Exercise 23:

Foundations of Mathematical, WS24

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This is **exercise** 23 for Foundations of Mathematical, WS24. Generated on 2025-04-28 with 10 problems per section.

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1. Problems

1.1. Vector Arithmetic

1.1.1. Addition

Find the sum of the following vectors $\mathbf{u}$ and $\mathbf{v}$

1. $\mathbf{u} = \begin{bmatrix} 6 \\ 10 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -8 \\ -7 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} 4 \\ 7 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} 2 \\ 10 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ -9 \\ 4 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 10 \\ 5 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 8 \\ 5 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} -6 \\ 3 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 10 \\ 4 \\ 10 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} 5 \\ -5 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ -10 \\ -10 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -9 \\ 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ -3 \\ 10 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -9 \\ 8 \\ 1 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} 8 \\ -4 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} 10 \\ 6 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -9 \\ -3 \\ -9 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

1.1.2. Subtraction

Find the difference of the following vectors $\mathbf{u}$ and $\mathbf{v}$

1. $\mathbf{u} = \begin{bmatrix} -9 \\ 5 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} -5 \\ 9 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 9 \\ -8 \\ -5 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} 0 \\ -7 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -8 \\ -5 \\ 6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ 8 \\ -5 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \\ -4 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

6. $\mathbf{u} = \begin{bmatrix} -9 \\ -4 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 9 \\ 4 \\ -10 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} 1 \\ 8 \\ -10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ 9 \\ 1 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -8 \\ 6 \\ -6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} 10 \\ -9 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ 5 \\ -6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

1.1.3. Scalar Multiplication

Find the scalar product of the following vector $\mathbf{u}$ and scalar k

1. $\mathbf{u} = \begin{bmatrix} 9 \\ -10 \\ -8 \end{bmatrix}$ $-6\mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} -3 \\ 1 \\ 7 \end{bmatrix}$ $1\mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ $1\mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 3 \\ -8 \\ 3 \end{bmatrix}$ $3\mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 9 \\ -4 \\ 10 \end{bmatrix}$ $6\mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} 3 \\ -4 \\ 8 \end{bmatrix}$ $5\mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -10 \\ -3 \\ -7 \end{bmatrix}$ $-3\mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -4 \\ -10 \\ -2 \end{bmatrix}$ $3\mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} 9 \\ -3 \\ -9 \end{bmatrix}$ $6\mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} -6 \\ -10 \\ 3 \end{bmatrix}$ $4\mathbf{v}$.

1.2. Matrix Arithmetic

1.2.1. Addition

Find the sum of the following matrices A and B

1.
$$A = \begin{bmatrix} -4 & -2 & -9 \\ -4 & 7 & 7 \\ 3 & -7 & -7 \end{bmatrix} \tag{1}$$

and

$$B = \begin{bmatrix} -1 & -6 & -3 \\ 9 & -7 & -10 \\ 3 & 4 & 1 \end{bmatrix} \tag{2}$$

2.
$$A = \begin{bmatrix} 5 & 3 & 3 \\ 2 & 6 & 8 \\ 1 & -10 & 4 \end{bmatrix} \tag{3}$$

and

$$B = \begin{bmatrix} -9 & 0 & -9 \\ 7 & -1 & 7 \\ -3 & -2 & -3 \end{bmatrix} \tag{4}$$

3.
$$A = \begin{bmatrix} -8 & -10 & 4 \\ -2 & -7 & 9 \\ 8 & 3 & 6 \end{bmatrix} \tag{5}$$

and

$$B = \begin{bmatrix} 4 & 4 & 4 \\ -4 & -1 & 6 \\ 2 & 5 & 1 \end{bmatrix} \tag{6}$$

4.
$$A = \begin{bmatrix} -1 & 5 & -7 \\ -3 & -10 & 1 \\ 3 & 0 & -8 \end{bmatrix} \tag{7}$$

and

$$B = \begin{bmatrix} 8 & 5 & 3 \\ -6 & 5 & 4 \\ 3 & -1 & 7 \end{bmatrix} \tag{8}$$

5.
$$A = \begin{bmatrix} 7 & -5 & -5 \\ -2 & -6 & -7 \\ -1 & -4 & -2 \end{bmatrix} \tag{9}$$

and

$$B = \begin{bmatrix} -1 & -6 & 2 \\ 1 & -6 & 5 \\ -3 & -6 & -4 \end{bmatrix} \tag{10}$$

6.
$$A = \begin{bmatrix} -3 & 3 & -10 \\ -5 & 6 & -9 \\ 5 & -3 & 3 \end{bmatrix} \tag{11}$$

and

$$B = \begin{bmatrix} -7 & 3 & 8 \\ -4 & -7 & 6 \\ -10 & -9 & -2 \end{bmatrix} \tag{12}$$

7.

$$A = \begin{bmatrix} 2 & 8 & 9 \\ 6 & -3 & -2 \\ 8 & 9 & 9 \end{bmatrix} \tag{13}$$

and

$$B = \begin{bmatrix} -10 & -7 & 8 \\ -8 & -1 & 5 \\ 0 & -6 & -7 \end{bmatrix} \tag{14}$$

8.

