Exercise 14:

Foundations of Mathematical, WS24

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This is **exercise** 14 for Foundations of Mathematical, WS24. Generated on 2025-02-24 with 10 problems per section.

2025-04-21

1. Problems

1.1. Vector Arithmetic

1.1.1. Addition

Find the sum of the following vectors $\mathbf{u}$ and $\mathbf{v}$

1. $\mathbf{u} = \begin{bmatrix} -4 \\ -9 \\ -10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} 7 \\ -8 \\ -6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} -2 \\ 10 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} -8 \\ -3 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 7 \\ 8 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} -7 \\ -9 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 7 \\ 7 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ -1 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} 8 \\ -9 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 8 \\ -3 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -5 \\ -2 \\ -9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 9 \\ 8 \\ -2 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} -7 \\ -9 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -10 \\ -10 \\ 1 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -8 \\ -3 \end{bmatrix}$ $\mathbf{u} + \mathbf{v}$.

1.1.2. Subtraction

Find the difference of the following vectors $\mathbf{u}$ and $\mathbf{v}$

1. $\mathbf{u} = \begin{bmatrix} -9 \\ 6 \\ 10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -9 \\ -4 \\ 8 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \\ -6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ -3 \\ -3 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} -6 \\ -4 \\ 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -8 \\ 0 \\ 3 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 10 \\ 1 \\ -3 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

6. $\mathbf{u} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -5 \\ 1 \\ 10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ 2 \\ -6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -10 \\ 6 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} 9 \\ 8 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -8 \\ -5 \\ -10 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} -2 \\ -8 \\ -9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix}$ $\mathbf{u} - \mathbf{v}$.

1.1.3. Scalar Multiplication

Find the scalar product of the following vector $\mathbf{u}$ and scalar k

1. $\mathbf{u} = \begin{bmatrix} -10 \\ -4 \\ 5 \end{bmatrix}$ $-2\mathbf{v}$.
2. $\mathbf{u} = \begin{bmatrix} -2 \\ 7 \\ -7 \end{bmatrix}$ $-9\mathbf{v}$.
3. $\mathbf{u} = \begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix}$ $-7\mathbf{v}$.
4. $\mathbf{u} = \begin{bmatrix} 1 \\ 9 \\ 9 \end{bmatrix}$ $0\mathbf{v}$.
5. $\mathbf{u} = \begin{bmatrix} 0 \\ -7 \\ 10 \end{bmatrix}$ $1\mathbf{v}$.
6. $\mathbf{u} = \begin{bmatrix} 9 \\ -7 \\ -3 \end{bmatrix}$ $-5\mathbf{v}$.
7. $\mathbf{u} = \begin{bmatrix} -2 \\ -3 \\ -3 \end{bmatrix}$ $3\mathbf{v}$.
8. $\mathbf{u} = \begin{bmatrix} -3 \\ -7 \\ 0 \end{bmatrix}$ $-5\mathbf{v}$.
9. $\mathbf{u} = \begin{bmatrix} 8 \\ -8 \\ -10 \end{bmatrix}$ $6\mathbf{v}$.
10. $\mathbf{u} = \begin{bmatrix} -2 \\ 9 \\ 1 \end{bmatrix}$ $-2\mathbf{v}$.

1.2. Matrix Arithmetic

1.2.1. Addition

Find the sum of the following matrices A and B

1.

$$A = \begin{bmatrix} -1 & -9 & -8 \\ -8 & 3 & -10 \\ -1 & -3 & -1 \end{bmatrix} \tag{1}$$

and

$$B = \begin{bmatrix} 2 & -5 & 0 \\ 2 & 3 & -8 \\ 5 & -9 & 5 \end{bmatrix} \tag{2}$$

2.

$$A = \begin{bmatrix} 9 & 2 & -3 \\ 9 & 6 & 3 \\ -8 & 7 & -7 \end{bmatrix} \tag{3}$$

and

$$B = \begin{bmatrix} 1 & 2 & -3 \\ -6 & -2 & 5 \\ 8 & -3 & -7 \end{bmatrix} \tag{4}$$

3.

$$A = \begin{bmatrix} 8 & 0 & -2 \\ 4 & 5 & 7 \\ 1 & -6 & 1 \end{bmatrix} \tag{5}$$

and

$$B = \begin{bmatrix} -9 & 6 & -3 \\ 3 & -7 & 5 \\ -5 & -10 & 0 \end{bmatrix} \tag{6}$$

4.

$$A = \begin{bmatrix} -5 & -5 & 2 \\ -6 & -7 & -5 \\ -3 & 6 & -10 \end{bmatrix} \tag{7}$$

and

$$B = \begin{bmatrix} -3 & -4 & -9 \\ -1 & 9 & -2 \\ -6 & -9 & -2 \end{bmatrix} \tag{8}$$

5.

$$A = \begin{bmatrix} -2 & -5 & 4 \\ -5 & 6 & 9 \\ -1 & -6 & -2 \end{bmatrix} \tag{9}$$

and

$$B = \begin{bmatrix} 2 & 1 & -9 \\ -1 & -8 & 5 \\ 9 & 9 & -2 \end{bmatrix} \tag{10}$$

