{bmatrix}$
8. $A = \begin{bmatrix} 6 & 0 & 9 \\ -7 & 8 & 9 \\ 2 & 4 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -8 & -7 & -8 \\ 5 & -8 & 3 \\ 8 & 4 & -6 \end{bmatrix}$
9. $A = \begin{bmatrix} 1 & -9 & -9 \\ -3 & -9 & 3 \\ -7 & -10 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 & -5 \\ 0 & -3 & 2 \\ 0 & -1 & -10 \end{bmatrix}$
10. $A = \begin{bmatrix} -5 & 0 & 4 \\ 8 & 8 & -4 \\ 6 & 8 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -6 & -5 \\ -2 & -1 & -7 \\ -7 & 5 & 1 \end{bmatrix}$

1.2.2. Subtraction

Find the difference of the following matrices A and B

1. $A = \begin{bmatrix} -2 & -9 & 5 \\ 6 & 5 & 9 \\ 7 & -5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 7 & -4 \\ 8 & 4 & 5 \\ -9 & 8 & 6 \end{bmatrix}$
2. $A = \begin{bmatrix} -1 & -4 & 9 \\ 1 & 4 & 5 \\ -1 & -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 & -1 \\ 4 & -9 & -3 \\ 8 & -10 & -5 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & 6 & 2 \\ 0 & -4 & -1 \\ 5 & -10 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 5 \\ 5 & 5 & 4 \\ 5 & -4 & -3 \end{bmatrix}$
4. $A = \begin{bmatrix} 7 & -9 & 7 \\ -5 & -8 & -4 \\ -5 & -10 & -9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -10 & 3 \\ 7 & 2 & -3 \\ 8 & -1 & -4 \end{bmatrix}$
5. $A = \begin{bmatrix} 1 & 0 & -7 \\ -1 & -9 & -10 \\ -5 & -3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -10 & -6 & 2 \\ -1 & -2 & 0 \\ -1 & -6 & -8 \end{bmatrix}$
6. $A = \begin{bmatrix} -9 & -4 & -10 \\ -5 & -7 & -3 \\ -2 & -9 & -10 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 4 & -9 \\ -6 & 1 & 7 \\ -5 & -9 & 4 \end{bmatrix}$
7. $A = \begin{bmatrix} 4 & -2 & -9 \\ 9 & -7 & 8 \\ 4 & 3 & -10 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 1 & -5 \\ -10 & 2 & 2 \\ 7 & -1 & 0 \end{bmatrix}$
8. $A = \begin{bmatrix} 2 & 7 & -9 \\ 8 & -5 & 1 \\ 5 & 7 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 8 \\ -10 & -9 & -10 \\ -8 & -9 & 1 \end{bmatrix}$

$$\begin{aligned}
9. \quad A &= \begin{bmatrix} 7 & 1 & -2 \\ 8 & -5 & 1 \\ 2 & -1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -3 & -5 \\ -10 & 9 & 1 \\ 7 & 4 & -6 \end{bmatrix} \\
10. \quad A &= \begin{bmatrix} 0 & -6 & 1 \\ -3 & 6 & 9 \\ -10 & 6 & -7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & -4 \\ -5 & -1 & 6 \\ -4 & -1 & -10 \end{bmatrix}
\end{aligned}$$