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 1 & 7 \\ 0 & 7 & -10 \end{bmatrix} \tag{15}$$

and

$$B = \begin{bmatrix} 8 & -7 & -8 \\ 9 & 1 & 1 \\ -6 & -5 & 9 \end{bmatrix} \tag{16}$$

9.

$$A = \begin{bmatrix} -8 & 1 & -3 \\ 4 & -3 & 2 \\ -9 & 0 & 7 \end{bmatrix} \tag{17}$$

and

$$B = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -10 & -7 \\ -10 & 4 & 2 \end{bmatrix} \tag{18}$$

10.

$$A = \begin{bmatrix} 4 & -7 & -4 \\ 8 & -10 & 9 \\ 4 & -7 & -4 \end{bmatrix} \tag{19}$$

and

$$B = \begin{bmatrix} 2 & 9 & 7 \\ 7 & 8 & 8 \\ -3 & -10 & 7 \end{bmatrix} \tag{20}$$

1.2.2. Subtraction

Find the difference of the following matrices A and B

1.

$$A = \begin{bmatrix} 2 & -7 & 0 \\ 0 & 4 & 7 \\ -3 & 6 & -9 \end{bmatrix} \tag{21}$$

and

$$B = \begin{bmatrix} -6 & 6 & -4 \\ 7 & 7 & -1 \\ 5 & -4 & -7 \end{bmatrix} \tag{22}$$

2.

$$A = \begin{bmatrix} 9 & -1 & 4 \\ -3 & 1 & -9 \\ 8 & 8 & 6 \end{bmatrix} \tag{23}$$

and

$$B = \begin{bmatrix} -7 & -7 & -10 \\ -8 & 0 & -1 \\ 0 & 2 & -7 \end{bmatrix} \tag{24}$$

3.

$$A = \begin{bmatrix} 4 & 7 & 2 \\ 3 & -10 & 3 \\ -9 & -1 & -3 \end{bmatrix} \tag{25}$$

and

$$B = \begin{bmatrix} 9 & -8 & 0 \\ 2 & 2 & -7 \\ 5 & -5 & 2 \end{bmatrix} \tag{26}$$

4.

$$A = \begin{bmatrix} -5 & 2 & 5 \\ 1 & -3 & 6 \\ 7 & 4 & -2 \end{bmatrix} \tag{27}$$

and

$$B = \begin{bmatrix} -9 & 5 & -10 \\ 5 & -6 & 9 \\ -7 & 7 & -2 \end{bmatrix} \tag{28}$$

5.

$$A = \begin{bmatrix} -9 & 5 & -10 \\ 6 & 6 & 2 \\ -1 & 2 & 8 \end{bmatrix} \tag{29}$$

and

$$B = \begin{bmatrix} -5 & -4 & 7 \\ 0 & 9 & -5 \\ -4 & -8 & -8 \end{bmatrix} \tag{30}$$

6.

$$A = \begin{bmatrix} 5 & 8 & 9 \\ -2 & -1 & 4 \\ 8 & -7 & -6 \end{bmatrix} \tag{31}$$

and

$$B = \begin{bmatrix} 8 & 8 & 7 \\ -8 & 9 & 9 \\ -1 & 1 & -1 \end{bmatrix} \tag{32}$$

7.

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -8 & -2 & 1 \\ 0 & 5 & -10 \end{bmatrix} \tag{33}$$

and

$$B = \begin{bmatrix} 2 & -9 & -5 \\ 0 & 7 & 2 \\ 2 & -3 & -9 \end{bmatrix} \tag{34}$$

8.

$$A = \begin{bmatrix} 1 & -9 & -5 \\ 1 & 7 & 0 \\ -6 & 5 & -2 \end{bmatrix} \tag{35}$$

and

$$B = \begin{bmatrix} 8 & -3 & -1 \\ 5 & 8 & -9 \\ 8 & 2 & 6 \end{bmatrix} \tag{36}$$

9.

$$A = \begin{bmatrix} 9 & 3 & -7 \\ 0 & -6 & -10 \\ -4 & -10 & -3 \end{bmatrix} \tag{37}$$

and

$$B = \begin{bmatrix} 4 & -5 & 3 \\ 1 & 9 & 1 \\ -10 & -10 & -7 \end{bmatrix} \tag{38}$$

10.

$$A = \begin{bmatrix} 0 & 8 & 4 \\ -4 & 3 & -4 \\ 2 & 3 & -3 \end{bmatrix} \tag{39}$$

and

$$B = \begin{bmatrix} -5 & -3 & 0 \\ -10 & 3 & -7 \\ -10 & -9 & 1 \end{bmatrix} \tag{40}$$