6.

$$A = \begin{bmatrix} -9 & -9 & 3 \\ 4 & -4 & 1 \\ -4 & -6 & 5 \end{bmatrix} \tag{11}$$

and

$$B = \begin{bmatrix} -9 & 9 & -9 \\ -8 & 3 & 4 \\ -10 & -6 & -6 \end{bmatrix} \tag{12}$$

7.

$$A = \begin{bmatrix} 6 & -6 & -3 \\ -1 & -6 & -9 \\ -1 & -10 & -10 \end{bmatrix} \tag{13}$$

and

$$B = \begin{bmatrix} -5 & -1 & -7 \\ -4 & 2 & 4 \\ -3 & -4 & -1 \end{bmatrix} \tag{14}$$

8.

$$A = \begin{bmatrix} -9 & -5 & -3 \\ 8 & 7 & -4 \\ -7 & -1 & -4 \end{bmatrix} \tag{15}$$

and

$$B = \begin{bmatrix} -8 & -6 & -1 \\ -7 & -5 & -8 \\ 1 & 9 & -9 \end{bmatrix} \tag{16}$$

9.

$$A = \begin{bmatrix} 4 & 2 & -10 \\ -1 & -2 & -5 \\ 1 & -10 & 9 \end{bmatrix} \tag{17}$$

and

$$B = \begin{bmatrix} -7 & -8 & 9 \\ -10 & 5 & 2 \\ -6 & 0 & -7 \end{bmatrix} \tag{18}$$

10.

$$A = \begin{bmatrix} 9 & 3 & 2 \\ -10 & 2 & 0 \\ -9 & 4 & 6 \end{bmatrix} \tag{19}$$

and

$$B = \begin{bmatrix} -6 & -5 & 3 \\ -9 & -8 & 3 \\ 5 & -6 & 1 \end{bmatrix} \tag{20}$$

1.2.2. Subtraction

Find the difference of the following matrices A and B

1.
$$A = \begin{bmatrix} 6 & 7 & -1 \\ 5 & -10 & 7 \\ 5 & -4 & -1 \end{bmatrix} \tag{21}$$

and

$$B = \begin{bmatrix} 9 & 0 & -8 \\ -10 & -1 & -9 \\ 7 & -7 & -8 \end{bmatrix} \tag{22}$$

2.
$$A = \begin{bmatrix} -4 & 4 & -10 \\ 7 & -8 & -7 \\ -8 & -7 & -7 \end{bmatrix} \tag{23}$$

and

$$B = \begin{bmatrix} -3 & 0 & 7 \\ -9 & 7 & -8 \\ 6 & 2 & 1 \end{bmatrix} \tag{24}$$

3.
$$A = \begin{bmatrix} -8 & 1 & -10 \\ 8 & 9 & 2 \\ -3 & 1 & -6 \end{bmatrix} \tag{25}$$

and

$$B = \begin{bmatrix} 9 & -7 & 9 \\ 2 & 8 & 5 \\ 3 & 8 & 9 \end{bmatrix} \tag{26}$$

4.
$$A = \begin{bmatrix} -2 & 1 & -9 \\ 6 & -9 & 3 \\ -7 & -9 & -10 \end{bmatrix} \tag{27}$$

and

$$B = \begin{bmatrix} -5 & -7 & -3 \\ 9 & 8 & -9 \\ -1 & 9 & 4 \end{bmatrix} \tag{28}$$

5.
$$A = \begin{bmatrix} -4 & 9 & 5 \\ -1 & -8 & 5 \\ -8 & -8 & 2 \end{bmatrix} \tag{29}$$

and

$$B = \begin{bmatrix} 7 & -3 & -1 \\ -3 & 1 & -2 \\ 6 & -5 & 4 \end{bmatrix} \tag{30}$$

6.
$$A = \begin{bmatrix} -2 & -10 & -2 \\ -6 & 0 & -6 \\ 6 & 0 & 1 \end{bmatrix} \tag{31}$$

and

$$B = \begin{bmatrix} -6 & 3 & 5 \\ -8 & 4 & -10 \\ 1 & 4 & 1 \end{bmatrix} \tag{32}$$

7.

$$A = \begin{bmatrix} 5 & 5 & -4 \\ -6 & -1 & 5 \\ -3 & -1 & 6 \end{bmatrix} \tag{33}$$

and

$$B = \begin{bmatrix} 2 & 8 & 8 \\ -2 & -8 & 9 \\ -7 & -1 & -4 \end{bmatrix} \tag{34}$$

8.

$$A = \begin{bmatrix} -9 & 8 & -10 \\ -10 & -3 & 2 \\ -8 & -9 & -8 \end{bmatrix} \tag{35}$$

and

$$B = \begin{bmatrix} -4 & -9 & 6 \\ 3 & -2 & 9 \\ -3 & -2 & 5 \end{bmatrix} \tag{36}$$

9.