1.2.3. Multiplication

Find the product of the following matrices A and B

$$\begin{aligned}
1. \quad A &= \begin{bmatrix} -6 & -6 & -8 \\ 1 & 1 & 0 \\ 3 & 9 & -10 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & -6 & -5 \\ 6 & -8 & 7 \\ -4 & 9 & -2 \end{bmatrix} \\
2. \quad A &= \begin{bmatrix} 4 & 0 & 8 \\ 4 & 0 & -1 \\ -6 & 9 & -7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 6 & -1 \\ 2 & -7 & 3 \\ 5 & 6 & -9 \end{bmatrix} \\
3. \quad A &= \begin{bmatrix} 2 & -5 & -9 \\ -6 & -3 & -5 \\ 9 & -8 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 4 & 3 \\ -9 & -6 & -4 \\ -6 & 5 & -3 \end{bmatrix} \\
4. \quad A &= \begin{bmatrix} 7 & 6 & 8 \\ -9 & -6 & -5 \\ 4 & -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -10 & -3 & 3 \\ -1 & 2 & 4 \\ 1 & 4 & -1 \end{bmatrix} \\
5. \quad A &= \begin{bmatrix} -6 & 0 & -1 \\ -4 & 3 & -2 \\ -9 & -3 & -10 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 5 & 4 \\ 6 & 1 & -6 \\ -10 & 1 & 5 \end{bmatrix} \\
6. \quad A &= \begin{bmatrix} 0 & 9 & -7 \\ -4 & 5 & -5 \\ 7 & -9 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 3 & 2 \\ 6 & -9 & -7 \\ 6 & 0 & 7 \end{bmatrix} \\
7. \quad A &= \begin{bmatrix} 9 & 9 & -7 \\ -5 & 8 & 1 \\ -8 & 9 & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -9 & 8 & 1 \\ 6 & -1 & 1 \\ -10 & 0 & 0 \end{bmatrix} \\
8. \quad A &= \begin{bmatrix} 3 & -3 & -5 \\ 1 & 5 & -6 \\ 2 & 1 & -9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 0 & 8 \\ -1 & -4 & 7 \end{bmatrix} \\
9. \quad A &= \begin{bmatrix} 1 & -9 & -10 \\ 3 & 5 & -7 \\ -9 & -4 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & -4 & -5 \\ -6 & 8 & -6 \\ -3 & 3 & 1 \end{bmatrix} \\
10. \quad A &= \begin{bmatrix} -4 & -6 & -1 \\ -5 & 5 & 7 \\ 2 & 3 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 7 & 9 \\ -4 & 2 & -8 \\ 7 & -7 & -9 \end{bmatrix}
\end{aligned}$$

1.3. Matrix Properties

1.3.1. Properties

For each matrix A , find:

- $\text{rank}(A)$
- $\text{nullity}(A)$
- $\det(A)$
- A^{-1} (if exists)
- basis of $\ker(A)$

$$1. \quad A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}$$

2. $A = \begin{bmatrix} 1 & 3 & 0 \\ 4 & 13 & 4 \\ -6 & -20 & -7 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$
4. $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 1 \\ 1 & 5 & 4 \end{bmatrix}$
5. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 2 \\ -1 & -2 & 2 \end{bmatrix}$
6. $A = \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$
7. $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ -2 & -4 & 5 \end{bmatrix}$
8. $A = \begin{bmatrix} 1 & 0 & -2 \\ -4 & 1 & 9 \\ 2 & 0 & -3 \end{bmatrix}$
9. $A = \begin{bmatrix} -1 & 0 & -4 \\ -1 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$
10. $A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix A

1. $A = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
4. $A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$
5. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$
6. $A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
7. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$
8. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$9. \quad A = \begin{bmatrix} -1 & 2 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$10. \quad A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Solutions

2.1. Vector Arithmetic

2.1.1. Addition

$$\begin{bmatrix} 1 \\ -3 \\ -9 \end{bmatrix} + \begin{bmatrix} 15 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \\ -14 \end{bmatrix}$$
$$\begin{bmatrix} -4 \\ 0 \\ -12 \end{bmatrix} + \begin{bmatrix} -7 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 12 \\ -11 \end{bmatrix}$$

2.1.2. Subtraction

$$\begin{bmatrix} 6 \\ 1 \\ 10 \end{bmatrix} - \begin{bmatrix} -7 \\ -12 \\ -11 \end{bmatrix} = \begin{bmatrix} 13 \\ 13 \\ 21 \end{bmatrix}$$
$$\begin{bmatrix} -14 \\ 7 \\ 5 \end{bmatrix} - \begin{bmatrix} -4 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 9 \end{bmatrix}$$