1.2.3. Multiplication

Find the product of the following matrices A and B

1.

$$A = \begin{bmatrix} -3 & 6 & -9 \\ 3 & -7 & 0 \\ -3 & -10 & -6 \end{bmatrix} \tag{41}$$

and

$$B = \begin{bmatrix} 7 & -3 & -3 \\ -6 & 1 & 0 \\ -5 & 0 & 8 \end{bmatrix} \tag{42}$$

2.

$$A = \begin{bmatrix} -3 & 9 & -3 \\ -5 & 5 & 3 \\ -9 & -9 & -9 \end{bmatrix} \tag{43}$$

and

$$B = \begin{bmatrix} -3 & 0 & -5 \\ 7 & -2 & -9 \\ -3 & -6 & -6 \end{bmatrix} \tag{44}$$

3.

$$A = \begin{bmatrix} -6 & 0 & -2 \\ 3 & 4 & 3 \\ -3 & -6 & 8 \end{bmatrix} \tag{45}$$

and

$$B = \begin{bmatrix} -3 & 5 & 0 \\ -6 & 0 & 5 \\ -6 & -6 & -7 \end{bmatrix} \tag{46}$$

4.

$$A = \begin{bmatrix} 7 & 7 & -10 \\ -10 & -8 & 4 \\ 5 & 0 & 2 \end{bmatrix} \tag{47}$$

and

$$B = \begin{bmatrix} 8 & -6 & -6 \\ 0 & -3 & 0 \\ 0 & -7 & 2 \end{bmatrix} \tag{48}$$

5.

$$A = \begin{bmatrix} 5 & 0 & 7 \\ 1 & -8 & 6 \\ -7 & -2 & 2 \end{bmatrix} \tag{49}$$

and

$$B = \begin{bmatrix} 9 & -7 & 0 \\ -10 & 8 & -6 \\ -4 & -1 & 9 \end{bmatrix} \tag{50}$$

6.

$$A = \begin{bmatrix} -4 & 0 & -2 \\ 3 & 7 & -3 \\ -4 & -9 & -4 \end{bmatrix} \tag{51}$$

and

$$B = \begin{bmatrix} 9 & 8 & -9 \\ 3 & 1 & -2 \\ -2 & 3 & 4 \end{bmatrix} \tag{52}$$

7.

$$A = \begin{bmatrix} -10 & 5 & 5 \\ -8 & -10 & -10 \\ -6 & -3 & -2 \end{bmatrix} \tag{53}$$

and

$$B = \begin{bmatrix} -4 & 0 & -3 \\ 1 & -7 & -9 \\ -2 & -4 & 7 \end{bmatrix} \tag{54}$$

8.

$$A = \begin{bmatrix} -10 & 4 & -8 \\ -1 & -2 & -7 \\ -8 & -4 & 5 \end{bmatrix} \tag{55}$$

and

$$B = \begin{bmatrix} 5 & -2 & -5 \\ 8 & 7 & -2 \\ 7 & -9 & 1 \end{bmatrix} \tag{56}$$

9.

$$A = \begin{bmatrix} 4 & 4 & -1 \\ 8 & -4 & 1 \\ 2 & -7 & -5 \end{bmatrix} \tag{57}$$

and

$$B = \begin{bmatrix} -8 & -4 & 4 \\ -5 & 9 & -2 \\ -4 & 1 & -4 \end{bmatrix} \tag{58}$$

10.

$$A = \begin{bmatrix} 8 & -1 & -1 \\ 6 & -1 & 8 \\ 5 & 2 & -9 \end{bmatrix} \tag{59}$$

and

$$B = \begin{bmatrix} -4 & 1 & 8 \\ -3 & 3 & -1 \\ 8 & 8 & 8 \end{bmatrix} \tag{60}$$

1.3. Matrix Properties

1.3.1. Properties

For each matrix A, find:

a) $\operatorname{rank}(A)$

b) nullity(A)

c) det(A)

d) A^{-1} (if exists)

e) basis of ker(A)

1.
$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \tag{61}$$

2.
$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{62}$$

3.
$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \tag{63}$$

4.
$$A = \begin{bmatrix} 3 & -5 & -4 \\ 5 & -8 & -7 \\ -6 & 10 & 8 \end{bmatrix} \tag{64}$$

5.
$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{65}$$

6.
$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{66}$$

7.
$$A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \tag{67}$$

8.
$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 6 \\ 0 & 1 & 7 \end{bmatrix} \tag{68}$$

9.
$$A = \begin{bmatrix} 1 & -1 & 0 \\ 5 & 0 & 8 \\ -2 & 0 & -3 \end{bmatrix} \tag{69}$$

10.
$$A = \begin{bmatrix} 1 & -1 & -2 \\ 4 & -3 & -9 \\ 2 & -1 & -5 \end{bmatrix} \tag{70}$$