$$A = \begin{bmatrix} 9 & -9 & 1 \\ -8 & 0 & -4 \\ -6 & -4 & -4 \end{bmatrix} \tag{37}$$

and

$$B = \begin{bmatrix} -10 & 7 & -6 \\ -10 & 4 & -1 \\ 4 & 3 & -10 \end{bmatrix} \tag{38}$$

10.

$$A = \begin{bmatrix} 7 & 2 & -5 \\ 7 & -8 & -9 \\ 2 & 6 & -4 \end{bmatrix} \tag{39}$$

and

$$B = \begin{bmatrix} -6 & -8 & 1 \\ 6 & -8 & -6 \\ 9 & -6 & 9 \end{bmatrix} \tag{40}$$

1.2.3. Multiplication

Find the product of the following matrices A and B

1.

$$A = \begin{bmatrix} 4 & -5 & -7 \\ 9 & -5 & -4 \\ 0 & 4 & 0 \end{bmatrix} \tag{41}$$

and

$$B = \begin{bmatrix} 1 & 6 & 1 \\ 3 & 5 & -8 \\ -9 & 0 & -9 \end{bmatrix} \tag{42}$$

2.

$$A = \begin{bmatrix} 8 & -10 & -6 \\ -5 & 4 & 6 \\ 4 & 4 & 1 \end{bmatrix} \tag{43}$$

and

$$B = \begin{bmatrix} 0 & -9 & -1 \\ -9 & -1 & 6 \\ 4 & 0 & 5 \end{bmatrix} \tag{44}$$

3.

$$A = \begin{bmatrix} 2 & 5 & 9 \\ 4 & -5 & -8 \\ 3 & -5 & 3 \end{bmatrix} \tag{45}$$

and

$$B = \begin{bmatrix} 2 & -4 & 5 \\ 7 & 4 & 2 \\ 7 & 7 & 5 \end{bmatrix} \tag{46}$$

4.

$$A = \begin{bmatrix} -1 & -3 & 4 \\ 0 & 2 & 1 \\ -3 & 7 & 5 \end{bmatrix} \tag{47}$$

and

$$B = \begin{bmatrix} -5 & 1 & 3 \\ 4 & 5 & -8 \\ -3 & -3 & 8 \end{bmatrix} \tag{48}$$

5.

$$A = \begin{bmatrix} 9 & 2 & -9 \\ 7 & 3 & 7 \\ 5 & -3 & -7 \end{bmatrix} \tag{49}$$

and

$$B = \begin{bmatrix} 8 & 4 & -9 \\ 0 & -3 & 3 \\ -10 & -1 & -4 \end{bmatrix} \tag{50}$$

6.

$$A = \begin{bmatrix} -5 & -4 & 3 \\ 9 & -9 & -6 \\ -5 & 1 & -5 \end{bmatrix} \tag{51}$$

and

$$B = \begin{bmatrix} -10 & 5 & -7 \\ 5 & 7 & 9 \\ -9 & 2 & -7 \end{bmatrix} \tag{52}$$

7.
$$A = \begin{bmatrix} -4 & 3 & 1 \\ -8 & -10 & 7 \\ -6 & 4 & -3 \end{bmatrix} \tag{53}$$

and

$$B = \begin{bmatrix} -9 & 3 & -1 \\ -2 & 7 & 9 \\ 0 & -1 & 1 \end{bmatrix} \tag{54}$$

8.
$$A = \begin{bmatrix} 7 & 9 & 7 \\ -10 & 0 & 8 \\ 0 & -7 & -2 \end{bmatrix} \tag{55}$$

and

$$B = \begin{bmatrix} 7 & -5 & 2 \\ -4 & 6 & -9 \\ -3 & 6 & 1 \end{bmatrix} \tag{56}$$

9.
$$A = \begin{bmatrix} -2 & 1 & -5 \\ 4 & -10 & -4 \\ -1 & -9 & 1 \end{bmatrix} \tag{57}$$

and

$$B = \begin{bmatrix} -2 & 7 & -5 \\ 3 & 0 & -2 \\ -7 & -8 & -5 \end{bmatrix} \tag{58}$$

10.
$$A = \begin{bmatrix} 2 & 0 & -9 \\ -4 & -10 & 5 \\ 9 & -10 & -8 \end{bmatrix} \tag{59}$$

and

$$B = \begin{bmatrix} 3 & -1 & -4 \\ -1 & 2 & 6 \\ 9 & 1 & -5 \end{bmatrix} \tag{60}$$

1.3. Matrix Properties

1.3.1. Properties

For each matrix A, find:

a) $\mathrm{rank}(A)$

b) nullity(A)

c) det(A)

d) A^{-1} (if exists)

e) basis of ker(A)

1.
$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{61}$$

2.
$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix} \tag{62}$$

3.
$$A = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -1 \\ 1 & -2 & 2 \end{bmatrix} \tag{63}$$

4.
$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 4 \\ 1 & -2 & 1 \end{bmatrix} \tag{64}$$

5.
$$A = \begin{bmatrix} 1 & -5 & -14 \\ -1 & 4 & 11 \\ 0 & 2 & 5 \end{bmatrix} \tag{65}$$

6.
$$A = \begin{bmatrix} -3 & -5 & -1 \\ 0 & 1 & -1 \\ -2 & -3 & -1 \end{bmatrix} \tag{66}$$

7.
$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & -6 \\ -2 & -2 & -5 \end{bmatrix} \tag{67}$$

8.
$$A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{68}$$

9.
$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 2 & 2 \\ -3 & 5 & 5 \end{bmatrix} \tag{69}$$

10.
$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \tag{70}$$

1.3.2. RREF

Find the Reduced Row Echelon Form of the following matrix A

1.
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{71}$$

2.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (72)$$

3.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (73)$$

4.