2.1.3. Scalar Multiplication

$$1: \begin{bmatrix} -16 \\ 6 \\ -6 \end{bmatrix} \quad 2: \begin{bmatrix} -40 \\ 0 \\ -50 \end{bmatrix} \quad 3: \begin{bmatrix} 0 \\ 18 \\ -3 \end{bmatrix} \quad 4: \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} \quad 5: \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$6: \begin{bmatrix} 16 \\ 2 \\ 10 \end{bmatrix} \quad 7: \begin{bmatrix} -60 \\ 70 \\ 100 \end{bmatrix} \quad 8: \begin{bmatrix} -4 \\ 10 \\ 1 \end{bmatrix} \quad 9: \begin{bmatrix} -10 \\ 90 \\ -10 \end{bmatrix} \quad 10: \begin{bmatrix} -70 \\ -49 \\ 14 \end{bmatrix}$$

2.2. Matrix Arithmetic

2.2.1. Addition

$$1: \begin{bmatrix} -1 & -1 & -3 \\ -4 & 6 & 10 \\ -1 & -3 & 1 \end{bmatrix} \quad 2: \begin{bmatrix} -8 & 11 & -4 \\ 15 & -7 & 5 \\ -19 & 0 & 2 \end{bmatrix} \quad 3: \begin{bmatrix} 3 & -4 & 8 \\ -12 & -7 & 14 \\ 7 & -14 & -6 \end{bmatrix} \quad 4: \begin{bmatrix} 4 & -5 & 5 \\ -11 & 12 & -1 \\ -13 & 2 & 2 \end{bmatrix} \quad 5: \begin{bmatrix} 1 & -8 & -7 \\ 8 & -18 & 4 \\ -12 & 5 & -1 \end{bmatrix}$$
$$6: \begin{bmatrix} -9 & -5 & -1 \\ -5 & 15 & 0 \\ 9 & -5 & 11 \end{bmatrix} \quad 7: \begin{bmatrix} -6 & -8 & -1 \\ 4 & -9 & 8 \\ -2 & 0 & 17 \end{bmatrix} \quad 8: \begin{bmatrix} -2 & -7 & 1 \\ -2 & 0 & 12 \\ 10 & 8 & -2 \end{bmatrix} \quad 9: \begin{bmatrix} 0 & -11 & -14 \\ -3 & -12 & 5 \\ -7 & -11 & -10 \end{bmatrix} \quad 10: \begin{bmatrix} -8 & -6 & -1 \\ 6 & 7 & -11 \\ -1 & 13 & 9 \end{bmatrix}$$

2.2.2. Subtraction

$$1: \begin{bmatrix} -1 & -16 & 9 \\ -2 & 1 & 4 \\ 16 & -13 & -6 \end{bmatrix} \quad 2: \begin{bmatrix} -5 & -10 & 10 \\ -3 & 13 & 8 \\ -9 & 7 & 6 \end{bmatrix} \quad 3: \begin{bmatrix} -2 & 7 & -3 \\ -5 & -9 & -5 \\ 0 & -6 & -5 \end{bmatrix} \quad 4: \begin{bmatrix} 4 & 1 & 4 \\ -12 & -10 & -1 \\ -13 & -9 & -5 \end{bmatrix} \quad 5: \begin{bmatrix} 11 & 6 & -9 \\ 0 & -7 & -10 \\ -4 & 3 & 15 \end{bmatrix}$$
$$6: \begin{bmatrix} -4 & -8 & -1 \\ 1 & -8 & -10 \\ 3 & 0 & -14 \end{bmatrix} \quad 7: \begin{bmatrix} -4 & -3 & -4 \\ 19 & -9 & 6 \\ -3 & 4 & -10 \end{bmatrix} \quad 8: \begin{bmatrix} -2 & 2 & -17 \\ 18 & 4 & 11 \\ 13 & 16 & -4 \end{bmatrix} \quad 9: \begin{bmatrix} 8 & 4 & 3 \\ 18 & -14 & 0 \\ -5 & -5 & 6 \end{bmatrix} \quad 10: \begin{bmatrix} 0 & -6 & 5 \\ 2 & 7 & 3 \\ -6 & 7 & 3 \end{bmatrix}$$