1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix A

1.
$$A = \begin{bmatrix} 5 & 10 & 13 \\ 0 & 1 & 1 \\ -2 & -4 & -5 \end{bmatrix} \tag{71}$$

2.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{72}$$

3.
$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{73}$$

4.
$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{74}$$

5.
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} \tag{75}$$

6.
$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{76}$$

7.
$$A = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \tag{77}$$

8.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \tag{78}$$

9.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{79}$$

10.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \tag{80}$$

1.4. Calculus

1.4.1. Limit

Calculate the following limits

1. Calculate the limit of the following expression:

$$\lim_{x \to -1} 4x^2 + 4x - 3 \tag{81}$$

2. Calculate the limit of the following expression:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} \tag{82}$$

3. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{83}$$

4. Calculate the limit of the following expression:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} \tag{84}$$

5. Calculate the limit of the following expression:

$$\lim_{x \to -3} -4x^2 - 3x + 3 \tag{85}$$

6. Calculate the limit of the following expression:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} \tag{86}$$

7. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{87}$$

8. Calculate the limit of the following expression:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} \tag{88}$$

9. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{89}$$

10. Calculate the limit of the following expression:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} \tag{90}$$

1.4.2. Derivative

Calculate the derivatives of the following expressions

1. Calculate the derivative of the following expression:

$$x^4 \tag{91}$$

2. Calculate the derivative of the following expression:

$$e^{x^2+2} \tag{92}$$

3. Calculate the derivative of the following expression:

$$x \log(x) \tag{93}$$

4. Calculate the derivative of the following expression:

$$x \log(x) \tag{94}$$

5. Calculate the derivative of the following expression:

$$\frac{x^2}{x^2 + 1} \tag{95}$$

6. Calculate the derivative of the following expression:

$$\log(x^2 - 1) \tag{96}$$

7. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \tag{97}$$

8. Calculate the derivative of the following expression:

$$x^4 \tag{98}$$

9. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \tag{99}$$

10. Calculate the derivative of the following expression:

$$\frac{x^3}{x^2 + 1} \tag{100}$$

1.4.3. Integral

Calculate the indefinite and definite integrals of the following expressions

1. the indefinite integral and evaluate from 2 to 3:

$$\int e^{\sin(x)} \cos(x) dx \tag{101}$$

2. Evaluate the improper integral:

$$\int_1^{oo} e^{-x} dx \tag{102}$$

3. the indefinite integral and evaluate from 1 to 3:

$$\int \frac{e^x}{x} dx \tag{103}$$

4. the indefinite integral and evaluate from 1 to 2:

$$\int \frac{1}{x \log(x)} dx \tag{104}$$

5. the indefinite integral and evaluate from 2 to 2:

$$\int x \sqrt{x^2 + 1} dx \tag{105}$$

6. the indefinite integral and evaluate from 2 to 4:

$$\int \frac{1}{x \log(x)} dx \tag{106}$$

7. the indefinite integral and evaluate from 2 to 4:

$$\int \frac{1}{\sqrt{1-x^2}} dx \tag{107}$$

8. the indefinite integral and evaluate from 2 to 5:

$$\int x\sqrt{x^2+1} dx \tag{108}$$

9. the indefinite integral and evaluate from 3 to 3:

$$\int x^3 \log(x) dx \tag{109}$$

10. the indefinite integral and evaluate from 2 to 3:

$$\int \sqrt{4-x^2} dx \tag{110}$$

1.4.4. Partial Derivative

Calculate the partial derivatives of the following expressions

1. the partial derivatives of the function:

$$f(x, y) = (x+y)e^{x^2+y^2} \tag{111}$$

   $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

2. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \tag{112}$$

   where $f = f(u, v)$

3. the second order partial derivative of:

$$f(x, y) = x^4 y^3 + 3x^2 y^4 \tag{113}$$

   $\frac{\partial^2 f}{\partial x^2}$

4. Given the implicit function:

$$x^2 y + xy^2 - xy = 0 \tag{114}$$

   $\frac{\partial y}{\partial x}$

5. the partial derivatives of the function:

$$f(x, y) = x^3y^2 - 3x^2y + 2xy^3 \tag{115}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

6. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \tag{116}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

7. the mixed partial derivative of:

$$f(x, y) = x^3y^2 + xy^4 \tag{117}$$

$\frac{\partial^2 f}{\partial x \partial y}$

8. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \tag{118}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

9. the partial derivatives of the function:

$$f(x, y) = (x + y)e^{x^2+y^2} \tag{119}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

10. the partial derivatives of the function:

$$f(x, y) = x^3y^2 - 3x^2y + 2xy^3 \tag{120}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