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad (74)$$

5.

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (75)$$

6.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad (76)$$

7.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (77)$$

8.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (78)$$

9.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (79)$$

10.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 0 \end{bmatrix} \quad (80)$$

1.4. Calculus

1.4.1. Limit

Calculate the following limits

1. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x + 1)}{x} \quad (81)$$

2. Calculate the limit of the following expression:

$$\lim_{x \to -1} 3 - 2x \quad (82)$$

3. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{83}$$

4. Calculate the limit of the following expression:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} \tag{84}$$

5. Calculate the limit of the following expression:

$$\lim_{x \to -3} -4x - 2 \tag{85}$$

6. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x + 1)}{x} \tag{86}$$

7. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x + 1)}{x} \tag{87}$$

8. Calculate the limit of the following expression:

$$\lim_{x \to 1} -4x^2 + x + 2 \tag{88}$$

9. Calculate the limit of the following expression:

$$\lim_{x \to 0} \frac{\log(x + 1)}{x} \tag{89}$$

10. Calculate the limit of the following expression:

$$\lim_{x \to oo} \left(1 + \frac{1}{x}\right)^x \tag{90}$$

1.4.2. Derivative

Calculate the derivatives of the following expressions

1. Calculate the derivative of the following expression:

$$x^2 e^x \tag{91}$$

2. Calculate the derivative of the following expression:

$$x^3 \log(x) \tag{92}$$

3. Calculate the derivative of the following expression:

$$x^3 \log(x) \tag{93}$$

4. Calculate the derivative of the following expression:

$$\log(x + 1) + \log(x^2 + 1) \tag{94}$$

5. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \tag{95}$$

6. Calculate the derivative of the following expression:

$$e^{x} \tag{96}$$

7. Calculate the derivative of the following expression:

$$e^{2x} + e^{x^2} \tag{97}$$

8. Calculate the derivative of the following expression:

$$e^{x^2+1} \tag{98}$$

9. Calculate the derivative of the following expression:

$$\log(x+1) + \log(x^2+1) \tag{99}$$

10. Calculate the derivative of the following expression:

$$x^2 \tag{100}$$

1.4.3. Integral

Calculate the indefinite and definite integrals of the following expressions

1. the indefinite integral and evaluate from 2 to 5:

$$\int 2x^3 - 3x^2 - x - 5dx \tag{101}$$

2. Evaluate the improper integral:

$$\int_1^{oo} e^{-x}dx \tag{102}$$

3. Evaluate the improper integral:

$$\int_1^{oo} e^{-x}dx \tag{103}$$

4. the indefinite integral and evaluate from 2 to 5:

$$\int \frac{3x+2}{x^2-4}dx \tag{104}$$

5. the indefinite integral and evaluate from 1 to 5:

$$\int -3x^3 + 4x^2 - x + 3dx \tag{105}$$

6. the indefinite integral and evaluate from 3 to 5:

$$\int \frac{1}{x^2+1} dx \tag{106}$$

7. Evaluate the improper integral:

$$\int_1^{oo} \frac{1}{x^2} dx \tag{107}$$

8. Evaluate the improper integral:

$$\int_1^{oo} e^{-x} dx \tag{108}$$

9. the indefinite integral and evaluate from 2 to 3:

$$\int \frac{3x+2}{x^2-4} dx \tag{109}$$

10. the indefinite integral and evaluate from 2 to 3:

$$\int \frac{1}{x^2+1} dx \tag{110}$$

1.4.4. Partial Derivative

Calculate the partial derivatives of the following expressions

1. the partial derivatives of the function:

$$f(x,y) = (x+y)e^{x^2+y^2} \tag{111}$$

   $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

2. Given $u = u(x,y)$ and $v = v(x,y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \tag{112}$$

   where $f = f(u,v)$

3. the partial derivatives of the function:

$$f(x,y) = x^3y^2 - 3x^2y + 2xy^3 \tag{113}$$

   $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

4. the partial derivatives of the function:

$$f(x,y) = -\log(xy) + \log(x^3+y^3) \tag{114}$$

   $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

5. Given the implicit function:

$$x^2y + xy^2 - xy = 0 \tag{115}$$

$\frac{\partial y}{\partial x}$

6. the third order partial derivative of:

$$f(x, y) = x^4 y^3 + 3x^2 y^4 \tag{116}$$

$\frac{\partial^3 f}{\partial y^3}$

7. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \tag{117}$$

where $f = f(u, v)$

8. the partial derivatives of the function:

$$f(x, y) = x^3 y^2 - 3x^2 y + 2xy^3 \tag{118}$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