2.2.3. Multiplication

$$1: \begin{bmatrix} 8 & 12 & 4 \\ 4 & -14 & 2 \\ 88 & -180 & 68 \end{bmatrix} \quad 2: \begin{bmatrix} 28 & 72 & -76 \\ -17 & 18 & 5 \\ 1 & -141 & 96 \end{bmatrix} \quad 3: \begin{bmatrix} 107 & -7 & 53 \\ 33 & -31 & 9 \\ 60 & 124 & 35 \end{bmatrix} \quad 4: \begin{bmatrix} -68 & 23 & 37 \\ 91 & -5 & -46 \\ -35 & -4 & 1 \end{bmatrix} \quad 5: \begin{bmatrix} 28 & -31 & -29 \\ 50 & -19 & -44 \\ 109 & -58 & -68 \end{bmatrix}$$
$$6: \begin{bmatrix} 12 & -81 & -112 \\ -28 & -57 & -78 \\ -5 & 102 & 77 \end{bmatrix} \quad 7: \begin{bmatrix} 43 & 63 & 18 \\ 83 & -48 & 3 \\ 186 & -73 & 1 \end{bmatrix} \quad 8: \begin{bmatrix} 14 & 11 & -56 \\ 9 & 21 & -1 \\ 15 & 30 & -53 \end{bmatrix} \quad 9: \begin{bmatrix} 81 & -106 & 39 \\ -18 & 7 & -52 \\ 66 & -11 & 64 \end{bmatrix} \quad 10: \begin{bmatrix} 9 & -33 & 21 \\ 19 & -74 & -148 \\ 27 & -15 & -51 \end{bmatrix}$$

2.3. Matrix Properties

2.3.1. Properties

Solution

Row Operations:

$$\text{Step 1: } r_3 := r_3 - (-1)r_1 \begin{bmatrix} 1 & -2 & -3 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & -1 & -1 & | & 1 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_1 := r_1 - (-2)r_2 \begin{bmatrix} 1 & 0 & 1 & | & 1 & 2 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & -1 & -1 & | & 1 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_3 := r_3 - (-1)r_2 \begin{bmatrix} 1 & 0 & 1 & | & 1 & 2 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 1 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_1 := r_1 - r_3 \begin{bmatrix} 1 & 0 & 0 & | & 0 & 1 & -1 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 1 & 1 \end{bmatrix}$$

$$\text{Step 5: } r_2 := r_2 - (2)r_3 \begin{bmatrix} 1 & 0 & 0 & | & 0 & 1 & -1 \\ 0 & 1 & 0 & | & -2 & -1 & -2 \\ 0 & 0 & 1 & | & 1 & 1 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (4)r_1 \begin{bmatrix} 1 & 3 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 4 & | & -4 & 1 & 0 \\ -6 & -20 & -7 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_3 := r_3 - (-6)r_1 \begin{bmatrix} 1 & 3 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 4 & | & -4 & 1 & 0 \\ 0 & -2 & -7 & | & 6 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_1 := r_1 - (3)r_2 \begin{bmatrix} 1 & 0 & -12 & | & 13 & -3 & 0 \\ 0 & 1 & 4 & | & -4 & 1 & 0 \\ 0 & -2 & -7 & | & 6 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_3 := r_3 - (-2)r_2 \begin{bmatrix} 1 & 0 & -12 & | & 13 & -3 & 0 \\ 0 & 1 & 4 & | & -4 & 1 & 0 \\ 0 & 0 & 1 & | & -2 & 2 & 1 \end{bmatrix}$$