2. Solutions

2.1. Vector Arithmetic

2.1.1. Addition

$\begin{bmatrix} 7 \\ 2 \\ -2 \end{bmatrix}$ $\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$ $\begin{bmatrix} 6 \\ 13 \\ 13 \end{bmatrix}$ $\begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 11 \\ -15 \\ -14 \end{bmatrix}$ $\begin{bmatrix} -12 \\ -2 \\ 8 \end{bmatrix}$ $\begin{bmatrix} -13 \\ 6 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 14 \\ -3 \\ 6 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 3 \\ -13 \end{bmatrix}$

2.1.2. Subtraction

$\begin{bmatrix} -4 \\ 0 \\ 12 \end{bmatrix}$ $\begin{bmatrix} -14 \\ 17 \\ 8 \end{bmatrix}$ $\begin{bmatrix} 8 \\ -2 \\ -8 \end{bmatrix}$ $\begin{bmatrix} 8 \\ -3 \\ 6 \end{bmatrix}$ $\begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$

$\begin{bmatrix} -18 \\ -8 \\ 17 \end{bmatrix}$ $\begin{bmatrix} 7 \\ -1 \\ -11 \end{bmatrix}$ $\begin{bmatrix} 8 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$ $\begin{bmatrix} 16 \\ -14 \\ 9 \end{bmatrix}$

2.1.3. Scalar Multiplication

1: $\begin{bmatrix} -54 \\ 60 \\ 48 \end{bmatrix}$ 2: $\begin{bmatrix} -3 \\ 1 \\ 7 \end{bmatrix}$ 3: $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ 4: $\begin{bmatrix} 9 \\ -24 \\ 9 \end{bmatrix}$ 5: $\begin{bmatrix} 54 \\ -24 \\ 60 \end{bmatrix}$

6: $\begin{bmatrix} 15 \\ -20 \\ 40 \end{bmatrix}$ 7: $\begin{bmatrix} 30 \\ 9 \\ 21 \end{bmatrix}$ 8: $\begin{bmatrix} -12 \\ -30 \\ -6 \end{bmatrix}$ 9: $\begin{bmatrix} 54 \\ -18 \\ -54 \end{bmatrix}$ 10: $\begin{bmatrix} -24 \\ -40 \\ 12 \end{bmatrix}$

2.2. Matrix Arithmetic

2.2.1. Addition

1:

$$\begin{bmatrix} -5 & -8 & -12 \\ 5 & 0 & -3 \\ 6 & -3 & -6 \end{bmatrix} \tag{121}$$

1:

$$\begin{bmatrix} -4 & 3 & -6 \\ 9 & 5 & 15 \\ -2 & -12 & 1 \end{bmatrix} \tag{122}$$

1:

$$\begin{bmatrix} -4 & -6 & 8 \\ -6 & -8 & 15 \\ 10 & 8 & 7 \end{bmatrix} \tag{123}$$

1:

$$\begin{bmatrix} 7 & 10 & -4 \\ -9 & -5 & 5 \\ 6 & -1 & -1 \end{bmatrix} \tag{124}$$

1:

$$\begin{bmatrix} 6 & -11 & -3 \\ -1 & -12 & -2 \\ -4 & -10 & -6 \end{bmatrix} \tag{125}$$

1:

$$\begin{bmatrix} -10 & 6 & -2 \\ -9 & -1 & -3 \\ -5 & -12 & 1 \end{bmatrix} \tag{126}$$

1:

$$\begin{bmatrix} -8 & 1 & 17 \\ -2 & -4 & 3 \\ 8 & 3 & 2 \end{bmatrix} \tag{127}$$

1:

$$\begin{bmatrix} 9 & -3 & -7 \\ 13 & 2 & 8 \\ -6 & 2 & -1 \end{bmatrix} \tag{128}$$

1:

$$\begin{bmatrix} -10 & -2 & -2 \\ 1 & -13 & -5 \\ -19 & 4 & 9 \end{bmatrix} \tag{129}$$

1:

$$\begin{bmatrix} 6 & 2 & 3 \\ 15 & -2 & 17 \\ 1 & -17 & 3 \end{bmatrix} \tag{130}$$

2.2.2. Subtraction

1:

$$\begin{bmatrix} 8 & -13 & 4 \\ -7 & -3 & 8 \\ -8 & 10 & -2 \end{bmatrix} \tag{131}$$

1:

$$\begin{bmatrix} 16 & 6 & 14 \\ 5 & 1 & -8 \\ 8 & 6 & 13 \end{bmatrix} \tag{132}$$

1:

$$\begin{bmatrix} -5 & 15 & 2 \\ 1 & -12 & 10 \\ -14 & 4 & -5 \end{bmatrix} \tag{133}$$

1:

$$\begin{bmatrix} 4 & -3 & 15 \\ -4 & 3 & -3 \\ 14 & -3 & 0 \end{bmatrix} \tag{134}$$

1:

$$\begin{bmatrix} -4 & 9 & -17 \\ 6 & -3 & 7 \\ 3 & 10 & 16 \end{bmatrix} \tag{135}$$