9. Given the implicit function:

$$x^2 y + xy^2 - xy = 0 \tag{119}$$

$\frac{\partial y}{\partial x}$

10. Given $u = u(x, y)$ and $v = v(x, y)$, use the chain rule to find:

$$\frac{\partial f}{\partial x} \tag{120}$$

where $f = f(u, v)$

2. Solutions

2.1. Vector Arithmetic

2.1.1. Addition

$\begin{bmatrix}-6\\-10\\-9\end{bmatrix}$ $\begin{bmatrix}6\\-5\\-5\end{bmatrix}$ $\begin{bmatrix}-6\\15\\8\end{bmatrix}$ $\begin{bmatrix}0\\4\\4\end{bmatrix}$ $\begin{bmatrix}-1\\-2\\8\end{bmatrix}$

$\begin{bmatrix}-5\\4\\2\end{bmatrix}$ $\begin{bmatrix}16\\-1\\2\end{bmatrix}$ $\begin{bmatrix}4\\6\\-11\end{bmatrix}$ $\begin{bmatrix}-17\\-19\\8\end{bmatrix}$ $\begin{bmatrix}0\\-5\\5\end{bmatrix}$

2.1.2. Subtraction

$\begin{bmatrix}0\\10\\2\end{bmatrix}$ $\begin{bmatrix}-4\\4\\6\end{bmatrix}$ $\begin{bmatrix}10\\8\\9\end{bmatrix}$ $\begin{bmatrix}2\\-4\\6\end{bmatrix}$ $\begin{bmatrix}-20\\2\\4\end{bmatrix}$

$\begin{bmatrix}-6\\6\\7\end{bmatrix}$ $\begin{bmatrix}0\\-1\\16\end{bmatrix}$ $\begin{bmatrix}-2\\9\\-2\end{bmatrix}$ $\begin{bmatrix}17\\13\\3\end{bmatrix}$ $\begin{bmatrix}3\\-9\\-14\end{bmatrix}$

2.1.3. Scalar Multiplication

1: $\begin{bmatrix}20\\8\\-10\end{bmatrix}$ 2: $\begin{bmatrix}18\\-63\\63\end{bmatrix}$ 3: $\begin{bmatrix}21\\-7\\-42\end{bmatrix}$ 4: $\begin{bmatrix}0\\0\\0\end{bmatrix}$ 5: $\begin{bmatrix}0\\-7\\10\end{bmatrix}$

6: $\begin{bmatrix}-45\\35\\15\end{bmatrix}$ 7: $\begin{bmatrix}-6\\-9\\-9\end{bmatrix}$ 8: $\begin{bmatrix}15\\35\\0\end{bmatrix}$ 9: $\begin{bmatrix}48\\-48\\-60\end{bmatrix}$ 10: $\begin{bmatrix}4\\-18\\-2\end{bmatrix}$

2.2. Matrix Arithmetic

2.2.1. Addition

1:

$$\begin{bmatrix}1 & -14 & -8\\-6 & 6 & -18\\4 & -12 & 4\end{bmatrix} \tag{121}$$

1:

$$\begin{bmatrix}10 & 4 & -6\\3 & 4 & 8\\0 & 4 & -14\end{bmatrix} \tag{122}$$

1:

$$\begin{bmatrix}-1 & 6 & -5\\7 & -2 & 12\\-4 & -16 & 1\end{bmatrix} \tag{123}$$

1:

$$\begin{bmatrix}-8 & -9 & -7\\-7 & 2 & -7\\-9 & -3 & -12\end{bmatrix} \tag{124}$$

1:

$$\begin{bmatrix} 0 & -4 & -5 \\ -6 & -2 & 14 \\ 8 & 3 & -4 \end{bmatrix} \tag{125}$$

1:

$$\begin{bmatrix} -18 & 0 & -6 \\ -4 & -1 & 5 \\ -14 & -12 & -1 \end{bmatrix} \tag{126}$$

1:

$$\begin{bmatrix} 1 & -7 & -10 \\ -5 & -4 & -5 \\ -4 & -14 & -11 \end{bmatrix} \tag{127}$$

1:

$$\begin{bmatrix} -17 & -11 & -4 \\ 1 & 2 & -12 \\ -6 & 8 & -13 \end{bmatrix} \tag{128}$$

1:

$$\begin{bmatrix} -3 & -6 & -1 \\ -11 & 3 & -3 \\ -5 & -10 & 2 \end{bmatrix} \tag{129}$$

1:

$$\begin{bmatrix} 3 & -2 & 5 \\ -19 & -6 & 3 \\ -4 & -2 & 7 \end{bmatrix} \tag{130}$$