$$\text{Step 5: } r_1 := r_1 - (-12)r_3 \begin{bmatrix} 1 & 0 & 0 & | & -11 & 21 & 12 \\ 0 & 1 & 4 & | & -4 & 1 & 0 \\ 0 & 0 & 1 & | & -2 & 2 & 1 \end{bmatrix}$$

$$\text{Step 6: } r_2 := r_2 - (4)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 21 & 12 \\ 0 & 1 & 0 & 4 & -7 & -4 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -3 & -2 \\ -1 & 2 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := r_1 - (2)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} =$ does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_3 := r_3 - r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_1 := r_1 - (4)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -4 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_3 := r_3 - r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -4 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

Step 1: $r_2 := r_2 - r_1$ $\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{array} \right]$

Step 2: $r_3 := r_3 - (-1)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right]$

Step 3: $r_1 := r_1 - (2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -7 & 3 & -2 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right]$

Step 4: $r_1 := r_1 - (-7)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -2 & 7 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right]$

Step 5: $r_2 := r_2 - (3)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -2 & 7 \\ 0 & 1 & 0 & -4 & 1 & -3 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right]$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 4 & -1 & 2 \\ -3 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_1 := r_1 - (4)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -4 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_3 := r_3 - (-2)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_1 := r_1 - (2)r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_1 := r_1 - (-2)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_2 := r_2 - (-4)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4 & 1 & 0 \\ 2 & 0 & -3 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step 2: } r_3 := r_3 - (2)r_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right]$$

$$\text{Step 3: } r_1 := r_1 - (-2)r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 2 \\ 0 & 1 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right]$$

$$\text{Step 4: } r_2 := r_2 - r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 2 \\ 0 & 1 & 0 & 6 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right]$$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := -1r_1 \begin{bmatrix} 1 & 0 & 4 & | & -1 & 0 & 0 \\ -1 & -1 & -2 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_2 := r_2 - (-1)r_1 \begin{bmatrix} 1 & 0 & 4 & | & -1 & 0 & 0 \\ 0 & -1 & 2 & | & -1 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_2 := -1r_2 \begin{bmatrix} 1 & 0 & 4 & | & -1 & 0 & 0 \\ 0 & 1 & -2 & | & 1 & -1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

$$\text{Step 1: } r_1 := -1r_1 \begin{bmatrix} 1 & -1 & -1 & | & -1 & 0 & 0 \\ 2 & -3 & -3 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 2: } r_2 := r_2 - (2)r_1 \begin{bmatrix} 1 & -1 & -1 & | & -1 & 0 & 0 \\ 0 & -1 & -1 & | & 2 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 3: } r_2 := -1r_2 \begin{bmatrix} 1 & -1 & -1 & | & -1 & 0 & 0 \\ 0 & 1 & 1 & | & -2 & -1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Step 4: } r_1 := r_1 - (-1)r_2 \begin{bmatrix} 1 & 0 & 0 & | & -3 & -1 & 0 \\ 0 & 1 & 1 & | & -2 & -1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) A^{-1} = does not exist

e) $\ker(A) = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$

2.3.2. RREF

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 + (-1)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 - (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 - r_2$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_1 := r_1 - (-2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_2 := r_2 + (-2)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 - r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 - (2)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$(2) \ r_3 := r_3 - (2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_1 := r_1 + (-1)r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 - r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$(3) \ r_3 := r_3 - r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_2 := r_2 - (2)r_1$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-2)r_2$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_1 := r_1 - (2)r_2$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 + (-2)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$(2) \ r_3 := r_3 + (-2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$(3) \ r_3 := r_3 + (-2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_3 := r_3 + (-1)r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 - r_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_3 := r_3 - r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_2 := r_2 + (-2)r_1$$

$$\begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_1 := r_1 + (-2)r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_2 := r_2 - r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

$$(1) \ r_1 := r_1 + (-2)r_3$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2) \ r_2 := r_2 + (-1)r_3$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3) \ r_1 := r_1 - r_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$