1:

$$\begin{bmatrix} -3 & 0 & 2 \\ 6 & -10 & -5 \\ 9 & -8 & -5 \end{bmatrix} \tag{136}$$

1:

$$\begin{bmatrix} -4 & 9 & 6 \\ -8 & -9 & -1 \\ -2 & 8 & -1 \end{bmatrix} \tag{137}$$

1:

$$\begin{bmatrix} -7 & -6 & -4 \\ -4 & -1 & 9 \\ -14 & 3 & -8 \end{bmatrix} \tag{138}$$

1:

$$\begin{bmatrix} 5 & 8 & -10 \\ -1 & -15 & -11 \\ 6 & 0 & 4 \end{bmatrix} \tag{139}$$

1:

$$\begin{bmatrix} 5 & 11 & 4 \\ 6 & 0 & 3 \\ 12 & 12 & -4 \end{bmatrix} \tag{140}$$

2.2.3. Multiplication

1:

$$\begin{bmatrix} -12 & 15 & -63 \\ 63 & -16 & -9 \\ 69 & -1 & -39 \end{bmatrix} \tag{141}$$

1:

$$\begin{bmatrix} 81 & 0 & -48 \\ 41 & -28 & -38 \\ -9 & 72 & 180 \end{bmatrix} \tag{142}$$

1:

$$\begin{bmatrix} 30 & -18 & 14 \\ -51 & -3 & -1 \\ -3 & -63 & -86 \end{bmatrix} \tag{143}$$

1:

$$\begin{bmatrix} 56 & 7 & -62 \\ -80 & 56 & 68 \\ 40 & -44 & -26 \end{bmatrix} \tag{144}$$

1:

$$\begin{bmatrix} 17 & -42 & 63 \\ 65 & -77 & 102 \\ -51 & 31 & 30 \end{bmatrix} \tag{145}$$

1:

$$\begin{bmatrix} -32 & -38 & 28 \\ 54 & 22 & -53 \\ -55 & -53 & 38 \end{bmatrix} \tag{146}$$

1:

$$\begin{bmatrix} 35 & -55 & 20 \\ 42 & 110 & 44 \\ 25 & 29 & 31 \end{bmatrix} \tag{147}$$

1:

$$\begin{bmatrix} -74 & 120 & 34 \\ -70 & 51 & 2 \\ -37 & -57 & 53 \end{bmatrix} \tag{148}$$

1:

$$\begin{bmatrix} -48 & 19 & 12 \\ -48 & -67 & 36 \\ 39 & -76 & 42 \end{bmatrix} \tag{149}$$

1:

$$\begin{bmatrix} -37 & -3 & 57 \\ 43 & 67 & 113 \\ -98 & -61 & -34 \end{bmatrix} \quad (150)$$

2.3. Matrix Properties

2.3.1. Properties

Solution

Row Operations:

Step 1: $r_3 := r_3 - r_1$ $\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$

Step 2: $r_1 := r_1 - (-1)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} =$ does not exist

e) $\ker(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

Step 1: $r_2 := r_2 - (2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_1 := r_1 - r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 3 & -1 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} =$ does not exist

e) $\ker(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

Step 1: $r_1 := r_1 - (3)r_2$ $\begin{bmatrix} 1 & 0 & 0 & | & 1 & -3 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & -3 & -3 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_3 := r_3 - (-3)r_2$ $\begin{bmatrix} 1 & 0 & 0 & | & 1 & -3 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 3 & 1 \end{bmatrix}$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} =$ does not exist

e) $\ker(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

Step 1: $r_1 := 1/3r_1$ $\begin{bmatrix} 1 & -5/3 & -4/3 & | & 1/3 & 0 & 0 \\ 5 & -8 & -7 & | & 0 & 1 & 0 \\ -6 & 10 & 8 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_2 := r_2 - (5)r_1$ $\begin{bmatrix} 1 & -5/3 & -4/3 & | & 1/3 & 0 & 0 \\ 0 & 1/3 & -1/3 & | & -5/3 & 1 & 0 \\ -6 & 10 & 8 & | & 0 & 0 & 1 \end{bmatrix}$

Step 3: $r_3 := r_3 - (-6)r_1$ $\begin{bmatrix} 1 & -5/3 & -4/3 & | & 1/3 & 0 & 0 \\ 0 & 1/3 & -1/3 & | & -5/3 & 1 & 0 \\ 0 & 0 & 0 & | & 2 & 0 & 1 \end{bmatrix}$

Step 4: $r_2 := 3r_2$ $\begin{bmatrix} 1 & -5/3 & -4/3 & | & 1/3 & 0 & 0 \\ 0 & 1 & -1 & | & -5 & 3 & 0 \\ 0 & 0 & 0 & | & 2 & 0 & 1 \end{bmatrix}$