2.2.2. Subtraction

1:

$$\begin{bmatrix} -3 & 7 & 7 \\ 15 & -9 & 16 \\ -2 & 3 & 7 \end{bmatrix} \tag{131}$$

1:

$$\begin{bmatrix} -1 & 4 & -17 \\ 16 & -15 & 1 \\ -14 & -9 & -8 \end{bmatrix} \tag{132}$$

1:

$$\begin{bmatrix} -17 & 8 & -19 \\ 6 & 1 & -3 \\ -6 & -7 & -15 \end{bmatrix} \tag{133}$$

1:

$$\begin{bmatrix} 3 & 8 & -6 \\ -3 & -17 & 12 \\ -6 & -18 & -14 \end{bmatrix} \tag{134}$$

1:

$$\begin{bmatrix} -11 & 12 & 6 \\ 2 & -9 & 7 \\ -14 & -3 & -2 \end{bmatrix} \tag{135}$$

1:

$$\begin{bmatrix} 4 & -13 & -7 \\ 2 & -4 & 4 \\ 5 & -4 & 0 \end{bmatrix} \tag{136}$$

1:

$$\begin{bmatrix} 3 & -3 & -12 \\ -4 & 7 & -4 \\ 4 & 0 & 10 \end{bmatrix} \tag{137}$$

1:

$$\begin{bmatrix} -5 & 17 & -16 \\ -13 & -1 & -7 \\ -5 & -7 & -13 \end{bmatrix} \tag{138}$$

1:

$$\begin{bmatrix} 19 & -16 & 7 \\ 2 & -4 & -3 \\ -10 & -7 & 6 \end{bmatrix} \tag{139}$$

1:

$$\begin{bmatrix} 13 & 10 & -6 \\ 1 & 0 & -3 \\ -7 & 12 & -13 \end{bmatrix} \tag{140}$$

2.2.3. Multiplication

1:

$$\begin{bmatrix} 52 & -1 & 107 \\ 30 & 29 & 85 \\ 12 & 20 & -32 \end{bmatrix} \tag{141}$$

1:

$$\begin{bmatrix} 66 & -62 & -98 \\ -12 & 41 & 59 \\ -32 & -40 & 25 \end{bmatrix} \tag{142}$$

1:

$$\begin{bmatrix} 102 & 75 & 65 \\ -83 & -92 & -30 \\ -8 & -11 & 20 \end{bmatrix} \tag{143}$$

1:

$$\begin{bmatrix} -19 & -28 & 53 \\ 5 & 7 & -8 \\ 28 & 17 & -25 \end{bmatrix} \tag{144}$$

1:

$$\begin{bmatrix} 162 & 39 & -39 \\ -14 & 12 & -82 \\ 110 & 36 & -26 \end{bmatrix} \tag{145}$$

1:

$$\begin{bmatrix} 3 & -47 & -22 \\ -81 & -30 & -102 \\ 100 & -28 & 79 \end{bmatrix} \tag{146}$$

1:

$$\begin{bmatrix} 30 & 8 & 32 \\ 92 & -101 & -75 \\ 46 & 13 & 39 \end{bmatrix} \tag{147}$$

1:

$$\begin{bmatrix} -8 & 61 & -60 \\ -94 & 98 & -12 \\ 34 & -54 & 61 \end{bmatrix} \tag{148}$$

1:

$$\begin{bmatrix} 42 & 26 & 33 \\ -10 & 60 & 20 \\ -32 & -15 & 18 \end{bmatrix} \tag{149}$$

1:

$$\begin{bmatrix} -75 & -11 & 37 \\ 43 & -11 & -69 \\ -35 & -37 & -56 \end{bmatrix} \tag{150}$$

2.3. Matrix Properties

2.3.1. Properties

Solution

Row Operations:

Step 1: $r_1 \leftrightarrow r_2$ $\begin{bmatrix} 1 & 2 & 1 & | & 0 & 1 & 0 \\ 0 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_2 := -1r_2$ $\begin{bmatrix} 1 & 2 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & -1 & | & -1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 3: $r_1 := r_1 - (2)r_2$ $\begin{bmatrix} 1 & 0 & 3 & | & 2 & 1 & 0 \\ 0 & 1 & -1 & | & -1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 4: $r_1 := r_1 - (3)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 2 & 1 & -3 \\ 0 & 1 & -1 & | & -1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Step 5: $r_2 := r_2 - (-1)r_3$ $\begin{bmatrix} 1 & 0 & 0 & | & 2 & 1 & -3 \\ 0 & 1 & 0 & | & -1 & 0 & 1 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 0

d) $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_1 := r_1 - (-2)r_2$ $\begin{bmatrix} 1 & 0 & -2 & | & 1 & 2 & 0 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 2 & -4 & | & 0 & 0 & 1 \end{bmatrix}$

Step 2: $r_3 := r_3 - (2)r_2$ $\begin{bmatrix} 1 & 0 & -2 & | & 1 & 2 & 0 \\ 0 & 1 & -2 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & -2 & 1 \end{bmatrix}$