Step 5: $r_1 := r_1 - (-5/3)r_2$ $\begin{bmatrix} 1 & 0 & -3 & | & -8 & 5 & 0 \\ 0 & 1 & -1 & | & -5 & 3 & 0 \\ 0 & 0 & 0 & | & 2 & 0 & 1 \end{bmatrix}$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} =$ does not exist

e) $\ker(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

Step 1: $r_1 := r_1 - (3)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_1 := r_1 - (2)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_1 := r_1 - (2)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_2 := r_2 - r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_1 := r_1 - (-2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_3 := r_3 - (2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$

Step 3: $r_1 := r_1 - (3)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 8 & -3 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$

Step 4: $r_2 := r_2 - (2)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 8 & -3 \\ 0 & 1 & 0 & 0 & 5 & -2 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 0

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_1 := r_1 - (2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -10 & 1 & -2 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 \\ 0 & 1 & 7 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_3 := r_3 - r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -10 & 1 & -2 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

Step 3: $r_1 := r_1 - (-10)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -12 & 10 \\ 0 & 1 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

Step 4: $r_2 := r_2 - (6)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -12 & 10 \\ 0 & 1 & 0 & 0 & 7 & -6 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 0

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -4 \\ 0 & -1 & 1 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_2 := r_2 - (5)r_1$ $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 5 & 8 & -5 & 1 & 0 \\ -2 & 0 & -3 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_3 := r_3 - (-2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 5 & 8 & -5 & 1 & 0 \\ 0 & -2 & -3 & 2 & 0 & 1 \end{array}\right]$

Step 3: $r_2 := 1/5 r_2$ $\begin{bmatrix} 1 & -1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 8/5 & | & -1 & 1/5 & 0 \\ 0 & -2 & -3 & | & 2 & 0 & 1 \end{bmatrix}$

Step 4: $r_1 := r_1 - (-1) r_2$ $\begin{bmatrix} 1 & 0 & 8/5 & | & 0 & 1/5 & 0 \\ 0 & 1 & 8/5 & | & -1 & 1/5 & 0 \\ 0 & -2 & -3 & | & 2 & 0 & 1 \end{bmatrix}$

Step 5: $r_3 := r_3 - (-2) r_2$ $\begin{bmatrix} 1 & 0 & 8/5 & | & 0 & 1/5 & 0 \\ 0 & 1 & 8/5 & | & -1 & 1/5 & 0 \\ 0 & 0 & 1/5 & | & 0 & 2/5 & 1 \end{bmatrix}$

Step 6: $r_3 := 5 r_3$ $\begin{bmatrix} 1 & 0 & 8/5 & | & 0 & 1/5 & 0 \\ 0 & 1 & 8/5 & | & -1 & 1/5 & 0 \\ 0 & 0 & 1 & | & 0 & 2 & 5 \end{bmatrix}$

Step 7: $r_1 := r_1 - (8/5) r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 0 & -3 & -8 \\ 0 & 1 & 8/5 & | & -1 & 1/5 & 0 \\ 0 & 0 & 1 & | & 0 & 2 & 5 \end{bmatrix}$

Step 8: $r_2 := r_2 - (8/5) r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 0 & -3 & -8 \\ 0 & 1 & 0 & | & -1 & -3 & -8 \\ 0 & 0 & 1 & | & 0 & 2 & 5 \end{bmatrix}$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 0

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_2 := r_2 - (4) r_1$ $\begin{bmatrix} 1 & -1 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -4 & 1 & 0 \\ 2 & -1 & -5 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_3 := r_3 - (2) r_1$ $\begin{bmatrix} 1 & -1 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -4 & 1 & 0 \\ 0 & 1 & -1 & | & -2 & 0 & 1 \end{bmatrix}$

Step 3: $r_1 := r_1 - (-1) r_2$ $\begin{bmatrix} 1 & 0 & -3 & | & -3 & 1 & 0 \\ 0 & 1 & -1 & | & -4 & 1 & 0 \\ 0 & 1 & -1 & | & -2 & 0 & 1 \end{bmatrix}$

Step 4: $r_3 := r_3 - r_2$ $\begin{bmatrix} 1 & 0 & -3 & | & -3 & 1 & 0 \\ 0 & 1 & -1 & | & -4 & 1 & 0 \\ 0 & 0 & 0 & | & 2 & -1 & 1 \end{bmatrix}$

Results:

a) rank(A) = 2

b) nullity(A) = 1

c) $\det(A) = 0$

d) $A^{-1} = \text{does not exist}$

e) $\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

2.3.2. RREF

Solution

Elementary Row Operations:

(1) $r_1 := r_1 - (2)r_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ -2 & -4 & -5 \end{bmatrix}$$

(2) $r_3 := r_3 - (2)r_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) $r_1 := r_1 + (-2)r_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_3 := r_3 + (-1)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 - (2)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_1 := r_1 - r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 + (-1)r_3$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 - r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_2 := r_2 - r_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_2 := r_2 + (-1)r_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_2 := r_2 + (-1)r_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_3 := r_3 - (2)r_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

(2) $r_3 := r_3 + (-1)r_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_2 := r_2 + (-1)r_3$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 + (-1)r_3$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_1 := r_1 - r_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 + (-1)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(2) $r_3 := r_3 - (2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) $r_2 := r_2 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_2 := r_2 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

(2) $r_2 := r_2 - (2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

(3) $r_3 := r_3 - r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 - (2)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 + (-1)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_1 := r_1 + (-1)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_3 := r_3 - (2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) $r_2 := r_2 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.4. Calculus

2.4.1. Limit

The limit is:

$$-3 \tag{151}$$

The limit is:

$$2 \tag{152}$$

The limit is:

$$e \tag{153}$$

The limit is:

$$2 \tag{154}$$

The limit is:

$$-24 \tag{155}$$

The limit is:

$$2 \tag{156}$$

The limit is:

$$e \tag{157}$$

The limit is:

$$2 \tag{158}$$

The limit is:

$$e \tag{159}$$

The limit is:

$$2 \tag{160}$$

2.4.2. Derivative

The derivative is:

$$4x^3 \tag{161}$$

The derivative is:

$$2xe^{x^2+2} \tag{162}$$

The derivative is:

$$\log(x) + 1 \tag{163}$$

The derivative is:

$$\log(x) + 1 \tag{164}$$

The derivative is:

$$-\frac{2x^3}{\left(x^2+1\right)^2} + \frac{2x}{x^2+1} \tag{165}$$

The derivative is:

$$\frac{2x}{x^2-1} \tag{166}$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \tag{167}$$

The derivative is:

$$4x^3 \tag{168}$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \tag{169}$$

The derivative is:

$$-\frac{2x^4}{\left(x^2+1\right)^2} + \frac{3x^2}{x^2+1} \tag{170}$$

2.4.3. Integral

The indefinite integral is:

$$e^{\sin(x)} \tag{171}$$

Definite integral from 2 to 3:

$$-e^{\sin(2)} + e^{\sin(3)} \tag{172}$$

The improper integral converges to:

$$e^{-1} \tag{173}$$

The indefinite integral is:

$$\operatorname{Ei}\left(x\right) \tag{174}$$

Definite integral from 1 to 3:

$$- \operatorname{Ei}{\left(1 \right)} + \operatorname{Ei}{\left(3 \right)} \tag{175}$$

The indefinite integral is:

$$\log{\left(\log{\left(x \right)} \right)} \tag{176}$$

Definite integral from 1 to 2:

$$\infty \tag{177}$$

The indefinite integral is:

$$\frac{x^{2} \sqrt{x^{2} + 1}}{3} + \frac{\sqrt{x^{2} + 1}}{3} \tag{178}$$

Definite integral from 2 to 2:

$$0 \tag{179}$$

The indefinite integral is:

$$\log{\left(\log{\left(x \right)} \right)} \tag{180}$$

Definite integral from 2 to 4:

$$\log{\left(\log{\left(4 \right)} \right)} - \log{\left(\log{\left(2 \right)} \right)} \tag{181}$$

The indefinite integral is:

$$\operatorname{asin}{\left(x \right)} \tag{182}$$

Definite integral from 2 to 4:

$$\operatorname{asin}{\left(4 \right)} - \operatorname{asin}{\left(2 \right)} \tag{183}$$

The indefinite integral is:

$$\frac{x^{2} \sqrt{x^{2} + 1}}{3} + \frac{\sqrt{x^{2} + 1}}{3} \tag{184}$$

Definite integral from 2 to 5:

$$- \frac{5 \sqrt{5}}{3} + \frac{26 \sqrt{26}}{3} \tag{185}$$

The indefinite integral is:

$$\frac{x^{4} \log{\left(x \right)}}{4} - \frac{x^{4}}{16} \tag{186}$$

Definite integral from 3 to 3:

$$0 \tag{187}$$

The indefinite integral is:

$$\frac{x\sqrt{4-x^2}}{2} + 2\ \mathrm{asin}\left(\frac{x}{2}\right) \tag{188}$$

Definite integral from 2 to 3:

$$-\pi + 2\ \mathrm{asin}\left(\frac{3}{2}\right) + \frac{3\sqrt{5}i}{2} \tag{189}$$

2.4.4. Partial Derivative

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{190}$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{191}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \tag{192}$$

$$\frac{\partial^2 f}{\partial x^2} = 6y^3(2x^2+y) \tag{193}$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \tag{194}$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \tag{195}$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \tag{196}$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{197}$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{198}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y\left(3x^2 + 2y^2\right) \tag{199}$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{200}$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{201}$$

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{202}$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{203}$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \tag{204}$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \tag{205}$$