Results:

a) rank(A) = 2

b) nullity(A) = 1

c) $\det(A) = 0$

d) $A^{-1} = \text{does not exist}$

e) $\ker(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

Solution

Row Operations:

Step 1: $r_3 := r_3 - r_1$ $\left[\begin{array}{ccc|ccc} 1 & -5 & 4 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array}\right]$

Step 2: $r_1 := r_1 - (-5)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 5 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array}\right]$

Step 3: $r_3 := r_3 - (3)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 5 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -3 & 1 \end{array}\right]$

Step 4: $r_1 := r_1 - (-1)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -3 & 1 \end{array}\right]$

Step 5: $r_2 := r_2 - (-1)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 & -3 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_3 := r_3 - r_1$ $\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

Step 2: $r_1 := r_1 - (-2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

Step 3: $r_1 := r_1 - (8)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 2 & -8 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

Step 4: $r_2 := r_2 - (4)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 2 & -8 \\ 0 & 1 & 0 & 4 & 1 & -4 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 0

d) $A^{-1} = \begin{bmatrix} 3 & 0 & -2 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_2 := r_2 - (-1)r_1$ $\left[\begin{array}{ccc|ccc} 1 & -5 & -14 & 1 & 0 & 0 \\ 0 & -1 & -3 & 1 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_2 := -1r_2$ $\left[\begin{array}{ccc|ccc} 1 & -5 & -14 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & -1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

Step 3: $r_1 := r_1 - (-5)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -4 & -5 & 0 \\ 0 & 1 & 3 & -1 & -1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

Step 4: $r_3 := r_3 - (2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -4 & -5 & 0 \\ 0 & 1 & 3 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 2 & 1 \end{array}\right]$

Step 5: $r_3 := -1r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -4 & -5 & 0 \\ 0 & 1 & 3 & -1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{array}\right]$

Step 6: $r_1 := r_1 - r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 3 & -1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{array}\right]$

Step 7: $r_2 := r_2 - (3)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & 5 & 5 & 3 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{array}\right]$

Results:

a) rank(A) = 3

b) nullity(A) = 0

c) det(A) = 0

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix}$

e) ker(A) = $\{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_1 := -1/3r_1$ $\left[\begin{array}{ccc|ccc} 1 & 5/3 & 1/3 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -2 & -3 & -1 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_3 := r_3 - (-2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 5/3 & 1/3 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1/3 & -1/3 & -2/3 & 0 & 1 \end{array}\right]$

Step 3: $r_1 := r_1 - (5/3)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1/3 & -5/3 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1/3 & -1/3 & -2/3 & 0 & 1 \end{array}\right]$

Step 4: $r_3 := r_3 - (1/3)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1/3 & -5/3 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2/3 & -1/3 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} =$ does not exist

e) $\ker(A) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$

Solution

Row Operations:

Step 1: $r_2 := r_2 - (-2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & -4 & 2 & 1 & 0 \\ -2 & -2 & -5 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_3 := r_3 - (-2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & -4 & 2 & 1 & 0 \\ 0 & -2 & -3 & 2 & 0 & 1 \end{array}\right]$

Step 3: $r_2 := -1/3r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4/3 & -2/3 & -1/3 & 0 \\ 0 & -2 & -3 & 2 & 0 & 1 \end{array}\right]$

Step 4: $r_3 := r_3 - (-2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4/3 & -2/3 & -1/3 & 0 \\ 0 & 0 & -1/3 & 2/3 & -2/3 & 1 \end{array}\right]$

Step 5: $r_3 := -3r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4/3 & -2/3 & -1/3 & 0 \\ 0 & 0 & 1 & -2 & 2 & -3 \end{array}\right]$

Step 6: $r_1 := r_1 - r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 3 \\ 0 & 1 & 4/3 & -2/3 & -1/3 & 0 \\ 0 & 0 & 1 & -2 & 2 & -3 \end{array}\right]$

Step 7: $r_2 := r_2 - (4/3)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 3 \\ 0 & 1 & 0 & 2 & -3 & 4 \\ 0 & 0 & 1 & -2 & 2 & -3 \end{array}\right]$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & -2 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

Solution

Row Operations:

Step 1: $r_1 := r_1 - (-2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} = \text{does not exist}$

e) $\ker(A) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$

Solution

Row Operations:

Step 1: $r_1 := -1r_1$ $\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ -3 & 5 & 5 & 0 & 0 & 1 \end{array}\right]$

Step 2: $r_2 := r_2 - (-1)r_1$ $\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ -3 & 5 & 5 & 0 & 0 & 1 \end{array}\right]$

Step 3: $r_3 := r_3 - (-3)r_1$ $\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & -3 & 0 & 1 \end{array}\right]$

Step 4: $r_1 := r_1 - (-1)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & -3 & 0 & 1 \end{array}\right]$

Step 5: $r_3 := r_3 - (2)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 2$

b) $\text{nullity}(A) = 1$

c) $\det(A) = 0$

d) $A^{-1} = \text{does not exist}$

e) $\ker(A) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$

Solution

Row Operations:

Step 1: $r_3 := r_3 - r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

Step 2: $r_1 := r_1 - (2)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

Step 3: $r_2 := r_2 - (-2)r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

Results:

a) $\text{rank}(A) = 3$

b) $\text{nullity}(A) = 0$

c) $\det(A) = 0$

d) $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

e) $\ker(A) = \{\mathbf{0}\}$

2.3.2. RREF

Solution

Elementary Row Operations:

(1) $r_1 := r_1 + (-1)r_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) $r_1 := r_1 + (-2)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_2 := r_2 - r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_2 := r_2 - (2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_1 := r_1 - r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 + (-2)r_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 - r_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 - (2)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

(2) $r_3 := r_3 + (-2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_1 := r_1 - r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_2 := r_2 + (-1)r_3$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_2 := r_2 + (-1)r_3$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_1 := r_1 - r_2$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 - r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(2) $r_3 := r_3 + (-2)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_3 := r_3 + (-1)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_2 := r_2 + (-1)r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_2 := r_2 + (-1)r_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 - (2)r_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_1 := r_1 + (-1)r_3$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 + (-2)r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(3) $r_1 := r_1 - r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Elementary Row Operations:

(1) $r_3 := r_3 - (2)r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(2) $r_1 := r_1 - r_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(3) $r_3 := r_3 + (-1)r_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.4. Calculus

2.4.1. Limit

The limit is:

$$1 \tag{151}$$

The limit is:

$$5 \tag{152}$$

The limit is:

$$e \tag{153}$$

The limit is:

$$2 \tag{154}$$

The limit is:

$$10 \tag{155}$$

The limit is:

$$1 \tag{156}$$

The limit is:

$$1 \tag{157}$$

The limit is:

$$-1 \tag{158}$$

The limit is:

$$1 \tag{159}$$

The limit is:

$$e \tag{160}$$

2.4.2. Derivative

The derivative is:

$$x^2 e^x + 2xe^x \tag{161}$$

The derivative is:

$$3x^2 \log(x) + x^2 \tag{162}$$

The derivative is:

$$3x^2 \log(x) + x^2 \tag{163}$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \tag{164}$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \tag{165}$$

The derivative is:

$$e^x \tag{166}$$

The derivative is:

$$2xe^{x^2} + 2e^{2x} \tag{167}$$

The derivative is:

$$2xe^{x^2+1} \tag{168}$$

The derivative is:

$$\frac{2x}{x^2 + 1} + \frac{1}{x + 1} \tag{169}$$

The derivative is:

$$2x \tag{170}$$

2.4.3. Integral

The indefinite integral is:

$$\frac{x^4}{2} - x^3 - \frac{x^2}{2} - 5x \tag{171}$$

Definite integral from 2 to 5:

$$162 \tag{172}$$

The improper integral converges to:

$$e^{-1} \tag{173}$$

The improper integral converges to:

$$e^{-1} \tag{174}$$

The indefinite integral is:

$$2\log(x-2)+\log(x+2) \tag{175}$$

Definite integral from 2 to 5:

$$\infty \tag{176}$$

The indefinite integral is:

$$-\frac{3x^4}{4}+\frac{4x^3}{3}-\frac{x^2}{2}+3x \tag{177}$$

Definite integral from 1 to 5:

$$-\frac{908}{3} \tag{178}$$

The indefinite integral is:

$$\operatorname{atan}\,(x) \tag{179}$$

Definite integral from 3 to 5:

$$-\operatorname{atan}\,(3)+\operatorname{atan}\,(5) \tag{180}$$

The improper integral converges to:

$$1 \tag{181}$$

The improper integral converges to:

$$e^{-1} \tag{182}$$

The indefinite integral is:

$$2\log(x-2)+\log(x+2) \tag{183}$$

Definite integral from 2 to 3:

$$\infty \tag{184}$$

The indefinite integral is:

$$\operatorname{atan}(x) \tag{185}$$

Definite integral from 2 to 3:

$$-\operatorname{atan}(2) + \operatorname{atan}(3) \tag{186}$$

2.4.4. Partial Derivative

$$\frac{\partial f}{\partial x} = 2x(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{187}$$

$$\frac{\partial f}{\partial y} = 2y(x+y)e^{x^2+y^2} + e^{x^2+y^2} \tag{188}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \tag{189}$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \tag{190}$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \tag{191}$$

$$\frac{\partial f}{\partial x} = \frac{3x^2}{x^3+y^3} - \frac{1}{x} \tag{192}$$

$$\frac{\partial f}{\partial y} = \frac{3y^2}{x^3+y^3} - \frac{1}{y} \tag{193}$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \tag{194}$$

$$\frac{\partial^3 f}{\partial y^3} = 6x^2(x^2 + 12y) \tag{195}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \tag{196}$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 6xy + 2y^3 \tag{197}$$

$$\frac{\partial f}{\partial y} = 2x^3y - 3x^2 + 6xy^2 \tag{198}$$

$$\frac{\partial y}{\partial x} = \frac{-2xy - y^2 + y}{x^2 + 2xy - x} \tag{199}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \tag{200}$$